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Characters and q-series in $\mathbb{Q}(\sqrt{2})$

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Abstract

In 1988, G. Andrews, F. Dyson, and D. Hickerson related the arithmetic of $\mathbb{Q}(\sqrt{6})$ to certain q-series. We have found q-series that relate in a similar way to $\mathbb{Q}(\sqrt{2})$. In addition to proving analogous results, including an explicit formula for a partition function, we also obtain a generating function for the values of a particular L-function.

MSC: 11P81; 11M41

1. Introduction and statement of results

In [3], Andrews et al., studied the relationship between the arithmetic of $\mathbb{Q}(\sqrt{6})$ and certain partition functions. This connection allowed them to prove new results about combinatorial objects by taking a non-combinatorial perspective. They were interested in the following q-series:

$$R(q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n(n+1)/2}}{(1+q)(1+q^2)\cdots(1+q^n)} = 1 + q - q^2 + 2q^3 - 2q^4 + \cdots .$$

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$$L(q) = 2\sum_{n=1}^{\infty} \frac{(-1)^n q^{n^2}}{(1-q)(1-q^3)\cdots(1-q^{2n-1})} = -2q - 2q^2 - 2q^3 + 2q^7 + \cdots .$$

They showed that the coefficients of $R(q)$ and $L(q)$ are determined by the coefficients of a certain Hecke L-function associated with the quadratic field $\mathbb{Q}(\sqrt{6})$. Using the arithmetic of $\mathbb{Q}(\sqrt{6})$, the combinatorics of q-series, and basic hypergeometric series, they proved a number of results about the coefficients of

$$qR(q^{24}) - \frac{1}{q}L(q^{24}),$$

including multiplicativity and lacunarity. They also showed that the coefficients attain every integer infinitely often. Examples of q-series with these properties are rare and surprising. In the words of Dyson [6],

> This pair of functions $R(q)$ and $L(q)$ is today an isolated curiosity. But I am convinced that, like so many other beautiful things in Ramanujan's garden, it will turn out to be a special case of a broader mathematical structure. There probably exist other sets of two or more functions with coefficients related by cross-multiplicativity, satisfying identities similar to those which Ramanujan discovered for his $R(q)$. I have a hunch that such sets of cross-multiplicative functions will form a structure within which the mock theta-functions will also find a place. But this hunch is not backed up by any solid evidence. I leave it to the ladies and gentlemen of the audience to find the connections if they exist.

In this paper we find q-series analogous to $R(q)$ and $L(q)$, associated in a similar way to $\mathbb{Q}(\sqrt{2})$. We relate a sum of these basic hypergeometric series with a Hecke L-function, using the machinery of Bailey pairs. We prove analogous combinatorial results to those in [3]; using the arithmetic of $\mathbb{Q}(\sqrt{2})$, we establish combinatorial properties of a certain partition function. In addition, we find a generating function for values of the associated L-function.

Throughout the paper we employ the standard notation

$$(a)_n := (a;q)_n := \prod_{k=0}^{n-1} (1 - aq^k).$$

Let $O_K = \mathbb{Z}[\sqrt{2}]$ be the ring of integers of $K = \mathbb{Q}(\sqrt{2})$. In O_K define the norm of any ideal $\mathfrak{a} = (x + y\sqrt{2})$ as $N(\mathfrak{a}) := |x^2 - 2y^2|$.

Define the q-series $W_1(q)$ and $W_2(q)$ as

$$W_1(q) := \sum_{n\geqslant 0} \frac{(q)_n (-1)^n q^{\binom{n+1}{2}}}{(-q)_n} = 1 - q + 2q^2 - q^3 - 2q^5 + 3q^6 + \cdots, \tag{1.1}$$

$$W_2(q) := \sum_{n\geqslant 1} \frac{(-1;q^2)_n(-q)^n}{(q;q^2)_n} = -2q - 2q^3 + 2q^4 + 2q^6 + 2q^8 - 2q^9 + \cdots . \quad (1.2)$$

Let χ be the character

$$\chi(\mathfrak{a}) := \begin{cases} 1 & N(\mathfrak{a}) \equiv \pm 1 \bmod 16, \\ -1 & N(\mathfrak{a}) \equiv \pm 7 \bmod 16, \\ 0 & \text{otherwise} \end{cases} \quad (1.3)$$

and define $a(n)$ for any positive integer n by

$$a(n) := \sum_{\substack{\mathfrak{a}\subset O_K \\ N(\mathfrak{a})=n}} \chi(\mathfrak{a}). \quad (1.4)$$

Theorem 1.1. *We have*

$$qW_1(q^8) + \frac{1}{q} W_2(q^8) = \sum_{n\geqslant 0} a(n)q^n. \quad (1.5)$$

Remark. The $a(n)$'s are constructed such that the following holds ($\Re(s)>1$):

$$L(\chi, s) := \sum_{\mathfrak{a}\subset O_K} \frac{\chi(\mathfrak{a})}{N(\mathfrak{a})^s} = \sum_{n\geqslant 1} \frac{a(n)}{n^s}. \quad (1.6)$$

In particular, $L(\chi, s)$ is a standard Hecke L-function which is well known to have an analytic continuation to $\mathbb{C}$ [2].

Corollary 1.2. *The following identity is true*:

$$qW_1(-q^8) + \frac{1}{q} W_2(-q^8) = \sum_{\substack{n\geqslant 1 \\ n \text{ odd}}} b(n)q^n,$$

where the $b(n)$'*s are defined by*

$$b(n) := \sum_{\substack{n \text{ odd} \\ \mathfrak{a}\subset O_K \\ N(\mathfrak{a})=n}} 1.$$

Remark. The $b(n)$'s are constructed such that the following holds ($\Re(s)>1$):

$$\zeta_K^*(s) := \sum_{\substack{\mathfrak{a}\subset O_K \\ N(\mathfrak{a}) \text{ odd}}} \frac{1}{N(\mathfrak{a})^s} = \sum_{\substack{n\geqslant 1 \\ n \text{ odd}}} \frac{b(n)}{n^s}.$$

Notice $\zeta_K^*(s)$ is essentially the usual Dedekind ζ-function, but the only difference is the omission of the Euler factor corresponding to the prime ideal above 2. Here $\zeta_K^*(s)$ has an analytic continuation to $\mathbb{C}$ with the exception of a simple pole at $s=1$ (for example, see [8]).

Consider the q-series identity in (1.5) with $q=e^{-t}$. This gives a well-defined t-series, since the substitution of $e^{-t}=\sum_{n=0}^{\infty}\frac{(-t)^n}{n!}$ into (1.1) amounts to performing formal operations (addition, multiplication, and taking positive integral powers) of power series.

Theorem 1.3. *The following is a generating function for L-values.*

$$e^{-t}W_1(e^{-8t})-e^{t}\sum_{n\geqslant 0}\frac{(e^{-8t};e^{-16t})_n}{(-e^{-16t};e^{-16t})_n}=\sum_{n\geqslant 0}L(\chi,-n)\frac{(-1)^{n+1}t^n}{n!}$$
$$=-10t-\frac{7949}{3}t^3-\frac{26765521}{12}t^5-\cdots .$$

Theorem 1.1 is proven in two steps. In Section 2, using the theory of Bailey pairs, we find alternate expressions for $W_1(q)$ and $W_2(q)$, and in Section 3 we prove the theorem by revealing the connection to $\mathbb{Q}(\sqrt{2})$ of these other representations. In Section 4 we prove Corollary 1.2. In Section 5 we find an explicit formula for the coefficients of our q-series, and provide combinatorial results. In Section 6 we establish the generating function for L-values.

2. Hecke identities

Here, we employ the theory of Bailey pairs to obtain alternate q-series expressions for $W_1(q)$ and $W_2(q)$.

Definition 2.1. Two sequences α_n and β_n, form a Bailey pair relative to a if for all $n\geqslant 0$

$$\beta_n=\sum_{r=0}^{n}\frac{\alpha_r}{(q)_{n-r}(aq)_{n+r}}.$$

Theorem 2.2 (Bailey's Lemma). *If α_n and β_n form a Bailey pair relative to a, then*

$$\sum_{n\geqslant 0}\frac{(\rho_1)_n(\rho_2)_n(aq/\rho_1\rho_2)^n}{(aq/\rho_1)_n(aq/\rho_2)_n}\alpha_n=\frac{(aq)_\infty(aq/\rho_1\rho_2)_\infty}{(aq/\rho_1)_\infty(aq/\rho_2)_\infty}\sum_{n\geqslant 0}(\rho_1)_n(\rho_2)_n(aq/\rho_1\rho_2)^n\beta_n,$$

provided that both sums converge absolutely.

A proof can be found in [1].

Theorem 2.3. *The following identity is true*:

$$W_1(q) = \sum_{\substack{n\geqslant 0\\ |j|\leqslant n}} (-1)^{n+j} q^{2n^2+n-j^2}(1-q^{2n+1}). \tag{2.1}$$

Proof. Recall that

$$W_1(q) := \sum_{n\geqslant 0} \frac{(q)_n(-1)^n q^{\binom{n+1}{2}}}{(-q)_n}.$$

In Bailey's Lemma, let $\rho_1 \to \infty$, $\rho_2 = q$ and $a = q$. Note that when $\rho_1 \to \infty$ then $(\rho_1)_n\left(\frac{1}{\rho_1}\right)^n \to (-1)^n q^{\binom{n}{2}}$. This yields

$$\sum_{n\geqslant 0} (-1)^n q^{\binom{n+1}{2}} \alpha_n = \frac{1}{1-q} \sum_{n\geqslant 0} (-1)^n q^{\binom{n}{2}} (q)_n q^n \beta_n. \tag{2.2}$$

By [4], the following form a Bailey pair relative to $a = q$:

$$\alpha_n = \frac{q^{(3n^2+n)/2}(1-q^{2n+1})}{1-q} \sum_{j=-n}^{n} (-1)^j q^{-j^2} \quad \text{and} \quad \beta_n = \frac{1}{(-q)_n}.$$

Substitution into (2.2) gives the result. □

Theorem 2.4. *The following identity is true*:

$$W_2(q) = \sum_{\substack{n\geqslant 1\\ -n\leqslant j\leqslant n-1}} (-1)^n q^{n(2n-1)-(j^2-j)}(1+q^{2n}). \tag{2.3}$$

Proof. Recall that

$$W_2(q) := \sum_{n\geqslant 1} \frac{(-1;q^2)_n(-q)^n}{(q;q^2)_n}.$$

Make the substitution $q \to \sqrt{q}$ and shift the sums via $n \to n+1$. The left-hand side becomes

$$\sum_{n\geqslant 0} \frac{(-1)_{n+1}(-\sqrt{q})^{n+1}}{(\sqrt{q})_{n+1}} = \frac{-2\sqrt{q}}{1-\sqrt{q}} \sum_{n\geqslant 0} \frac{(-q)_n(-\sqrt{q})^n}{(q^{3/2})_n}.$$

The right-hand side becomes

$$-\sum_{n\geqslant 0} (-1)^n q^{(2n^2+3n+1)/2}(1+q^{n+1})\left(\sum_{j=0}^{n} q^{-j(j+1)/2} + \sum_{j=-n-1}^{-1} q^{-j(j+1)/2}\right).$$

Flip the last sum by taking $i = -(j+1)$ to get

$$-\sum_{n\geqslant 0}(-1)^n q^{(2n^2+3n+1)/2}(1+q^{n+1})\left(\sum_{j=0}^{n} q^{-j(j+1)/2} + \sum_{i=0}^{n} q^{-i(i+1)/2}\right),$$

and then combine sums

$$-\sum_{n\geqslant 0}(-1)^n q^{(2n^2+3n+1)/2}(1+q^{n+1})\left(2\sum_{j=0}^{n} q^{-j(j+1)/2}\right).$$

It remains to show

$$-2\sqrt{q}\sum_{n\geqslant 0}(-1)^n q^{n^2+3n/2}(1+q^{n+1})\left(\sum_{j=0}^{n} q^{-j(j+1)/2}\right) = \frac{-2\sqrt{q}}{1-\sqrt{q}}\sum_{n\geqslant 0}\frac{(-q)_n(-\sqrt{q})^n}{(q^{3/2})_n}.$$

The following is a Bailey pair relative to $a = q^2$:

$$\alpha_n = \frac{q^{n^2+n}(1-q^{2n+2})}{(1-q^2)}\sum_{j=0}^{n} q^{-j(j+1)/2} \quad \text{and} \quad \beta_n = \frac{(-q)_n}{(q)_n(-q^{3/2})_n(q^{3/2})_n},$$

as can be seen by taking $b = -q^{1/2}$ and $c = q^{1/2}$ in Theorem 2.2 in [4]. Apply Bailey's lemma to this pair, choosing $\rho_1 = -q^{3/2}$ and $\rho_2 = q$, to obtain

$$\frac{1}{(1+q)}\sum_{n\geqslant 0}(-1)^n q^{n^2+3n/2}(1+q^{n+1})\sum_{j=0}^{n} q^{-j(j+1)/2} = \frac{(1+\sqrt{q})}{(1-q^2)}\sum_{n\geqslant 0}\frac{(-\sqrt{q})^n(-q)_n}{(q^{3/2})_n}.$$

Multiplying both sides by $-2\sqrt{q}(1+q)$ and simplifying yields the identity. $\square$

3. Proof of Theorem 1.1

Theorem 1.1 will follow from (2.1) and (2.3) once we know that the only ideals $\mathfrak{a}$ with $\chi(\mathfrak{a}) \neq 0$ have $N(\mathfrak{a}) \equiv \pm 1 \bmod 8$. The following lemma establishes that.

Lemma 3.1. *There are no ideals of norm* $\pm 3 \bmod 8$ *in* O_K.

Proof. Consider any ideal $\mathfrak{a} = \left(x + y\sqrt{2}\right)$ with $x^2 - 2y^2 = 8n + 3$ for some $n \in \mathbb{Z}$. Look mod 2 to see x must be odd, $x = 2k+1$. Then $4k^2 + 4k + 1 - 2y^2 = 8n + 3$, so $2k^2 + 2k - y^2 = 4n + 1$. Looking mod 2 again shows y must also be odd, $y = 2m + 1$. Then $2k^2 + 2k - 4m^2 - 4m - 1 = 4n + 1$, so $k(k+1) - 2m^2 - 2m = 2n + 1$. If we look mod 2 again, we have that $k(k+1)$ is odd. But that is impossible. The proof for $N(\mathfrak{a}) = -3 \bmod 8$ is similar. $\square$

The next two theorems complete the proof of Theorem 1.1.

Theorem 3.2. *The following identity is true*:

$$qW_1(q^8) = \sum_{\substack{n\geqslant 0 \\ n\equiv 1 \bmod 8}} a(n)q^n. \tag{3.1}$$

Proof. The fundamental solution of $x^2 - 2y^2 = 1$ (the solution with x and y minimal positive) is (3,2). From [4, Lemma 3, p. 396], we know that we choose a unique representative of each ideal $\mathfrak{a} = (x + y\sqrt{2})$ in O_K by restricting $x\geqslant 0$ and $-\frac{2}{3+1}x < y \leqslant \frac{2}{3+1}x$.

Suppose $x^2 - 2y^2 = 8m + 1$. Looking mod 2, we see x is odd. Write $x = 2k + 1$. The inequalities become $k\geqslant 0$ and $|y|\leqslant k$. Note that since $N(\mathfrak{a}) \equiv 1 \bmod 8$, from (1.3) we can say $\chi(\mathfrak{a}) = (-1)^{\frac{N(\mathfrak{a})-1}{8}}$. This gives the following:

$$\sum_{\substack{n\geqslant 0 \\ n\equiv 1 \bmod 8}} a(n)q^n = \sum_{\substack{k\geqslant 0 \\ |y|\leqslant k}} (-1)^{\frac{k^2+k}{2}-\frac{y^2}{4}} q^{(2k+1)^2-2y^2}.$$

Now we split into two sums, corresponding to the cases $k = 2n + 1$ and $2n$. Since y must always be even, take $y = 2j$.

$$\sum_{\substack{n\geqslant 0 \\ |j|\leqslant n}} (-1)^{n+j+1} q^{(4n+3)^2-8j^2} + \sum_{\substack{n\geqslant 0 \\ |j|\leqslant n}} (-1)^{n+j} q^{(4n+1)^2-8j^2}.$$

Combining these two sums we get the result:

$$\sum_{\substack{n\geqslant 0 \\ |j|\leqslant n}} (-1)^{n+j} q^{(4n+1)^2-8j^2}(1 - q^{8(2n+1)}). \qquad \square$$

Theorem 3.3. *The following identity is true*:

$$\frac{1}{q}W_2(q^8) = \sum_{\substack{n\geqslant 0 \\ n\equiv -1 \bmod 8}} a(n)q^n. \tag{3.2}$$

Proof. Suppose $x^2 - 2y^2 = 8m - 1$. From (1.3), $\chi(\mathfrak{a}) = (-1)^{\frac{N(\mathfrak{a})+1}{8}}$. Again, x must be odd, $x = 2k + 1$, and now y is also odd, $y = 2j + 1$. To ensure a unique representative of each ideal, we use the inequalities above, $k\geqslant 0$ and $|y|\leqslant k$. Consider the two sums, $k = 2n + 1$ and $k = 2n$.

$$\sum_{\substack{n\geqslant 0 \\ n\equiv -1 \bmod 8}} a(n)q^n = \sum_{\substack{n\geqslant 0 \\ -n-1\leqslant j\leqslant n}} (-1)^{n+1} q^{(4n+3)^2-2(2j+1)^2} + \sum_{\substack{n\geqslant 0 \\ -n\leqslant j\leqslant n-1}} (-1)^{n} q^{(4n+1)^2-2(2j+1)^2}.$$

Shifting the first sum and combining them we get the result,

$$\sum_{\substack{n\geqslant 1\\ -n\leqslant j\leqslant n-1}} (-1)^n q^{(4n-1)^2-2(2j+1)^2}(1+q^{16n}). \qquad \square$$

4. Proof of Corollary 1.2

Corollary 1.2 gives the result of Theorem 1.1 on the trivial character

$$|\chi|(\mathfrak{a}) = \begin{cases} 1, & N(\mathfrak{a}) \equiv \pm 1, \pm 7 \bmod 16, \\ 0 & \text{otherwise} \end{cases}$$

with the particularly simple associated L-function $\zeta_K^*(s)$. Instead of repeating the methods used to prove Theorem 1.1, however, we can use Theorem 1.1 more directly.

Proof of Corollary 1.2. Let $\gamma := e^{2\pi i/16}$, be a primitive 16th root of unity. Substitute $q \to \gamma q$ in (3.1):

$$\gamma q W_1((\gamma q)^8) = \sum_{\substack{\mathfrak{a}\subset O_K\\ N(\mathfrak{a})\equiv 1 \bmod 8}} \chi(\mathfrak{a})(\gamma q)^{N(\mathfrak{a})}.$$

Dividing through by γ shows

$$qW_1(-q^8) = \sum_{\substack{\mathfrak{a}\subset O_K\\ N(\mathfrak{a})\equiv 1 \bmod 8}} \chi(\mathfrak{a})\gamma^{N(\mathfrak{a})-1}q^{N(\mathfrak{a})}.$$

Recall from (1.3) that $\chi(\mathfrak{a}) = (-1)^{\frac{N(\mathfrak{a})-1}{8}}$ when $N(\mathfrak{a}) \equiv 1 \pmod 8$, thus

$$qW_1(-q^8) = \sum_{\substack{\mathfrak{a}\subset O_K\\ N(\mathfrak{a})\equiv 1 \bmod 8}} q^{N(\mathfrak{a})} = \sum_{n\equiv 1 \bmod 8} b(n)q^n.$$

Substitute $q \to \gamma q$ in (3.2),

$$\frac{1}{\gamma q} W_2(\gamma q) = \sum_{\substack{n\geqslant 0\\ n\equiv -1 \bmod 8}} a(n)(\gamma q)^n.$$

Multiplying through by γ gives

$$qW_2(-q^8) = \sum_{\substack{\mathfrak{a}\subset O_K\\ N(\mathfrak{a})\equiv -1 \bmod 8}} \chi(\mathfrak{a})\gamma^{N(\mathfrak{a})+1}q^{N(\mathfrak{a})}.$$

Similarly, $\chi(\mathfrak{a}) = (-1)^{\frac{N(\mathfrak{a})+1}{8}}$ when $N(\mathfrak{a}) \equiv -1 \bmod 8$, thus

$$qW_2(-q^8) = \sum_{\substack{\mathfrak{a} \subset O_K \\ N(\mathfrak{a}) \equiv -1 \bmod 8}} q^{N(\mathfrak{a})} = \sum_{n \equiv -1 \bmod 8} b(n)q^n.$$

Since there are no ideals of norm $\pm 3 \bmod 8$ in O_K, the result follows. $\square$

5. Combinatorial interpretation

The q-series $W_1(q)$ has interesting combinatorial properties. It is related to the Rogers–Ramanujan-type identity [10, Eq. (8)]:

$$\sum_{n=0}^{\infty} \frac{(-q)_n q^{\binom{n+1}{2}}}{(q)_n} = \frac{(-q^2; q^2)_\infty}{(q; q^2)_\infty}.$$

It is also a generating function for certain types of partitions. If

$$W_1(q) := \sum_{n \geqslant 0} \frac{(q)_n (-1)^n q^{\binom{n+1}{2}}}{(-q)_n} = \sum_{n \geqslant 0} A(n)q^n,$$

then $A(n)$ counts the number of colored partitions of n into *quasi-distinct* parts where the largest yellow part is less than or equal to the number of purple parts, weighted by $(-1)^{P+Y}$ where P is the largest purple part and Y is the number of yellow parts. Here, *quasi-distinct* means no two parts can have both the same value and color, but there may be two parts of the same value and different colors. Notice from (3.1) that $A(n) = a(8n+1)$.

Example. When $n = 4$, the colored partitions of this type are 4 and $3 + 1'$ with weight 1, and $3 + 1$ and $2 + 1 + 1'$ with weight -1 (unprimed numbers are purple parts, primed numbers are yellow parts). So $A(4) = 0$. There are no ideals of norm 33 in O_K, so $a(8 \cdot 4 + 1) = 0$ as well.

Example. When $n = 5$, the colored partitions of this type are $4 + 1$ and $3 + 1 + 1'$ with weight 1; and 5, $4 + 1'$, $3 + 2$, and $2 + 2' + 1$ with weight -1. So $A(5) = -2$. The ideals of norm 41 in O_K are $(7 + 2\sqrt{2})$ and $(7 - 2\sqrt{2})$, and since χ is -1 for both these ideals because $41 \equiv -7 \bmod 16$, we also have $a(41) = -2$.

The following two results establish a general formula for the $a(n)$'s, which we use to study $A(n)$.

Lemma 5.1. *The $a(n)$'s are multiplicative. That is, if* $\gcd(n, m) = 1$ *then* $a(nm) = a(n)a(m)$.

Proof. Recall the definition of $a(n)$

$$a(n) := \sum_{\substack{\mathfrak{a} \subset O_K \\ N(\mathfrak{a})=n}} \chi(\mathfrak{a}).$$

Suppose we have an ideal $\mathfrak{a}$ with $N(\mathfrak{a}) = nm$. It is well known that $\mathbb{Z}[\sqrt{2}]$ is a UFD, so factor the ideal $\mathfrak{a} = \mathfrak{p}_1\mathfrak{p}_2\cdots\mathfrak{p}_k$. Then $nm = N(\mathfrak{p}_1)N(\mathfrak{p}_2)\cdots N(\mathfrak{p}_k)$, since the norm is multiplicative. Because n and m are coprime, there must be a (set theoretic) partition $\{n_1, \dots, n_r\} \cup \{m_1, \dots, m_s\} = \{1, \dots, k\}$ such that $n = N(\mathfrak{p}_{n_1})N(\mathfrak{p}_{n_2})\cdots N(\mathfrak{p}_{n_r})$ and $m = N(\mathfrak{p}_{m_1})N(\mathfrak{p}_{m_2})\cdots N(\mathfrak{p}_{m_s})$. Let $\mathfrak{b} = \mathfrak{p}_{n_1}\mathfrak{p}_{n_2}\cdots\mathfrak{p}_{n_r}$ and $\mathfrak{c} = \mathfrak{p}_{m_1}\mathfrak{p}_{m_2}\cdots\mathfrak{p}_{m_s}$. Then $\mathfrak{a} = \mathfrak{b}\mathfrak{c}$ and $N(\mathfrak{b}) = n$ and $N(\mathfrak{c}) = m$. So

$$\begin{aligned} a(nm) &= \sum_{\substack{\mathfrak{a} \subset O_K \\ N(\mathfrak{a})=nm}} \chi(\mathfrak{a}) = \sum_{\substack{\mathfrak{b},\mathfrak{c} \subset O_K \\ N(\mathfrak{b})=n \\ N(\mathfrak{c})=m}} \chi(\mathfrak{b})\chi(\mathfrak{c}) \\ &= \left(\sum_{\substack{\mathfrak{b} \subset O_K \\ N(\mathfrak{b})=n}} \chi(\mathfrak{b}) \right) \left(\sum_{\substack{\mathfrak{c} \subset O_K \\ N(\mathfrak{c})=m}} \chi(\mathfrak{c}) \right) = a(n)a(m). \qquad \square \end{aligned}$$

Theorem 5.2. *If p is prime and $e \geqslant 0$, then*

$$a(p^e) = \begin{cases} (e+1) & \text{if } a(p) = 2 \text{ and } p \equiv \pm 1 \bmod 8, \\ (-1)^e(e+1) & \text{if } a(p) = -2 \text{ and } p \equiv \pm 1 \bmod 8, \\ (-1)^{e/2} & \text{if } e \text{ is even and } p \equiv \pm 3 \bmod 8, \\ 0 & \text{if } p = 2 \text{ or } e \text{ is odd and } p \equiv \pm 3 \bmod 8. \end{cases} \tag{5.1}$$

Proof. Since $\chi(\mathfrak{a}) = 0$ when $N(\mathfrak{a})$ is even, we have $a(2^e) = 0$. For p an odd prime, 2 is a quadratic residue mod p if and only if $p \equiv \pm 1 \bmod 8$, and it is exactly in this case that (p) splits in O_K.

In the splitting case, let p factor as $\alpha\beta$. Since α and β are the only elements of norm p, the elements of norm p^e are exactly the $e+1$ elements of the form $\alpha^k\beta^l$ where $k + l = e$ and $\chi(\alpha^k\beta^l) = \chi(\alpha)^k\chi(\beta)^l$. Since α and β are conjugate, and hence have the same norm, $\chi(\alpha) = \chi(\beta)$, and so $\chi(\alpha^k\beta^l) = \chi(\alpha)^e$. When $a(p) = 2$, then $\chi(\alpha) = 1$, and when $a(p) = -2$, then $\chi(\alpha) = -1$. There are no other possibilities for $a(p)$ since $\chi(\alpha) = \chi(\beta)$. This gives the first two cases.

Now suppose $p \equiv \pm 3 \bmod 8$. There are no ideals of norm p^e when e is odd by Lemma 3.1, because $p^e \equiv \pm 3 \bmod 8$.

When e is even, the only ideal of norm p^e is $(p^{e/2})$, with factorization $(p)^{e/2}$, since p does not split. Here (p) is the unique ideal of norm p^2 and $\chi(p) = -1$; since $p^2 \equiv 9 \bmod 16$ when $p \equiv \pm 3, \pm 5 \bmod 16$. Thus $\chi(p^{e/2}) = (-1)^{e/2}$. $\square$

Remark. It is well-known that in a number field with degree greater than 1 over $\mathbb{Q}$, the number of positive integers that are norms of ideals has density 0 [9]. This immediately gives that $A(n)$ is almost always 0.

Corollary 5.3. *$A(n)$ hits every integer infinitely many times.*

Proof. Given any integer $k \geqslant 2$ consider any prime $p \equiv 1 \bmod 8$. Then $p^{k-1} \equiv 1 \bmod 8$ and $9p^{k-1} \equiv 1 \bmod 8$. Let $n = (p^{k-1} - 1)/8$ and $m = 9(p^{k-1} - 1)/8$. If $a(p) = 2$ then $A(n) = a(8n + 1) = a(p^{k-1}) = k$ and $A(m) = a(8m + 1) = a(9p^{k-1}) = -k$. If $a(p) = -2$ then $A(n) = a(8n + 1) = a(p^{k-1}) = (-1)^{k+1}k$ and $A(m) = a(8m + 1) = a(9p^{k-1}) = (-1)^k k$. Since there are infinitely many primes $p \equiv 1 \bmod 8$, there must be infinitely many p in at least one of these two cases. Thus $A(n)$ hits $\pm k$ infinitely many times.

For the $|k| = 1$ case, consider any $p \equiv \pm 3 \bmod 8$. For any even e, we have $p^e \equiv 1 \bmod 8$. Let $n = (p^e - 1)/8$, then $A(n) = a(8n + 1) = a(p^e) = (-1)^{e/2}$. So $A(n)$ hits ± 1 infinitely many times. $\square$

6. Proof of Theorem 1.3

We prove the generating function for L-values (Theorem 1.3) in two steps. Theorem 6.1 is a corollary to Theorem 1.1 which proves the existence and gives an explicit form of the asymptotic expansion of $\sum_{n=1}^{\infty} a(n)e^{-nt}$. Then, independent of Theorem 1.1, we prove that an asymptotic expansion of $\sum_{n=1}^{\infty} a(n)e^{-nt}$ is in fact a generating function for L-values.

Theorem 6.1. *As $t \searrow 0$ we have*

$$\sum_{n=1}^{\infty} a(n)e^{-nt} \sim e^{-t} W_1(e^{-8t}) - e^t \sum_{n \geqslant 0} \frac{(e^{-8t}; e^{-16t})_n}{(-e^{-16t}; e^{-16t})_n}.$$

Proof. Recall (Theorem 1.1) that

$$\sum_{n \geqslant 1} a(n)q^n = qW_1(q^8) + \frac{1}{q} W_2(q^8). \tag{6.1}$$

We will make the specialization $q = e^{-t}$ and then demonstrate convergence of the resulting t-series. In the first term

$$W_1(e^{-8t}) = \sum_{n \geqslant 0} \frac{e^{-8tn(n+1)/2}(-1)^n (e^{-8t}; e^{-8t})_n}{(-e^{-8t}; e^{-8t})_n}$$

is a convergent t-series since $(e^{-8t}; e^{-8t})_n \to 0$. For the second term, it can be seen that $W_2(e^{-8t})$ is asymptotically, as $t \searrow 0$, equal to the following convergent t-series when we let $t = q, q = q^2$, and $a = -q^2$ in Theorem 1.1 of [5]:

$$W_2(e^{-8t}) \sim \sum_{n \geqslant 0} \left(\frac{(e^{-8t}; e^{-16t})_\infty}{(-e^{-16t}; e^{-16t})_\infty} - \frac{(e^{-8t}; e^{-16t})_n}{(-e^{-16t}; e^{-16t})_n} \right).$$

The first term in the sum goes to 0 as $t \searrow 0$. The result is now just a matter of substituting $q = e^{-t}$ in (6.1), and applying the above observations. $\square$

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. The proof is analogous to the proof of Proposition 3.1 in [7]. Note that $L(\chi, s)$ has an analytic continuation to $\mathbb{C}$. Suppose the asymptotic expansion as $t \searrow 0$ is given by

$$\sum_{n \geqslant 1} a(n) e^{-nt} \sim \sum_{n \geqslant 0} c(n) t^n. \tag{6.2}$$

Consider the following integral (assume $\Re(s) > 1$):

$$\begin{aligned} \int_0^\infty \left(\sum_{n \geqslant 1} a(n) e^{-nt} \right) t^{s-1}\, dt &= \sum_{n \geqslant 1} a(n) \int_0^\infty e^{-nt} t^{s-1}\, dt \\ &= \sum_{n \geqslant 1} \frac{a(n)}{n^s} \int_0^\infty e^{-T} T^{s-1}\, dT = \Gamma(s) L(\chi, s), \end{aligned} \tag{6.3}$$

where for the second equality we have made the substitution $T = nt$. We can switch the order of integration and summation in the first equality because we have absolute convergence, which follows from the following linear bound on the $a(n)$'s:

Lemma 6.2. *For all n, $a(n) \leqslant n$.*

Proof. It is easily seen by induction that for all $m \in \mathbb{N}$, $m + 1 \leqslant 2^m$, and hence $m + 1 \leqslant p^m$ for all primes p.

Factor n as $p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k}$. Then, by the results of Section 5, we see $|a(n)| \leqslant |a(p_1^{m_1}) a(p_2^{m_2}) \cdots a(p_k^{m_k})| \leqslant |(m_1 + 1)(m_2 + 1) \cdots (m_k + 1)| \leqslant |p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k}| = n$. $\square$

For any $N > 0$, (6.3) combined with the asymptotic expansion (6.2) implies that for some $\varepsilon > 0$,

$$\begin{aligned} \Gamma(s) L(\chi, s) &= \int_0^\infty \left(\sum_{n \geqslant 1} a(n) e^{-nt} \right) t^{s-1}\, dt \\ &= \int_0^\varepsilon \left(\sum_{n \geqslant 0} c(n) t^n \right) t^{s-1}\, dt + \int_\varepsilon^\infty \left(\sum_{n \geqslant 1} a(n) e^{-nt} \right) t^{s-1}\, dt. \end{aligned} \tag{6.4}$$

We truncate our asymptotic expansion to break up the first part of the integral as

$$\int_0^{\varepsilon} \left(\sum_{n \geqslant 0} c(n) t^n \right) t^{s-1}\, dt = \int_0^{\varepsilon} \sum_{n=0}^{N} c(n) t^{n+s-1}\, dt + \int_0^{\varepsilon} O(t^{n+s-1})\, dt$$
$$= \sum_{n=0}^{N} c(n) \frac{\varepsilon^{n+s}}{n+s} + F(s).$$

That $f = O(t^{N+s-1})$ means that for some M, we have $f \leqslant M t^{M+s-1}$. We then have that

$$|F(s)| \leqslant |M| \left| \int_0^{\varepsilon} t^{N+s-1}\, dt \right| = |M| \left| \frac{t^{N+s}}{N+s} \right| \Bigg|_{t=0}^{t=\varepsilon}$$

which is finite for $\Re(s) > -N$. So $F(s)$ is analytic for $\Re(s) > -N$.

Now consider the second half of (6.4), $G(s) = \int_{\varepsilon}^{\infty} (\sum_{n \geqslant 1} a(n) e^{-nt}) t^{s-1}\, dt$. By Lemma 6.2, again, the integrand is bounded for any s, and so $G(s)$ is analytic.

So (6.4) becomes

$$\Gamma(s) L(\chi, s) = \sum_{n=0}^{N} c(n) \frac{\varepsilon^{n+s}}{n+s} + F(s) + G(s),$$

where $F(s) + G(s)$ is analytic. Taking residues of both sides, we find

$$c(n) = \frac{(-1)^n}{n!} L(\chi, -n). \qquad \square$$

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