ANALOGS OF DIRICHLET L-FUNCTIONS IN CHROMATIC HOMOTOPY THEORY

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Abstract. The relation between Eisenstein series and the J-homomorphism is an important topic in chromatic homotopy theory at height 1. Both sides are related to the special values of the Riemann ζ-function. Number theorists have studied the twistings of the Riemann ζ-functions and Eisenstein series by Dirichlet characters.

Motivated by the Dirichlet equivariance of these Eisenstein series, we introduce the Dirichlet J-spectra in this paper. The homotopy groups of the Dirichlet J-spectra are related to the special values of the Dirichlet L-functions. Moreover, we find Brown-Comenetz duals of the Dirichlet J-spectra, whose formulas resemble functional equations of the corresponding Dirichlet L-functions. In this sense, the Dirichlet J-spectra we constructed are analogs of Dirichlet L-functions in chromatic homotopy theory.

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Bernoulli numbers show up in many seemingly unrelated areas of mathematics, as observed in [Maz08]. They are the special values of the Riemann ζ-function at negative integers:

$$\zeta(1-k)=-\frac{B_k}{k}.$$

Another two such occasions are q-expansions of normalized Eisenstein series in number theory

$$E_{2k}(q)=1-\frac{4k}{B_{2k}}\sum_{n\geq 1}\sigma_{2k-1}(n)q^n,$$

and the images of the J-homomorphisms in the stable homotopy groups of spheres in algebraic topology:

$$\operatorname{Im}(J_{4k-1})\simeq\mathbb{Z}/D_{2k},\ D_{2k}=\text{the denominator of } B_{2k}/4k.$$

The connections between the congruences of the normalized Eisenstein series E_{2k} and images of the J_{4k-1} have been explained in [Bak99; Lau99; Hop02; Beh09] in different ways since the invention of elliptic cohomology and topological modular forms (TMF).

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Number theorists have studied the twistings of the Riemann ζ-functions and Eisenstein series by Dirichlet characters. Let $\chi : (\mathbb{Z}/N)^\times \to \mathbb{C}^\times$ be a primitive Dirichlet character of conductor N. Leopoldt defined generalized Bernoulli numbers $B_{k,\chi}$ associated to χ (1.1.3) in [Leo58]. These numbers are algebraic numbers in $\mathbb{Q}(\chi)$. Moreover, they are related to the special values of the Dirichlet L-functions $L(s,\chi)$ at negative integers:

$$L(1-k;\chi) = -\frac{B_{k,\chi}}{k}.$$

As in the classical case, $B_{k,\chi}$ appears in the q-expansion of $E_{k,\chi}$ (1.2.7), the normalized Eisenstein series assoicated to χ when $(-1)^k = \chi(-1)$:

$$E_k(q;\chi) = 1 - \frac{2k}{B_{k,\chi}} \sum_{n=1}^{\infty} \sigma_{k-1,\chi}(n) q^n.$$

Denote the ideal of $\mathbb{Z}[\chi] \coloneqq \mathbb{Z}[\operatorname{Im}\chi]$ generated by the denominator of $\frac{B_{k,\chi}}{2k}$ by $\mathcal{D}_{k,\chi}$ when $(-1)^k = \chi(-1)$.[1] One may now wonder what is the object in homotopy theory that completes the analogy below:

L-functions	Modular forms	Homotopy theory
$\zeta(1-2k) = -\frac{B_{2k}}{2k}$	$E_{2k} \equiv 1 \mod D_{2k}$	$\operatorname{Im} J_{4k-1} \simeq \mathbb{Z}/D_{2k}$
$L(1-k;\chi) = -\frac{B_{k,\chi}}{k}$	$E_{k,\chi} \equiv 1 \mod \mathcal{D}_{k,\chi}$	?

Table 1. Analogy of L-functions, modular forms and homotopy theory

In this paper, we construct analogs of Dirichlet L-functions in homotopy theory, called the **Dirichlet J-spectra**, that fit in the table above. We further compute their homotopy groups and study their properties. The relations between homotopy groups of the Dirichlet J-spectra and congruences of $E_{k,\chi}$ will be explained in a subsequent paper in preparation.

The motivation of our construction of the Dirichlet J-spectra is the Dirichlet equivariance of the Eisenstein series $E_{k,\chi}$. This Eisenstein series is a modular form of weight k and level $\Gamma_1(N)$. Moreover, it satisfies an automorphic equation (1.2.4) for a larger congruence subgroup $\Gamma_0(N)$ that translates into a Dirichlet equivariance with respect to the action of the quotient group $\Gamma_0(N)/\Gamma_1(N) \simeq (\mathbb{Z}/N)^\times$:

$$E_{k,\chi} \in \operatorname{Hom}_{(\mathbb{Z}/N)^\times\text{-rep}}(\mathbb{C}_{\chi^{-1}}, H^0(\mathcal{M}_{ell}(\Gamma_1(N)), \boldsymbol{\omega}^{\otimes k})).$$

Imitating this formula, we define the Dirichlet J-spectrum in Construction 3.4.1 by

$$J(N)^{h\chi} \coloneqq \operatorname{Map}\left(M(\mathbb{Z}[\chi]), J(N)\right)^{h(\mathbb{Z}/N)^\times}.$$

In this formula,

- The notation $(-)^{h\chi}$ stands for the "homotopy χ-eigen-spectrum".
- $\mathbb{Z}[\chi]$ is the $\mathbb{Z}$-subalgebra of $\mathbb{C}$ generated by the image of χ. The character χ induces a $(\mathbb{Z}/N)^\times$-action on $\mathbb{Z}[\chi]$ where $a \in (\mathbb{Z}/N)^\times$ acts by multiplication by $\chi(a)$. $M(\mathbb{Z}[\chi])$ is the Moore spectrum of $\mathbb{Z}[\chi]$ with a $(\mathbb{Z}/N)^\times$-action such that the induced $(\mathbb{Z}/N)^\times$-action on π_0 is equivalent to that on $\mathbb{Z}[\chi]$. The existence of such actions on the Moore spectra is non-trivial since the taking Moore spectra is NOT functorial. In Section 3.3, we give an explicit construction of $M(\mathbb{Z}[\chi])$ with $(\mathbb{Z}/N)^\times$-action suggested by Charles Rezk.

[1] A priori, the denominator of $\frac{B_{k,\chi}}{2k}$ is not well-defined since the ring $\mathbb{Z}[\chi]$ is in general not a unique factorization domain and has non-trivial unit group. But since $\mathbb{Z}[\chi]$ is a Dedekind domain, its fractional ideals have unique factorizations. As a result, the principal fractional ideal generated by $\frac{B_{k,\chi}}{2k}$ can be uniquely written as the difference of two actual ideals of $\mathbb{Z}[\chi]$. Thus the "denominator ideal" makes sense.

- $J(N)$ is the "J-spectrum with μ_N-level structure". It is defined as the homotopy pullback of the arithmetic fracture square (3.2.8):

$$\begin{tikzcd} J(N) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \prod_p S^0_{KU/p}\left(p^{v_p(N)}\right) \arrow[d, "\text{Rationalization}"] \\ S^0_{\mathbb{Q}} \arrow[r, "\text{Hurewicz}"] & \left(\prod_p S^0_{KU/p}\left(p^{v_p(N)}\right)\right)_{\mathbb{Q}} \end{tikzcd}$$

Here, $S^0_{KU/p}(p^v) := \left(KU_p^\wedge\right)^{h(1+p^v\mathbb{Z}_p)}$ is a $(\mathbb{Z}/p^v)^\times$-Galois extension of the $K(1)$-local sphere $S^0_{KU/p}$. $J(N)$ is endowed with a $(\mathbb{Z}/N)^\times$-action by assembling the Galois actions of $\left(\mathbb{Z}/p^{v_p(N)}\right)^\times$ for each prime $p \mid N$.

In particular, $J := J(1)$ is equivalent to S^0_{KU}, the Bousfield localization of the sphere spectrum S^0 at KU, as discussed in [Bou79]. We call it the J-spectrum, because its Hurewicz map detects the image of the stable J-homomorphism. The details of this construction are explained in Section 3.2.

Proposition. (3.4.7) *There is a variant of the homotopy fixed point spectral sequence to compute $\pi_*(J(N)^{h\chi})$:*

$$E_2^{s,t} \simeq \operatorname{Ext}^s_{\mathbb{Z}[(\mathbb{Z}/N)^\times]}(\mathbb{Z}[\chi], \pi_t(J(N))) \Longrightarrow \pi_{t-s}\left(J(N)^{h\chi}\right).$$

As the E_2-page consists of derived χ-eigenspaces of $\pi_*(J(N))$, it is appropriate to call this spectral sequence the "homotopy eigen(-spectrum) spectral sequence".

This computation is carried out p-adically. For a p-adic Dirichlet character $\chi : (\mathbb{Z}/N)^\times \to \mathbb{C}_p^\times$, we construct the Dirichlet $K(1)$-local sphere $S^0_{K(1)}(p^v)^{h\chi}$ in a similar fashion. We show in Proposition 3.5.3 that the p-completion of $J(N)^{h\chi}$ decomposes into a wedge sum of Dirichlet $K(1)$-local spheres. When $N = p > 2$ or 4, the summands in this decomposition represent elements of finite order in the $K(1)$-local Picard group, first defined in [HMS94]. Moreover, we notice the definitions of the Dirichlet J-spectra and $K(1)$-local spheres depend on the group actions on the Moore spectra. In the case when $N = 4$ and $p = 2$, we observe in Remark 4.2.11 that the Dirichlet $K(1)$-local spheres constructed using different group actions on the Moore spectra are differed by the the exotic element in the $K(1)$-local Picard group at $p = 2$.

The homotopy groups of these Dirichlet $K(1)$-local spheres are computed by a homotopy fixed point spectral sequence (HFPSS), whose E_2-page consists of continuous group cohomology.

Corollary. (3.5.7) *Write $N = p^v \cdot N'$, where $p \nmid N'$. Then χ factorizes as $\chi = \chi_p \cdot \chi'$ where χ_p and χ' have conductors p^v and N', respectively. Then there is a HFPSS:*

$$E_2^{s,2t} = \operatorname{Ext}^s_{\mathbb{Z}_p[\![\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times]\!]}(\mathbb{Z}_p[\chi], \mathbb{Z}_p^{\otimes t}) \simeq H_c^s(\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times; \mathbb{Z}_p^{\otimes t}[\chi^{-1}]) \Longrightarrow \pi_{2t-s}\left(S^0_{K(1)}(p^v)^{h\chi}\right),$$

where $\mathbb{Z}_p^{\otimes t}[\chi]$ is the representation associated to the character $\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times \xrightarrow{(a,b)\mapsto \chi_p(a)\chi'(b)a^t} (\mathbb{Z}_p[\chi])^\times$.

In a subsequent paper, we will relate the group cohomology $H_c^1(\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times; \mathbb{Z}_p^{\otimes k}[\chi^{-1}])$ in Corollary 3.5.7 to congruences of the p-adic Eisenstein series $E_{k,\chi^{-1}}$, using Dieudonné theory of height 1 formal groups and formal A-modules.

Assembling the computations of homotopy groups of the Dirichlet $K(1)$-local spheres in Section 4, we record the homotopy groups of the Dirichlet J-spectra in Theorem 5.1.1. These homotopy groups are related to the special values of the corresponding Dirichlet L-functions:

Theorem. (5.1.2) *Assume $N = p^v > 1$. For all integers k satisfying $(-1)^k = \chi(-1)$, we have*

$$\pi_{2k-1}\left(J(p^v)^{h\chi}\left[\frac{1}{\ell(\chi)}\right]\right) \simeq \mathbb{Z}[\chi]/\mathcal{I}_{|k|,\chi^{-1}}, \quad \text{where } \ell(\chi) = \begin{cases} \ell, & \text{if } |\operatorname{Im}(\chi)| \text{ is a power of a prime } \ell \neq p; \\ 1, & \text{otherwise.} \end{cases}$$

where the possible multiplicative difference of the ideals $\mathcal{I}_{k,\chi}$ *and* $\mathcal{D}_{k,\chi}$ *of* $\mathbb{Z}[\chi]$ *contains the principal ideal* (2) *in* $\mathbb{Z}[\chi]$*.*

This computation of Dirichlet J-spectra allows us to compare the spectrum $J(N)$ with the Dedekind ζ-function attached to $\mathbb{Q}(\zeta_N)$. The comparison does not work directly, as the latter has only zero special values. Instead, we focus on *totally real* abelian extensions of $\mathbb{Q}$.

Definition. (5.2.4) Let $\mathbb{K}/\mathbb{Q}$ be a finite abelian extension and N be the smallest integer such that $\mathbb{K} \subseteq \mathbb{Q}(\zeta_N)$. We define:

$$J(\mathbb{K}) := J(N)^{h\mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{K})}$$

Here, we identify $\mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{K}) \subseteq \mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q}) \simeq (\mathbb{Z}/N)^{\times}$.

Theorem. (5.2.6) *Let* $\mathbb{K}/\mathbb{Q}$ *be a totally real finite abelian extension and* p^v *be the smallest integer such that* $\mathbb{K} \subseteq \mathbb{Q}(\zeta_{p^v})$*. Denote the Galois group* $\mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{K})$ *by* G*. Then*

$$\pi_{4t-1}\left(J(\mathbb{K})\left[\frac{1}{|G|}\right]\right) = \mathbb{Z}\left[\frac{1}{|G|}\right]\Big/ D_{\mathbb{K},2t},$$

where $D_{\mathbb{K},2t} \in \mathbb{Z}_{>0}$ *is the denominator of* $\zeta_{\mathbb{K}}(1-2t)$*.*

The special values of $\zeta_{\mathbb{K}}$ are closely related to the algebraic K-theory of $\mathcal{O}_{\mathbb{K}}$, the ring of integers of $\mathbb{K}$. The precise formula of this connection is given by the Lichtenbaum-Quillen Conjecture, which is proved by Voevodsky-Rost. By comparing the spectra $J(\mathbb{K})$ and $K(\mathbb{O}_{\mathbb{K}})$, we propose the following questions:

Question. (5.2.10) *Let* $\mathbb{K}/\mathbb{Q}$ *be a finite abelian extension. Is there a natural* $\mathrm{Gal}(\mathbb{K}/\mathbb{Q})$*-equivariant map of* KU*-local* $\mathbb{E}_\infty$*-ring spectra* $h(\mathbb{K}): J(\mathbb{K}) \to L_{KU}K(\mathcal{O}_{\mathbb{K}})$ *extending the* KU*-local Hurewicz map* h_{KU}*?*

$$\begin{tikzcd} J(\mathbb{K}) \arrow[r, dashed, "\exists h(\mathbb{K})?"] & L_{KU}K(\mathcal{O}_{\mathbb{K}}) \\ J = S^0_{KU} \arrow[u] \arrow[r, "h_{KU}"] & L_{KU}K(\mathbb{Z}) \arrow[u] \end{tikzcd}$$

In addition, for an arbitrary number field $\mathbb{K}$*, how can we extract a "*J*-spectrum" from* $L_{KU}K(\mathcal{O}_{\mathbb{K}})$*?*

Moreover, we find the Brown-Comenetz duals of the Dirichlet J-spectra and $K(1)$-local spheres in Section 5.3. This duality phenomenon resembles functional equations of the corresponding Dirichlet L-functions.

Theorem. (5.3.14) *Let* p *be an odd prime. Then we have:*

$$I_{KU}\left(J(p^v)^{h\chi}\left[\frac{1}{\ell(\chi)}\right]\right) \simeq \Sigma^2 J(p^v)^{h\chi^{-1}}\left[\frac{1}{\ell(\chi)}\right] \implies \pi_t\left(J(p^v)^{h\chi}\left[\frac{1}{\ell(\chi)}\right]\right) \simeq \pi_{-2-t}\left(J(p^v)^{h\chi^{-1}}\left[\frac{1}{\ell(\chi)}\right]\right).$$

We also find the Brown-Comenetz dual of $J(N)$. The formula resembles the functional equation of the corresponding Dedekind ζ-function.

Proposition. (5.3.16,5.3.18) $I_{KU}J(N) \simeq \Sigma^2\mathcal{E}_{KU} \wedge J(N) \wedge M(\widehat{\mathbb{Z}})$*, where* $\mathcal{E}$ *is a finite CW-spectrum*

$$\mathcal{E} := \Sigma^{-2}(S^{-1} \cup_2 e^0 \cup_\eta e^2),$$

and $M(\widehat{\mathbb{Z}})$ *is the Moore spectrum of the group of profinite integers* $\widehat{\mathbb{Z}}$*. Moreover, we have*

$$I_{KU}J(4N) \simeq \Sigma^2 J(4N) \wedge M(\widehat{\mathbb{Z}}) \implies \pi_t(J(4N))^\wedge \simeq \mathrm{Hom}_{\mathbb{Z}}(\pi_{-2-t}(J(4N)), \mathbb{Q}/\mathbb{Z}),$$

where $(-)^\wedge$ *is the profinite completion of an abelian group.*

It is because of these observations that the Dirichlet J-spectra constructed in this paper are analogs of Dirichlet L-functions in chromatic homotopy theory. We end the introduction with a table of analogy between homotopy theory and L-functions.

Chromatic Homotopy Theory	**L-functions**
$J := S^0_{KU}$	$\zeta(s)$
$J(\mathbb{K}) := J(N)^{h\mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{K})}$	$\zeta_{\mathbb{K}}(s)$
$J(N)^{h\chi^{-1}}$	$L(s;\chi)$
$S^0_{K(1)}(p^v)^{h\chi^{-1}}$	$L_p(s;\chi)$
Homotopy Groups	Denominators of Special Values
Brown-Comenetz Duality	Functional Equation

TABLE 2. Comparisons of chromatic homotopy theory and L-functions

Notations and conventions.

- Denote the Teichmüller character by the Greek letter ω and denote the sheaf of invariant differentials on various stacks by the boldface version of the same Greek letter $\boldsymbol{\omega}$.
- C_n is the cyclic group of order n and σ is the sign representation of C_2.
- Denote the suspension spectrum $\Sigma^\infty X_+$ of a based space X_+ also by X_+.
- X_E and $L_E X$ are the Bousfield localization of a spectrum X at a homology theory E. In particular, we denote $L_{E(1)}$ by L_1. $\mathbf{Sp}_E$ is the category of E-local spectra.
- KU is the topological complex K-theory, KO is the topological real K-theory, and $K\mathbb{R}$ is Atiyah's genuine C_2-equivariant Real K-theory in [Ati66]. $K(R)$ is the algebraic K-theory spectrum for a ring R.
- $K(1)$ is the Morava K-theory of height 1 at a prime p. When the prime needs to be specified, we write $X_{KU/p}$ for the $K(1)$-localization of X.[2]
- We write S^0_p for the p-complete sphere spectrum. When $p > 2$, we write $S^0_{\omega^a}$ for the p-complete sphere spectrum with an action of $(\mathbb{Z}/p)^\times$ induced by the character ω^a, where $0 \le a \le p-2$.
- $\mathbb{C}_p$ is the analytic completion of $\overline{\mathbb{Q}_p}$, the algebraic closure of the rational p-adics.

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1. DIRICHLET CHARACTERS AND MODULAR FORMS

1.1. Dirichlet L-functions and Dedekind ζ-functions.

Definitions and statements in this subsection are from [Iwa72, §1, §2], unless otherwise specified.

Definition 1.1.1. A multiplicative map $\chi : \mathbb{Z} \to \mathbb{C}$ is called a **Dirichlet character** of modulus N if it is nonzero only at integers coprime to N and it only depends on the residue class modulo N. Alternatively,

[2] When $p = 2$, $K(1) \simeq KU/2$. When p is odd, $K(1)$ and KU/p are related by the Adams splitting : $KU/p \simeq \bigvee_{i=0}^{p-2} \Sigma^{2i} K(1)$. As a result, $K(1)$ and KU/p are Bousfield equivalent.

a Dirichlet character is equivalent to a group homomorphism $\chi : (\mathbb{Z}/N)^{\times} \to \mathbb{C}^{\times}$. A Dirichlet character $\chi : \mathbb{Z} \to \mathbb{C}$ of modulus N is said to be **primitive** if it is not of modulus M for any $M < N$. This N is called the **conductor** of χ. Denote the trivial Dirichlet character that maps every nonzero integer to 1 by χ^0.

The **Dirichlet L-function** associated to χ is defined to be the series:

$$L(s;\chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

By definition, $L(s;\chi^0) = \zeta(s)$. Like the Riemann ζ-function, $L(s;\chi)$ has a Euler factorization:

$$L(s;\chi) = \prod_p (1 - \chi(p)p^{-s})^{-1}.$$

As a function of s, $L(s,\chi)$ converges absolutely for all s with $\mathrm{Re}(s) > 0$ and non-absolutely for $\mathrm{Re}(s) > 0$ when $\chi \neq \chi^0$. Thus $L(s;\chi)$ defines a holomorphic function on the half plane $\mathrm{Re}(s) > 0$ ($\mathrm{Re}(s) > 1$ if $\chi = \chi^0$) and it admits an analytic continuation to the whole complex plane (minus $s = 1$ if $\chi = \chi^0$). Just as the Riemann ζ function, $L(s;\chi)$ takes special values at negative integers. These values are related to the **generalized Bernoulli numbers**.

Definition 1.1.2. The ordinary Bernoulli numbers are defined to by

$$F(t) = \frac{te^t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}.$$

Let χ be a Dirichlet character with conductor N. We define the generalized Bernoulli numbers associated to χ by setting

$$F_\chi(t) = \sum_{a=1}^{N} \frac{\chi(a)te^{at}}{e^{Nt} - 1} = \sum_{n=0}^{\infty} B_{k,\chi} \frac{t^k}{k!}. \tag{1.1.3}$$

Remark 1.1.4. Notice that the conductor of the trivial character χ^0 is 1. So we have $F_{\chi^0}(t) = F(t)$ and $B_{k,\chi^0} = B_k$.

Proposition 1.1.5. *$B_{k,\chi} = 0$ unless $(-1)^k = \chi(-1)$. In particular, $B_k = 0$ when k is odd.*

Proposition 1.1.6. *Let k be a positive integer. For any Dirichlet character $\chi : (\mathbb{Z}/N)^{\times} \to \mathbb{C}^{\times}$, we have*

$$L(1-k;\chi) = -\frac{B_{k,\chi}}{k}.$$

It now follows from (1.1.3) that $L(1-k;\chi) \in \mathbb{Q}(\chi)$, where $\mathbb{Q}(\chi)$ is the field extension of $\mathbb{Q}$ by the image of χ. In particular, $\zeta(1-k) \in \mathbb{Q}$.

Arithmetic properties of B_k and $B_{k,\chi}$ are summarized below:

Theorem 1.1.7 (Clausen-von Staudt, von-Staudt)**.** [MS74, Theorem B.3, B.4]

(1) The denominator of B_k, expressed as a fraction in the lowest term is equal to the product of all primes p with $(p-1) \mid 2k$.

(2) A prime divides the denominator of $\frac{B_k}{2k}$ if and only if it divides the denominator of B_k.

Theorem 1.1.8. [Car59, Theorem 1 and 3] *Let $\chi : (\mathbb{Z}/N)^{\times} \to \mathbb{C}^{\times}$ be a primitive Dirichlet character of conductor N.*

(1) If N is divisible by at least two distinct prime numbers, then $\frac{B_{k,\chi}}{k}$ is an algebraic integer. When $N = p^v$, the ideal of $\mathbb{Z}[\chi]$ generated by the denominator of $\frac{B_{k,\chi}}{k}$ contains only prime ideal factors of (p).

(2) If $N = p^v, p > 2$, let g be a primitive $\phi(N)$-th root of unity mod p. $\frac{B_{k,\chi}}{k}$ is integral unless $\mathfrak{p} = (p, 1 - \chi(g)g^k) \neq (1)$. In this case, when $v = 1$,

$$pB_{k,\chi} \equiv p - 1 \mod \mathfrak{p}^{v_p(k)+1}; \tag{1.1.9}$$

when $v > 1$,

$$(1 - \chi(1+p))\frac{B_{k,\chi}}{k} \equiv 1 \mod \mathfrak{p}. \tag{1.1.10}$$

(3) If $N = 4$, then

$$\frac{B_{k,\chi}}{k} \equiv \frac{k}{2} \mod 1. \tag{1.1.11}$$

If $N = 2^v, v > 2$, then $\frac{B_{k,\chi}}{k}$ is an algebraic integer.

We also define Dedekind ζ-functions attached to number fields.

Definition 1.1.12. [Lan94, page 160] Let $\mathbb{K}$ be a number field. We define

$$\zeta_{\mathbb{K}}(s) = \prod_{\mathfrak{p}} \frac{1}{1 - |\mathcal{O}_{\mathbb{K}}/\mathfrak{p}|^{-s}},$$

where $\mathfrak{p}$ ranges over all nonzero prime ideals of $\mathcal{O}_{\mathbb{K}}$.

When $\mathbb{K}/\mathbb{Q}$ is a finite abelian extension, $\mathbb{K} \subseteq \mathbb{Q}(\zeta_N)$ for some N by the Kronecker-Weber Theorem. $\mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{K})$ is a subgroup of $\mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$. The latter is isomorphic to $(\mathbb{Z}/N)^\times$ by Lemma A.2.1.

Theorem 1.1.13. [Was97, Theorem 4.3] *Let $\mathbb{K}/\mathbb{Q}$ be a finite abelian extension. Then $\zeta_{\mathbb{K}}$ and Dirichlet L-functions are related by:*

$$\zeta_{\mathbb{K}}(s) = \prod_{\substack{\chi:(\mathbb{Z}/N)^\times \to \mathbb{C}^\times \\ \mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{K}) \subseteq \ker \chi}} L(s, \chi).$$

1.2. **Eisenstein series.** One way to study the Dirichlet L-functions is through modular forms, more precisely the Eisenstein series. Here, we give a brief review of the basic theory of modular forms from [Sil94].

Definition 1.2.1. A subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ is called a **congruence subgroup** if it contains all matrices congruent to NI_2 in $\mathrm{SL}_2(\mathbb{Z})$ for some integer $N > 0$. Examples of congruence subgroups are

- $\Gamma(N) = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid a \equiv d \equiv 1, b \equiv c \equiv 0 \mod N\right\}$,
- $\Gamma_0(N) = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \mod N\right\}$,
- $\Gamma_1(N) = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid a \equiv d \equiv 1, c \equiv 0 \mod N\right\}$.

Let $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup. $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ when $N = 1$. A modular form of level Γ and weight k is a holomorphic function over the complex upper half plane $\mathfrak{h}$ satisfying the functional equation:

$$f(\gamma z) = (cz + d)^k f(z) \quad \text{for all } \gamma = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \Gamma, \mathrm{Im}\, z > 0. \tag{1.2.2}$$

and is holomorphic at all cusps. The space of such modular forms is denoted by $M_k(\Gamma)$, where Γ is omitted if it is $\mathrm{SL}_2(\mathbb{Z})$.

Recall that the classical Eisenstein series of weight k attached to a lattice $\Lambda \subseteq \mathbb{C}$ is defined by

$$G_k(\Lambda) = \sum_{w \in \Lambda \backslash \{0\}} \frac{1}{w^k}.$$

This formal power series is absolutely convergent when $k > 2$. Let $z \in \mathfrak{h}$ be a complex number in the upper half plane and denote the lattice $(z\mathbb{Z} \oplus \mathbb{Z}) \subseteq \mathbb{C}$ by $\Lambda(z)$. Define

$$G_k(z) \coloneqq G_k(\Lambda(z)) = \sum_{(m,n) \neq (0,0)} \frac{1}{(mz+n)^k}.$$

This is a modular function of weight k and level $\mathrm{SL}_2(\mathbb{Z})$. It is easy to see $G_k(z) = 0$ when k is odd. As $G_{2k}(z+1) = G_{2k}(z)$ by (1.2.2), G_{2k} is a function of $q = e^{2\pi i z}$:

$$G_{2k}(q) = 2\zeta(2k) + \frac{(2\pi i)^{2k}}{(2k-1)!} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n, \text{ where } \sigma_m(n) = \sum_{0<d|n} d^m.$$

This is the **q-expansion** of G_{2k}. As $G_{2k}(q)$ is a power series of q, it is holomorphic at the only cusp $q = 0$ and thus a modular form. Dividing G_{2k} by the constant term in its q-expansion, we get the **normalized Eisenstein series** E_{2k} of weight $2k$:

$$E_{2k}(q) \coloneqq \frac{G_{2k}(q)}{2\zeta(2k)} = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n.$$

Let $\chi : (\mathbb{Z}/N)^{\times} \to \mathbb{C}^{\times}$ be a primitive Dirichlet character of conductor N. We are now going to introduce the twisting of G_k by χ following [Hid93, §5.1].

Definition 1.2.3. The Eisenstein series associated χ of weight k is defined to be

$$G_k(z;\chi) \coloneqq \sum_{(m,n)\neq(0,0)} \frac{\chi^{-1}(n)}{(mNz+n)^k}.$$

This series is nonzero only when $\chi(-1) = (-1)^k$. It is not hard to see $G_k(z;\chi) \in M_k(\Gamma_1(N))$. Moreover, it also satisfies an automorphic equation for $\gamma \in \Gamma_0(N)$:

$$G_k(\gamma \cdot z;\chi) = \chi(d)(cz+d)^k G_k(z;\chi), \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N). \tag{1.2.4}$$

Definition 1.2.5. $M_k(\Gamma_0(N),\chi) = \{f \in M_k(\Gamma_1(N)) \mid f \text{ satisfies (1.2.4)}\}$. In particular, $M_k(\Gamma_0(N),\chi^0) = M_k(\Gamma_0(N))$.

Proposition 1.2.6. *Set $q = e^{2\pi i z}$ and assume $(-1)^k = \chi(-1)$. The q-expansion of $G_{k,\chi}$ is*

$$G_{k,\chi}(q) = 2L(k,\chi^{-1}) + 2N^{-k}\left(\sum_{l=1}^{N} \chi^{-1}(l) e^{\frac{2\pi i l}{N}}\right) \frac{(-2\pi i)^k}{(k-1)!} \left(\sum_{\substack{m\geq 0, n\geq 0 \\ (n,N)=1}} \chi(n) n^{k-1} q^{nm}\right).$$

When χ is primitive or $\chi = \chi^0$, one can use the functional equation of $L(s;\chi^{-1})$ to normalize the constant term of $G_{k,\chi}(z)$. We define

$$E_{k,\chi}(q) \coloneqq \frac{G_{k,\chi}(z;q)}{2L(k,\chi^{-1})} = 1 - \frac{2k}{B_{k,\chi}} \sum_{n=1}^{\infty} \sigma_{k-1,\chi}(n) q^n, \text{ where } \sigma_{m,\chi}(n) = \sum_{0<d|n} \chi(d) d^m. \tag{1.2.7}$$

Remark 1.2.8. E_{2k} and $E_{k,\chi}$ can be expressed in terms of z as:

$$E_{2k}(z) = \sum_{(m,n)=1, m>0} \frac{1}{(mz+n)^{2k}}, \quad E_k(z;\chi) = \sum_{(m,n)=1,m>0} \frac{\chi^{-1}(n)}{(mNz+n)^k}.$$

It is straight forward to check from these formulas that

$$G_{2k}(z) = 2\zeta(2k)E_{2k}(z), \quad G_k(z;\chi) = 2L(k,\chi^{-1})E_k(z;\chi).$$

1.3. **Moduli interpretations of modular forms.** Modular forms are closely related to moduli stacks of elliptic curves with level structures over $\mathbb{C}$.

Definitions 1.3.1. Let $\mathcal{M}_{ell}$ be the moduli stack of **generalized elliptic curves** over $\mathbb{C}$. That is, cubic curves with possible nodal singularities. Let N be a positive integer. Define the following moduli stacks:

- $\mathcal{M}_{ell}(\Gamma_0(N))$ is the moduli stack for the pairs (C,H), where C is a generalized elliptic curve and $H \subseteq C$ is a subgroup of order N.
- $\mathcal{M}_{ell}(\Gamma_1(N))$ is the moduli stack for the triples (C,H,η), where C is a generalized elliptic curve, $H \subseteq C$ is a subgroup of order N, and $\eta : \mathbb{Z}/N \xrightarrow{\sim} H$ is an isomorphism.

Remark 1.3.2. $\mathcal{M}_{ell}(\Gamma) = \mathcal{M}_{ell}$ when $N = 1$.

Proposition 1.3.3. *For the stacks above, denote the sheaves of invariant differentials by $\boldsymbol{\omega}$. Then we have*

$$M_k(\Gamma) \simeq H^0(\mathcal{M}_{ell}(\Gamma), \boldsymbol{\omega}^{\otimes k}).$$

It is not hard to see the forgetful map $\mathcal{M}_{ell}(\Gamma_1(N)) \to \mathcal{M}_{ell}(\Gamma_0(N))$ is a $(\mathbb{Z}/N)^\times$-torsor: $g \in (\mathbb{Z}/N)^\times \simeq \mathrm{Aut}(\mathbb{Z}/N)$ acts by $(C,H,\eta) \mapsto (C,H,\eta\circ g)$. As a result, there is a natural action of $(\mathbb{Z}/N)^\times$ on

$$H^0(\mathcal{M}_{ell}(\Gamma_1(N)), \boldsymbol{\omega}^{\otimes k}) \simeq M_k(\Gamma_1(N)).$$

Proposition 1.3.4. *Let $\chi : (\mathbb{Z}/N)^\times \to \mathbb{C}^\times$ be a Dirichlet character. $M_k(\Gamma_0(N),\chi)$ defined in Definition 1.2.5 is isomorphic to* $\mathrm{Hom}_{(\mathbb{Z}/N)^\times\text{-rep}}(\mathbb{C}_{\chi^{-1}}, M_k(\Gamma_1(N)))$.

Proof. It suffices to rephrase the automorphic equation (1.2.4) in terms of the $(\mathbb{Z}/N)^\times$-action on the moduli stack $\mathcal{M}_{ell}(\Gamma_1(N))$. Consider the lattice $\Lambda(z) = z\mathbb{Z}\oplus\mathbb{Z}$. There is a triple (C,H,η) associated to $\Lambda(z)$:

$$C = \mathbb{C}/\Lambda(z), H = \Lambda(z/N)/\Lambda(z) \subseteq C, \eta : (\mathbb{Z}/N) \xrightarrow{\sim} H, 1 \mapsto z/N.$$

For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, its actions on the lattices are:

$$\begin{aligned} \Lambda(z) &\mapsto \mathbb{Z}(az+b)\oplus\mathbb{Z}(cz+b) = \Lambda(z), \\ \Lambda(z/N) &\mapsto \mathbb{Z}(az/N+b)\oplus\mathbb{Z}(cz/N+b) \equiv \Lambda(az/N) \equiv \Lambda(z/N) && \mod \Lambda(z), \\ z/N &\mapsto az/N + b \equiv az/N && \mod \Lambda(z). \end{aligned}$$

Here the second line uses the facts $c \equiv 0 \mod N$ and a is invertible mod N. From this formula, the action of γ is trivial when $a \equiv 1 \mod N$, i.e. $\gamma \in \Gamma_1(N)$. For $[\gamma] \in \Gamma_0(N)/\Gamma_1(N) \simeq (\mathbb{Z}/N)^\times$, its action on the triple (C,H,η) is:

$$(C,H,\eta : 1 \mapsto z/N) \longmapsto (C,H,\eta\circ[\gamma] : 1 \mapsto a \mapsto az/N).$$

Thus for $f(z) \in M_k(\Gamma_0(N),\chi) \simeq \mathrm{Hom}_{(\mathbb{Z}/N)^\times\text{-rep}}(\mathbb{C}_{\chi^{-1}}, M_k(\Gamma_1(N)))$, we have

$$f(\gamma\cdot z) = \chi^{-1}(a)(cz+d)^k f(z) = \chi(d)(cz+d)^k f(z).$$

□

2. From the J-homomorphism to the $K(1)$-local sphere

2.1. **The J-homomoprhism and the e-invariant.** The J-homomorphism is a group homomorphism $J_{k,n} : \pi_k(\mathrm{SO}(n)) \to \pi_{n+k}(S^n)$. This map passes to a stable J-homomorphism $J_k : \pi_k(\mathrm{SO}) \to \pi_k(S^0)$.

Definitions 2.1.1. The (unstable) J-homomoprhism is defined in the following ways:

(1) Loop spaces. An linear isometry of $\mathbb{R}^n$ restricts to a boundary preserving isometry of the unit ball D^n and thus induces a selfmap $S^n \to S^n$. From this, we get a continuous map $g_n : \mathrm{SO}(n) \to \Omega^n S^n$. We define
$$J_{k,n} \coloneqq \pi_k(g_n) : \pi_k(\mathrm{SO}(n)) \longrightarrow \pi_k(\Omega^n S^n) \simeq \pi_{n+k}(S^n).$$
(2) Framed cobordism. Geometrically, the image of the J-homomorphism identifies the framed k-dimensional submanifolds of S^{n+k} whose underlying submanifolds are S^k. As the normal bundle of $S^k \hookrightarrow S^{n+k}$ is trivial, a framing of this embedding is equivalent a map $f : S^k \to O(n)$. One can further show two framings of the embedding $S^k \hookrightarrow S^{n+k}$ are equivalent iff the associated maps are homotopical. Thus we get a map $J_{k,n} : \pi_k(O(n)) \to \pi_{n+k}(S^n)$.
(3) Thom space. A map $f \in \pi_k(\mathrm{SO}(n)) \simeq \pi_{k+1}(B\mathrm{SO}(n))$ induces a n-dimensional oriented vector bundle ξ_f over S^{k+1}. The Thom space of ξ_f is a two-cell complex $\mathrm{Th}(\xi_f) = S^n \cup e^{n+k+1}$. Define $J_{k,n}(f)$ to be the gluing map of $\mathrm{Th}(\xi_f)$, i.e.
$$S^{n+k} = \partial e^{n+k+1} \xrightarrow{J_{k,n}(f)} S^n \longrightarrow \mathrm{Th}(\xi_f).$$

Proposition 2.1.2. *The definitions above are equivalent up to a sign.*

Proposition 2.1.3. *The J-homomorphisms $J_{k,n}$ are compatible under stabilization. More precisely, let $i_n : \mathrm{SO}(n) \hookrightarrow \mathrm{SO}(n+1)$ be the map that sends an $n \times n$ orthogonal matrix A to $\begin{pmatrix} A & \\ & 1 \end{pmatrix}$. The following diagram commutes:*
$$\begin{tikzcd} \pi_k(\mathrm{SO}(n)) \arrow[r, "J_{k,n}"] \arrow[d, "\pi_k(i_n)"] & \pi_{n+k}(S^k) \arrow[d, "\Sigma"] \\ \pi_k(\mathrm{SO}(n+1)) \arrow[r, "J_{k,n+1}"] & \pi_{n+k+1}(S^{k+1}) \end{tikzcd}$$

Definition 2.1.4. We define the stable J-homomorphism to be the colimit:
$$J_k = \operatorname*{colim}_n J_{k,n} : \pi_k(\mathrm{SO}) \longrightarrow \pi_k(S^0)$$

Remark 2.1.5. $J_{k,n}$ stabilizes when $n > k+1$.

Remark 2.1.6. The definitions of the J-homomorphism above can be phrased stably:

(1) The colimit of the maps g_n in the first definition is a map $g : \mathrm{SO} \longrightarrow \Omega^\infty S^\infty$. The induced map
$$\pi_k(g) : \pi_k(\mathrm{SO}) \longrightarrow \pi_k(\Omega^\infty S^\infty) \simeq \pi_k(S^0)$$
is then the k-th stable J-homomorphism.
(2) In terms of framed cobordism, the stable homotopy group $\pi_k(S^0)$ classifies the framed-cobordism classes of k-dimensional manifolds with a framing on its stable normal bundle, when embedded in $\mathbb{R}^\infty$. A framing on the stable normal bundle of S^k is then a map $f : S^k \to \mathrm{SO}$. Again if $f_1, f_2 : S^k \to \mathrm{SO}$ are homotopic, then the corresponding stably framed k-dimensional manifolds are framed cobordant. From this point view we get the stable J-homomorphism $J_k : \pi_k(\mathrm{SO}) \to \pi_k(S^0)$.

(3) $f \in \pi_k(\mathrm{SO}) \simeq \pi_{k+1}(B\mathrm{SO})$ induces a virtual vector bundle ξ_f of dimensional 0 on S^{k+1}. The Thom space of ξ_f is a two-cell complex $\mathrm{Th}(\xi_f) = e^0 \cup e^{k+1}$. Again, $J(f)$ is defined to be the gluing map of the stable two-cell complex $\mathrm{Th}(\xi_f)$.

Remark 2.1.7. The three definitions of the J-homomorphisms above lead to different directions in homotopy theory. (1) leads to the units of ring spectra, studied in [ABG+14]. (2) is related to the work of Kervaire and Milnor in [KM63]. (3) leads to the computation of the image of the J-homomorphism by Adams in [Ada66], which we explain below.

Define the e-invariant of a stable map $f : S^{2k-1} \to S^0$ as below. Consider the cofiber sequence:

$$S^0 \longrightarrow S^0 \cup_f e^{2k} \longrightarrow S^{2k}.$$

Apply complex K-theory homology KU to this sequence. As KU_* is concentrated in even degrees, we get a short exact sequence:

$$0 \longrightarrow KU_0(S^0) \longrightarrow KU_0(S^0 \cup_f e^{2k}) \longrightarrow KU_0(S^{2k}) \longrightarrow 0.$$

This is not only an extension of abelian groups, but also of KU_0KU-comodules. As such, this short exact sequence corresponds to an element

$$e(f) \in \mathrm{Ext}^1_{KU_0KU}(KU(S^0), KU(S^{2k})).$$

This is the e**-invariant** of $f : S^{2k-1} \to S^0$.

Remark 2.1.8. KU_*KU is computed in [AHS71, Theorem 2.3]:

$$KU_*KU \simeq \left\{ f(u,v) \in \mathbb{Q}((u,v)) \,\middle|\, f(ht,kt) \in \mathbb{Z}\left[t, t^{-1}, \frac{1}{hk}\right], \forall h,k \in \mathbb{Z} \right\},$$

where $t \in KU_2(KU)$. In particular,

$$KU_0KU \simeq \{ f(w) \in \mathbb{Q}((w)) \mid f(\mathbb{Z}) \subseteq \mathbb{Z} \}.$$

Theorem 2.1.9. [Ada66, Theorem 1.1–1.6] *The image of the stable J-homomorphism $J_k : \pi_k(\mathrm{SO}) \to \pi_k(S^0)$ is described below:*

(1) J_k is injective when $k \equiv 0, 1 \mod 8$.
(2) The image of J_{8k+3} is a cyclic group of order D_{4k+2}, the denominator of $\frac{B_{4k+2}}{8k+4}$. The image of J_{8k-1} is a cyclic group of order D_{4k} or $2D_{4k}$.
(3) The image of J_{4k-1} in $\pi_{4k-1}(S^0)$ is a direct summand. The direct sum splitting is accomplished by the homomorphism $e' \circ J_{4k-1} : \pi_{4k-1}(\mathrm{SO}) \twoheadrightarrow \mathbb{Z}/D_{2k}$ associated to the e-invariant.

2.2. **K-theory and formal groups of height 1.** In this subsection, we will discuss the relation between complex K-theory and formal groups of height 1. In the end, we will identify $\mathrm{Ext}^1_{KU_0KU}(KU(S^0), KU(S^{2k}))$ to a group cohomology. References on formal groups and chromatic homotopy theory can be found in [Ada95; Hop99; Lur10].

Definition 2.2.1. A cohomology theory E is called **complex oriented** if it is multiplicative and it satisfies the Thom isomorphism theorem for complex vector bundles. It is **even periodic** if E_* is concentrated in even degrees and there is a $\beta \in E^{-2}(\mathrm{pt})$ such that β is invertible in E_*.

Proposition 2.2.2. *Let E be a complex oriented evenly periodic cohomology theory, then*

(1) $E^*(\mathbb{CP}^\infty) \simeq E_*[\![t]\!]$ where $t \in E^2(\mathbb{CP}^\infty)$ is the first Chern class of the tautological line bundle ξ over $\mathbb{CP}^\infty$.
(2) Let $p_i : \mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$ be the projection map of the i-th component for $i = 1, 2$. Then $E^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \simeq E_*[\![t_1, t_2]\!]$, where $t_i = p_i^* c_1(\xi)$.

*(3) The tensor product of line bundles over $\mathbb{CP}^\infty$ induces a E_0-**formal group** structure on $\operatorname{Spf} E(\mathbb{CP}^\infty)$. Denote this formal group associated to a complex-oriented cohomology theory E by $\widehat{G}_E$.*

(4) $E(S^{2k})$ is identified with $\boldsymbol{\omega}^{\otimes k}$, the k-th tensor power of the sheaf of invariant differentials on $\widehat{G}_E$.

Examples 2.2.3. Here are two examples of complex oriented cohomology theories and their associated formal groups:

(1) For ordinary cohomology theory H, $\widehat{G}_H \simeq \widehat{G}_a$ is the additive formal group.
(2) For complex K-theory, $\widehat{G}_{KU} \simeq \widehat{G}_m$ is the multiplicative formal group.

Theorem 2.2.4 (Quillen)**.** *The formal group associated to the periodic complex cobordism $MP := \bigvee_{i\in\mathbb{Z}} \Sigma^{2i} MU$ is the universal formal group. More precisely, the pair $(MP_0, MP_0(MP))$ classifies formal groups and isomorphisms between formal groups.*

As $\widehat{G}_{MP}$ is the universal formal group, one might wonder given a formal group over a ring R classified by a map $MP_0 \to R$, is $MP_*(-) \otimes_{MP_0} R$ a cohomology theory? The answer is yes when the map $MP_0 \to R$ satisfies certain flatness conditions. In particular, we have

Theorem 2.2.5 (Conner-Floyd)**.** *Let $\theta : MP_0 \to KU_0$ be the map that classifies $\widehat{G}_m$. Then $KU_*(X) \simeq MP_0(X) \otimes_{MP_0} KU_*$ and*

$$KU_0 KU \simeq KU_0 \otimes_{MP_0} MP_0(MP) \otimes_{MP_0} KU_0.$$

The map of Hopf algebroids $\theta : (MP_0, MP_0(MP)) \to (KU_0, KU_0KU)$ induces a map of comodule ext-groups:

$$\theta_* : \operatorname{Ext}^1_{MP_0MP}(MP(S^0), MP(S^{2k})) \to \operatorname{Ext}^1_{KU_0KU}(KU(S^0), KU(S^{2k}))$$

The e-invariant lives in the target and the source is on the E_2-page of the **Adams-Novikov spectral sequence** (ANSS):

$$E_2^{s,t} = \operatorname{Ext}^s_{MP_0MP}(MP(S^0), MP(S^t)) \Longrightarrow \pi_{t-s}(S^0).$$

Theorem 2.2.6. *The e-invariant map $e : \pi_{2k-1}(S^0) \to \operatorname{Ext}^1_{KU_0KU}(KU(S^0), KU(S^{2k}))$ factors through θ_*. Moreover, θ_* is an isomorphism when restricted the image of the J-homomorphism.*

Remark 2.2.7. The computation of the 1-line in the ANSS and its comparison with the images of the J-homomorphisms can be found in [Rav86, Section 5.3].

Thus, the image of the J-homomorphism is computed by its image under the e-invariant map in the KU_0KU-Ext groups. Completed at a prime p, these Ext-groups are identified with group cohomology.

Corollary 2.2.8. *As $MP_0(MP)$ classifies isomorphisms between formal group, $\operatorname{Spec} KU_0KU$ is isomorphic to the group scheme $\operatorname{Aut}(\widehat{G}_m)$ over $\mathbb{Z}$.*

Theorem 2.2.9. [Hov02] *Let (A, Γ) be a Hopf algebroid.*

(1) $(\operatorname{Spec} A, \operatorname{Spec} \Gamma)$ is a groupoid scheme.
(2) There is an equivalence of abelian categories between (A, Γ)-comodules and quasicoherent sheaves over the quotient stack $\operatorname{Spec} A // \operatorname{Spec} \Gamma$.

Corollary 2.2.10. *The stack associated to the pair (KU_0, KU_0KU) is the classifying stack*

$$B\operatorname{Aut}(\widehat{G}_m) := \operatorname{Spec} \mathbb{Z} // \operatorname{Aut}(\widehat{G}_m).$$

As a result, the e-invariant lives in

$$\begin{aligned}\operatorname{Ext}^1_{KU_0KU}(KU(S^0), KU(S^{2k})) &\simeq R^1\operatorname{Hom}_{\operatorname{Qcoh}(B\operatorname{Aut}(\widehat{G}_m))}(\mathcal{O}, \boldsymbol{\omega}^{\otimes k}) \\ &\simeq H^1(B\operatorname{Aut}(\widehat{G}_m), \boldsymbol{\omega}^{\otimes k}).\end{aligned}$$

The group scheme $\mathrm{Aut}(\widehat{G}_m)$ is not a constant group scheme over $\mathbb{Z}$. However, it becomes one when restricted to the closed points $\operatorname{Spec}\mathbb{F}_p \in \operatorname{Spec}\mathbb{Z}$. This is even true over $\operatorname{Spf}\mathbb{Z}_p$, the formal neighborhood of $\operatorname{Spec}\mathbb{F}_p$ in $\operatorname{Spec}\mathbb{Z}$.

Lemma 2.2.11. *Over $\mathbb{F}_p$ or $\mathbb{Z}_p$, $\mathrm{Aut}(\widehat{G}_m) \simeq \underline{\mathbb{Z}_p^\times}$ as a constant pro-group scheme.*

Thus for the p-adic e-invariant, it suffices to compute

$$e \in H^1(B\mathrm{Aut}(\widehat{G}_m)_p^\wedge, \boldsymbol{\omega}^{\otimes k}) \simeq H^1\left(B\mathbb{Z}_p^\times, \boldsymbol{\omega}^{\otimes k}\right) \simeq H_c^1(\mathbb{Z}_p^\times; \left(KU_p^\wedge\right)_{2k}), \tag{2.2.12}$$

where $KU_p^\wedge$ is the p-completion of the complex K-theory and $\mathbb{Z}_p^\times$ acts on $(KU_p^\wedge)_{2k}$ by the k-th power map.

2.3. **The homotopy fixed point spectral sequence.** Let G be a finite group. Recall that the group cohomology of G is the derived functor of G-fixed points. If G acts on a spectrum E, then the group cohomology of G with coefficients in $\pi_*(E)$ computes homotopy groups of E^{hG}, the **homotopy fixed point spectrum** of E under the G-action.

Definition 2.3.1. Let $G_+^{\wedge\bullet} \wedge E$ be the group action cosimpicial spectrum. The homotopy fixed points of this action is defined to be the totalization of this cosimplicial spectrum:

$$E^{hG} := \mathrm{Map}(\Sigma^\infty EG_+, E)^G \simeq \left(\mathrm{Tot}\left[\mathrm{Map}(G_+^\bullet, E)\right]\right)^G.$$

The Bousfield-Kan spectral sequence associated to this cosimpicial spectrum is called the **homotopy fixed point spectral sequence** (HFPSS), whose E_2-page is identified with

$$E_2^{s,t} = H^s(G; \pi_t(E)) \Longrightarrow \pi_{t-s}(E^{hG}). \tag{2.3.2}$$

In (2.2.12), we showed that the p-adic e-invariant is in $H^1\left(\mathbb{Z}_p^\times; \left(KU_p^\wedge\right)_{2k}\right)$, where $\mathbb{Z}_p^\times$ acts on the p-adic K-theory spectrum by the Adams operations. In [DH04], Devinatz and Hopkins defined E^{hG} for *pro-finite* groups and showed that the E_2-page of the associated HFPSS consists of *continuous* group cohomology of G. Moreover, they proved

Theorem 2.3.3. *Let $\mathbb{Z}_p^\times$ acts on the p-adic K-theory spectrum by Adams operation. Then the homotopy fixed points $\left(KU_p^\wedge\right)^{h\mathbb{Z}_p^\times}$ is equivalent to $S^0_{K(1)}$, the $K(1)$-**local sphere**. Here, $S^0_{K(1)}$ is the Bousfield localization of the sphere spectrum S^0 at the Morava K-theory $K(1) := KU/p$.*

For a purpose of this paper, we need to study finite Galois extensions of $S^0_{K(1)}$ in the sense of [Rog08].

Definition 2.3.4. Define $S^0_{K(1)}(p^v)$ to be the homotopy fixed point spectrum $\left(KU_p^\wedge\right)^{h(1+p^v\mathbb{Z}_p)}$ under the Adams operations. This notation was used in [LN12, Definition 5.10].

$S^0_{K(1)}(p^v)$ is a $(\mathbb{Z}/p^v)^\times$-Galois extension of $S^0_{K(1)}$. This shows that there is a Galois correspondence between open subgroups of $\mathbb{Z}_p^\times$ and finite Galois extensions of $S^0_{K(1)}$. We consider the following family of open subgroups of $\mathbb{Z}_p^\times$ nested in a descending chain for $p > 2$:

$$\mathbb{Z}_p^\times \supsetneq 1 + p\mathbb{Z}_p \supsetneq 1 + p^2\mathbb{Z}_p \supsetneq 1 + p^3\mathbb{Z}_p \supsetneq \cdots,$$

and for $p = 2$:

$$\mathbb{Z}_2^\times = 1 + 2\mathbb{Z}_p \supsetneq 1 + 2^2\mathbb{Z}_p \supsetneq 1 + 2^3\mathbb{Z}_p \supsetneq \cdots.$$

Now we are going to compute $\pi_*\left(S^0_{K(1)}(p^v)\right)$ using HFPSS, whose E_2-page is

$$E_2^{s,t} = H_c^s\left(1 + p^v\mathbb{Z}_p; \left(KU_p^\wedge\right)_t\right) \Longrightarrow \pi_{t-s}\left(S^0_{K(1)}(p^v)\right). \tag{2.3.5}$$

One reference of this computation (and also the HFPSS at height n) is [Hen17]. There are two cases.

Case I: $p>2$ or $p=2$ and $v\geq 2$. In this case, $\mathbb{Z}_p^\times$ and $1+4\mathbb{Z}_2$ are pro-cyclic. Let g be a topological generator in $\mathbb{Z}_p^\times$ for $p>2$ and in $1+4\mathbb{Z}_2$ for $p=2$. Then for $p>2$, $1+p^v\mathbb{Z}_p=\left\langle g^{(p-1)p^{v-1}}\right\rangle$ and for $p=2$, $1+2^v\mathbb{Z}_2=\left\langle g^{2^{v-2}}\right\rangle$. Let $n=1$ if $G=\mathbb{Z}_p^\times$ and $n=(p-1)p^{v-1}$ if $G=1+p^v\mathbb{Z}_p$ for $p>2$, and $n=2^{v-2}$ if $G=1+2^v\mathbb{Z}_2$. The minimal continuous projective resolution for $\mathbb{Z}_p$ in $\mathbb{Z}_p[\![G]\!]$ is

$$0\longrightarrow \mathbb{Z}_p[\![G]\!]\xrightarrow{1-g^n}\mathbb{Z}_p[\![G]\!]\xrightarrow{g^n\mapsto 1}\mathbb{Z}_p\longrightarrow 0. \tag{2.3.6}$$

Since the length of the resolution is 1, the HFPSS collapses on E_2-page. The p-adic Adams operations on $KU_p^\wedge$ realize $\left(KU_p^\wedge\right)_{2t}$ as the t-th power representation of G. From this we get when $G=\mathbb{Z}_p^\times$ for $p>2$:

$$H_c^s(\mathbb{Z}_p^\times;\left(KU_p^\wedge\right)_t)=\begin{cases}\mathbb{Z}_p, & s=0,1 \text{ and } t=0;\\ \mathbb{Z}/p^{v_p(k)+1}, & s=1\text{ and } t=2(p-1)k;\\ 0, & \text{otherwise.}\end{cases} \tag{2.3.7}$$

$$\Longrightarrow \pi_i\left(S^0_{K(1)}\right)=\begin{cases}\mathbb{Z}_p, & i=0,-1;\\ \mathbb{Z}/p^{v_p(k)+1}, & i=2(p-1)k-1;\\ 0, & \text{otherwise.}\end{cases} \tag{2.3.8}$$

and when $G=1+p^v\mathbb{Z}_p$ ($v>1$ if $p=2$):

$$H_c^s(1+p^v\mathbb{Z}_p;\left(KU_p^\wedge\right)_t)=\begin{cases}\mathbb{Z}_p, & s=0,1 \text{ and } t=0;\\ \mathbb{Z}/p^{v_p(k)+v}, & s=1\text{ and } t=2k\neq 0;\\ 0, & \text{otherwise.}\end{cases}$$

$$\Longrightarrow \pi_i\left(S^0_{K(1)}(p^v)\right)=\begin{cases}\mathbb{Z}_p, & i=0,-1;\\ \mathbb{Z}/p^{v_p(k)+v}, & i=2k-1\neq -1;\\ 0, & \text{otherwise.}\end{cases} \tag{2.3.9}$$

Case II: $p=2$ and $G=\mathbb{Z}_2^\times$. In this case, $\mathbb{Z}_2^\times$ is not pro-cyclic. Rather, we have

$$\mathbb{Z}_2^\times\simeq\{\pm1\}\times(1+4\mathbb{Z}_2).$$

Notice $(KU_2^\wedge)^{h\mathbb{Z}/2}\simeq KO_2^\wedge$, where $\mathbb{Z}/2$ acts by complex conjugation on $KU_2^\wedge$. The homotopy groups of $KO_2^\wedge$ are given by:

(2.3.10)

$i \mod 8$	0	1	2	3	4	5	6	7
$\pi_i(KO_2^\wedge)$	$\mathbb{Z}_2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}_2$	0	0	0

Let $g\in 1+4\mathbb{Z}_2$ be a topological generator. g acts on π_{4l} by multiplication by g^{2l} and on π_{8l+1} and π_{8l+2} by identity. The E_2-page of the HFPSS is

$$E_2^{s,t}=H_c^s(1+4\mathbb{Z}_2;\pi_t(KO_2^\wedge))=\begin{cases}\mathbb{Z}_2, & s=0,1\text{ and } t=0;\\ \mathbb{Z}/2, & s=0,1\text{ and } t\equiv 1,2 \mod 8;\\ \mathbb{Z}/2^{v_2(k)+3}, & s=1\text{ and } t=4k\neq 0;\\ 0, & \text{otherwise.}\end{cases} \tag{2.3.11}$$

Proposition 2.3.12. *The extension problems of this spectral sequence are trivial.*

Proof. We need to solve the extension problems when $t-s=0$ or $t-s\equiv 1 \mod 8$. The following explanation is from Mark Behrens.

The extension when $t-s=0$ is trivial, because there is no non-trivial extension of $\mathbb{Z}/2$ by $\mathbb{Z}_2$.

When $t-s\equiv 1 \mod 8$, we recall that the Hopf element $\eta\in\pi_1\left(S^0\right)$ has order 2. η is represented in (2.3.11) by the non-zero element of $H^0(1+4\mathbb{Z}_2;\pi_1(KO_2^\wedge))=\mathbb{Z}/2$. If the extension at $t-s=1$ were nontrivial, then

$\pi_1\left(S^0_{K(1)}\right) \simeq \mathbb{Z}/4$. From the short exact sequence

$$0 \longrightarrow H^1_c(1+4\mathbb{Z}_2; \pi_0(KO_2^\wedge)) \longrightarrow \pi_1\left(S^0_{K(1)}\right) \longrightarrow H^0_c(1+4\mathbb{Z}_2; \pi_1(KO_2^\wedge)) \longrightarrow 0,$$

η would then have order 4 in $\pi_1\left(S^0_{K(1)}\right)$. This contradicts the fact that the order of $\eta \in \pi_1(S^0)$ is 2.

For the general $t-s = 8k+1$ case, replace η by $\beta^k \cdot \eta \in \pi_{8k+1}(KO)$ in the argument above, where $\beta \in \pi_8(KO)$ is the Bott element. $\square$

In conclusion, we get when $p = 2$,

$$\pi_i\left(S^0_{K(1)}\right) = \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}/2, & i = 0; \\ \mathbb{Z}_2, & i = -1; \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2, & i \equiv 1 \mod 8; \\ \mathbb{Z}/2, & i \equiv 0, 2 \mod 8 \text{ and } i \neq 0; \\ \mathbb{Z}/2^{v_2(k)+3}, & i = 4k-1 \neq -1; \\ 0, & \text{otherwise.} \end{cases} \tag{2.3.13}$$

Alternatively, we can apply HFPSS on $G = \mathbb{Z}_2^\times$ directly. The E_2-page is computed using the **Hochschild-Serre spectral sequence** (HSSS) whose E_2-page is

$$E_2^{p,q} = H^p_c(1+4\mathbb{Z}_2; H^q(\mathbb{Z}/2; (KU_2^\wedge)_t)) \Longrightarrow H^{p+q}_c(\mathbb{Z}_2^\times; (KU_2^\wedge)_t). \tag{2.3.14}$$

This spectral sequence collapses on the E_2-page and we have

$$H^s_c(\mathbb{Z}_2^\times; (KU_2^\wedge)_t) = \begin{cases} \mathbb{Z}_2, & s = 0, 1 \text{ and } t = 0; \\ \mathbb{Z}/2^{v_2(k)+3}, & s = 1 \text{ and } t = 4k \neq 0; \\ \mathbb{Z}/2, & s = 1 \text{ and } t = 4k+2; \\ \mathbb{Z}/2, & s \geq 2 \text{ and } t \text{ even}; \\ 0, & \text{otherwise.} \end{cases}$$

3. Constructions of the Dirichlet J-spectra and $K(1)$-local spheres

Let $\chi : (\mathbb{Z}/N)^\times \to \mathbb{C}^\times$ be a primitive Dirichlet character of conductor N. In this section, we construct $J(N)^{h\chi}$, the Dirichlet J-spectrum in three steps:

(1) Identify an integral model of the J-spectrum, a ring spectrum whose Hurewicz map detects the image of the J-homomorphism in $\pi_*(S^0)$.
(2) Define $J(N)$, "the J-spectrum with μ_N-level structure" using local structures of the finite group scheme μ_N and the Hopkins-Miller theorem. $J(N)$ comes with a natural $(\mathbb{Z}/N)^\times$-action by assembling the $(\mathbb{Z}/p^v)^\times$-Galois action at each prime.
(3) Construct a Moore spectrum $M(\mathbb{Z}[\chi])$ with a $(\mathbb{Z}/N)^\times$-action that lifts the $(\mathbb{Z}/N)^\times$-action on $\mathbb{Z}[\chi]$ induced by χ. Here $\mathbb{Z}[\chi]$ is the subalgebra of $\mathbb{C}$ generated by the image of χ. This construction is non-trivial since taking Moore spectrum is not functorial. We give an explicit construction of the Moore spectra with group actions suggested by Charles Rezk.

From these data, we define the Dirichlet J-spectrum associated to χ by

$$J(N)^{h\chi} := \operatorname{Map}\left(M(\mathbb{Z}[\chi]), J(N)\right)^{h(\mathbb{Z}/N)^\times}.$$

This definition leads to a spectral sequence whose E_2-page consists of derived χ-eigenspaces of $\pi_*(J(N))$:

$$E_2^{s,t} \simeq \operatorname{Ext}^s_{\mathbb{Z}[(\mathbb{Z}/N)^\times]}(\mathbb{Z}[\chi], \pi_t(J(N))) \Longrightarrow \pi_{t-s}\left(J(N)^{h\chi}\right).$$

The actual computation of $J(N)^{h\chi}$ is carried out by studying its local structures. Rationally, the Dirichlet J-spectra are contractible unless χ is trivial. Completed at each prime, the $J(N)^{h\chi}$ splits into a wedge sum of Dirichlet $K(1)$-local spheres. The Dirichlet $K(1)$-local spheres are constructed in a similar way as the Dirichlet J-spectra, but the p-adic Moore spectra with a prescribed $(\mathbb{Z}/N)^\times$-action induced χ is constructed by Cooke's obstruction theory in [Coo78]. This splitting of p-completion of integral Moore spectra uses the uniqueness part of Cooke's obstruction theory.

3.1. **An integral model of the J-spectrum.** In the previous section, we have explained the relations between the images of the stable J-homomorphisms and the $K(1)$-local spheres:

$$\mathrm{Im}(J_{4k-1})^\wedge_p \simeq \pi_{4k-1}\left(S^0_{K(1)}\right), k > 0.$$

We are now going to define an integral J-spectrum by assembling the $K(1)$-local spheres at each prime.

Theorem 3.1.1. [Bou79, Corollary 4.5, 4.6] *Let $J = S^0_{KU}$, the Bousfield localization of the sphere spectrum S^0 at complex K-theory.*

(1) The J-spectrum and the KU/p-local spheres are related by the arithmetic fracture square:

$$\begin{tikzcd} J := S^0_{KU} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \prod_p S^0_{KU/p} \arrow[d, "L_{\mathbb{Q}}"] \\ S^0_{\mathbb{Q}} \arrow[r, "h_{\mathbb{Q}}"'] & \left(\prod_p S^0_{KU/p}\right)_{\mathbb{Q}} \end{tikzcd} \tag{3.1.2}$$

Here $h_{\mathbb{Q}}$ is the rational Hurewicz map and $L_{\mathbb{Q}}$ is the rationalization map.

(2) Denote the denominator of $B_{2k}/4k$ by D_{2k}. We have:

$$\pi_i(J) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}/2, & i = 0; \\ \mathbb{Q}/\mathbb{Z}, & i = -2; \\ \mathbb{Z}/D_{|2k|}, & i = 4k-1 \neq -1; \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2, & i \equiv 1 \mod 8; \\ \mathbb{Z}/2, & i \equiv 0, 2 \mod 8 \text{ and } i \neq 0; \\ 0, & \text{otherwise.} \end{cases} \tag{3.1.3}$$

Corollary 3.1.4. *$J^\wedge_p \simeq S^0_{KU/p}$ and $J_{(p)} \simeq S^0_{E(1)}$ is the Bousfield localization of S^0 at $E(1) := BP\langle 1\rangle$.*

Remark 3.1.5. $J := S^0_{KU}$ is an $\mathbb{E}_\infty$-ring spectrum since it is the localization of an $\mathbb{E}_\infty$-ring spectrum by [EKM+97].

Proof. (3.1.2) is the almost same homotopy pullback diagram for S^0_{KU} as in the proof of [Bou79, Corollary 4.7], except for the lower left corner – the rationalization of S^0_{KU} is a priori $S^0_{KU\mathbb{Q}}$, where $KU\mathbb{Q} := KU \wedge M\mathbb{Q}$ is the rational K-spectrum. Now it remains to show $KU\mathbb{Q}$ and $H\mathbb{Q}$ are Bousfield equivalent. This follows from the facts that $KU\mathbb{Q}$ and the periodic $HP\mathbb{Q} := \bigvee_i \Sigma^{2i} H\mathbb{Q}$ are equivalent cohomology theories via the Chern character map and that $HP\mathbb{Q}$ is Bousfield equivalent to $H\mathbb{Q}$.

The computation of $\pi_*(J)$ is the integral version of that of the $\pi_*\left(S^0_{E(1)}\right)$ in [Lur10, Theorem 6, Lecture 35]. The arithmetic fracture square (3.1.2) induces a long exact sequence of homotopy groups:

$$\cdots \to \pi_i(J) \longrightarrow \pi_i\left(S^0_{\mathbb{Q}}\right) \oplus \prod_p \pi_i\left(S^0_{KU/p}\right) \longrightarrow \left(\prod_p \pi_i\left(S^0_{KU/p}\right)\right) \otimes \mathbb{Q} \longrightarrow \pi_{i-1}(J) \to \cdots$$

Notice that $\left(\prod_p \pi_i\left(S^0_{KU/p}\right)\right) \otimes \mathbb{Q} = 0$ unless $i = 0$ or -1 and $\pi_i\left(S^0_{\mathbb{Q}}\right) = 0$ unless $i = 0$, we have $\pi_i(J) \simeq \prod_p \pi_i\left(S^0_{KU/p}\right)$ unless $i \in \{-2,-1,0\}$. In those three cases, there is an exact sequence:

$$0 \to \pi_0(J) \to \mathbb{Q} \oplus \prod_p \mathbb{Z}_p \oplus \mathbb{Z}/2 \xrightarrow{h_0} \prod_p \mathbb{Q}_p \to \pi_{-1}(J) \to \prod_p \mathbb{Z}_p \xrightarrow{h_{-1}} \prod_p \mathbb{Q}_p \to \pi_{-2}(J) \to 0.$$

As h_0 is surjective and h_{-1} is injective, we have

$$\pi_0(J) \simeq \mathbb{Z} \oplus \mathbb{Z}/2, \quad \pi_{-1}(J) = 0, \quad \pi_{-2}(J) \simeq \mathbb{Q}/\mathbb{Z}.$$

For $i \neq 0,-1,-2$, we recover $\pi_i(J)$ from Section 2.3 and Theorem 1.1.7. □

Remark 3.1.6. We call S^0_{KU} the J-spectrum because the Hurewicz map (also the KU-localization map) $S^0 \longrightarrow S^0_{KU}$ detects the image of J_{4k-1}. But $\pi_k(J)$ is not the same as the image of the stable J-homomorphism in general. The spectrum J is non-connective and has an extra $\mathbb{Z}/2$-summand in $\pi_0(J)$ and $\pi_{8k+1}(J)$ for $k > 0$. For details, see [Ada66].

3.2. ***J*-spectra with level structures.** We will now add level structures to the J-spectrum. Let μ_N be the N-torsion sub-group scheme of $\widehat{G}_m$. Define $\mathcal{M}_{mult}(N)$ to be the moduli stack of globally height 1 formal groups with μ_N-level structures. R-points of $\mathcal{M}_{mult}(N)$ are given by:

$$\mathcal{M}_{mult}(N)(R) := \left\{ \left(\widehat{G}, \eta : \mu_N \xrightarrow{\sim} \widehat{G}[N]\right) \middle| \begin{array}{c} \widehat{G} \text{ is a formal group over } R \\ \text{that has height 1 at all primes} \end{array} \right\}.$$

The local structures of $\mathcal{M}_{mult}(N)$ are determined by the local behaviors of μ_N.

Lemma 3.2.1. *$\widehat{G}_m$ has no non-trivial finite subgroup over $\mathbb{Q}$. Over $\mathbb{Z}_p$, finite subgroups of $\widehat{G}_m$ are of the form μ_{p^v} for some $v \geq 0$. As a result, $(\mu_N)_{\mathbb{Q}} \simeq 0$ for all N and $(\mu_N)^\wedge_p \simeq \mu_{p^v}$, where $v = v_p(N)$.*

Proof. This follows from the facts that $\mathrm{End}_{\mathbb{Q}}(\widehat{G}_m) \simeq \mathbb{Q}$ and $\mathrm{End}_{\mathbb{Z}_p}(\widehat{G}_m) \simeq \mathbb{Z}_p$. □

Proposition 3.2.2. *$(\mathcal{M}_{mult}(N))_{\mathbb{Q}} \simeq (\mathcal{M}_{mult})_{\mathbb{Q}}$. Fix a prime p and let $v = v_p(N)$, we have*

$$\mathcal{M}_{mult}(N)^\wedge_p \simeq \mathcal{M}_{mult}(p^v)^\wedge_p \simeq B(1 + p^v\mathbb{Z}_p).$$

Corollary 3.2.3. *$\mathcal{M}_{mult}(N) \simeq \mathcal{M}_{mult}(2N)$ for any odd number N.*

Proof. This follows from the fact $(\mathbb{Z}/2N)^\times$ is canonically isomorphic $(\mathbb{Z}/N)^\times$ if N is odd. □

Theorem 3.2.4 (Hopkins-Miller, Goerss-Hopkins)**.** [Rez98, Theorem 2.1] *Let $\mathcal{FG}$ denote the category whose objects are pairs (κ, Γ) where Γ is a finite height formal group over a finite field k of characteristic p and whose morphisms are pairs of maps $(i, f) : (\kappa_1, \Gamma_1) \to (\kappa_2, \Gamma_2)$, where $i : \kappa_1 \to \kappa_2$ is a ring homomorphism and $f : \Gamma_1 \xrightarrow{\sim} i^*\Gamma_2$ is an isomorphism of formal groups.*

Then there exists a functor $(\kappa, \Gamma) \to E_{\kappa,\Gamma}$ from $\mathcal{FG}^{\mathrm{op}}$ to the category of $\mathbb{E}_\infty$-ring spectra, such that

(1) $E_{\kappa,\Gamma}$ is a commutative ring spectra.
(2) There is a unit in $\pi_2(E_{\kappa,\Gamma})$.
(3) $\pi_{\mathrm{odd}} E_{\kappa,\Gamma} = 0$, which implies $E_{\kappa,\Gamma}$ is complex-oriented.
(4) The formal group associated to $E_{\kappa,\Gamma}$ is the universal deformation of (κ, Γ).

Proposition 3.2.5. *There is a sheaf $\mathcal{O}^{top}_{K(1)}$ of $K(1)$-local $\mathbb{E}_\infty$-ring spectra over the stack $\widehat{\mathcal{H}(1)} \simeq B\mathbb{Z}_p^\times := \mathrm{Spf}\,\mathbb{Z}_p//\mathbb{Z}_p^\times$ such that*

$$\Gamma\left(\mathcal{O}^{top}_{K(1)}, B\mathbb{Z}_p^\times\right) \simeq S^0_{K(1)}, \quad \Gamma\left(\mathcal{O}^{top}_{K(1)}, B(1 + p^v\mathbb{Z}_p)\right) \simeq S^0_{K(1)}(p^v) := \left(KU^\wedge_p\right)^{h(1+p^v\mathbb{Z}_p)}.$$

Remark 3.2.6. Let $\widehat{\mathcal{H}(h)}$ be the moduli stack of formal groups over p-complete local rings with height h reductions modulo the maximal ideal. The Hopkins-Miller theorem and the Goerss-Hopkins theorem imply there is a sheaf of $K(h)$-local $\mathbb{E}_\infty$-ring spectra $\mathcal{O}^{top}_{K(h)}$ over $\widehat{\mathcal{H}(h)}$ whose global section is the $K(h)$-local sphere $S^0_{K(h)}$. For the algebro-geometric properties of the stack $\widehat{\mathcal{H}(h)}$, see [Goe08, Chapter 7].

Corollary 3.2.9 implies $\mathcal{M}_{mult}(N)^\wedge_p \simeq \mathcal{M}_{mult}(p^v)^\wedge_p \to (\mathcal{M}_{mult})^\wedge_p$ is a $(\mathbb{Z}/p^v)^\times$-torsor for each prime p. Thus by Proposition 3.2.5 we can define $J(N)$, the **J-spectrum with μ_N-level structure** by setting $J(N)^\wedge_p := \mathcal{O}^{top}_{K(1)}(\mathcal{M}_{mult}(p^v)) \simeq S^0_{KU/p}(p^v)$ and $J(N)_\mathbb{Q} = S^0_\mathbb{Q}$ as follows:

Construction 3.2.7. $J(N)$ is the homotopy pullback of the following arithmetic fracture square as in (3.1.2):

$$\begin{tikzcd} J(N) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \prod_p S^0_{KU/p}\left(p^{v_p(N)}\right) \arrow[d, "L_\mathbb{Q}"] \\ S^0_\mathbb{Q} \arrow[r, "h_\mathbb{Q}"'] & \left(\prod_p S^0_{KU/p}\left(p^{v_p(N)}\right)\right)_\mathbb{Q} \end{tikzcd} \tag{3.2.8}$$

Here $h_\mathbb{Q}$ is the rational Hurewicz map and $L_\mathbb{Q}$ is the rationalization map. $h_\mathbb{Q}$ exists because the lower right corner in the diagram is a rational ring spectrum.

The $J(N)$ defined above satisfies the prescribed local properties:

Corollary 3.2.9. *$J(N)_\mathbb{Q} \simeq S^0_\mathbb{Q}$ for all N and $J(N)^\wedge_p \simeq S^0_{K(1)}(p^v)$, where $v = v_p(N)$. Moreover, $J(N) \simeq J(2N)$ for any odd number N.*

Proposition 3.2.10. *$J(N)$ admits a natural $(\mathbb{Z}/N)^\times$-action such that*
- *$(\mathbb{Z}/N)^\times$ acts on $J(N)_\mathbb{Q}$ trivially.*
- *$(\mathbb{Z}/N)^\times$ acts on $J(N)^\wedge_p \simeq S^0_{K(1)}(p^v)$ by the Galois action of its quotient group $(\mathbb{Z}/p^v)^\times$.*

Proof. Since the spectrum $S^0_{K(1)}(p^v)$ is a $(\mathbb{Z}/p^v)^\times$-Galois extension of $S^0_{K(1)}$, it admits a natural $(\mathbb{Z}/p^v)^\times$-action. As a result the product $\prod_p S^0_{KU/p}\left(p^{v_p(N)}\right)$ admits a natural $(\mathbb{Z}/N)^\times \simeq \prod_{p\mid N}(\mathbb{Z}/p^v)^\times$-action. (When $p \nmid N$, $(\mathbb{Z}/N)^\times$ acts on $S^0_{KU/p}$ trivially). The spectrum $\left(\prod_p S^0_{KU/p}\left(p^{v_p(N)}\right)\right)_\mathbb{Q}$ in the lower right corner of (3.2.8) then inherits a $(\mathbb{Z}/N)^\times$-action from that on $\prod_p S^0_{KU/p}\left(p^{v_p(N)}\right)$.

We now need to check the rational Hurewicz map $h_\mathbb{Q}$ in (3.2.8) is $(\mathbb{Z}/N)^\times$-equivariant. As both spectra are rational, it suffices to check the induced maps on homotopy groups are equivariant by Cooke's obstruction theory (see Section 3.3). Since $\pi_*(S^0_\mathbb{Q})$ is concentrated in π_0 and $(\mathbb{Z}/N)^\times$ acts on it trivially, it reduces to checking $(\mathbb{Z}/N)^\times$ acts $\pi_0\left(S^0_{KU/p}\left(p^{v_p(N)}\right)_\mathbb{Q}\right)$ trivially. Recall from Definition 2.3.4, $S^0_{KU/p}(p^v) := \left(KU^\wedge_p\right)^{h(1+p^v\mathbb{Z}_p)}$. The HFPSS in Section 2.3 shows

$$\pi_0\left(S^0_{KU/p}(p^v)_\mathbb{Q}\right) \simeq H^0_c\left(1+p^v\mathbb{Z}_p; \pi_0\left(KU^\wedge_p\right)\right) \otimes \mathbb{Q}.$$

As the Adams operation ψ^a acts on $\pi_0\left(KU^\wedge_p\right)$ trivially for all $a \in \mathbb{Z}_p^\times$, the residual $(\mathbb{Z}/p^v)^\times$-action on the group cohomology $H^*\left(1+p^v\mathbb{Z}_p; \pi_0\left(KU^\wedge_p\right)\right)$ is also trivial. Hence $(\mathbb{Z}/p^v)^\times$ acts trivially on $\pi_0\left(S^0_{KU/p}(p^v)_\mathbb{Q}\right)$.

We have shown the rational Hurewicz map $h_\mathbb{Q}$ is $(\mathbb{Z}/N)^\times$-equivariant. Then $J(N)$ as the homotopy pullback in (3.2.8) of a diagram of $(\mathbb{Z}/N)^\times$-equivariant maps of spectra has a natural $(\mathbb{Z}/N)^\times$-action with the prescribed local properties. ∎

Proposition 3.2.11. *$J(N)$ is a KU-local $\mathbb{E}_\infty$-ring spectrum, with $(\mathbb{Z}/N)^\times$ acting on it by $\mathbb{E}_\infty$-ring automorphisms as described in Proposition 3.2.10.*

Proof. This proposition contains three parts:

(1) $J(N)$ is an $\mathbb{E}_\infty$-ring spectrum since it is the homotopy pullback of a diagram of $\mathbb{E}_\infty$ maps between $\mathbb{E}_\infty$-ring spectra.
(2) $J(N)$ is KU-local since $J(N)_p^\wedge \simeq S^0_{KU/p}\left(p^{v_p(N)}\right)$ is KU/p-local for all primes p by Corollary 3.2.9.
(3) The action of $\left(\mathbb{Z}/p^{v_p(N)}\right)^\times$ on $J(N)_p^\wedge \simeq S^0_{KU/p}\left(p^{v_p(N)}\right)$ is $\mathbb{E}_\infty$ by the Goerss-Hopkins theorem. Thus the action of $(\mathbb{Z}/N)^\times \simeq \prod_{p|N}\left(\mathbb{Z}/p^{v_p(N)}\right)^\times$ is $\mathbb{E}_\infty$ on the upper right corner of (3.2.8). This implies the induced $(\mathbb{Z}/N)^\times$-action on lower right corner is also $\mathbb{E}_\infty$. The trivial $(\mathbb{Z}/N)^\times$-action on $S^0_{\mathbb{Q}}$ is $\mathbb{E}_\infty$. We conclude $(\mathbb{Z}/N)^\times$ acts by $\mathbb{E}_\infty$-ring maps on $J(N)$ in Proposition 3.2.10, since the action is assembled from $\mathbb{E}_\infty$-actions on the other three corners of (3.2.8).

$\square$

Remark 3.2.12. The homotopy fixed points $J(N)^{h(\mathbb{Z}/N)^\times}$ is not equivalent to J, unless N is a power of 2. As a result, $J(N)$ is in general NOT a $(\mathbb{Z}/N)^\times$-Galois extension of J. One explicit example is when $N=3$, we have

$$\left(J(3)^{h(\mathbb{Z}/3)^\times}\right)_2^\wedge \simeq \left(S^0_{KU/2}\right)^{h(\mathbb{Z}/3)^\times} \simeq \left(S^0_{KU/2}\right)_{h(\mathbb{Z}/3)^\times} \simeq (B\Sigma_2)_{KU/2} \not\simeq S^0_{KU/2} \simeq J_2^\wedge.$$

Here we use the following facts:

- Homotopy fixed points commute with p-completion.
- $J(3)_2^\wedge \simeq S^0_{KU/2}$ by Corollary 3.2.9.
- Homotopy fixed points of finite group actions in $\mathbf{Sp}_{K(1)}$ are equivalent to homotopy orbits.
- $(\mathbb{Z}/3)^\times$ acts on $S^0_{KU/2}$ trivially and $(\mathbb{Z}/3)^\times \simeq C_2 \simeq \Sigma_2$.
- $(B\Sigma_p)_+ \simeq S^0_{KU/p} \times S^0_{KU/p}$ in $\mathbf{Sp}_{K(1)}$ by [Hop14, Lemma 3.1].

In general, $J(N)^{h(\mathbb{Z}/N)^\times}$ is equivalent to J after inverting $\prod_{p|N}(p-1)$.

Parallel to (3.1.3), we now compute $\pi_*(J(N))$.

Proposition 3.2.13. *The computation of $\pi_*(J(N))$ has two cases: $4 \mid N$ and N is odd (since $J(N) \simeq J(2N)$ for odd N). Define $D_{2k,N}$ by*

$$D_{2k,N} = \begin{cases} ND_{2k}/(2\Pi), & \text{if } 4 \mid N; \\ ND_{2k}/\Pi, & \text{if } 2 \nmid N, \end{cases} \quad \text{where } \Pi = \prod_{p|N,(p-1)|(2k)} p.$$

When $4 \mid N$, we get

$$\pi_i(J(N)) = \begin{cases} \mathbb{Z}, & i = 0; \\ \mathbb{Q}/\mathbb{Z}, & i = -2; \\ \mathbb{Z}/D_{|2k|,N}, & i = 4k-1 \neq -1; \\ \mathbb{Z}/N, & i \equiv 1 \mod 4; \\ 0, & \text{otherwise.} \end{cases} \tag{3.2.14}$$

When N is odd, we get

$$\pi_i(J(N)) = \begin{cases} \mathbb{Z}\oplus\mathbb{Z}/2, & i=0;\\ \mathbb{Q}/\mathbb{Z}, & i=-2;\\ \mathbb{Z}/D_{|2k|,N}, & i=4k-1\neq -1;\\ \mathbb{Z}/N\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2, & i\equiv 1 \mod 8;\\ \mathbb{Z}/N, & i\equiv 5 \mod 8;\\ \mathbb{Z}/2, & i\equiv 0,2 \mod 8 \text{ and } i\neq 0;\\ 0, & \text{otherwise.}\end{cases}$$

Remark 3.2.15. One can check from (3.2.14) that

$$\mathrm{Hom}(\pi_i(J(4N)),\mathbb{Q}/\mathbb{Z}) \simeq (\pi_{-2-i}(J(4N)))^{\wedge}$$

holds for all N and i, where $(-)^{\wedge}$ is the profinite completion of a group. The formula is true up to summands of $\mathbb{Z}/2$ for $J(N)$ when N is odd. We will see in Corollary 5.3.18 that this isomorphism is the result of Brown-Comenetz duality $I_{KU}(J(4N))\simeq\Sigma^2 J(4N)\wedge M(\widehat{\mathbb{Z}})$. In particular, $\pi_{4k-1}(J(4))\simeq\pi_{4k-1}(J)=\mathbb{Z}/D_{|2k|}$, whose order is equal to the denominator of $\zeta(1-2k)$ (expressed as a fraction in lowest terms). The suggested Brown-Comenetz duality for $J(4)$ is similar to the functional equation of the Riemann ζ-function:

$$\zeta(2k)=\frac{(2\pi i)^{2k}}{2(2k-1)!}\cdot\zeta(1-2k).$$

3.3. **Constructing Moore spectra with group actions.** Another ingredient needed to construct the Dirichlet J-spectra and $K(1)$-local spheres is a Moore spectrum with a $(\mathbb{Z}/N)^\times$-action induced by a (p-adic) Dirichlet character $\chi:(\mathbb{Z}/N)^\times\to\mathbb{C}^\times$ (or $\mathbb{C}_p^\times$). The first observation is following:

Lemma 3.3.1. *There is a unique number n such that χ factorizes as*

$$\chi:(\mathbb{Z}/N)^\times \twoheadrightarrow C_n \hookrightarrow (\mathbb{Z}[\zeta_n])^\times \hookrightarrow \mathbb{C}^\times, \qquad \textit{when } \chi \textit{ is } \mathbb{C}\textit{-valued;}$$
$$\chi:(\mathbb{Z}/N)^\times \twoheadrightarrow C_n \hookrightarrow (\mathbb{Z}_p[\zeta_n])^\times \hookrightarrow \mathbb{C}_p^\times, \qquad \textit{when } \chi \textit{ is } \mathbb{C}_p\textit{-valued,}$$

where C_n is the cyclic group of order n and the second maps send a generator $g\in C_n$ to a primitive n-th root of unity ζ_n.

Then it suffices to construct the Moore spectra $M(\mathbb{Z}[\zeta_n])$ and $M(\mathbb{Z}_p[\zeta_n])$ with C_n-actions such that the induced C_n-action on H_0 (equivalently π_0) is equivalent to that on $\mathbb{Z}[\zeta_n]$ and $\mathbb{Z}_p[\zeta_n]$. The latter is called the integral/p-adic cyclotomic representation of C_n. Properties of such representations needed in this subsection are summarized in Appendix A.

We can further reduce to cases $n=p^v$ by noting from Lemma A.1.2:

$$\begin{array}{ccc} & \mathbb{Z}[\zeta_n]\simeq\bigotimes_{p|n}\mathbb{Z}[\zeta_{p^{v_p(n)}}] & \mathbb{Z}_p[\zeta_n]\simeq\bigotimes_{q|n}\mathbb{Z}_p[\zeta_{q^{v_q(n)}}],\\ \xRightarrow{\text{non-equivariantly}} & M(\mathbb{Z}[\zeta_n])\simeq\bigwedge_{p|n}M(\mathbb{Z}[\zeta_{p^{v_p(n)}}]) & M(\mathbb{Z}_p[\zeta_n])\simeq\bigwedge_{q|n}M(\mathbb{Z}_p[\zeta_{q^{v_q(n)}}]).\end{array}$$

The constructions now split into three cases:

(1) In the integral case, we give an explicit construction suggested by Charles Rezk.
(2) The p-adic case where $n=p^v$ is the p-completion of the corresponding integral case.
(3) The p-adic case where $(n,p)=1$ uses Cooke's obstruction theory [Coo78] to lift group actions on homotopy groups to the homotopy category of spectra. The comparison of this case with the integral case uses the obstruction theory to uniqueness of the lifting.

The integral case.

Construction 3.3.2 (Charles Rezk)**.** From the short exact sequence of C_{p^v}-representations in Lemma A.1.3:

$$0 \longrightarrow \mathbb{Z}[\zeta_{p^v}] \longrightarrow \mathbb{Z}[C_{p^v}] \longrightarrow \mathbb{Z}[C_{p^{v-1}}] \longrightarrow 0, \tag{3.3.3}$$

we define $M(\mathbb{Z}[\zeta_{p^v}])$ as the de-suspension of the cofiber of the quotient map $C_{p^v} \twoheadrightarrow C_{p^{v-1}}$. That is, there is a cofiber sequence:

$$S^0 \wedge (C_{p^v})_+ \longrightarrow S^0 \wedge (C_{p^{v-1}})_+ \longrightarrow \Sigma M(\mathbb{Z}[\zeta_{p^v}]). \tag{3.3.4}$$

$M(\mathbb{Z}[\zeta_{p^v}])$ inherits a natural $(\mathbb{Z}/p^v)^\times$-action from its suspension as the cofiber of a C_{p^v}-equivariant map.

Proposition 3.3.5. *$M(\mathbb{Z}[\zeta_{p^v}])$ constructed above is a Moore spectrum for $\mathbb{Z}[\zeta_{p^v}]$. The induced $(\mathbb{Z}/p^v)^\times$-action on $H_0(M(\mathbb{Z}[\zeta_{p^v}]);\mathbb{Z})$ is equivalent to the cyclotomic action of C_{p^v} on $\mathbb{Z}[\zeta_{p^v}]$.*

Proof. Applying $H_*(-;\mathbb{Z})$ to the cofiber sequence (3.3.4), we can show that $M(\mathbb{Z}[\zeta_{p^v}])$ is a Moore spectrum. The rest follows from (3.3.3). $\square$

Below are some examples of the C_{p^v}-equivariant cell structures of $\Sigma M(\mathbb{Z}[\zeta_{p^v}])$:

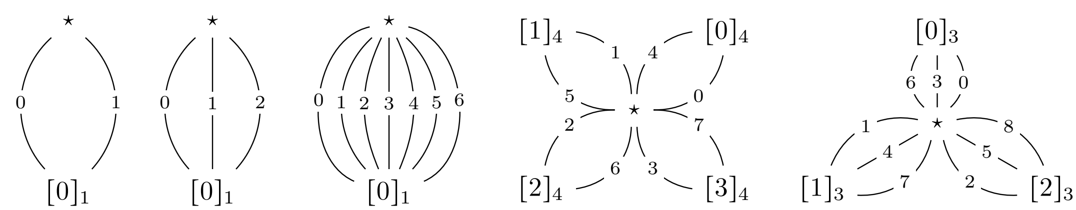


FIGURE 1. C_{p^v}-cell structures of $\Sigma M(\mathbb{Z}[\zeta_{p^v}])$ for $p^v = 2, 3, 7, 8, 9$

- $\star$ is the base point and is fixed by the C_n-action.
- $[a]_b := (a \mod b)$ is the label of (non-equivariant) 0-cells.
- $a := (a \mod n)$ is the label of (non-equivariant) 1-cells.
- $g \in C_n \simeq \mathbb{Z}/n$ acts on the labels by mapping $(a \mod b)$ to $(a + g \mod b)$.

Here is another description of this construction:

(1) $M(\mathbb{Z}[\zeta_2]) \simeq S^{\sigma-1}$, where σ is the sign representation of C_2.

(2) C_n acts on $\mathbb{C}$ by multiplication by n-th roots of unity. Denote the associated C_n-representation by ρ_{cyclo} and the representation sphere by $S^{\rho_{\text{cyclo}}}$. When $n = p$, the C_p-cell structure of $\Sigma M(\mathbb{Z}[\zeta_p])$ above shows

$$S^{\rho_{\text{cyclo}}} \simeq \Sigma M(\mathbb{Z}[\zeta_p]) \cup (C_p \times D^2).$$

As a result, $M(\mathbb{Z}[\zeta_p])$ is the 1-skeleton in this equivariant cell structure of the representation sphere $S^{\rho_{\text{cyclo}}}$.

(3) Foling Zou has observed and proved the following relation between $M(\mathbb{Z}[\zeta_{p^v}])$ and $M(\mathbb{Z}[\zeta_p])$ via private conversations with the author:

Proposition 3.3.6 (Foling Zou)**.** *There is a C_{p^v}-equivariant equivalence:*

$$M(\mathbb{Z}[\zeta_{p^v}]) \simeq (C_{p^v})_+ \bigwedge_{C_p} M(\mathbb{Z}[\zeta_p]),$$

where $a \in \mathbb{Z}/p \simeq C_p$ acts on $\mathbb{Z}/p^v \simeq C_{p^v}$ by sending $(b \mod p^v)$ to $(b + ap^{v-1} \mod p^v)$.

Proof. Notice that $C_{p^{v-1}} \simeq C_{p^v}/C_p$, we can rewrite this quotient as pointed sets by

$$(C_{p^{v-1}})_+ \simeq S^0 \bigwedge_{C_p} (C_{p^v})_+,$$

where C_p acts on C_{p^v} as described in the proposition. From this we get:

$$\begin{aligned}
\Sigma M(\mathbb{Z}[\zeta_{p^v}]) :=& \mathrm{Cofib}\left(S^0 \bigwedge (C_{p^v})_+ \longrightarrow S^0 \bigwedge (C_{p^{v-1}})_+\right) \\
\simeq& \mathrm{Cofib}\left(S^0 \bigwedge (C_p)_+ \bigwedge_{C_p} (C_{p^v})_+ \longrightarrow S^0 \bigwedge S^0 \bigwedge_{C_p} (C_{p^v})_+\right) \\
\simeq& \mathrm{Cofib}\left(S^0 \bigwedge (C_p)_+ \longrightarrow S^0 \bigwedge S^0\right) \bigwedge_{C_p} (C_{p^v})_+ \\
\simeq& \Sigma M(\mathbb{Z}[\zeta_p]) \bigwedge_{C_p} (C_{p^v})_+.
\end{aligned}$$

□

Taking external smash product of $M(\mathbb{Z}[\zeta_{p^v}])$ with the prescribed C_{p^v}-actions over all $p \mid n$, we have constructed a Moore spectrum $M(\mathbb{Z}[\zeta_n])$ with a C_n-action such that the induced action on $H^0(-;\mathbb{Z})$ is equivalent to the cyclotomic action of C_n. We now give an explicit description of the C_n-equivariant simplicial structure of $M(\mathbb{Z}[\zeta_n])$.

Write $n = p_1^{v_1} \cdots p_m^{v_m}$. $X_n := \Sigma^m M(\mathbb{Z}[\zeta_n])$ is constructed as follows:

(1) Set the 0-th skeleton by $\mathrm{sk}_0 X_n := \star \coprod C_n/C_{p_1 \cdots p_m}$, where $\star$ is the base point fixed by the $(\mathbb{Z}/N)^\times$-action.

(2) Assuming we have defined $\mathrm{sk}_{k-1} X_n$ for $k < m$, then define the k-th skeleton to be:

$$\mathrm{sk}_k X_n := \mathrm{sk}_{k-1} X_n \bigcup \left(\coprod_{i_1 < \cdots < i_{m-k}} C_n/C_{p_{i_1} \cdots p_{i_{m-k}}} \right) \times \Delta^k.$$

The attaching map of an equivariant k-simplex $C_n/C_{p_{i_1} \cdots p_{i_{m-k}}} \times \Delta^k$ is described by the following:

- The 0-th face $C_n/C_{p_{i_1} \cdots p_{i_{m-k}}} \times \Delta^k_{[0]}$ is attached to the base point $\star$.
- Let $\{j_1 < \cdots < j_k\}$ be the complement of $\{i_1, \cdots i_{m-k}\} \subseteq \{1, \cdots, m\}$. Then the l-th face $C_n/C_{p_{i_1} \cdots p_{i_{m-k}}} \times \Delta^k_{[l]}$ for $1 \le l \le k$ is attached to the equivariant $(k-1)$-complex

$$C_n/C_{p_{i_1} \cdots p_{i_{m-k}} \cdot p_{j_l}} \times \Delta^{k-1}$$

via the quotient map of orbits.

(3) The top simplex is $C_n \times \Delta^m$. The 0-th face $C_n \times \Delta^m_{[0]}$ is attached to the base point $\star$. The l-th face $C_n \times \Delta^m_{[l]}$ for $1 \le l \le m$ is attached to the $(m-1)$-equivariant simplex $C_n/C_{p_l} \times \Delta^{m-1}$ via the quotient map $C_n \twoheadrightarrow C_n/C_{p_l}$.

Remark 3.3.7. The non-equivariant Euler number of $X_n = \Sigma^m M(\mathbb{Z}[\zeta_n])$ is equal to $1 + (-1)^m \phi(n)$ since it is non-equivariantly a wedge sum of $\phi(n)$ many copies of S^m. On the other hand, by counting the number of non-equivariant simplices in each dimension from the above construction, we get

$$\begin{aligned}
1 + (-1)^m \phi(n) = e(X_n) = 1 + \sum_{k=0}^{m-1} \left((-1)^k \sum_{i_1 < \cdots < i_{m-k}} \frac{n}{p_{i_1} \cdots p_{i_{m-k}}} \right) + (-1)^m n \\
\Longrightarrow \qquad \phi(n) = n + \sum_{k=1}^{m} \left((-1)^k \sum_{i_1 < \cdots < i_k} \frac{n}{p_{i_1} \cdots p_{i_k}} \right).
\end{aligned}$$

This is precisely the formula of $\phi(n) := |\{a \in \mathbb{N} \mid 1 \le a \le n, (a,n) = 1\}|$ via the Inclusion and Exclusion Principle.

Remark 3.3.8. The construction above is not unique. For example when $n = 2$, $M(\mathbb{Z}[\zeta_2])$ is by definition S^0 with a C_2-action such that the induced action of C_2 on $\pi_*(S^0)$ is the sign representation in all degrees. Figure 1 shows our model for $M(\mathbb{Z}[\zeta_2])$ is $S^{\sigma-1}$. But one can check $S^{(2k-1)(\sigma-1)}$ also satisfies the assumptions for all $k \in \mathbb{Z}$ and these are non-equivalent C_2-actions on S^0.

The p-adic case when $n = p^v$. By Corollary A.3.1, $(\mathbb{Z}[\zeta_{p^v}])^\wedge_p \simeq \mathbb{Z}_p[\zeta_{p^v}]$. From this we can simply define the Moore spectrum with a C_{p^v}-action by setting

$$M(\mathbb{Z}_p[\zeta_{p^v}]) := M(\mathbb{Z}[\zeta_{p^v}])^\wedge_p. \tag{3.3.9}$$

The p-adic case when $p \nmid n$. In this case, Proposition A.2.3 implies that $(\mathbb{Z}[\zeta_n])^\wedge_p \not\simeq \mathbb{Z}_p[\zeta_n]$, since the two sides have different ranks as $\mathbb{Z}_p$-modules. As a result, the construction in the $n = p^v$ case does not apply. Instead, we use Cooke's obstruction theory in [Coo78] to lift the C_n-action on $\mathbb{Z}_p[\zeta_n] = \pi_0(M(\mathbb{Z}_p[\zeta_n]))$ to the Moore spectrum $M(\mathbb{Z}[\zeta_n])$.

Let X be a spectrum and $h\mathrm{Aut}(X)$ be the group of self-homotopy equivalences of X. $h\mathrm{Aut}(X)$ is an associative H-space. Then $\pi_0(h\mathrm{Aut}(X))$ is the group of homotopy classes of homotopy equivalences of X. Denote the identity component of $h\mathrm{Aut}(X)$ by $h\mathrm{Aut}_1(X)$. There is an short exact sequence of H-spaces:

$$1 \longrightarrow h\mathrm{Aut}_1(X) \longrightarrow h\mathrm{Aut}(X) \longrightarrow \pi_0(h\mathrm{Aut}(X)) \longrightarrow 1.$$

This induces a fiber sequence by taking classifying spaces:

$$Bh\mathrm{Aut}_1(X) \longrightarrow Bh\mathrm{Aut}(X) \longrightarrow B\pi_0(h\mathrm{Aut}(X)).$$

An action of a group G on $\pi_0(X)$ is then a group homomorphism $\alpha : G \to \pi_0(h\mathrm{Aut}(X))$.

Theorem 3.3.10. [Coo78, Theorem 1.1] *There is an obstruction theory to lift α to an action on X:*

$$\begin{tikzcd} & Bh\mathrm{Aut}(X) \arrow[d] \\ BG \arrow[ur, dashed] \arrow[r, "B\alpha"'] & B\pi_0(h\mathrm{Aut}(X)). \end{tikzcd}$$

The obstruction classes to the existence of such liftings live in

$$H^n(G; \{\pi_{n-2}(h\mathrm{Aut}_1(X))\}), \quad n \geq 3.$$

In particular, one can always lift a G-action on $\pi_*(X)$ to X if G is finite and $|G|$ is invertible in $\pi_n(h\mathrm{Aut}_1(X))$ for all $n \geq 1$.

Corollary 3.3.11. *When $p \nmid n$, any of C_n-action on π_* of a p-complete spectrum can be lifted to an action on the spectrum itself.*

Proof. As n is invertible in $\mathbb{Z}_p$, group cohomology of C_n with coefficients in $\mathbb{Z}_p$-modules vanishes in positive degrees. As a result, the obstruction classes in Theorem 3.3.10 all vanish. ∎

As a result, there exists a C_n-action on the p-adic Moore spectrum $M(\mathbb{Z}_p[\zeta_n])$ such that the induced action on π_0 agrees with p-adic cyclotomic representation of C_n.

One last thing to check is the compatibility of the constructions in the integral and p-adic cases when $p \nmid n$. Fix an embedding $\iota : \mathbb{Z}[\zeta_n] \hookrightarrow \mathbb{Z}_p[\zeta_n]$. ι induces a map of Galois groups:

$$\iota^* : \mathrm{Gal}(\mathbb{Q}_p(\zeta_n)/\mathbb{Q}_p) \hookrightarrow \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}).$$

By Proposition A.3.4, there is an equivalence of p-adic C_n-representations:

$$\mathbb{Z}[\zeta_n] \otimes \mathbb{Z}_p \simeq \bigoplus_{[\sigma] \in \operatorname{Coker} \iota^*} (\mathbb{Z}_p[\zeta_n])_{\iota\circ\sigma}, \tag{3.3.12}$$

where C_n acts on the summand $(\mathbb{Z}_p[\zeta_n])_{\iota\circ\sigma}$ by

$$C_n \hookrightarrow (\mathbb{Z}[\zeta_n])^\times \xrightarrow{\sigma} (\mathbb{Z}[\zeta_n])^\times \xrightarrow{\iota} (\mathbb{Z}_p[\zeta_n])^\times.$$

By Corollary 3.3.11, there is a C_n-action on $M(\mathbb{Z}_p[\zeta_n])^{\vee|\operatorname{Coker}\iota^*|}$ such that the induced C_n-action on π_0 agrees with the right hand side of (3.3.12). On the other hand, the C_n-action $M(\mathbb{Z}[\zeta_n])^\wedge_p$ induces an equivalent C_n-representation on π_0. To check the two C_n-actions on the p-adic Moore spectrum are equivalent, we use the uniqueness part of Cooke's obstruction theory.

Proposition 3.3.13. *In Theorem 3.3.10, the obstruction classes to the uniqueness of the liftings live in*

$$H^n(G; \{\pi_{n-1}(h\mathrm{Aut}_1(X))\}), \quad n \geq 2.$$

Corollary 3.3.14. *Let X be a p-complete spectrum. When $p \nmid n$, any two lifts of a C_n-action from $\pi_*(X)$ to X are C_n-equivariantly equivalent.*

As a result, there is a C_n-equivalence:

$$M(\mathbb{Z}[\zeta_n])^\wedge_p \simeq \bigvee_{[\sigma]\in\operatorname{Coker}\iota^*} (M(\mathbb{Z}_p[\zeta_n]))_{\iota\circ\sigma}.$$

Remark 3.3.15. When $n = p^v$, there could be non-equivalent C_{p^v}-actions on $M(\mathbb{Z}_p[\zeta_{p^v}])$ inducing the same action on π_0. One counterexample in the integral case is C_2-equivariant spheres $S^{2\sigma-2}$ and S^0 – both induce trivial action on the homotopy groups.

Pre-composing with the map $(\mathbb{Z}/N)^\times \twoheadrightarrow C_n$ in Lemma 3.3.1, we have shown in this subsection:

Theorem 3.3.16. *Let $\chi : (\mathbb{Z}/N)^\times \to \mathbb{C}^\times$ or $\mathbb{C}_p^\times$ be a Dirichlet character.*

(1) There is a Moore spectrum $M(\mathbb{Z}[\chi])$ or $M(\mathbb{Z}_p[\chi])$ with a $(\mathbb{Z}/N)^\times$-action such that the induced action on π_0 is equivalent to that induced by χ.

(2) Let $\iota : \mathbb{Z}[\chi] \hookrightarrow \mathbb{Z}_p[\chi]$ be an embedding. There is a $(\mathbb{Z}/N)^\times$-equivariant equivalence:

$$M(\mathbb{Z}[\chi])^\wedge_p \simeq \bigvee_{[\sigma]\in\operatorname{Coker}\iota^*} M(\mathbb{Z}_p[\iota\circ\sigma\circ\chi]). \tag{3.3.17}$$

3.4. **The homotopy eigen spectra.** Now we are ready to twist the J-spectrum and the $K(1)$-local spheres with a Dirichlet character. Analogous to Proposition 1.3.4, the twisting is realized as the "homotopy χ-eigen-spectrum".

Construction 3.4.1. Let $\chi : (\mathbb{Z}/N)^\times \to \mathbb{C}^\times$ be a primitive Dirichlet character of conductor N. We define the **Dirichlet J-spectrum** by:

$$J(N)^{h\chi} := \mathrm{Map}(M(\mathbb{Z}[\chi]), J(N))^{h(\mathbb{Z}/N)^\times}, \tag{3.4.2}$$

Let $\chi : (\mathbb{Z}/N)^\times \to \mathbb{C}_p^\times$ be a primitive p-adic Dirichlet character of conductor N and set $v = v_p(N)$. We define the **Dirichlet $K(1)$-local sphere** to be

$$S^0_{K(1)}(p^v)^{h\chi} := \mathrm{Map}_{\mathbb{Z}_p}\left(M(\mathbb{Z}_p[\chi]), S^0_{K(1)}(p^v)\right)^{h(\mathbb{Z}/N)^\times}. \tag{3.4.3}$$

The $(\mathbb{Z}/N)^\times$-actions on the Moore spectrum and $J(N)$ are described in Theorem 3.3.16 and Proposition 3.2.10, respectively. $(\mathbb{Z}/N)^\times$ acts on $S^0_{K(1)}(p^v)$ through the Galois action of its quotient group $(\mathbb{Z}/p^v)^\times$.

Remark 3.4.4. The spectra $J(N)^{h\chi}$ and $S^0_{K(1)}(p^v)^{h\chi}$ depend on the constructions of the $(\mathbb{Z}/N)^\times$-actions on $M(\mathbb{Z}[\chi])$ and $M(\mathbb{Z}_p[\chi])$, which is not unique in general as illustrated in Remark 3.3.8. When $N=4, p=2$ and $\chi:(\mathbb{Z}/4)^\times \simeq C_2 \to \mathbb{C}_2^\times$, different models of $M(\mathbb{Z}_2[\chi])$ lead to different $S^0_{K(1)}(4)^{h\chi}$. We will explain the differences in more detail in Remark 4.2.11.

One immediate consequence of this construction is

Proposition 3.4.5. *If χ_1 and χ_2 are Dirichlet characters of conductor N with isomorphic induced representations, then $J(N)^{h\chi_1} \simeq J(N)^{h\chi_2}$. In particular, $J(N)^{h\chi} \simeq J(N)^{h(\sigma\circ\chi)}$ for any $\sigma \in \mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})$.*

Remark 3.4.6. As $S^0_{K(1)}(p^v)$ is $K(1)$-local, we have

$$S^0_{K(1)}(p^v)^{h\chi} \simeq \mathrm{Map}_{K(1)\text{-loc}}\left(M(\mathbb{Z}_p[\chi])_{K(1)}, S^0_{K(1)}(p^v)\right)^{h(\mathbb{Z}/N)^\times}$$

is also $K(1)$-local.

Proposition 3.4.7. *The E_2-pages of the HFPSS to compute $\pi_*\left((J(N))^{h\chi}\right)$ and $\pi_*\left(S^0_{K(1)}(p^v)^{h\chi}\right)$ are identified with*

$$E_2^{s,t} \simeq \mathrm{Ext}^s_{\mathbb{Z}[(\mathbb{Z}/N)^\times]}\left(\mathbb{Z}[\chi], \pi_t(J(N))\right) \Longrightarrow \pi_{t-s}\left(J(N)^{h\chi}\right) \tag{3.4.8}$$

$$E_2^{s,t} \simeq \mathrm{Ext}^s_{\mathbb{Z}_p[(\mathbb{Z}/N)^\times]}\left(\mathbb{Z}_p[\chi], \pi_t\left(S^0_{K(1)}(p^v)\right)\right) \Longrightarrow \pi_{s-t}\left(S^0_{K(1)}(p^v)^{h\chi}\right) \tag{3.4.9}$$

where $a \in (\mathbb{Z}/N)^\times$ acts on $\mathbb{Z}[\chi]$ and $\mathbb{Z}_p[\chi]$ by multiplication by $\chi(a)$.

Proof. We give a proof of (3.4.8). The proof of (3.4.9) is similar. By construction, the E_2-page of the HFPSS for (3.4.2) is

$$E_2^{s,t} = H^s((\mathbb{Z}/N)^\times; \pi_t(\mathrm{Map}(M(\mathbb{Z}[\chi]), J(N)))).$$

Denote the rank of $\mathbb{Z}[\chi]$ as a free $\mathbb{Z}$-module by r. Then $M(\mathbb{Z}[\chi])$ is non-equivariantly equivalent to $\left(S^0\right)^{\vee r}$. The Atiyah-Hirzebruch spectral sequence:

$$E_2^{s,t} = H^s(M(\mathbb{Z}[\chi]); \pi_t(J(N))) \Longrightarrow \pi_{s+t}(\mathrm{Map}(M(\mathbb{Z}[\chi]), J(N)))$$

collapses on the E_2-page since $H^*(M(\mathbb{Z}[\chi]); -)$ is concentrated in degree 0. Together with the universal coefficient theorem, this implies:

$$\begin{aligned} \pi_t(\mathrm{Map}(M(\mathbb{Z}[\chi]), J(N))) \simeq& H^0(M(\mathbb{Z}[\chi]); \pi_t(J(N))) \\ \simeq& \mathrm{Hom}_{\mathbb{Z}}(H^0(M(\mathbb{Z}[\chi]); \mathbb{Z}), \pi_t(J(N))) \\ \simeq& \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[\chi], \pi_t(J(N))). \end{aligned}$$

By Theorem 3.3.16, $(\mathbb{Z}/N)^\times$ acts on $\mathbb{Z}[\chi] \simeq H^0(M(\mathbb{Z}[\chi]); \mathbb{Z})$ by χ. Since $\mathbb{Z}[\chi]$ is a finite free $\mathbb{Z}$-module, the Grothendieck spectral sequence

$$E_2^{s,t} = H^s((\mathbb{Z}/N)^\times; \mathrm{Ext}^t_{\mathbb{Z}}(\mathbb{Z}[\chi], \pi_t(J(N)))) \Longrightarrow \mathrm{Ext}^{s+t}_{\mathbb{Z}[(\mathbb{Z}/N)^\times]}(\mathbb{Z}[\chi], \pi_t(J(N)))$$

collapses on the E_2-page, yielding

$$H^s((\mathbb{Z}/N)^\times; \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[\chi], \pi_t(J(N)))) \simeq \mathrm{Ext}^s_{\mathbb{Z}[(\mathbb{Z}/N)^\times]}(\mathbb{Z}[\chi], \pi_t(J(N)))\,.$$

□

Remark 3.4.10. The E_2-page of (3.4.8) consists of the derived χ-eigenspaces of $\pi_*(J(N))$. Moreover, $J(N)^{h\chi}$ is defined as the **homotopy χ-eigen-spectrum** of $J(N)$. In this sense, we will call (3.4.8) the **homotopy eigen spectral sequence** (HESS). [3]

[3] The alternative name "homotopy eigen-spectrum spectral sequence" would be too redundant.

3.5. **Local structures of the Dirichlet J-spectra.** While it is not hard to compute the E_2-page of (3.4.8) directly, the differentials are non-trivial as the cohomological dimension of $(\mathbb{Z}/N)^\times$ with coefficients in $\mathbb{Z}$-modules is infinite. Instead, we will compute $\pi_*\left(J(N)\right)^{h\chi}$ rationally and completed at each prime p.

Over $\mathbb{Q}$, the spectral sequence is concentrated in the 0-th line, since $(\mathbb{Z}/N)^\times$ is a finite group. By Corollary 3.2.9, $J(N)_\mathbb{Q} \simeq S^0_\mathbb{Q}$ and $(\mathbb{Z}/N)^\times$ acts on it trivially. We conclude from these facts:

Proposition 3.5.1. *The homotopy groups of $\left(J(N)^{h\chi}\right)_\mathbb{Q}$ are given by*

$$\pi_i\left(\left(J(N)^{h\chi}\right)_\mathbb{Q}\right) \simeq \begin{cases} \mathbb{Q}, & i=0 \text{ and } \chi=\chi^0; \\ 0, & \text{otherwise.}\end{cases}$$

Corollary 3.5.2. *$\left(J(N)^{h\chi}\right)_\mathbb{Q}$ is contractible unless $\chi=\chi^0$ is trivial. In that case, $N=0$ and $J(N)^{h\chi}_\mathbb{Q} \simeq J_\mathbb{Q} \simeq S^0_\mathbb{Q}$.*

Proof. By Corollary 3.2.9, $J(N)_\mathbb{Q}\simeq S^0_\mathbb{Q}$. Then $E_2^{s,t}\otimes\mathbb{Q}=0$ for all $(s,t)\neq(0,0)$ (3.4.8). The remaining entry $E_2^{0,0}\simeq\mathbb{Q}(\chi^{-1})^{(\mathbb{Z}/N)^\times}$ is zero unless $\chi=\chi^0$ is trivial, yielding the claim. □

Proposition 3.5.3. *Fix an embedding $\iota:\mathbb{Q}(\chi)\hookrightarrow\mathbb{C}_p$. The p-completion of the Dirichlet J-spectrum decomposes as*

$$\left(J(N)^{h\chi}\right)^\wedge_p \simeq \bigvee_{[\sigma]\in\operatorname{Coker}\iota^*} S^0_{K(1)}(p^v)^{h(\iota\circ\sigma\circ\chi)},$$

where $\iota^*:\operatorname{Gal}(\mathbb{Q}_p(\zeta_n)/\mathbb{Q}_p)\hookrightarrow\operatorname{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ is defined in (A.3.3).

Proof. Since homotopy fixed points and p-completions commute and that the p-completion of $J(N)$ is $S^0_{K(1)}(p^v)$

$$\left(J(N)^{h\chi}\right)^\wedge_p \simeq \operatorname{Map}_{\mathbb{Z}_p}\left(M(\mathbb{Z}[\chi])^\wedge_p, S^0_{K(1)}(p^v)\right)^{h(\mathbb{Z}/N)^\times}$$

The rest follows from (3.3.17). □

Now we give explicit descriptions of how $\left(J(N)^{h\chi}\right)^\wedge_p$ decomposes when $N=p^v$.

Examples 3.5.4. Let $\chi:(\mathbb{Z}/N)^\times\to\mathbb{C}^\times$ be a Dirichlet character of conductor $N=p^v$. Fix an embedding $\iota:\mathbb{Z}[\chi]\hookrightarrow\mathbb{C}_p$. There are two cases.

- $p=2$. The $v=1$ case is trivial. For $v>1$, $(\mathbb{Z}/2^v)^\times\simeq\{\pm1\}\times\mathbb{Z}/2^{v-2}$. When $v=2$, χ is primitive when it is non-trivial, i.e. $\chi(-1)=-1$. When $v>2$, χ is primitive of conductor 2^v iff $\mathbb{Z}[\chi]\simeq\mathbb{Z}[\zeta_{2^{v-2}}]$. In both cases, we have by Proposition A.2.3, $(\mathbb{Z}[\zeta_{2^{v-2}}])^\wedge_2\simeq\mathbb{Z}_2[\zeta_{2^{v-2}}]$. As a result,

$$\left(J(2^v)^{h\chi}\right)^\wedge_2 \simeq S^0_{K(1)}(2^v)^{h(\iota\circ\chi)}.$$

  Notice for any two 2-adic Dirichlet characters χ_1 and χ_2 of conductor 2^v with the same parity, there is a $\sigma\in\operatorname{Gal}(\mathbb{Q}_2(\zeta_{2^{v-2}})/\mathbb{Q}_2)$ such that $\chi_1=\sigma\circ\chi_2$. By Proposition 3.4.5, the above isomorphism does not depend on ι, since $\iota\circ\chi(-1)$ is independent of the choice of ι.
- $p>2$. In this case, $(\mathbb{Z}/p^v)^\times\simeq(\mathbb{Z}/p)^\times\times\mathbb{Z}/p^{v-1}$. When $v=1$, χ is primitive iff it is non-trivial. When $v>1$, χ is primitive iff $\zeta_{p^{v-1}}\in\mathbb{Z}[\chi]$, i.e. $\chi|_{\mathbb{Z}/p^{v-1}}$ is injective. By Corollary A.3.6, there is an isomorphism of p-adic $(\mathbb{Z}/p^v)^\times$-representations:

$$(\mathbb{Z}[\chi])^\wedge_p \simeq \bigoplus_{\substack{0\le a\le p-2\\ \ker\omega^a=\ker\chi|_{(\mathbb{Z}/p)^\times}}} \mathbb{Z}_p[\chi_a],$$

where $\chi_a = \omega^a \cdot (\iota \circ \chi|_{\mathbb{Z}/p^{v-1}})$ and $\omega : (\mathbb{Z}/p)^\times \to \mathbb{Z}_p^\times$ is the **Teichmüller** character. This implies a decomposition of the p-completion of the Dirichlet J-spectrum as in Proposition 3.5.3:

$$\left(J(p^v)^{h\chi}\right)_p^\wedge \simeq \bigvee_{\substack{0\le a\le p-2\\ \ker\omega^a=\ker\chi|_{(\mathbb{Z}/p)^\times}}} S^0_{K(1)}(p^v)^{h\chi_a}. \tag{3.5.5}$$

Now we need to compute the homotopy groups of the Dirichlet $K(1)$-local spheres. Like the integral case, while the E_2-page of (3.4.9) are not hard to compute in general, there are infinitely many differentials unless $p \nmid \phi(N) = |(\mathbb{Z}/N)^\times|$. We now set up another spectral sequence that we will use in Section 4.

Proposition 3.5.6. *Let $\chi : (\mathbb{Z}/N)^\times \to \mathbb{C}_p^\times$ be a p-adic Dirichlet character of conductor N. Write $N = p^v \cdot N'$ where $p \nmid N'$. There is an equivalence of $K(1)$-local spectra:*

$$S^0_{K(1)}(p^v)^{h\chi} \simeq \mathrm{Map}_{\mathbb{Z}_p}\left(M(\mathbb{Z}_p[\chi]), KU_p^\wedge\right)^{h(\mathbb{Z}_p^\times\times(\mathbb{Z}/N')^\times)},$$

where $\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times$ acts on the Moore spectra through the action of $(\mathbb{Z}/N)^\times$ described in Section 3.3 via the quotient map $\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times \twoheadrightarrow (\mathbb{Z}/p^v)^\times \times (\mathbb{Z}/N')^\times \cong (\mathbb{Z}/N)^\times$; and on $KU_p^\wedge$ via the Adams operations by the factor $\mathbb{Z}_p^\times$.

Proof. Recall from Definition 2.3.4 that $S^0_{K(1)}(p^v) := \left(KU_p^\wedge\right)^{h(1+p^v\mathbb{Z}_p)}$. From this, we have

$$\begin{aligned} S^0_{K(1)}(p^v)^{h\chi} :=& \mathrm{Map}_{\mathbb{Z}_p}\left(M(\mathbb{Z}_p[\chi]), S^0_{K(1)}(p^v)\right)^{h(\mathbb{Z}/N)^\times}\\ \simeq& \mathrm{Map}_{\mathbb{Z}_p}\left(M(\mathbb{Z}_p[\chi]), \left(KU_p^\wedge\right)^{h(1+p^v\mathbb{Z}_p)}\right)^{h(\mathbb{Z}/N)^\times}\\ \simeq& \mathrm{Map}_{\mathbb{Z}_p}\left(M(\mathbb{Z}_p[\chi]), KU_p^\wedge\right)^{hG}\end{aligned}$$

where G is an extension of $1+p^v$ by $(\mathbb{Z}/N)^\times$. The subgroup $1+p^v\mathbb{Z}_p$ acts trivially on the Moore spectrum and by the Adams operations on $KU_p^\wedge$. Notice that the $(\mathbb{Z}/p^v)^\times$-action on $S^0_{K(1)}(p^v) := \left(KU_p^\wedge\right)^{h(1+p^v\mathbb{Z}_p)}$ is via the residual Adams operations by viewing $(\mathbb{Z}/p^v)^\times$ as a group group of $\mathbb{Z}_p^\times$. The group G is then isomorphic to $\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times$, with actions on $M(\mathbb{Z}_p[\chi])$ and $KU_p^\wedge$ as claimed. ☐

Corollary 3.5.7. *Write $\chi = \chi_p \cdot \chi'$ where χ_p and χ' have conductors p^v and N', respectively. There is another spectral sequence to compute homotopy groups of Dirichlet $K(1)$-local spheres:*

$$E_2^{s,2t} = \mathrm{Ext}^s_{\mathbb{Z}_p[\![\mathbb{Z}_p^\times\times(\mathbb{Z}/N')^\times]\!]}(\mathbb{Z}_p[\chi], \mathbb{Z}_p^{\otimes t}) \simeq H_c^s(\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times; \mathbb{Z}_p^{\otimes t}[\chi^{-1}]) \Longrightarrow \pi_{2t-s}\left(S^0_{K(1)}(p^v)^{h\chi}\right), \tag{3.5.8}$$

where $\mathbb{Z}_p^{\otimes t}[\chi]$ is the $\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times$-representation associated to the character $\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times \xrightarrow{(a,b)\mapsto\chi_p(a)\chi'(b)a^t} (\mathbb{Z}_p[\chi])^\times$.

Remark 3.5.9. When $N = p > 2$ or $N = 4$ and $p = 2$, $M(\mathbb{Z}_p[\chi])$ is non-equivariantly equivalent to S_p^0 and $\chi = \omega^a$ for some a. Proposition 3.5.6 identifies the Dirichlet $K(1)$-local spheres $S^0_{K(1)}(p)^{h\omega^a}$ and $S^0_{K(1)}(4)^{h\omega}$ with elements of finite order in the $K(1)$-local Picard group $\mathrm{Pic}_{K(1)}$, first defined in [HMS94].

4. Computations of the Dirichlet $K(1)$-local spheres

In this section, we compute homotopy groups of the Dirichlet $K(1)$-local spheres and J-spectra. By Proposition 3.5.3, we can recover the p-primary parts of the homotopy groups of Dirichlet J-spectra from the corresponding summands of Dirichlet $K(1)$-local spheres. Let $\chi : (\mathbb{Z}/N)^\times \to \mathbb{C}_p^\times$ be a p-adic Dirichlet character of conductor N. The computations of $\pi_*\left(S^0_{K(1)}(p^v)^{h\chi}\right)$ break up into four cases:

(1) $N = 1$.
(2) $N = p^v$ and $p > 2$.
(3) $N = 2^v$.
(4) N is not a p-power.

In the $N = 1$ case, we recover the classical $K(1)$-local sphere, whose homotopy groups are computed in (2.3.8) when $p > 2$, and in (2.3.13) when $p = 2$. When N is power of p, we use HFPSS/HESS (3.4.9) and (3.5.8) to compute homotopy groups of the Dirichlet $K(1)$-local spheres. When N has prime factors other than p, the character χ factorizes as a product $\chi = \chi_p \cdot \chi'$, where χ_p has conductor $p^{v_p(N)}$. The Dirichlet $K(1)$-local spheres are contractible when $|\operatorname{Im}\chi'|$ is not a power of p. When $|\operatorname{Im}\chi'|$ is a power of p, we compute the homotopy groups of the Dirichlet $K(1)$-local spheres from its construction.

4.1. **The $N = p^v$ and $p > 2$ case.** Let's start with the $N = p > 2$ case. We will compute $\pi_*\left(S^0_{K(1)}(p)^{h\chi}\right)$ when $p > 2$ first using the homotopy eigen spectral sequence (HESS) (3.4.9). The E_2-page of this spectral sequence is:

$$E_2^{s,t} = \operatorname{Ext}^s_{\mathbb{Z}_p[(\mathbb{Z}/p)^\times]}\left((\mathbb{Z}_p)_\chi, \pi_t\left(S^0_{K(1)}(p)\right)\right) \Longrightarrow \pi_{t-s}\left(S^0_{K(1)}(p)^{h\chi}\right), \tag{4.1.1}$$

where $a \in (\mathbb{Z}/p)^\times$ acts on $(\mathbb{Z}_p)_\chi$ by multiplication by $\chi(a)$.

Remark 4.1.2. When χ is the trivial character χ^0, we recover the HFPSS in (2.3.5).

Let $g \in (\mathbb{Z}/p)^\times$ be a generator. A projective resolution of $(\mathbb{Z}_p)_\chi$ as a $\mathbb{Z}_p[(\mathbb{Z}/p)^\times]$-module is

$$\cdots \longrightarrow \mathbb{Z}_p[(\mathbb{Z}/p)^\times] \xrightarrow{\times(\sum \chi(g)^{-i}g^i)} \mathbb{Z}_p[(\mathbb{Z}/p)^\times] \xrightarrow{\times(g-\chi(g))} \mathbb{Z}_p[(\mathbb{Z}/p)^\times] \xrightarrow{g\mapsto\chi(g)} (\mathbb{Z}_p)_\chi.$$

By (2.3.8), the homotopy groups of $S^0(p)$ are

$$\pi_t\left(S^0_{K(1)}(p)\right) = \begin{cases} \mathbb{Z}_p, & t = 0 \text{ or } -1; \\ \mathbb{Z}/p^{v_p(k)+1}, & t = 2k-1 \neq -1; \\ 0, & \text{otherwise.} \end{cases}$$

Descending from the Adams operations on $\left(KU_p^\wedge\right)_t$, $(\mathbb{Z}/p)^\times$ acts trivially on π_0 and π_{-1} and by $\chi = \omega^k$ on π_{2k-1} of $S^0_{K(1)}(p)$. A direct computation shows

Proposition 4.1.3. *When $\chi = \omega^a$, $a \neq 0$, the E_2-page of (4.1.1) is*

$$E_2^{s,t} = \begin{cases} \mathbb{Z}/p^{v_p(k)+1}, & s = 0, t = 2k-1, \text{ and } (p-1) \mid (k-a); \\ 0, & \text{otherwise.} \end{cases}$$

As the spectral sequence collapses on the E_2-page, we conclude

$$\pi_t\left(S^0_{K(1)}(p)^{h\omega^a}\right) = \begin{cases} \mathbb{Z}/p^{v_p(k)+1}, & t = 2k-1, \text{ and } (p-1) \mid (k-a); \\ 0, & \text{otherwise.} \end{cases} \tag{4.1.4}$$

We can also use the HFPSS in (3.5.8) to compute $\pi_*\left(S^0_{K(1)}(p)^{h\omega^a}\right)$. The E_2-page of this spectral sequence is:

$$E_2^{s,2t} = \operatorname{Ext}^s_{\mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!]}\left((\mathbb{Z}_p)_{\widetilde{\chi}}, \mathbb{Z}_p^{\otimes t}\right) \simeq H^s_c(\mathbb{Z}_p^\times; \mathbb{Z}_p^{\otimes t}[\chi^{-1}]) \Longrightarrow \pi_{2t-s}\left(S^0_{K(1)}(p)^{h\chi}\right), \tag{4.1.5}$$

where $(\mathbb{Z}_p)_{\widetilde{\chi}}$ is the $\mathbb{Z}_p^\times$-representation associated to the character $\widetilde{\chi}$:

$$\widetilde{\chi}: \mathbb{Z}_p^\times \twoheadrightarrow (\mathbb{Z}/p)^\times \xrightarrow{\chi} \mathbb{Z}_p^\times.$$

The two approaches to compute $\pi_*\left(S^0_{K(1)}(p)^{h\chi}\right)$ are related by the diagram:

$$\begin{tikzcd} \mathrm{Ext}^r_{\mathbb{Z}_p[(\mathbb{Z}/p)^\times]}\left((\mathbb{Z}_p)_\chi, H^s_c\left(1+p\mathbb{Z}_p;\left(KU^\wedge_p\right)_t\right)\right) \arrow[r, Rightarrow, "\mathrm{HSSS}"] \arrow[d, Rightarrow, "\mathrm{HFPSS}"'] & \mathrm{Ext}^{r+s}_{\mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!]}\left((\mathbb{Z}_p)_{\widetilde{\chi}}, \left(KU^\wedge_p\right)_t\right) \arrow[d, Rightarrow, "(4.1.5)"] \\ \mathrm{Ext}^r_{\mathbb{Z}_p[(\mathbb{Z}/p)^\times]}\left((\mathbb{Z}_p)_\chi, \pi_{t-s}\left(S^0_{K(1)}(p)\right)\right) \arrow[r, Rightarrow, "(4.1.1)"'] & \pi_{t-r-s}\left(S^0_{K(1)}(p)^{h\chi}\right) \end{tikzcd} \tag{4.1.6}$$

Here, the top line is a Hochschild-Serre spectra sequence. Retrospectively from this diagram, we get when $\chi = \omega^a$, $a \neq 0$:

$$\mathrm{Ext}^s_{\mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!]}\left((\mathbb{Z}_p)_{\widetilde{\chi}}, \left(KU^\wedge_p\right)_{2t}\right) \simeq H^s_c(\mathbb{Z}_p^\times; \mathbb{Z}_p^{\otimes t}[\chi^{-1}]) = \begin{cases} \mathbb{Z}/p^{v_p(t)+1}, & s = 1, (p-1) \mid (t-a); \\ 0, & \text{otherwise.} \end{cases} \tag{4.1.7}$$

When $N = p^v > p > 2$, we compute the homotopy groups of the Dirichlet $K(1)$-local spheres using (3.5.8). The other spectral sequence (3.4.9) does not quite work in this case. This is because $\mathrm{cd}_p\left((\mathbb{Z}/p^v)^\times\right) = \infty$ when $v > 1$, whereas $\mathrm{cd}_p\left(\mathbb{Z}_p^\times\right) = 1$. As in Proposition 3.5.6, there is an identification:

$$S^0_{K(1)}(p^v)^{h\chi} \simeq \mathrm{Map}_{\mathbb{Z}_p}\left(M(\mathbb{Z}_p[\chi]), KU^\wedge_p\right)^{h\mathbb{Z}_p^\times},$$

Using the resolution in (2.3.6), we get the E_2-page of the HESS:

$$E_2^{s,2t} = H^s_c(\mathbb{Z}_p^\times; \mathbb{Z}_p^{\otimes t}[\chi^{-1}]) = \begin{cases} \mathbb{Z}_p[\chi]/\left(\chi(g) - g^t\right), & s = 1; \\ 0, & \text{otherwise,} \end{cases} \tag{4.1.8}$$

where g is a topological generator of $\mathbb{Z}_p^\times$.

Lemma 4.1.9. *Let $\chi|_{(\mathbb{Z}/p)^\times} = \omega^a$. Then*

$$\mathbb{Z}_p[\chi]/\left(\chi(g) - g^t\right) = \begin{cases} \mathbb{Z}/p, & t \equiv a \mod (p-1); \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Since χ is primitive, we have $\chi(g) = \chi|_{(\mathbb{Z}/p)^\times}(g) \cdot \zeta_{p^{v-1}} = \omega^a(g)\zeta_{p^{v-1}}$. Rewrite $\chi(g) - g^t$ as

$$g^t - \chi(g) = g^t - \omega^a(g)\zeta_{p^{v-1}} = \omega^a(g)(1 - \zeta_{p^{v-1}}) + g^t - \omega^a(g).$$

As $1 - \zeta_{p^{v-1}}$ is a uniformizer of $\mathbb{Z}_p[\chi] \simeq \mathbb{Z}_p[\zeta_{p^{v-1}}]$, $g^t - \chi(g)$ is invertible whenever $g^t - \omega^a(g)$ is. This happens when $t \not\equiv a \mod (p-1)$. When $t \equiv a \mod (p-1)$, $v_p(g^t - \omega^a(g)) \geq 1 > v_p(1 - \zeta_{p^{v-1}})$, yielding

$$(g^t - \chi(g)) = (1 - \zeta_p^{v-1}) \implies \mathbb{Z}_p[\chi]/\left(\chi(g) - g^t\right) \simeq \mathbb{Z}/p.$$

∎

Again let $\chi|_{(\mathbb{Z}/p)^\times} = \omega^a$. The spectral sequence collapses at the E_2-page and we conclude:

$$\pi_i\left(S^0_{K(1)}(p^v)^{h\chi}\right) = \begin{cases} \mathbb{Z}/p, & i = 2(a + k(p-1)) - 1; \\ 0, & \text{otherwise.} \end{cases} \tag{4.1.10}$$

We can also compute $\pi_*\left(S^0_{K(1)}(p^v)^{h\chi}\right)$ from the following identification:

Proposition 4.1.11. $S^0_{K(1)}(p^v)^{h\chi} \simeq \mathrm{Cofib}\left(S^0_{K(1)}(p^{v-1}) \to S^0_{K(1)}(p^v)\right)^{h\chi|_{(\mathbb{Z}/p)^\times}}$.

Proof. As χ is primitive of conductor p^v, $\mathbb{Z}_p[\chi] = \mathbb{Z}_p[\zeta_{p^{v-1}}]$ as an algebra. Recall from (3.3.9), $M((\mathbb{Z}_p[\zeta_{p^{v-1}}])$ is the p-completion of the integral Moore spectrum $M(\mathbb{Z}[\zeta_{p^{v-1}}])$ with a prescribed $C_{p^{v-1}}$-action. As the Moore spectrum is defined by

$$M(\mathbb{Z}[\zeta_{p^{v-1}}]) := \Sigma^{-1}\mathrm{Cofib}\left(S^0 \bigwedge (C_{p^{v-1}})_+ \longrightarrow S^0 \bigwedge (C_{p^{v-2}})_+\right),$$

we have

$$\begin{aligned}
S^0_{K(1)}(p^v)^{h\chi} :=& \mathrm{Map}_{\mathbb{Z}_p}\left(M(\mathbb{Z}_p[\chi]), S^0_{K(1)}(p^v)\right)^{h(\mathbb{Z}/p^v)^\times}\\
\simeq& \mathrm{Map}_{\mathbb{Z}_p}\left(S^0_{\chi|_{(\mathbb{Z}/p)^\times}}, \mathrm{Map}_{\mathbb{Z}_p}\left(M(\mathbb{Z}_p[\zeta_{p^{v-1}}]), S^0_{K(1)}(p^v)\right)^{h\mathbb{Z}/p^{v-1}}\right)^{h(\mathbb{Z}/p)^\times}\\
\simeq& \left(\mathrm{Cofib}\left(\mathrm{Map}((C_{p^{v-2}})_+, S^0_{K(1)}(p^v)) \to \mathrm{Map}((C_{p^{v-1}})_+, S^0_{K(1)}(p^v))\right)^{h\mathbb{Z}/p^{v-1}}\right)^{h\chi|_{(\mathbb{Z}/p)^\times}}\\
\simeq& \mathrm{Cofib}\left(S^0_{K(1)}(p^{v-1}) \to S^0_{K(1)}(p^v)\right)^{h\chi|_{(\mathbb{Z}/p)^\times}}.
\end{aligned}$$

In the last step, we used the facts that

$$\begin{aligned}
\mathrm{Map}((C_{p^{v-2}})_+, S^0_{K(1)}(p^v))^{h\mathbb{Z}/p^{v-1}} &\simeq S^0_{K(1)}(p^v)^{h(\mathbb{Z}/p^{v-1})/(\mathbb{Z}/p^{v-2})} \simeq S^0_{K(1)}(p^v)^{h\mathbb{Z}/p} \simeq S^0_{K(1)}(p^{v-1})\\
\mathrm{Map}((C_{p^{v-1}})_+, S^0_{K(1)}(p^v))^{h\mathbb{Z}/p^{v-1}} &\simeq S^0_{K(1)}(p^v)^{h(\mathbb{Z}/p^{v-1})/(\mathbb{Z}/p^{v-1})} \simeq S^0_{K(1)}(p^v)
\end{aligned}$$

□

Using the long exact sequence associated to this cofibration, we can recover (4.1.10) from $\pi_*\left(S^0_{K(1)}(p^v)\right)$ in (2.3.9) and the spectral sequence (4.1.1). Another consequence of this identification is the following:

Corollary 4.1.12. $\left(KU^\wedge_p\right)_*\left(S^0_{K(1)}(p^v)^{h\chi}\right) \simeq \mathrm{Hom}_{\mathbb{Z}_p}\left(\mathbb{Z}_p[\chi], \left(KU^\wedge_p\right)_*\right)$ *as* $\mathbb{Z}_p^\times$-$\left(KU^\wedge_p\right)_*$*-modules.*

Let χ_1 and χ_2 be two p-adic Dirichlet characters of conductors p^{v_1} and p^{v_2}, respectively. From (4.1.10), $\pi_i\left(S^0_{K(1)}(p^{v_1})^{h\chi_1}\right) \simeq \pi_i\left(S^0_{K(1)}(p^{v_2})^{h\chi_2}\right)$ whenever $\chi_1|_{(\mathbb{Z}/p)^\times} = \chi_2|_{(\mathbb{Z}/p)^\times}$ and $v_1, v_2 > 1$. But this does NOT imply $S^0_{K(1)}(p^{v_1})^{h\chi_1} \simeq S^0_{K(1)}(p^{v_2})^{h\chi_2}$ as spectra.

Proposition 4.1.13. $S^0_{K(1)}(p^{v_1})^{h\chi_1} \simeq S^0_{K(1)}(p^{v_2})^{h\chi_2}$ *iff* $\chi_1|_{(\mathbb{Z}/p)^\times} = \chi_2|_{(\mathbb{Z}/p)^\times}$ *and* $v_1 = v_2$.

Proof. This follows from Corollary 4.1.12 and the lemma below. □

Lemma 4.1.14. *Let X and Y be two $K(1)$-local spectra. X and Y are equivalent iff* $\left(KU^\wedge_p\right)_* X \simeq \left(KU^\wedge_p\right)_* Y$ *as* $\mathbb{Z}_p^\times$-$\left(KU^\wedge_p\right)_*$*-modules.*

Proof. The only if direction is clear. Let $f : \left(KU^\wedge_p\right)_* X \xrightarrow{\sim} \left(KU^\wedge_p\right)_* Y$ be an isomorphism of $\mathbb{Z}_p^\times$-$\left(KU^\wedge_p\right)_*$-modules. There is a HFPSS to compute $\pi_*\left(\mathrm{Map}(X,Y)\right)$:

$$E_2^{s,t} = H_c^s\left(\mathbb{Z}_p^\times; \left(KU^\wedge_p\right)_t\left(\mathrm{Map}(X,Y)\right)\right) \Longrightarrow \pi_{t-s}\left(\mathrm{Map}(X,Y)\right).$$

The isomorphism f is an element of $E_2^{0,0}$, since it is isomorphic to $\mathrm{Hom}_{\left(KU^\wedge_p\right)_*}\left(\left(KU^\wedge_p\right)_* X, \left(KU^\wedge_p\right)_* Y\right)^{\mathbb{Z}_p^\times}$. The HFPSS collapses on the E_2-page, as $\mathrm{cd}(\mathbb{Z}_p^\times) = 1$. This implies $f \in E_2^{0,0}$ is a permanent cycle, and is represented by a map of spectra $\alpha : X \to Y$ such that $\left(KU^\wedge_p\right)_* \alpha = f$.

We claim α is a weak equivalence. As $f = \left(KU^\wedge_p\right)_* \alpha$ is an isomorphism of $\mathbb{Z}_p^\times$-$\left(KU^\wedge_p\right)_*$-modules, α induces an isomorphism on the E_2-page of the HFPSS to compute $\pi_*(X)$ and $\pi_*(Y)$. It now follows from [Boa99, Theorem 5.3] that α is a weak equivalence. □

4.2. **The $N = 2^v$ case.** We start with the $N = 4$ case, when the only non-trivial 2-adic Dirichlet character of conductor 4 is the Teichmüller character $\omega : (\mathbb{Z}/4)^\times \to \mathbb{Z}_2^\times$. By Proposition 3.5.6, the Dirichlet $K(1)$-local sphere is identified with

$$\begin{aligned} S^0_{K(1)}(4)^{h\omega} &\simeq \mathrm{Map}_{\mathbb{Z}_2}(M(\mathbb{Z}_2[\omega]), KU_2^\wedge)^{h\mathbb{Z}_2^\times} \\ &\simeq \left(\mathrm{Map}_{\mathbb{Z}_2}(M(\mathbb{Z}_2[\omega]), KU_2^\wedge)^{h\{\pm1\}}\right)^{h(1+4\mathbb{Z}_2)} \\ &= \left((KU_2^\wedge)^{h\omega}\right)^{h(1+4\mathbb{Z}_2)}. \end{aligned} \tag{4.2.1}$$

Parallel to the computation of the classical $K(1)$-local sphere at $p = 2$ in Section 2.3, we will first identify $(KU_2^\wedge)^{h\omega}$ geometrically.

Proposition 4.2.2. *Let σ be the sign representation of C_2 on $\mathbb{Z}$ Define $K\mathbb{R}^{h\sigma}$ to be the homotopy σ-eigen-spectrum of Atiyah's C_2-equivariant $K\mathbb{R}$-spectrum in* [Ati66]*. Then we have an identification:*

$$K\mathbb{R}^{h\sigma} := \mathrm{Map}(M(\mathbb{Z}[\sigma]), K\mathbb{R})^{hC_2} \simeq \Sigma^2 KO.$$

Proof. By Figure 1, $M(\mathbb{Z}[\sigma])$ is C_2-equivariantly equivalent to $S^{\sigma-1}$. Now using the $(1+\sigma)$-periodicity of $K\mathbb{R}$ [Ati66, Theorem 2.1], we have a C_2-equivalence

$$\mathrm{Map}(S^{\sigma-1}, K\mathbb{R}) \simeq \Sigma^{1-\sigma} K\mathbb{R} \simeq \Sigma^2 K\mathbb{R}.$$

The claim now follows from the equivalence $K\mathbb{R}^{hC_2} \simeq KO$. □

Remark 4.2.3. This statement depends on the actual model of $M(\mathbb{Z}[\sigma])$. If we started with $S^{1-\sigma}$, where C_2 also acts by the sign representation on $\pi_*(S^0)$, we would have

$$\mathrm{Map}(S^{1-\sigma}, K\mathbb{R})^{hC_2} \simeq \Sigma^{-2} KO.$$

In terms of the HFPSS computations, the E_2-pages of $\mathrm{Map}(S^{\sigma-1}, K\mathbb{R})^{hC_2}$ and $\mathrm{Map}(S^{1-\sigma}, K\mathbb{R})^{hC_2}$ are the same. The difference is the d_3-differentials, which are invisible in algebra. Likewise, one can check the HFPSS for

$$\mathrm{Map}(S^{2\sigma-2}, K\mathbb{R})^{hC_2} \simeq \Sigma^4 KO \simeq KSp$$

has the same E_2-page as that for $K\mathbb{R}^{hC_2} \simeq KO$. Again the difference is the d_3-differentials that are invisible in algebra.

Remark 4.2.4. A more explicit construction is the following. For any compact space X, $K\mathbb{R}^{h\sigma}(X)$ consists of virtual complex vector bundles $[E]$ over X such that $\psi^{-1}([E]) = [\overline{E}] = -[E]$. For any such virtual vector bundle, its tensor product with the complexification of a real vector also satisfies this condition. As a result, $K\mathbb{R}^{h\sigma}$ is a KO-module spectrum.

Let ξ be the tautological complex line bundle over $\mathbb{CP}^1 \simeq S^2$. Then $[\xi]-[\overline{\xi}] \in K\mathbb{R}^{h\sigma}(S^2)$. Proposition 4.2.2 implies the external tensor product with $\xi - \overline{\xi}$ induces an isomorphism:

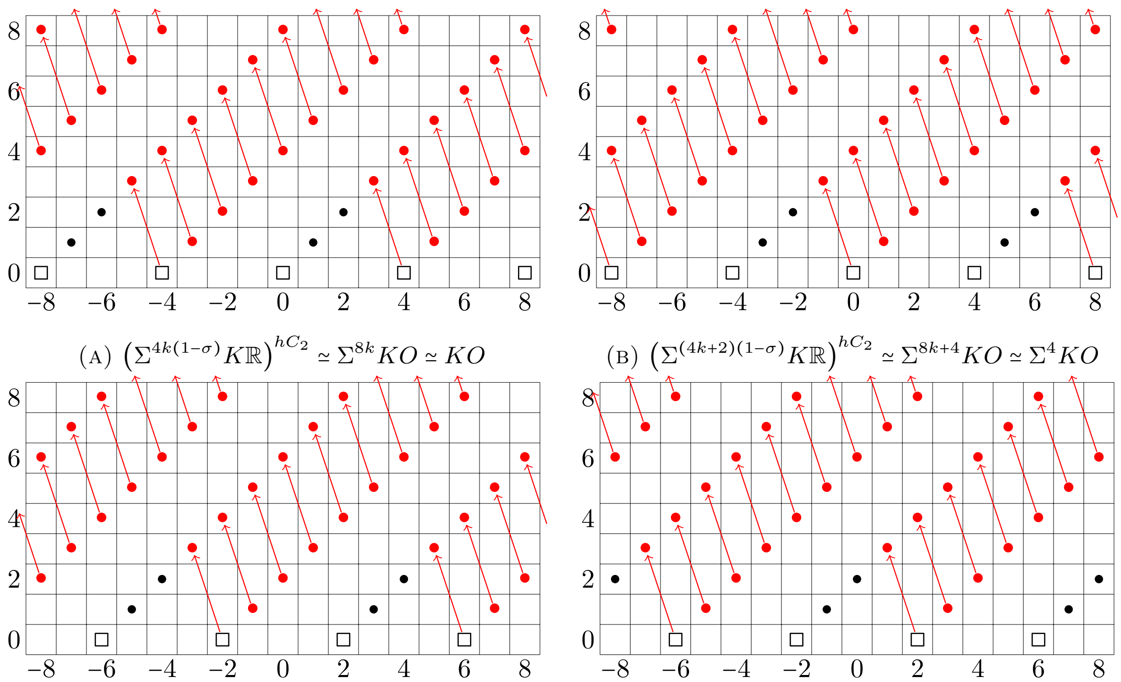
$$(\xi - \overline{\xi}) \boxtimes (-)_{\mathbb{C}} : KO(X) \xrightarrow{\sim} K\mathbb{R}^{h\sigma}(S^2 \times X).$$

As elements in $K\mathbb{R}^{h\sigma}(X)$ satisfy $[\overline{E}] = -[E]$, $K\mathbb{R}^{h\sigma}$ can be thought of as the *purely imaginary* K-theory, compared to the real K-theory $KO \simeq K\mathbb{R}^{hC_2}$.

Corollary 4.2.5. $(KU_2^\wedge)^{h\omega} \simeq \Sigma^2 KO_2^\wedge$ *and its homotopy groups are given by:*

$i \mod 8$	0	1	2	3	4	5	6	7
$\pi_i\left((KU_2^\wedge)^{h\omega}\right)$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}_2$	0

Remark 4.2.6. The equivalence $(KU_2^\wedge)^{h\omega} \simeq \Sigma^2 KO_2^\wedge$ is NOT $(1+4\mathbb{Z}_2)$-equivariant.



(A) $\left(\Sigma^{4k(1-\sigma)}K\mathbb{R}\right)^{hC_2} \simeq \Sigma^{8k}KO \simeq KO$

(B) $\left(\Sigma^{(4k+2)(1-\sigma)}K\mathbb{R}\right)^{hC_2} \simeq \Sigma^{8k+4}KO \simeq \Sigma^4 KO$

(C) $\left(\Sigma^{(4k+1)(1-\sigma)}K\mathbb{R}\right)^{hC_2} \simeq \Sigma^{8k+2}KO \simeq \Sigma^2 KO$

(D) $\left(\Sigma^{(4k+3)(1-\sigma)}K\mathbb{R}\right)^{hC_2} \simeq \Sigma^{8k+6}KO \simeq \Sigma^6 KO$

FIGURE 2. d_3-differentials in the HFPSS for different C_2-actions on the K-theory spectrum (Adams grading. $\square = \mathbb{Z}$ and $\bullet = \mathbb{Z}/2$. (A) and (B) are the same as Figures 3 and 6 in [HS14].)

The next step is to compute the HFPSS:

$$E_2^{s,t} = H_c^s\left(1+4\mathbb{Z}_2;\pi_t\left(\left(KU_2^\wedge\right)^{h\omega}\right)\right) \Longrightarrow \pi_{t-s}\left(S^0_{K(1)}(4)^{h\omega}\right).$$

Let $g \in 1+4\mathbb{Z}_2$ be a topological generator. Descending the Adams operations on $KU_2^\wedge$ to $\left(KU_2^\wedge\right)^{h\omega}$, we get g acts on $\pi_{4t+2}\left(\left(KU_2^\wedge\right)^{h\omega}\right)$ by g^{2t+1}. The actions on the $\mathbb{Z}/2$-terms are trivial since $\mathbb{Z}/2$ has only trivial automorphism. Using the continuous resolution (2.3.6), we compute the E_2-page of the HFPSS:

$$E_2^{s,t} = H_c^s\left(1+4\mathbb{Z}_2;\pi_t\left(\left(KU_2^\wedge\right)^{h\omega}\right)\right) = \begin{cases} \mathbb{Z}/4, & s=1, t\equiv 2 \mod 4; \\ \mathbb{Z}/2, & s=0,1, t\equiv 3,4 \mod 8; \\ 0, & \text{otherwise.} \end{cases} \tag{4.2.7}$$

Proposition 4.2.8. *The extension problems of this spectral sequence are trivial.*

Proof. We need solve the extension problems at $t-s \equiv 3 \mod 8$. The argument here is analogous to Proposition 2.3.12. As $\left(KU_2^\wedge\right)^{h\omega} \simeq \Sigma^2 KO_2^\wedge$ is a $KO_2^\wedge$-module spectrum, we denote the non-zero element in $\pi_3\left(\left(KU_2^\wedge\right)^{h\omega}\right)$ by $\Sigma^2\eta$. This is an element of order 2 and represents a permanent cycle in $E_2^{0,1}$ of (4.2.7). As $\Sigma^2\eta$ represents an element of order 2 in $\pi_3\left(S^0_{K(1)}(4)^{h\omega}\right)$, the extension problem is trivial. For general $t-s=8k+3$, replace $\Sigma^2\eta$ by $\beta^t\cdot\Sigma^2\eta$ in the argument above, where $\beta\in\pi_8(KO_2^\wedge)$ is the Bott element. $\square$

From this, we conclude:

$$\pi_i\left(S^0_{K(1)}(4)^{h\omega}\right)=\begin{cases}\mathbb{Z}/4, & i\equiv 1 \mod 4;\\ \mathbb{Z}/2, & i\equiv 2,4 \mod 8;\\ \mathbb{Z}/2\oplus\mathbb{Z}/2, & i\equiv 3 \mod 8;\\ 0, & \text{otherwise,}\end{cases}\tag{4.2.9}$$

We also record the E_2-page of the HESS associated to (4.2.1):

$$\operatorname{Ext}^s_{\mathbb{Z}_2[\![\mathbb{Z}_2^\times]\!]}\left((\mathbb{Z}_2)_{\tilde\omega},(KU_2^\wedge)_t\right)=\begin{cases}\mathbb{Z}/4, & s=1,t\equiv 2 \mod 4;\\ \mathbb{Z}/2, & s>1,t\equiv 2 \mod 4;\\ \mathbb{Z}/2, & s>0,4\mid t;\\ 0, & \text{otherwise.}\end{cases}\tag{4.2.10}$$

Remark 4.2.11. As explained in Remark 3.3.8, we could have chosen $M(\mathbb{Z}[\zeta_2])=S^{1-\sigma}$ when defining the Dirichlet J-spectra and $K(1)$-local spheres. Denote the resulting homotopy eigen-spectra by

$$X^{h'\omega}:=\operatorname{Map}_{\mathbb{Z}_2}(S^{1-\sigma},X)^{hC_2},$$

where $\omega: C_2\simeq(\mathbb{Z}/4)^\times\to\mathbb{Z}_2^\times$ is the 2-adic Teichmüller character. Then by Remark 4.2.3, $(KU_2^\wedge)^{h'\omega}\simeq\Sigma^{-2}KO_2^\wedge$. A similar computation as above yields:

$$\pi_i\left(S^0_{K(1)}(4)^{h'\omega}\right)=\begin{cases}\mathbb{Z}/4, & i\equiv 1 \mod 4;\\ \mathbb{Z}/2, & i\equiv -2,0 \mod 8;\\ \mathbb{Z}/2\oplus\mathbb{Z}/2, & i\equiv -1 \mod 8;\\ 0, & \text{otherwise,}\end{cases}$$

Note that $\pi_{2k-1}\left(S^0_{K(1)}(4)^{h\chi}\right)=\pi_{2k-1}\left(S^0_{K(1)}(4)^{h'\chi}\right)$ when $(-1)^k=\chi(-1)$.

Both $S^0_{K(1)}(4)^{h\omega}$ and $S^0_{K(1)}(4)^{h'\omega}$ are elements of order 4 in the $K(1)$-local Picard group $\operatorname{Pic}_{K(1)}$ at prime 2. Their difference in $\operatorname{Pic}_{K(1)}$ is the **exotic $K(1)$-local sphere $\mathcal{E}_{K(1)}$**, an element whose HFPSS has the same E_2-page as that for the $K(1)$-local sphere.[4] One construction of this element is given in [HS14, Section 9]. It is also the $K(1)$-localization of a finite CW-spectra $\mathcal{E}$, described in Theorem 5.3.6. By identifying $\mathcal{E}_{K(1)}$ with $(KSp_2^\wedge)^{h(1+4\mathbb{Z}_2)}$, we can compute its homotopy groups as in (2.3.13):

$$\pi_i\left(\mathcal{E}_{K(1)}\right)=\begin{cases}\mathbb{Z}_2, & i=0,-1;\\ \mathbb{Z}/2, & i\equiv 4,6 \mod 8;\\ \mathbb{Z}/2\oplus\mathbb{Z}/2, & i\equiv 5 \mod 8;\\ \mathbb{Z}/2^{v_2(k)+3}, & i=4k-1\neq -1;\\ 0, & \text{otherwise.}\end{cases}\tag{4.2.12}$$

When $p=2$ and $N=2^v>4$, we apply Proposition 3.5.6 as before:

$$S^0_{K(1)}(2^v)^{h\chi}\simeq\operatorname{Map}_{\mathbb{Z}_2}\left(M(\mathbb{Z}_2[\chi]),KU_2^\wedge\right)^{h\mathbb{Z}_2^\times}.$$

Lemma 4.2.13. $S^0_{K(1)}(2^v)^{h\chi}\simeq\operatorname{Map}_{\mathbb{Z}_2}\left(M(\mathbb{Z}_2[\chi]),(KU_2^\wedge)^{h\chi|_{(\mathbb{Z}/4)^\times}}\right)^{h(1+4\mathbb{Z}_2)}$.

[4]The spectrum $\mathcal{E}_{K(1)}$ has been referred to by different letters (like P) in the literature. Here we use the letter $\mathcal{E}$ since it stands for exotic. In Theorem 5.3.9, we will see it is the "error term" in the $K(1)$-local Brown-Comenetz duality at prime 2.

Proof. We prove the claim by breaking the $\mathbb{Z}_2^\times$-homotopy fixed points into two steps.

$$\begin{aligned}S^0_{K(1)}(2^v)^{h\chi} \simeq & \operatorname{Map}_{\mathbb{Z}_2}\left(M(\mathbb{Z}_2[\chi]), KU_2^\wedge\right)^{h\mathbb{Z}_2^\times}\\ \simeq & \operatorname{Map}_{\mathbb{Z}_2}\left(M(\mathbb{Z}_2[\chi]), \operatorname{Map}_{\mathbb{Z}_2}\left(S^0_{\chi|_{(\mathbb{Z}/4)^\times}}, KU_2^\wedge\right)^{h(\mathbb{Z}/4)^\times}\right)^{h(1+4\mathbb{Z}_2)}\\ \simeq & \operatorname{Map}_{\mathbb{Z}_2}\left(M(\mathbb{Z}_2[\chi]), (KU_2^\wedge)^{h\chi|_{(\mathbb{Z}/4)^\times}}\right)^{h(1+4\mathbb{Z}_2)}.\end{aligned}$$

Here, $S^0_\omega := S^{\sigma-1}$ and $S^0_{\omega^0} := S^0$. In the third line, we used the fact $\chi\cdot\chi|_{(\mathbb{Z}/4)^\times}$ is trivial when restricted to $(\mathbb{Z}/4)^\times$ and is equal to $\widetilde{\chi}$ when restricted to $1+4\mathbb{Z}_2$. ☐

Let g be a topological generator of $1+4\mathbb{Z}_2$. Denote by $\operatorname{Ann}(\widetilde{\chi}(g)-1)$ the ideal of annihilators of $\widetilde{\chi}(g)-1$ in $\mathbb{Z}_2[\chi]/(2)$. The computation now splits into two subcases depending on the parity of χ:

- When $\chi(-1)=1$, $(KU_2^\wedge)^{h\chi|_{(\mathbb{Z}/4)^\times}} \simeq KO_2^\wedge$. By (2.3.6) and (2.3.10), E_2-page of the HESS is:

$$\begin{aligned}E_2^{s,t} =& \operatorname{Ext}^s_{\mathbb{Z}_2[\![1+4\mathbb{Z}_2]\!]}\left(\mathbb{Z}_2[\chi], \pi_t\left(KO_2^\wedge\right)\right)\\ =& \begin{cases}\mathbb{Z}_2[\chi]/\left(\widetilde{\chi}(g)-g^{2k}\right), & s=1, t=4k;\\ \operatorname{Ann}\left(\widetilde{\chi}(g)-1\right), & s=0, t\equiv 1,2 \mod 8;\\ \mathbb{Z}_2[\chi]/\left(2,\widetilde{\chi}(g)-1\right), & s=1, t\equiv 1,2 \mod 8;\\ 0, & \text{otherwise.}\end{cases}\end{aligned}\tag{4.2.14}$$

- When $\chi(-1)=-1$, $(KU_2^\wedge)^{h\chi|_{(\mathbb{Z}/4)^\times}} \simeq \Sigma^2 KO_2^\wedge$ by Proposition 4.2.2. The E_2-page of the HESS is:

$$\begin{aligned}E_2^{s,t} =& \operatorname{Ext}^s_{\mathbb{Z}_2[\![1+4\mathbb{Z}_2]\!]}\left(\mathbb{Z}_2[\chi], \pi_t\left(\Sigma^2 KO_2^\wedge\right)\right)\\ =& \begin{cases}\mathbb{Z}_2[\chi]/\left(\widetilde{\chi}(g)-g^{2k+1}\right), & s=1, t=4k+2;\\ \operatorname{Ann}\left(\widetilde{\chi}(g)-1\right), & s=0, t\equiv 3,4 \mod 8;\\ \mathbb{Z}_2[\chi]/\left(2,\widetilde{\chi}(g)-1\right), & s=1, t\equiv 3,4 \mod 8;\\ 0, & \text{otherwise.}\end{cases}\end{aligned}\tag{4.2.15}$$

In both cases, the spectral sequences collapse at the E_2-pages. Analogous to Proposition 2.3.12 (Proposition 4.2.8), the extension problems at $t-s\equiv 1 \mod 8$ ($t-s\equiv 3 \mod 8$, resp.) are trivial. We further simplify the formulas using the following facts about $\mathbb{Z}_2[\chi]$ from Proposition A.2.3.

Lemma 4.2.16. *Let χ be a primitive 2-adic Dirichlet character of conductor $2^v\geq 8$. Let g be a topological generator of $1+4\mathbb{Z}_2$.*

(1) $\mathbb{Z}_2[\chi]$ is a totally ramified extension of $\mathbb{Z}_2$ of ramification index 2^{v-3}.
(2) $1-\widetilde{\chi}(g)$ is a uniformizer of $\mathbb{Z}_2[\chi]$ and $\mathbb{Z}_2[\chi]/(1-\widetilde{\chi}(g))\simeq\mathbb{Z}/2$.
(3) The ideal of annihilators of $\widetilde{\chi}(g)-1\in\mathbb{Z}_2[\chi]/(2)$ is isomorphic to $\mathbb{Z}/2$.
(4) $\mathbb{Z}_2[\chi]/(\widetilde{\chi}(g)-g^k)=\mathbb{Z}/2$ for any k.

Proof. Only (4) needs a proof. $\widetilde{\chi}(g)=\zeta_{2^{v-2}}$ since χ is primitive. Write $\widetilde{\chi}(g)-g^k=\widetilde{\chi}(g)-1+1-g^k$. By (2), $\widetilde{\chi}(g)-1$ is a uniformizer. On the other hand $v_2(1-g^k)\geq 2 > v_2(\widetilde{\chi}(g)-1)$, since $g\equiv 1 \mod 4$. This implies:

$$(\widetilde{\chi}(g)-g^k)=(\widetilde{\chi}(g)-1)\implies \mathbb{Z}_2[\chi]/(\widetilde{\chi}(g)-g^k)=\mathbb{Z}/2.$$

☐

Proposition 4.2.17. *When $\chi(-1)=1$, we have*

$$\pi_i\left(S^0_{K(1)}(2^v)^{h\chi}\right)=\begin{cases}\mathbb{Z}/2, & i\equiv 0,2,3,7 \mod 8;\\ \mathbb{Z}/2\oplus\mathbb{Z}/2, & i\equiv 1 \mod 8;\\ 0, & \text{otherwise.}\end{cases}\tag{4.2.18}$$

When $\chi(-1) = -1$*, we have*

$$\pi_i\left(S^0_{K(1)}(2^v)^{h\chi}\right) = \begin{cases} \mathbb{Z}/2, & i \equiv 1,2,4,5 \mod 8; \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2, & i \equiv 3 \mod 8; \\ 0, & \text{otherwise.} \end{cases} \tag{4.2.19}$$

Remark 4.2.20. The computations above depend on the actual model of the C_2-actions on the Moore spectra:

- When $\chi(-1) = 1$, if we choose $S^{2-2\sigma}$ as a model for the C_2-action on S^0 with trivial induced action on π_*, (4.2.18) becomes:
$$\pi_i\left(S^0_{K(1)}(2^v)^{h'\chi}\right) = \begin{cases} \mathbb{Z}/2, & i \equiv 3,4,6,7 \mod 8; \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2, & i \equiv 5 \mod 8; \\ 0, & \text{otherwise.} \end{cases}$$
- When $\chi(-1) = -1$, if we choose $S^{1-\sigma}$ as a model for the C_2-action on S^0 that induces sign representations on π_*, (4.2.19) becomes:
$$\pi_i\left(S^0_{K(1)}(2^v)^{h'\chi}\right) = \begin{cases} \mathbb{Z}/2, & i \equiv 0,1,5,6 \mod 8; \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2, & i \equiv 7 \mod 8; \\ 0, & \text{otherwise.} \end{cases}$$

Note that $\pi_{2k-1}\left(S^0_{K(1)}(2^v)^{h\chi}\right) = \pi_{2k-1}\left(S^0_{K(1)}(2^v)^{h'\chi}\right)$ when $(-1)^k = \chi(-1)$.

Like the odd prime case, we can recover the results in Proposition 4.2.17 using the following identification:

Proposition 4.2.21. *Let* χ *be a primitive* 2*-adic Dirichlet character of conductor* $2^v > 4$*. When* $\chi(-1) = 1$*, we have*

$$S^0_{K(1)}(2^v)^{h\chi} \simeq \operatorname{Cofib}\left((KO_2^\wedge)^{h(1+2^{v-1}\mathbb{Z}_2)} \to (KO_2^\wedge)^{h(1+2^v\mathbb{Z}_2)}\right).$$

If $\chi(-1) = -1$*, then*

$$S^0_{K(1)}(2^v)^{h\chi} \simeq \operatorname{Cofib}\left(\left((KU_2^\wedge)^{h\omega}\right)^{h(1+2^{v-1}\mathbb{Z}_2)} \to \left((KU_2^\wedge)^{h\omega}\right)^{h(1+2^v\mathbb{Z}_2)}\right).$$

Proof. This is the same as proof of Proposition 4.1.11. □

Corollary 4.2.22. *There is an equivalence of* $(1+4\mathbb{Z}_2)$*-*$(KO_2^\wedge)_*$*-modules:*

$$(KO_2^\wedge)_*\left(S^0_{K(1)}(2^v)^{h\chi}\right) \simeq \begin{cases} \operatorname{Hom}\left(\mathbb{Z}_2[\chi], (KO_2^\wedge)_*\right) & \chi(-1) = 1; \\ \operatorname{Hom}\left(\mathbb{Z}_2[\chi], (KU_2^\wedge)_*^{h\omega}\right) & \chi(-1) = -1, \end{cases}$$

This computation leads to a $p = 2$ version of Proposition 4.1.13. Let χ_1 and χ_2 be two 2-adic Dirichlet characters of conductors 2^{v_1} and 2^{v_2}, respectively. From Proposition 4.2.17, $\pi_i\left(S^0_{K(1)}(2^{v_1})^{h\chi_1}\right) \simeq \pi_i\left(S^0_{K(1)}(2^{v_2})^{h\chi_2}\right)$ whenever $\chi_1(-1) = \chi_2(-1)$ and $v_1, v_2 > 2$. But this DOES NOT imply $S^0_{K(1)}(2^{v_1})^{h\chi_1} \simeq S^0_{K(1)}(2^{v_2})^{h\chi_2}$ as spectra. We have the $p = 2$ version of Proposition 4.1.13:

Proposition 4.2.23. $S^0_{K(1)}(2^{v_1})^{h\chi_1} \simeq S^0_{K(1)}(2^{v_2})^{h\chi_2}$ *iff* $\chi_1(-1) = \chi_2(-1)$ *and* $v_1 = v_2$*.*

Proof. This follows from Corollary 4.2.22 and the $p = 2$ version of Lemma 4.1.14. □

Lemma 4.2.24. *Let* X *and* Y *be two* $K(1)$*-local spectra and* $p = 2$*. Then* $X \simeq Y$ *iff* $(KO_2^\wedge)_* X \simeq (KO_2^\wedge)_* Y$ *as* $(1+4\mathbb{Z}_2)$*-*$(KO_2^\wedge)_*$*-modules.*

4.3. **The N is not a p-power case.** Write $N = p^v \cdot N'$, where $p \nmid N' > 1$. In this case, a primitive Dirichlet character $\chi : (\mathbb{Z}/N)^\times \to \mathbb{C}_p^\times$ factorizes into a product $\chi = \chi_p \cdot \chi'$, where χ_p has conductor p^v and χ' has conductor N'. The subgroup $(\mathbb{Z}/N')^\times$ of $(\mathbb{Z}/N)^\times$ acts trivially on $S^0_{K(1)}(p^v)$.

Proposition 4.3.1. *$S^0_{K(1)}(p^v)^{h\chi}$ is contractible when $|\operatorname{Im}\chi'|$ is not a power of p.*

Proof. We claim that when $|\operatorname{Im}\chi'|$ is not a power of p, the E_2-page of spectral sequence (3.5.8) to compute homotopy groups of $S^0_{K(1)}(p^v)^{h\chi}$ is zero. That is, the group cohomology

$$H^s_c(\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times; \mathbb{Z}_p^{\otimes t}[\chi^{-1}])$$

is zero for all $s \geq 0$ and $t \in \mathbb{Z}$. Suppose $\zeta_n \in \operatorname{Im}\chi'$ and $p \nmid n$. The $(\mathbb{Z}/N')^\times$ contains a subgroup C_n such that $\chi'|_{C_n}$ is injective. We have a Hochschild-Serre spectral sequence to compute this group cohomology:

$$E_2^{r,s,t} = H^r_c(\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times / C_n; H^s(C_n; \mathbb{Z}_p^{\otimes t}[\chi^{-1}])) \Longrightarrow H^s_c(\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times; \mathbb{Z}_p^{\otimes t}[\chi^{-1}]). \tag{4.3.2}$$

The group cohomology of C_n vanishes in positive degrees since its order is invertible in $\mathbb{Z}_p$. As a generator $g \in C_n$ acts on $\mathbb{Z}_p^{\otimes t}[\chi^{-1}] = \mathbb{Z}_p[\chi]$ by multiplication by ζ_n^{-1}, the fixed points of this group action is zero. This shows $E_2^{r,s,t} = 0$ for all $s \geq 0$ in (4.3.2). Consequently, the E_2-page of (3.5.8) to compute $\pi_*\left(S^0_{K(1)}(p^v)^{h\chi}\right)$ is zero and the Dirichlet $K(1)$-local sphere is contractible. □

Corollary 4.3.3. *Suppose χ is primitive character of conductor $N = p^v \cdot N'$ as above. $S^0_{K(1)}(p^v)^{h\chi}$ is contractible when $p \nmid \phi(N')$, in particular when:*
(1) $N = q \neq p$ is a prime such that $p \nmid (q-1)$.
(2) $N = q^v > 2q$ for any prime q not equal to p.

Proof. χ' is a primitive character since χ is. This guarantees $\operatorname{Im}\chi'$ is not trivial. The assumption $p \nmid \phi(N')$ implies that $\operatorname{Im}\chi'$ contains no p-power roots of unity. The claim now follows from Proposition 4.3.1. □

When $\operatorname{Im}\chi'$ contains only p-power roots of unity, the spectral sequence (3.5.8) does not collapse on the E_2-page. Instead, we compute $\pi_*\left(S^0_{K(1)}(p^v)^{h\chi}\right)$ by identifying the spectrum from its construction.

Theorem 4.3.4. *Suppose $\operatorname{Im}\chi' \simeq C_{p^n}$ for some $n \geq 1$. Then we have:*

$$S^0_{K(1)}(p^v)^{h\chi} \simeq \begin{cases} \Sigma\left(S^0_{K(1)}(2p)^{h\chi|_{(\mathbb{Z}/2p)^\times}}\right)^{\vee p}, & \text{if } p^v \leq 2p; & \text{(Case I)} \\ \Sigma\left(S^0_{K(1)}(p)^{h\chi|_{(\mathbb{Z}/p)^\times}}\right)^{\vee p}, & \text{if } n \geq v-1 > 0 \text{ and } p > 2; & \text{(Case II)} \\ \Sigma\left(S^0_{K(1)}(4)^{h\chi|_{(\mathbb{Z}/4)^\times}}\right)^{\vee p}, & \text{if } n \geq v-2 > 0 \text{ and } p = 2; & \text{(Case II')} \\ \Sigma\left(S^0_{K(1)}(p^{v-n})^{h\chi|_{(\mathbb{Z}/p)^\times}}\right)^{\vee p}, & \text{if } n < v-1 \text{ and } p > 2; & \text{(Case III)} \\ \Sigma\left(S^0_{K(1)}(2^{v-n})^{h\chi|_{(\mathbb{Z}/4)^\times}}\right)^{\vee p}, & \text{if } n < v-2 \text{ and } p = 2. & \text{(Case III')} \end{cases}$$

Here $(\mathbb{Z}/2p)^\times$ is thought of as a subgroup of $(\mathbb{Z}/N)^\times$ via the inclusions $(\mathbb{Z}/2p)^\times \subseteq (\mathbb{Z}/p^v)^\times \subseteq (\mathbb{Z}/N)^\times$.

Proof. We prove the cases when $p > 2$. The $p = 2$ cases are similar. Recall from Construction 3.4.1 that

$$S^0_{K(1)}(p^v)^{h\chi} := \operatorname{Map}_{\mathbb{Z}_p}\left(M(\mathbb{Z}_p[\chi]), S^0_{K(1)}(p^v)\right)^{h(\mathbb{Z}/N)^\times}.$$

We need to compare $\mathbb{Z}_p[\chi_p]$, $\mathbb{Z}_p[\chi']$, and $\mathbb{Z}_p[\chi]$. As χ_p is primitive, $\mathbb{Z}_p[\chi_p] = \mathbb{Z}_p[\zeta_{p^{v-1}}]$ as an algebra. Our assumption says $\operatorname{Im}\chi' \simeq C_{p^n}$ for some $n \geq 1$. As a result, $\mathbb{Z}_p[\chi] = \mathbb{Z}_p[\zeta_{p^{\max\{v-1,n\}}}]$. In particular, when $p^v \leq p$, $\mathbb{Z}_p[\chi_p] = \mathbb{Z}_p$ and $\mathbb{Z}_p[\chi] = \mathbb{Z}_p[\chi']$. So there are three cases depending on n and $v-1$ ($v-2$ when $p = 2$).

In **Case I** when $p^v \leq p$ and $p > 2$, we have the following identification:

$$\begin{aligned}
S^0_{K(1)}(p^v)^{h\chi} :=& \mathrm{Map}_{\mathbb{Z}_p}\left(M(\mathbb{Z}_p[\chi]), S^0_{K(1)}(p)\right)^{h((\mathbb{Z}/p)^\times \times (\mathbb{Z}/N')^\times)} \\
\simeq& \mathrm{Map}_{\mathbb{Z}_p}\left(M(\mathbb{Z}_p[\zeta_{p^n}]), \mathrm{Map}\left(S^0_{\chi|_{(\mathbb{Z}/p)^\times}}, S^0_{K(1)}(p)\right)\right)^{h((\mathbb{Z}/p)^\times \times (\mathbb{Z}/N')^\times)} \\
(*) \quad \simeq& \mathrm{Map}_{\mathbb{Z}_p}\left(M(\mathbb{Z}_p[\zeta_{p^n}])_{h(\mathbb{Z}/N')^\times}, \mathrm{Map}\left(S^0_{\chi|_{(\mathbb{Z}/p)^\times}}, S^0_{K(1)}(p)\right)^{h(\mathbb{Z}/p)^\times}\right) \\
\simeq& \mathrm{Map}_{\mathbb{Z}_p}\left(M(\mathbb{Z}_p[\zeta_{p^n}])_{h(\mathbb{Z}/N')^\times}, S^0_{K(1)}(p)^{h\chi|_{(\mathbb{Z}/p)^\times}}\right) \\
\simeq& \mathrm{Map}_{K(1)}\left((M(\mathbb{Z}_p[\zeta_{p^n}])_{h(\mathbb{Z}/N')^\times})_{K(1)}, S^0_{K(1)}(p)^{h\chi_p}\right).
\end{aligned}$$

In (*), we used the facts that $(\mathbb{Z}/p)^\times$ acts trivially on the source, and that $(\mathbb{Z}/N')^\times$ acts trivially on the target. Also, notice $S^0_{K(1)}(p)^{h\chi_p} \simeq S^0_{K(1)}$ when χ_p is trivial. We now show:

$$(M(\mathbb{Z}_p[\zeta_{p^n}])_{h(\mathbb{Z}/N')^\times})_{K(1)} \simeq \left(S^{-1}_{K(1)}\right)^{\vee p}.$$

In (3.3.9), $M((\mathbb{Z}_p[\zeta_{p^n}])$ is defined to be the p-completion of the integral Moore spectrum $M(\mathbb{Z}[\zeta_{p^n}])$. The p-completion commutes with the taking homotopy orbits, since it is equivalent to smashing with $M(\mathbb{Z}_p)$ in this case. As a result, we should first find $M(\mathbb{Z}[\zeta_{p^n}])_{h(\mathbb{Z}/N')^\times}$. By (3.3.4), we have:

$$\begin{aligned}
& M(\mathbb{Z}[\zeta_{p^n}]) := \Sigma^{-1}\mathrm{Cofib}\left((C_{p^n})_+ \longrightarrow (C_{p^{n-1}})_+\right) \\
(4.3.5) \qquad \Longrightarrow\ & M(\mathbb{Z}[\zeta_{p^n}])_{h(\mathbb{Z}/N')^\times} \simeq \Sigma^{-1}\mathrm{Cofib}\left(((C_{p^n})_+)_{h(\mathbb{Z}/N')^\times} \longrightarrow \left((C_{p^{n-1}})_+\right)_{h(\mathbb{Z}/N')^\times}\right).
\end{aligned}$$

Lemma 4.3.6. *Let H be a closed subgroup of G. Then $(G/H)_{hG} \simeq BH$.*

The Lemma implies that $((C_{p^n})_+)_{h(\mathbb{Z}/N')^\times} \simeq (B\ker\chi')_+$ and $\left((C_{p^{n-1}})_+\right)_{h(\mathbb{Z}/N')^\times} \simeq \left(B\chi'^{-1}(C_p)\right)_+$. From the short exact sequence of abelian groups:

$$0 \longrightarrow \ker\chi' \longrightarrow \chi'^{-1}(C_p) \longrightarrow C_p \longrightarrow 0,$$

we get a fiber sequence of classifying spaces:

$$B\ker\chi' \longrightarrow B\chi'^{-1}(C_p) \longrightarrow BC_p.$$

Together with (4.3.5), we have shown $M(\mathbb{Z}[\zeta_{p^n}])_{h(\mathbb{Z}/N')^\times} \simeq \Sigma^{-1}(BC_p)_+$ as a spectrum. It now remains to identify $(BC_p)_+$ in $\mathbf{Sp}_{K(1)}$.

Lemma 4.3.7. *Let A be a finite abelian group and $A_{(p)}$ be its Sylow p-subgroup. Then*

$$(BA_+)_{K(1)} \simeq \left(S^0_{K(1)}\right)^{\vee|A_{(p)}|}.$$

Proof. By [HKR00, Corollary 5.10], $\left(KU^\wedge_p\right)_*(BA) \simeq \mathrm{Fun}(A_{(p)}, \left(KU^\wedge_p\right)_*)$, where $\mathbb{Z}_p^\times$ acts on $A_{(p)}$ trivially. The claim now follows from Lemma 4.1.14. □

Corollary 4.3.8. $(M(\mathbb{Z}_p[\zeta_{p^n}])_{h(\mathbb{Z}/N')^\times})_{K(1)} \simeq \left(S^{-1}_{K(1)}\right)^{\vee p}$.

In **Case II** when $p > 2$ and $n \geq v-1 > 0$, $\mathbb{Z}_p[\chi] = \mathbb{Z}_p[\zeta_{p^n}]$. From this, we have:

$$\begin{aligned}
S^0_{K(1)}(p^v)^{h\chi} :=& \mathrm{Map}_{\mathbb{Z}_p}\left(M(\mathbb{Z}_p[\chi]), S^0_{K(1)}(p^v)\right)^{h((\mathbb{Z}/p^v)^\times \times (\mathbb{Z}/N')^\times)}\\
\simeq& \mathrm{Map}_{\mathbb{Z}_p}\left(M(\mathbb{Z}_p[\zeta_{p^n}])_{h(\mathbb{Z}/N')^\times}, S^0_{K(1)}(p^v)\right)^{h(\mathbb{Z}/p^v)^\times}\\
\simeq& \mathrm{Map}_{\mathbb{Z}_p}\left(\Sigma^{-1}(BC_p)_+, S^0_{K(1)}(p^v)\right)^{h(\mathbb{Z}/p^v)^\times}\\
\simeq& \mathrm{Map}_{K(1)}\left(\left(S^{-1}_{K(1)}\right)^{\vee p}, S^0_{K(1)}(p^v)\right)^{h(\mathbb{Z}/p^v)^\times}.
\end{aligned}$$

The subgroup $(\mathbb{Z}/p)^\times \subseteq (\mathbb{Z}/p^v)^\times$ acts on $\left(S^{-1}_{K(1)}\right)^{\vee p} \simeq \left(M(\mathbb{Z}_p[\zeta_{p^n}])_{h(\mathbb{Z}/N')^\times}\right)_{K(1)}$ by $\chi_p|_{(\mathbb{Z}/p)^\times}$. We claim the other summand $\mathbb{Z}/p^{v-1} \subseteq (\mathbb{Z}/p^v)^\times$ acts on the homotopy orbit trivially. This is because both actions of $(\mathbb{Z}/N')^\times$ and $\mathbb{Z}/p^{v-1}$ on the Moore spectrum $M(\mathbb{Z}_p[\zeta_{p^n}])$ factors through the action by C_{p^n}. As $(\mathbb{Z}/N')^\times$ surjects onto C_{p^n} via χ', the action of $\mathbb{Z}/p^{v-1}$ on $M(\mathbb{Z}_p[\zeta_{p^n}])_{h(\mathbb{Z}/N')^\times}$ is trivial, yielding:

$$\begin{aligned}
S^0_{K(1)}(p^v)^{h\chi} \simeq& \mathrm{Map}_{K(1)}\left(\left(S^{-1}_{K(1)}\right)^{\vee p}, S^0_{K(1)}(p^v)\right)^{h(\mathbb{Z}/p^v)^\times}\\
\simeq& (S^1)^{\vee p} \wedge \mathrm{Map}\left(S^0_{\chi|_{(\mathbb{Z}/p)^\times}}, \left(S^0_{K(1)}(p^v)\right)^{h\mathbb{Z}/p^{v-1}}\right)^{h(\mathbb{Z}/p)^\times}\\
\simeq& \Sigma\left(S^0_{K(1)}(p)^{h\chi|_{(\mathbb{Z}/p)^\times}}\right)^{\vee p}.
\end{aligned}$$

In **Case III** when $p > 2$ and $v-1 > n$, $\mathbb{Z}_p[\chi] = \mathbb{Z}_p[\zeta_{p^{v-1}}]$. By Proposition 3.3.6, there is a $C_{p^{v-1}}$-equivalence:

$$M(\mathbb{Z}_p[\zeta_{p^n}]) \simeq (C_{p^{v-1}})_+ \bigwedge_{C_{p^n}} (C_{p^n})_+ \bigwedge_{C_p} M(\mathbb{Z}_p[\zeta_p]) \simeq (C_{p^{v-1}})_+ \bigwedge_{C_{p^n}} M(\mathbb{Z}_p[\zeta_{p^n}]).$$

We have the identification:

$$\begin{aligned}
S^0_{K(1)}(p^v)^{h\chi} :=& \mathrm{Map}_{\mathbb{Z}_p}\left(M(\mathbb{Z}_p[\chi]), S^0_{K(1)}(p^v)\right)^{h((\mathbb{Z}/p^v)^\times \times (\mathbb{Z}/N')^\times)}\\
\simeq& \left(\mathrm{Map}_{\mathbb{Z}_p}\left(M(\mathbb{Z}_p[\zeta_{p^{v-1}}]), S^0_{K(1)}(p^v)\right)^{h(\mathbb{Z}/N')^\times}\right)^{h(\mathbb{Z}/p^v)^\times}\\
\simeq& \left(\mathrm{Map}_{\mathbb{Z}_p}\left((C_{p^{v-1}})_+ \bigwedge_{C_{p^n}} M(\mathbb{Z}_p[\zeta_{p^n}]), S^0_{K(1)}(p^v)\right)^{h(\mathbb{Z}/N')^\times}\right)^{h(\mathbb{Z}/p^v)^\times}\\
\simeq& \mathrm{Map}_{\mathbb{Z}_p}\left((C_{p^{v-1}}/C_{p^n})_+ \bigwedge M(\mathbb{Z}_p[\zeta_{p^n}])_{h(\mathbb{Z}/N')^\times}, S^0_{K(1)}(p^v)\right)^{h(\mathbb{Z}/p^v)^\times}\\
\simeq& \mathrm{Map}_{K(1)}\left((C_{p^{v-1}}/C_{p^n})_+ \bigwedge (S^{-1}_{K(1)})^{\vee p}, S^0_{K(1)}(p^v)\right)^{h(\mathbb{Z}/p^v)^\times}.
\end{aligned}$$

Like in the previous cases, the subgroup $(\mathbb{Z}/p)^\times \subseteq (\mathbb{Z}/p^v)^\times$ acts on $\left(M(\mathbb{Z}_p[\zeta_{p^{v-1}}])_{h(\mathbb{Z}/N')^\times}\right)_{K(1)}$ by $\chi_p|_{(\mathbb{Z}/p)^\times}$. The other summand $\mathbb{Z}/p^{v-1} \subseteq (\mathbb{Z}/p^v)^\times$ acts on the source $(C_{p^{v-1}}/C_{p^n})_+ \wedge (S^{-1}_{K(1)})^{\vee p}$ via the projection $\mathbb{Z}/p^{v-1} \simeq C_{p^{v-1}} \twoheadrightarrow C_{p^{v-1}}/C_{p^n}$, and on the target $S^0_{K(1)}(p^v)$ by the Galois action. As the latter action is free, we have

$$\begin{aligned}
S^0_{K(1)}(p^v)^{h\chi} \simeq& \mathrm{Map}_{K(1)}\left((C_{p^{v-1}}/C_{p^n})_+ \bigwedge (S^{-1}_{K(1)})^{\vee p}, S^0_{K(1)}(p^v)\right)^{h(\mathbb{Z}/p^v)^\times}\\
\simeq& \Sigma\left(\mathrm{Map}_{K(1)}\left(S^0_{\chi|_{(\mathbb{Z}/p)^\times}}, S^0_{K(1)}(p^v)^{h\mathbb{Z}/p^n}\right)^{h(\mathbb{Z}/p)^\times}\right)^{\vee p}\\
\simeq& \Sigma\left(S^0_{K(1)}(p^{v-n})^{h\chi|_{(\mathbb{Z}/p)^\times}}\right)^{\vee p}.
\end{aligned}$$

This completes the proof. $\square$

In Cases I and II (II') in Theorem 4.3.4, we can now compute $\pi_*\left(S^0_{K(1)}(p^v)^{h\chi}\right)$ using (2.3.8) and (4.1.4) when $p>2$, and (2.3.13) and (4.2.9) when $p=2$, respectively. Computations in Case III (III') are similar. Here we list the results below.

Corollary 4.3.9. *When $p>2$, suppose $\chi|_{(\mathbb{Z}/p)^\times}=\omega^a$ for some $0\le a\le p-2$. We have:*

$$\pi_i\left(S^0_{K(1)}(p^v)^{h\chi}\right)=\begin{cases}\mathbb{Z}_p^{\oplus p}, & a=0 \text{ and } i=0 \text{ or } 1;\\ \left(\mathbb{Z}/p^{v_p(k)+1}\right)^{\oplus p}, & n\ge v-1, i=2k\neq 0, \text{ and } (p-1)\mid(k-a);\\ \left(\mathbb{Z}/p^{v_p(k)+v-n}\right)^{\oplus p}, & n<v-1, i=2k\neq 0, \text{ and } (p-1)\mid(k-a);\\ 0, & \text{otherwise.}\end{cases}$$

When $p=2$, if $\chi|_{(\mathbb{Z}/4)^\times}$ is trivial, then

$$\pi_i\left(S^0_{K(1)}(2^v)^{h\chi}\right)=\begin{cases}\mathbb{Z}_2^{\oplus 2}, & i=0;\\ (\mathbb{Z}_2\oplus\mathbb{Z}/2)^{\oplus 2}, & i=1;\\ (\mathbb{Z}/2\oplus\mathbb{Z}/2)^{\oplus 2}, & i\equiv 2 \mod 8;\\ (\mathbb{Z}/2)^{\oplus 2}, & i\equiv 1,3 \mod 8 \text{ and } i\neq 1;\\ \left(\mathbb{Z}/2^{v_2(k)+3}\right)^{\oplus 2} & n\ge v-2 \text{ and } i=4k\neq 0;\\ \left(\mathbb{Z}/2^{v_2(k)+v-n+1}\right)^{\oplus 2} & n<v-2 \text{ and } i=4k\neq 0;\\ 0, & \text{otherwise.}\end{cases}$$

If $\chi|_{(\mathbb{Z}/4)^\times}=\omega$, then

$$\pi_i\left(S^0_{K(1)}(2^v)^{h\chi}\right)=\begin{cases}(\mathbb{Z}/2)^{\oplus 2}, & i\equiv 3,5 \mod 8 \text{ and } i\neq 1;\\ (\mathbb{Z}/2\oplus\mathbb{Z}/2)^{\oplus 2}, & i\equiv 4 \mod 8;\\ (\mathbb{Z}/4)^{\oplus 2} & n\ge v-2 \text{ and } i\equiv 2 \mod 4;\\ (\mathbb{Z}/2^{v-n})^{\oplus 2} & n<v-2 \text{ and } i\equiv 2 \mod 4;\\ 0, & \text{otherwise.}\end{cases}$$

5. Comparisons of J-spectra and L-functions

In this section, we first compute homotopy groups of the Dirichlet J-spectra by assembling the results in Section 4. From there, we compare these homotopy groups with the special values of the corresponding Dirichlet L-functions in Theorem 5.1.2. This comparison allows to relate the spectrum $J(N)$ and Dedekind ζ-functions in Section 5.2. In addition, we find the Brown-Comenetz duals of the Dirichlet J-spectra and $K(1)$-spheres, as well as $J(N)$ in Section 5.3. This duality phenomenon is the similar to the functional equations of the corresponding L-functions.

5.1. Dirichlet J-spectra and L-functions.

Theorem 5.1.1. *Let χ be a primitive Dirichlet character $(\mathbb{Z}/N)^\times\to\mathbb{C}^\times$ of conductor N.*

(1) When $N=p>2$, if $|\operatorname{Im}\chi|>1$ is not a prime power, then

$$J(p)^{h\chi}\simeq\bigvee_{\substack{0\le a\le p-2\\ \ker\omega^a=\ker\chi}}S^0_{K(1)}(p)^{h\omega^a}\implies \pi_i\left(J(p)^{h\chi}\right)=\begin{cases}\mathbb{Z}/p^{v_p(k)+1}, & i=2k-1 \text{ and } \ker\omega^k=\ker\chi;\\ 0, & \text{otherwise.}\end{cases}$$

If $|\operatorname{Im}\chi| > 1$ *is a power of a prime* ℓ*, then*

$$J(p)^{h\chi} \simeq \left(\Sigma\left(S^0_{KU/\ell}\right)^{\vee\ell}\right) \vee \bigvee_{\substack{0\le a\le p-2\\ \ker\omega^a=\ker\chi}} S^0_{KU/p}(p)^{h\omega^a}.$$

When $\ell > 2$*, we have*

$$\pi_i\left(J(p)^{h\chi}\right) = \begin{cases} \mathbb{Z}_\ell^{\oplus\ell}, & i = 0;\\ \mathbb{Z}_\ell^{\oplus\ell}, & i = 1 \text{ and } \ker\chi \neq 0;\\ \mathbb{Z}/p\oplus\mathbb{Z}_\ell^{\oplus\ell}, & i = 1 \text{ and } \ker\chi = 0;\\ \mathbb{Z}/p^{v_p(k)+1}, & i = 2k-1 \neq 1 \text{ and } \ker\omega^k = \ker\chi;\\ \left(\mathbb{Z}/\ell^{v_\ell(k)+1}\right)^{\oplus\ell}, & i = 2k \neq 0, \text{ and } (\ell-1) \mid k;\\ 0, & \text{otherwise.}\end{cases}$$

When $\ell = 2$ *(in particular whenever* $p = 2^{2^n}+1$ *is a Fermat prime), we have*

$$\pi_i\left(J(p)^{h\chi}\right) = \begin{cases} \mathbb{Z}_2^{\oplus 2}, & i = 0;\\ (\mathbb{Z}_2\oplus\mathbb{Z}/2)^{\oplus 2}, & i = 1 \text{ and } \ker\chi \neq 0;\\ \mathbb{Z}/p\oplus(\mathbb{Z}_2\oplus\mathbb{Z}/2)^{\oplus 2}, & i = 1 \text{ and } \ker\chi = 0;\\ (\mathbb{Z}/2\oplus\mathbb{Z}/2)^{\oplus 2}, & i \equiv 2 \mod 8;\\ \mathbb{Z}/p^{v_p(k)+1}\oplus(\mathbb{Z}/2)^{\oplus 2}, & i = 2k-1\neq 1, i\equiv 1,3 \mod 8, \text{ and } \ker\omega^k = \ker\chi;\\ \mathbb{Z}/p^{v_p(k)+1}, & i = 2k-1, i\equiv 5,7 \mod 8, \text{ and } \ker\omega^k = \ker\chi;\\ \left(\mathbb{Z}/2^{v_2(k)+3}\right)^{\oplus 2} & i = 4k \neq 0;\\ 0, & \text{otherwise.}\end{cases}$$

(2) *When* $N = p^v$*,* $v > 1$ *and* $p > 2$*, we have*

$$J(p^v)^{h\chi} \simeq \bigvee_{\substack{0\le a\le p-2\\ \ker\omega^a=\ker\chi|_{(\mathbb{Z}/p)^\times}}} S^0_{K(1)}(p^v)^{h\chi_a} \implies \pi_i\left(J(p^v)^{h\chi}\right) = \begin{cases} \mathbb{Z}/p, & i = 2k-1 \text{ and } \ker\omega^k = \ker\chi|_{(\mathbb{Z}/p)^\times};\\ 0, & \text{otherwise.}\end{cases}$$

where $\chi_a = \omega^a\cdot(\iota\circ\chi|_{\mathbb{Z}/p^{v-1}})$ *and* $\iota : \mathbb{Q}(\chi) \hookrightarrow \mathbb{C}_p$ *is an embedding as in Examples 3.5.4.*

(3) *When* $N = 4$*, the only non-trivial character satisfies* $\chi(-1) = -1$*. We have:*

$$J(4)^{h\chi} \simeq S^0_{K(1)}(4)^{h\omega} \implies \pi_i\left(J(4)^{h\chi}\right) = \begin{cases} \mathbb{Z}/4, & i = 4k+1;\\ \mathbb{Z}/2, & i\equiv 2,4 \mod 8;\\ \mathbb{Z}/2\oplus\mathbb{Z}/2, & i \equiv 3 \mod 8;\\ 0, & \text{otherwise.}\end{cases}$$

(4) *When* $N = 2^v > 4$*,* $J(4)^{h\chi} \simeq S^0_{K(1)}(2^v)^{h(\iota\circ\chi)}$*, where* $\iota : \mathbb{Q}(\chi) \hookrightarrow \mathbb{C}_2$ *is an embedding. If* $\chi(-1) = 1$*, then*

$$\pi_i\left(J(2^v)^{h\chi}\right) = \begin{cases} \mathbb{Z}/2, & i \equiv 0,2,3,7 \mod 8;\\ \mathbb{Z}/2\oplus\mathbb{Z}/2, & i\equiv 1 \mod 8;\\ 0, & \text{otherwise.}\end{cases}$$

If $\chi(-1) = -1$*, then*

$$\pi_i\left(J(2^v)^{h\chi}\right) = \begin{cases} \mathbb{Z}/2, & i \equiv 1,2,4,5 \mod 8;\\ \mathbb{Z}/2\oplus\mathbb{Z}/2, & i\equiv 3 \mod 8;\\ 0, & \text{otherwise.}\end{cases}$$

(5) *Suppose* N *has more than one prime factors.*
   (a) $J(N)^{h\chi}$ *is contractible unless there is a prime* p *such that* $|\operatorname{Im}\chi|_{(\mathbb{Z}/N')^\times}| = p^n$ *where* $N' = N/p^{v_p(N)}$*. In particular,* $J(N)^{h\chi}$ *is contractible whenever* $v_\ell(N) \geq 2$ *for at least two distinct primes* ℓ*.*

(b) *When there is such a prime p, then $J(N)^{h\chi} \simeq \left(J(N)^{h\chi}\right)^\wedge_p$. When p is odd, we have*

$$\pi_i\left(J(N)^{h\chi}\right) = \begin{cases} \mathbb{Z}_p^{\oplus p}, & \chi|_{(\mathbb{Z}/p)^\times} \text{ is trivial and } i = 0 \text{ or } 1; \\ \left(\mathbb{Z}/p^{v_p(k)+1}\right)^{\oplus p}, & n \geq v-1, i = 2k \neq 0, \text{ and } \ker \omega^k = \ker \chi|_{(\mathbb{Z}/p)^\times}; \\ \left(\mathbb{Z}/p^{v_p(k)+v-n}\right)^{\oplus p}, & n < v-1, i = 2k \neq 0, \text{ and } \ker \omega^k = \ker \chi|_{(\mathbb{Z}/p)^\times}; \\ 0, & \text{otherwise.} \end{cases}$$

When $p = 2$, if $\chi|_{(\mathbb{Z}/4)^\times}$ is trivial, then

$$\pi_i\left(J(N)^{h\chi}\right) = \begin{cases} \mathbb{Z}_2^{\oplus 2}, & i = 0; \\ (\mathbb{Z}_2 \oplus \mathbb{Z}/2)^{\oplus 2}, & i = 1; \\ (\mathbb{Z}/2 \oplus \mathbb{Z}/2)^{\oplus 2}, & i \equiv 2 \mod 8; \\ (\mathbb{Z}/2)^{\oplus 2}, & i \equiv 1, 3 \mod 8 \text{ and } i \neq 1; \\ \left(\mathbb{Z}/2^{v_2(k)+3}\right)^{\oplus 2} & n \geq v-2 \text{ and } i = 4k \neq 0; \\ \left(\mathbb{Z}/2^{v_2(k)+v-n+1}\right)^{\oplus 2} & n < v-2 \text{ and } i = 4k \neq 0; \\ 0, & \text{otherwise.} \end{cases}$$

If $\chi|_{(\mathbb{Z}/4)^\times} = \omega$, we have

$$\pi_i\left(J(N)^{h\chi}\right) = \begin{cases} (\mathbb{Z}/2)^{\oplus 2}, & i \equiv 3, 5 \mod 8 \text{ and } i \neq 1; \\ (\mathbb{Z}/2 \oplus \mathbb{Z}/2)^{\oplus 2}, & i \equiv 4 \mod 8; \\ (\mathbb{Z}/4)^{\oplus 2} & n \geq v-2 \text{ and } i \equiv 2 \mod 4; \\ (\mathbb{Z}/2^{v-n})^{\oplus 2} & n < v-2 \text{ and } i \equiv 2 \mod 4; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. By Proposition 3.5.1, $J(N)^{h\chi}_{\mathbb{Q}}$ is contractible unless χ is trivial. As a result, we can compute $\pi_*\left(J(N)^{h\chi}\right)$ by assembling computations of the Dirichlet $K(1)$-local spheres in Section 4, via Proposition 3.5.3. □

Theorem 5.1.2. *Let $\mathcal{D}_{k,\chi}$ be the ideal of $\mathbb{Z}[\chi]$ generated by the denominator of $\frac{B_{k,\chi}}{2k} \in \mathbb{Q}(\chi)$. Set $\mathcal{D}_{k,\chi} = (1)$ when $(-1)^k \neq \chi(-1)$ (i.e. when $B_{k,\chi} = 0$).*

(1) *Assume $N = p > 2$ or $N = 4$ when $p = 2$. For all integers k satisfying $(-1)^k = \chi(-1)$, we have*

$$\pi_{2k-1}\left(J(N)^{h\chi}\left[\frac{1}{\ell(\chi)}\right]\right) \simeq \mathbb{Z}[\chi]/\mathcal{D}_{|k|,\chi^{-1}}, \quad \text{where } \ell(\chi) = \begin{cases} \ell, & \text{if } |\mathrm{Im}(\chi)| \text{ is a power of a prime } \ell; \\ 1, & \text{otherwise.} \end{cases}$$

(2) *When $N = p^v > 2p$, $\pi_{2k-1}\left(J(p^v)^{h\chi}\right) \simeq \mathbb{Z}[\chi]/\mathcal{I}_{k,\chi^{-1}}$, where $I_{k,\chi}$ is an ideal of $\mathbb{Z}[\chi]$ such that its multiplicative difference with $\mathcal{D}_{k,\chi}$ contains the principal ideal (2) in $\mathbb{Z}[\chi]$.*

Remark 5.1.3. By Remark 4.2.11 and Remark 4.2.20, the statements above are independent of the models of $M(\mathbb{Z}[\chi])$ when $(-1)^k = \chi(-1)$.

Proof. In the first four cases in Theorem 5.1.1, the Dirichlet J-spectra are equivalent to their p-completions after inverting $\ell(\chi)$ by Corollary 3.5.2, Proposition 3.5.3 and Corollary 4.3.3. The only thing remains to check is π_{2k-1} where $(-1)^k = \chi(-1)$ and $N = p^v > 1$. For that, it suffices to compare the arithmetic properties of $B_{k,\chi}$ in Theorem 1.1.8 with computations in Section 4.1.

(1) $N = p > 2$. Comparing the decomposition in Examples 3.5.4 and computation in (4.1.4) with Theorem 1.1.8, we need to check the following:

- Let g be a primitive $(p-1)$-st root of unity mod p. The ideal $\mathfrak{p} := (p, 1-\chi(g)g^k)$ of $\mathbb{Z}[\chi]$ is not equal to (1) iff $\ker\chi = \ker\omega^{-k}$. To see this, notice by Corollary A.3.5, there is an isomorphism of $(\mathbb{Z}/p)^\times$-representations:

$$\mathbb{Z}[\chi]/p \simeq \bigoplus_{\substack{0\le a\le p-2\\ \ker\omega^a=\ker\chi}} (\mathbb{Z}/p)_{\omega^a} \simeq \bigoplus_{\substack{0\le a\le p-2\\ \ker\omega^a=\ker\chi}} (\mathbb{Z}/p)^{\otimes a}.$$

  Then $1-\chi(g)g^k$ is invertible in $\mathbb{Z}[\chi]/p$ iff $1 \equiv g^a\cdot g^k \mod p$ for some a satisfying $0\le a\le p-2$ and $\ker\chi = \ker\omega^a$. Since g is a primitive $(p-1)$-st root of unity mod p, this condition is further equivalent to saying $(p-1)\mid(a+k)$ for such an a. From this we conclude $\ker\chi = \ker\omega^{-k}$.
- When $\mathfrak{p}\neq(1)$, the congruence (1.1.9) $pB_{k,\chi}\equiv p-1 \mod \mathfrak{p}^{v_p(k)+1}$ implies $\mathbb{Z}[\chi]/\mathcal{D}_{k,\chi}\simeq\mathbb{Z}/p^{v_p(k)+1}$. It suffices to check this formula holds p-adically and 2-adically since the denominator ideal of $\frac{B_{k,\chi}}{k}$ is p-primary by Theorem 1.1.8. As $2\mid(p-1)$, $\mathcal{D}_{k,\chi}$ has no 2-primary factors by (1.1.9). p-adically, $\mathfrak{p}$ is the same as (p) when it is not (1). Now (1.1.9) becomes

$$pB_{k,\omega^a}\equiv p-1 \mod p^{v_p(k)+1} \implies \frac{B_{k,\omega^a}}{2k}\equiv\frac{p-1}{2pk} \mod \mathbb{Z}_p,$$

  where a satisfies $\ker\omega^a=\ker\chi$ and $(p-1)\mid(k+a)$. This implies

$$\mathbb{Z}[\chi]/\mathcal{D}_{k,\chi^{-1}}\simeq\mathbb{Z}/p^{v_p(k)+1}\simeq\pi_{2k-1}\left(J(p)^{h\chi}\left[\frac{1}{\ell(\chi)}\right]\right).$$

(2) $N=p^v$, $v>1$ and $p>2$. By Lemma 4.1.9, $\mathfrak{p}=(p,1-\chi(g)g^k)\neq(1)$ when $\ker\chi|_{(\mathbb{Z}/p)^\times}=\ker\omega^{-k}$. In that case, $\mathfrak{p}=(1-\zeta_{p^{v-1}},p)=(1-\zeta_{p^{v-1}})$. On the other hand, since $1+p$ is a generator of the subgroup $\mathbb{Z}/p^{v-1}\subseteq(\mathbb{Z}/p^v)^\times$ and χ is primitive, $\chi(1+p)$ is also a primitive p^{v-1}-th root of unity. As a result, (1.1.9) translates into

$$(1-\chi(p+1))\frac{B_{k,\chi}}{k}\equiv 1 \mod \mathfrak{p} \implies \frac{B_{k,\chi}}{k}\equiv\frac{1}{1-\zeta_{p^{v-1}}} \mod \mathbb{Z}_p[\zeta_{p^{v-1}}].$$

Thus $\mathcal{D}_{k,\chi}$ is either $(1-\zeta_{p^{v-1}})$ or $(2(1-\zeta_{p^{v-1}}))$. Whereas by Theorem 5.1.1, $\pi_{2k-1}\left(J(p^v)^{h\chi}\right)\simeq\mathbb{Z}/p\simeq\mathbb{Z}[\chi]/(1-\zeta^{p_{v-1}})$.

(3) $N=4$. In this case $\chi=\chi^{-1}$ since $(\mathbb{Z}/4)^\times\simeq C_2$. By (1.1.11), we have when k is odd:

$$\frac{B_{k,\chi}}{k}\equiv\frac{1}{2} \mod 1 \implies \frac{B_{k,\chi}}{2k}\equiv\pm\frac{1}{4} \mod 1.$$

Thus $\mathcal{D}_{k,\chi}=\mathcal{D}_{k,\chi^{-1}}$ is equal to the ideal (4) of $\mathbb{Z}[\chi]\simeq\mathbb{Z}$. This matches the computation in (4.2.9) that $\pi_{2k-1}\left(S^0_{K(1)}(4)^{h\omega}\right)\simeq\mathbb{Z}/4$ when k is odd.

(4) $N=2^v>4$. Theorem 1.1.8 says $\frac{B_{k,\chi}}{k}$ is an algebraic integer. As a result, $\mathcal{D}_{k,\chi}$ the denominator ideal of $\frac{B_{k,\chi}}{2k}$ contains (2) as a sub-ideal. By Theorem 5.1.1, $\pi_{2k-1}\left(J(2^v)^{h\chi}\right)\simeq\mathbb{Z}/2\simeq\mathbb{Z}[\chi]/(1-\zeta_{2^{v-2}})$. As both $\mathcal{D}_{k,\chi}$ and $\mathcal{I}_{k,\chi}$ contain the ideal (2) in $\mathbb{Z}[\chi]$, their difference contains the ideal (2).

□

5.2. **$J(N)$ and the Dedekind ζ-function of $\mathbb{Q}(\zeta_N)$.** In this subsection, we compare the spectrum $J(N)$ with Dedekind ζ-function of the field $\mathbb{Q}(\zeta_N)$. We will focus on the case when $N=p^v$ for some prime p. By Theorem 1.1.13, Dedekind ζ-functions of $\mathbb{Q}(\zeta_N)$ and Dirichlet L-functions are related by:

$$\zeta_{\mathbb{Q}(\zeta_N)}(s)=\prod_{\chi:(\mathbb{Z}/N)^\times\to\mathbb{C}^\times}L(s,\chi). \tag{5.2.1}$$

When $N = p^v$, this yields:

$$\frac{\zeta_{\mathbb{Q}(\zeta_{p^v})}(s)}{\zeta_{\mathbb{Q}(\zeta_{p^{v-1}})}(s)} = \prod_{\substack{\chi:(\mathbb{Z}/p^v)^\times \to \mathbb{C}^\times \\ \text{primitive}}} L(s,\chi).$$

The formula above reminds us of Proposition 4.1.11 and Proposition 4.2.21.

Proposition 5.2.2. *Let $p > 2$ be a prime and $\chi : (\mathbb{Z}/p^v)^\times \to \mathbb{C}_p^\times$ be any primitive p-adic Dirichlet character of conductor p^v.*

$$\mathrm{Cofib}(J(p^{v-1}) \to J(p^v)) \simeq \begin{cases} \bigvee_{a=1}^{p-2} S^0_{K(1)}(p)^{h\omega^a}, & v = 1; \\ \bigvee_{a=0}^{p-2} S^0_{K(1)}(p^v)^{h\chi_a}, & v > 1, \end{cases}$$

where $\chi_a = \omega^a \cdot (\chi|_{\mathbb{Z}/p^{v-1}})$.

Proof. By Corollary 3.2.9, $\mathrm{Cofib}(J(p^{v-1}) \to J(p^v))$ is equivalent to its p-completion, since the map is an equivalence rationally, and when completed at a prime other than p. At prime p, we have

$$\mathrm{Cofib}(J(p^{v-1}) \to J(p^v)) \simeq \mathrm{Cofib}(J(p^{v-1}) \to J(p^v))^\wedge_p \simeq \mathrm{Cofib}\left(S^0_{K(1)}(p^{v-1}) \to S^0_{K(1)}(p^v)\right).$$

When $v = 1$, taking this cofiber removes the $a = 0$ summand in the Adams splitting of $S^0_{K(1)}(p)$. When $v > 1$, the claim follows from Proposition 4.1.13 and the Adams splittings. ☐

Corollary 5.2.3. *Notations as above. When $p > 2$ and $v > 1$, we have*

$$\mathrm{Cofib}(J(p^{v-1}) \to J(p^v)) \simeq \bigvee_{a \in [0,p-2]/\sim} J(p^v)^{h\chi_a},$$

where $a \sim b$ if $\ker \omega^a = \ker \omega^b$.

Proof. This follows from Proposition 5.2.2 and Case (2) in Theorem 5.1.1. ☐

One might now wonder if there is a connection between special values of $\zeta_{\mathbb{Q}(\zeta_N)}$ and homotopy groups of $J(N)$ as in Theorem 5.1.2. Notice in (5.2.1), both even and odd characters show up on the right hand side. Also recall $L(1-k,\chi) = 0$ unless $(-1)^k = \chi(-1)$. This implies $\zeta_{\mathbb{Q}(\zeta_N)}(1-k) = 0$ for all positive integers k. As a result, a direct analogy of Theorem 5.1.2 does not exist in this case.

There are two ways one might try to fix this. The first one is to exclude odd characters in the product formula (5.2.1). Let $\mathbb{K}$ be a *totally real* finite abelian extension of $\mathbb{Q}$, and N be the smallest integer such that $\mathbb{K} \subseteq \mathbb{Q}(\zeta_N)$. $\mathbb{K}$ being totally real is equivalent to $\mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{K})$ containing complex conjugation, which is identified with $-1 \in (\mathbb{Z}/N)^\times$ via Lemma A.2.1. This means in the product formula Theorem 1.1.13

$$\zeta_{\mathbb{K}}(s) = \prod_{\substack{\chi:(\mathbb{Z}/N)^\times \to \mathbb{C}^\times \\ \mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{K}) \subseteq \ker\chi}} L(s,\chi),$$

only even characters show up on the right hand side and $\zeta_{\mathbb{K}}$ has non-zero special values as $s = 1 - 2k$.

Definition 5.2.4. Let $\mathbb{K}/\mathbb{Q}$ be a finite abelian extension and let N be the smallest integer such that $\mathbb{K} \subseteq \mathbb{Q}(\zeta_N)$. We define

$$J(\mathbb{K}) \coloneqq J(N)^{h\mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{K})}.$$

Here, we identify $\mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{K}) \subseteq \mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q}) \simeq (\mathbb{Z}/N)^\times$. The action of $(\mathbb{Z}/N)^\times$ on $J(N)$ was described in Proposition 3.2.11.

Remark 5.2.5. The construction $J(\mathbb{K})$ does not satisfy Galois descent, as observed in Remark 3.2.12.

Theorem 5.2.6. *Let $\mathbb{K}/\mathbb{Q}$ be a totally real finite abelian extension and p^v be the smallest integer such that $\mathbb{K} \subseteq \mathbb{Q}(\zeta_{p^v})$. Denote the Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{K})$ by G. Then*

$$\pi_{4t-1}\left(J(\mathbb{K})\left[\frac{1}{|G|}\right]\right) = \mathbb{Z}\left[\frac{1}{|G|}\right]\Big/ D_{\mathbb{K},2t},$$

where $D_{\mathbb{K},2t} \in \mathbb{Z}_{>0}$ is the denominator of $\zeta_{\mathbb{K}}(1-2t)$.

Proof. It suffices to compare the sides at primes not dividing $|G|$. We first show $G \subseteq (\mathbb{Z}/p)^\times \subseteq (\mathbb{Z}/p^v)^\times \simeq \mathrm{Gal}(\mathbb{Q}(\zeta_{p^v})/\mathbb{Q})$. This is true when $v = 1$ since $\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q}) \simeq (\mathbb{Z}/p)^\times$. Now assume $v > 1$ and suppose G contains an element of order p. Then we have $\mathrm{Gal}(\mathbb{Q}(\zeta_{p^v})/\mathbb{Q}(\zeta_{p^{v-1}})) \simeq C_p \subseteq G$. By Galois correspondence between subfields of $\mathbb{Q}(\zeta_{p^v})$ and subgroups of $(\mathbb{Z}/p^v)^\times$, this would imply $\mathbb{K} \subseteq \mathbb{Q}(\zeta_{p^{v-1}})$, contradicting our assumption.

It follows that $p \nmid |G|$. Let ℓ be a prime such that $\ell \nmid |G|$ and $\ell \neq p$. By Corollary 3.2.9, we have

$$J(\mathbb{K})^\wedge_\ell = \left(J(p^v)^{hG}\right)^\wedge_\ell \simeq S^0_{KU/\ell}$$

This shows $\pi_{4t-1}\left(J(\mathbb{K})^\wedge_\ell\right) = \pi_{4t-1}\left(S^0_{KU/\ell}\right) = \mathbb{Z}_\ell/D_{2t}$. Using the product formula Theorem 1.1.13 and Carlitz's Theorem 1.1.8, we get $\pi_{4t-1}\left(J(\mathbb{K})^\wedge_\ell\right) \simeq \mathbb{Z}_\ell/D_{2t} = \mathbb{Z}_\ell/D_{\mathbb{K},2t}$.

Completed at the prime p, we have by Corollary 3.2.9 and Theorem 5.1.1

$$J(\mathbb{K})^\wedge_p = \left(J(p^v)^{hG}\right)^\wedge_p \simeq S^0_{K(1)}(p^v)^{hG} \simeq \bigvee_{\substack{0\leq a\leq p-2\\ G\subseteq \ker\omega^a}} S^0_{K(1)}(p^v)^{h\omega^a} \simeq \bigvee_{\substack{\chi\in\mathrm{Hom}((\mathbb{Z}/p)^\times,\mathbb{C}^\times)/\sim\\ G\subseteq\ker\chi}} \left(J(p^v)^{h\chi}\right)^\wedge_p$$

where $\chi_1 \sim \chi_2$ iff they have the same images (thus differ by an element in the Galois group). Now we can prove the claim by comparing Theorem 1.1.13 and Proposition 4.1.11 via Theorem 5.1.2. ☐

A second approach to relate $J(N)$ to $\zeta_{\mathbb{Q}(\zeta_N)}$ is to consider the Taylor expansion of the Dedekind ζ-functions at zero special values.

Definition 5.2.7. For any number field $\mathbb{K}$, we denote by $\zeta^*_{\mathbb{K}}(1-k)$ the coefficient of the first non-zero term in the Taylor expansion of $\zeta_{\mathbb{K}}(s)$ at $s = 1-k$.

From the definition, $\zeta^*_{\mathbb{K}}(1-k) = \zeta_{\mathbb{K}}(1-k)$ when the latter is not zero. This special value $\zeta^*_{\mathbb{K}}(1-k)$ is closely related to the **algebraic K-theory** of $\mathcal{O}_{\mathbb{K}}$, the ring of integers of $\mathbb{K}$.

Theorem 5.2.8 (Lichtenbaum-Quillen Conjecture, Voevodsky-Rost)**.** [Kol04, pages 199 – 200] *For all $k \geq 2$:*

$$\zeta^*_{\mathbb{K}}(1-k) = \pm\frac{|K_{2k-2}(\mathcal{O}_{\mathbb{K}})|}{|K_{2k-1}(\mathcal{O}_{\mathbb{K}})_{\mathrm{tors}}|}\cdot R^B_k(\mathbb{K}),$$

up to powers of 2, where $R^B_k(\mathbb{K})$ is the k-th Borel regulator of $\mathbb{K}$.

Let $\mathbb{K}/\mathbb{Q}$ be a finite abelian extension. In the case when $\mathbb{K}$ is totally real, both $\pi_{2k-1}(J(\mathbb{K}))$ and $K_{2k-1}(\mathcal{O}_{\mathbb{K}})$ capture the denominator of $\zeta^*_{\mathbb{K}}(1-k)$. We now want to compare $J(\mathbb{K})$ and $K(\mathcal{O}_{\mathbb{K}})$. Recall in Proposition 3.2.11, we showed $J(N)$ is a KU-local $\mathbb{E}_\infty$-ring spectrum, with a natural $(\mathbb{Z}/N)^\times$-action by $\mathbb{E}_\infty$-maps. This implies $J(\mathbb{K})$ is a KU-local $\mathbb{E}_\infty$-ring spectrum, with a natural $\mathrm{Gal}(\mathbb{K}/\mathbb{Q})$-action by $\mathbb{E}_\infty$-maps.

While $K(\mathcal{O}_{\mathbb{K}})$ is not a KU-local spectrum (since it is connective and is not a wedge sum of Eilenberg-MacLane spectra), it is very close to one in the following sense:

Theorem 5.2.9 (Waldhausen)**.** [Mit94, Conjecture 11.5] *Fix a prime ℓ. Assuming Lichtenbaum-Quillen Conjecture holds at ℓ, the $E(1)$-localization map $K(\mathcal{O}_{\mathbb{K}}) \to L_1K(\mathcal{O}_{\mathbb{K}})$ is an ℓ-local isomorphism on π_n for all $n \geq 1$.*

This shows the KU-localization map to $K(\mathcal{O}_{\mathbb{K}}) \to L_{KU}K(\mathcal{O}_{\mathbb{K}})$ is an equivalence on the 1-connective covers of the two spectra. $K(\mathcal{O}_{\mathbb{K}})$ is an $\mathbb{E}_\infty$-ring spectrum by a result of May [Mit94, 2.8], Since Quillen's construction of algebraic K-theory spectra is functorial, there is a natural $\mathrm{Gal}(\mathbb{K}/\mathbb{Q})$-action on $K(\mathcal{O}_{\mathbb{K}})$ by $\mathbb{E}_\infty$-maps. So just like $J(\mathbb{K})$, $L_{KU}K(\mathcal{O}_{\mathbb{K}})$ is a KU-local $\mathbb{E}_\infty$-ring spectrum, with a natural $\mathrm{Gal}(\mathbb{K}/\mathbb{Q})$-action by $\mathbb{E}_\infty$-maps. When $\mathbb{K} = \mathbb{Q}$, $J(\mathbb{Q}) = J$ and $L_{KU}(K(\mathbb{Z}))$ is connected by the KU-local Hurewicz map:

$$h_{KU} : J = S^0_{KU} \longrightarrow L_{KU}K(\mathbb{Z}).$$

A natural question to ask here is:

Question 5.2.10. *Let $\mathbb{K}/\mathbb{Q}$ be a finite abelian extension. Is there a natural* $\mathrm{Gal}(\mathbb{K}/\mathbb{Q})$*-equivariant map of KU-local $\mathbb{E}_\infty$-ring spectra $h(\mathbb{K}) : J(\mathbb{K}) \to L_{KU}K(\mathcal{O}_{\mathbb{K}})$ extending the KU-local Hurewicz map h_{KU}?*

$$\begin{tikzcd} J(\mathbb{K}) \arrow[r, dashed, "\exists h(\mathbb{K})?"] & L_{KU}K(\mathcal{O}_{\mathbb{K}}) \\ J = S^0_{KU} \arrow[u] \arrow[r, "h_{KU}"] & L_{KU}K(\mathbb{Z}) \arrow[u] \end{tikzcd}$$

In addition, for an arbitrary number field $\mathbb{K}$, how can we extract a "J-spectrum" from $L_{KU}K(\mathcal{O}_{\mathbb{K}})$?

5.3. **Brown-Comenetz duality.** From Remark 3.2.15 and the computations in Section 4 and Theorem 5.1.1, we observe the following duality phenomena hold in many (but not all) cases:

$$\pi_t(J(N)) \simeq \pi_{-2-t}(J(N)), \quad \pi_t\left(J(N)^{h\chi}\right) \simeq \pi_{-2-t}\left(J(N)^{h\chi^{-1}}\right).$$

This duality resembles the functional equations of the Dedekind ζ-functions and Dirichlet L-functions. Let $\chi : (\mathbb{Z}/N)^\times \to \mathbb{C}^\times$ be a primitive Dirichlet character of conductor N and k is a positive integer such that $(-1)^k = \chi(-1)$. Then we have the following functional equation of $L(k;\chi)$:

$$L(k;\chi) = \frac{\tau(\chi)}{2(k-1)!} \cdot \left(\frac{2\pi i}{N}\right)^k \cdot L(1-k;\chi^{-1}), \text{ where } \tau(\chi) = \sum_{a=1}^{N} \chi(a) e^{\frac{2\pi i a}{N}}.$$

The duality in homotopy groups we observed is a result of **Brown-Comenetz duality** of the spectra. In this subsection, we find the Brown-Comenetz duals for the Dirichlet J-spectra, $K(1)$-local spheres, and the spectra $J(N)$. Let's first review the setup following [HG94].

Definitions 5.3.1. Let I be the spectrum that represents the cohomology theory $X \mapsto \mathrm{Hom}_{\mathbb{Z}}(\pi_0(X), \mathbb{Q}/\mathbb{Z})$. The Brown-Comenetz dual of a finite type spectrum X is defined to be

$$IX := \mathrm{Map}(X, I) \simeq I \wedge DX,$$

where $DX := \mathrm{Map}(X, S^0)$ is the **Spanier-Whitehead dual** of X. In particular, $I = IS^0$ is the Brown-Comenetz dual of S^0.

It follows from the definition that $\pi_t(IX) \simeq \mathrm{Hom}_{\mathbb{Z}}(\pi_{-t}(X), \mathbb{Q}/\mathbb{Z})$.

Definition 5.3.2. Let $I_1 := \mathrm{Map}(L_1S^0, I)$ be the Brown-Comenetz dual of the $E(1)$-local sphere L_1S^0. The $K(1)$-local Brown-Comenetz dual of S^0 is defined by $I_{K(1)} := L_{K(1)}I_1$. Define the $E(1)$-local and $K(1)$-local duals of a spectrum X to be

$$I_1X := \mathrm{Map}(X, I_1) \simeq I_1 \wedge D_{E(1)}X, \qquad I_{K(1)}X := \mathrm{Map}(X, I_{K(1)}) \simeq I_{K(1)} \wedge D_{K(1)}(X).$$

Proposition 5.3.3. *Homotopy groups of I_1X and $I_{K(1)}X$ are computed by:*

$$\pi_t(I_1X) \simeq \mathrm{Hom}_{\mathbb{Z}_{(p)}}\left(\pi_{-t}(L_1X), \mathbb{Q}/\mathbb{Z}_{(p)}\right), \tag{5.3.4}$$

$$\pi_t(I_{K(1)}X) \simeq \mathrm{Hom}_{\mathbb{Z}_p}(\pi_{-t}(M_1X), \mathbb{Q}_p/\mathbb{Z}_p), \tag{5.3.5}$$

where $M_1X \coloneqq \mathrm{hoFib}(L_1X \to L_0X)$.

Proof. (5.3.4) follows from the definition. (5.3.5) is in [HG94, page 84]. $\square$

Theorem 5.3.6 (Hopkins)**.** [Dev90, Remark 1.5], [MR99, Corollary 9.6] *When* $p>2$, $I_1 \simeq \Sigma^2 L_1\left(S^0_p\right)$, *where* S^0_p *is the* p*-complete sphere. When* $p=2$, $I_1 \simeq \Sigma^2 L_1\left(\mathcal{E}^\wedge_2\right)$, *where* $\mathcal{E}$ *is a finite CW-spectrum defined by*

$$\mathcal{E} \coloneqq \Sigma^{-2}(S^{-1}\cup_2 e^0 \cup_\eta e^2).$$

Remark 5.3.7. $E(1)$-localization does NOT commute with p-completion. On one hand, we have $L_1\left(S^0_p\right) \simeq M(\mathbb{Z}_p)\wedge L_1S^0$, since L_1 is smashing by [Rav92, Theorem 7.5.6]. One the other hand, $\left(L_1S^0\right)^\wedge_p \simeq S^0_{K(1)}$. We can then show $M(\mathbb{Z}_p)\wedge L_1S^0 \not\simeq S^0_{K(1)}$ by comparing their homotopy groups.

Localized at an odd prime p, $\mathcal{E}$ is equivalent to the sphere spectrum. As a result, the formula $I_1 \simeq \Sigma^2 L_1(\mathcal{E}^\wedge_p)$ holds for all primes p. This suggests:

Corollary 5.3.8. $I_{KU} \coloneqq \mathrm{Map}(L_{KU}S^0, I) \simeq \Sigma^2 L_{KU}(\mathcal{E}^\wedge)$, *where* $\mathcal{E}^\wedge$ *is the profinite completion of* $\mathcal{E}$.

Theorem 5.3.9 (Gross-Hopkins)**.** *When* $p>2$, $I_{K(1)} \simeq S^2_{K(1)}$. *When* $p=2$, $I_{K(1)} \simeq \Sigma^2\mathcal{E}_{K(1)}$.

The Dirichlet J-spectra and $K(1)$-local spheres constructed in this paper are KU-local and $K(1)$-local, respectively. The analysis above shows:

$$I_{KU}\left(J(N)^{h\chi}\right) \simeq \Sigma^2 D_{KU}\left(J(N)^{h\chi}\right)\wedge L_{KU}(\mathcal{E}^\wedge)$$

$$I_{K(1)}\left(S^0_{K(1)}(p^v)^{h\chi}\right) \simeq \begin{cases} \Sigma^2 D_{K(1)}\left(S^0_{K(1)}(p^v)^{h\chi}\right), & p>2;\\ \Sigma^2 D_{K(1)}\left(S^0_{K(1)}(p^v)^{h\chi}\right)\wedge\mathcal{E}_{K(1)}, & p=2.\end{cases}$$

To find Brown-Comenetz duals of these spectra, it now remains to identify their Spanier-Whitehead duals in $\mathbf{Sp}_{KU}$ and $\mathbf{Sp}_{K(1)}$, respectively. We start with the $K(1)$-local cases, which the KU-local cases depend on.

Proposition 5.3.10. *Let* χ *be a* p*-adic Dirichlet character of conductor* p^v. *The Spanier-Whitehead dual of the Dirichlet* $K(1)$*-local sphere attached to* χ *is*

$$D_{K(1)}\left(S^0_{K(1)}(p^v)^{h\chi}\right) \simeq \begin{cases} S^0_{K(1)}(2^v)^{h\chi^{-1}}\wedge\mathcal{E}_{K(1)}, & p=2 \text{ and } \chi(-1)=-1;\\ S^0_{K(1)}(p^v)^{h\chi^{-1}}, & \text{otherwise.}\end{cases}$$

Proof. When $p>2$, by Lemma 4.1.14, it suffices to check the $\mathbb{Z}_p^\times$-action on the $KU^\wedge_p$-homology of the Dirichlet $K(1)$-local spheres. In Corollary 4.1.12, we computed:

$$\left(KU^\wedge_p\right)_*\left(S^0_{K(1)}(p^v)^{h\chi}\right) \simeq \mathrm{Hom}_{\mathbb{Z}_p}\left(\mathbb{Z}_p[\chi], \left(KU^\wedge_p\right)_*\right),$$

where $\mathbb{Z}_p^\times$ acts on $\mathbb{Z}_p[\chi]$ through the character χ, and on $\left(KU^\wedge_p\right)_*$ by the Adams operations. The dual $\mathbb{Z}_p^\times$-representation of $\mathbb{Z}_p[\chi]$ is $\mathbb{Z}_p[\chi^{-1}]$. Also notice $\left(KU^\wedge_p\right)_*$ is self dual as graded $\mathbb{Z}_p^\times$-representations. We have:

$$\mathrm{Hom}_{\left(KU^\wedge_p\right)_*}\left(\left(KU^\wedge_p\right)_*\left(S^0_{K(1)}(p^v)^{h\chi}\right), \left(KU^\wedge_p\right)_*\right) \simeq \mathrm{Hom}_{\mathbb{Z}_p}\left(\mathbb{Z}_p[\chi^{-1}], \left(KU^\wedge_p\right)_*\right) \simeq \left(KU^\wedge_p\right)_*\left(S^0_{K(1)}(p^v)^{h\chi^{-1}}\right).$$

This implies $D_{K(1)}\left(S^0_{K(1)}(p^v)^{h\chi}\right) \simeq S^0_{K(1)}(p^v)^{h\chi^{-1}}$ when $p>2$.

When $p=2$, by Lemma 4.2.24, we need to check the $(1+4\mathbb{Z}_2)$-action on the $KO^\wedge_2$-homology of the Dirichlet $K(1)$-local spheres. By Corollary 4.2.22, we have

$$(KO^\wedge_2)_*\left(S^0_{K(1)}(2^v)^{h\chi}\right) \simeq \begin{cases} \mathrm{Hom}\left(\mathbb{Z}_2[\chi], (KO^\wedge_2)_*\right), & \chi(-1)=1;\\ \mathrm{Hom}\left(\mathbb{Z}_2[\chi], (KU^\wedge_2)^{h\omega}_*\right), & \chi(-1)=-1.\end{cases}$$

Here $1+4\mathbb{Z}_2$ acts by the character $\chi|_{\mathbb{Z}/2^{v-2}}$ on $\mathbb{Z}_2[\chi]$ and by Adams operations on $(KO_2^\wedge)_*$ and $(KU_2^\wedge)_*^{h\omega}$. Notice that $\mathbb{Z}_2[\chi]$ is dual to $\mathbb{Z}_2[\chi^{-1}]$, $(KO_2^\wedge)_*$ is self-dual, and $(KU_2^\wedge)_*^{h\omega}$ is dual to $(KU_2^\wedge)_*^{h'\omega}$, where $(KU_2^\wedge)^{h'\omega} := \mathrm{Map}\left(S^{1-\sigma}, K\mathbb{R}_2^\wedge\right)^{h\{\pm1\}}$. From this, we get

$$\begin{aligned}\mathrm{Hom}_{(KO_2^\wedge)_*}\left((KO_2^\wedge)_*\left(S^0_{K(1)}(2^v)^{h\chi}\right),(KO_2^\wedge)_*\right) &\simeq \begin{cases} \mathrm{Hom}\left(\mathbb{Z}_2[\chi^{-1}],(KO_2^\wedge)_*\right), & \chi(-1)=1;\\ \mathrm{Hom}\left(\mathbb{Z}_2[\chi^{-1}],(KU_2^\wedge)_*^{h'\omega}\right), & \chi(-1)=-1.\end{cases}\\ &\simeq \begin{cases} (KO_2^\wedge)_*\left(S^0_{K(1)}(2^v)^{h\chi^{-1}}\right), & \chi(-1)=1;\\ (KO_2^\wedge)_*\left(S^0_{K(1)}(2^v)^{h\chi^{-1}}\wedge \mathcal{E}_{K(1)}\right), & \chi(-1)=-1.\end{cases}\end{aligned}$$

This implies the claim at $p=2$. □

Theorem 5.3.11. *When the conductor of χ is a power of p, the Brown-Comenetz dual of the Dirichlet $K(1)$-local sphere attached to χ is:*

$$I_{K(1)}\left(S^0_{K(1)}(p^v)^{h\chi}\right)\simeq \begin{cases} \Sigma^2 S^0_{K(1)}(2^v)^{h\chi^{-1}}\wedge\mathcal{E}_{K(1)}, & p=2 \text{ and } \chi(-1)=1;\\ \Sigma^2 S^0_{K(1)}(p^v)^{h\chi^{-1}}, & \text{otherwise.}\end{cases}$$

Proof. We used the fact $\mathcal{E}_{K(1)}\wedge\mathcal{E}_{K(1)}\simeq S^0_{K(1)}$ in the case when $p=2$ and $\chi(-1)=-1$. □

From computations in Section 4, we know $L_0\left(S^0_{K(1)}(p^v)^{h\chi}\right)\simeq *$ whenever the conductor N of χ is a power of p. This implies $M_1S^0_{K(1)}(p^v)^{h\chi}=S^0_{K(1)}(p^v)^{h\chi}$. Also, as the homotopy groups of the Dirichlet $K(1)$-local spheres are finite p-groups, they are (non-canonically) isomorphic to their p-adic Pontryagin duals. Now plugging Theorem 5.3.11 into (5.3.5), we get

$$\begin{aligned}\pi_t\left(S^0_{K(1)}(p^v)^{h\chi}\right) &\simeq \mathrm{Hom}_{\mathbb{Z}_p}\left(\pi_t\left(S^0_{K(1)}(p^v)^{h\chi}\right),\mathbb{Q}_p/\mathbb{Z}_p\right) \qquad \text{(non-canonically)}\\ &\simeq \mathrm{Hom}_{\mathbb{Z}_p}\left(\pi_t\left(M_1S^0_{K(1)}(p^v)^{h\chi}\right),\mathbb{Q}_p/\mathbb{Z}_p\right)\\ &\simeq \begin{cases} \pi_{-2-t}\left(S^0_{K(1)}(2^v)^{h\chi^{-1}}\wedge\mathcal{E}_{K(1)}\right), & p=2 \text{ and } \chi(-1)=1;\\ \pi_{-2-t}\left(S^0_{K(1)}(p^v)^{h\chi^{-1}}\right), & \text{otherwise.}\end{cases}\end{aligned}$$

When the conductor N of the p-adic Dirichlet character χ is not a p-power, the Brown-Comenetz dual of the Dirichlet $K(1)$-local is slightly different.

Corollary 5.3.12. *Let χ be a p-adic Dirichlet character $N=p^v\cdot N'$. Write $\chi=\chi_p\cdot\chi'$ as before. If $|\mathrm{Im}\,\chi'|>1$ is not a p-power, then*

$$I_{K(1)}\left(S^0_{K(1)}(p^v)^{h\chi}\right)\simeq I_{K(1)}(*)\simeq *.$$

If $|\mathrm{Im}\,\chi'|>1$ is a p-power, then

$$I_{K(1)}\left(S^0_{K(1)}(p^v)^{h\chi}\right)\simeq \begin{cases} S^0_{K(1)}(2^v)^{h\chi^{-1}}\wedge\mathcal{E}_{K(1)}, & p=2 \text{ and } \chi|_{(\mathbb{Z}/4)^\times} \text{ is trivial};\\ S^0_{K(1)}(p^v)^{h\chi^{-1}}, & \text{otherwise.}\end{cases}$$

Proof. When $|\mathrm{Im}\,\chi'|>1$ is not a p-power, the claim follows from Proposition 4.3.1. The other cases follow from Theorem 4.3.4 and Theorem 5.3.11. □

Now we identify the KU-local Spanier-Whitehead dual of the Dirichlet J-spectra $J(N)^{h\chi}$. Similar to (5.3.4), we have in $\mathbf{Sp}_{KU}$:

$$\pi_t(I_{KU}X)\simeq \mathrm{Hom}_{\mathbb{Z}}\left(\pi_{-t}(L_{KU}X),\mathbb{Q}/\mathbb{Z}\right). \tag{5.3.13}$$

In this case, our strategy is to assemble duality formulas via the arithmetic fracture squares for KU-local spectra. By Proposition 3.5.1, $J(N)^{h\chi}_{\mathbb{Q}}$ is contractible unless χ is trivial. In Proposition 3.5.3, we described how $J(N)^{h\chi}$ decomposes upon p-completion. Now from Theorem 5.3.11 and Corollary 5.3.12, we have:

Theorem 5.3.14. *Let $\chi : (\mathbb{Z}/N)^\times \to \mathbb{C}^\times$ be a primitive Dirichlet character of conductor N.*

(1) Suppose $N = p^v$ and p is odd. If $|\operatorname{Im}\chi| > 1$ is not a power of another prime (in particular whenever $v > 1$), then

$$I_{KU}\left(J(p^v)^{h\chi}\left[\frac{1}{\ell(\chi)}\right]\right) \simeq \Sigma^2 J(p^v)^{h\chi^{-1}}\left[\frac{1}{\ell(\chi)}\right] \quad\Longrightarrow\quad \pi_t\left(J(p^v)^{h\chi}\left[\frac{1}{\ell(\chi)}\right]\right) \simeq \pi_{-2-t}\left(J(p^v)^{h\chi^{-1}}\left[\frac{1}{\ell(\chi)}\right]\right),$$

where $\ell(\chi)$ is as in Theorem 5.1.2:

$$\ell(\chi) = \begin{cases} \ell, & \text{if } |\operatorname{Im}(\chi)| \text{ is a power of a prime } \ell \neq p; \\ 1, & \text{otherwise.}\end{cases}$$

(2) If $N = 2^v \geq 4$, then

$$I_{KU}\left(J(2^v)^{h\chi}\right) \simeq \begin{cases} \Sigma^2 J(2^v)^{h\chi^{-1}} \wedge \mathcal{E}_{KU}, & \chi(-1) = 1; \\ \Sigma^2 J(2^v)^{h\chi^{-1}}, & \chi(-1) = -1.\end{cases}$$

$$\Longrightarrow \pi_t\left(J(2^v)^{h\chi}\right) \simeq \begin{cases} \pi_{-2-t}\left(J(2^v)^{h\chi^{-1}} \wedge \mathcal{E}_{KU}\right), & \chi(-1) = 1; \\ \pi_{-2-t}\left(J(2^v)^{h\chi^{-1}}\right), & \chi(-1) = -1.\end{cases}$$

Remark 5.3.15. In (1) above, it is necessary to invert $\ell(\chi)$. This is because the degrees of suspensions are different in Theorem 5.3.11 and Corollary 5.3.12.

We now identify the Brown-Comenetz dual of $J(N)$. It is KU-local by Proposition 3.2.11. This means $I_{KU}(J(N)) \simeq \Sigma^2 L_{KU}(\mathcal{E}^\wedge) \wedge D_{KU}(J(N))$ by Corollary 5.3.8.

Proposition 5.3.16. *$J(N)$ is Spanier-Whitehead self-dual in $\mathbf{Sp}_{KU}$. It follows that*

$$I_{KU}(J(N)) \simeq \Sigma^2 L_{KU}(\mathcal{E}^\wedge) \wedge J(N) \simeq \Sigma^2 \mathcal{E}_{KU} \wedge J(N) \wedge M(\widehat{\mathbb{Z}}).$$

Proof. This is because $J(N)_{\mathbb{Q}} \simeq S^0_{\mathbb{Q}}$ and $J(N)^\wedge_p \simeq S^0_{K(1)}\left(p^{v_p(N)}\right)$ are both Spanier-Whitehead self-dual in $\mathbf{Sp}_{\mathbb{Q}}$ and $\mathbf{Sp}_{K(1)}$, respectively. □

Lemma 5.3.17. $J(4N) \wedge \mathcal{E}_{KU} \simeq J(4N)$.

Proof. Since $\mathcal{E} \simeq S^0$ when 2 is inverted, the equivalence holds rationally, and completed at an odd prime. At prime 2, $J(4N)^\wedge_2 \simeq S^0_{K(1)}(2^{v+2})$ by Corollary 3.2.9, where $v = v_2(N)$. The claim now follows from the fact that $S^0_{K(1)}(4) \wedge \mathcal{E}_{K(1)} \simeq S^0_{K(1)}(4)$. □

Corollary 5.3.18. *$I_{KU}(J(4N)) \simeq \Sigma^2 J(4N) \wedge M(\widehat{\mathbb{Z}})$. By (5.3.13) and the Universal Coefficient Theorem,*

$$\pi_t(J(4N))^\wedge \simeq \pi_t(J(4N) \wedge M(\widehat{\mathbb{Z}})) \simeq \pi_{t+2}(I_{KU}(J(4N))) \simeq \operatorname{Hom}_{\mathbb{Z}}(\pi_{-2-t}(J(4N)), \mathbb{Q}/\mathbb{Z}),$$

as observed in Remark 3.2.15.

Appendix A. Cyclotomic representations of cyclic groups

In the appendix, we study the integral and p-adic cyclotomic representations of the cyclic group C_n.

A.1. **Integral cyclotomic representations.** Let $\Phi_n(t)$ be the n-th cyclotomic polynomial, i.e. the minimal polynomial of a primitive n-th root of unity ζ_n over $\mathbb{Q}$. The integral cyclotomic representation of C_n has underlying abelian group $\mathbb{Z}[\zeta_n] \simeq \mathbb{Z}[t]/\Phi(t)$ and $g \in C_n$ acts by multiplication by a primitive n-th root of unity (or $t \in \mathbb{Z}[t]/\Phi(t)$). The rank of $\mathbb{Z}[\zeta_n]$ as a free abelian group is equal to $\deg \Phi_n(t) = \phi(n)$.

Examples A.1.1. We consider the following examples:

(1) When $n = 5$, $\mathbb{Z}[\zeta_5]$ is a free $\mathbb{Z}$-module of rank 4 as $\phi(5) = 4$. $\{1, \zeta_5, \zeta_5^2, \zeta_5^3\}$ form a basis of $\mathbb{Z}[\zeta_5]$. The minimal polynomial of ζ_5 is $\Phi_5(t) = t^4 + t^3 + t^2 + t + 1$. Let $g \in C_5$ be a generator that acts on $\mathbb{Z}[\zeta_5]$ by multiplication by ζ_n. Then the matrix representation of $g \in C_5$ with respect the basis $\{1, \zeta_5, \zeta_5^2, \zeta_5^3\}$ of $\mathbb{Z}[\zeta_5]$ is
$$g = \begin{pmatrix} & & & -1 \\ 1 & & & -1 \\ & 1 & & -1 \\ & & 1 & -1 \end{pmatrix}.$$

(2) When $n = 6$, $\mathbb{Z}[\zeta_6]$ is a free $\mathbb{Z}$-module of rank 2 as $\phi(6) = 2$. $\{1, \zeta_6\}$ form a basis of $\mathbb{Z}[\zeta_6]$. The minimal polynomial of ζ_6 is $\Phi_6(t) = t^2 - t + 1$. Let $g \in C_6$ be a generator that acts on $\mathbb{Z}[\zeta_6]$ by multiplication by ζ_n. Then the matrix representation of $g \in C_6$ with respect the basis $\{1, \zeta_6\}$ of $\mathbb{Z}[\zeta_6]$ is
$$g = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}.$$

Lemma A.1.2. *The cyclotomic representation of C_n is equivalent to the external tensor product of the cyclotomic representations of $C_{p^{v_p(n)}}$, i.e. there is an equivalence of C_n-representations:*
$$\mathbb{Z}[\zeta_n] \simeq \bigotimes_{p|n} \mathbb{Z}\left[\zeta_{p^{v_p(n)}}\right]$$

Lemma A.1.3. *There is a short exact sequence of C_{p^v}-representations:*
$$0 \longrightarrow \mathbb{Z}[\zeta_{p^v}] \longrightarrow \mathbb{Z}[C_{p^v}] \longrightarrow \mathbb{Z}[C_{p^{v-1}}] \longrightarrow 0 \tag{A.1.4}$$
where C_{p^v} acts on $\mathbb{Z}[C_{p^{v-1}}]$ via the quotient map $C_{p^v} \twoheadrightarrow C_{p^{v-1}}$.

Proof. This follows from the observations that $\Phi_{p^v}(t) = \frac{t^{p^v}-1}{t^{p^{v-1}}-1}$ and $\mathbb{Z}[C_n] \simeq \mathbb{Z}[t]/(t^n - 1)$. ☐

A.2. **p-adic cyclotomic representations.** From now on, let $\chi : (\mathbb{Z}/N)^\times \to \mathbb{C}_p^\times$ be a p-adic Dirichlet character of conductor N and $\mathbb{Z}_p[\chi]$ be the $\mathbb{Z}_p$-subalgebra of $\mathbb{C}_p$ generated by the image of χ. Again, $\mathbb{Z}_p[\chi] = \mathbb{Z}_p[\zeta_n]$ for some n. Write $n = p^v \cdot n'$ with $p \nmid n'$, we have $\mathbb{Z}_p[\zeta_n] \simeq \mathbb{Z}_p[\zeta_{p^v}] \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\zeta_{n'}]$. Now it suffices to analyze C_n-actions on $\mathbb{Z}_p[\zeta_n]$ in the cases when $n = p^v$ or $p \nmid n$. Let's first recall some basic facts of cyclotomic extensions of $\mathbb{Q}$:

Lemma A.2.1. [Was97, Theorem 2.5, 2.6] *We recall the following basic facts of the cyclotomic extension $\mathbb{Q}(\zeta_n)/\mathbb{Q}$.*

(1) $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ is a Galois extension of degree $\phi(n)$ and $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \simeq (\mathbb{Z}/n)^\times$, with $a \in (\mathbb{Z}/n)^\times$ acting by $\zeta_n \mapsto \zeta_n^a$.

(2) The ring of integers of $\mathbb{Q}(\zeta_n)$ is $\mathbb{Z}[\zeta_n]$. Consequently, for any $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$, $\sigma(\mathbb{Z}[\zeta_n]) = \mathbb{Z}[\zeta_n]$.

As a result of this lemma, we can extract the action of $(\mathbb{Z}/N)^\times$ on $\mathbb{Z}[\zeta_n]$ from that on $\mathbb{Q}(\zeta_n)$.

Proposition A.2.2. *For any $\sigma \in \mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})$, the $(\mathbb{Z}/N)^\times$-representation induced by the Dirichlet character $\sigma \circ \chi$ is isomorphic to that induced by χ.*

Proof. Let $\mathbb{Z}[\chi] = \mathbb{Z}[\zeta_n]$, where ζ_n is a primitive n-th root of unity. For any $\sigma \in \mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})$, $\sigma(\zeta_n)$ is also a primitive n-th root of unity. As a result, the minimal polynomials of ζ_n and $\sigma(\zeta_n)$ are both $\Phi_n(t)$. It follows that the matrix representations of χ and $\sigma \circ \chi$ are differed by a change of basis induced by σ. Thus, the integral representations induced by χ and $\sigma \circ \chi$ are isomorphic. □

Proposition A.2.3. *Write $n = p^v \cdot n'$, where $p \nmid n'$ and let m be the multiplicative order of p mod n', i.e.*

$$m = \min\{k > 0 \mid p^k \equiv 1 \mod n'\}.$$

Then $\mathbb{Q}_p(\zeta_n)/\mathbb{Q}_p$ is a Galois extension of local fields of residue index m and ramification index $\phi(p^v)$. Moreover,

$$\mathrm{Gal}(\mathbb{Q}_p(\zeta_n)/\mathbb{Q}_p) \simeq \mathrm{Gal}(\mathbb{Q}_p(\zeta_{n'})/\mathbb{Q}_p) \times \mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^v})/\mathbb{Q}_p) \simeq (\mathbb{Z}/m) \times (\mathbb{Z}/p^v)^\times,$$

where a generator $\varphi \in \mathbb{Z}/m$ acts on $\mathbb{Q}_p(\zeta_{n'})$ by the lift of the Frobenius (p-th power map) from $\mathbb{Z}_p[\zeta_{n'}]/(p) \simeq \mathbb{F}_{p^m}$ to $\mathbb{Q}_p(\zeta_{n'}) \simeq \mathbb{W}(\mathbb{F}_{p^m})$. In particular, $\varphi(\zeta_{n'}) = \zeta_{n'}^p$.

A.3. **p-completions of integral cyclotomic representations.** We conclude this appendix with a discussion on how $\mathbb{Z}[\chi]$ decomposes upon p-completion. The simplest case is

Corollary A.3.1. $\mathbb{Z}_p[\zeta_{p^v}] \simeq \mathbb{Z}[\zeta_{p^v}] \otimes_{\mathbb{Z}} \mathbb{Z}_p \simeq (\mathbb{Z}[\zeta_{p^v}])^\wedge_p$.

Proof. By Proposition A.2.3, $\mathbb{Q}_p(\zeta_n)/\mathbb{Q}_p$ is a totally ramified extension of local fields of rank $\phi(p^v)$. This means $\mathbb{Z}_p[\zeta_{p^v}]$ is a free $\mathbb{Z}_p$-module of rank $\phi(p^v)$, which is equal to the rank of $\mathbb{Z}[\zeta_{p^v}]$ as a free $\mathbb{Z}$-module. This implies $\mathbb{Z}[\zeta_{p^v}]$ does not split upon p-completion. □

Comparing Lemma A.2.1 and Proposition A.2.3, we have shown:

Proposition A.3.2. *Fix an embedding $\iota : \mathbb{Q}[\zeta_n] \hookrightarrow \mathbb{C}_p$. For any $\sigma \in \mathrm{Gal}(\mathbb{Q}_p(\zeta_n)/\mathbb{Q}_p)$, $\sigma \circ \iota(\mathbb{Q}(\zeta_n)) = \iota(\mathbb{Q}(\zeta_n))$. In addition, the restriction map on the Galois group induced by ι*

$$\iota^* : \mathrm{Gal}(\mathbb{Q}_p(\zeta_n)/\mathbb{Q}_p) \longrightarrow \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \tag{A.3.3}$$

is injective. More precisely, rewrite $\mathbb{Q}(\zeta_n) = \mathbb{Q}(\zeta_{p^v}) \otimes_{\mathbb{Q}} \mathbb{Q}(\zeta_{n'})$ and $\iota = \iota_p \otimes \iota_{n'}$, where

$$\iota_p : \mathbb{Q}(\zeta_{p^v}) \hookrightarrow \mathbb{C}_p, \quad \iota_{n'} : \mathbb{Q}(\zeta_{n'}) \hookrightarrow \mathbb{C}_p.$$

Then we have

- $\iota_p^* : \mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^v})/\mathbb{Q}_p) \xrightarrow{\sim} \mathrm{Gal}(\mathbb{Q}(\zeta_{p^v})/\mathbb{Q})$ *is an isomorphism.*
- $\iota_{n'}^* : \mathrm{Gal}(\mathbb{Q}_p(\zeta_{n'})/\mathbb{Q}_p) \hookrightarrow \mathrm{Gal}(\mathbb{Q}(\zeta_{n'})/\mathbb{Q})$ *is the inclusion of the subgroup of $(\mathbb{Z}/n')^\times$ generated by $p \in (\mathbb{Z}/n')^\times$.*

Proposition A.3.4. *Pick a representative $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ for each coset in*

$$\mathrm{Coker}\,\iota^* = \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})/\mathrm{Gal}(\mathbb{Q}_p(\zeta_n)/\mathbb{Q}_p).$$

$\mathbb{Z}[\zeta_n] \otimes \mathbb{Z}_p$ decomposes as a $\mathbb{Z}_p$-algebra by

$$\mathbb{Z}[\zeta_n] \otimes_{\mathbb{Z}} \mathbb{Z}_p \xrightarrow[\sim]{\prod(\iota\circ\sigma)\otimes 1} \prod_{[\sigma]\in\mathrm{Coker}\,\iota^*} \mathbb{Z}_p[\zeta_n] \simeq \bigoplus_{[\sigma]\in\mathrm{Coker}\,\iota^*} \mathbb{Z}_p[\zeta_n].$$

Proof. The minimal polynomial of ζ_n over $\mathbb{Z}$ is

$$\Phi_n(t) = \prod_{\sigma\in\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})} (t - \sigma(\zeta_n)).$$

We have an isomorphism $\mathbb{Z}[\zeta_n] \otimes_{\mathbb{Z}} \mathbb{Z}_p \simeq \mathbb{Z}_p[t]/(\Phi_n(t))$. Over $\mathbb{Z}_p$, $\Phi_n(t)$ factorizes as

$$\Phi_n(t) = \prod_{[\sigma]\in\mathrm{Coker}\,\iota^*} \Phi_{n,\sigma}(t), \quad \text{where } \Phi_{n,\sigma}(t) := \prod_{\tau\in\mathrm{Gal}(\mathbb{Q}_p(\zeta_n)/\mathbb{Q}_p)} (t - \tau\circ\iota\circ\sigma(\zeta_n)).$$

For each $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$, $\Phi_{n,\sigma}(t)$ is the minimal polynomial of $\iota\circ\sigma(\zeta_n)$ over $\mathbb{Z}_p$. As $\Phi_{n,\sigma}(t)$ are coprime to each other for different cosets $[\sigma]\in\operatorname{Coker}\iota^*$ and $\mathbb{Z}_p[t]/(\Phi_{n,\sigma}(t))\simeq\mathbb{Z}_p[\zeta_n]$ for all σ, the claim now follows from the Chinese Reminder Theorem. $\square$

Corollary A.3.5. *Let $\chi:(\mathbb{Z}/N)^\times\to\mathbb{C}^\times$ be a Dirichlet character with $\mathbb{Z}[\chi]=\mathbb{Z}[\zeta_n]$. $\mathbb{Z}[\chi]\otimes_{\mathbb{Z}}\mathbb{Z}_p$ decomposes as a p-adic $(\mathbb{Z}/N)^\times$-representation by*

$$\mathbb{Z}[\chi]\otimes_{\mathbb{Z}}\mathbb{Z}_p\simeq\bigoplus_{[\sigma]\in\operatorname{Coker}\iota^*}\mathbb{Z}_p[\iota\circ\sigma\circ\chi],$$

where $\iota\circ\sigma\circ\chi$ is the p-adic Dirichlet character defined by

$$(\mathbb{Z}/N)^\times\xrightarrow{\chi}(\mathbb{Z}[\chi])^\times\xrightarrow{\sigma}(\mathbb{Z}[\chi])^\times\xhookrightarrow{\iota}\mathbb{C}_p^\times.$$

Proof. This is done by forcing the isomorphism in Proposition A.3.4 to be $(\mathbb{Z}/N)^\times$-equivariant. $\square$

Corollary A.3.6. *When $\chi:(\mathbb{Z}/N)^\times\to\mathbb{C}^\times$ is a primitive Dirichlet character of conductor $N=p^v$ and $p>2$, there is an equivalence of $(\mathbb{Z}/p^v)^\times$-representations:*

$$\mathbb{Z}[\chi]^\wedge_p\simeq\bigoplus_{\substack{0\le a\le p-2\\ \ker\omega^a=\ker\chi|_{(\mathbb{Z}/p)^\times}}}\mathbb{Z}_p[\chi_a],$$

where $\chi_a=\omega^a\cdot(\iota\circ\chi|_{\mathbb{Z}/p^{v-1}})$ and $\omega:(\mathbb{Z}/p)^\times\to\mathbb{Z}_p^\times$ is the Teichmüller character.

Proof. By Corollary A.3.5, we need show the following two sets of characters are the same:

$$\{\iota\circ\sigma\circ\chi\mid[\sigma]\in\operatorname{Coker}\iota^*\}=\{\omega^a\cdot(\iota\circ\chi|_{\mathbb{Z}/p^{v-1}})\mid 0\le a\le p-2,\ker\omega^a=\ker\chi|_{(\mathbb{Z}/p)^\times}\}.\tag{A.3.7}$$

We first prove the $v=1$ case. A p-adic character of conductor p is necessarily of the form ω^a for some a, since $\mathbb{Z}_p$ contains all $(p-1)$-st roots of unity. As ι and σ are injections, $\ker\iota\circ\sigma\circ\chi=\ker\chi$. Now it suffices to check the two sets have the same size. Since $\mathbb{Z}_p[\iota\circ\chi]=\mathbb{Z}_p$, we have $|\operatorname{Coker}\iota^*|=|\mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})|=\operatorname{rank}_{\mathbb{Z}}(\mathbb{Z}[\chi])$. χ factorizes as $(\mathbb{Z}/p)^\times\twoheadrightarrow C_{n'}\hookrightarrow(\mathbb{Z}[\zeta_{n'}])^\times$ for some $n'|(p-1)$. Then $\mathbb{Z}[\chi]$ has rank $\phi(n')$. Let $g\in(\mathbb{Z}/p)^\times$ be a generator, then $\ker\chi$ is the subgroup of $(\mathbb{Z}/p)^\times$ generated by $g^{n'}$. We have

$$\{a\mid 0\le a\le p-2,\ker\omega^a=\ker\chi=\langle g^{n'}\rangle\subseteq(\mathbb{Z}/p)^\times\}=\{a\mid 0\le a\le p-2,\ \text{the order of } a\in(\mathbb{Z}/p)^\times \text{ is } (p-1)/n'\}.$$

The size of this set is $\phi(n')$, which is equal to $|\operatorname{Coker}\iota^*|$, from which we conclude the two sets of characters in (A.3.7) are the same when $v=1$.

When $v>1$, write $\mathbb{Z}[\chi]=\mathbb{Z}[\chi|_{(\mathbb{Z}/p)^\times}]\otimes\mathbb{Z}[\chi|_{\mathbb{Z}/p^{v-1}}]$. χ being primitive implies $\chi|_{\mathbb{Z}/p^{v-1}}$ is injective and $\mathbb{Z}[\chi|_{\mathbb{Z}/p^{v-1}}]=\mathbb{Z}[\zeta_{p^{v-1}}]$. By Corollary A.3.1, $\mathbb{Z}[\chi|_{\mathbb{Z}/p^{v-1}}]^\wedge_p=\mathbb{Z}_p[\iota\circ\chi|_{\mathbb{Z}/p^{v-1}}]$. On the other hand , write $\iota=\iota_{n'}\cdot\iota_p$ as in Proposition A.3.2, where $\iota_p:\mathbb{Q}(\zeta_{p^{v-1}})\hookrightarrow\mathbb{C}_p$ is a field extension. Proposition A.3.2 says ι_p^* is an isomorphism, which implies $\operatorname{Coker}\iota^*=\operatorname{Coker}\iota^*_{n'}$. The analysis above shows:

$$\begin{aligned}\mathbb{Z}[\chi]^\wedge_p\simeq&\,\mathbb{Z}[\chi|_{(\mathbb{Z}/p)^\times}]^\wedge_p\bigotimes_{\mathbb{Z}_p}\mathbb{Z}_p[\iota_p\circ\chi|_{\mathbb{Z}/p^{v-1}}]\\ \bigoplus_{[\sigma]\in\operatorname{Coker}\iota^*}\mathbb{Z}_p[\iota\circ\sigma\circ\chi]\simeq&\left(\bigoplus_{[\sigma]\in\operatorname{Coker}\iota^*_{n'}}\mathbb{Z}_p[\iota_{n'}\circ\sigma\circ\chi|_{(\mathbb{Z}/p)^\times}]\right)\bigotimes_{\mathbb{Z}_p}\mathbb{Z}_p[\iota_p\circ\chi|_{\mathbb{Z}/p^{v-1}}]\end{aligned}$$

Now we have reduced this case to the $v=1$ situation for the character $\chi|_{(\mathbb{Z}/p)^\times}$. $\square$

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