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On $\mathbb{Z}_2$-equivariant loop spaces

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Dedicated to Prof. Samuel Gitler on occasion of his 70th birthday

Abstract. Let V finite dimensional $\mathbb{Z}_2$-representation which is the sum of the sign plus a trivial representation. We describe the homotopy type of the space of equivariant loops $(\Omega^V \Sigma^V X)^{\mathbb{Z}_2}$, where $\Omega^V \Sigma^V X$ denotes the space of maps $S^V \to \Sigma^V X = X \wedge S^V$ and X is a $\mathbb{Z}_2$-space with a fixed base point. The main result was suggested by the analysis of both, the James' construction $J(X)$ and J.P. May's construction $C_n X$, under the appropriate $\mathbb{Z}_2$-action.

1. Introduction

Let X be a $\mathbb{Z}_2$-space and V a real n-dimensional representation of $\mathbb{Z}_2$, which is the sum of one copy of the sign representation ζ and the $(n-1)$-dimensional trivial representation. The purpose of this article is to describe the homotopy type of the space of equivariant loops $(\Omega^V \Sigma^V X)^{\mathbb{Z}_2}$, as defined below. The case $n = 1$ has been partially analyzed by S. Rybicki in [**Ryb92**], where he introduced an equivariant version of the James construction and proved that there is a weak homotopy equivalence $J(X)^{\mathbb{Z}_2} \simeq (\Omega^V \Sigma^V X)^{\mathbb{Z}_2}$. We show here that the fixed point set $J(X)^{\mathbb{Z}_2}$ admits an easy description, namely in section 2 we prove that there is homeomorphism $J(X)^{\mathbb{Z}_2} \cong J(X) \times X^{\mathbb{Z}_2}$. Consequently, there is weak homotopy equivalence

$$(\Omega^V \Sigma^V X)^{\mathbb{Z}_2} \simeq (\Omega \Sigma X) \times X^{\mathbb{Z}_2} \tag{1.1}$$

A generalisation of Rybicki's approximation theorem for $n \geq 2$ will be given in section 3. This requires the use J.P. May's little-cubes construction $C_n X$ to replace $J(X)$. With the appropriate $\mathbb{Z}_2$-action on $C_n X$, the corresponding is

Theorem 1.1. *Let X be a $\mathbb{Z}_2$-space which is $\mathbb{Z}_2$-connected. Then there is a weak homotopy equivalence*

$$\alpha_n^{\mathbb{Z}_2} : (C_n X)^{\mathbb{Z}_2} \longrightarrow (\Omega^V \Sigma^V X)^{\mathbb{Z}_2}$$

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By further analysis of the model, one observes that there is a a homotopy equivalence: $(C_nX)^{\mathbb{Z}_2} \simeq (C_nX)\times C_{n-1}(X^{\mathbb{Z}_2})$ which immediately implies an equivalence of the corresponding function spaces, in the case when X is $\mathbb{Z}_2$-connected. However, it is possible to see that the splitting actually takes place at the level of function spaces. Thus in section 4 we prove that there is a weak homotopy equivalence

$$(\Omega^V\Sigma^VX)^{\mathbb{Z}_2} \simeq (\Omega^n\Sigma^nX) \times \Omega^{n-1}\Sigma^{n-1}(X^{\mathbb{Z}_2}). \tag{1.2}$$

We point out here that both equivalences (1.1) and (1.2) are particular cases of a more general result proven by Hauschild, see [**Haus**] Korollar 3.6. Their proofs are rather elementary and they are included here for completeness.

Notation. All our spaces will be in $\mathcal{T}$, the category of based, compactly generated, weak Hausdorff spaces and based maps. Mapping spaces are always to be given the compactly generated topology associated to the compact-open topology.

By a $\mathbb{Z}_2$-space X we will understand a $\mathbb{Z}_2$-space X whose based point $*$ is $\mathbb{Z}_2$-fixed. Base points are required to be non-degenerate in the sense that $(X, \{*\})$ is a $\mathbb{Z}_2$-equivariant NDR-pair. A $\mathbb{Z}_2$-space X is said to be $\mathbb{Z}_2$-connected if X and $X^{\mathbb{Z}_2}$ are path-connected.

For a G-representation V, let S^V denote its one-point compactification, where G acts trivially at the point at infinity. For a G-space X we write $\Sigma^VX = X \wedge S^V$ and $\Omega^VX = Map(S^V, X)$. Given two G-spaces X and Y, the space of continuous based maps $Map(X, Y)$ is also acted upon G by conjugation. The fixed point set $Map(X, Y)^G$ is the space of G-equivariant maps.

2. The one-dimensional case

Let X be a $\mathbb{Z}_2$-space and $V = \zeta$ the real one-dimensional sign representation of $\mathbb{Z}_2$. An equivalent way to describe Ω^VX is the following. Let Γ^VX to be the space of all continuous maps $\gamma : [-1, 1] \to X$ and consider the subspace of equivariant paths

$$(\Gamma^VX)^{\mathbb{Z}_2} = \{\, \gamma \in \Gamma^VX \;\mid\; \gamma(-s) = \tau \cdot \gamma(s) \quad \forall s \,\}.$$

Thus, if $\gamma \in (\Gamma^VX)^{\mathbb{Z}_2}$ then $\gamma(0) \in X^{\mathbb{Z}_2}$. Now, for $s \in [-1, 1]$ let $ev_s : (\Gamma X)^{\mathbb{Z}_2} \to X$ be the evaluation map sending γ to $\gamma(s)$. One observes the following basic results.

Lemma 2.1. *For a $\mathbb{Z}_2$ space X,*

(a) The evaluation map $ev_0 : (\Gamma^VX)^{\mathbb{Z}_2} \to X^{\mathbb{Z}_2}$ is a homotopy equivalence. Moreover, $X^{\mathbb{Z}_2}$ is a deformation retract of $(\Gamma^VX)^{\mathbb{Z}_2}$.

(b) The map $ev_1 : (\Gamma^VX)^{\mathbb{Z}_2} \to X$ is a fibration. The fibre over $*$ is the subspace of maps $\gamma \in (\Gamma^VX)^{\mathbb{Z}_2}$ which satisfy $\gamma(-1) = \gamma(1) = *$, and it is homotopy equivalent to $(\Omega^VX)^{\mathbb{Z}_2}$.

Remember that for a pointed space X, the James' construction $J(X)$ is defined to be the free topological monoid with basis X. This is, $J(X) = \varinjlim J_k(X)$, where $J_k(X) = X^k/\sim$. Here $\sim$ is the equivalence relation generated by

$$(x_1, \ldots, x_{j-1}, *, x_{j+1}, \ldots, x_k) \sim (x_1, \ldots, x_{j-1}, x_{j+1}, *, x_{j+2}, \ldots, x_k)$$

and the inclusion $J_k(X) \hookrightarrow J_{k+1}(X)$ is given by $[x_1, \ldots, x_k] \longmapsto [x_1, \ldots, x_k, *]$. In the case when X is also a $\mathbb{Z}_2$-space, there is a natural extension of the $\mathbb{Z}_2$-action to $J(X)$, see [**Ryb92**]. Namely, let τ denote the generator of $\mathbb{Z}_2$ and set $\tau \cdot [x_1, \ldots, x_k] = [\tau x_k, \ldots, \tau x_1]$.

Then, the main theorem in [**Ryb92**] can be stated as follows

Theorem 2.2. *Let X be a path-connected $\mathbb{Z}_2$-space and V as above. Then the James' map $\lambda : J(X) \to \Omega^V \Sigma^V X$ is $\mathbb{Z}_2$-equivariant. Moreover, its restriction to the fixed point set $\lambda^{\mathbb{Z}_2} : J(X)^{\mathbb{Z}_2} \to (\Omega^V \Sigma^V X)^{\mathbb{Z}_2}$ is a weak homotopy equivalence.*

In the rest of the section we identify the homotopy type of $(\Omega^V \Sigma^V X)^{\mathbb{Z}_2}$. For this purpose, the fixed point set $J(X)^{\mathbb{Z}_2}$ is easily described next. First, the following Lemma is immediate:

Lemma 2.3. *For a $\mathbb{Z}_2$-space X and $w \in J(X)$:*

(a) If $w = [x_1, \ldots, x_{2k}] \in J_{2k}(X) - J_{2k-1}(X)$, then $w \in J(X)^{\mathbb{Z}_2}$ if and only if $x_i = \tau x_{2k-i+1}$ for $i = 1, \ldots, k$,

(b) If $w = [x_1, \ldots, x_{2k+1}] \in J_{2k+1}(X) - J_{2k}(X)$, then $w \in J(X)^{\mathbb{Z}_2}$ if and only if $x_i = \tau x_{2k-i+2}$ for $i = 1, \ldots, k$ and $x_{k+1} \in X^{\mathbb{Z}_2}$.

As a consequence, we have the following result:

Theorem 2.4. *There is a homeomorphism $\varphi : J(X) \times X^{\mathbb{Z}_2} \longrightarrow J(X)^{\mathbb{Z}_2}$.*

Proof. The intersections $J_{2k}(X) \cap J(X)^{\mathbb{Z}_2}$ and $J_{2k+1}(X) \cap J(X)^{\mathbb{Z}_2}$ will be denoted by $J_{2k}(X)^{\mathbb{Z}_2}$ and $J_{2k+1}(X)^{\mathbb{Z}_2}$ respectively. We define the map φ by the following formula

$$\varphi([x_1, \ldots, x_k]\, , y) = [x_1, \ldots x_k,\, y,\, \tau x_k, \ldots, \tau x_1]$$

Clearly φ is continuous and it is a bijection by the above remarks. We construct a continuous inverse as follows: For a fixed k, let $\pi_k : X^{2k+1} \to J_{2k+1}(X)$ be the canonical projection and $W_k = \pi_k^{-1} J_{2k+1}(X)^{\mathbb{Z}_2}$. Notice that $J_{2k+1}(X)^{\mathbb{Z}_2}$ is closed in $J_{2k+1}(X)$ and therefore $\pi_k' := \pi_k \mid_{W_k}$ is also a quotient map.

Now, for $i = 1, \ldots, 2k+1$, let $V_i \subset X^{2k+1}$ be the subspace consisting of all $(2k+1)$-tuples $(z_1, \ldots, z_{2k+1})$ such that:

(1) $z_i \in X^{\mathbb{Z}_2}$,

(2) $(z_1, \ldots, z_{i-1}, \widehat{z_i}, z_{i+1}, \ldots, z_{2k+1}) = (x_1, \ldots x_k, \tau x_k, \ldots, \tau x_1)$ for some $x_1, \ldots, x_k \in X$, and

(3) $[z_1, \ldots, z_{2k+1}] = [x_1, \ldots x_k, z_i, \tau x_k, \ldots, \tau x_1]$ in $J_{2k+1}(X)$.

Then V_i is a closed subspace of X^{2k+1} and $W_k = \bigcup_{i=1}^{2k+1} V_i$. Now, define a map $\theta_k : V_i \to J_k(X) \times X^{\mathbb{Z}_2}$ by $\theta_k(z_1, \ldots, z_{2k+1}) = (\,[x_1, \ldots x_k]\,,\, z_i\,)$. Then θ_k agrees on all intersections $V_{i_1} \cap \ldots \cap V_{i_\ell}$ and thus, it defines a map $\theta_k : W_k \to J_k(X) \times X^{\mathbb{Z}_2}$ which is constant on the fibers of π_k and so it passes to the quotient,

$$\begin{array}{ccccc} X^{2k+1} & \hookleftarrow & W_k & \xrightarrow{\theta_k} & J_k(X) \times X^{\mathbb{Z}_2} \\ \downarrow{\scriptstyle \pi_k} & & \downarrow{\scriptstyle \pi'_k} & \nearrow{\scriptstyle \psi_k} & \\ J_{2k+1}(X) & \hookleftarrow & J_{2k+1}(X)^{\mathbb{Z}_2} & & \end{array}$$

Finally, the maps ψ_k are compatible with the filtration. That is, for every $k \geq 1$ the following diagram commutes:

$$\begin{array}{ccc} J_{2k+1}(X)^{\mathbb{Z}_2} & \xrightarrow{\psi_k} & J_k(X) \times X^{\mathbb{Z}_2} \\ \downarrow & & \downarrow \\ J_{2k+3}(X)^{\mathbb{Z}_2} & \xrightarrow{\psi_{k+1}} & J_{k+1}(X) \times X^{\mathbb{Z}_2} \end{array}$$

This defines a continuous map $\psi : J(X)^{\mathbb{Z}_2} \longrightarrow J(X) \times X^{\mathbb{Z}_2}$ which is the desired inverse.

□

Corollary 2.5. *For a path-connected $\mathbb{Z}_2$-space X, there is a weak homotopy equivalence* $(\Omega^V \Sigma^V X)^{\mathbb{Z}_2} \simeq (\Omega^V \Sigma^V X) \times X^{\mathbb{Z}_2}$.

3. The $\mathbb{Z}_2$-action on C_nX and the approximation map

We begin by recalling the definition of the space of little n-cubes. Let us denote by J the open interval $(-1,1) \subset \mathbb{R}$.

DEFINITION 3.1. A little n-cube c, is an orientation preserving affine embedding $c\colon J^n \to J^n$ of the form $c = f_1 \times \cdots \times f_n$, where each $f_i : J \to J$ is a map of the form $f_i(t) = a_i t + b_i$ whith $a_i > 0$, $-1 \le b_i - a_i$ and $b_i + a_i \le 1$.

The space of k little n-cubes, $\mathcal{C}_n(k)$, is the space of ordered k-tuples of little n-cubes whose images are mutually disjoint.

DEFINITION 3.2. Let X be a space in $\mathcal{T}$. We define

$$C_nX = (\coprod_{k\ge 0} \mathcal{C}_n(k) \times_{\Sigma_k} X^k)/\approx$$

where $\approx$ is the equivalence relation generated by

$$[\langle c_1,\dots,c_k\rangle; x_1,\dots,x_k] \approx [\langle c_1,\dots,c_{k-1}\rangle; x_1,\dots,x_{k-1}]$$

if and only if $x_k = *$, the base point.

The most important property of $C_n(X)$ to be used in this work is stated in the following theorem. The reader is referred to [**May72**] for the proof.

THEOREM 3.3. *For $n \ge 1$ and X path-connected, there is a functorial weak homotopy equivalence*

$$\alpha_n : C_n(X) \longrightarrow \Omega^n\Sigma^n X.$$

The map α_n is defined as follows: Identify S^n with the quotient of $[-1,1]^n$ by its boundary. Then, for every $[c\,;x] = [\,\langle c_1,\dots,c_k\rangle\ ;\ x_1,\dots,x_k] \in \mathcal{C}_n(k) \times_{\Sigma_k} X^k$, let $\alpha_n([c\,;x])$ be the map $S^n \to X \wedge S^n$ given by:

$$[v] \longmapsto \begin{cases} x_i \wedge [c_i^{-1}(v)] & \text{if } v_i \in im(c_i) \\ * & \text{otherwise} \end{cases}$$

REMARK 3.4. In [**May72**], $\mathcal{C}_n(k)$ was defined using $[0,1]$ instead of $(-1,1)$. This definition is equivalent to ours, but the later is more useful when handling $\mathbb{Z}_2$-actions on $J^n \cong \mathbb{R}^n$.

For $n \ge 2$ consider the $\mathbb{Z}_2$-representation $V = (n-1) \oplus \zeta$ which is the direct sum of one copy of the real one dimensional sign representation plus a copy of the trivial $n-1$ dimensional representation. Identifying V with $\mathbb{R}^n$, the action is given by reflection through a fixed hyperplane, say $\tau(v_1,\dots,v_n) = (v_1,\dots,v_{n-1},-v_n)$. The fixed-point set in this case is the hyperplane $H = \{v \in V \mid v_n = 0\} \cong \mathbb{R}^{n-1}$.

This induces the diagonal $\mathbb{Z}_2$-action on the configuration space $F(V,k)$ and also on the space of little cubes $\mathcal{C}_n(k)$, such that the natural map

$$\begin{aligned} \mathcal{C}_n(k) &\longrightarrow F(V,k) \\ \langle c_1,\dots,c_k\rangle &\longmapsto (c_1(0),\dots,c_k(0)) \end{aligned} \tag{3.1}$$

is $\mathbb{Z}_2$-equivariant homotopy equivalence. Namely if $c\in\mathcal{C}_n(1)$, write $c=c'\times c''$ where $c'\in\mathcal{C}_{n-1}(1)$, $c''\in\mathcal{C}_1(1)$, with $c''(t)=Rt+b$. Set $\tau c=c'\times(\tau c'')$ where $(\tau c'')(t)=Rt-b=-[R(-t)+b]$. We can now extend the action to $\mathcal{C}_n(k)$ in the natural way:

$$\tau\langle c_1,\dots,c_k\rangle=\langle\tau c_1,\dots,\tau c_k\rangle.$$

Then $\mathbb{Z}_2$ acts diagonally on $\mathcal{C}_n(k)\times X^k$ and this action is Σ_k-equivariant, so it passes to the quotient $\mathcal{C}_n(k)\times_{\Sigma_k}X^k$. Also, since $*\in X^{\mathbb{Z}_2}$, the action is compatible with the base point relation and thus, it induces an action on C_nX. Thus, the following result is now clear:

Theorem 3.5. *For a path connected $\mathbb{Z}_2$-space X, the weak homotopy equivalence given in Theorem 3.3 $\alpha_n: C_nX\to\Omega^V\Sigma^VX$ is $\mathbb{Z}_2$-equivariant with respect to the actions described above.*

The fixed point set $(C_nX)^{\mathbb{Z}_2}$ admits the following simple description: A point $\xi\in C_nX$ is in $(C_nX)^{\mathbb{Z}_2}$ if and only if by the use of permutations and the base point relation, it can be written in the form: $\xi=[\langle c,\tau c,d\rangle,x,\tau x,y]$ where: $c\in\mathcal{C}_n(i)$, $x\in(X-*)^i$, $d\in\mathcal{C}_n(j)^{\mathbb{Z}_2}$, $y\in(X^{\mathbb{Z}_2}-*)^j$ and $\langle c,\tau c,d,\rangle\in\mathcal{C}_n(2i+j)$ for some $i,j\geq 0$.

A complete analysis of the fixed point set $(C_nX)^{\mathbb{Z}_2}$ was carried out in [**Xi97**], using orbit configuration spaces and led us to prove the following:

Theorem 3.6. *For a path-connected $\mathbb{Z}_2$-spaces X, there is a weak equivalence*

$$\alpha_n^{\mathbb{Z}_2}:(C_nX)^{\mathbb{Z}_2}\longrightarrow(\Omega^V\Sigma^VX)^{\mathbb{Z}_2}$$

Also, we obtained a product splitting

$$(C_nX)^{\mathbb{Z}_2}\simeq(C_nX)\times C_{n-1}(X^{\mathbb{Z}_2})$$

which implied in turn a weak homotopy equivalence between the corresponding function spaces:

$$(\Omega^V\Sigma^VX)^{\mathbb{Z}_2}\simeq(\Omega^n\Sigma^nX)\times\Omega^{n-1}\Sigma^{n-1}(X^{\mathbb{Z}_2}). \tag{3.2}$$

This last splitting will proven in section 4.

Remark 3.7. It was actually during the write up of [**Xi97**] that the author came across the splitting (3.2) for the first time. Fortunately, an easier and less combinatorial proof is available now. Thus, to prove (3.2), we will work directly at the level of function spaces, rather than going trough the methods of [**Xi97**].

4. A product splitting for $(\Omega^V \Sigma^V X)^{\mathbb{Z}_2}$

Let $V = (n-1) \oplus \zeta$ be and n-dimensional $\mathbb{Z}_2$-representation with a one-dimensional non-trivial summand, as in the introduction. The we have

THEOREM 4.1. *For $n \geq 2$ and every $\mathbb{Z}_2$-space X which $\mathbb{Z}_2$-connected, there is a weak homotpy equivalence:*

$$(\Omega^V \Sigma^V X)^{\mathbb{Z}_2} \simeq (\Omega^n \Sigma^n X) \times \Omega^{n-1} \Sigma^{n-1} (X^{\mathbb{Z}_2})$$

PROOF. Let $G = \mathbb{Z}_2$ and set $m = n-1$. Notice that the inclusion of the fixed point set $S^0 = (S^\zeta)^G$ into S^ζ yields an equivariant cofibre sequence

$$S^0 \longrightarrow S^\zeta \longrightarrow G_+ \wedge S^1$$

Smashing with with S^m and applying the functor $Map(-, \Sigma^{m\oplus\zeta} X)$ one obtains an equivariant fibration

$$Map(G_+ \wedge S^{m+1}, \Sigma^{m\oplus\zeta} X) \longrightarrow \Omega^{m\oplus\zeta} \Sigma^{m\oplus\zeta} X \xrightarrow{\xi} \Omega^m \Sigma^{m\oplus\zeta} X$$

Passing to fixed points, one obtains the non-equivariant fibration

$$Map(G_+ \wedge S^{m+1}, \Sigma^{m\oplus\zeta} X)^G \longrightarrow (\Omega^{m\oplus\zeta} \Sigma^{m\oplus\zeta} X)^G \xrightarrow{\xi^G} (\Omega^m \Sigma^{m\oplus\zeta} X)^G$$

Observe that:

(1) Change of groups gives a standard homeomorphism (See [**May96**]):

$$\theta \,:\, \Omega^{m+1} \Sigma^{m+1} X \xrightarrow{\cong} Map(G_+ \wedge S^{m+1}, \Sigma^{m\oplus\zeta} X)^G$$

(2) Since the sphere S^m in $\Omega^m \Sigma^{m\oplus\zeta} X = Map(S^m, \Sigma^{m\oplus\zeta} X)$ has the trivial G-action, the fixed point set in this case is easily seen to be homeomorphic to $\Omega^m \Sigma^m (X^G)$. In fact, the $(\Sigma^\zeta, \Omega^\zeta)$ -adjunction together with the inclusion $X^G \subset X$ gives a G-map

$$\eta : \Omega^m \Sigma^m (X^G) \longrightarrow \Omega^{m\oplus\zeta} \Sigma^{m\oplus\zeta} X$$

Composing this map with the map with the map $\xi : \Omega^{m\oplus\zeta} \Sigma^{m\oplus\zeta} X \longrightarrow (\Omega^m \Sigma^{m\oplus\zeta})^G$ and passing to fixed points gives a homeomorphism

$$\gamma \,:\, \Omega^m \Sigma^m (X^G) \xrightarrow{\cong} (\Omega^m \Sigma^{m\oplus\zeta} X)^G$$

Finally, there is a map

$$\phi : \Omega^m\Sigma^m(X^G) \times \Omega^{m+1}\Sigma^{m+1}X \longrightarrow (\Omega^{m\oplus\zeta}\Sigma^{m\oplus\zeta}X)^G$$

such that the top square in the diagram

$$\begin{array}{ccc}
\Omega^{m+1}\Sigma^{m+1}X & \xrightarrow[\cong]{\theta} & Map(G_+ \wedge S^{m+1}, \Sigma^{m\oplus\zeta}X)^G \\
\downarrow{\scriptstyle \iota_2} & & \downarrow \\
\Omega^m\Sigma^m(X^G) \times \Omega^{m+1}\Sigma^{m+1}X & \xrightarrow{\phi} & (\Omega^{m\oplus\zeta}\Sigma^{m\oplus\zeta}X)^G \\
\downarrow{\scriptstyle \pi_1} & & \downarrow \\
\Omega^m\Sigma^m(X^G) & \xrightarrow[\cong]{\gamma} & (\Omega^m\Sigma^{m\oplus\zeta}X)^G
\end{array}$$

commutes up to homotopy and the bottom square commutes on the nose. Here ι_2 and π_1 are the obvious inclusions into, and projection out of the product. Since both columns are fibrations and the top and bottom horizontal maps are homeomorphisms, it follows that the middle horizontal map is a weak homotopy equivalence.

To define ϕ, identify S^ζ with the unit circle in the complex plane carrying the conjugation G-action. Identifying the three complex roots of unity to a point, gives a G-map $\rho : S^\zeta \to S^\zeta \vee (G_+ \wedge S^1)$. Here, the S^ζ is the image of the arc $\{e^{i\theta} \mid \frac{2\pi}{3} \leq \theta \leq \frac{4\pi}{3}\}$ under the quotient map, and the two circles in $G_+ \wedge S^1$ come from the two arcs of the unit circle between 1 and the two nontrivial cube roots of 1. If $\sigma \in \Omega^m\Sigma^m(X^G)$ and $\omega \in \Omega^{m+1}\Sigma^{m+1}X$, then $\phi(\sigma, \omega)$ is the composite:

$$S^{m\oplus\zeta} \xrightarrow{\Sigma^m\rho} S^{m\oplus\zeta} \vee (G_+ \wedge S^{m+1}) \xrightarrow{(\tilde{\sigma},\tilde{\omega})} \Sigma^{m\oplus\zeta}X$$

in which $\tilde{\sigma}$ is the composite

$$S^{m\oplus\zeta} \xrightarrow{\Sigma^\zeta\sigma} \Sigma^{m\oplus\zeta}(X^G) \subset \Sigma^{m\oplus\zeta}X$$

and $\tilde{\omega}$ is the map

$$G_+ \wedge S^{m+1} \longrightarrow \Sigma^{m\oplus\zeta}X$$

which is the the image of ω under θ. With this definition of ϕ is easy to see that the bottom square of the diagram above commutes, and that the top square commutes up to homotopy. □

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