THE Ω-SPECTRUM FOR BROWN-PETERSON COHOMOLOGY PART II.

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Introduction. Let BP denote the spectrum for the Brown-Peterson cohomology theory, $BP^*(\cdot)$. [2,5,12] We have $BP^k(X) \cong [X, BP_k]$ where $BP = \{BP_k\}$ as an Ω-spectrum, i.e. $\Omega BP_k \cong BP_{k-1}$. [4] In Part I [20] we determined the structure of the cohomology of BP_k. In this part we study the homotopy type of BP_k.

The structure of each BP_k is very nice and gives some insight into the cohomology theory. In particular, using it, we obtain a new proof of Quillen's theorem that $BP^*(X)$ is generated by non-negative degree elements as a module over $BP^*(S^\circ)$. [11] (X is a pointed finite CW complex.)

Let $Z_{(p)}$ be the integers localized at p, the prime associated with BP. We explicitly construct spaces Y_k which are the smallest possible $k-1$ connected H-spaces with π_* and H_* free over $Z_{(p)}$. The Y_k are the building blocks for BP_n, i.e., $BP_n \cong \Pi_i Y_{k_i}$. In fact, one of our main theorems states that for any H-space X with π_* and H_* free over $Z_{(p)}$, then $X \cong \Pi_i Y_{k_i}$. (This is not as H-spaces, see section 6.) To understand the spaces Y_k we need a sequence of homology theories:

$$BP_*(X) \cong BP\langle\infty\rangle_*(X) \to \cdots \to BP\langle n+1\rangle_*(X) \to BP\langle n\rangle_*(X)$$

$$\to \cdots \to BP\langle 0\rangle_*(X) = H_*(X, Z_{(p)})$$

These are constructed using Sullivan's theory of manifolds with singularities. $BP_*(S^0) \cong Z_{(p)}[x_1, x_2 \ldots]$ with degree of $x_i = 2(p^i - 1)$. $BP\langle n\rangle_*(S^0) = Z_{(p)}[x_1, \ldots, x_n]$ as a graded group. Let $BP\langle n\rangle = \{BP\langle n\rangle_k\}$ be the Ω-spectrum for $BP\langle n\rangle_*(\cdot)$. For $k > 2(p^{n-1} + \cdots + p + 1)$, the space $BP\langle n\rangle_k$ cannot be broken down as a product $BP\langle n\rangle_k \cong Y \times X$ with both X and Y non-trivial. For $k \leqslant 2(p^n + \cdots + p + 1)$, $H^*(BP\langle n\rangle_k, Z_{(p)})$ has no torsion. So, for k between these two numbers we get $Y_k \cong BP\langle n\rangle_k$.

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Main Theorem. For $2(p^{n-1}+\cdots+p+1)<k\leqslant 2(p^n+\cdots+p+1)$

$$BP_k \cong BP\langle n\rangle_k \times \prod_{j>n} BP\langle j\rangle_{k+2(p^j-1)}$$

and cannot be broken down further.

The proof of this theorem exploits the fact from [20] that the $Z_{(p)}$ cohomology of BP_k has no torsion.

We begin by constructing the theories $BP\langle n\rangle_*(\cdot)$. In section 2 we review what we need about Postnikov systems. Section 3 is devoted to preliminary necessities for the proof of the main theorems in section 4. Then we state the main results and prove Quillen's theorem (section 5) and a general decomposition theorem for spaces which are p-torsion free and H-spaces when localized at p. (section 6)

In a paper with Dave Johnson these results are applied to study the homological dimension of $BP^*(X)$ over $BP^*(S^0)$. [21]

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1. Construction of $BP\langle n\rangle$.

This section deals with Sullivan's theory of manifolds with singularities. [19] The approach we take is due to Nils Baas. This section is not intended to be an exposition on the Baas-Sullivan theory, for we only wish to use it to construct certain specific homology theories, the general case being covered in detail in [3]. Even the definitions we give will be missing major ingredients, in all cases we refer to [3].

If we dealt with the case of one singularity, P, then a manifold with singularity P would be a space $V=N\cup_{P\times M} cP\times M$ where N is a manifold with $\partial N=P\times M$ and cP is the cone on P. One can make a bordism group of a space using such objects in place of manifolds. An element of the bordism would be represented by a map $f: V\to X$. So, as far as bordism is concerned, one might just as well consider only the manifold N and insist that maps $f: N\to X$, when restricted to $\partial N=P\times M$, factor through the projection $P\times M\to M$. This is the approach Baas takes. When more than one singularity is considered, the definitions become quite technical. From [3]

Definition. V *is a closed decomposed manifold if there exist submanifolds* $\partial_1 V,\ldots,\partial_n V$ *such that* $\partial V=\partial_1 V\cup\cdots\cup\partial_n V$ *where union means identification along common part of boundary such that* $\partial(\partial_i V)=(\partial_1 V\cap\partial_i V)$

$\cup \cdots \cup (\partial_{i-1} V \cap \partial_i V) \cup \varnothing \cup \cdots \cup (\partial_n V \cap \partial_i V)$, *which gives* $\partial_i V$ *the structure of a decomposed manifold. Continue, defining* $\partial_k(\partial_j(\partial_i V))$, *etc.*

Let $S^n = \{P_1, P_2, \ldots, P_n\}$ be a fixed class of manifolds. Very loosely, A is a closed manifold of singularity type S^n if for each subset $\omega \subset \{1, 2, \ldots, n\}$ there is a decomposed manifold $A(\omega)$ such that $A(\varnothing) = A$, $\partial_i A(\omega) \cong A(\omega, i) \times P_i$ if $i \not\in \omega$, $\partial_i A(\omega) \cong \varnothing$ if $i \in \omega$. A singular S^n manifold in X is a map $g: A \to X$ such that $g | \partial_i A(\omega) \cong A(\omega, i) \times P_i$ factors through the projection $A(\omega, i) \times P_i \to A(\omega, i)$.

More generally, singular manifolds with boundary, singular manifolds in a pair, and a concept of bordism are all defined. (rigorously) These bordism groups are shown to give generalized homology theories, $MS^n{}_*(\cdot)$. One of the most important aspects of these theories is the relationship between $MS^n{}_*(\cdot)$ and $MS^{n+1}{}_*(\cdot)$. This will be a major tool throughout the paper. There is an exact sequence

$$\begin{array}{ccc} MS^n{}_*(X) & \xrightarrow{\beta} & MS^n{}_*(X) \\ & {}_{\delta}\nwarrow \quad \swarrow_{\gamma} & \\ & MS^{n+1}{}_*(X) & \end{array}$$

The product of an S^n manifold with a closed manifold N gives an S^n manifold by: $(N \times A)(\omega) = N \times A(\omega)$. On a representative element $A \to X$, β is $P_{n+1} \times A \to A \to X$. Any S^n manifold A can be considered as an S^{n+1} manifold by setting $A(\omega, n+1) = \varnothing$. So $\gamma(A \to X) = (A \to X)$. For an S^{n+1} manifold A we see that $A(n+1)$ is an S^n manifold, so $\delta(f: A \to X) = f | A(n+1) \to X$. The degrees of these maps are: degree β = dimension P_{n+1}, degree $\gamma = 0$, degree $\delta = -$dimension $P_{n+1} - 1$. In our one singularity example, $\partial N = P \times M$, δ just restricts to M. Baas of course defines these maps rigorously, shows they are well defined and proves the exactness theorem.

Above we remarked that the product of a manifold and an S^n manifold is again an S^n manifold. This gives us a map, $MS^0{}_*(X) \otimes MS^n{}_*(Y) \to MS^n{}_*(X \times Y)$.This is precisely the condition that tells us the spectrum associated with $MS^n{}_*(\cdot)$ is a module spectrum over the spectrum for the standard bordism theory $MS^0{}_*(\cdot)$. Further, $MS^n{}_*(X)$ is a module over $MS^0{}_*(S^0)$ and the above maps, β, γ, δ are all $MS^0{}_*(S^0)$ module maps.

We now get on to our applications. All manifolds considered above could be taken with some extra structure, and we assume them all to be U manifolds. So $MS^0{}_*(\cdot)$ is $MU_*(\cdot)$ the standard complex bordism homology theory for finite complexes. Now $MU_*(S^0) = \pi^S{}_*(MU) = Z[x_2, \ldots, x_{2i}, \ldots]$ where degree $x_{2j} = 2j$. We choose a representative manifold P_i for x_{2i}. Fix a prime p. $S(n, m) =$

$\{P_i | i \leqslant m, i \neq p^j - 1, j \leqslant n\}$. Then by all of the above, we have a homology theory $MUS(n,m)_*(\cdot)$ made from U manifolds with singularity type $S(n,m)$. For large m we have an exact sequence:

$$\begin{array}{ccc} MUS(n,m)_*(X) & \xrightarrow{\times P_{p^n-1}} & MUS(n,m)_*(X) \\ \nwarrow_{\delta} & & \swarrow_{\gamma} \\ & MUS(n-1,m)_*(X) & \end{array}$$

From these exact sequences and the homotopy of MU we see that $MUS(n,m)_*(S^0) = \pi^S{}_*(MU)/[S(n,m)]$ where $[S(n,m)]$ is the ideal generated by $S(n,m)$. We define the homology theory $MUS(n)_*(\cdot) = \lim(m \to \infty) MUS(n,m)_*(\cdot)$. $MUS(n)_*(\cdot) \otimes Z_{(p)}$ is a homology theory which we will denote by $BP\langle n\rangle_*(\cdot)$ and the corresponding spectrum by $BP\langle n\rangle$. The reason for the notation is that if $BP \to MU_{(p)}$ is Quillen's map ([12]), then $BP \to MU_{(p)} \to BP\langle\infty\rangle$ clearly gives an isomorphism on homotopy and so $BP \cong BP\langle\infty\rangle$. Thus $BP\langle n\rangle$ is a module spectrum over BP and we have:

$$\begin{array}{ccc} BP\langle n\rangle_*(X) & \xrightarrow{\beta} & BP\langle n\rangle_*(X) \\ \nwarrow_{\delta} & & \swarrow_{\gamma} \\ & BP\langle n-1\rangle_*(X) & \end{array} \qquad (1.1)$$

with degree of $\beta = 2(p^n - 1)$, degree $\gamma = 0$, degree $\delta = -2p^n + 1$. $BP_*(S^0) = Z_{(p)}[x_{2(p-1)}, \ldots, x_{2(p^i-1)}, \ldots]$. $BP\langle n\rangle_* = BP\langle n\rangle_*(S^0) = BP_*(S^0)/[x_{2(p^i-1)} | i > n]$ as a module over BP_*. BP_* acts on $BP\langle n\rangle_*(X)$, it is known that $x_{2(p^i-1)}$ acts trivially for $i > n$.

Every spectrum can be represented as an Ω-spectrum. [4] Let $BP\langle n\rangle = \{BP\langle n\rangle_k\}$ be the Ω-spectra, i.e. $\Omega BP\langle n\rangle_k \cong BP\langle n\rangle_{k-1}$ and $BP\langle n\rangle_k$ is $k-1$ connected for $k > 0$. This means that $BP\langle n\rangle^k(X) = [X, BP\langle n\rangle_k]$ where $BP\langle n\rangle^*(\cdot)$ is the cohomology theory given by $BP\langle n\rangle$.

The theories $BP\langle n\rangle$ are independent of choice of manifolds P_i representing x_{2i} but seemingly dependent on the choice of generators $x_{2(p^i-1)}$ chosen for $\pi^S{}_*(MU)$. However, the results we obtain are independent of the choice of even these generators because the spaces $BP\langle n\rangle_k$ for different choices become homotopy equivalent when k is small enough. In addition, in [21] we show that $BP\langle 1\rangle$ is independent of choice of x_{2i}. In fact, $BP\langle 1\rangle$ is just the irreducible part of connective K-theory when localized at p.

We now permanently reindex the $x_{2(p^i-1)}$ to x_i with degree $2(p^i-1)$. From 1.1 we have a split exact sequence:

$$0 \longrightarrow BP\langle n\rangle_* \xrightarrow{x_n} BP\langle n\rangle_* \longrightarrow BP\langle n-1\rangle_* \longrightarrow 0 \tag{1.2}$$

$BP\langle n\rangle_* = Z_{(p)}[x_1, \ldots, x_n]$ as a group. Again, from 1.1 for finite complexes we get a cofibration ([1]):

$$\begin{array}{ccc} S^{2i}BP\langle n\rangle & \xrightarrow{\beta} & BP\langle n\rangle \\ & & \downarrow \gamma \\ i = p^n - 1 & & BP\langle n-1\rangle \end{array} \tag{1.3}$$

For the spaces in the Ω-spectrum this becomes a fibration:

$$\begin{array}{ccc} BP\langle n\rangle_{k+j} & \xrightarrow{\beta} & BP\langle n\rangle_k \\ & & \downarrow \gamma \\ j = 2(p^n - 1) & & BP\langle n-1\rangle_k \end{array} \tag{1.4}$$

If M is a graded module let s^kM be the graded module $(s^kM)_{k+q} = M_q$. Then,

$$\pi_*(BP\langle n\rangle_k) = s^k(BP\langle n\rangle_*) \qquad k \geqslant 0 \tag{1.5}$$

From 1.3 we have an exact sequence:

$$H^*(BP\langle n\rangle) \xleftarrow{\beta^*} H^*(BP\langle n\rangle) \xleftarrow{\gamma^*} H^*(BP\langle n-1\rangle), \qquad \partial^*: H^*(BP\langle n\rangle) \to H^*(BP\langle n-1\rangle) \tag{1.6}$$

For most of the paper, unless otherwise noted, all coefficient groups will be Z_p where p is the fixed prime associated with the $BP\langle n\rangle$. Let A be the mod p Steenrod algebra and Q_i the Milnor elements. [9]

PROPOSITION 1.7. $H^*(BP\langle n\rangle) \cong A/A(Q_0, Q_1, \ldots, Q_n) = A_n$

Note. Baas and Madsen have a more general result which includes this, however, as this special case has a much more elementary proof we give it here.

Proof. $\pi^S_*(BP\langle 0\rangle) = Z_{(p)}$, so $BP\langle 0\rangle = K(Z_{(p)})$ and $H^*(BP\langle 0\rangle) = A/A(Q_0)$. We prove the result by induction on n using 1.6. Let 1 denote the lowest dimensional class of each spectrum, then $\gamma^*(1) = 1$. Assume $H^*(BP\langle n-$

$1\rangle)=A_{n-1}$. If $\gamma^*(Q_n 1)=0$, then for dimensional reasons, $\partial^*(1)=\lambda Q_n 1$, $0 \neq \lambda \in Z_p$. If $a \in A_n$, then $0 \neq aQ_n 1 = \partial^*(a1)$ in A_{n-1} because $A_{n-1}=A_n(1)\oplus A_n(Q_n 1)$. Therefore $a1 \neq 0$ in $H^*(BP\langle n\rangle)$. This takes care of exactness at $H^*(BP\langle n-1\rangle)$. So now $H^*(BP\langle n\rangle)=A_n \oplus X$ with $\beta^*: X \to X$ an isomorphism, but the degree of $\beta^* \neq 0$ so $X=0$.

All we need now is $Q_n 1 = 0 \in H^*(BP\langle n\rangle)$. Our map $BP \to BP\langle n\rangle$ is an isomorphism on homotopy below dimension $2(p^{n+1}-1)$ and therefore an isomorphism on cohomology in this range. $H^*(BP)\cong A/A(Q_0, Q_1, \ldots)$. [5] The dimension of Q_n is $2p^n-1$ so $Q_n 1=0$.

2. Postnikov Systems.

We collect here the results we need about Postnikov systems. We assume X is a simply connected CW complex. We start with the standard diagram:

$$
\begin{array}{ccccc}
 & & \vdots & & \\
 & & \downarrow & & \\
 & & X^n & & \\
 & \overset{\rho_n}{\nearrow} & \downarrow g_n & & \\
X & \underset{\rho_{n-1}}{\longrightarrow} & X^{n-1} & \overset{k_n}{\longrightarrow} & K(\pi_n(X), n+1) \\
 & & \downarrow & & \\
 & & \vdots & & \\
 & & \downarrow & & \\
 & & X^1 = \text{point} & &
\end{array}
$$

2.1. Definition and existence. [16] A Postnikov system for X is a sequence of spaces $\{X^n\}$ and maps, $\{g_n: X^n \to X^{n-1}\}$, $\{\rho_n: X \to X^n\}$ such that $\rho_{n-1} \cong g_n \cdot \rho_n$ and the fibre of g_n is $K(\pi_n(X), n)$, the Eilenberg-MacLane space. The fibration $g_n: X^n \to X^{n-1}$ is induced by a map $k_n: X^{n-1} \to K(\pi_n(X), n+1)$ from the path space of $K(\pi_n(X), n+1)$. Thus $k_n \cdot g_n \cong 0$ and $k_n \in H^{n+1}(X^{n-1}, \pi_n(X))$. k_n is called the n-th k-invariant of X. Postnikov systems for simply connected CW complexes always exist and $(\rho_n)_\#: \pi_k(X) \to \pi_k(X^n)$ is an isomorphism for $k \leqslant n$ and $\pi_k(X^n)=0$ for $k > n$.

2.2. Induced maps. [8] Given $f: X \to Y$ then we have $\{f^n: X^n \to Y^n\}$

such that $f^{n-1}\cdot g_n(X) \cong g_n(Y)\cdot f^n$, $f^n\cdot \rho_n(X) \cong \rho_n(Y)\cdot f$ and $f_\#(k_n(X)) = (f^{n-1})^*(k_n(Y))$.

2.3. Loop spaces. The Postnikov system for ΩX is given by: $(\Omega X)^n = \Omega X^{n+1}$, $\rho_n(\Omega X) = \Omega \rho_{n+1}(X)$, $g_n(\Omega X) = \Omega g_{n+1}(X)$, $k_n(\Omega X) = \Omega k_{n+1}(X)$, so $k_n(\Omega X) = s^*(k_{n+1}(X)) \in H^{n+1}(\Omega X^n, \pi_n(\Omega X))$ where s^* is the cohomology suspension defined by $\delta^{-1}\cdot p^*$.

$$H^*(\Omega X, G) \underset{\cong}{\overset{\delta}{\longrightarrow}} H^{*+1}(PX, \Omega X, G) \overset{p^*}{\longleftarrow} H^{*+1}(X, pt, G)$$

PX is the path space fibration over X.

2.4. Product spaces. A Postnikov system for $X \times Y$ is given by $\{X^n \times Y^n\}$ with k-invariants $\{k_n(X) \times k_n(Y)\}$.

2.5. H-spaces. [8] If X is an H-space, then each X^n is an H-space, ρ_n and g_n are maps of H-spaces and $k_n \in H^{n+1}(X^{n-1}, \pi_n(X))$ is torsion and is primitive in the Hopf algebra structure induced on H^* by the multiplication in X^{n-1}. Also, if X^{n-1} is an H-space and k_n is primitive, then X^n is an H-space. If all k-invariants are primitive, then X is an H-space.

2.6. Obstruction theory. [16] If Y is CW, and we have $f_{n-1}: Y \to X^{n-1}$, then f_{n-1} lifts to $f_n: Y \to X^n$ iff $(f_{n-1})^*(k_n(X)) = 0 \in H^{n+1}(Y, \pi_n(X))$. If there exist maps $\{f_n: Y \to X^n\}$ such that $g_n(X)\cdot f_n \cong f_{n-1}$ then there exists $f: Y \to X$ with $\rho_n(X)\cdot f \cong f_n$.

2.7. Construction of spaces. [16] Given a sequence of fibrations $g_n: X^n \to X^{n-1}$ with fibre $K(\pi_n, n)$ and $X^1 = pt$, then there exists a CW complex X and maps $\rho_n: X \to X^n$ such that $\{X^n\}$ is a Postnikov system for X.

2.8. Independent k-invariants. Assume for the rest of this section that $\pi_*(X) \otimes Z_{(p)}$ is free over $Z_{(p)}$ and the k-invariants $k_n(X)$ are torsion elements. This will always be the case in our applications. From the Serre spectral sequence of a fibration we obtain the following natural ladder of exact sequences:

$$\begin{array}{ccccccc}
0 \longrightarrow & H^s(X^{s-1}, Z_{(p)}) & \overset{(g_s)^*}{\longrightarrow} & H^s(X^s, Z_{(p)}) & \longrightarrow & H^s(K(\pi_s(X), s), Z_{(p)}) \\
& \downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow & H^s(X^{s-1}) & \overset{(\bar{g}_s)^*}{\longrightarrow} & H^s(X^s) & \longrightarrow & H^s(K(\pi_s(X), s))
\end{array}$$

2.9.

$$\begin{array}{ccccccc} \xrightarrow{\tau} & H^{s+1}(X^{s-1}, Z_{(p)}) & \xrightarrow{(g_s)^*} & H^{s+1}(X^s, Z_{(p)}) & \longrightarrow 0 \\ & \downarrow & & \downarrow & \\ \xrightarrow{\bar{\tau}} & H^{s+1}(X^{s-1}) & \xrightarrow{(\bar{g}_s)^*} & H^{s+1}(X^s) & \longrightarrow 0 \end{array}$$

τ is the transgression. Also we obtain

2.10. $H^k(X^s, G) \cong H^k(X, G)$ for $k \leqslant s$. In the dimension of our ladder, the transgression is related to the k-invariant map k_s by $\tau \cdot s^* = k_s^*$. This motivates the following definitions.

For $x \in H^s(K(\pi_s(X), s), Z_{(p)})$, a free generator, $\tau(x)$ will be called a k-invariant of X. If $\tau(x) = 0$, it is called dependent. The k-invariant $\tau(x)$ is independent and hits a p-torsion generator if and only if $\rho \cdot \tau(x) = \bar{\tau} \cdot \rho(x) \neq 0$ where ρ is the mod$p$ reduction. If the k-invariants, $\tau(x)$, of ΩX, hit p-torsion generators, then there is a y with $s^*(y) = x$, and so $s^*(\tau(y)) = \tau(x)$ showing that the k-invariants $\tau(y)$ of X also hit p-torsion generators. (Remember that we have restricted ourselves to spaces with torsion k-invariants.) If $H^*(X, Z_{(p)})$ has no p torsion, then all p torsion generators of $H^{s+1}(X^{s-1}, Z_{(p)})$ are hit by k-invariants. This is true because the coker $\tau \cong H^{s+1}(X^s, Z_{(p)}) \subset H^{s+1}(X^{s+1}, Z_{(p)}) \cong H^{s+1}(X, Z_{(p)})$ which is free, all by 2.9 and 2.10.

2.11. Localization. [18] Usually we will work with localized spaces, i.e. spaces with $\pi_*(X)$ a $Z_{(p)}$ module. For simply connected spaces or H-spaces, the localization $X_{(p)}$ and a mod$p$ equivalence $X \to X_{(p)}$ can be built by 2.7 using $\pi_*(X) \otimes Z_{(p)}$ for homotopy groups and $k_n(X) \otimes Z_{(p)}$ as the k-invariants. We get that

$$H_*(X, Z) \otimes Z_{(p)} \cong H_*(X, Z_{(p)}) \cong H_*(X_{(p)}, Z_{(p)}) \cong H_*(X_{(p)}, Z).$$

2.12. Irreducible spaces. If a space cannot be written as a non-trivial product of spaces it will be called irreducible (indecomposible). If X is connected and ΩX is irreducible, then X must also be irreducible. If X is a localized space with $\pi_*(X)$ free (over $Z_{(p)}$) then if X^{s-1} in the Postnikov system for X is irreducible and all of the s k-invariants are independent, then X^s is also irreducible.

3. The Map.

Before we can prove the main theorem,

$$BP_k \cong BP\langle n\rangle_k \times \prod_{j>n} BP\langle j\rangle_{k+2(p^j-1)}$$

for $k \leq 2(p^n + \cdots + p + 1)$, we need the maps $BP_k \to BP\langle j\rangle_{k+2(p^j-1)}$. The natural transformation $BP_*(\cdot) \to BP\langle n\rangle_*(\cdot)$ gives us the map $BP_k \to BP\langle n\rangle_k$ which is onto in homotopy. If we obtain the map $BP_k \to BP\langle j\rangle_{k+2(p^j-1)}$ for $k = 2(p^{j-1} + \cdots p + 1)$ then we have it for all $k \leq 2(p^{j-1} + \cdots + p + 1)$ by taking the loop map. We can then combine these maps to give a map

$$BP_k \to BP\langle n\rangle_k \times \prod_{j>n} BP\langle j\rangle_{k+2(p^j-1)} \qquad \text{for} \quad k \leq 2(p^n + \cdots + p + 1).$$

We fix $k = 2(p^{j-1} + \cdots + p + 1)$ and construct a map $BP_k \to BP\langle j\rangle_{k+2(p^j-1)}$ by the following series of lemmas.

LEMMA 3.1. *There is an element* $x_j \in H^{k+2(p^j-1)}(BP\langle j\rangle_k, Z_{(p)})$ *such that* $x_j : BP\langle j\rangle_k \to K(Z_{(p)}, k + 2(p^j - 1))$ *is onto in homotopy,* $k \leq 2(p^{j-1} + \cdots + p + 1)$.

Before proceeding, we need to state a lemma which we will prove later. Let i_k be the generator of $H^k(BP\langle j-1\rangle_k)$.

LEMMA 3.2. *For* $k > 2(p^{j-1} + \cdots + p + 1)$, $Q_j i_k \neq 0$ *in* $H^*(BP\langle j-1\rangle_k)$. *For* $k = 2(p^{j-1} + \cdots + p + 1)$, $H^i(BP\langle j-1\rangle_k) = 0$ *for* $i = k + 2p^j - 1 =$ *dimension* $Q_j i_k = pk + 1$.

Proof of 3.1. We go to the fibration 1.4.

$$\begin{array}{ccc} BP\langle j\rangle_s & \xrightarrow{\beta} & BP\langle j\rangle_k \\ s = k + 2(p^j - 1) & & \gamma\downarrow \\ k = 2(p^{j-1} + \cdots + p + 1) & & BP\langle j-1\rangle_k \end{array} \tag{3.3}$$

$BP\langle j\rangle_s$ is $s-1$ connected and $\pi_s(BP\langle j\rangle_s) \cong H^s(BP\langle j\rangle_s, Z_{(p)}) \cong Z_{(p)}$.

To show β^* is onto in dimension s we look at the Serre spectral sequence for the fibration 3.3. In this range we have the Serre exact sequence:

$$H^s(BP\langle j\rangle_k, Z_{(p)}) \xrightarrow{\beta^*} H^s(BP\langle j\rangle_s, Z_{(p)}) \longrightarrow H^{s+1}(BP\langle j-1\rangle_k, Z_{(p)}) \tag{3.4}$$

We have $k = 2(p^{j-1} + \cdots + p + 1)$ and so $s + 1 = k + 2(p^j - 1) + 1$. By 3.2 and the numbers we are using, the last term is zero and so β^* is onto. If $x_j : \in H^s(BP\langle j\rangle_k, Z_{(p)})$ is such that $\beta^*(x_j)$ is the generator, and $S^s \to BP\langle j\rangle_s$ represents $1 \in \pi_s(BP\langle j\rangle_s) = Z_{(p)}$, then the composition $S^s \to BP\langle j\rangle_s \to BP\langle j\rangle_k \xrightarrow{x_j} K(Z_{(p)}, s)$ induces an isomorphism on H^s, so therefore x_j is onto in homotopy.

LEMMA 3.5. *There is a map* $f_j : BP_k \to BP\langle j\rangle_{k+2(p^j-1)}$ *for* $k \leqslant 2(p^{j-1}+\cdots+p+1)$, *such that* $(f_j)_\#$ *is onto,* $(k \geqslant -2(p^j-1))$

$$(f_j)_\# : \pi_{k+2(p^j-1)}(BP_k) \to \pi_{k+2(p^j-1)}(BP\langle j\rangle_{k+2(p^j-1)}) \cong Z_{(p)}$$

Proof. It is enough to prove this for $k=2(p^{j-1}+\cdots+p+1)$. We have a map $BP_k \to BP\langle j\rangle_k \to K(Z_{(p)}, k+2(p^j-1))$ from lemma 3.1. Each of these maps is onto in homotopy so the composite is too. $K(Z_{(p)}, k+2(p^j-1))$ is the first non-trivial term of the Postnikov system for $BP\langle j\rangle_{k+2(p^j-1)}$. We know that the k-invariants of this space are torsion by 2.5 and that its homotopy is free over $Z_{(p)}$ by construction. (1.5) The main theorem of [20] gives us that $H^*(BP_k, Z_{(p)})$ has no torsion. Obstructions to lifting the map to a map of the type we want are therefore torsion elements in $H^{q+1}(BP_k, \pi_q(BP\langle j\rangle_k))$, (2.6) which has no torsion. Therefore we see that we can lift the map.

COROLLARY 3.6. *For* $k \leqslant 2(p^n+\cdots+p+1)$ *there is a map* $BP_k \to BP\langle n\rangle_k \times \coprod_{j>n} BP\langle j\rangle_{k+2(p^j-1)}$ *which composed with projections is onto in homotopy for* $\pi_*(BP\langle n\rangle_k)$ *and* $\pi_{k+2(p^j-1)}(BP\langle j\rangle_{k+2(p^j-1)})$.

Before proving 3.2 we will make an observation which we need in the next section.

Consider the map $\beta : BP\langle j\rangle_s \to BP\langle j\rangle_k$, $s=k+2(p^j-1)$. We have $\beta_\# : K(Z_{(p)}, s) \to K(\pi_s(BP\langle j\rangle_k), s)$. $(\beta_\#)^*$ is onto in $Z_{(p)}$ cohomology. Pick a generator $x \in H^s(K(\pi_s(BP\langle j\rangle_k), s), Z_{(p)})$ such that $(\beta_\#)^*(x)$ is a generator. We wish to study the k-invariant $\tau(x)$. Above we showed that for $k \leqslant 2(p^{j-1}+\cdots+p+1)$ there was such a k-invariant which was dependent. Here we wish to show the following lemma.

LEMMA 3.7. *For* $k > 2(p^{j-1}+\cdots+p+1)$, *the above k-invariant* $\tau(x)$ *is independent and hits a p-torsion generator.*

Proof. Using the naturality of the mod$p$ version of 2.9 we have:

$$\begin{array}{ccccccc}
 & & Z_p \cong H^s(BP\langle j\rangle_s) & & & & \\
 & & \uparrow\cong & & & & \\
0 & \longrightarrow & H^s(K(Z_{(p)}, s)) & \xrightarrow{\cong} & H^s(K(Z_{(p)}, s)) & \xrightarrow{\bar\tau} & 0 \\
 & & \beta^*\uparrow & & (\beta_\#)^*\uparrow & & \\
 & \longrightarrow & H^s((BP\langle j\rangle_k)^s) & \longrightarrow & H^s(K(\pi_s(BP\langle j\rangle_k), s)) & \xrightarrow{\bar\tau} & \\
 & & \downarrow\cong & & & & \\
 & & H^s(BP\langle j\rangle_k) & & & &
\end{array} \tag{3.8}$$

As in 2.8, $\tau(x)$ is independent and hits a p-torsion generator iff $\bar{\tau}(\rho(x))\neq 0$, ρ the mod$p$ reduction. Because $(\beta_{\#})^*(x)$ is a generator, this is equivalent to β^* not being onto in 3.8. Again we go to the Serre exact sequence for 1.4.

$$H^s(BP\langle j\rangle_k)\xrightarrow{\beta^*}H^s(BP\langle j\rangle_s)\xrightarrow{\bar{\tau}}H^{s+1}(BP\langle j-1\rangle_k) \tag{3.9}$$

We know from the proof of 1.7 that for k very large $\bar{\tau}(i_s)=\lambda Q_j i_k, \lambda\neq 0$. By 3.2, for $k>2(p^{j-1}+\cdots+p+1)$ we know $Q_j i_k\neq 0$ so $\bar{\tau}(i_s)=\lambda Q_j i_k\neq 0$ in this range. So, in 3.9 we see that β^* is not onto and $\tau(x)$ for such an x is an independent p-torsion generating k-invariant.

We will need the following in our proof of 3.2.

LEMMA 3.10. $Q_{n+1}=\lambda\beta P^{p^n+\cdots+p+1}(\mathrm{mod}\,A(Q_0,\ldots,Q_n))$, $\lambda\neq 0\in Z_p$.

Note. For $p=2$, just consider $P^i=Sq^{2i}$.

Proof. The lowest non-zero odd dimensional element of $A/A(Q_0,\ldots,Q_n)$ is Q_{n+1}. From [9], $Q_i=[P^{p^{i-1}},Q_{i-1}]$, so $Q_{n+1}=P^{p^n}Q_n-Q_nP^{p^n}=-Q_nP^{p^n}=-(P^{p^{n-1}}Q_{n-1}-Q_{n-1}P^{p^{n-1}})P^{p^n}=Q_{n-1}P^{p^{n-1}}P^{p^n}$ (as the dimension of $Q_{n-1}P^{p^n}$ is less than the dimension of Q_{n+1} and also odd, so it is zero)$=\cdots=(-1)^{n+1}Q_0P^1\cdots P^{p^{n-1}}P^{p^n}(\mathrm{mod}\,A(Q_0,\ldots,Q_n))$. Let $k_n=1+p+\cdots+p^{n-1}$, all that is left to show is: (note that $Q_0=\beta$)

Claim. $P^{k_n}P^{p^n}=\lambda P^{k_{n+1}}$, $\lambda\neq 0\in Z_p$.

Proof. By the Adem relations,

$$P^{k_n}P^{p^n}=\sum_{t=0}^{k_{n-1}}(-1)^{k_n+t}P^{k_{n+1}-t}P^t\binom{(p-1)(p^n-t)-1}{k_n-pt}$$

So all we need is for the binomial coefficient to be zero mod$p$ for $0<t\leqslant k_{n-1}$ and $\neq 0$ for $t=0$. First we reindex, let $s+t=k_{n-1}$. Then $(p-1)(p^n-t)-1=(p-1)(p^n-k_{n-1}+s)-1=(p-1)p^n-(p-1)k_{n-1}+(p-1)s-1$. Now $(p-1)k_{n-1}=p^{n-1}-1$, so this is $(p-1)p^n-p^{n-1}+1+(p-1)s-1=(p-2)p^n+(p-1)p^{n-1}+(p-1)s$. $k_n-pt=k_n-pk_{n-1}+ps=1+ps$. So our coefficient is:

$$\binom{(p-2)p^n+(p-1)p^{n-1}+(p-1)s}{1+ps}$$

We want to show this is 0 for $0\leqslant s<k_{n-1}$ and $\neq 0$ for $s=k_{n-1}$.

From [17], if $a=\Sigma a_ip^i$, $b=\Sigma b_ip^i$, a_i and $b_i<p$, then mod$p$ $\binom{a}{b}=\Pi_i\binom{a_i}{b_i}$.

So for $s<p^{n-2}$ our binomial coefficient is $\binom{p-2}{0}\binom{p-1}{0}\binom{(p-1)s}{1+ps}$ but $(p-1)s<1+ps$, so it is zero for $s<p^{n-2}$. Set $s=p^{n-2}+s_1\geqslant p^{n-2}$. We get:

$$\binom{(p-2)p^n+(p-1)p^{n-1}+(p-1)p^{n-2}+(p-1)s_1}{p^{n-1}+1+ps_1}$$

$$=\binom{p-2}{0}\binom{p-1}{1}\binom{p-1}{0}\binom{(p-1)s_1}{1+ps_1}$$

for $s_1<p^{n-3}$ this is zero again. Let $s_1=p^{n-3}+s_2$. Continue like this until we get:

$$\binom{(p-2)p^n+(p-1)p^{n-1}+\cdots+(p-1)p+(p-1)s_{n-2}}{p^{n-1}+p^{n-2}+\cdots+p^2+1+ps_{n-2}}$$

where $0\leqslant s_{n-2}\leqslant 1$. For $s_{n-2}=0$, this is zero again as it is

$$=\binom{p-2}{0}\binom{p-1}{1}\cdots\binom{p-1}{1}\binom{p-1}{0}\binom{0}{1}$$

For $s_{n-2}=1$, we get

$$\binom{p-2}{0}\binom{p-1}{1}\cdots\binom{p-1}{1}\binom{p-1}{1}=(-1)^n$$

This finishes the proof of 3.10.

Proof of 3.2. For large k, $Q_j i_k\neq 0$ in $H^*(BP\langle j-1\rangle_k)$ because $H^*(BP\langle j-1\rangle)=A/A(Q_0,\ldots,Q_{j-1})$. (1.7) The Eilenberg-Moore spectral sequence [15]

$$\mathrm{Tor}_{H^*(BP\langle j-1\rangle_{k+1})}(Z_p,Z_p)=>H^*(BP\langle j-1\rangle_k) \tag{3.11}$$

collapses in dimensions $<pk$ and on indecomposibles, $s^*:QH^*(BP\langle j-1\rangle_{k+1})\to QH^*(BP\langle j-1\rangle_k)$ is an isomorphism in this range. For $k>2(p^{j-1}+\cdots+p+1)$, dimension $Q_j i_k<pk$ so $Q_j i_k\neq 0$.

For $k=2(p^{j-1}+\cdots+p+1)$, the E_2 term of 3.10 has one element in dimension $pk+1$ (for $p=2$, none), $s^{-1}(Q_j i_{k+1})$. All $Q_i i_{k+1}=0$ for $i<j$ so by 3.10 this is $s^{-1}(\lambda\beta P^{p^{j-1}+\cdots+p+1}i_{k+1})$. s^{-1} corresponds to the cohomology suspension ([14]). $s^*(\beta P^{p^{j-1}+\cdots+p+1}i_{k+1})=\beta P^{k/2}i_k=\beta(i_k)^p=0$, so $s^{-1}(Q_j i_{k+1})$ is hit by a differential and the result follows.

4. Proofs.

In the last section we constructed a map (3.6)

$$BP_k \longrightarrow BP\langle n\rangle_k \times \prod_{j>n} BP\langle j\rangle_{k+2(p^j-1)} \qquad \text{for} \quad k \leqslant 2(p^n + \cdots + p + 1) \quad (4.1)$$

If this map is a homotopy equivalence for some $k>0$ then it is a homotopy equivalence for all $k \leqslant 2(p^n + \cdots + p+1)$. To see this, look at the diagram for $f: X \to Y$

$$\begin{array}{ccc} \pi_*(\Omega X) & \cong & \pi_{*+1}(X) \\ \downarrow \Omega f_\# & & \downarrow f_\# \\ \pi_*(\Omega Y) & \cong & \pi_{*+1}(Y) \end{array}$$

If either $\Omega f_\#$ or $f_\#$ is an isomorphism then so is the other and then they are both homotopy equivalences because our spaces are the homotopy type of *CW* complexes.

We will prove the homotopy equivalences for the

$$k = k_n = 2(p^{n-1} + \cdots + p + 1) + 1, \ (k_0 = 1)$$

by induction on the Postnikov system. As a plausibility argument, as well as the fact that we need it, we prove the following lemma.

LEMMA 4.2. *The homotopy is the same on both sides of* 4.1.

Proof. $\pi_*(BP_k) \cong s^k(Z_{(p)}[x_1, x_2, \ldots])$.

$$\pi_*\left(BP\langle n\rangle_k \times \prod_{j>n} BP\langle j\rangle_{k+2(p^j-1)}\right) = \pi_*(BP\langle n\rangle_k) \bigoplus_{j>n} \pi_*(BP\langle j\rangle_{k+2(p^j-1)})$$

$$= s^k(Z_{(p)}[x_1, \ldots, x_n]) \bigoplus_{j>n} s^{k+2(p^j-1)}(Z_{(p)}[x_1, \ldots, x_j]) \quad (1.5)$$

Our isomorphism takes a $Z_{(p)}$ generator on the right-hand side, $s^{k+2(p^j-1)}[(x_1)^{i_1} \cdots (x_j)^{i_j}]$ to $s^k[(x_1)^{i_1} \cdots (x_{j-1})^{i_{j-1}}(x_j)^{i_j+1}]$.

Recall that $k_n = 2(p^{n-1} + \cdots + p + 1) + 1$ $(k_0 = 1)$.

Statement P(n,s). $i = k_n + 2(p^j - 1)$

$$f^{k_n+s}: (BP_{k_n})^{k_n+s} \longrightarrow (BP\langle n\rangle_{k_n})^{k_n+s} \times \prod_{j>n} (BP\langle j\rangle_i)^{k_n+s}$$

is a homotopy equivalence.

4.3. Statement $P(n,s)$ implies a similar statement for any $k<k_{n+1}$ replacing k_n.

Statement $K(n,s)$. All k-invariants $\tau(x)$ in

$$H^{k_n+s+2}\Big(\big(BP\langle n\rangle_{k_n}\big)^{k_n+s}, Z_{(p)}\Big)$$

are independent and hit p-torsion generators. (see 2.8)

4.4. $K(n,s)$ implies that all k-invariants $\tau(x)$ in $H^{k+s+2}((BP\langle n\rangle_k)^{k+s}, Z_{(p)})$ are independent and hit p-torsion generators for $k \geqslant k_n$.

Statement A.

$$\left.\begin{matrix} P(n,s) & s \leqslant m \\ K(n,s) & s \leqslant m \end{matrix}\right\} => K(n+1,m)$$

Statement B.

$$\begin{matrix} (1) & K(n+j,s) & s \leqslant m \quad j \geqslant 0 \\ (2) & P(n,m) & \\ & & => P(n,m+1) \end{matrix}$$

4.5. Now, to get things started, observe that statement $P(n,0)$ is true for all n as it just reduces to $K(Z_{(p)},k_n) \xrightarrow{\cong} K(Z_{(p)},k_n)$. Also, statement $K(0,s)$ is trivially true for all s because $BP\langle 0\rangle_{k_0=1}$ is just the circle localized at p and has no k-invariants.

LEMMA 4.6. *Statements A and B imply statements $P(n,s)$ and $K(n,s)$ for all n and s.*

Proof. Claim (t). (a) $P(n,m)$ is true for $m \leqslant t$, all n.
(b) $K(n,m)$ is true for $m < t$, all n.

Claim (t) is true for $t=0$ by 4.5. We will show claim $(t) =>$ claim $(t+1)$. By 4.5 we know $K(0,t)$ is true, applying statement A n times we have $K(n,t)$, therefore we have $K(n,t)$ for all n giving us b) of claim $(t+1)$. Now, applying statement B we obtain $P(n,t+1)$ for all n. This proves claim $(t+1)$, so, by induction, claim (t) is true for all t and we are done.

Now we will prove statements A and B. In the next section we will explore some of the consequences of $P(n,s)$ and $K(n,s)$.

Proof of statement A. Consider the fibration 1.4

$$\begin{array}{ccc} BP\langle n+1\rangle_i & \xrightarrow{\beta} & BP\langle n+1\rangle_k \\ & & \downarrow \gamma \qquad k=k_{n+1} \\ i=k_{n+1}+2(p^{n+1}-1) & & BP\langle n\rangle_k \end{array}$$

and the induced maps on the Postnikov systems: $q=k+s+1$

$$\begin{array}{ccccc} (BP\langle n+1\rangle_i)^{k+s} & \xrightarrow{\beta^{q-1}} & (BP\langle n+1\rangle_k)^{k+s} & \xrightarrow{\gamma^{q-1}} & (BP\langle n\rangle_k)^{k+s} \\ \downarrow k' & & \downarrow k & & \downarrow k'' \\ K(\pi_q(BP\langle n+1\rangle_i),q+1) & \xrightarrow{\beta_\#} & K(\pi_q(BP\langle n+1\rangle_k),q+1) & \xrightarrow{\gamma_\#} & K(\pi_q(BP\langle n\rangle_k),q+1) \end{array} \tag{4.7}$$

$\beta_\#$ and $\gamma_\#$ give the split short exact sequence 1.2, 1.5. We know that the k-invariants in $H^{q+1}((BP\langle n\rangle_k)^{k+s}, Z_{(p)})$ are independent and hit p torsion generators for $s\leqslant m$ by statement $K(n,s)$, $s\leqslant m$ of A and comment 4.4; equivalently, $(\bar\tau'')_q$ is injective:

$$\begin{array}{ccc} H^q(K(\pi_q(BP\langle n\rangle_k),q)) & \xrightarrow{(\bar\tau'')_q} & H^{q+1}((BP\langle n\rangle_k)^{k+s}) \\ (\gamma_\#)^*\downarrow & & (\gamma^{q-1})^*\downarrow \\ H^q(K(\pi_q(BP\langle n+1\rangle_k),q)) & \xrightarrow{(\bar\tau)_q} & H^{q+1}((BP\langle n+1\rangle_k)^{k+s}) \\ (\beta_\#)^*\downarrow & & (\beta^{q-1})^*\downarrow \\ H^q(K(\pi_q(BP\langle n+1\rangle_i),q)) & \xrightarrow{(\bar\tau')_q} & H^{q+1}((BP\langle n+1\rangle_i)^{k+s}) \end{array} \tag{4.8}$$

Assume for a moment that γ^{q-1} pulls these k-invariants in $H^{q+1}((BP\langle n\rangle_k)^{k+s}, Z_{(p)})$ back to independent p-torsion generating k-invariants in $H^{q+1}((BP\langle n+1\rangle_k)^{k+s}, Z_{(p)})$, i.e. $(\bar\tau)_q\cdot(\gamma_\#)^*=(\gamma^{q-1})^*\cdot(\bar\tau'')_q$ is injective in 4.8. Then the first possible dependent k-invariant is of the type discussed in 3.7. There, it was shown to be an independent p-torsion generating k-invariant. Assume for some minimum $s\leqslant m$ that we have a dependent k-invariant, or one which is not a p-torsion generator, equivalently, assume there is an $x\in\ker(\bar\tau)_q$, therefore $q>i$. By what we have assumed about the k-invariants pulling back, x is not in the image of $(\gamma_\#)^*$. Thus, by the split exactness of homotopy, and therefore the $(\gamma_\#)^*$, $(\beta_\#)^*$ sequence of 4.8, $(\beta_\#)^*(x)=y\neq 0$. Now using the result 2.3 about the k-invariants of loop spaces, $(s^*)^r\cdot(\bar\tau')_q=(\bar\tau)_{q-r}\cdot(s^*)^r$, $r=i-k$ $=2(p^{n+1}-1)$. By our minimality assumption on s, $(\bar\tau)_{q-r}$ is injective so $0\neq(\bar\tau)_{q-r}\cdot(s^*)^r(y)=(s^*)^r\cdot(\bar\tau')_q(y)=(s^*)^r\cdot(\bar\tau')_q\cdot(\beta_\#)^*(x)=(s^*)^r\cdot(\beta^{q-1})^*\cdot(\bar\tau)_q(x)$, contradicting $(\bar\tau)_q(x)=0$.

All we need now is to show that $(\bar{\tau})_q \cdot (\gamma_\#)^* = (\gamma^{q-1})^* \cdot (\bar{\tau}'')_q$ is injective. We have the maps $(k+s+1=q)$

$$(BP_k)^{k+s} \xrightarrow{F^{q-1}} (BP\langle n+1\rangle_k)^{k+s} \xrightarrow{\gamma^{q-1}} (BP\langle n\rangle_k)^{k+s}$$

If we show that $(F^{q-1})^* \cdot (\gamma^{q-1})^* \cdot (\bar{\tau}'')_q$ is injective, we will be through. Using statement $P(n,s)$, $s \leqslant m$, from our given in A, we see that this is true if $G^* \cdot (\bar{\tau}'')_{k_n+s+1}$ is injective (as $k = k_{n+1} > k_n$), G the projection:

$$(BP\langle n\rangle_{k_n})^{k_n+s} \times \prod_{j>n} (BP\langle j\rangle_k)^{k_n+s} \to (BP\langle n\rangle_{k_n})^{k_n+s}$$

This follows trivially from statement $K(n,s)$, $s \leqslant m$.

Proof of Statement B. By (1) of statement B and 2.4 on k-invariants of product spaces, all of the k-invariants on the right hand side of $P(n,m)$ are independent and hit p-torsion generators except possibly a zero k-invariant if $m = 2p^j - 3$ which corresponds by construction (3.6) to a dependent k-invariant on the left hand side of $P(n,m)$. Now by $P(n,m)$, $(f^{k_n+m})^*$ is an isomorphism and so pulls back all of the independent p-torsion generating k-invariants to independent p-torsion generating k-invariants in $(BP_{k_n})^{k_n+m}$. This determines all of the k-invariants on the left hand side because we know the homotopy is the same on both sides (4.2). So by this, (and 3.6 if $m = 2p^j - 3$) $f_\#$ on $\pi_{k_n} + m + 1$ must be an isomorphism. Thus $(f^{k_n+m+1})_\#$ is an isomorphism on π_* giving us $P(n, m+1)$.

5. Statement of Results.

In section 4 we proved the main theorem: $k \leqslant 2(p^n + \cdots + p + 1)$

$$BP_k \cong BP\langle n\rangle_k \times \prod_{j>n} BP\langle j\rangle_{k+2(p^j-1)}$$

The main theorem of [20] says: The $Z_{(p)}$ (co)homology of the connected part of BP_k has no torsion and is a polynomial algebra for k even and an exterior algebra for k odd. The map above is a map of H-spaces for $k < 2(p^n + \cdots + p + 1)$ so we have the following corollary.

COROLLARY 5.1. *For $k < 2(p^n + \cdots + p + 1)$, the $Z_{(p)}$ (co)homology of the connected part of $BP\langle n\rangle_k$ has no torsion and is a polynomial algebra for k even and an exterior algebra for k odd. For $k = 2(p^n + \cdots + p + 1)$, $H^*(BP\langle n\rangle_k, Z_{(p)})$ has no torsion and is a polynomial algebra. (Note that for $k > 0$ or k odd < 0, $BP\langle n\rangle_k$ is connected.)*

Note. For $k=2(p^n+\cdots+p+1)$, $H_*(BP\langle n\rangle_k, Z_{(p)})$ is not a polynomial algebra.

At the rationals, the space $BP\langle n\rangle_k$ is just a product of Eilenberg-MacLane spaces. So, since there is no torsion, the number of generators over $Z_{(p)}$ is the same as over Q. As an example, for k even, $0<k\leqslant 2(p^n+\cdots+p+1)$, we have

$$H^*(BP\langle n\rangle_k, Z_{(p)}) = Z_{(p)}[s^k\pi_*^S(BP\langle n\rangle)] = Z_{(p)}[s^k(Z_{(p)}[x_1,\ldots,x_n])]$$

For k even and less than $2(p^n+\cdots+p+1)$, the (co)homology Hopf algebras of 5.1 are bipolynomial, that is, both it and its dual are polynomial algebras. Such Hopf algebras are studied in [13]. There, such a Hopf algebra is shown to be isomorphic to a tensor product of the Hopf algebras $B_{(p)}[x,2d]$ studied in [7]. $B_{(p)}[x,2d]$, as an algebra, is a polynomial algebra over $Z_{(p)}$ on generators $a_k(x)$ of degree $2p^kd$. As a Hopf algebra it is isomorphic to its own dual.

Letting $R(n,k)$ be the set of all n-tuples of non-negative integers, $R=(r_1,\ldots,r_n)$ with $d(R)=2k+\Sigma 2(p^i-1)r_i$. R is called prime if it cannot be written $R=pR'+(k,0,\ldots,0)$ with $R'\in R(n,k)$. Then, as a further example, we have the following corollary from [13] and the counting done above.

COROLLARY 5.2. *For $0<k<p^n+\cdots+p+1$ as Hopf algebras:*

$$H^*(BP\langle n\rangle_{2k}, Z_{(p)}) \cong \bigotimes_{\substack{R\in R(n,k)\\ R\text{ prime}}} B_{(p)}[x_R, d(R)]$$

We now utilize statement $K(n,s)$; all k-invariants $\tau(x)$ in $H^{k_n+s+2}((BP\langle n\rangle_{k_n})^{k_n+s}, Z_{(p)})$ are independent and hit p torsion generators, $k_n=2(p^{n-1}+\cdots+p+1)+1$. This implies that $BP\langle n\rangle_{k_n}$ cannot be written as a non-trivial product. (2.12)

COROLLARY 5.3. *For $k>2(p^{n-1}+\cdots+p+1)$, $BP\langle n\rangle_k$ is irreducible.*

Using the fact that $k_n+2(p^j-1)\geqslant k_j$ for $j>n$ we have now completed the proof of the main theorem.

THEOREM 5.4. *For $k\leqslant 2(p^n+\cdots+p+1)$*

$$BP_k \cong BP\langle n\rangle_k \times \prod_{j>n} BP\langle j\rangle_{k+2(p^j-1)}$$

and for $k>2(p^{n-1}+\cdots+p+1)$, this decomposition is as irreducibles.

Note. For $k<2(p^n+\cdots+p+1)$ this is as H-spaces.

Now letting $k\leqslant 2(p^{n-1}+\cdots+p+1)$ and using two versions of 5.4 we have

$$BP_k \cong BP\langle n\rangle_k \times \text{OTHER}$$

and

$$BP_k \cong BP\langle n-1\rangle_k \times BP\langle n\rangle_{k+2(p^n-1)} \times \text{OTHER}$$

From this we get the following corollary.

COROLLARY 5.5. *For* $k \leqslant 2(p^{n-1}+\cdots+p+1)$

$$BP\langle n\rangle_k \cong BP\langle n-1\rangle_k \times BP\langle n\rangle_{k+2(p^n-1)}$$

Note. For $k<2(p^{n-1}+\cdots+p+1)$ this is as H-spaces.

This gives us the point where the fibration 1.4 becomes trivial. Again, using $BP_k \cong BP\langle n\rangle_k \times \text{OTHER}$ for $k \leqslant 2(p^n+\cdots+p+1)$ and the fact that for finite complexes $BP\langle n\rangle^k(X)=0$ for high k we get 5.6.

COROLLARY 5.6. (i) $BP^k(X) \to BP\langle n\rangle^k(X)$ *is onto for* $k \leqslant 2(p^n+\cdots+p+1)$. (ii) $BP^*(X) \to BP\langle n\rangle^*(X)$ *is onto in all but a finite number of dimensions.*

We now apply 5.6 to prove Quillen's Theorem. The problem was first studied in [6].

THEOREM 5.7 (Quillen). *Let X be a finite CW complex, then $BP^*(X)$ is generated as a $BP^*(S^0)$ module by elements of non-negative degree.*

Proof. If $u \in BP^k(X)$ and $k<0$, we will show u is a finite sum $\Sigma_{i>0} x_i u_i = u$, $u_i \in BP^{k+2(p^i-1)}(X)$ and $x_i \in BP^*(S^0) = Z_{(p)}[x_1,\ldots,x_i,\ldots]$ of degree $-2(p^i-1)$. By downward induction on the degree of u we will be done.

Consider the maps

$$BP^*(X) \xrightarrow{g_n} BP\langle n\rangle^*(X) \xrightarrow{f_n} BP\langle n-1\rangle^*(X)$$

Find n such that $g_n(u) \neq 0$ but $f_n \cdot g_n(u) = g_{n-1}(u) = 0$. Such an n exists because $n=0$ gives $g_0(u) \in H^k(X, Z_{(p)}) = 0$ as $k<0$, and for n high enough $BP^k(X) \cong BP\langle n\rangle^k(X)$, by the finiteness of X.

Dual to 1.1 we have an exact sequence and commuting diagram:

$$\begin{array}{ccccc}
BP^{k+2(p^n-1)}(X) & \xrightarrow{x_n} & BP^k(X) & & \\
{\scriptstyle g_n}\downarrow & & {\scriptstyle g_n}\downarrow & & \\
BP\langle n\rangle^{k+2(p^n-1)}(X) & \xrightarrow{x_n} & BP\langle n\rangle^k(X) & \xrightarrow{f_n} & BP\langle n-1\rangle^k(X) \\
& & u' \longrightarrow g_n(u) \longrightarrow 0 & &
\end{array}$$

As $f_n(g_n(u))=0$ there exists u' with $x_n u'=g_n(u)$ by exactness. But now, by 5.6 and $2(p^n+\cdots+p+1)\geqslant k+2(p^n-1)$ for $k<0$ we have that g_n is onto in dimension $k+2(p^n-1)$ and so pick $u_n\in BP^{k+2(p^n-1)}(X)$ with $g_n(u_n)=u'$. Then by commutativity, $g_n(x_n u_n)=g_n(u)$. Now continue this process using $u-x_n u_n$. By the finiteness of X, $BP^{k+2(p^j-1)}(X)$ will be zero for large j and we will get our finite sum $u=\sum_{i>0}x_i u_i$ and be done.

The spaces $BP\langle n\rangle_k$ are most useful in the range $2(p^{n-1}+\cdots+p+1)<k\leqslant 2(p^n+\cdots+p+1)$ where they are both irreducible and torsion free. In the next section, we will identify these with spaces that have perhaps a more tangible description.

6. Torsion free H-spaces.

All modules will be over $Z_{(p)}$, and, until further notice, all coefficients will be $Z_{(p)}$. In this section we will study torsion free H-spaces. Our immediate goal is to construct and study the following spaces.

PROPOSITION 6.1. *There exists an irreducible $k-1$ connected H-space Y_k which has $H^*(Y_k)$ and $\pi_*(Y_k)$ both free over $Z_{(p)}$ and such that each stage of the Postnikov system is irreducible.*

Proof. We will build up a Postnikov system for Y_k and use 2.7. We drop the subscript k. Clearly we must start the Postnikov system with $Y^k=K(Z_{(p)},k)$. We will now just build up a Postnikov system by killing off the torsion in cohomology as efficiently as possible. $\pi_*(Y^k)$ is free over $Z_{(p)}$ and $H^j(Y^k)$ has no torsion for $j\leqslant k+1$. Y^k is an H-space. Assume we have constructed the $s-1$ stage, Y^{s-1} for $s>k$ such that $\pi_*(Y^{s-1})$ is free and $H^j(Y^{s-1})$ has no torsion for $j\leqslant s$. Assume also that Y^{s-1} has an H-space structure. $H^{s+1}(Y^{s-1})\cong F\oplus T$ where F is the free part and T is the torsion part. It is finitely generated so it is isomorphic to $(Z_{(p)})^{n_0}\oplus_{i>0}(Z_{p^i})^{n_i}$, where $(G)^n=G\oplus\cdots\oplus G$ n times. Using the torsion generators, this isomorphism determines a map:

$$Y^{s-1}\longrightarrow \prod_{n=\sum_{i>0} n_i \text{ times}} K(Z_{(p)},s+1)=K(F_n,s+1),\qquad F_n=(Z_{(p)})^n$$

Let this map be the s k-invariant, k_s. This constructs the space Y^s as the induced fibration. k_s is torsion and so it is primitive because there is no torsion in lower dimensions, therefore by 2.5, Y^s is an H-space. Recall the $Z_{(p)}$

sequence 2.9

$$0 \longrightarrow H^s(Y^{s-1}) \xrightarrow{(g_s)^*} H^s(Y^s) \longrightarrow H^s(K(F_n, s))$$

$$\xrightarrow{\tau} H^{s+1}(Y^{s-1}) \longrightarrow H^{s+1}(Y^s) \longrightarrow 0$$

Using $k_s^* = \tau \cdot s^*$ we see that all of our "k-invariants" $\tau(x)$ are independent and hit p-torsion generators by construction. Coker $(g_s)^*$ is a subgroup of a free group and so is free giving us:

$$0 \longrightarrow H^s(Y^{s-1}) \xrightarrow{(g_s)^*} H^s(Y^s) \longrightarrow \mathrm{coker}(g_s)^* \longrightarrow 0$$

with both ends free by our induction hypothesis. Therefore $H^s(Y^s)$ is free. $H^{s+1}(Y^s)$ is coker $\tau = \mathrm{coker}(k_s)^*$ which by construction is F, so free. By the isomorphism 2.10, $H^j(Y^s) \cong H^j(Y^{s-1})$, $j < s$, we have $H^j(Y^s)$ is free for $j \leqslant s+1$. Also $\pi_*(Y^s)$ is free by construction. Because we have used the minimum number of $Z_{(p)}$'s for $\pi_*(Y^s)$, if Y^{s-1} is irreducible, then so is Y^s. (2.12)

THEOREM 6.2. *If X is a simply connected CW H-space with $\pi_*(X)$ and $H^*(X, Z_{(p)})$ free and locally finitely generated over $Z_{(p)}$, then $X \cong \Pi_i Y_{k_i}$.*

Remark 1. The simply connected assumption is not necessary because one can just split off a bunch of circles localized at p. $Y_1 = (S^1)_{(p)}$. Then, what is left is still an H-space, see the next remark.

Remark 2. The reason for the H-space hypothesis is that we want torsion k-invariants (2.5). Since spaces with π_* and H^* free are H-spaces if their k-invariants are torsion we could have used the hypothesis that X must have torsion k-invariants instead. Note that our homotopy equivalence is not as H-spaces.

Proof of 6.2. As always, we do everything by induction on the Postnikov system, but first we need the map $X \to \Pi_i Y_{k_i}$. The construction is similar to that for the main theorem except easier because X is only a theoretical space. We revert back to mod p cohomology for the proof. We start with the mod p version of the sequence 2.9.

$$0 \longrightarrow H^s(X^{s-1}) \xrightarrow{g^*} H^s(X^s) \xrightarrow{i^*} H^s(K(\pi_s(X), s))$$

$$\xrightarrow{\bar{\tau}} H^{s+1}(X^{s-1}) \longrightarrow H^{s+1}(X^s) \longrightarrow 0 \qquad (6.3)$$

Choose $V^s \subset H^s(X^s) = H^s(X)$ such that $i^* : V^s \to \ker \bar{\tau}$. Let r_s be the rank of V^s. This determines a map $f_s' : X \to K((Z_p)^{r_s}, s)$. $H^*(X, Z_{(p)})$ has no p torsion so f_s' lifts first into the product of r_s copies of $K(Z_{(p)}, s)$ and then into the product of r_s copies of Y_s, denoted rY_s. (It lifts by 2.6 because Y_s has p-torsion k-invariants and free homotopy.) So we have $f_s : X \to rY_s$ such that the image of $(f_s)^*$ in dimension s is V^s. Let $f = \prod_s f_s : X \to \prod_s rY_s = Y$.

Claim. f is a homotopy equivalence.

Proof. By induction on the Postnikov system assume $f^s : X^s \to Y^s = \prod_{k \leqslant s}(rY_k)^s$ is a homotopy equivalence. ($X^1 = Y^1 = pt$) Let $f_\#$ be the induced map $K(\pi_{s+1}(X), s+1) \to K(\pi_{s+1}(Y), s+1)$. f^s is a homotopy equivalence, if $f_\#$ is too, then ${f^{s+1}}_\# : \pi_*(X^{s+1}) \to \pi_*(Y^{s+1})$ is an isomorphism and so f^{s+1} is a homotopy equivalence. $f_\#$ is a homotopy equivalence iff $(f_\#)^* : H^{s+1}(K(\pi_{s+1}(Y), s+1)) \to H^{s+1}(K(\pi_{s+1}(X), s+1))$ is an isomorphism. Now

$$K(\pi_{s+1}(Y), s+1) = K\left(\pi_{s+1}\left(\prod_{k \leqslant s} rY_k\right), s+1\right) \times K\left((Z_{(p)})^{r_{s+1}}, s+1\right) = K \times K'$$

and $H^{s+1}(K \times K') = H^{s+1}(K) \oplus H^{s+1}(K')$. $\operatorname{Ker} \bar{\tau}_Y = H^{s+1}(K')$ by the construction of the Y_k, i.e., all k-invariants are independent and hit p-torsion generators. Using the naturality of 2.9 we have:

$$\begin{array}{ccccccc}
\longrightarrow & H^{s+1}(X^{s+1}) & \xrightarrow{i_X^*} & H^{s+1}(K(\pi_{s+1}(X), s+1)) & \xrightarrow{\bar{\tau}} & H^{s+2}(X^s) & \longrightarrow \\
& \uparrow {\scriptstyle f^* = (f^{s+1})^*} & & {\scriptstyle (f_\#)^*}\uparrow & & \uparrow {\scriptstyle (f^s)^*} & \\
\longrightarrow & H^{s+1}(Y^{s+1}) & \xrightarrow{i_Y^*} & H^{s+1}(K(\pi_{s+1}(Y), s+1)) & \xrightarrow{\bar{\tau}} & H^{s+2}(Y^s) & \longrightarrow \\
& \cong\uparrow & & \cong\uparrow & & & \\
& H^{s+1}\left(\prod_{k \leqslant s} rY_s\right) \oplus H^{s+1}(rY_{s+1}) & \longrightarrow & H^{s+1}(K) \oplus H^{s+1}(K') & \longrightarrow & &
\end{array} \qquad (6.4)$$

Now $(i_Y)^* : H^{s+1}(rY_{s+1}) \to \ker \bar{\tau}_Y = H^{s+1}(K')$ and by construction of f_{s+1}, $f^* = (f^{s+1})^* : H^{s+1}(rY_{s+1}) \to V^{s+1}$. By commutativity, $(f_\#)^* : \ker \bar{\tau}_Y \to \ker \bar{\tau}_X$. $\bar{\tau}_Y | H^{s+1}(K)$ is injective and by our construction of the Y_k it hits every possible element in cohomology, i.e., all that reduce from torsion elements in $Z_{(p)}$ cohomology. f^s is a homotopy equivalence by induction so by commutativity of 6.4 $\bar{\tau}_X$ also hits all possible elements and we have isomorphism on the ends of diagram 6.5 giving us the desired isomorphism by the five lemma.

$$\begin{array}{ccccccc}
0 \longrightarrow & V^{s+1} & \longrightarrow & H^{s+1}(K(\pi_{s+1}(X), s+1)) & \longrightarrow & \text{image } \bar{\tau}_X & \longrightarrow 0 \\
& \uparrow\cong & & {\scriptstyle (f_\#)^*}\uparrow & & {\scriptstyle (f^s)^*}\uparrow & \\
0 \longrightarrow & H^{s+1}(rY_{s+1}) & \longrightarrow & H^{s+1}(K') \oplus H^{s+1}(K) & \longrightarrow & \text{image } \bar{\tau}_Y & \longrightarrow 0
\end{array} \qquad (6.5)$$

COROLLARY 6.6. *Y_k as in 6.1 is unique up to homotopy type.*

COROLLARY 6.7. *Any map $Y_k \to Y_k$ which induces an isomorphism on $\pi_k(Y_k) = Z_{(p)}$ is a homotopy equivalence.*

COROLLARY 6.8. *For $2(p^{n-1} + \cdots + p + 1) < k \leqslant 2(p^n + \cdots + p + 1)$, $BP\langle n \rangle_k \simeq Y_k$.*

Proof. $\pi_*(BP\langle n \rangle_k)$ is free by construction. (1.5) $H^*(BP\langle n \rangle_k)$ has no torsion for $k \leqslant 2(p^n + \cdots + p + 1)$ by 5.1. For $k > 2(p^{n-1} + \cdots + p + 1)$ $BP\langle n \rangle_k$ is irreducible by 5.3. Now, for k in this range just apply 6.2.

COROLLARY 6.9. *For $k > 2(p^{n-1} + \cdots + p + 1)$, any map $BP\langle n \rangle_k \to BP\langle n \rangle_k$ which induces an isomorphism on π_k is a homotopy equivalence.*

Note that $Y_\infty \simeq BP$ and we get an unpublished result of F. P. Peterson.

COROLLARY 6.10 (*Peterson*). *Given a spectrum X with $H^*(X, Z_{(p)})$ and $\pi_*^S(X)$ bounded below, locally finitely generated, and free over $Z_{(p)}$, then*

$$X \simeq V_i S^{k_i} BP$$

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