The Ω-Spectrum for Brown-Peterson Cohomology. Part I

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Introduction

BP denotes the spectrum for the Brown-Peterson cohomology, $BP^*(\cdot)$, associated with the prime p [1, 3, 11]. The spectrum can be given as an Ω-spectrum $BP=\{BP_k\}$, [2, 16], i.e. $\Omega BP_k \simeq BP_{k-1}$ and BP_k is $k-1$ connected for $k>0$. We have $BP^k(\cdot)\simeq[\cdot, BP_k]$, the unstable homotopy classes of maps. The usual way of viewing $BP^*(\cdot)$ is $BP^*(\cdot)\simeq\{\cdot, BP\}^*$, the stable homotopy classes of maps of the suspension spectrum of a space into BP. We will study the Brown-Peterson cohomology theory from an unstable point of view by studying the BP_k.

Interest in the Brown-Peterson theory stems from the fact that it is a "small" cohomology theory which determines the complex cobordism theory localized at the prime p and that all of the nice properties of complex cobordism carry over to $BP^*(\cdot)$, such as knowledge of the operation ring. Historically, everything about the Brown-Peterson theory has been as nice as could be hoped for. We will push on further in that direction. $Z_{(p)}$ is the integers localized at p, i.e., rationals with denominator prime to p.

MAIN THEOREM (3.3). *The $Z_{(p)}$ (co)homology of the zero component of* BP_k *has no torsion and is a polynomial algebra for k even and an exterior algebra for k odd. (k can be less than zero.)* #

Using the main result of [12], the above theorem determines the Hopf algebra structure of the (co)homology. (see section 3) We begin by reviewing Larry Smith's result on the Eilenberg-Moore spectral sequence for stable Postnikov systems. [14] We combine this with Brown and Peterson's original construction of BP([3]) to calculate $H^*(BP_{2k+1}, Z_p)$ assuming a technical lemma which we prove in section 2. In section 3 we prove the main theorem and some miscellaneous items such as lifting our result to MU.

In Part II we determine the homotopy type of the BP_k using the main theorem here.

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Section 1

For the remainder of the paper all coefficient rings are assumed to be $Z_p = Z/pZ$ unless stated otherwise. In this section we show $H^*(\mathrm{BP}_{2k+1})$ is an exterior algebra on odd dimensional generators. $H^*(\mathrm{BP}_{2k+1})$ is a Hopf algebra, so for odd primes having odd dimensional generators is equivalent to being an exterior algebra. The general reference for Hopf algebras is [10]. We quote what we need from [14].

Let K be a product of Eilenberg-MacLane spaces. We will be concerned with the situation

$$\begin{array}{ccc} Y & \rightarrow & PK \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ X & \underset{f}{\rightarrow} & K \end{array} \tag{A}$$

where all spaces are infinite loop spaces and all maps are infinite loop maps. π is the fibration induced by f from the path space PK over K. All cohomologies are thus cocommutative Hopf algebras and $H^*(K)\backslash\backslash f^*$ and $H^*(X)//f^*$, the kernel and cokernel of f^* in the category of Hopf algebras are defined.

There is a natural map $PH \rightarrow QH$, where P and Q denote the primitives and indecomposibles respectively of a Hopf algebra H. When this is onto, H is called primitive.

LEMMA 1.1 ([14, p. 69]). *$H' \subset H$ a subHopf algebra over Z_p, H primitive, then H' is primitive.* #

If V is a graded module, let $s^q V$ be the graded module $(s^q V)_{n+q} = V_n$. Let V^- denote the elements of odd degree. From [14, p. 95] we have a filtration of $H^*(Y)$ of diagram A such that

$$E_0 H^*(Y) \simeq H^*(X)//f^* \otimes E[\ldots] \otimes \frac{P[s^{-1}((Q(H^*(K)\backslash\backslash f^*))^-)]}{[s^{-1}((Q(H^*(K)\backslash\backslash f^*))^-)]^p} \tag{1.2}$$

as Hopf algebras. E and P denote exterior and polynomial algebras generated by odd and even dimensional elements respectively. $E[\ldots]$ is determined by $H^*(K)\backslash\backslash f^*$.

$H^*(K)$ is primitive because it is generated by cohomology operations on fundamental classes, therefore, $H^*(K)\backslash\backslash f^*$ is primitive by 1.1. So for $x \in Q(H^*(K)\backslash\backslash f^*)$ we have $x' \rightarrow x$, $x' \in P(H^*(K)\backslash\backslash f^*)$ and thus $x' \rightarrow x'' \in PH^*(K)$. For x of odd degree, x' and thus x'', are determined uniquely by x. Let $i: \Omega K \rightarrow Y$ be the inclusion of the fibre.

LEMMA 1.3 ([14, p. 86 and p. 110]). $i^*(s^{-1}(x)) = s^*(x'')$, *$s^*$ the cohomology suspension, $s^*: H^*(K) \rightarrow H^*(\Omega K)$.* #

Note that if x is of odd degree then $s^*(x'')\neq 0$ by the following lemma.

LEMMA 1.4. *$a\in PH^m(K)$, if $s^*(a)=0$, then $a=P^t x_{2t}$ or $a=\beta P^k x_{2k+1}$ where $P^i\in A$ is the i-th reduced p-th power, A is the Steenrod algebra and x_i is of degree i. ($p=2, P^i=Sq^{2i}$ and $\beta=Sq^1$).* #

Proof. It is enough to consider $K=K(Z_{(p)}, n)$ and $a=P^I i_n$ where $P^I\in A$ is an Adem basis element. The kernel of $s^*: QH^*(K(Z_{(p)}, n))\to PH^*(K(Z_{(p)}, n-1))$ is of the type $\beta P^k x_{2k+1}$. The proof is an argument on the excess of I and can be found in [13]. The kernel of $PH^*(K)\to QH^*(K)$ is of the type $P^t x_{2t}=(x_{2t})^p$. The degree of $P^t x_{2t}=2pt$ and the degree of $\beta P^k x_{2k+1}=2pk+2$ so the two terms cannot occur in the same dimension. #

Brown and Peterson [3] construct BP by a series of fibrations which we now describe. Let $\mathscr{R}$ be the set of sequences of non-negative integers $(r_1, r_2,\ldots)$ which are almost all zero. Define $d(R)=\sum 2r_i(p^i-1)$, $l(R)=\sum r_i$ and let Δ_i be the R with $r_i=1$ and zeros everywhere else. Let V_j be the graded abelian group, free over $Z_{(p)}$, generated by $R\in\mathscr{R}$ with $l(\mathscr{R})=j$ and graded by $d(R)$. Then we have the generalized Eilenberg-MacLane spectrum $K(V_j)=\bigvee_{l(R)=j} S^{d(R)}K(Z_{(p)})$. BP$=$inverse limit X^j where we have the fibrations

$$\begin{array}{l} K(V_j)\xrightarrow{i_j} X^j \\ \qquad\qquad\downarrow \\ \qquad\qquad X^{j-1}\xrightarrow{k_{j-1}} SK(V_j) \end{array} \qquad (*)$$

induced by k_{j-1}. We have an $A/A(Q_0)$ resolution for $A/A(Q_0, Q_1,\ldots)=H^*(\mathrm{BP})$, $d_j: M_j\to M_{j-1}$ with $H^*(K(V_j))=M_j$ and $(i_j)^*\cdot(k_j)^*=d_{j+1}$. The Q_i are the Milnor primitives [8]. (For $p=2, Q_i=P^{\Delta_{i+1}}$ in the Milnor basis.) For an $A/A(Q_0)$ generator $i_R\in H^*(K(V_j))$, $d_j(i_R)=\sum_i Q_i i_{R-\Delta_i}$.

The spectrum $K(V_j)$ can be given as an Ω-spectrum, $\{K(V_j, k)=\bigtimes_{l(R)=j} K\times\times(Z_{(p)}, d(R)+k)\}$. The entire diagram $(*)$ can be turned into Ω-spectra and maps of Ω-spectra. From this we get a sequence of fibrations with $\mathrm{BP}_k=$inverse limit X^j.

$$\begin{array}{l} K(V_j, k)\xrightarrow{i_j} X^j \\ \qquad\qquad\quad\downarrow \\ \qquad\qquad\quad X^{j-1}\xrightarrow{k_{j-1}} K(V_j, k+1). \end{array} \qquad (**)$$

We suppress the k in the notation for X^j, i_j and k_j. Note that k can be less than zero. We have $(i_j)^*\cdot(k_j)^*\cdot s^*=s^*\cdot(i_j)^*\cdot(k_j)^*$ where the i_j and k_j on the right are for BP_k and on the left for BP_{k-1}. This is because k_j for BP_{k-1} is the loop map of the k_j for BP_k. Similarly for i_j. The iterated cohomology suspension gives a map $s^*: M_j\to H^*\times\times(K(V_j, k))$ which has as its image the primitives, $PH^*(K(V_j, k))$. In general we will denote the iterated suspension by s^* and it should be clear when we mean only one. We have the following commutative diagram.

$$\begin{array}{ccc} M_j & \xleftarrow{d_{j+1}} & M_{j+1} \\ {\scriptstyle s^*}\downarrow & & {\scriptstyle s^*}\downarrow \\ H^*(K(V_j, k)) & \xleftarrow[(i_j)^*\cdot(k_j)^*]{} & H^*(K(V_{j+1}, k+1)). \end{array}$$

We will often use $s^*(d_{j+1})$ for $(i_j)^*\cdot(k_j)^*$. It is given by the same formula $\sum Q_i i_{R-\Delta_i}$. In the next section we prove the following lemma.

LEMMA 1.5(j). *For k odd, if $a\in PH^{2i+1}(K(V_j, k+1))$ such that $(k_{j-1})^*(a)=0$, then there exists $b\in PH^*(K(V_{j+1}, k+1))$ such that $(i_j)^*\cdot(k_j)^*(b)=s^*(a)\neq 0$.* #

We use this to prove the next proposition.

PROPOSITION 1.6(j). *For k odd, $H^*(X^j)//(k_j)^*$ has no even dimensional* generators. *(For $p=2$ it is an exterior algebra.)* #

Proof. For $j=0$, $X^0=K(Z_{(p)}, k)$ and all generators of $H^*(X^0)$ are in the image of $s^*: M_0=A/A(Q_0)\to H^*(K(Z_{(p)}, k))$. So if x is an even dimensional generator of $H^*(K(Z_{(p)}, k))$ and k is odd, then there is an odd dimensional $x'\in M_0$ with $s^*(x')=x$. We have the exact sequence

$$M_1 \xrightarrow{d_1} A/A(Q_0) = M_0 \xrightarrow{\varepsilon} A/A(Q_0, Q_1, \ldots) \to 0.$$

Thus there exists $x''\in M_1$ with $d_1(x'')=x'$ as $\varepsilon(x')=0$ because $\varepsilon(x')$ is an odd dimensional element in $A/A(Q_0, Q_1, \ldots)$ which only has even degree elements. So $s^*(x'')\times \times\in H^*(K(V_1, k+1))$ and $(k_0)^*(s^*(x''))=s^*(d_1)\cdot s^*(x'')=s^*(d_1x'')=s^*(x)=x$ and the even dimensional generator $x\in H^*(X^0)$ goes to zero in $H^*(X^0)//(k_0)^*$. (For $p=2$ and x an odd dimensional generator, then $x^2=Sq^{\deg x}x$ is killed by the same argument, so we have an exterior algebra.)

By induction, assume proposition 1.6(j-1). By 1.2 we have:

$$E_0H^*(X^j) \simeq H^*(X^{j-1})//(k_{j-1})^* \otimes E[\ldots] \otimes \frac{P[s^{-1}(Q(H^*(K(V_j, k+1))\backslash\backslash(k_{j-1})^*)^-)]}{[s^{-1}(Q(H^*(K(V_j, k+1))\backslash\backslash(k_{j-1})^*)^-)]^p}.$$

Now by our induction assumption, all even dimensional generators look like $s^{-1}(x)$ where $x\in QH^*(K(V_j, k+1))\backslash\backslash(k_{j-1})^{*-}$. These elements map injectively to the cohomology of the fibre, see 1.3 and the remark after it. As discussed above (before 1.3), x can be represented by an $a\in PH^*(K(V_j, k+1))$ with $(k_{j-1})^*(a)=0$. Now, as a is of odd degree, from 1.5(j), there exists b such that $(i_j)^*\cdot(k_j)^*(b)=s^*(a)\neq 0$. But by 1.3, $(i_j)^*(s^{-1}(x))=s^*(a)$ and $(i_j)^*$ is injective on these even degree indecomposibles giving that $(k_j)^*(b)=s^{-1}(x)+$decomposibles. Therefore, the generator $s^{-1}(x)$ goes to a decomposible in $H^*(X^j)//(k_j)^*$ and we are done. #

COROLLARY 1.7. *For k odd, $H^*(\mathrm{BP}_k)$ is an exterior algebra on odd dimensional generators.* #

Proof. Because $K(V_j, k)$ is highly connected for high j we have $H^*(\mathrm{BP}_k)=$direct limit $H^*(X^j)//(k_j)^*$. Because we are working with Hopf algebras, odd dimensional generators for odd primes means we have an exterior algebra. The direct limit is achieved in a finite number of stages so we have the result using 1.6. #

Section 2

We will now prove lemma 1.5(j). We have already seen that that $s^*(a)\neq 0$. (1.4)

Let A be the mod p Steenrod algebra. We define a filtration: $A=F^0A\supset F^1A\supset F^2A\supset \ldots$ by giving a basis for F^sA. Given an Adem basis element, $\beta^{\varepsilon_0}P^{i_1}\beta^{\varepsilon^1}\ldots P^{i_n}\beta^{\varepsilon_n}$, it is basis element for F^sA if $s\leqslant\sum\varepsilon_i$. Also, we give a basis for B_s by taking all Adem basis elements with $s=\sum\varepsilon_i$. For $p=2$, $P^i=Sq^{2i}$. We do not define B_s for $p=2$ using the Adem basis.

For our purposes it is usually more convenient to work in the Adem basis, however, the Milnor basis is a necessary excursion for $p=2$. For odd primes, a Milnor basis element Q^IP^R $(Q^I=Q_0^{\varepsilon_0}Q_1^{\varepsilon_1}\ldots)$ is a basis element for F^sA if $s\leqslant\sum\varepsilon_i$. For $p=2$, a Milnor basis element P^R is a basis element for F^sA if $R=(r_1, r_2, \ldots)$ has s or more odd r_i. Again, a basis element for B_s has $s=\sum\varepsilon_i$ $(p=2$, s odd $r_i)$.

CLAIM 1. i) *The two definitions of F^sA and B_s are the same.*

ii) *If $a\in F^sA$ and $b\in F^tA$, then $ab\in F^{s+t}A$.*

iii) $F^sA=B_s\oplus F^{s+1}A$.

Sketch proof. Milnor's $Q_i=P^{\Delta_i}\beta-\beta P^{\Delta_i}$. For odd primes P^{Δ_i} is in the algebra of reduced p-th powers and so can be written in the Adem basis without any β's, similarly for all P^R in the Milnor basis. The Adem relations for p odd preserve the number of β's exactly, so we see that $Q_i\in B_1\subset F^1A$. If we were to rewrite a Milnor basis element Q^IP^R in the Adem basis we would still have $\sum\varepsilon_i$ β's.

The proof of the second part just uses the fact that the Adem relations never decrease the number of Bocksteins.

The proof for $p=2$ is slightly more complicated and is left for the reader. iii) is elementary. #

Given $a\in PH^*(K(V_j, k))$, (any k), it can be written as $a=\sum_{l(R)=j}a_Ri_R$ where i_R is the fundamental class of $K(Z_{(p)}, d(R)+k)$ and $a_R\in A$. If it can be written like this with each $a_R\in F^nA$, then we say a is with n Bocksteins ($w.n\beta$'s). If $n=1$, we just say $w.\beta$'s. If a is with n Bocksteins but not with $n+1$ β's we say a is with exactly n β's. As discussed above, Q_i is with exactly one Bockstein. Therefore by the definition of d_j and the above claim, if a is with $n\beta$'s, then $s^*(d_j)(a)$ is with $n+1$ β's. Recall that by our notation $s^*(d_j)=(i_{j-1})^*\cdot(k_{j-1})^*$.

CLAIM 2. *If $a=s^*(d_j)(b)$ and a is with 2 Bocksteins, then there is a b' with β's such that $s^*(d_j)(b')=a$.* #

Proof. First for odd primes; write $b=\sum_{l(R)=j} a_R i_R$ with $a_R\in A$. $A=B_0\oplus F^1A$, so write $a_R=b_R+c_R$ with $b_R\in B_0$ and $c_R\in F^1A$. $b_RQ_i\in B_1$ and $c_RQ_i\in F^2A$ so $s^*(d_j)(\sum b_Ri_R)=0$. Let $b'=\sum c_Ri_R$.

For prime 2 we have $Q_{i-1}=P^{\Delta i}$ and for a Milnor basis element P^R we have $P^RP^{\Delta i}=\sum_{r_{i+j}\text{ even}} P^{R-2^i\Delta_j+\Delta_{i+j}}$, thus $b_RP^{\Delta i}\in B_1$ and $c_RP^{\Delta i}\in F^2A$ and same proof works. #

PROPOSITION 2.1(j). *Given $a\in PH^*(K(V_j,k))$, a with β's such that $s^*(d_j)(a)=0$, then there exists $\bar a\in M_j$ such that $s^*(\bar a)=a$ and $d_j(\bar a)=0$.* #

Proof of 1.5(j) For k and a odd, then a is with β's in $PH^*(K(V_j,k+1))$ for dimensional reasons, i.e., all of the Steenrod algebra elements used are odd dimensional, and all odd dimensional elements have β's. $(k_{j-1})^*(a)=0$ implies $s^*(d_j)(a)=0$ and we can apply proposition 2.1(j) to get $\bar a$ such that $s^*(\bar a)=a$ and $d_j(\bar a)=0$. By exactness, there exists $\bar b\in M_{j+1}$ such that $d_{j+1}(\bar b)=\bar a$. Then $b'=s^*(\bar b)\in PH^*(K\times\times(V_{j+1},k+2))$ has $s^*(d_{j+1})(b')=s^*(d_{j+1})(s^*(\bar b))=s^*(d_{j+1}(\bar b))=s^*(\bar a)=a$. So let $b=s^*(b')$, then $s^*(a)=s^*(d_{j+1})(b)$ which is what we want. #

PROPOSITION 2.2(j). *Given an a as in* 2.1(j), *then there exists $b\in PH^*(K\times\times(V_{j+1},k+1))$ such that $s^*(d_{j+1})(b)=a$.* #

Proof. See proof of 1.5(j). #

Remark. Proposition 2.2(j) is really the essential feature that makes everything work. It means that exactness still holds in the unstable range for primitives with β's.

We need proposition 2.2(j-1) in the induction argument for the proof of proposition 2.1(j).

Proof of 2.1(j). This follows at once from the next proposition, just lift a up one step at a time until it is in the stable range. #

PROPOSITION 2.3(j). *Given a with β's in $PH^*(K(V_j,k))$ (any k) such that $s^*(d_j)(a)=0$, then there exists $\bar a$ with β's in $PH^*(K(V_j,k+1))$ such that $s^*(\bar a)=a$ and $s^*(d_j)(\bar a)=0$. (For $j=0$, $s^*(d_0)(a)=0$ is a vacuous condition).* #

Proof. $j=0$, trivial. For $j=1$ the arguments is the same as for $j>1$ except easier, so assume $j>1$. Now, trivially, there exists a' with β's such that $s^*(a')=a$. (Let $a'=\tau(a)$.) Now $s^*(d_j)(a')\in\ker s^*$ by commutativity of the following diagram.

$$\begin{tikzcd}
PH^*(K(V_{j-1},k)) \arrow[d, "s^*"'] & PH^*(K(V_j,k+1)) \arrow[l, "s^*(d_j)"'] \arrow[d, "s^*"'] & a' \arrow[dd] \\
PH^{*-1}(K(V_{j-1},k-1)) & PH^{*-1}(K(V_j,k)) \arrow[l, "s^*(d_j)"'] & \\
0 & & a \arrow[ll]
\end{tikzcd}$$

By 1.4, $s^*(d_j)\,(a')=0$, $P^n x_{2n}$, or $\beta P^t x_{2t+1}$ in $PH^*(K(V_{j-1},k))$.

Case 1. If $s^*(d_j)\,(a')=0$ we are done.

Case 2. If $s^*(d_j)\,(a')=P^n x_{2n}$ then $s^*(d_{j-1})\,(P^n x_{2n})=0$ because $d_{j-1}\cdot d_j=0$. $0=s^*(d_{j-1})\,(P^n x_{2n})=s^*(d_{j-1})\,(x_{2n})^p=[s^*(d_{j-1})\,(x_{2n})]^p$. $H^*(K(V_{j-2},k-1))$ is a free commutative algebra so this implies $s^*(d_{j-1})\,(x_{2n})=0$. Now a' is with β's so $s^*(d_j)\,(a')$ is with 2 β's. This gives us that $P^n x_{2n}$ is with 2 β's. If $x_{2n}=\sum_R(\sum_i \lambda_i b_i)i_R$ with $\lambda_i \neq 0 \in Z_p$ and b_i Adem basis elements, then $P^n x_{2n}=\sum_R(\sum_i \lambda_i P^n b_i)i_R$ and for dimensional reasons $P^n b_i$ is in Adem basis form. Since each $P^n b_i$ is with 2 β's, each b_i is with 2 β's and so x_{2n} is with 2 β's. x_{2n} is also in the kernel of $s^*(d_{j-1})$ so we can apply 2.2(j-1) to produce a $y_{2n} \in PH^{2n}(K(V_j,k+1))$ with $s^*(d_j)\,(y_{2n})=x_{2n}$. By claim 2 we can choose y_{2n} to be with β's. $a'-P^n y_{2n}$ is with β's and has $s^*(d_j)\,(a'-P^n y_{2n})=0$ and $s^*(a'-P^n y_{2n})=s^*(a')=a$, so we are done.

Case 3. If $s^*(d_j)\,(a')=\beta P^t x_{2t+1}$ the proof is similar to case 2. We sketch the differences. βP^t is injective on $PH^{2t+1}(K(V_{j-2},k-1))$ because it is for any product of Eilenberg-MacLane spaces [13]. So we get x_{2t+1} is in the kernel of $s^*(d_{j-1})$. If $\sum_R(\sum_i \lambda_i b_i i_R))=x_{2t+1}$ b_i Adem basis elements, then each $\beta P^t b_i$ is also in Adem basis form and since $\beta P^t x_{2t+1}$ must be with 2 β's, each b_i is with one and so x_{2t+1} must be with β's. Use 2.2(j-1) again to produce y_{2t+1} with $s^*(d_j)\,(y_{2t+1})=x_{2t+1}$. Now $a'-\beta P^t x_{2t+1}$ has the desired property. #

Section 3

Our first objective is to compute the (co)homology of BP_{2k}. The bar construction ([4]) gives a spectral sequence of Hopf algebras: (k odd)

$$\mathrm{Tor}^{H^*(\mathrm{BP}_k)}(Z_p, Z_p) => E_0 H_*\,(\text{zero component of } \mathrm{BP}_{k+1}).$$

Now $H_*(\mathrm{BP}_k)$ is an exterior algebra on odd dimensional generators $QH_*(\mathrm{BP}_k)$. (Cor. 1.7) A standard computation (see [14]) gives: $\mathrm{Tor}^{H^*(\mathrm{BP}_k)}\,(Z_p, Z_p)=\Gamma\,(s^1(QH_* \times \times(\mathrm{BP}_k)))$ where Γ denotes the Hopf algebra dual to the polynomial algebra. Now all elements in $\Gamma(s^1\,(QH_*(\mathrm{BP}_k)))$ are of even degree and the differentials change degree by one, so our spectral sequence collapses and we have: $H^*(\text{zero component of } \mathrm{BP}_{k+1})= [E_0 H_*(\text{zero component of } \mathrm{BP}_{k+1})]^*=[\mathrm{Tor}^{H^*(\mathrm{BP}_{k+1})}\,(Z_p, Z_p)]^*=[\Gamma\,(s^1(QH_*(\mathrm{BP}_k)))]^*$ $=$polynomial algebra.

We will now show $H_*(\mathrm{BP}_{k-1})$ is a polynomial algebra for k odd. Using the

Eilenberg-Moore spectral sequence ([6, 14]) we have $\mathrm{Tor}_{H*(\mathrm{BP}_k)}(Z_p, Z_p) => E_0H^*(\mathrm{BP}_{k-1})$ if BP_k is simply connected. Assume it is, then the same argument just given shows $H_*(\mathrm{BP}_{k-1})$ is a polynomial algebra. The only modification is:

$$\mathrm{Tor}_{H^*(\mathrm{BP}_k)}(Z_p, Z_p) = \Gamma(s^{-1}(QH^*(\mathrm{BP}_k))).$$

If BP_k, k odd, is not simply connected, then it is easy to see that one can get a splitting $\mathrm{BP}_k \simeq (\times S^1)_{(p)} \times X$ where X is simply connected. This is because BP_k is an H-space with $Z_{(p)}$ free homotopy. Its k-invariants are therefore torsion and primitive, but $(\times S^1)_{(p)}$ has no torsion in $Z_{(p)}$ cohomology. Thus we have a spectral sequence of Hopf algebras:

$$\mathrm{Tor}_{H^*(X)}(Z_p, Z_p) = E_0H^*(\text{zero component of } \mathrm{BP}_{k-1})$$

and our argument goes through. We have proved the following proposition.

PROPOSITION 3.1. *The* mod$p$ *(co)homology of the zero component of* BP_k *is a polynomial algebra on even dimensional generators for k even, and an exterior algebra on odd dimensional generators for k odd.* (Note that for k odd, BP_k is connected.)#

PROPOSITION 3.2. *The* $Z_{(p)}$ *(co)homology of* BP_k *has no torsion.*#

Proof. For k even this is trivial because $H^*(\mathrm{BP}_k)$ has no elements in odd degrees. For k odd we view the Bockstein spectral sequence as a spectral sequence of Hopf algebras. The differentials are the higher order Bocksteins. Let β_s be the first non-trivial differential and let x be the minimum degree generator that β_s acts non-trivially on. $\beta_s(x)$ is an even dimensional primitive, contradiction, so all differentials are zero. #

We can now prove the main theorem.

THEOREM 3.3. *The* $Z_{(p)}$ *(co)homology of* BP_{2k+1} *is an exterior algebra and the* $Z_{(p)}$ *(co)homology of the zero component of* BP_{2k} *is a polynomial algebra.*#

Proof. We will do the case for polynomial algebras, the exterior case being similar. From 3.2 we know the (co)homology is free over $Z_{(p)}$ and so we can lift the mod$p$ generators (3.1) up to it. These lifted elements generate the $Z_{(p)}$ (co)homology ring because there is no torsion and their mod$p$ reductions generate the Z_p (co)homology. By considering the rank we can see there can be no relations and we have a polynomial algebra.#

We can now lift our result to MU. Normally the spectrum MU is given by $\{\mathrm{MU}(n)\}$, the Thom complexes, and maps $S^2\mathrm{MU}(n) \to \mathrm{MU}(n+1)$. [9, 15] However, if $M_n = \lim(k\to\infty)\Omega^{2k-n}\mathrm{MU}(k)$, then $\Omega M_n \simeq M_{n-1}$ and for finite complexes $\mathrm{MU}^n(X) = \lim(k\to\infty)[S^{2k-n}X, \mathrm{MU}(k)] = \lim(k\to\infty)[X, \Omega^{2k-n}\mathrm{MU}(k)] = [X, M_n]$. Thus, $\{M_n\} = \mathrm{MU}$ as an Ω-spectrum.

COROLLARY 3.4. *The integer (co)homology of the zero component of M_n has no torsion and is a polynomial algebra over Z for n even and an exterior algebra for n odd.* #

Proof. From [3] we have $MU_{(p)} \simeq \bigvee_i S^{2n_i}BP$ and so $(M_n)_{(p)} \simeq \prod_i BP_{n+2n_i}$. By 3.3 for n even $H_*(M_n, Z) \otimes Z_{(p)} \simeq H_*(M_n, Z_{(p)}) \simeq H_*((M_n)_{(p)}, Z_{(p)}) \simeq$ polynomial algebra over $Z_{(p)}$. Thus the integer homology has no torsion, and localized at every prime it is a polynomial algebra, so it is a polynomial algebra over Z. Similarly for n odd. Since there is no torsion, the same thing works for cohomology. #

Remark 1. A completely analogous theorem is true for MSO if the ring $Z(1/2)$ is used.

Remark 2. There are several ways to determine the number of generators for 3.1, 3.3, and 3.4. The spaces BP_n and M_n are just products of rational Eilenberg-MacLane spaces when localized at $\mathbf{Q}$. (This is because their k-invariants are torsion.) Because there is no torsion, the number of generators is the same as for the rationals. As examples we have $\pi_*^S(BP) = Z_{(p)}[x_{2(p-1)}, \ldots, x_{2(p^i-1)}, \ldots]$ so for $2n > 0$, $H^*(BP_{2n}, Z_{(p)}) \simeq Z_{(p)}[s^{2n}\pi_*^S(BP)]$ and $\pi_*^S(MU) = Z[x_2, \ldots, x_{2i}, \ldots]$ so for $2n > 0$, $H^*(M_{2n}, Z) \simeq Z[s^{2n}\pi_*^S(MU)]$.

We have shown that both the cohomology and homology of the zero component of BP_{2n} are polynomial algebras. This is a very strong statement, in fact, it determines the Hopf algebra structure of the (co)homology.

DEFINITION. *A connected bicommutative Hopf algebra is called bipolynomial if both it and its dual are polynomial algebras.* #

There is a bipolynomial Hopf algebra $B_{(p)}[x, 2n]$ over $Z_{(p)}$ (or Z_p) which has generators $a_k(x)$ of degree $2p^k n$ [7]. It is isomorphic as Hopf algebras to its own dual.

In [12] we prove the following proposition.

PROPOSITION 3.5. *If H is a bipolynomial Hopf algebra over $Z_{(p)}$ (or Z_p), then $H \simeq \otimes_j B_{(p)}[x_j, 2d_j]$. (For* $p=2$ and *Z_2, replace $2d_j$ by d_j).* #

Using this and the counting argument of remark 2 we can just write down the Hopf algebra structure for BP_{2n}. As an example, we will do this for $n > 0$. Let $\mathscr{R}_n$ be the set of sequences of non-negative integers $R = (r_1, r_2, \ldots)$ with almost all $r_i = 0$. Let $d(R) = 2n + \sum 2(p^i - 1) r_i$ for our fixed prime p. We say R is prime if it cannot be written $R = pS + (n, 0, 0, \ldots)$, $S \in \mathscr{R}_n$.

PROPOSITION 3.6. *For $n > 0$, $H^*(BP_{2n}, Z_{(p)}) \simeq \bigotimes_{\substack{R \in \mathscr{R}_n \\ R \text{ prime}}} B_{(p)}[x_R, d(R)]$.* #

If we work over the integers and let $B[x, 2d]$ be the bipolynomial Hopf algebra on generators $c_n(x)$ of degree $2dn$ with coproduct $c_n(x) \to \sum c_{n-j}(x) \oplus c_j(x)$ ([7]) then we have an analogous proposition. [12]

PROPOSITION 3.7. *If H is a bipolynomial Hopf algebra over Z, then $H \simeq \otimes_j B[x_j, 2d_j]$.* #

We can now apply this to $\mathrm{MU} = \{M_n\}$. Let I_n be the set of sequences of non-negative integers $I = (i_1, i_2, \ldots)$ with $i_1 \geqslant n$ and almost all $i_j = 0$, $(n > 0)$. Let $d(I) = \sum_j 2ji_j$. We say I is prime if it cannot be written $I = kJ$, where $k > 1$ and $J \in I_n$.

PROPOSITION 3.8. *If $\{M_k\}$ is the Ω-spectrum for* MU, *then for $n > 0$, $H^*(M_{2n}, Z) \simeq \otimes_{I \text{ prime} \in I_m} B[x_I, d(I)]$ as Hopf algebras.* #

Proof. Just use 3.7 and the counting done in remark 2. #

Let S be the sphere spectrum and let $i: S \to \mathrm{BP}$ represent $1 \in \pi_0^S(\mathrm{BP})$. $S = \{QS^n\}$ as an Ω-spectrum where $QX = \lim \Omega^n S^n X$. i induces maps $i_n: QS^n \to \mathrm{BP}_n$. $H_*(QS^n)$ is given in terms of homology operations on the n dimensional generator [5].

PROPOSITION 3.9. *Let $n > 0$, the kernel of $(i_n)_*: H_*(QS^n) \to H_*(BP_n)$ is generated by homology operations on the n-dimensional class which have Bocksteins in them.* #

PROPOSITION 3.10. *Let $n > 0$, if $j_n: \mathrm{BP}_n \to K(Z_{(p)}, n)$ represents the generator of $H^n(\mathrm{BP}_n, Z_{(p)})$, then the kernel of $(j_n)^*: H^*(K(Z_{(p)}, n)) \to H^*(\mathrm{BP}_n)$ is generated by cohomology operations on the n-dimensional class which have Bocksteins in them.* #

Proof of 3.9. By 3.2, any homology operation which has Bocksteins in it goes to zero. Let u be a homology operation with no β's such that $ux_n \neq 0$ in $H_*(QS^n)$. As u has no β's, $u(s_*)^k x_n$ is a p-th power for some k. So $u(s_*)^k x_n = ux_{n+k} = (u'x_{n+k})^p$. Now by induction on the degree of u, $i_*(u'x_{n+k}) \neq 0$ in $H_*(\mathrm{BP}_{n+k})$ and $n+k$ is even since we have a p-th power. $H_*(\mathrm{BP}_{n+k})$ is a polynomial algebra and so $[i_*(u'x_{n+k})]^p \neq 0$ and is $= i_*[u'x_{n+k}]^p = i_* u(s_*)^k x_n = i_*(s_*)^k ux_n = (s_*)^k i_*(ux_n)$ and so $i_*(ux_n) \neq 0$. #

The proof of 3.10 is similar.

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