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W. STEPHEN WILSON

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An Introduction and Sampler

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BROWN-PETERSON HOMOLOGY: AN INTRODUCTION AND SAMPLER

by

W. STEPHEN WILSON

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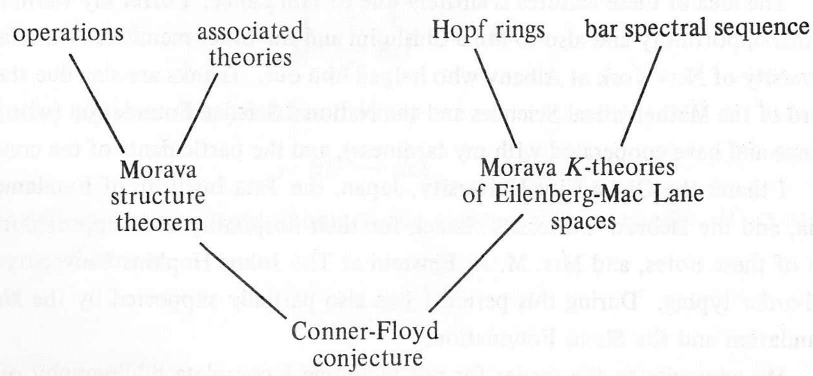
Contents

Introduction

A few years ago, I hoped to have the opportunity to write lecture notes such as these. It was then possible to do a complete survey of the post-Stong **[St]** research in the direction of Brown-Peterson homology. When finally confronted with this outlet, I found it impossible to do justice to the entire field. Although grand schemes such as another book like Stong's or an exposition of Morava's work did occur to me, I had only three months to prepare the lectures. The nature of C. B. M. S. Regional Conferences dictated that the lectures should expose nonspecialists to the area (although few were present). The format became an introduction to the basic facts about BP, with proofs, and a highly personal sampler of further material, mostly without proofs. These notes are fairly true to the lectures; I talked too much. (I have added §8 and expanded §11.) I have tried to retain an informal style and where clarity means lying, the notes are clear.

An introduction to BP is a construction of BP and a study of the stable operations. In the sampler I discuss a number of distinct tools: operations, associated homology theories, formal groups, and Hopf rings; and show how each applies in several ways to interact with the others. Throughout, I have emphasized the transition from sophisticated internal theorems to the necessary applications; the moral: support basic research. The Conner-Floyd conjecture emerges as a central figure in these notes; it depends on almost everything else and therefore is difficult to discuss outside of a series of lectures like these. A diagram of dependencies for the Conner-Floyd conjecture is given as:



Each item in the diagram has other uses as well. Also, since these lectures, serious computations for BP_*X have been done which depend on the Conner-Floyd conjecture.

I begin by assuming the reader is familiar with the homology and homotopy of the spectrum *MU*. From this I develop the basic facts about complex bordism cooperations (§1), introduce the minimal necessary formal group machinery (§2), construct *BP* (§3), and describe its cooperations (§3).

The sampler is begun by going into more depth on the cooperations and applying them to stable homotopy (§4). The algebra discussed arises again and again in later sections. Next, I introduce the sequences of associated homology theories which contribute so much to the rich and plentiful internal theorems in the subject (§5). Operations and associated homology theories are combined to describe Morava's little structure theorem (§6). A little knowledge of the Morava *K*-theories of Eilenberg-MacLane spaces is fed into the Morava structure theorem to prove the Conner-Floyd conjecture (§6). To obtain this "little knowledge" it is necessary to study Ω-spectra, $\underline{G}_*$. The main tool is Hopf rings and they are inserted into the bar spectral sequence (§7). In order to complete the proof of the Conner-Floyd conjecture, the first application is to the Morava *K*-theories of Eilenberg-MacLane spaces (§7). As a detailed example of Hopf rings and the bar spectral sequence at work, the homology of the Eilenberg-MacLane spaces is computed (§8). Hopf rings and formal groups are then combined to study the Ω-spectrum for *BP* (§9). This provides the opportunity to give Chan's proof of no torsion (§10) and leads naturally to the study of unstable operations (§11). The material on unstable operations is previously unpublished. For an introduction, see the beginning of §11.

Although there are many new ways to do the material, §§1–3 follow Adams [$\mathbf{A}_2$]. The required properties are established as quickly as possible; the student and researcher alike will need to read Adams's book for the wealth of results which do not lie on the geodesic to *BP*. Mike Boardman made the inclusion of the Hazewinkel generators possible by supplying the simple proof used here.

§§4–10 owe debts to all of the coauthors and colleagues mentioned in the text. §8 is new and is joint work with Douglas C. Ravenel.

The acknowledgements for §11 are in the introduction to that section.

The idea of these lectures is entirely due to Tim Lance. I offer my warm thanks to him for this opportunity and also to Mike Chisholm and the other members of the faculty of State University of New York at Albany who helped him out. Thanks are also due the Conference Board of the Mathematical Sciences and the National Science Foundation (who funded the conference and have cooperated with my tardiness), and the participants of the conference.

I thank the Osaka City University, Japan, the Tata Institute of Fundamental Research, India, and the Hebrew University, Israel, for their hospitality and support during the production of these notes, and Mrs. M. A. Einstein at The Johns Hopkins University for difficult mail-order typing. During this period I was also partially supported by the National Science Foundation and the Sloan Foundation.

My apologies to the reader for not including a complete bibliography of the subject, but I found it too difficult to attempt away from my own files.

Finally I want to thank Michael Boardman, David Johnson and Robert Stong for numerous comments on the manuscript which allowed me to correct errors and improve the exposition.

Part I. An Introduction

1. Complex bordism. Bordism begins with geometry. Manifolds are well-studied objects; bordism exploits the reservoir of knowledge about manifolds to attack homotopy theory for more general spaces. This exploitation is analogous to the use of the geometry of Euclidean cells by standard homology. In classical homology we quickly subordinate the geometry to the rich algebraic structure which it induces. Likewise with bordism. We must, however, keep the geometry in mind, not just for philosophical reasons, but because we often return to bordism's geometrical roots to derive new algebra.

For the correct details for the first part of this section we suggest Stong's book **[St]**; for the second part, Adams's book **[A_2]**.

Let

$$(1.1)\qquad \begin{array}{c} \nu \\ \downarrow \\ M^n \end{array}$$

be the stable normal bundle for a manifold M^n of dimension n. Then ν is the pull-back of the universal bundle ξ over BO by a map f (which is unique up to homotopy):

$$(1.2)\qquad \begin{array}{ccc} \nu & \longrightarrow & \xi \\ \downarrow & & \downarrow \\ M^n & \xrightarrow[f]{} & BO \end{array}$$

The inclusion $U(n) \longrightarrow O(2n)$ induces the map

$$(1.3)\qquad r\colon BU \longrightarrow BO.$$

This map pulls back the universal real bundle to the universal unitary bundle. If we can choose a homotopy lifting g,

$$(1.4)\qquad \begin{array}{ccc} & & BU \\ & {\scriptstyle g}\nearrow & \downarrow{\scriptstyle r} \\ M^n & \xrightarrow[f]{} & BO \end{array}$$

we say that M^n has a U-structure g. It can have several U-structures or none. A pair (M^n, g) of a manifold and a U-structure is called a *U-manifold*. There is also an "opposite" U-structure on M^n, $(-M^n, g)$ (cf. **[CF$_3$]**).

We define the nth complex bordism group of a space X, $MU_n X$. We consider all triples (M^n, g, f), where (M^n, g) is a U-manifold of dimension n and f is a map

$$f\colon M^n \longrightarrow X. \tag{1.5}$$

As with chains in homology, there are far too many of these to take seriously the first time around. (They can, however, be useful.) So we put an equivalence relation on these triples by $(M_1^n, g_1, f_1) \sim (M_2^n, g_2, f_2)$ if there is a triple (W^{n+1}, h, k) with $\partial(W^{n+1}, h) = (M_1^n, g_1) \cup -(M_2^n, g_2)$ and $k|\partial W^{n+1} = f_1 \cup f_2$:

$$\begin{array}{ccc} & M_1^n & \\ & \searrow^{f_1} & \\ W^{n+1} & \xrightarrow{\;k\;} & X, \qquad \partial W^{n+1} = M_1^n \cup -M_2^n. \\ & \nearrow_{f_2} & \\ & M_2^n & \end{array} \tag{1.6}$$

Under this equivalence relation we have an abelian group $MU_n X$, where addition is given by disjoint union. Using manifolds with boundary one can define bordism for a pair (X, A) and verify geometrically that $MU_*(X, A)$ is a generalized homology theory **[CF$_1$, CF$_3$, C]**. Generalized homology theories like this are always obtained from spectra **[Br, Wh]**. The spectrum for complex bordism is just the Thom spectrum for the unitary groups:

$$MU = \{MU(n)\}_{n>0} \quad \text{with maps } S^2 \wedge MU(n-1) \longrightarrow MU(n). \tag{1.7}$$

The space $MU(n)$ is the Thom space of the universal n-dimensional complex bundle ξ_n over $BU(n)$. The maps are induced by the maps

$$\begin{array}{ccc} \xi_{n-1} \oplus \mathbf{C} & \longrightarrow & \xi_n \\ \downarrow & & \downarrow \\ BU(n-1) & \longrightarrow & BU(n) \end{array} \tag{1.8}$$

We have an isomorphism:

$$MU_n X \simeq \lim_{k \to \infty} [S^{n+2k}, MU(k) \wedge X] \equiv \{S^n, MU \wedge X\} \equiv \{S^0, MU \wedge X\}_n. \tag{1.9}$$

(We are making free use of Boardman's stable homotopy category **[V]**.) This isomorphism is proven using transversality as in Thom's original paper **[T]**; see **[CF$_1$, CF$_3$, C]**. We can use the right-hand side of (1.9) to define complex bordism. We see that all of the information of complex bordism is contained in the spectrum MU. (We have now reduced to homotopy theory the geometry we wish to use to study homotopy theory to solve the geometric problems which have been reduced to homotopy theory.) To study bordism we need to know about MU. We will ignore the geometry as we develop the basic properties of MU.

Where there is a generalized homology theory, there is a generalized cohomology theory. In this case we have complex cobordism defined for finite complexes by

$$(1.10) \qquad MU^n X \equiv \lim_{k\to\infty} [S^{2k-n}X, MU(k)] \equiv \{S^{-n}X, MU\} \equiv \{X, MU\}^n.$$

The Whitney sum of bundles gives a map

$$(1.11) \qquad BU(n) \times BU(m) \longrightarrow BU(n+m),$$

which, when we pass to the Thom spaces, gives

$$(1.12) \qquad MU(n) \wedge MU(m) \longrightarrow MU(n+m).$$

This pairing turns the spectrum MU into a ring spectrum,

$$(1.13) \qquad m\colon MU \wedge MU \longrightarrow MU.$$

This makes MU^*X an algebra over $MU^* \equiv MU^*(\text{point})$ and MU_*X a module over $\pi_*MU = MU_* = MU_*(\text{point}) = MU^{-*}$. This algebra follows from the geometric Cartesian product of U-manifolds.

The ring structure turns

$$(1.14) \qquad H_*MU = \lim_{k\to\infty} \widetilde{H}_{*+2k}MU(k)$$

into an algebra with a simple description. We combine the homotopy equivalence $MU(1) \simeq CP^\infty$ and the map used in defining the spectrum:

$$(1.15) \qquad x\colon CP^\infty \simeq MU(1) \longrightarrow MU.$$

Applying homology we obtain a map

$$(1.16) \qquad \widetilde{H}_{*+2}CP^\infty \longrightarrow H_*MU.$$

The group $\widetilde{H}_{2i+2}CP^\infty$ is free on one generator denoted by β_{i+1}. We define

$$(1.17) \qquad b_i \in H_{2i}MU$$

to be the image of β_{i+1} under this map. We have

$$(1.18) \qquad H_*MU \simeq Z[b_1, b_2, \ldots],$$

a polynomial algebra on the b's. Since these generators, b_i, come from CP^∞, the comodule structure over the dual of the Steenrod algebra can be computed. Using this, the homotopy of MU can be computed with the Adams spectral sequence to obtain

$$(1.19) \qquad MU_* = \pi_*MU \simeq Z[x_2, x_4, \ldots],$$

a polynomial algebra on even degree generators. An element $x_{2n} \in \pi_{2n}MU \subset H_{2n}MU$ is a polynomial generator if $x_{2n} = \lambda b_n + \cdots$ where $\lambda = \pm p$ if $n = p^i - 1$ for some prime p and $\lambda = \pm 1$ otherwise. We assume that the reader is familiar with these basic homology and homotopy facts about MU. They are readily accessible. See Stong's book **[St]**, Switzer's book **[Sw]**, or the original papers of Milnor **[Mi$_1$]** and Novikov **[N]**.

We can consider the Hurewicz homomorphism

(1.20) $$MU_* = \pi_* MU \subset H_* MU.$$

The groups MU_* are just the equivalence classes of U-manifolds under the bordism relation (1.6). Denote the equivalence class of $\mathbf{C}P^n$ with the standard U-structure by $[\mathbf{C}P^n]$. We have

(1.21) $$[\mathbf{C}P^n] \in \pi_* MU \subset H_* MU.$$

These elements form rational generators. We need to identify these elements in homology. This is one of the key connections between the geometry and the algebra. In the power series ring $H_*MU[[s]]$, define

(1.22) $$\exp s = \sum_{i \geq 0} b_i s^{i+1}, \qquad b_0 = 1.$$

Define

(1.23) $$\log s = \sum_{n \geq 0} m_n s^{n+1},$$

by

(1.24) $$\log(\exp(s)) = s = \exp(\log(s)).$$

The m_n $(n > 0)$ are new polynomial generators for H_*MU and from Miščenko [N, Appendix 1],

(1.25) $$[\mathbf{C}P^n] = (n+1) m_n.$$

This is proven using characteristic number arguments and is part of the assumed knowledge of the homology and homotopy of MU.

We see that the Atiyah-Hirzebruch spectral sequences for $MU_*\mathbf{C}P^\infty$ and $MU^*\mathbf{C}P^\infty$ both collapse (they are in even degrees). Thus

(1.26) $$MU^*[[x]] \simeq MU^*\mathbf{C}P^\infty \simeq \hom_{MU_*}(MU_*\mathbf{C}P, MU_*)$$

where $x \in MU^2\mathbf{C}P^\infty$ is the map (1.15) (recall (1.10)). This all follows from the compatibility of the spectral sequence with the usual pairing of (generalized) homology and cohomology. We also need the obvious fact that x reduces to the standard cohomology algebra generator. To describe the Kronecker pairing above, let

(1.27) $$y\colon S \longrightarrow MU \wedge X \in MU_*X \quad \text{and} \quad z\colon X \longrightarrow MU \in MU^*X.$$

Then

(1.28) $$\langle z, y \rangle \in MU_*$$

is

(1.29) $$S \xrightarrow{y} MU \wedge X \xrightarrow{I \wedge z} MU \wedge MU \xrightarrow{m} MU$$

(good for any ring spectrum) where m is (1.13). Define

(1.30) $$\beta_i \in MU_{2i}CP^\infty$$

by

(1.31) $$\langle x^i, \beta_j \rangle = \delta^j_i.$$

This definition is valid because of the similar duality for homology and the compatibility of the spectral sequence with the pairing. These β's reduce to the standard homology β's under the map

(1.32) $$MU_*X \longrightarrow H_*X$$

which we have from our knowledge of H^0MU.

Using our map (1.15), we define

(1.33) $$x_*(\beta_{i+1}) = b_i^{MU} \in MU_{2i}MU$$

just as with homology (1.17).

The Atiyah-Hirzebruch spectral sequence

(1.34) $$H_*(MU; MU_*) \Rightarrow MU_*MU \simeq \pi_*(MU \wedge MU)$$

collapses (it is even degree), giving

(1.35) $$MU_*MU \simeq MU_*[b_1, b_2, \dots].$$

(The upper MU on b_i^{MU} will appear only when it seems necessary.) This is the continuous dual of the Landweber-Novikov operations MU^*MU [$\mathbf{L_1}$, $\mathbf{N}$]. (The topology is obtained from the finite skeleta of MU.)

Using

(1.36) $$1: S^0 \longrightarrow MU \in MU_*$$

to get the map

(1.37) $$X \simeq S^0 \wedge X \longrightarrow MU \wedge X,$$

we can apply $MU_*(-)$ to get

(1.38) $$MU_*X \longrightarrow MU_*(MU \wedge X).$$

We need a standard lemma to proceed.

LEMMA 1.39. *Let $E_*(-)$ be a reduced generalized homology theory. Let E_*X be free over E_*. The exterior product gives a Künneth isomorphism*:

$$E_*X \otimes_{E_*} E_*Y \xrightarrow{\simeq} E_*(X \wedge Y). \quad \square$$

PROOF. Both $E_*X \otimes_{E_*} E_*(-)$ and $E_*(X \wedge -)$ are generalized homology theories. The exterior product provides a natural transformation between them. They agree on a point so the E^2 terms of the Atiyah-Hirzebruch spectral sequences are isomorphic and we are done. $\square$

By (1.38) and Lemma 1.39 we have

$$\psi_x: MU_*X \longrightarrow MU_*(MU \wedge X) \xleftarrow{\simeq} MU_*MU \otimes_{MU_*} MU_*X. \tag{1.40}$$

$MU_*MU = \pi_*(MU \wedge MU)$ has two distinct MU_* module structures: a left and a right (see below). From (1.35) we know the left module structure is free; so, by symmetry, the right is free. In the tensor product of (1.40) we are using the right module structure.

Let $X = MU$ and we see that MU_*MU is a "Hopf algebra". For a general X, MU_*X is a comodule over this Hopf algebra. We know the algebra structure of MU_*MU already from (1.35). We give the other structure maps; see **[A₁]**.

We have

$$\epsilon: MU_*MU \longrightarrow MU_*; \tag{1.41}$$

just apply $\pi_*(-)$ to (1.13). The left and right units,

$$\eta_L, \eta_R: MU_* \longrightarrow MU_*MU, \tag{1.42}$$

are obtained by applying $\pi_*(-)$ to

$$MU \wedge S^0 \longrightarrow MU \wedge MU \quad \text{and} \quad S^0 \wedge MU \longrightarrow MU \wedge MU \tag{1.43}$$

respectively. Next we have the conjugation

$$c: MU_*MU \circlearrowleft \tag{1.44}$$

which comes in the same way from the switch map

$$MU \wedge MU \circlearrowleft. \tag{1.45}$$

We consider $MU_* \subset H_*MU$, (1.20), and describe our maps accordingly. Note also that

$$MU_*MU \subset H_*MU[b_1, b_2, \ldots]. \tag{1.46}$$

Use b_i^{MU} to define exp and log series. As with homology, we obtain m_i^{MU}; see (1.22)–(1.24). At times we write

$$m = \sum_{i\geq 0} m_i, \quad m^{MU} = \sum_{i\geq 0} m_i^{MU},$$
$$\beta = \sum_{i\geq 0} \beta_i, \quad b = \sum_{i\geq 0} b_i, \quad \text{etc.} \tag{1.47}$$

When we use these symbols in equations, we mean the equation is valid degree by degree.

THEOREM 1.48. (a) $\epsilon(b_i) = 0$, $i > 0$, *i.e.* $\epsilon(b) = 1$.
(b) $\eta_L(m) = m$ *and* $\eta_R(m) = \Sigma_{i\geq 0} m_i(m^{MU})^{i+1}$.
(c) $c^2 = I$, $c\eta_L = \eta_R$, $c\eta_R = \eta_L$, *and* $c(b^{MU}) = m^{MU}$.
(d) *The coproduct* (1.40) *is* $\psi_{MU}(b) = \Sigma_{j\geq 0} b^{j+1} \otimes b_j$.
(e) *The coproduct for* MU_*CP^∞ *is* $\psi_{CP^\infty}(\beta) = \Sigma_{j\geq 0} b^j \otimes \beta_j$. □

These are formulas which we shall use later to obtain the corresponding formulas for BP. Because this is an introduction to the subject we will give the details of the proofs.

Part (a). $\epsilon(b_i) = \langle 1, b_i \rangle = \langle 1, x_* \beta_{i+1} \rangle = \langle x^*(1), \beta_{i+1} \rangle = \langle x, \beta_{i+1} \rangle = 0, \; i > 0.$

The remaining formulas are a corollary of one result. By mapping $MU \longrightarrow MU \wedge MU$ into the left or right factor as in (1.43) we can define elements $x^L, x^R, \beta_i^L, \beta_i^R$ in

(1.49) $\qquad (MU \wedge MU)^* \mathbf{C}P^\infty \simeq (MU \wedge MU)^*[[x]] \quad \text{and} \quad (MU \wedge MU)_* \mathbf{C}P^\infty,$

respectively. Here

(1.50) $\qquad (MU \wedge MU)^{-*} = (MU \wedge MU)_* = \pi_*(MU \wedge MU) = MU_*MU.$

LEMMA 1.51. $x^R = \Sigma_{i \geq 0} b_i^{MU} (x^L)^{i+1}$. □

Before we begin the proof, we need:

LEMMA 1.52. *The following diagram commutes*:

$$\begin{array}{ccc} MU^*\mathbf{C}P^\infty & \xrightarrow{\quad R \quad} & (MU \wedge MU)^*\mathbf{C}P^\infty \\ & {\scriptstyle\alpha}\searrow \qquad \swarrow{\scriptstyle \simeq\, P} & \\ & \hom_{MU_*}(MU_*\mathbf{C}P^\infty, MU_*MU) & \end{array}$$

and P *is an isomorphism. Here* R *is the Boardman map induced by*

$$MU = S^0 \wedge MU \longrightarrow MU \wedge MU, \qquad \alpha(f)(\beta_i) = f_*(\beta_i),$$

and

$$P(g)(\beta_i) = \langle g, \beta_i \rangle$$

where $\langle\ ,\ \rangle$ *is defined because* $MU \wedge MU$ *is an* MU *module spectrum by the map*

$$MU \wedge (MU \wedge MU) \simeq (MU \wedge MU) \wedge MU \xrightarrow{m \wedge I} MU \wedge MU. \quad \square$$

PROOF. $\alpha(f)(\beta_i)$ is $S \longrightarrow^{\beta_i} MU \wedge \mathbf{C}P^\infty \longrightarrow^{I \wedge f} MU \wedge MU$.

$R(f)$ is $\mathbf{C}P^\infty \longrightarrow^f MU \simeq S^0 \wedge MU \longrightarrow^R MU \wedge MU$.

So, $P(R(f))(\beta_i) = \langle R(f), \beta_i \rangle$ is

$$S \xrightarrow{\beta_i} MU \wedge \mathbf{C}P^\infty$$

$$\xrightarrow{f} MU \wedge MU \simeq MU \wedge S^0 \wedge MU \xrightarrow{I \wedge R} MU \wedge MU \wedge MU \xrightarrow{m \wedge I} MU \wedge MU.$$

The last maps can be ignored because they give the identity. We see that we are left with $\alpha(f)(\beta_i)$, thus proving the diagram commutes.

The fact that P is an isomorphism is really the more general fact that $F^*\mathbf{C}P^\infty \simeq \hom_{E_*}(E_*\mathbf{C}P^\infty, F_*)$ when F is a module spectrum over E and the Atiyah-Hirzebruch spectral sequence collapses. □

PROOF OF 1.51. We have $\alpha(x)(\beta_{i+1}) = b_i^{MU}$ by definition. We also have

$$P(x^L)^j(\beta_i) = \langle (x^L)^j, \beta_i\rangle = \delta_i^j.$$

To show this, we write down the map for the element $\langle (x^L)^j, \beta_i\rangle$:

$$S \xrightarrow{\beta_i} MU \wedge \mathbf{C}P^\infty \xrightarrow{I\wedge(x^L)^j} MU \wedge MU \wedge MU \xrightarrow{m\wedge I} MU \wedge MU$$

and observe that it factors

$$\begin{array}{ccccccc} S \xrightarrow{\beta_i} MU \wedge \mathbf{C}P^\infty & \xrightarrow{I\wedge(x^L)^j} & MU \wedge MU \wedge MU & \xrightarrow{m\wedge I} & MU \wedge MU \\ \big\downarrow{\scriptstyle I\wedge x^j} & & \big\uparrow & & \big\uparrow \\ MU \wedge MU & \xleftarrow{\simeq} & MU \wedge MU \wedge S^0 & \xrightarrow{m} & MU \wedge S^0 \end{array}$$

This is already δ_i^j in the lower right-hand corner. Since

$$(MU \wedge MU)^*\mathbf{C}P^\infty \simeq (MU \wedge MU)^*[[x^L]],$$

we know we can write $x^R = \Sigma_{i\geqslant 0} a_i (x^L)^{i+1}$. Then

$$\begin{aligned} a_i &= \left\langle \sum_{j\geqslant 0} a_j(x^L)^{j+1}, \beta_{i+1}\right\rangle = \langle x^R, \beta_{i+1}\rangle = P(x^R)(\beta_{i+1}) \\ &= P(R(x))(\beta_{i+1}) = \alpha(x)(\beta_{i+1}) = b_i^{MU}. \quad \square \end{aligned}$$

We return to the proof of 1.48.

Part (c). The first three parts are obvious from definitions. For the last, apply c to 1.51 to obtain

$$x^L = \sum_{i\geqslant 0} c(b_i^{MU})(x^R)^{i+1}.$$

Thus it is the inverse series to 1.51 which defines the m_i^{MU} so we have $c(b_i^{MU}) = m_i^{MU}$.

Part (b). $\eta_L(m) = m$ follows because η_L is a left module map. For η_R, let H be the integral Eilenberg-MacLane spectrum. We reduce the formula for x^L as follows and obtain two formulas for x^H:

$$\begin{array}{ccc} x^L = \displaystyle\sum_{i\geqslant 0} m_i^{MU}(x^R)^{i+1} & \in & (MU \wedge MU)^*\mathbf{C}P^\infty \\ \big\downarrow & & \big\downarrow \\ x^H = \displaystyle\sum_{i\geqslant 0} m_i x^{i+1} & \in & (H \wedge MU)^*\mathbf{C}P^\infty \\ \big\downarrow\;\big\downarrow & & \searrow\;\searrow \\ x^H = \left\{\begin{array}{l} \displaystyle\sum_{i\geqslant 0} m_i(x^L)^{i+1} \\ \displaystyle\sum_{i\geqslant 0} \eta_R(m_i)(x^R)^{i+1} \end{array}\right\} & \in & (H \wedge MU \wedge MU)^*\mathbf{C}P^\infty. \end{array}$$

So

$$\sum_{i\geq 0} \eta_R(m_i)(x^R)^{i+1} = \sum_{i\geq 0} m_i(x^L)^{i+1} = \sum_{i\geq 0} m_i\left(\sum_{j\geq 0} m_j^{MU}(x^R)^{j+1}\right)^{i+1},$$

or $\eta_R(m) = \Sigma_{i\geq 0} m_i(m^{MU})^{i+1}$.

Part (e). The coproduct for $\mathbf{CP}^\infty$ is defined by

$$\begin{array}{ccc} MU_*\mathbf{CP}^\infty & \longrightarrow & MU_*MU \otimes_{MU_*} MU_*\mathbf{CP}^\infty \\ & {\scriptstyle L}\searrow \quad \swarrow{\scriptstyle \simeq} & \\ & (MU \wedge MU)_*\mathbf{CP}^\infty & \end{array}$$

The element β_i goes to β_i^L, i.e. $\psi_{CP}(\beta_i) = \beta_i^L$, and $1 \otimes \beta_i$ is just β_i^R. We need to write β_i^L in terms of β_i^R, say $\beta_i^L = \Sigma_{k=0}^i a_{i-k} \otimes \beta_k$. We recall $x^R = \Sigma_{i\geq 0} b_i^{MU}(x^L)^{i+1}$ which gives

$$(x^R)^j = \sum_{k\geq 0} (b^{MU})_k^j (x^L)^{j+k}$$

where $(b^{MU})_k^j$ means the $2k$ degree component of $(b^{MU})^j$. We have

$$\begin{aligned} a_{i-j} &= \left\langle (x^R)^j, \sum_{k=0}^{i} a_{i-k} \otimes \beta_k \right\rangle = \langle (x^R)^j, \beta_i^L \rangle \\ &= \left\langle \sum_{k\geq 0} (b^{MU})_k^j (x^L)^{j+k}, \beta_i^L \right\rangle = (b^{MU})_{i-j}^j. \end{aligned}$$

So $\psi_{CP^\infty}(\beta_i) = \Sigma_{0\leq j\leq i}(b^{MU})_{i-j}^j \otimes \beta_j$, or

$$\psi_{CP^\infty}(\beta) = \sum_{j\geq 0} (b^{MU})^j \otimes \beta_j.$$

Part (d). This follows at once from part (e). □

We will need one more result about MU.

LEMMA 1.53. *There is a* 1-1 *correspondence between maps of ring spectra, g: $MU \longrightarrow MU$, and power series*

$$f(x) = \sum_{i\geq 0} d_i x^{i+1} \in MU^2\mathbf{CP}^\infty, \quad d_0 = 1.$$

The correspondence is given by $g_*(x) = f(x)$. □

PROOF.

$$MU^*MU \simeq \hom_{MU_*}(MU_*MU, MU_*),$$

$$MU^*(MU \wedge MU) \simeq \hom_{MU_*}(MU_*(MU \wedge MU), MU_*),$$

$$MU_*(MU \wedge MU) \simeq MU_*MU \otimes_{MU_*} MU_*MU.$$

Let $\theta_g \in \hom_{MU_*}(MU_*MU, MU_*)$ correspond to $g \in MU^*MU$. The diagram

$$\begin{array}{ccc} MU \wedge MU & \xrightarrow{g\wedge g} & MU \wedge MU \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle m} \\ MU & \xrightarrow{\quad g \quad} & MU \end{array}$$

shows that θ_g is a map of algebras if and only if g is a map of ring spectra. Let $\theta_g(b_i) = d_i \in MU_{2i}$, $i > 0$ $(\theta_g(b_0 = 1) = 1)$, and let $g_*(x) = \Sigma_{i \geq 0} a_i x^{i+1}$.

$$\begin{aligned} d_i &= \theta_g(b_i) = \theta_g(x_*\beta_{i+1}) = \langle g, x_*\beta_{i+1}\rangle = \langle x^*g, \beta_{i+1}\rangle \\ &= \langle g_*x, \beta_{i+1}\rangle = \left\langle \sum_{j\geq 0} a_j x^{j+1}, \beta_{i+1}\right\rangle = a_i. \quad \square \end{aligned}$$

2. **Formal groups.** A great time can be had with formal groups, especially in connection with MU and BP. In fact it could easily be made the main theme of lecture notes with the same title as these. However, our theme must be elsewhere and the reader will regrettably be exposed only to the necessary minimum.

A *commutative formal group law* over a graded ring R_* is a power series

(2.1) $$F(x, y) = \sum_{i,j} a_{ij} x^i y^j \in R_*[[x, y]], \qquad a_{ij} \in R_{2(i+j-1)},$$

with:

$F(x, y) = F(y, x)$, commutativity,
$F(F(x, y), z) = F(x, F(y, z))$, associativity, and
$F(x, 0) = x = F(0, x)$, identity.

Many relations on the a_{ij} follow. Some simple ones are:

(2.2) $$a_{ij} = a_{ji}; \qquad a_{k,0} = 0, \quad k \neq 1; \qquad a_{10} = 1.$$

The cohomology theory $MU^*(-)$ gives rise to a formal group law over MU_*. Just apply $MU^*(-)$ to the standard map

(2.3) $$\mu\colon \mathbf{C}P^\infty \times \mathbf{C}P^\infty \longrightarrow \mathbf{C}P^\infty$$

to get

(2.4) $$\mu^*\colon MU^*\mathbf{C}P^\infty \longrightarrow MU^*(\mathbf{C}P^\infty \times \mathbf{C}P^\infty) \simeq MU^*\mathbf{C}P^\infty \hat{\otimes} MU^*\mathbf{C}P^\infty.$$

We obtain the formal group law

(2.5) $$F(x_1, x_2) = \mu^*(x) = \sum_{i,j} a_{ij} x_1^i \hat{\otimes} x_2^j.$$

Its properties follow from the properties of μ: homotopy commutativity, associativity, etc.

As usual we can best describe F by considering

(2.6) $$MU^*\mathbf{C}P^\infty \subset (H \wedge MU)^*\mathbf{C}P^\infty.$$

Applying c to 1.51 we have

$$x^L = \sum_{i \geq 0} m_i^{MU}(x^R)^{i+1} \in (MU \wedge MU)^2 CP^\infty. \tag{2.7}$$

Reducing to $(H \wedge MU)^* CP^\infty$ we have

$$x^H = \sum_{i \geq 0} m_i x^{MU} = \log x. \tag{2.8}$$

The element x^H is primitive, so apply μ^*:

$$\log F(x_1, x_2) = x_1^H + x_2^H = \log x_1 + \log x_2. \tag{2.9}$$

Now apply exp(–) to obtain:

$$F(x_1, x_2) = \exp(\log x_1 + \log x_2). \tag{2.10}$$

LEMMA 2.11. *The primitives of* $MU_Q^* CP^\infty$ *are* MU_Q^* *free on* $\log x^{MU}$. □

PROOF. Because $H_*(CP^\infty, Q)$ is a polynomial algebra on a two dimensional element, $MU_*^Q CP^\infty$ is a polynomial algebra over MU_*^Q on a two dimensional element. By duality there is only one primitive free over MU^* rationally. We know $\log x^{MU}$ is a primitive and a quick check of its leading coefficient shows it is the generator we want. □

The ring for a *universal formal group law* can be constructed using arbitrary a_{ij}'s and the relations from (2.1). Lazard has determined the structure of this ring. The theory of rational formal group laws is trivial; (2.10)–(2.12).

One of the strongest connections between bordism and formal groups is displayed in the following theorem.

THEOREM 2.12 (QUILLEN [Q_2], OR SEE [A_2]). *The formal group law for complex cobordism is isomorphic to the universal formal group law of Lazard.* □

Given any formal group law $F(x, y)$ over R_*, there is a ring map g: $MU_* \longrightarrow R_*$ which induces the formal group; that is

$$g(a_{ij}^{MU}) = a_{ij}^R. \tag{2.13}$$

The result can be made to appear even stronger because a topological analogue is true. It follows, however, from a slight generalization of 1.53, even without Quillen's result, and it does not imply 2.12.

THEOREM 2.14. *If $F(x, y)$ is a formal group law over E_* which comes from a generalized cohomology theory $E^*(-)$ and the map (2.3), then there is a map of ring spectra, $MU \longrightarrow E$, which induces the formal group law.* □

We define the *formal group sum*:

$$x +_F y = F(x, y) = \exp(\log x + \log y). \tag{2.15}$$

We define

(2.16) $$[1](x) = x = \exp(\log x),$$

and, inductively,

$$[n](x) = [n-1](x) +_F x = \exp(n \log x), \qquad n > 1.$$

We have

(2.17) $$[-1](x) = \iota(x) = \exp(-\log x)$$

which implies

(2.18) $$F(x, \iota(x)) = 0.$$

To prove this is well defined, let

(2.19) $$\iota(x) = \sum_{k \geq 0} c_k x^{k+1}.$$

Then

(2.20) $$F(x, \iota(x)) = \sum_{i,j} a_{ij} x^i \left(\sum_{k \geq 0} c_k x^{k+1} \right)^j = 0$$

can be solved inductively for c_n since $a_{01} = 1$.

This allows us to do formal group subtraction:

(2.21) $$x -_F y = x +_F ([-1](y)).$$

We need only one more elementary formal group fact for the construction of *BP*.

LEMMA 2.22. *The power series* $[1/d](x) = \exp(1/d \log x)$ *is defined over* $MU_* \otimes Z_{[1/d]}$.

PROOF. Clearly,

$$[d]([1/d](x)) = \exp(d \log(\exp(1/d \log x))) = \exp(d\, 1/d \log x) = x.$$

Let $[1/d](x) = \Sigma_{j \geq 0} c_j x^{j+1}$ and $[d](x) = \Sigma_{i \geq 0} a_i x^{i+1}$; then

$$x = \sum_{i \geq 0} a_i \left(\sum_{j \geq 0} c_j x^{j+1} \right)^{i+1}$$

can be solved inductively for c_n. We get that $c_0 = a_0^{-1} = 1/d$ and c_n is obtained in terms of a_i, c_j, $j < n$, and a_0^{-1}; so when d has been inverted, the power series $[1/d](x)$ is defined. □

3. Brown-Peterson homology. Brown and Peterson constructed *BP*, known to them as *X*, in **[BP]**. They built *BP* using a generalized Postnikov system. Novikov also gave a construction **[N]**. *BP* was not computationally useful until a description of its operation ring was available. This was produced by Quillen **[Q$_1$]** and written up by Adams **[A$_2$]**.

We localize the spectrum MU at a prime p and find a multiplicative idempotent ϵ of $MU_{(p)}$. The image of a multiplicative idempotent in $MU_*(X)_{(p)}$ is a natural direct summand and so gives rise to the multiplicative generalized homology theory called Brown-Peterson homology: BP_*X. The representing spectrum is called BP. One of the problems in the field is that there are no pointwise models for BP, only homotopy constructions.

The idempotent of interest is the composition of a family of commuting idempotents. We describe these idempotents.

LEMMA 3.1. *Let q be a prime. There exists a multiplicative idempotent ϵ_q on $MU_{[1/q]}$ such that on $H_*(MU_{[1/q]})$*

$$\epsilon_q(m_n) = \begin{cases} m_n, & n+1 \not\equiv 0\ (q), \\ 0, & n+1 \equiv 0\ (q). \end{cases}$$

The ϵ_q commute. □

From this we have:

THEOREM 3.2 (QUILLEN [Q₁]). *The multiplicative idempotent*

$$\epsilon = \prod_{q \neq p} \epsilon_q \in MU^*MU_{(p)}$$

is well defined. On $H_*(MU_{(p)})$

$$\epsilon(m_n) = \begin{cases} 0, & n \neq p^i - 1, \\ m_n, & n = p^i - 1. \end{cases}$$

The image of ϵ_* in $MU_*(X)_{(p)}$ is a multiplicative generalized homology theory denoted by BP_*X, with

$$Z_{(p)}[v_1, v_2, \ldots] \simeq \pi_*BP \subset H_*BP \simeq Z_{(p)}[m_{p-1}, m_{p^2-1}, \ldots]. \quad \square$$

PROOF OF 3.2. The product $\Pi_{q \neq p} \epsilon_q$ is convergent in the topology on $MU^*MU_{(p)}$ because for large primes ϵ_q is the identity for large skeleta. The commuting of the ϵ_q makes ϵ an idempotent. The rest follows from the lemma and above. □

Lemma 1.53 on multiplicative maps is true after localization. The same proof applies. Multiplicative maps are identified there in terms of what they do on x^{MU}. Our ϵ_q, however, is described in terms of what it does to $H_*MU_{[1/q]}$. We need to go from one type of information to the other type. Let $f(x^{MU})$ be a power series representing a multiplicative operation g. We define

$$\text{mog}\, x = \sum_{n \geq 0} g_*(m_n) x^{n+1}. \tag{3.3}$$

Since $\log x^{MU}$ is primitive, (2.11), by checking the first coefficient and using the fact that $g_*(\log x^{MU})$ must still be primitive, we have

$$g_*(\log x^{MU}) = \log x^{MU}. \tag{3.4}$$

Also,

$$g_*(\log x^{MU}) = \operatorname{mog} f(x^{MU}). \tag{3.5}$$

Let $f^{-1}(x)$ be the inverse power series, i.e. $f^{-1}(f(x)) = x = f(f^{-1}(x))$. From (3.4)–(3.5)

$$\log x^{MU} = \operatorname{mog} f(x^{MU}), \tag{3.6}$$

so

$$\log f^{-1}(x) = \operatorname{mog} x \tag{3.7}$$

and by applying $\exp(-)$ we have

$$f^{-1}(x) = \exp(\operatorname{mog} x). \tag{3.8}$$

We have computed $f(x)$ in terms of $g_*(m_n)$ (and vice versa).

PROOF OF 3.1. We have a chosen mog for ϵ_q. Our only problem is to see that the corresponding $f(x)$ has coefficients in MU_* after inverting q. This is done indirectly. Let ξ_i be the qth roots of unity, $q \geqslant i > 0$. Since $x^q - 1 = \Pi_{i>0}^q (x - \xi_i)$, most symmetric functions are zero. In particular,

$$\xi_1^k + \cdots + \xi_q^k = \begin{cases} 0, & k \not\equiv 0 \ (q), \\ q, & k \equiv 0 \ (q). \end{cases}$$

We see that we can rewrite our desired mog x as

$$\operatorname{mog} x = \log x - \frac{1}{q}(\log \xi_1 x + \cdots + \log \xi_q x).$$

Our formal group nonsense comes in handy here. This tells us, by (3.8),

$$f^{-1}(x) = \exp(\operatorname{mog} x) = x -_F \left[\frac{1}{q}\right](\xi_1 x +_F \cdots +_F \xi_q x).$$

Because this is obtained using formal group sums and the $[1/q]$ sequence, we see that the coefficients of $f^{-1}(x)$ are in $MU_*[\xi_1, \ldots, \xi_q]_{[1/q]}$. However, all terms involving the ξ_i are symmetric, and the symmetric functions in the ξ_i are integral. Thus $f^{-1}(x)$ has coefficients in $MU_{*[1/q]}$ and since its first term is x, the inverse $f(x)$ exists and has coefficients in $MU_{*[1/q]}$.

This gives a multiplicative map with the desired properties. The commuting of the ϵ_q can be checked by evaluating in $\hom_{MU_{*[1/q]}}(MU_* MU_{[1/q]}, MU_{*[1/q]})$ which is, rationally, the same as $\hom_{\mathbf{Q}}(H_*(MU; \mathbf{Q}), H_*(MU; \mathbf{Q}))$. Likewise for $\epsilon_q \circ \epsilon_q = \epsilon_q$. □

We develop the basic properties of BP_*BP. Just as with MU, we will do our computations with

$$BP_*BP \subset (H \wedge BP)_*BP. \tag{3.9}$$

We occasionally change the indexing on the m's so that

$$|m_n| = 2(p^n - 1). \tag{3.10}$$

THEOREM 3.11 (QUILLEN [Q_1]). (i) *There are* $t_i \in BP_{2(p^i-1)}BP$, $t_0 = 1$, *such that* $\eta_R(m_k) = \Sigma_{i=0}^{k} m_i t_{k-i}^{p^i}$.

(ii) $BP_*BP \simeq BP_*[t_1, t_2, \ldots]$.

(iii) c *is given by*

$$m_k = \sum_{0 \leq i+j \leq k} m_i t_j^{p^i} c(t_{k-i-j})^{p^{i+j}} \quad or \quad 1 = \sum_{n,j\geq 0}^{F} t_n c(t_j)^{p^n}.$$

(iv) *The counit* ϵ *has* $\epsilon(1) = 1$, $\epsilon(t_i) = 0$, $i > 0$.

(v) *The coproduct* ψ *is computed by*

$$\sum_{i=0}^{k} m_i (\psi(t_{k-i}))^{p^i} = \sum_{h+i+j=k} m_h t_i^{p^h} \otimes t_j^{p^{h+i}}$$

or

$$\sum_{i\geq 0}^{F} \psi(t_i) = \sum_{i,j\geq 0}^{F} t_i \otimes t_j^{p^i}. \quad \square$$

Ravenel first showed us the nice formal group sum versions of these formulas. Note the similarities with the formulas for the dual of the Steenrod algebra where formal group addition is just regular addition and you use the conjugates of the normal generators for the dual of the Steenrod algebra.

PROOF OF 3.11. *Parts* (i) *and* (ii). The formula in (i) can be used to define the t's. The question is whether or not they actually lie in BP_*BP, since they were defined in $(H \wedge BP)_*BP$. We recall 1.48(b) for MU:

$$\eta_R(m) = \sum_{i\geq 0} m_i (m^{MU})^{i+1}.$$

Apply the idempotent of 3.2 to obtain $\Sigma_{i\geq 0}\eta_R(m_{p^i-1}) = \Sigma_{i\geq 0} m_{p^i-1}(m^{BP})^{p^i}$. Rewrite the left-hand side using the formula defining the t's in (i):

$$\sum_{i\geq 0} \eta_R(m_{p^i-1}) = \sum_{k,j\geq 0} m_{p^j-1} t_k^{p^j}.$$

Using these two expressions we have $\Sigma_{k\geq 0} \log^{BP} t_k = \log^{BP} m^{BP}$. Apply $\exp^{BP}(-)$ to obtain

(3.12)
$$\sum_{k\geq 0}^{F} t_k = m^{BP}.$$

Since $m^{BP} \in BP_*BP$ it is simple to prove, by induction on k, that t_k is in BP_*BP. Furthermore, it shows that $t_k = m_{p^k-1}^{BP}$ modulo lower m's. Since these m's reduce to the generating m's in homology we have part (ii).

Part (iii). Apply c to (i),

$$\eta_R(m_k) = \sum_{i=0}^{k} m_{p^i-1} t_{k-i}^{p^i} = \sum_{i+j=k} m_i t_j^{p^i},$$

to obtain

$$m_k = \sum_{i+j=k} \eta_R(m_i)c(t_j)^{p^i},$$

which is

$$= \sum_{s+n+j=k} m_s t_n^{p^s} c(t_j)^{p^{s+n}} \quad \text{by (i).}$$

Add these over k and apply exp to get the formal group sum version of the formula.

Part (iv). The proof is by induction. Apply ϵ to (i),

$$\eta_R(m_k) = \sum_{i+j=k} m_i t_j^{p^i},$$

to get $m_k = m_k + \epsilon(t_k)$, so $\epsilon(t_k) = 0$.

Part (v). Apply ψ to part (i), $\eta_R(m_k) = \Sigma_{i+j=k} m_i t_j^{p^i}$, to get

$$1 \otimes \eta_R(m_k) = \sum_{i+j=k} m_i \psi(t_j)^{p^i}.$$

The left-hand side, by part (i), is

$$1 \otimes \sum_{i+j=k} m_i t_j^{p^i} = \sum_{i+j=k} \eta_R(m_i) \otimes t_j^{p^i},$$

and again by part (i),

$$= \sum_{h+s+j=k} m_h t_s^{p^h} \otimes t_j^{p^{h+s}}.$$

This proves the first formula. Just add over k to get

$$\sum_{j\geq 0} \log \psi(t_j) = \sum_{s,j\geq 0} \log t_s \otimes t_j^{p^s}.$$

Apply exp to obtain the formal group sum version. □

With our new generators t_i, it seems that we have lost track of the coproduct on CP^∞. It can be recovered. This is an example of how much information remains in $[\mathbf{A}_2]$. Several people have studied operations on BP^*CP^∞ without noticing the following simple formula (which, unfortunately for the author, was first conjectured by computations).

THEOREM 3.13 [**RW**$_1$]. *The coaction* $\psi_{CP^\infty}: BP_*CP^\infty \longrightarrow BP_*BP \otimes_{BP_*} BP_*CP^\infty$ *is given by*

$$\psi_{CP^\infty}(\beta^{BP}) = \sum_{i\geq 0} \left(c\left(\sum_{n\geq 0}^{F} t_n\right)\right)^i \otimes \beta_i^{BP}. \quad \square$$

PROOF. Apply the idempotent to the MU_* coproduct 1.48(e) to obtain

$$\psi_{CP^\infty}(\beta^{BP}) = \sum_{i\geq 0} (b^{BP})^i \otimes \beta_i^{BP}.$$

Substitute $b^{BP} = c(m^{BP})$ from 1.48(c) and $m^{BP} = \Sigma^F_{n \geq 0} t_n$ from (3.12). □

For future use we record the formula used in the proof above.

LEMMA 3.14. $b^{BP} = c(\Sigma^F_{n\geq 0} t_n)$. □

Even the most naive reader should realize there are problems with viewing BP_* only as a subring of H_*BP. Historically, general computations with BP had to wait not only for Quillen's description of BP_*BP, but also for explicit generators for BP_*. In retrospect, many results do not need explicit generators to be proven but they were essential for the first proofs.

Liulevicius constructed generators for BP at $p = 2$ in **[Li]**. Then Hazewinkel constructed generators at all primes **[H₁, H₂]**. Other generators followed; **[K, Ar₁]**, etc. Unfortunately, Hazewinkel's generators came after Adams's lecture notes **[A₂]** and were not included.

THEOREM 3.15 (HAZEWINKEL **[H₁, H₂]**). *Generators for*

$$\begin{array}{ccc} \pi_*BP & \subset & H_*BP \\ \updownarrow \simeq & & \updownarrow \simeq \\ Z_{(p)}[v_1, v_2, \ldots] & \subset & Z_{(p)}[m_1, m_2, \ldots] \end{array}$$

$$|v_n| = 2(p^n - 1) = |m_n|$$

are given by

$$v_n = pm_n - \sum_{i=1}^{n-1} m_i v_{n-i}^{p^i}. \quad \square$$

PROOF. From (1.19) and the properties of the idempotent, it is clear that an element $w_n \in BP_{2(p^n-1)}$ is a generator if and only if $w_n = \lambda m_n + \cdots$, $p|\lambda$, $\lambda \not\equiv 0\ (p^2)$. Our formula for v_n meets this requirement if v_n lies in BP_*.

LEMMA 3.16. $\exp(px) \in pBP_*[[x]]$. □

PROOF. From (1.25) we know that $p^n m_n = [CP^{p^n-1}] \in BP_*$. Thus $p^k H_{2k(p-1)}BP \subset \pi_{2k(p-1)}BP$, the worst case being m_1^k, so $p^n b_n$ is clearly in BP_*. Thus

$$\exp(px) = \sum_{i\geq 0} b_i (px)^{i+1} \in pBP_*[[x]]. \quad \square$$

Rewrite the formula for v_n as $pm_n = \Sigma_{i=0}^{n-1} m_i v_{n-i}^{p^i}$ and add over all $n > 0$:

$$pm - p = \sum_{\substack{0 \leq i+j \\ 0<j}} m_i v_j^{p^i}, \qquad m = \sum_{m\geq 0} m_n.$$

Write p as $\log(\exp(p))$. The above is now $p\log(1) = \log(\exp(p)) + \Sigma_{j>0} \log v_j$. Apply exp:

$$[p](1) = \exp(p) +_F \sum_{j>0}^F v_j.$$

Since $\exp(p) \in pBP_*$ and $[p](1)$ has coefficients in BP_* we can easily show, by induction on j, that $v_j \in BP_*$. □

In the above proof we showed:

LEMMA 3.17. $[p](x) \equiv \Sigma^F_{j>0} v_j x^{p^j} \bmod (p)$. □

This is a useful formula. Among other things, it says that the coefficients of x^{p^n} in the $[p]$-sequence are generators. This was surely known to Morava and probably Quillen before other people were even interested. We see, however, that the result can be obtained from **[CF$_1$]**, which existed before the question could be formulated. Araki's generators **[Ar$_1$]** have the property that the formula in 3.17 holds without going mod (p).

We have introduced the basic formulas for *BP*. The reader is now equipped to use *BP* as a tool. The formulas, however, can be difficult to use. They are inductive. In principle, one needs to know everything in lower dimensions in order to compute the next. This is in sharp contrast to the closed formulas in the Adem relations. In practice it is impossible to keep track of all of the information forever. Consequently, much of the job of a beeper (*BP*'er) is to be able to skim some information from the formulas without having to deal with it all and come out alive. There is strong motivation to do this. The formulas contain a vast amount of information.

The Brown-Peterson spectrum sits midway between the sphere spectrum and the Eilenberg-MacLane spectrum. It has a nice balance between its homotopy and homology:

(3.18) $$S^0_{(p)} \longrightarrow BP \longrightarrow HZ_{(p)}$$

homology:	good	okay	bad
homotopy:	bad	okay	good

It can be expected to be useful because a tremendous amount of information has already been used in its construction.

It has the advantage of small size over *MU*. Anyone who has worked with a polynomial algebra prefers one whose generators go up exponentially in degree (like *BP*) to one where they go up linearly (like *MU*).

The mod (p) cohomology of *BP* is just the reduced pth powers, i.e., the Steenrod algebra modulo the left-right ideal generated by the Bockstein, or the left ideal generated by the Milnor Bocksteins, Q_n. The k-invariants of *BP* are given by all of the higher order operations which come from the Milnor Q_n's. Philosophically, using *BP* instead of mod (p) homology should have the effect of changing this higher order information from the Q_n's into simple BP_* module structure information. Thus it is reasonable to expect BP_* module structures which contain information which is difficult to gain access to from a standard homology point of view. On the other hand, *BP* can be expected to be a useful tool for only certain types of problems: those where there is higher order Q_n information and it has been turned into accessible BP_* module structure information. The next section on stable homotopy gives an example of the success of this principle. However, the process of trading inaccessible higher order information for inductive formulas does not make life easy.

A useful way of viewing Quillen's splitting is as **[Q₁]**

$$BP_* \otimes_{MU_*} MU_*X \simeq BP_*X, \qquad MU_{*(p)} \otimes_{BP_*} BP_*X \simeq MU_*X_{(p)}. \tag{3.19}$$

We end this section with a few examples of formulas taken from Giambalvo's paper **[G]**. There are more terms and formulas in his paper.

$p = 2$.

$$\begin{aligned}[2](x) = {} & 2x - v_1x^2 + 2v_1^2x^3 - (7v_2 + 8v_1^3)x^4 + (30v_1v_2 + 26v_1^4)x^5 \\ & - (111v_1^2v_2 + 84v_1^5)x^6 + (502v_1^3v_2 + 300v_1^6 + 112v_2^2)x^7 \\ & - (127v_3 + 960v_1v_2^2 + 2299v_1^4v_2 + 1140v_1^7)x^8 \\ & + (766v_1v_3 + 5414v_1^2v_2^2 + 9958v_1^5v_2 + 4334v_1^8)x^9 + \dots,\end{aligned}$$

$$\eta_R(v_1) = v_1 + 2t_1, \qquad \eta_R(v_2) = v_2 + 2t_2 - 5v_1t_1^2 - 4t_1^3 - 3v_1^2t_1,$$

$$c(t_1) = -t_1, \qquad c(t_2) = -t_2 - v_1t_1^2 - t_1^3,$$

$$\psi(t_1) = t_1 \otimes 1 + 1 \otimes t_1, \qquad \psi(t_2) = t_2 \otimes 1 + 1 \otimes t_2 + t_1 \otimes t_1^2 - v_1t_1 \otimes t_1.$$

$p = 3$.

$$\begin{aligned}[3](x) = {} & 3x - 8v_1x^3 + 72v_1^2x^5 - 840v_1^3x^7 - (6560v_2 - 9000v_1^4)x^9 \\ & + (216504v_1v_2 - 88992v_1^5)x^{11} - (5360208v_1^2v_2 - 658776v_1^6)x^{13} \\ & + (119105576v_1^3v_2 + 1199088v_1^7)x^{15} + \dots,\end{aligned}$$

$$\eta_R(v_1) = v_1 + 3t_1,$$

$$\eta_R(v_2) = v_2 - 4v_1^3t_1 - 18v_1^2t_1^2 - 35v_1t_1^3 - 27t_1^4 + 3t_3,$$

$$c(t_1) = -t_1, \qquad c(t_2) = -t_2 + t_1^4,$$

$$\psi(t_1) = t_1 \otimes 1 + 1 \otimes t_1,$$

$$\psi(t_2) = t_2 \otimes 1 + 1 \otimes t_2 - v_1t_1^2 \otimes t_1 - v_1t_1 \otimes t_1^2 + t_1 \otimes t_1^3.$$

Part II. A Sampler

4. Cooperations and stable homotopy. We begin our sampling of BP with a closer look at BP_*BP with an eye to applications. There is an obvious interesting object:

$$\text{Ext}^{**}_{BP_*BP}(BP_*, A) \equiv H^{**}(A), \tag{4.1}$$

where A is a BP_*BP comodule. We will concentrate mostly on simple cases of

$$A = BP_*/I \tag{4.2}$$

where I is an *invariant ideal*, i.e., an ideal which is also a subcomodule of BP_*. Since the coproduct (1.40)

$$BP_* \longrightarrow BP_*BP \otimes_{BP_*} BP_* \simeq BP_*BP \tag{4.3}$$

is a left module map, I is invariant if and only if

$$IBP_*BP \simeq BP_*BPI. \tag{4.4}$$

If we want to verify I is an invariant ideal, by symmetry it is enough to check that

$$\eta_R(I) \subset IBP_*BP. \tag{4.5}$$

Thus it would be useful to know $\eta_R(v_n)$ (since η_R is an algebra homomorphism). We give a formula which we shall use many times.

THEOREM 4.6 (RAVENEL [**R**$_1$]).

$$\sum_{\substack{i\geq 0\\ j>0}}^F t_i\eta_R(v_j)^{p^i} \equiv \sum_{\substack{i>0\\ j\geq 0}}^F v_i t_j^{p^i} \quad \text{mod}(p). \quad \square$$

PROOF. From 3.15, $pm_n = \Sigma_{0\leq i<n} m_i v_{n-i}^{p^i}$. Apply η_R,

$$p\eta_R(m_n) = \sum_{0\leq i<n} \eta_R(m_i)\eta_R(v_{n-i})^{p^i},$$

and 3.11(i),

$$p\sum_{i+j=n} m_i t_j^{p^i} = \sum_{0\leq j+k<n} m_j t_k^{p^j}\eta_R(v_{n-j-k})^{p^{j+k}}.$$

Add over all n, substituting 3.15 in the left-hand side,

$$p \sum_{n \geq 0} t_n + \sum_{\substack{h+k+j=n \\ k>0}} m_h v_k^{p^h} t_j^{p^{h+k}} = \sum_{\substack{k \geq 0 \\ s>0}} \log t_k \eta_R(v_s)^{p^k}.$$

Rewrite this as

$$\log \exp\left(p\left(\sum_{n \geq 0} t_n\right)\right) + \sum_{\substack{k>0 \\ j \geq 0}} \log v_k t_j^{p^k} = \sum_{\substack{k \geq 0 \\ s>0}} \log t_k \eta_R(v_s)^{p^k}.$$

Apply exp,

$$\exp\left(p\left(\sum_{n \geq 0} t_n\right)\right) +_F \sum_{\substack{k>0 \\ j \geq 0}}^F v_k t_j^{p^k} = \sum_{\substack{k \geq 0 \\ s>0}}^F t_k \eta_R(v_s)^{p^k}.$$

Reducing mod(p) we are through by 3.16. □

This is a useful formula for extracting information about $\eta_R(-)$.

LEMMA 4.7. *The ideal* $I_n = (p, v_1, \ldots, v_{n-1}) \subset BP_*$ *is invariant.* □

PROOF. We need only show that $\eta_R(v_j) \in I_n BP_*BP$, $0 \leq j < n$. Inductively it is enough to show $\eta_R(v_{n-1}) \in I_n BP_*BP$. We will show a little more.

LEMMA 4.8. $\eta_R(v_n) = v_n \bmod I_n$. □

PROOF. Inductively we can assume I_n is invariant and we have $I_n BP_*BP = BP_*BPI_n$ so working modulo I_n makes sense. Use 4.6 in degree $2(p^n - 1)$ modulo I_n; 4.8 follows. □

This shows 4.7 inductively. □

Using Ravenel's theorem on Lemmas 4.7–4.8 is overpowering.

THEOREM 4.9 (LANDWEBER [L_2], MORAVA [Mo_1]). *The invariant prime ideals are* I_n, $0 \leq n \leq \infty$. □

SKETCH OF THE PROOF. We know I_n is invariant and prime. Assume I is an invariant prime ideal. Inductively assume $I_n \subset I$; we show that if $I_n \neq I$, then $I_{n+1} \subset I$. Let $y \in I - I_n$. If we could show

$$\eta_R(y) = a v_n^k t^J + \cdots, \qquad a \not\equiv 0 \ (p), \tag{4.10}$$

then we would have $v_n^k \in I$ and since I is prime, v_n as well; thus $I_{n+1} \subset I$. A proof of (4.10) is not difficult at this point; however, it does require some serious bookkeeping to know how to choose $t^J = t_1^{j_1} t_2^{j_2} \cdots$ given y in terms of v's. We refer the reader to the original proofs or the more recent [JW_2]. □

Having meandered into operations only a little way, we are already in a position to state a rather nice internal structure theorem on BP_*X.

LANDWEBER DECOMPOSITION THEOREM 4.11 (LANDWEBER [$\mathbf{L_3}$]). *For X a finite complex, there exists a finite sequence of subcomodules over BP_*BP, $0 = M_0 \subset M_1 \subset \cdots \subset M_n = BP_*X$, with*

$$M_{i+1}/M_i \simeq BP_*/I_{n_i}, \quad i \geqslant 0, \infty > n_i \geqslant 0. \quad \square$$

The problem of 4.1 arises naturally from results like 4.7–4.9. Let I be an invariant ideal; then

$$H^0 BP_*/I \simeq \hom_{BP_*BP}(BP_*, BP_*/I) \subset BP_*/I. \tag{4.12}$$

Let $a \in BP_*/I$ determine such a map. Then we have the commuting diagram

$$\begin{array}{ccc} BP_* & \longrightarrow & BP_*BP \otimes_{BP_*} BP_* \\ \Big\downarrow{\scriptstyle a} & & \Big\downarrow{\scriptstyle 1 \otimes a} \\ BP_*/I & \longrightarrow & BP_*BP \otimes_{BP_*} BP_*/I \end{array} \tag{4.13}$$

where, since the comodule maps are BP_* module maps we have

$$\begin{array}{ccc} 1 & \longrightarrow & 1 \otimes 1 \\ \Big\downarrow & & \Big\downarrow \\ a & \longrightarrow & a \otimes 1 = 1 \otimes a \end{array} \tag{4.14}$$

so

$$\eta_R(a) \equiv \eta_L(a) \quad \text{modulo } I \tag{4.15}$$

determines $H^0 BP_*/I$.

THEOREM 4.16 (LANDWEBER [$\mathbf{L_2}$]).

$$H^0 BP_* \simeq Z_{(p)}, \qquad H^0 BP_*/I_n \simeq \mathbf{F}_p[v_n], n > 0. \quad \square$$

SKETCH OF THE PROOF. By 4.8 we have $\mathbf{F}_p[v_n] \subset H^0 BP_*/I_n$. For other $a \in BP_* - I_n$, by (4.10), $\eta_R(a) \not\equiv \eta_L(a) \bmod I_n$. $\square$

This makes $H^k BP_*/I_n$ into an $\mathbf{F}_p[v_n]$ module. Since $\mathbf{F}_p[v_n]$ is a P.I.D. we know that $H^k BP_*/I_n$ is a direct sum of free copies of $\mathbf{F}_p[v_n]$ and cyclic modules, $\mathbf{F}_p[v_n]/(v_n^i)$. This develops a craving for the next step, $H^1 BP_*/I_n$, which, for reasons that will be clear in a moment, seems accessible. Before we move on to the higher Ext groups we should have some method of computation [**M**].

Let

$$\Omega^n(BP_*BP, A) \equiv \underbrace{BP_*BP \otimes_{BP_*} \cdots \otimes_{BP_*} BP_*BP}_{n \text{ copies}} \otimes_{BP_*} A \tag{4.17}$$

be the cobar resolution with

$$d(\gamma_1|\cdots|\gamma_n|a) = 1|\gamma_1|\cdots|\gamma_n|a + \sum_{i=1}^{n}\sum(-1)^i\gamma_1|\cdots|\gamma_i'|\gamma_i''|\cdots|\gamma_n|a \tag{4.18}$$
$$+ (-1)^{n+1}\sum\gamma_1|\cdots|\gamma_n|a'|a''$$

where

$$\psi(\gamma) = \sum\gamma'\otimes\gamma'' \quad\text{and}\quad \psi(a) = \sum a'\otimes a''. \tag{4.19}$$

The homology is $H^{**}A$.

For $A = BP_*/I$, we have $\Omega^0 A = BP_*/I$. If $a \in A$, then $\psi(a) = \eta_L(a)$, so

$$d(a) = 1|a - a|1 = \eta_R(a) - \eta_L(a). \tag{4.20}$$

Just as in (4.15), we find H^0BP_*/I is the kernel of

$$a \longrightarrow \eta_R(a) - \eta_L(a). \tag{4.21}$$

Observe that the resolution has no elements in internal degree $q \neq 0 \bmod 2(p-1)$. This property is called "sparseness", and is quite nice [$\mathbf{TZ}_2$].

The short exact sequence

$$0 \longrightarrow BP_*/I_{n-1} \xrightarrow{v_{n-1}} BP_*/I_{n-1} \longrightarrow BP_*/I_n \longrightarrow 0 \tag{4.22}$$

gives rise to a "Bockstein" long exact sequence:

$$\begin{array}{ccc} H^*BP_*/I_{n-1} & \xrightarrow{v_{n-1}} & H^*BP_*/I_{n-1} \\ & {}_{\delta_{n-1}}\nwarrow \quad \swarrow & \\ & H^*BP_*/I_n & \end{array} \tag{4.23}$$

This begins

$$\begin{array}{ccccccc} & & \mathbf{F}_p[v_{n-1}] & & \mathbf{F}_p[v_{n-1}] & & \mathbf{F}_p[v_n] \\ & & \updownarrow\simeq & & \updownarrow\simeq & & \updownarrow\simeq \\ 0 & \longrightarrow & H^0BP_*/I_{n-1} & \xrightarrow{v_{n-1}} & H^0BP_*/I_{n-1} & \longrightarrow & H^0BP_*/I_n \\ & & & & & \delta_{n-1} \swarrow & \\ & & H^1BP_*/I_{n-1} & \xrightarrow{v_{n-1}} & H^1BP_*/I_{n-1} & \longrightarrow & H^1BP_*/I_n \longrightarrow \end{array} \tag{4.24}$$

We see that $\delta_{n-1}(v_n^k) \neq 0$, $k > 0$, and gives all elements in H^1BP_*/I_{n-1} which are killed by v_{n-1}. The structure theorem for modules over the P.I.D. $\mathbf{F}_p[v_{n-1}]$ gives hope for finding the entire group. All we must do is compute the $\mathbf{F}_p[v_{n-1}]$ free part and find module generators, x_j, such that $v_{n-1}^{q-1}x_j = \delta_{n-1}(v_n^j)$. The free part will be discussed later. Finding generators x_j is just a matter of dividing $\delta_{n-1}(v_n^j)$ by v_{n-1} as much as possible. We give a simple example. To compute δ_{n-1}, we need to know $\eta_R(v_n)$ modulo I_{n-1}, as

$$\delta_{n-1}(v_n^k) = (v_{n-1})^{-1}(\eta_R(v_n^k) - \eta_L(v_n^k)) \mod I_{n-1}. \tag{4.25}$$

LEMMA 4.26 (MILLER-WILSON **[MW]**, RAVENEL'S PROOF **[R₁]**).

$$\eta_R(v_n) \equiv v_n + v_{n-1}t_1^{p^{n-1}} - v_{n-1}^p t_1 \mod I_{n-1}. \quad \square$$

This is a good example of skimming off information from the inductive formulas. This formula contains a tremendous amount of information and can be pushed a long way for results; see **[MW]**.

PROOF. We use Ravenel's equation 4.6. In the degree of $\eta_R(v_n)$, the left-hand side is, modulo I_{n-1}, $\eta_R(v_{n-1}) +_F \eta_R(v_n) +_F t_1\eta_R(v_{n-1})^p$, which is simply $\eta_R(v_n) + t_1\eta_R(v_{n-1})^p$, and by 4.8, this is $\eta_R(v_n) + v_{n-1}^p t_1$.

The right-hand side of 4.6 is $v_{n-1} +_F v_{n-1}t_1^{p^{n-1}} +_F v_n$ which is $v_n + v_{n-1}t_1^{p^{n-1}}$. $\square$

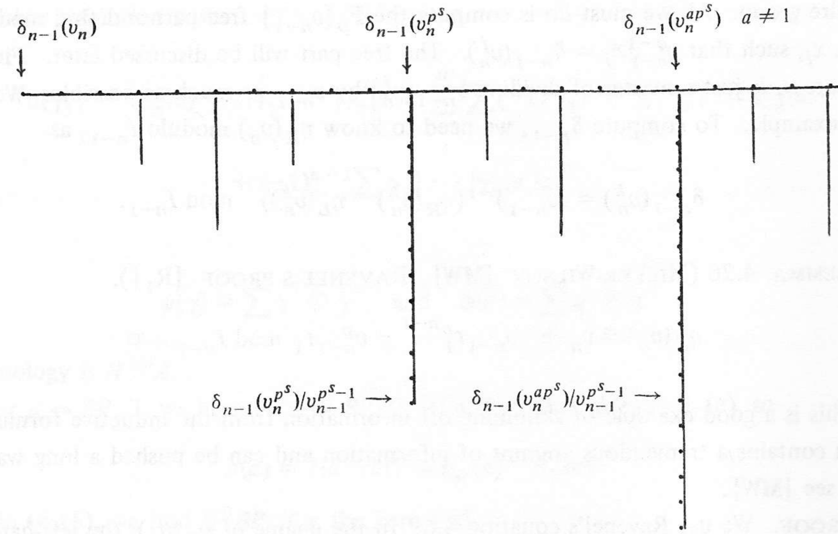
Now we can present our example of some dividing by v_{n-1}. Write $k = ap^s$, $a \not\equiv 0\ (p)$. We know

$$\begin{aligned} 0 \neq \delta_{n-1}(v_n^k) &= \delta_{n-1}(v_n^{ap^s}) \\ &= (v_{n-1})^{-1}[\eta_R(v_n^{ap^s}) - \eta_L(v_n^{ap^s})] \mod I_{n-1} \\ &= (v_{n-1})^{-1}[(v_n + v_{n-1}t_1^{p^{n-1}} - v_{n-1}^p t_1)^{ap^s} - v_n^{ap^s}] \mod I_{n-1} \\ &= (v_{n-1})^{-1}[(v_n^{p^s} + v_{n-1}^{p^s}t_1^{p^{n-1+s}} - v_{n-1}^{p^{s+1}}t_1^{p^s})^a - v_n^{ap^s}] \mod I_{n-1}. \end{aligned} \tag{4.27}$$

Clearly $v_{n-1}^{p^s-1}$ divides this, so the element

$$\delta_{n-1}(v_n^{ap^s})/v_{n-1}^{p^s-1} \tag{4.28}$$

is nonzero and defines for us an injection of $\mathbf{F}_p[v_{n-1}]/(v_{n-1}^{p^s})$. We have easily come up with a large number of elements in H^1BP_*/I_{n-1}. These elements, for $n = 2$, were found independently by several (6) people, but first by Zahler **[Zh]**. For awhile it was believed this was all of the v_{n-1} torsion. Haynes Miller gave an example of a new element for $n = 2$, and eventually the entire structure fell **[MW]**. When the above $a = 1$, there are no more elements, but when $a \neq 1$, one can divide (a homologous cycle) by v_{n-1} more times. A favorite pictorial representation for the v_{n-1} torsion in H^1BP_*/I_{n-1} is



(4.29)

where multiplication by v_{n-1} is vertical and the top line gives all of the elements $\delta_{n-1}(v_n^k)$ killed by v_{n-1}.

Let us do a little more computing. Let p be odd. For $H^1 BP_*$ we have

$$\eta_R(v_1) = v_1 + pt_1. \tag{4.30}$$

Then

$$\delta_0(v_1^{ap^s}) = \frac{1}{p}[(v_1 + pt_1)^{ap^s} - v_1^{ap^s}] \tag{4.31}$$

is divisible by p^s, and $\delta_0(v_1^{ap^s})/p^s$ reduces to $av_1^{ap^s-1}t_1$, $a \not\equiv 0$ (p), in $H^1 BP_*/(p)$. The v_1 free part of $H^1 BP_*/(p)$ is generated by t_1 (and there are no $Z_{(p)}$ summands of $H^1 BP_*$). Therefore,

$$H^{1,2(p-1)p^s a} BP_* \simeq Z/(p^{s+1}). \tag{4.32}$$

This looks like the image of J at an odd prime. Seeing this, we know it is time to leave our discussion of rather sophisticated internal theorems about the cohomology of BP_*BP and start to look for applications. We need the Adams-Novikov spectral sequence.

Define $\overline{BP}$ by the stable cofibration:

$$S^0_{(p)} \longrightarrow BP \longrightarrow \overline{BP}. \tag{4.33}$$

We form a sequence of stable cofibration triangles with the top maps all changing degree:

$$\begin{array}{ccccccc} S^0_{(p)} & \leftarrow \overline{BP} & \longleftarrow & \wedge^2 \overline{BP} & \longleftarrow & \wedge^3 \overline{BP} & \longleftarrow \cdots \\ \searrow\nearrow & & \searrow\nearrow & & \searrow\nearrow & & \searrow \\ BP & & BP \wedge \overline{BP} & & BP \wedge^2 \overline{BP} & & \cdots \end{array} \tag{4.34}$$

Smash this with X and apply $\pi_*(-)$. The resulting spectral sequence has

$$E_2 = \mathrm{Ext}^{**}_{BP_*BP}(BP_*, BP_*X) \simeq H^{**}BP_*X \Rightarrow \pi_*X_{(p)}. \tag{4.35}$$

For $H^*BP_* \Rightarrow \pi_*S^0_{(p)}$, $p > 2$, the calculation of H^1BP_* really does correspond to the image of J [**N**]. Other parts of the previous algebraic material correspond to geometric situations. For the first example, Larry Smith generalized the elements of order p in the image of J. He considered spaces $V(n)$ with mod (p) cohomology $E(Q_0, Q_1, \ldots, Q_n)$, or equivalently [**Sm$_1$**]

$$BP_*V(n) \simeq BP_*/I_{n+1}. \tag{4.36}$$

This is the situation discussed at the end of the last section as most likely to be accessible by BP techniques. Unfortunately, the $V(n)$ spaces are only known to exist for very small values of n, but in these cases they have produced a great deal of homotopy information.

Letting $S^0 = V(-1)$, the $V(n)$ are constructed inductively as cofibrations,

$$\Sigma^{2(p^n-1)}V(n-1) \longrightarrow V(n-1) \longrightarrow V(n), \tag{4.37}$$

which realize the exact sequences

$$0 \longrightarrow BP_*/I_n \xrightarrow{v_n} BP_*/I_n \longrightarrow BP_*/I_{n+1} \longrightarrow 0. \tag{4.38}$$

The space $V(0)$ exists for all p, $V(1)$ exists for $p > 2$ [**A$_3$**], $V(2)$ for $p > 3$ [**Sm$_2$**, **To$_1$**], and $V(3)$ for $p > 5$ [**To$_1$**].

The elements of order p in the image of J are constructed by the composition

$$\Sigma^{2(p-1)k}S^0 \longrightarrow \Sigma^{2(p-1)k}V(0) \xrightarrow{\alpha^k} V(0) \longrightarrow S^1 \tag{4.39}$$

obtained by iterating the map in (4.37). The map to $V(0)$ is represented by

$$v_1^k \in H^0BP_*/(p) \tag{4.40}$$

in the Adams-Novikov spectral sequence. The elements of order p in the image of J are represented by

$$\delta_0(v_1^k) \in H^1BP_*. \tag{4.41}$$

Larry Smith constructed and detected the elements β_k [**Sm$_2$**] defined by the composition

$$S^{2(p^2-1)k} \longrightarrow \Sigma^{2(p^2-1)k}V(1) \xrightarrow{\beta^k} V(1) \longrightarrow \Sigma^{2(p-1)+1}V(0) \longrightarrow S^{2(p-1)+2}. \tag{4.42}$$

The map to $V(1)$ is represented by

$$v_2^k \in H^0BP_*/(p, v_1). \tag{4.43}$$

The nontriviality of this composition is determined by the nontriviality of

$$\delta_0\delta_1(v_2^k) \in H^2BP_*. \tag{4.44}$$

We have seen that $\delta_1(v_2^k) \neq 0$. We then have an exact sequence

$$\to H^1BP_* \to H^1BP_*/(p) \xrightarrow{\delta_0} H^2BP_* \to \tag{4.45}$$

where we know the first two groups and can easily verify that $\delta_0\delta_1(v_2^k) \neq 0$. See **[JMWZ]** or these notes after (4.31).

The element γ_k is defined by the composition

$$S^{2(p^3-1)k} \to \Sigma^{2(p^3-1)k}V(2) \xrightarrow{\gamma^k} V(2) \to S^{2(p^2-1)+2(p-1)+3}. \tag{4.46}$$

The map to $V(2)$ is represented by

$$v_3^k \in H^0BP_*/(p, v_1, v_2). \tag{4.47}$$

It is enough to show that

$$0 \neq \delta_0\delta_1\delta_2(v_3^k) \in H^3BP_*. \tag{4.48}$$

There is a result involved in making the leap from algebra to geometry. The most accessible place to find it is in **[JMWZ]**. We already know that

$$0 \neq \delta_2(v_3^k) \in H^1BP_*/(p, v_1). \tag{4.49}$$

We consider the sequence

$$\to H^1BP_*/(p) \to H^1BP_*/(p, v_1) \xrightarrow{\delta_1} H^2BP_*/(p) \to. \tag{4.50}$$

We know the first two groups and so we can verify that

$$0 \neq \delta_1\delta_2(v_3^k) \in H^2BP_*/(p). \tag{4.51}$$

Our next exact sequence is

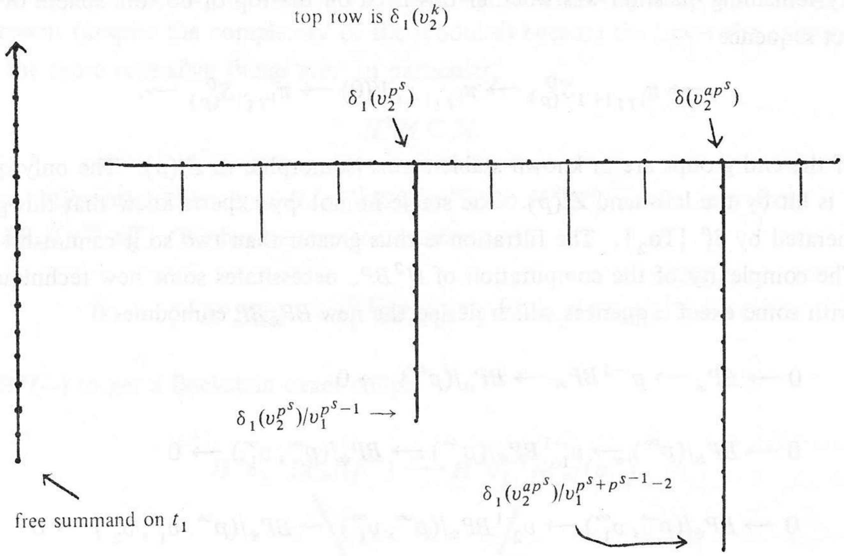
$$\to H^2BP_* \to H^2BP_*/(p) \xrightarrow{\delta_0} H^3BP_* \to. \tag{4.52}$$

To prove that

$$0 \neq \delta_0\delta_1\delta_2(v_3^k) \in H^3BP_* \tag{4.53}$$

it turns out to be necessary to compute H^2BP_*, but fortunately not $H^2BP_*/(p)$. There is a pictorial representation for H^2BP_* corresponding to that for H^1BP_*/I_n given already. We picture $H^1BP_*/(p)$ more explicitly:

(4.54)



v_1 multiplication is vertical

H^2BP_* is all p torsion. Using (4.31) to compute (4.45), we see that the image of H^1BP_* in $H^1BP_*/(p)$ is the free tower on t_1. Thus all of the v_1 torsion part of $H^1BP_*/(p)$ maps to H^2BP_* and gives the elements of order p in H^2BP_*. All that remains is to divide these elements by p as much as possible. The exact divisibility can be found in **[MRW]**. The v_1 torsion part of the picture for $H^1BP_*/(p)$ is shifted to H^2BP_*. The main elements are Smith's $\beta_k = \delta_0\delta_1(v_2^k)$. From β_k with $k = ap^s$, $s > 0$, there are hairy stalactites hanging down. The hair represents the division by powers of p.

Before proceeding to the description of the approach to the computation of H^2BP_* this seems like a good place to sketch a simple proof of the fact that $\gamma_1 \neq 0$ **[TZ$_1$, OT, A$_4$]**. This element generated a great deal of interest when the original papers of Oka-Toda and Thomas-Zahler came to opposite conclusions. The proof given here has evolved over the years with all of the advantages of hindsight with contributions from many people. Nothing is used that was not available to the combined forces of bordism and stable homotopy experts at the time of the original proofs. The bordism experts had known for some time that

(4.55) $$0 \neq \gamma^1 \in \pi_{|\gamma_1|+1} V(0)$$

in complex bordism (BP) filtration two, i.e.

(4.56) $$0 \neq \delta_1\delta_2(v_3) \in H^2BP_*/(p).$$

The only remaining question was whether this lived on the top or bottom sphere of $V(0)$ in the exact sequence

$$\longrightarrow \pi_{|\gamma_1|+1} S^0_{(p)} \longrightarrow \pi_{|\gamma_1|+1} V(0) \longrightarrow \pi_{|\gamma_1|} S^0_{(p)} \longrightarrow . \tag{4.57}$$

Both of the end groups are in known stable stems isomorphic to $Z/(p)$. The only question is if γ^1 is hit by the left-hand $Z/(p)$. The stable homotopy experts knew that this group was generated by β_1^p [To$_2$]. The filtration is thus greater than two so it cannot hit γ^1.

The complexity of the computation of H^2BP_* necessitates some new techniques. We begin with some exact sequences which define the new BP_*BP comodules:

$$\begin{array}{l} 0 \longrightarrow BP_* \longrightarrow p^{-1}BP_* \longrightarrow BP_*/(p^\infty) \longrightarrow 0 \\ 0 \longrightarrow BP_*/(p^\infty) \longrightarrow v_1^{-1}BP_*/(p^\infty) \longrightarrow BP_*/(p^\infty, v_1^\infty) \longrightarrow 0 \\ 0 \longrightarrow BP_*/(p^\infty, v_1^\infty) \longrightarrow v_2^{-1}BP_*/(p^\infty, v_1^\infty) \longrightarrow BP_*/(p^\infty, v_1^\infty, v_2^\infty) \longrightarrow 0 \\ \qquad\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \end{array} \tag{4.58}$$

Apply $H^*(-)$:

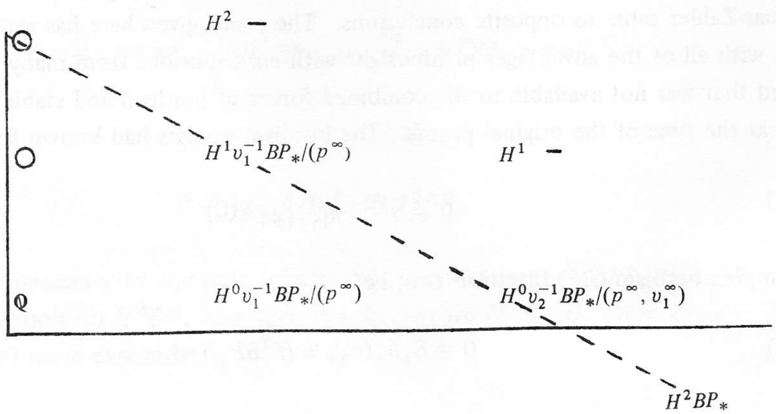
$$\begin{array}{ccccccc} H^*BP_* & \xleftarrow{\delta} & H^*BP_*/(p^\infty) & \xleftarrow{\delta} & H^*BP_*/(p^\infty, v_1^\infty) & \xleftarrow{\delta} & \cdots \\ & \searrow \nwarrow & & \searrow \nwarrow & & \searrow \nwarrow & \\ & H^*p^{-1}BP_* & & H^*v_1^{-1}BP_*/(p^\infty) & & H^*v_2^{-1}BP_*/(p^\infty, v_1^\infty) & \end{array} \tag{4.59}$$

This gives a spectral sequence

$$E_1^{s,t} = H^t v_s^{-1} BP_*/(p^\infty, \ldots, v_{s-1}^\infty) \Rightarrow H^*BP_*, \tag{4.60}$$

called the *chromatic spectral sequence* **[MRW]**. It breaks H^*BP_* up into its v_n-periodic components. Since $H^*p^{-1}BP_* \simeq Q$, we have

(4.61) $E_1 =$



To get H^2BP_* we need $H^1v_1^{-1}BP_*/(p^\infty)$ and $H^0v_2^{-1}BP_*/(p^\infty, v_1^\infty)$. This is already a great improvement (despite the complexity of the modules) because the lower the cohomological degree, the more accessible things are. In particular,

(4.62) $$H^0M \subset M.$$

The first differential kills all of $H^1v_1^{-1}BP_*/(p^\infty)$, so H^2BP_* comes only from $H^0v_2^{-1}BP_*/(p^\infty, v_1^\infty)$. We have an exact sequence

(4.63) $$0 \longrightarrow v_1^{-1}BP_*/(p) \longrightarrow v_1^{-1}BP_*/(p^\infty) \xrightarrow{p} v_1^{-1}BP_*/(p^\infty) \longrightarrow 0.$$

Apply $H^*(-)$ to get a Bockstein exact couple

(4.64) $$\begin{array}{ccc} H^*v_1^{-1}BP_*/(p^\infty) & \xrightarrow{p} & H^*v_1^{-1}BP_*/(p^\infty) \\ & \nwarrow \quad \swarrow & \\ & H^*v_1^{-1}BP_*/(p) & \end{array}$$

Likewise, we have

(4.65) $$0 \longrightarrow v_2^{-1}BP_*/(p, v_1^\infty) \longrightarrow v_2^{-1}BP_*/(p^\infty, v_1^\infty) \xrightarrow{p} v_2^{-1}BP_*/(p^\infty, v_1^\infty) \longrightarrow 0,$$
$$0 \longrightarrow v_2^{-1}BP_*/(p, v_1) \longrightarrow v_2^{-1}BP_*/(p, v_1^\infty) \xrightarrow{v_1} v_2^{-1}BP_*/(p, v_1^\infty) \longrightarrow 0.$$

Applying $H^*(-)$ we get two exact couples, the second beginning with

(4.66) $$H^*v_2^{-1}BP_*/(p, v_1) \simeq v_2^{-1}H^*BP_*/(p, v_1).$$

In general we start with the v_n torsion free part,

(4.67) $$H^*v_n^{-1}BP_*/I_n \simeq v_n^{-1}H^*BP_*/I_n,$$

and have n Bockstein spectral sequences to get to

(4.68) $$H^*v_n^{-1}BP_*/(p^\infty, v_1^\infty, \ldots, v_{n-1}^\infty).$$

The computability of (4.67) is a key motivating factor in this approach to H^*BP_*. This computability is part of the Morava structure theorem for complex cobordism. We give the Miller-Ravenel approach.

THEOREM 4.69 (RAVENEL [$\mathbf{R}_2$]). *Let* $K(n)_* = \mathbf{F}_p[v_n, v_n^{-1}]$.

$$K(n)_*K(n) \equiv K(n)_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} K(n)_* \simeq K(n)_*[t_1, t_2, \ldots]/(v_n^{p^i}t_i - v_n t_i^{p^n}). \quad \square$$

PROOF. It is elementary that

$$K(n)_*K(n) \simeq K(n)_*[t_1, t_2, \ldots]/(\eta_R(v_{n+1}), \eta_R(v_{n+2}), \ldots).$$

We go to Ravenel's formula 4.6

$$\sum_{\substack{i\geq 0\\ j>0}}^F t_i\eta_R(v_j)^{p^i} \equiv \sum_{\substack{i>0\\ j\geq 0}}^F v_i t_j^{p^i} \mod (p).$$

This reduces to $\Sigma^F_{i\geq 0} t_i \eta_R(v_n)^{p^i} = \Sigma^F_{j\geq 0} v_n t_j^{p^n}$. By induction on degree we have $t_i\eta_R(v_n)^{p^i} = v_n t_i^{p^n}$, but by 4.8, $\eta_R(v_n) = v_n$, and so

$$v_n^{p^i} t_i = v_n t_i^{p^n}. \quad \square$$

THEOREM 4.70 (MILLER-RAVENEL [MR]). $\mathrm{Ext}_{K(n)_*K(n)}(K(n)_*, K(n)_*) \simeq H^*v_n^{-1}BP_*/I_n \simeq v_n^{-1}H^*BP_*/I_n$. $\square$

Proving this is beyond the scope of these lecture notes.

Morava's approach (the original) is dual to this [$\mathbf{Mo}_1$, $\mathbf{Mo}_2$]. The "dual" of $K(n)_*K(n)$ is the "group-ring" for the p-adic Lie group (known as the Morava stabilizers) of endomorphisms of the height n formal group law over the closure of $\mathbf{F}_p$. The continuous cohomology of this Lie group determines the cohomology in 4.70. A good explanation of the details of this duality is in Ravenel [$\mathbf{R}_2$]. Both approaches give $H^1v_n^{-1}BP_*/I_n$ easily.

Another exciting motivating factor for the study made in **[MRW]** is the following important finiteness property which gives a parabolic vanishing curve (with spikes) for the chromatic spectral sequence.

THEOREM 4.71 (MORAVA [$\mathbf{Mo}_1$]). *If $p-1$ does not divide n, then*

$$H^tv_n^{-1}BP_*/I_n = 0 \quad \text{for } t > n^2. \quad \square$$

We are leaving the computations of cohomology groups over BP_*BP. The computation of H^1BP_*/I_n and H^2BP_* demonstrates a good grasp of the operations for BP. Furthermore, these computations have led to many concrete applications in stable homotopy.

5. Associated homology theories. There are many generalized homology theories associated with BP. Each family of theories takes us in a different direction, but the theories themselves have a common origin. For this origin we must revert to geometry.

Recall that MU_*X is given by equivalence classes of maps (1.6):

(5.1) $$M^n \longrightarrow X.$$

It is Sullivan's idea **[Su]** to kill manifolds selectively to form new generalized (bordism) homology theories. Baas **[Ba]** developed a good bookkeeping system for dealing with these theories. Let

(5.2) $$[M] = x_{2n} \in MU_* \simeq Z[x_2, x_4, \ldots].$$

The bordism theory $MU(x_{2n})_*X$ fits into an exact sequence

$$\begin{array}{ccc} MU_*X & \xrightarrow{x_{2n}} & MU_*X \\ & {\scriptstyle\delta}\nwarrow \quad \swarrow & \\ & MU(x_{2n})_*X & \end{array} \tag{5.3}$$

with $MU(x_{2n})_* \simeq MU_*/(x_{2n})$. A "manifold with singularity" in $MU(x_{2n})_*X$ is a U-manifold with boundary

$$V, \quad \partial V \simeq M \times P \tag{5.4}$$

and a map $V \longrightarrow X$ such that on ∂V the map factors through the projection

$$M \times P \longrightarrow P \longrightarrow X. \tag{5.5}$$

From Sullivan's point of view this is the same as allowing the cone on M to be a manifold with boundary M. This can be generalized to define "bordism" for pairs of spaces. The verification that we have a homology theory and the exact sequence (5.3) all come from the geometry **[Ba]**. Inductively we can define

$$MU(x_{2i_1}, \ldots, x_{2i_n})_*X \tag{5.6}$$

with coefficients

$$MU_*/(x_{2i_1}, \ldots, x_{2i_n}), \tag{5.7}$$

and all possible long exact sequences. The multiplicativity of these theories is dealt with in **[SY, Wu$_1$, Mo$_3$]**.

Choose an infinite sequence of x_{2i}'s which excludes all of the $x_{2(p^n-1)}$, all n, fixed prime p. In the limit, after localizing at p, we have constructed BP_*X! Kill off a few more generators and we can construct theories with coefficient rings

$$\begin{aligned} BP\langle n\rangle_* &\simeq Z_{(p)}[v_1, \ldots, v_n] \quad \textbf{[JW}_1\textbf{]}, \\ P(n)_* &\simeq \mathbf{F}_p[v_n, v_{n+1}, \ldots] \simeq BP_*/I_n \quad \textbf{[JW}_2\textbf{]}, \\ k(n)_* &\simeq \mathbf{F}_p[v_n] \quad \textbf{[JW}_2\textbf{, Wu}_1\textbf{]}. \end{aligned} \tag{5.8}$$

The last two are due to Morava. These theories come with exact sequences

$$\begin{array}{ccc} BP\langle n\rangle_*X & \xrightarrow{\ v_n\ } & BP\langle n\rangle_*X \\ {\scriptstyle\delta}\nwarrow & & \swarrow \\ & BP\langle n-1\rangle_*X & \end{array}$$

(5.9)
$$\begin{array}{ccc} P(n)_*X & \xrightarrow{\ v_n\ } & P(n)_*X \\ {\scriptstyle\delta}\nwarrow & & \swarrow \\ & P(n+1)_*X & \end{array}$$

$$\begin{array}{ccc} k(n)_*X & \xrightarrow{\ v_n\ } & k(n)_*X \\ {\scriptstyle\delta}\nwarrow & & \swarrow \\ & H_*(X; Z/(p)) & \end{array}$$

These long exact sequences all give Bockstein spectral sequences. We have seen this before.

The first direction we pursue using these associated theories has not really moved out of the internal theorem stage. However, there are many beautiful internal results and much has been written about it. See **[CS]** and **[JW$_1$]** for the basics.

We define

(5.10) $$\text{hom dim}_{BP_*} BP_*X$$

to be the minimum length of a free BP_* resolution for BP_*X:

(5.11) $$0 \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow BP_*X \longrightarrow 0.$$

The study of this number (for MU_*X) was initiated by Conner and Smith in **[CS]**, where the $n = 0$, 1 and 2 cases are investigated thoroughly. Unfortunately it turned out to be necessary to localize at a prime and go to BP in order to continue the study. This number measures the complexity of the BP_* module structure and so is sometimes referred to as the "ugliness number". The main characterization is in terms of the associated theories $BP\langle n\rangle_*X$.

Consider the sequence

$$BP_*X \longrightarrow \cdots \longrightarrow BP\langle n+1\rangle_*X \xrightarrow{\rho_{n+1}} BP\langle n\rangle_*X \xrightarrow{\rho_n} BP\langle n-1\rangle_*X \longrightarrow$$

$$\cdots \longrightarrow BP\langle 0\rangle_*X \xrightarrow{\rho_0} BP\langle -1\rangle_*X \tag{5.12}$$

$$\| \qquad\qquad \|$$

$$H_*(X; Z_{(p)}) \qquad H_*(X; Z/(p)).$$

Theorem 5.13 (Johnson-Wilson [$\mathbf{JW_1}$]). hom $\dim_{BP_*} BP_*X = n \Leftrightarrow \rho_n, \rho_{n+1}, \ldots$ *are all surjective and* $\rho_0, \rho_1, \ldots, \rho_{n-1}$ *all fail to be surjective.* □

Other related results are

Theorem 5.14 (Johnson-Wilson [$\mathbf{JW_1}$]). *If* $x \in BP_*X$ *is* v_n *torsion then* hom $\dim_{BP^*} BP_*X > n$. □

Although any expert tends to feel it should be obvious, it is only recently that a proof has been found for

Theorem 5.15 (Johnson-Yosimura [**JY**], see also [$\mathbf{L_5}$]). *If* $x \in BP_*X$ *is* v_n *torsion then it is* v_{n-1} *torsion.* □

The proof is not obvious.

Along this line is a possibility that any n dimensional class in BP_*X (for X a space) is not v_n torsion. David Johnson first raised this question. There is some minor evidence for it as a conjecture. It is a very strong statement about the possible unstable BP_* module structures.

In this direction the most interest has been centered on the theories

$$v_n^{-1}BP\langle n\rangle_*X \simeq v_n^{-1}BP\langle n\rangle_* \otimes_{BP_*} BP_*X, \tag{5.16}$$

where v_n^{-1} means localization with respect to $\{v_n^k\}$, i.e., inverting v_n, first studied in [$\mathbf{JW_1}$] and since studied in [$\mathbf{L_4}$, $\mathbf{R_3}$, **JY**]. The $n = 1$ case of (5.16) is the BP version of the Conner-Floyd theorem [$\mathbf{CF_2}$] showing how complex cobordism determines complex K-theory:

$$KU^*X \simeq KU^* \otimes_{MU^*} MU^*X. \tag{5.17}$$

The $n > 1$ cases of (5.16) generalize the Conner-Floyd theorem in one direction. These theories are v_n-periodic and one can see similarities between this type of thing and the chromatic spectral sequence. In particular we have the recurring theme of using one v_n at a time and building up slowly. We will see a lot more of this in the next section.

6. Morava's little structure theorem and the Conner-Floyd conjecture. We take another direction using associated homology theories. We have already mentioned Morava's structure theorem in the setting of computing Ext groups. The Morava structure theorem is a vast internal sequence of theorems by Morava about complex cobordism and related algebra. It is an internal theorem with long range consequences and applications in many directions. Here we will ignore Morava's work with operations and the relationship with stable homotopy and give Morava's little structure theorem and tie it in with a new direction in our study to give a proof of the Conner-Floyd conjecture. Because this calls for so much machinery it usually cannot be presented to an audience unless they have been properly warmed up to *BP*. However, it is one of the most beautiful and complex applications of the theory so it is a central theme in these lectures.

Published Morava references tend not to exist, but many of his results (in preprint form) date back to 1971 or so. For Morava today, see [**Mo$_1$**] and [**Mo$_2$**], both presently still preprints.

THEOREM 6.1 (MORAVA, SEE [**JW$_2$**]). *$B(n)_*X \equiv v_n^{-1}P(n)_*X$ is free over $B(n)_* \simeq$ $\mathbf{F}_p[v_n^{-1}, v_n, v_{n+1}, \ldots]$.* □

Plausibility argument. $P(n)_*X$ is approximately a comodule over BP_*BP/I_n. There are no nontrivial invariant ideals in $B(n)_*$ because, as in (4.10), v_n^k would be in such an ideal but it is a unit. So, it is plausible that $B(n)_*X$ is free. □

THEOREM 6.2 (MORAVA, SEE [**JW$_2$**]).

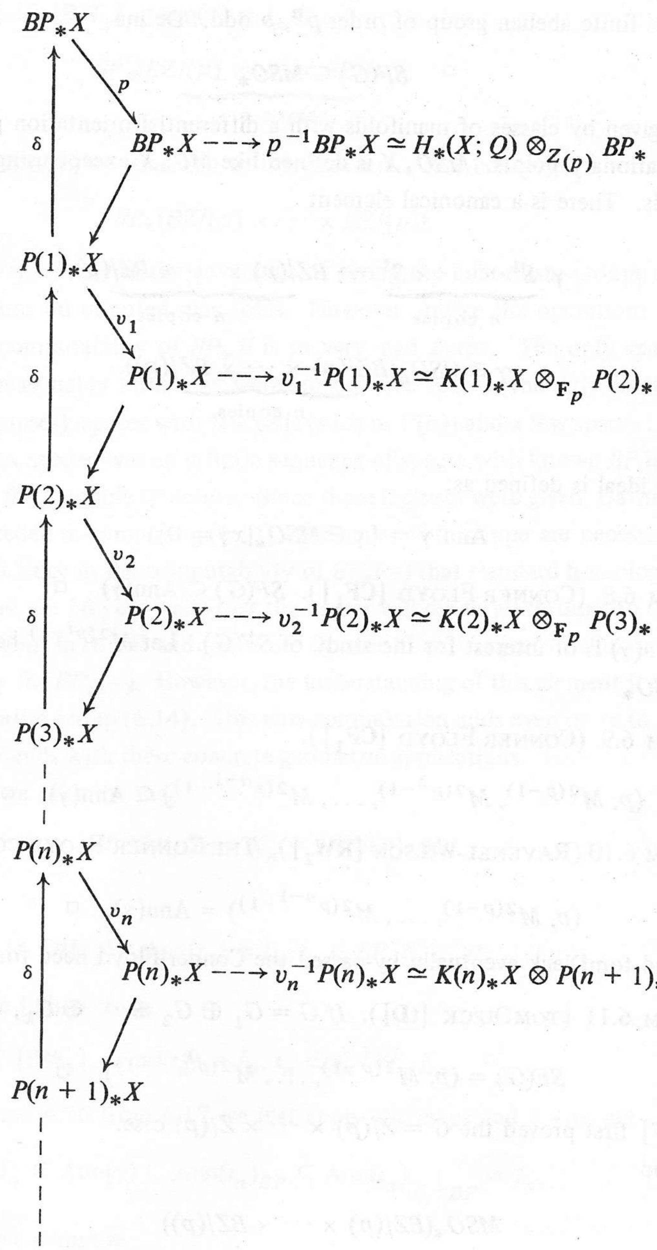
$$K(n)_*X \equiv v_n^{-1}k(n)_*X \simeq K(n)_* \otimes_{B(n)_*} B(n)_*X \simeq K(n)_* \otimes_{P(n)_*} P(n)_*X. \quad \square$$

Here is another generalization of the Conner-Floyd theorem (5.17) (the $n = 1$ case is just the mod (p) version of (5.16)).

COROLLARY 6.3 (MORAVA, SEE [**JW$_2$**]). *There is an unnatural isomorphism $B(n)_*X$ $\simeq K(n)_*X \otimes_{\mathbf{F}_p} P(n+1)_*$.* □

Würgler [**Wu$_2$**] has made this reversing process precise.

MORAVA'S LITTLE (NO OPERATIONS) STRUCTURE THEOREM 6.4. *The v_n torsion free part of $P(n)_*X$ is determined by $K(n)_*X$ (6.3). The Bockstein long exact sequence* (5.9) *relating $P(n)_*X$ and $P(n+1)_*X$ gives the v_n torsion. For BP_*X, the diagram is*



□

Although there are no operations in the statement, they are in the proof. Again we see v_n-periodicity in this structure theorem.

Up to this point the result is purely internal. An attempt to use this to compute BP_*X for some space X would convince most people that it really is only an internal result. However, we will use it to prove the Conner-Floyd conjecture which we now digress to describe.

Let G be a finite abelian group of order p^n, p odd. Define

(6.5) $$SF(G) \subset MSO_*$$

to be the ideal given by classes of manifolds with a differential orientation preserving action of G with no stationary points. MSO_*X is defined like MU_*X except using SO structures on the manifolds. There is a canonical element

(6.6)
$$\gamma\colon \underbrace{S^1 \times \cdots \times S^1}_{n \text{ copies}} \longrightarrow \underbrace{BZ/(p) \times \cdots \times BZ/(p)}_{n \text{ copies}},$$
$$\gamma \in MSO_n(\underbrace{BZ/(p) \times \cdots \times BZ/(p)}_{n \text{ copies}}).$$

The annihilator ideal is defined as:

(6.7) $$\text{Ann}\,\gamma = \{x \in MSO_* | x\gamma = 0\}.$$

THEOREM 6.8 (CONNER-FLOYD [$\mathbf{CF_1}$]). *$SF(G) \subset \text{Ann}(\gamma)$.* □

Thus, $\text{Ann}(\gamma)$ is of interest for the study of $SF(G)$. Let $M^{2(p^i-1)}$ be Milnor basis elements in MSO_*.

THEOREM 6.9 (CONNER-FLOYD [$\mathbf{CF_1}$]).

$$(p, M^{2(p-1)}, M^{2(p^2-1)}, \ldots, M^{2(p^{n-1}-1)}) \subset \text{Ann}(\gamma). \quad □$$

THEOREM 6.10 (RAVENEL-WILSON [$\mathbf{RW_2}$]). THE CONNER-FLOYD CONJECTURE [$\mathbf{CF_1}$].

$$(p, M^{2(p-1)}, \ldots, M^{2(p^{n-1}-1)}) = \text{Ann}(\gamma). \quad □$$

Floyd and tomDieck eventually by-passed the Conner-Floyd need for this with

THEOREM 6.11 (TOMDIECK [**tD**]). *If $G = G_1 \oplus G_2 \oplus \cdots \oplus G_k$, G_i cyclic, then*

$$SF(G) = (p, M^{2(p-1)}, \ldots, M^{2(p^{k-1}-1)}). \quad □$$

Floyd [**F**] first proved the $G = Z/(p) \times \cdots \times Z/(p)$ case.

The group

(6.12) $$MSO_*(BZ/(p) \times \cdots \times BZ/(p))$$

is the group of oriented manifolds with free actions of $Z/(p) \times \cdots \times Z/(p)$ on them, up to cobordism.

At odd primes the relationship between $MSO_*X_{(p)}$ and BP_*X is completely analogous to that for $MU_*X_{(p)}$. In particular, we have $MU_*X \otimes_{MU_*} MSO_{*(p)} \simeq MSO_*X_{(p)}$ induced by the map (1.3) (cf. [**St**, p. 180]), p odd. Thus the elements $[M^{2(p^i-1)}]$ come from MU_*.

The conjecture is completely a p-primary problem, so it is the same as:

COROLLARY 6.13 [$\mathbf{RW}_2$]. $\mathrm{Ann}(\gamma) = I_n$ *in*

$$BP_*(\underbrace{BZ/(p) \times \cdots \times BZ/(p)}_{n\ copies}). \quad \square$$

There is a rather obvious approach to this problem. Just compute

$$BP_*(BZ/(p) \times \cdots \times BZ/(p)) \tag{6.14}$$

and look at the answer. This has the advantage of giving the cobordism groups of $Z/(p) \times \cdots \times Z/(p)$ free actions on oriented manifolds. However, unlike the operations discussed in the last section, the computability of BP_*X is in very bad shape. The only spaces that could be dealt with reasonably until now were torsion free spaces (the Atiyah-Hirzebruch spectral sequence collapses), spaces with few cells (such as $V(n)$) and a few spaces like $BZ/(p)$. In particular, what was needed was an infinite sequence of spaces with known BP homology and increasingly complex BP_* module structure. Since these lectures were given, David Johnson and the author have succeeded in computing (6.14). Examples of this type are necessary in order to build the sort of confidence in the computability of $BP_*(-)$ that standard homology presently enjoys. The labor (and the 16 year wait) that the reader will see us go through to understand something of one element in (6.14) and prove the Conner-Floyd conjecture illustrates the poor state of computability for $BP_*(-)$. However, the understanding of this element is crucial to the computation of the entire group (6.14). This new computation adds even more to the complex chain of results which ends with these concrete geometric applications.

To prove 6.13 we take the map

$$\times^n BZ/(p) \longrightarrow \underline{K}_n = K(Z/(p), n) \tag{6.15}$$

which takes γ to ι_n.

COROLLARY 6.16 [$\mathbf{RW}_2$]. $\mathrm{Ann}(\iota_n) = I_n$, $\iota_n \in BP_n\underline{K}_n$. $\square$

All of the above follow from

THEOREM 6.17 [$\mathbf{RW}_2$]. $\mathrm{Ann}(\iota_n) = I_n$, $\iota_n \in v_n^{-1}BP_n\underline{K}_n$. $\square$

To prove 6.13 and 6.16 from 6.17 we just apply the maps and 6.9 to get

$$I_n \subset \mathrm{Ann}(\gamma) \subset \mathrm{Ann}(\iota_n)_{BP} \subset \mathrm{Ann}(\iota_n)_{v_n^{-1}BP} = I_n. \tag{6.18}$$

To prove 6.17 we compute

$$v_n^{-1}BP_*\underline{K}_n \tag{6.19}$$

in its entirety. To do this we use the Morava structure theorem. We start with the Morava K-theories of $\underline{K}_n$ (developed more later)

$$\begin{aligned} &\widetilde{K(j)}_*\underline{K}_n = 0, \quad j < n, \\ &\widetilde{K(n)}_*\underline{K}_n \simeq \text{a } K(n)_* \text{ free module on } p-1 \text{ even generators.} \end{aligned} \tag{6.20}$$

To compute, we invert v_n in the Morava Structure Theorem 6.4 using reduced theories in each place:

(6.21)

$$v_n^{-1}BP_*\underline{K}_n \xrightarrow{\ p\ } v_n^{-1}BP_*\underline{K}_n \dashrightarrow p^{-1}v_n^{-1}BP_*\underline{K}_n \simeq \overset{\simeq 0}{H_*(\underline{K}_n;Q)} \otimes v_n^{-1}BP_*$$

$$v_n^{-1}BP_*\underline{K}_n \longrightarrow v_n^{-1}P(1)_*\underline{K}_n \xrightarrow{\ \delta\ } v_n^{-1}BP_*\underline{K}_n$$

$$v_n^{-1}P(1)_*\underline{K}_n \xrightarrow{\ v_1\ } v_n^{-1}P(1)_*\underline{K}_n \dashrightarrow v_1^{-1}v_n^{-1}P(1)_*\underline{K}_n \simeq \overset{\simeq 0}{K(1)_*\underline{K}_n} \otimes v_n^{-1}P(2)_*$$

$$v_n^{-1}P(1)_*\underline{K}_n \longrightarrow v_n^{-1}P(2)_*\underline{K}_n \xrightarrow{\ \delta\ } v_n^{-1}P(1)_*\underline{K}_n$$

$$v_n^{-1}P(2)_*\underline{K}_n \quad \vdots \quad v_n^{-1}P(n-1)_*\underline{K}_n$$

$$v_n^{-1}P(n-1)_*\underline{K}_n \xrightarrow{\ v_{n-1}\ } v_n^{-1}P(n-1)_*\underline{K}_n \dashrightarrow v_{n-1}^{-1}v_n^{-1}P(n-1)_*\underline{K}_n \simeq \overset{\simeq 0}{K(n-1)_*\underline{K}_n} \otimes v_n^{-1}P(n)_*$$

$$v_n^{-1}P(n-1)_*\underline{K}_n \longrightarrow v_n^{-1}P(n)_*\underline{K}_n \xrightarrow{\ \delta\ } v_n^{-1}P(n-1)_*\underline{K}_n$$

$$v_n^{-1}P(n)_*\underline{K}_n \underset{\simeq}{\xrightarrow{\ v_n\ }} v_n^{-1}P(n)_*\underline{K}_n \xrightarrow{\ \simeq\ } v_n^{-1}P(n)_*\underline{K}_n \simeq K(n)_*\underline{K}_n \otimes P(n+1)_*$$

$$v_n^{-1}P(n)_*\underline{K}_n \longrightarrow v_n^{-1}P(n+1)_*\underline{K}_n \simeq 0 \xrightarrow{\ \delta\ } v_n^{-1}P(n)_*\underline{K}_n$$

After this localization we have reduced the computation process to a finite number of Bockstein exact couples. By (6.20) we have

(6.22) $\quad v_n^{-1}P(n)_*\underline{K}_n \simeq p-1$ copies of $v_n^{-1}BP_*/I_n$ on even degree generators.

We now compute $v_n^{-1}P(n-1)_*\underline{K}_n$. We see from (6.20) and (6.21) that this is all v_{n-1} torsion. We use

(6.23)
$$\begin{array}{ccc} v_n^{-1}P(n-1)_*\underline{K}_n & & \\ \uparrow \delta & \searrow v_{n-1} & \\ & & v_n^{-1}P(n-1)_*\underline{K}_n \\ & \swarrow \rho & \\ v_n^{-1}P(n)_*\underline{K}_n & & \end{array}$$

From (6.23) the v_{n-1} torsion must be in odd degrees because the degree of δ is odd, the degree of v_{n-1} is even, and $v_n^{-1}P(n)_*\underline{K}_n$ is in even degrees. Thus ρ is zero and (6.23) is a short exact sequence which is $p-1$ copies of

(6.24)
$$\begin{aligned} 0 \longrightarrow v_n^{-1}BP_*/I_n &\longrightarrow v_n^{-1}BP_*/(p, v_1, \ldots, v_{n-2}, v_{n-1}^{\infty}) \\ &\longrightarrow v_n^{-1}BP_*/(p, \ldots, v_{n-2}, v_{n-1}^{\infty}) \longrightarrow 0. \end{aligned}$$

So

(6.25)
$$v_n^{-1}P(n-1)_*\underline{K}_n \simeq p-1 \text{ copies of } v_n^{-1}BP_*/(p, \ldots, v_{n-2}, v_{n-1}^{\infty})$$
on odd degree generators.

Repeat this type of computation until we have

(6.26) $$v_n^{-1}BP_*\underline{K}_n \simeq p-1 \text{ copies of } v_n^{-1}BP_*/(p^{\infty}, \ldots, v_{n-1}^{\infty}).$$

Note (4.63)–(4.68) the familiar BP_* modules! To finish the Conner-Floyd conjecture we need to locate ι_n. This is just a homology computation.

The problem of computability is a very serious one. Consider the (time and location dependent) functor

(6.27) $$\text{topologists} \xrightarrow{F} \text{topological spaces}$$

which assigns to each topologist his favorite topological space. If an algebraic theory $E_*(-)$ is to be useful, then X must be able to compute $E_*(F(X))$. After all, if he wants to prove theorems about $F(X)$, $E_*(-)$ does no good if he cannot get his hands on the algebraic invariant $E_*(F(X))$. By this standard it is clear that $H_*(-)$ is fairly computable, but it is not clear why, since there is no workable algorithm for computing $H_*(F(X))$. As a substitute for an algorithm, Y has had to rely on the observation that X has computed $H_*(F(X))$ for all X older than Y. This supplies a certain faith that Y too can compute $H_*(F(Y))$, even if he must do it in an "ad hoc" fashion and be prepared to spend two years at it. With $H_*(-)$ he has the decades of development of homological algebra, the Steenrod algebra, homology operations, etc., to help him out. In general, although there are millions ($>>\infty$) of homology theories, very few are computationally useful. After $H_*(-)$ we find that people have been quite successful with K-theory. Now, with

$$K(n)_*K(Z/(p), j) \quad \text{and} \quad BP_*(BZ/(p) \times \cdots \times BZ/(p)) \tag{6.28}$$

computed there is some hope for $K(n)_*(-)$ and $BP_*(-)$. It may be that $BP_*(-)$ never becomes an easily computable theory. We see by the above example that $K(n)_*(-)$ is sometimes an acceptable substitute. It appears to be quite computable, due mainly to the Künneth isomorphism

$$K(n)_*(X \times Y) \simeq K(n)_*X \otimes_{K(n)_*} K(n)_*Y \tag{6.29}$$

which follows because $K(n)_*$ is a graded field. We discuss (6.28) more later.

7. Hopf rings, the bar spectral sequence and $K(n)_*\underline{K}_*$. The techniques for computing and describing $K(n)_*\underline{K}_j$ ($\underline{K}_j = K(Z/(p), j)$) are general and have several BP related applications (like Sullivan's theory of manifolds with singularities). We have already used the Morava structure theorem for the proof of the Conner-Floyd conjecture. In addition, we need all of the material in this section to compute the necessary $K(j)_*\underline{K}_n$ for the completion of the proof. After this discussion we can bring up some other examples of the technique and this will eventually lead us to unstable cohomology operations.

Let

$$\underline{G}_* = \{\underline{G}_k\}_k \tag{7.1}$$

be an Ω-spectrum, i.e.

$$\Omega \underline{G}_{k+1} \simeq \underline{G}_k. \tag{7.2}$$

This represents a generalized cohomology theory

$$G^*X \simeq [X, \underline{G}_*]. \tag{7.3}$$

We want to study

$$E_*\underline{G}_* = \{E_*\underline{G}_k\}_k \tag{7.4}$$

where $E_*(-)$ is a multiplicative homology theory. To apply our tools we need a Künneth isomorphism for the spaces $\underline{G}_k$. This condition seldom holds in general, but always holds for

$$E_*(-) = K(n)_*(-) \quad \text{or} \quad H_*(-; Z/(p)). \tag{7.5}$$

It also usually holds for

$$\underline{G}_* = \underline{MU}_* \quad \text{or} \quad \underline{BP}_*. \tag{7.6}$$

These techniques can be used to compute things like

$$K(n)_*\underline{K}_*, \quad E_*\underline{MU}_*, \quad E_*\underline{BP}_*, \quad H_*\underline{k(n)}_*, \quad H_*\underline{BP\langle n\rangle}_*, \quad \text{and} \quad H_*\underline{K}_*. \tag{7.7}$$

We feel that even more non-BP applications could be made.

We have that G^kX is an abelian group. Thus $\underline{G}_k$ must be a homotopy commutative H-space (not surprising since we already know it is an infinite loop space), or, in other words, an abelian group object in the homotopy category. By our Künneth isomorphism, $E_*(-)$ takes $\underline{G}_k$ to a coalgebra; it also takes products of $\underline{G}_k$ to products in the category of coalgebras (tensor product). Thus $E_*(-)$ takes the abelian group object $\underline{G}_k$ to an abelian group object in the category of coalgebras. This is just to say that $E_*\underline{G}_k$ is a (bi-) commutative Hopf algebra with conjugation. This "abelian group" structure, or the Hopf algebra multiplication, just comes from the product

(7.8) $$*: \underline{G}_k \times \underline{G}_k \longrightarrow \underline{G}_k$$

after applying $E_*(-)$:

(7.9) $$*: E_*\underline{G}_k \otimes_{E_*} E_*\underline{G}_k \longrightarrow E_*\underline{G}_k.$$

Considering everything at once, G^*X is a graded abelian group so, as above, $\underline{G}_*$ is a graded abelian group object in the homotopy category. Likewise,

(7.10) $$E_*\underline{G}_* = \{E_*\underline{G}_k\}_k$$

is a graded abelian group object in the category of E_* coalgebras.

Of course, when G is a ring spectrum, G^*X is more than just a graded group; it is a graded ring. Then the graded abelian group object $\underline{G}_*$ in the homotopy category becomes a graded ring object. The multiplication

(7.11) $$G^kX \times G^nX \longrightarrow G^{k+n}X$$

has a corresponding multiplication in $\underline{G}_*$:

(7.12) $$\circ: \underline{G}_k \times \underline{G}_n \longrightarrow \underline{G}_{k+n}.$$

Applying $E_*(-)$ we have

(7.13) $$\circ: E_*\underline{G}_k \otimes_{E_*} E_*\underline{G}_n \longrightarrow E_*\underline{G}_{k+n},$$

turning $E_*\underline{G}_*$ into a graded ring object in the category of coalgebras. This object "should" be called a "coalgebraic ring" but instead has taken on names such as "Hopf bialgebra" and the one we use, "Hopf ring". See **[RW₁]**.

A ring must have a distributive law, and ours is:

(7.14) $$x \circ (y * z) = \sum \pm (x' \circ y) * (x'' \circ z) \quad \text{where } \psi(x) = \sum x' \otimes x''.$$

Because of the importance of the relationship between the two products and the coproduct we will track it down from the distributive law in G^*X. We have that

$$\text{(7.15)}\qquad \begin{array}{ccc} G^nX \times G^kX \times G^kX & \xrightarrow{I \times *} & G^nX \times G^kX \\ \downarrow{\scriptstyle \text{diag} \times I \times I} & & \downarrow{\scriptstyle \circ} \\ G^nX \times G^nX \times G^kX \times G^kX & & G^{n+k}X \\ \downarrow{\scriptstyle I \times \text{switch} \times I} & & \uparrow{\scriptstyle *} \\ G^nX \times G^kX \times G^nX \times G^kX & \xrightarrow{\circ \times \circ} & G^{n+k}X \times G^{n+k}X \end{array}$$

is the distributive law in G^*X. In terms of classifying spaces it becomes

$$\text{(7.16)}\qquad \begin{array}{ccc} \underline{G}_n \times \underline{G}_k \times \underline{G}_k & \xrightarrow{I \times *} & \underline{G}_n \times \underline{G}_k \\ \downarrow{\scriptstyle \Delta \times I \times I} & & \downarrow{\scriptstyle \circ} \\ \underline{G}_n \times \underline{G}_n \times \underline{G}_k \times \underline{G}_k & & \underline{G}_{n+k} \\ \downarrow{\scriptstyle I \times \text{switch} \times I} & & \uparrow{\scriptstyle *} \\ \underline{G}_n \times \underline{G}_k \times \underline{G}_n \times \underline{G}_k & \xrightarrow{\circ \times \circ} & \underline{G}_{n+k} \times \underline{G}_{n+k} \end{array}$$

Apply $E_*(-)$ to obtain (7.14).

Because of the two products it is easy to construct many elements starting with just a few; this helps to describe answers in terms of Hopf rings. All of our examples will demonstrate this property in a strong way. We will also demonstrate some of the other benefits which allow the Hopf ring structure to be used seriously in proofs as well. For even deeper applications we will put the Hopf ring structure into the bar spectral sequence. To do this we review the bar construction.

Let σ^n be the geometric n-simplex and $\underline{G}'_k$ the zero component of $\underline{G}_k$. Then

$$\text{(7.17)}\qquad \underline{G}'_{k+1} \simeq B\underline{G}_k \simeq \coprod_{n \geq 0} \sigma^n \times \underbrace{\underline{G}_k \times \cdots \times \underline{G}_k}_{n \text{ copies}} /\sim$$

where $\sim$ indicates that there are identifications made **[Mg]** which we will not give explicitly in these lectures. $B\underline{G}_k$ is filtered by

$$\text{(7.18)}\qquad B_s\underline{G}_k \simeq \coprod_{s \geq n \geq 0} \sigma^n \times \underbrace{\underline{G}_k \times \cdots \times \underline{G}_k}_{n \text{ copies}} /\sim \subset B\underline{G}_k.$$

Apply $E_*(-)$ to this filtered space to get the bar spectral sequence **[RS]**. Since

$$\text{(7.19)}\qquad B_s\underline{G}_k / B_{s-1}\underline{G}_k \simeq \Sigma^s \wedge \underbrace{\underline{G}_k \wedge \cdots \wedge \underline{G}_k}_{s \text{ copies}},$$

we have

$$\text{(7.20)}\qquad E^1_{s,*} \simeq \widetilde{E}_*(B_s\underline{G}_k / B_{s-1}\underline{G}_k) \simeq \widetilde{E}_*(\Sigma^s) \otimes_{E^*} \otimes^s \widetilde{E}_*\underline{G}_k,$$

and

$$E^2_{*,*} \simeq \operatorname{Tor}^{E_*\underline{G}_k}_{*,*}(E_*, E_*) \Rightarrow E_*\underline{G}'_{k+1}. \tag{7.21}$$

This is a spectral sequence of Hopf algebras. To put the additional structure of the $\circ$ multiplication in the bar spectral sequence we look at

$$\begin{array}{ccc}
\circ\colon \underline{G}_{k+1} \times \underline{G}_n & \longrightarrow & \underline{G}_{k+n+1} \\
\cup & & \cup \\
\circ\colon B\underline{G}_k \times \underline{G}_n & \longrightarrow & B\underline{G}_{k+n} \\
\| & & \| \\
\circ\colon \left\{ \coprod_{s\geq 0} \sigma^s \times \underbrace{\underline{G}_k \times \cdots \times \underline{G}_k}_{s \text{ copies}}/\sim \right\} \times \underline{G}_n & \longrightarrow & \left\{ \coprod_{s\geq 0} \sigma^s \times \underbrace{\underline{G}_{k+n} \times \cdots \times \underline{G}_{k+n}}_{s \text{ copies}}/\sim \right\}
\end{array} \tag{7.22}$$

THEOREM 7.23 (THOMASON-WILSON [TW]). *The $\circ$ product factors as*

$$\begin{array}{ccc}
B_s\underline{G}_k \times \underline{G}_n & \longrightarrow & B_s\underline{G}_{k+n} \\
\cap & & \cap \\
\circ\colon B\underline{G}_k \times \underline{G}_n & \longrightarrow & B\underline{G}_{k+n}
\end{array}$$

and the map

$$\begin{array}{ccc}
B_s\underline{G}_k/B_{s-1}\underline{G}_k \times \underline{G}_n & \longrightarrow & B_s\underline{G}_{k+n}/B_{s-1}\underline{G}_{k+n} \\
\| & & \| \\
\Sigma^s \underbrace{\underline{G}_k \wedge \cdots \wedge \underline{G}_k}_{s \text{ copies}} \times \underline{G}_n & \longrightarrow & \Sigma^s \underbrace{\underline{G}_{k+n} \wedge \cdots \wedge \underline{G}_{k+n}}_{s \text{ copies}}
\end{array}$$

is described inductively as $(g_1, \ldots, g_s) \circ g = (g_1 \circ g, \ldots, g_s \circ g)$. □

This automatically implies:

THEOREM 7.24 (THOMASON-WILSON [TW]). *Let $E^*_{*,*}(E_*\underline{G}_k) \Rightarrow E_*\underline{G}'_{k+1}$ be the bar spectral sequence. Compatible with*

$$\circ\colon E_*\underline{G}'_{k+1} \otimes_{E_*} E_*\underline{G}_n \longrightarrow E_*\underline{G}'_{k+1+n}$$

is a pairing

$$E^r_{s,*}(E_*\underline{G}_k) \otimes_{E_*} E_*\underline{G}_n \longrightarrow E^r_{s,*}(E_*\underline{G}_{k+n}),$$

with $d^r(x) \circ y = d^r(x \circ y)$. *For* $r = 1$ *this pairing is given by*

$$(g_1 | \cdots | g_s) \circ g = \Sigma \pm (g_1 \circ g' | g_2 \circ g'' | \cdots | g_s \circ g^{(s)})$$

where $g \longrightarrow \Sigma g' \otimes g'' \otimes \cdots \otimes g^{(s)}$ *is the iterated reduced coproduct.* □

This result is easy to prove, but tremendously powerful. It allows one to identify elements in terms of $\circ$ products, compute differentials inductively, and solve extension problems using Hopf ring properties. This will all be demonstrated. The first example of the above pairing was given in [**RW**$_2$] with $\underline{G}_* = \underline{K}_*$. It is unnecessary to know how to prove this result in order to use it for computational purposes.

We wish to inject some formal group nonsense at this point. Our formal groups all come from the standard map (2.3):

$$\mu\colon \mathbf{C}P^\infty \times \mathbf{C}P^\infty \longrightarrow \mathbf{C}P^\infty, \tag{7.25}$$

which is unstable information. This is generally stabilized immediately, with the exception of Quillen's proof that MU^*X is generated over MU^* by nonnegative degree elements [**Q**$_2$]. We go even further in [**RW**$_1$] in extracting unstable information from formal groups. In $E_*\underline{G}_*$ we have two formal groups. Later we will get unstable relations from their interaction. For now, we concentrate on just one formal group. We assume E is complex orientable, i.e. $\mathbf{C}P^\infty$ has x^E and β_i^E just like MU in (1.26) and (1.30). Thus we have a formal group law for E and elements a_{ij}^E. This formal group is the coproduct in $E^*\mathbf{C}P^\infty$. Dual to this is the product in $E_*\mathbf{C}P^\infty$. For $E_*(-) = H_*(-; Z)$ we can let $\beta(s) = \Sigma_{i\geqslant 0}\beta_i s^i$ and describe this product as

$$\beta(s)\beta(t) = \beta(s+t). \tag{7.26}$$

We know $H_*(\mathbf{C}P^\infty, Z)$ is a divided power algebra. If we look at the coefficient of s^it^j we get

$$\beta_i\beta_j = (i, j)\beta_{i+j}, \tag{7.27}$$

so the notation has managed to hide the binomial coefficients. We have much more to hide. Let

$$\beta(s) = \sum_{i\geqslant 0} \beta_i^E s^i \in E_*\mathbf{C}P^\infty[[s]]. \tag{7.28}$$

THEOREM 7.29 (RAVENEL-WILSON [**RW**$_1$]). *In* $E_*\mathbf{C}P^\infty[[s, t]]$,

$$\beta(s)\beta(t) = \beta(s +_F t). \quad \square$$

PROOF. Write $\beta_i\beta_j = \Sigma_k c_k\beta_k$ and $(x_1 +_F x_2)^k = \Sigma_{i,j} a_{ij}^k x_1^i x_2^j$. Then

$$c_k = \left\langle x^k, \sum_{i\geqslant 0} c_i x^i \right\rangle = \langle x^k, \beta_i\beta_j\rangle = \langle x^k, \mu_*(\beta_i \otimes \beta_j)\rangle$$

$$= \langle \mu^* x^k, \beta_i \otimes \beta_j\rangle = \left\langle \sum_{ij} a_{ij}^k x_1^i \otimes x_2^j, \beta_i \otimes \beta_j \right\rangle = a_{ij}^k.$$

Then $\beta(s)\beta(t) = \Sigma_{i,j}\beta_i s^i \beta_j t^j = \Sigma_{i,j,k} a_{ij}^k s^i t^j \beta_k = \Sigma_k \beta_k \Sigma_{ij} a_{ij}^k s^i t^j = \Sigma_k \beta_k (s +_F t)^k = \beta(s +_F t)$. □

Iterating we have:

COROLLARY 7.30 **[RW₁]**. $\beta(s)^n = \beta([n]_F(s))$. □

On the surface, this appears to be another nonsense formula involving totally inaccessible coefficients. However, in the important case $E = BP$ we can extract an explicit formula. Let

(7.31) $$\beta_{(i)} \equiv \beta_{p^i}.$$

It is fairly easy to see that the other β's are decomposable.

THEOREM 7.32 (RAVENEL-WILSON **[RW₁]**). *In* $QBP_* CP^\infty$ mod (p),

$$0 \equiv \sum_{k=1}^{n} v_k^{p^{n-k}} \beta_{(n-k)}. \quad \square$$

PROOF. Recall 3.17: $[p](x) \equiv \Sigma_{n>0}^F v_n x^{p^n}$ mod (p). By 7.30,

$$\beta(s)^p = \beta([p](s)) = \beta\left(\sum_{n>0}^F v_n s^{p^n}\right) \quad \text{mod } (p)$$

$$= \prod_{n>0} \beta(v_n s^{p^n}) \quad \text{mod } (p) \quad \text{by iterating 7.29.}$$

Since β_0 is the multiplicative identity we have $\beta(s)^p = \beta_0$ and $\Pi_{n>0}\beta(v_n s^{p^n}) = \Sigma_{n>0}\beta(v_n s^{p^n})$ in $QBP_* CP$ mod (p). Just pick off the coefficient of s^{p^k}. □

We can now proceed to our discussion of the Morava K-theories of the mod (p) Eilenberg-MacLane spaces, $K(n)_* \underline{K}_*$.

THEOREM 7.33 (RAVENEL-WILSON **[RW₂]**). *In* $K(n)_* CP^\infty$

$$\beta_{(i)}^p = 0, \quad 0 \leq i < n-1,$$

$$\beta_{(i+n-1)}^p = v_n^{p^i} \beta_{(i)}, \quad 0 \leq i. \quad \square$$

PROOF. The formal group law for BP reduces to $K(n)$ to give

$$[p](s) = \sum_{k>0}^F v_k s^{p^k} = v_n s^{p^n}.$$

By 7.30, $\beta(s)^p = \beta(v_n s^{p^n})$, so $\beta(s)^p = \Sigma_{i\geq 0}\beta_i^p s^{pi}$, and

$$\beta(v_n s^{p^n}) = \sum_{i \geq 0} \beta_i v_n^i s^{ip^n}.$$

The coefficients of $s^{p^{i+n}}$ give the result. □

Recall that $\underline{K}_j = K(Z/(p), j)$.

THEOREM 7.34 [$\mathbf{RW_2}$]. *The standard map $\underline{K}_1 \longrightarrow CP^\infty$ induces a Hopf algebra inclusion $K(n)_*\underline{K}_1 \subset K(n)_*CP^\infty$, where $K(n)_*\underline{K}_1$ is free over $K(n)_*$ on $a_i \in K(n)_{2i}\underline{K}_1$, $0 \leqslant i < p^n$, which maps to β_i. The standard coproduct $a_k \longrightarrow \Sigma a_{k-i} \otimes a_i$ follows and*

$$a_{(n-1)}^{*p} = v_n a_{(0)},$$

$$a_{(i)}^{*p} = 0, \quad 0 \leqslant i < n-1. \quad \square$$

PROOF. Any attempt works. $\square$

THEOREM 7.35 (GLOBAL VERSION, RAVENEL-WILSON [$\mathbf{RW_2}$]). *$K(n)_*\underline{K}_*$ is the free Hopf ring on $K(n)_*\underline{K}_1$.* $\square$

We have to say what a free Hopf ring is. Given a graded Hopf algebra

(7.36) $$H_*(*) = \{H_*(k)\}_k,$$

$H_*(k)$ a (graded) Hopf algebra with conjugation, the free Hopf ring $FH_*(*)$ is a functorially assigned Hopf ring and map α such that any map f of our Hopf algebra to a Hopf ring factors through α:

(7.37) $$\begin{array}{ccc} H_*(*) & \xrightarrow{\ f\ } & R_*(*) \\ & \searrow^{\alpha} & \uparrow \\ & & FH_*(*) \end{array}$$

The Hopf ring $FH_*(*)$ is constructed by taking all finite $*$ products of all finite $\circ$ products of elements and then using the relations obtained from Hopf algebras and Hopf rings.

Since our "abelian groups" in our fancy ring are bicommutative Hopf algebras with conjugation, our "-1", denoted $[-1]$, is just conjugation. So, if we are in a Hopf ring $R_*(*)$ with

(7.38) $$x \in R_i(n), y \in R_j(k), \quad x \circ y = (-1)^{ij}[-1]^{nk} y \circ x.$$

In the case of $K(n)_*\underline{K}_1$, $[-1]a_{(i)} = -a_{(i)}$, so

(7.39) $$a_{(i)} \circ a_{(j)} = -a_{(j)} \circ a_{(i)},$$

which implies, by Hopf ring madness, that

(7.40) $$a_{(i)} \circ a_{(i)} = 0.$$

In $K(n)_*\underline{K}_*$, the longest $\circ$ product is

(7.41) $$a_I = a_{(0)} \circ a_{(1)} \circ \cdots \circ a_{(n-1)}.$$

By the distributivity law

(7.42) $$a_{(i+1)} \circ x^{*p} = (a_{(i)} \circ x)^{*p},$$

so

$$
\begin{aligned}
a_I^{*p} &= (a_{(0)} \circ \cdots \circ a_{(n-1)})^{*p} = a_{(1)} \circ (a_{(1)} \circ \cdots \circ a_{(n-1)})^{*p} \\
&= a_{(1)} \circ a_{(2)} \circ (a_{(2)} \circ \cdots \circ a_{(n-1)})^{*p} \\
&\quad \vdots \\
&= a_{(1)} \circ a_{(2)} \circ \cdots \circ a_{(n-1)} \circ (a_{(n-1)}^{*p}) \\
&= a_{(1)} \circ \cdots \circ a_{(n-1)} \circ v_n a_{(0)} \\
&= (-1)^{n-1} v_n a_I.
\end{aligned} \tag{7.43}
$$

These are all of the elements in $K(n)_*\underline{K}_n$, so as advertised in (6.20), $K(n)_*\underline{K}_n$ is free over $p-1$ even degree generators. Furthermore, any longer $\circ$ product must give zero, so since $K(n)_*\underline{K}_1$ generates everything, $\widetilde{K(n)}_*\underline{K}_j = 0, j > n$, as in (6.20). This is all we needed to prove the Conner-Floyd conjecture.

A detailed description (local version) and proof for $K(n)_*\underline{K}_*$ are available in **[RW$_2$]**, and for the more difficult $K(Z/(p^i), j)$ as well. In particular, we describe $K(n)_*\underline{K}_j$ completely as a Hopf algebra. This is necessary since we use the bar spectral sequence (7.21) to make the computation. Inductively we compute Tor. Using 7.24 we can name many elements, show which are infinite cycles, and compute the differentials, all inductively. Then we use the Hopf ring structure to solve extension problems. Theorem 7.35 follows from this detailed version.

This type of calculation, which relies heavily on the Künneth theorem, demonstrates the computability of $K(n)_*(-)$ to our satisfaction, and we recommend them to other homotopy theorists.

8. $H_*\underline{K}_*$ and the Steenrod algebra. Before we introduce the second formal group law into Hopf rings we want to do a complete calculation. We will do a simple one, just the mod (p) homology of the mod (p) Eilenberg-MacLane spaces $H_*\underline{K}_*$. Usually people do the $p = 2$ case, but we will do only the odd primes. They demonstrate the pairing 7.24 better. Letting

$$
H = \{\underline{K}_i\}_{i \geqslant 0}, \tag{8.1}
$$

when we have $H_*\underline{K}_*$ we will also have H_*H, the dual of the odd primary Steenrod algebra. Having computed H_*H it is nearly obligatory to give its structure as a Hopf algebra. This is no problem since we have already done all the work for MU_*MU.

This computation of $H_*\underline{K}_*$ represents joint work with Douglas Ravenel. We came across it in the early stages of our work with Hopf rings. Since that time we have promised many people that we would write it up for pedagogical purposes. We thank Doug Ravenel for allowing us to use this in these notes. We leave the $K(Z/(p^j), n)$ cases and Bockstein's, etc., for a potential future write-up.

The beauty of this approach is that neither chains nor Steenrod operations need to be introduced. Everything is done using standard homological algebra to keep track of the Hopf ring structure inductively using the pairing in 7.24.

To describe our answer we need notation for $H_*\underline{K}_1$ and H_*CP^∞. We have

$$(8.2) \qquad e_1 \in H_1\underline{K}_1, \qquad \alpha_i \in H_{2i}\underline{K}_1, \qquad \beta_i \in H_{2i}CP^\infty, \qquad i \geq 0.$$

The generators are

$$(8.3) \qquad e_1, \qquad \alpha_{(i)} = \alpha_{p^i}, \qquad \beta_{(i)} = \beta_{p^i}.$$

The coproduct is

$$(8.4) \qquad \psi(\alpha_n) = \sum_{i=0}^{n} \alpha_{n-i} \otimes \alpha_i, \qquad \psi(\beta_n) = \sum_{i=0}^{n} \beta_{n-i} \otimes \beta_i.$$

THEOREM 8.5 (GLOBAL VERSION). *$H_*\underline{K}_*$ is the free Hopf ring on $H_*\underline{K}_0 = H_*[Z/p]$, $H_*\underline{K}_1$, and $H_*CP^\infty \subset H_*\underline{K}_2$, subject to the relation that $e_1 \circ e_1 = \beta_1$.* □

Denote the height p truncated polynomial algebra and the exterior algebra respectively by

$$(8.6) \qquad TP_1(x) = Z/(p)[x]/(x^p), \qquad E(x) = Z/(p)[x]/(x^2).$$

For finite sequences

$$(8.7) \qquad I = (i_1, i_2, \ldots), \qquad 0 \leq i_1 < i_2 < \cdots$$

define

$$(8.8) \qquad \alpha_I = \alpha_{(i_1)} \circ \alpha_{(i_2)} \circ \cdots, \qquad \alpha_\phi = [1] - [0] \in H_0\underline{K}_0.$$

For finite sequences

$$(8.9) \qquad J = (j_0, j_1, \ldots), \qquad j_k \geq 0,$$

define

$$(8.10) \qquad \beta^J = \beta_{(0)}^{\circ j_0} \circ \beta_{(1)}^{\circ j_1} \circ \cdots, \qquad \beta^{(0)} = [1] - [0].$$

THEOREM 8.11 (LOCAL VERSION). *As an algebra*

$$H_*\underline{K}_* \simeq \otimes_{I,J} E(e_1 \circ \alpha_I \circ \beta^J) \otimes_{I,J} TP_1(\alpha_I \circ \beta^J),$$

where the tensor product is over all I, J as above and the coproduct follows by Hopf ring properties from the α's and β's. □

Anyone who knows $H^*\underline{K}_*$ can conclude 8.5 and 8.11 rapidly enough. We give this example to demonstrate the computation techniques in a familiar setting.

PROOF OF 8.5. Just as in (7.38)–(7.40), $\alpha_{(i)} \circ \alpha_{(i)} = 0$. The relation $e_1 \circ e_1 = \beta_1$ is obvious. All even generators must be truncated of height one because of Hopf rings and the fact that they are in $H_*\underline{K}_1$ and $H_*\mathbf{C}P^\infty$. There are no additional relations in 8.11, so 8.5 follows. □

Homology suspend $\beta_{(i)}$ to define

(8.12) $$\xi_i \in H_{2(p^i-1)}H,$$

and $\alpha_{(i)}$ to define

(8.13) $$\tau_i \in H_{2p^i-1}H.$$

From 8.11 we have

(8.14) $$H_*H \simeq E[\tau_0, \tau_1, \ldots] \otimes P[\xi_1, \xi_2, \ldots].$$

THEOREM 8.15 (MILNOR [Mi_2]). *The coproduct on the Hopf algebra H_*H is given by*

$$\psi(\xi_n) = \sum_{i=0}^{n} \xi_{n-i}^{p^i} \otimes \xi_i \quad \text{and} \quad \psi(\tau_n) = \sum_{i=0}^{n} \xi_{n-i}^{p^i} \otimes \tau_i + \tau_n \otimes 1. \quad \square$$

PROOF. We have our usual (stable) maps

$$\begin{array}{ccccccc} \mathbf{C}P^\infty & \longrightarrow & \underline{MU}_2 & \longrightarrow & \mathbf{C}P^\infty = K(Z,2) & \longrightarrow & \underline{K}_2 \\ \downarrow & & \downarrow & & & & \downarrow \\ MU & = & MU & & \longrightarrow & & H \end{array}$$

and commuting diagram. From this we see that $b = \Sigma_{i\geqslant 0} b_i \in MU_*MU$ reduces to $\xi = \Sigma_{i\geqslant 0}\xi_i \in H_*H$. (Recall that since β_i, $i \neq p^j$, is decomposable it suspends trivially to H_*H.) Theorem 1.48(e) now gives us the coaction on $\mathbf{C}P^\infty$, just by the reduction $MU_*MU \longrightarrow H_*H$. Part (d) reduces

$$\psi(\xi) = \sum_{j\geqslant 0} \xi^{p^j} \otimes \xi_j.$$

For the τ's we compare with the ξ's using the maps

$$\begin{array}{ccc} \underline{K}_1 & \longrightarrow & \mathbf{C}P^\infty \\ \downarrow & & \downarrow \\ H & & H \end{array}$$

This gives us all of the coproduct except the term ? ⊗ 1, which must be $\tau \otimes 1$ because we are in a Hopf algebra. To be fair we must reduce the entire proof of 1.48(e) from MU to H; easily done. □

The τ_k and ξ_k are seen to be Milnor's.

PROOF OF 8.11 (RAVENEL-WILSON). This proof is by induction on spaces but we will do all steps at once. We use the bar spectral sequence as in §7. First we compute

$$E^2_{*,*} = \mathrm{Tor}^{H_*\underline{K}_*}_{*,*}(Z/(p), Z/(p)) \Rightarrow H_*\underline{K}_{*+1}.$$

Tor preserves tensor products and depends only on the algebra structure. $\mathrm{Tor}^{E(x)} \simeq \Gamma(\sigma x)$, $\Gamma(y)$ the divided power algebra, free over $Z/(p)$ on $\gamma_n(y)$ with coproduct

$$\psi(\gamma_n) = \sum \gamma_{n-i} \otimes \gamma_i;$$

product $\gamma_i\gamma_j = (i, j)\gamma_{i+j}$ generated by $\gamma_{(n)} = \gamma_{p^n}$. We have $\gamma_1(y) = y$ and σ is homology suspension. We have

$$\sigma x \in \mathrm{Tor}^{E(x)}_{1,|x|}(Z/(p), Z/(p))$$

and so $\gamma_n(\sigma x) \in \mathrm{Tor}^{E(x)}_{n,n|x|}(Z/(p), Z/(p))$.

$$\mathrm{Tor}^{TP_1(x)}(Z/(p), Z/(p)) \simeq E(\sigma x) \otimes \Gamma(\phi(x)),$$

with $\sigma x \in \mathrm{Tor}^{TP_1(x)}_{1,|x|}(Z/(p), Z/(p))$ and the transpotence,

$$\phi(x) \in \mathrm{Tor}^{TP_1(x)}_{2,p|x|}(Z/(p), Z/(p))$$

represented by the cycles $x^i|x^{p-i}$ in the bar resolution, E^1_{**}.

$$\gamma_n(\phi(x)) \in \mathrm{Tor}^{TP_1(x)}_{2n,np|x|}(Z/(p), Z/(p)).$$

Thus

$$\mathrm{Tor}^{H_*\underline{K}_*}_{**} \simeq \otimes_{I,J} E(\sigma\alpha_I\beta^J) \otimes_{I,J} \Gamma(\phi(\alpha_I\beta^J)) \otimes_{I,J} \Gamma(\sigma e_1\alpha_I\beta^J).$$

Since suspension is just $\circ$ multiplication with e_1, this is

$$\otimes E(e_1\alpha_I\beta^J) \otimes \Gamma(\phi(\alpha_I\beta^J)) \otimes \Gamma(\alpha_I\beta^{J+\Delta_0}).$$

Define

$$s(I) = (i_1 + 1, i_2 + 1, \ldots) \quad \text{and} \quad s(J) = (0, j_0, j_1, \ldots).$$

CLAIM 8.16. Modulo decomposables, in the spectral sequence pairing 7.24, for

(a) $$B\underline{K}_1 \times \underline{K}_{*-1} \longrightarrow B\underline{K}_*,$$

$$\gamma_{p^i}(\beta_{(0)}) \circ \alpha_{s^i(I)} \circ \beta^{s^i(J)} = \gamma_{p^i}(\alpha_I\beta^{J+\Delta_0}),$$

and for

(b) $$B\underline{K}_0 \times \underline{K}_* \longrightarrow B\underline{K}_*,$$

$$\gamma_{p^i}(\phi([1] - [0])) \circ \alpha_{s^{i+1}(I)}\beta^{s^{i+1}(J)} = \gamma_{p^i}(\phi(\alpha_I\beta^J)). \quad \square$$

We will finish the proof of 8.11 before we prove the claim. For degree reasons the spectral sequence for $B\underline{K}_0$ collapses and we have already named the elements

$$\gamma_{p^i}(\phi([1]-[0])) = \alpha_{(i)}.$$

Letting $S^1 \longrightarrow \underline{K}_1$ and comparing the spectral sequences for $CP^\infty = BS^1 \longrightarrow B\underline{K}_1$, we see that the one for BS^1 collapses trivially, and so, like the $\alpha_{(i)}$, the

$$\gamma_{p^i}(\beta_{(0)}) = \beta_{(i)}$$

are permanent cycles. Thus by induction, the elements on the right in 8.16 (using the differentials in 7.24) are also permanent cycles. Since these are all of the even degree generators, the spectral sequence collapses. (The odd degree generators are in Tor_1 and so are permanent cycles.) Furthermore, since we have names for the elements on the left in 8.16, we see that $\gamma_{p^i}(\phi(\alpha_I\beta^J))$ represents $\alpha_{(i)} \circ \alpha_{s^{i+1}(I)} \circ \beta^{s^{i+1}(J)}$ and $\gamma_{p^i}(\alpha_I\beta^{J+\Delta_0})$ represents $\beta_{(i)} \circ \alpha_{s^i(I)} \circ \beta^{s^i(J)} = \alpha_{s^i(I)} \circ \beta^{s^i(J+\Delta_0)}$. We have now located and named all of the elements

$$e_1 \circ \alpha_I \circ \beta^J, \quad \alpha_{(i)} \circ \alpha_{s^{i+1}(I)} \circ \beta^{s^{i+1}(J)}, \quad \alpha_{s^i(I)} \circ \beta^{s^i(J+\Delta_0)}.$$

This is just an obscure way of writing $e_1 \circ \alpha_I \circ \beta^J$, $\alpha_I \circ \beta^J$ without $\alpha_\phi\beta^{(0)}$.

So far we have collapsing and the correct answer. All we need now is to show that there are no algebra extension problems. We give two proofs. This is easy to do without Hopf rings. The map multiplication by p, $p\colon \underline{K}_* \longrightarrow \underline{K}_*$, is homotopically trivial, so

$$0 = p_*\colon H_*\underline{K}_* \longrightarrow H_*\underline{K}_*.$$

To show there are no extension problems all that we need to do is show

$$(\alpha_I \circ \beta^J)^{*p} = 0,$$

but this is just $p_*(\alpha_{s(I)} \circ \beta^{s(J)})$. However, since we are trying to demonstrate Hopf rings we will give a Hopf ring proof. Each $\alpha_{I'} \circ \beta^{J'}$ can be written as $\pm\alpha_{(i)} \circ \alpha_I \circ \beta^J$ or $\alpha_I \circ \beta^J \circ \beta_{(i)}$. The proof is the same for both. By the distributive law (7.14) (as in (7.42)),

$$(\alpha_I \circ \beta^J \circ \beta_{(i)})^{*p} = \alpha_{s(I)} \circ \beta^{s(J)} \circ (\beta_{(i)}^{*p}) = 0.$$

But by the same calculation, if $\beta_{(i)}^{*p}$ were not zero (such as in $K(n)_*\underline{K}_*$) this Hopf ring technique would have solved our extension problem for us. □

Proof of Claim 8.16. There is nothing to prove for the $i = 0$ case of 8.16(a). For $i > 0$ compute the $p^i - 1$ times iterated reduced coproduct. The symmetric term from the right-hand side agrees with that on the left as $\alpha_I\beta^{J+\Delta_0}$. In fact, there are no other terms but the symmetric one. The terms in Tor_{p^i} with this symmetric term equal to zero are easily seen to be the decomposables. Thus the two sides of (a) are equal modulo decomposables. When we establish the $i = 0$ part of (b), the $i > 0$ part will follow in the same way. It is this $i = 0$ part that makes the odd primes a much better example of the spectral sequence pairing than $p = 2$. Part (a) was done just by using the homology suspension fact that $e_1 \circ e_1 = \beta_1$. Nothing more complicated ever happens for $p = 2$ because there is never the filtration 2 transpotence to worry about. We now get to use the $r = 1$ computation of the pairing in 7.24. Using 7.24, iterated coproducts, the distributive law and the facts that, in our cases, $[1] \circ x = x$ and $[0] \circ x = 0$, we have

$$\phi([1] - [0]) \circ \alpha_{s(I)}\beta^{s(J)} = (([1] - [0])^{*p-1}|[1] - [0]) \circ \alpha_{s(I)}\beta^{s(J)}$$
$$= (\alpha_I \beta^J)^{*p-1}|\alpha_I \beta^J$$

plus many terms with more $*$ products involved. We know what all cycles representing nontrivial elements in Tor_2 look like. They are products of elements in Tor_1 or transpotence elements. Thus the terms with more $*$ products contribute nothing. □

9. Two formal groups and $BP_*\underline{BP}_*$. In this section we bring in the second formal group law to add even more structure to the Hopf ring $E_*\underline{G}_*$. We assume that both E and G are complex orientable (see after (7.25)) so we have x^E, x^G, β_i^E, and β_i^G. We have already done a crucial step with one of the formal group laws by computing the multiplication in E_*CP^∞ in 7.29.

We need to define a few elements. For

(9.1) $$x^G \in G^2 CP^\infty \simeq [CP^\infty, \underline{G}_2]$$

define

(9.2) $$x_*^G(\beta_i^E) \equiv b_i \in E_{2i}\underline{G}_2.$$

To make your confusion specific, this b_i stabilizes to b_{i-1} for $E = MU = G$.

For

(9.3) $$a \in G^i \simeq [\mathrm{pt}, \underline{G}_i], \qquad 1 \in E_0(\mathrm{pt}) \simeq E_0,$$

define

(9.4) $$a_*(1) \equiv [a] \in E_0\underline{G}_i.$$

We now have a sub-Hopf ring

(9.5) $$E_*[G^*] \subset E_*\underline{G}_*$$

where $E_*[G^*]$ is the "ring-ring" (a group ring with some extra structure) on G^*.

We define a "formal group sum"

(9.6) $$z +_{[FG]} y = \mathop{*}_{i,j} [a_{ij}^G] \circ z^{\circ i} \circ y^{\circ j}$$

using the Hopf ring "addition" and "multiplication".

We are ready to give the "main relation" which uses the interplay between the two formal group laws and Hopf rings to give unstable information.

THEOREM 9.7 (RAVENEL-WILSON [**RW**$_1$]). *Let $b(s) = \Sigma_{i \geq 0} b_i s^i$. In $E_*G_*[[s, t]]$,*

$$b(s +_{F_E} t) = b(s) +_{[FG]} b(t). \quad □$$

REMARK 9.8. This does not require the Künneth isomorphism and so is not necessarily in a Hopf ring.

REMARK 9.9. For some E and G, such as $K(n)_*\underline{K}_*$, this is vacuous because $b_i = 0$, $i > 0$.

REMARK 9.10. The notation is almost necessary because when 9.7 is expanded out we have

$$\sum_i b_i\Big(\sum_{jk} a^E_{jk} s^j t^k\Big)^i = \underset{ij}{*}\,[a^G_{ij}] \circ \Big(\sum_n b_n s^n\Big)^{\circ i} \circ \Big(\sum_m b_m t^m\Big)^{\circ j} \tag{9.11}$$

and the actual coefficients of $s^i t^j$ are even more unpleasant.

PROOF OF 9.7. We compute the composition $CP^\infty \times CP^\infty \xrightarrow{\mu} CP^\infty \xrightarrow{x^G} \underline{G}_2$ in two different ways.

$$b(s +_F t) = x^G_*(\beta(s +_F t)) = x^G_*(\beta(s)\beta(t)) = x^G_*\mu_*(\beta(s) \otimes \beta(t))$$

$$= (x^G \circ \mu)_*(\beta(s) \otimes \beta(t)) = (\mu^*(x^G))_*(\beta(s) \otimes \beta(t))$$

$$= \Big(\sum_{ij} a^G_{ij}(x^G_1)^i \otimes (x^G_2)^j\Big)_*(\beta(s) \otimes \beta(t))$$

$$= \underset{ij}{*}\,[a^G_{ij}] \circ b(s)^{\circ i} \circ b(t)^{\circ j} = b(s) +_{[F_G]} b(t).$$

The second to the last step is because the sum goes to $*$, the a_{ij} to $[a_{ij}]$ and the product to $\circ$. □

We obtain the following corollary by iteration.

COROLLARY 9.12 [**RW**$_1$]. *In* $E_*\underline{G}_*[[s]]$,

$$b([n]_{F_E}(s)) = [n]_{[F_G]}(b(s)). \quad □$$

Again, as with 7.29, this appears to be a useless nonsense formula; but again, as in 7.32, when we restrict to BP we can obtain very useful explicit results. To do this we go to the mod (p) homology.

THEOREM 9.13 [**RW**$_1$]. *Let* $I = (p, v_1, v_2, \ldots)$. *In* $QH_*\underline{BP}_2/[I]^{\circ 2}QH_*\underline{BP}_*$,

$$0 \equiv \sum_{k=1}^{n} [v_k] \circ b^{\circ p^k}_{(n-k)}. \quad □$$

PROOF. Using 9.12 for $n = p$ we have

$$b_0 = b(ps) = [p]_{[F]_{BP}}(b(s)) = \sum_{n>0}{}^{[F]} [v_n] \circ b(s)^{\circ p^n} * [p]\circ(z), \quad \text{by 3.17.}$$

Since $[p] \circ (z) = b_0$ mod decomposables, this becomes, mod decomposables and $[I]^{\circ 2}$, $\Sigma_{n>0}[v_n] \circ b(s)^{\circ p^n}$. Picking out the coefficient of s^{p^n} gives the result. □

We still have not answered the question about how powerful the "main relation" 9.7 is. In the universal case $G = MU$, it is complete and gives all information.

THEOREM 9.14 (RAVENEL-WILSON [$\mathbf{RW_1}$]). *$E_*\underline{MU}_{2*}$ ($E_*\underline{BP}_{2*}$) is generated over E_* by $[MU^*]$ ($[BP^*]$) and the b's ($b_{(i)}$'s). The only relations come from 9.7.* □

The proof is difficult and will not be given here. To obtain

$$E_*\underline{MU}_* \quad \text{and} \quad E_*\underline{BP}_* \tag{9.15}$$

it is enough to add $e_1 \in E_1\underline{MU}_1$ ($E_1\underline{BP}_1$) and the relation

$$e_1 \circ e_1 = b_1. \tag{9.16}$$

A key step in the proof is to show that the integral homology of $\underline{BP}_*$ has no torsion. With this the Atiyah-Hirzebruch spectral sequence for $BP_*\underline{BP}_*$ ($MU_*\underline{MU}_*$) collapses. The above solves all $*$ and $\circ$ product extension problems. By duality we have a complete description of $MU^*\underline{MU}_*$ ($BP^*\underline{BP}_*$), the unstable operations. The last section is dedicated to developing this further.

The explicit formula 9.13 can be used to give a basis for $QH_*\underline{BP}_*$ and $PH_*\underline{BP}_*$ which then lifts to $QE_*\underline{BP}_*$ and $PE_*\underline{BP}_*$. $E_*\underline{BP}_k$ is an exterior algebra or a polynomial algebra for k odd and even respectively. Let

$$I = (i_1, i_2, \ldots) \tag{9.17}$$

and

$$J = (j_0, j_1, \ldots), \tag{9.18}$$

both nonnegative finite sequences. Define

$$[v^I]b^J = [v_1^{i_1}v_2^{i_2}\cdots] \circ b_{(0)}^{\circ j_0} \circ b_{(1)}^{\circ j_1} \circ \cdots. \tag{9.19}$$

THEOREM 9.20 [$\mathbf{RW_1}$]. *A basis for $QE_*\underline{BP}_{2*}$ is given by all $[v^I]b^J$ such that if J can be written as*

$$J = p\Delta_{k_1} + p^2\Delta_{k_2} + \cdots + p^n\Delta_{k_n} + J', \quad k_1 \leqslant k_2 \leqslant k_3 \leqslant \cdots,$$

J' a nonnegative sequence, then $i_n = 0$. □

This basis uses 9.13 to get the largest v's possible. This is useful in the study of the $\underline{BP\langle n\rangle}_k$ [Si] and the unstable Hurewicz homomorphism for $\underline{BP}_k$ [Ma]. It is also possible to use 9.13 to obtain a basis using the smallest v's possible. This basis is particularly useful for computations with unstable operations; see §11.

THEOREM 9.21 (BOARDMAN). *A basis for $QE_*\underline{BP}_{2*}$ is given by all $[v^I]b^J$ such that if I can be written as*

$$I = \Delta_{k_0} + \Delta_{k_1} + \cdots + \Delta_{k_n} + I', \quad k_0 \leqslant k_1 \leqslant \cdots,$$

I' a nonnegative sequence, then $j_n < p^{k_n}$. □

For $BP_*\underline{BP}_*$, the $b_{(i)}$ are not always the best elements to work with. Stably, in BP_*BP, the t_i are not everyone's choice either. The elements $c(t_i)$ will certainly do as alternative stable generators. Even better, they desuspend to elements

(9.22) $$\overline{t}_i \in BP_{2p^i}\underline{BP}_2.$$

For this reason many people prefer them. Thus in 9.20 and 9.21 the $b_{(i)}$ can be replaced by the $\overline{t}_i$ (by 3.14).

10. Chan's proof of no torsion in $H_*\underline{BP}_*$. As already mentioned, knowing there is no torsion in the homology of $\underline{BP}_*$ is very important. The first proof of this fact in **[W$_1$]** is very ugly and we suspect no one ever reads it. There the spaces $\underline{BP}_k$ are dismantled in an unpleasant way. Then every possible torsion element that could arise during reassembly is hunted down and killed with such individuality that the process borders on sadism. The result, even in this weak form, already has several applications. It allows one to study the entire homotopy type of $\underline{BP}_k$. The important spaces here are the $\underline{BP\langle n\rangle}_k$. This leads to a successful analysis of H-spaces with no torsion in π_* or H_*. See **[W$_2$]** for these results. The homotopy type analysis is important in the study of hom $\dim_{BP_*} BP_*X$ in **[JW$_1$]** and gives the upper bound from the dimension of X. (For X of dimension k, hom $\dim_{BP_*} BP_*X \leqslant n$ where $k < p^n + p^{n-1} + \cdots + p + 1$.) In **[Zb]**, Zabrodsky also uses these spaces.

Later, a second proof for the no torsion of $H_*\underline{BP}_*$ was given in **[RW$_1$]**. Unfortunately, it is still rather inaccessible because it essentially proves 9.14 simultaneously. However, it is nicer than the original and it gives more information. Again we suspect few people read the inner details.

It is with great delight that we sketch a recent proof by Ken Chan which we think is simple enough to convince most people that the theorem is true. Chan's proof can easily be improved to show $H_*\underline{BP}_*$ is generated by the $[v_i]$'s and $b_{(i)}$'s, but completeness of the relations still requires more work.

The no torsion follows immediately from the following statement because the Bockstein spectral sequence collapses. We use mod (p) homology. A bipolynomial Hopf algebra is one where both it and its dual are polynomial.

THEOREM 10.1 (**[W$_1$]**; SECOND PROOF **[RW$_1$]**; CHAN'S PROOF **[Ch]**). *$H_*\underline{BP}_{2k+1}$ is an exterior algebra and $H_*\underline{BP}'_{2k}$ is bipolynomial. We have, for $k + n > 0$,*

$$\text{rank } QH_{k+n}\underline{BP}'_n = \text{rank } \pi_k BP = \text{rank } H_k BP. \quad \square$$

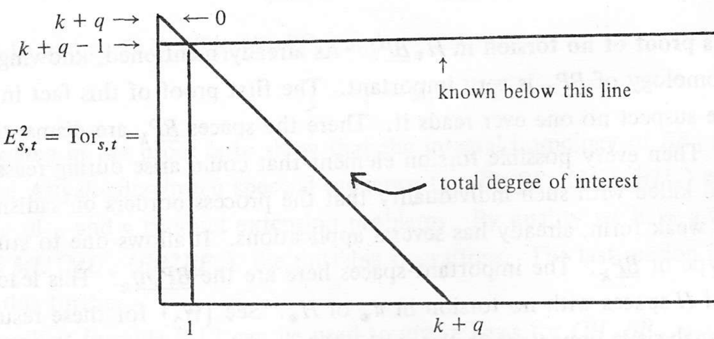
SKETCH OF CHAN'S PROOF. The proof is by induction. To ground the induction we have

$$H_1\underline{BP}'_n \simeq QH_1 BP'_n \simeq \pi_1\underline{BP}'_n \simeq \pi_{1-n}BP$$

which proves the result for degree $1 = k + n$. For our induction we assume the result for $H_{n+i}\underline{BP}'_n$, $0 \leqslant i < k$, all n, and $H_{n+k}\underline{BP}'_n$, $n < q$. We must show it for $H_{q+k}\underline{BP}'_q$. We use the bar spectral sequence

$$E^2_{**} = \mathrm{Tor}_{**}^{H_*\underline{BP}_{q-1}}(Z/(p), Z/(p)) \Rightarrow H_*\underline{BP}'_q.$$

For q odd, Tor of a polynomial algebra is just an exterior algebra on the suspensions of the generators. The spectral sequence obviously collapses and we are done. For q even, Tor of the exterior algebra is a divided power algebra Γ on the suspension of the generators. For both odd and even cases we compute



For q even, Γ is even degree so the spectral sequence collapses (in our range, by the diagram). Since Γ is dual to a polynomial algebra we have the cohomology part of bipolynomial. The rank of $QH^*\underline{BP}_q$ is also correct. We must now solve all extension problems in $E^2 = E^\infty = \Gamma$. We must show that it becomes a polynomial algebra of the correct rank. Since its dual is a polynomial algebra of the correct rank we know that if it is a polynomial algebra then its rank will be correct. Let A be the algebra after the extension problems are solved. If A is not polynomial then clearly rank $QA_{k+q} >$ rank $\pi_k BP$. By induction everything is okay in lower degrees and we are only concerned with degree $k + q$.

Now, use the bar spectral sequence to compute $H_{k+q+i}\underline{BP}'_{q+i}$, $i > 0$. (By induction we already know the lower degrees.) It is easy to see that this oversized rank will persist into the stable range, which is a contradiction because we know the stable homology. Thus A must be a polynomial algebra of the correct rank. □

This technique of "trapping" the homology of an Ω-spectrum between the known π_*E, in the form of $H_0\underline{E}_*$ and the known H_*E, has wide applications. This is the easiest case. It can be used in other instances to solve extension problems, to show collapsing, or to compute differentials.

Part III. Something New

11. Unstable operations. When I first showed, in 1971 [W_1], that $H_*\underline{BP}_k$ had no torsion, I realized it meant that unstable BP operations were accessible and abundant. It was hopeless to study them until a basis for $H_*\underline{BP}_k$ had been found [RW_1]. It is only in recent years that I have dabbled seriously with them. The usefulness of the only other truly unstable, additive operations, the Adams ψ^k in K-theory, led me to believe the BP operations carried a great deal of information. The recent work of Bendersky, Curtis, Miller and Ravenel (see [**BCM**] and later works) supports this conclusion. Since K-theory splits off cobordism [CF_2], we know that the unstable cobordism operations contain, at minimum, the same information as K-theory. However, a quick look at the homotopy type of $\underline{BP}_k$ [W_2] makes it clear that K-theory operations are only the surface of the operations available to cobordism. Although it is clear that BP operations contain a great deal of information, the old problem with BP is still there: how do you get information out and live to tell the tale? The machinery is now set up for straightforward computations, but, as yet, I have been unable to do a general computation with applications of interest. This has caused a delay in publishing these results. The reader has mercifully been spared the tortuous backwaters that months of calculations led me to. They have resulted in only a few conjectures to which I hope to return someday.

First I will turn the unstable operations into a "ring" and give a rigorous general nonsense definition of unstable modules, which is somewhat independent of the rest of the section. Second, I show that unstable operations are not just a simple divisibility problem for stable operations as I first suspected, but that unstable operations are very different animals despite their close relationship with stable operations. More explicitly, no multiple of a truly unstable operation is ever stable; although rationally, unstable and stable are the same! This is our main result. It is a fact first revealed by unspeakable computations of examples. This property is demonstrated in a familiar setting by showing that the Adams operations for complex cobordism are legitimate unstable operations. Just as with K-theory, ψ^k can be delooped twice for every power of k it is multiplied by. In order to prove this, formulas for computing

(11.1) $$r_*\colon BP_*\underline{BP}_n \longrightarrow BP_*\underline{BP}_k$$

are worked out for all unstable operations $r \in BP^k\underline{BP}_n = [\underline{BP}_n, \underline{BP}_k]$. The proofs in this section allow us to talk about stable operations, an aspect of BP neglected so far in these lectures.

Near the end of this section I have presented the gruesome details of an elementary application. The motivation is to show that straightforward calculation does produce, but this is not the way to do it. It is included for the student who really wants to learn the material.

The research related to this section took place over the years at The Institute for Advanced Study, Oxford University, The Johns Hopkins University, and the Centro de Investigacion del I.P.N. in Mexico City. In addition to the support of these institutions, the research was partially supported by the National Science Foundation and the Sloan Foundation. I am grateful to Luis Astey, Mike Boardman, Daciberg Goncalves, Don Davis, Vince Giambalvo, Sam Gitler, Dave Johnson, Peter Landweber, Dana Latch, Haynes Miller, John Moore, Bob Thomason, and surely, a few others I have forgotten; all of whom helped me out in one way or another.

Since this work was completed, Mike Boardman has developed another approach to unstable operations and he can reproduce most of these results and more. In particular he has proven the conjecture of mine that $QBP_*\underline{BP}_*$ is $[BP_*]$ free as well as BP_* free, a curious fact indeed.

We have described $BP_*\underline{BP}_*$ which gives, by duality, one description of the unstable BP operations. For our definition of unstable modules, we will go to a more general setting.

Let

(11.2) $$E^*(-) \simeq [-, \underline{E}_*]$$

be a generalized cohomology theory. The unstable cohomology operations are the natural transformation

(11.3) $$\begin{array}{ccc} E^kX & \longrightarrow & E^nX \\ \Vert & & \Vert \\ [X, \underline{E}_k] & \longrightarrow & [X, \underline{E}_n] \end{array}$$

which, by **[Br]**, are given by

(11.4) $$E^n\underline{E}_k \simeq [\underline{E}_k, \underline{E}_n].$$

We will restrict our attention to additive operations:

(11.5) $$r(x+y) = r(x) + r(y).$$

To do this effectively we need to assume that

(11.6) $$E^*(\underline{E}_k \times \underline{E}_k) \simeq E^*\underline{E}_k \hat{\otimes}_{E^*} E^*\underline{E}_k.$$

The "abelian group" structure of $\underline{E}_k$,

(11.7) $$\underline{E}_k \times \underline{E}_k \to \underline{E}_k,$$

which comes from the abelian group E^kX, gives rise to a "coproduct":

$$E^*\underline{E}_k \longrightarrow E^*\underline{E}_k \hat{\otimes}_{E^*} E^*\underline{E}_k, \tag{11.8}$$

and the "primitives", $PE^*\underline{E}_k$, i.e., those $r \in E^*\underline{E}_k$ such that

$$r \longrightarrow r \hat{\otimes} 1 + 1 \hat{\otimes} r, \tag{11.9}$$

are the additive unstable operations.

We want an unstable E^*E module to behave like E^*X for X a space. That is, we want a graded (topologized) E^* module, like E^*X, and pairings like

$$PE^n\underline{E}_k \otimes E^kX \longrightarrow E^nX, \tag{11.10}$$

giving a commuting diagram

$$\begin{array}{ccc} PE^i\underline{E}_n \otimes PE^n\underline{E}_k \otimes E^kX & \longrightarrow & PE^i\underline{E}_n \otimes E^nX \\ \downarrow & & \downarrow \\ PE^i\underline{E}_k \otimes E^kX & \longrightarrow & E^iX \end{array} \tag{11.11}$$

where the composition of maps gives the pairings (11.10) and

$$PE^i\underline{E}_n \otimes PE^n\underline{E}_k \longrightarrow PE^i\underline{E}_k. \tag{11.12}$$

We take the time now to make rigorous what we mean by the "ring" of unstable (additive) operations and modules over it. This part can be safely skipped by readers who need to do so.

It is understood that upper $*$ modules are topologized and that maps occur in the appropriate category. This will be suppressed throughout.

We have R algebras

$$E^* = E_{-*}, \qquad G^* = G_{-*}, \quad \text{etc.} \tag{11.13}$$

A *graded* $E^* - G_*$ *bimodule* is a bigraded R module

$$B^*_* = \{B^n_k\}, \tag{11.14}$$

where B^*_k is a left E^* module, B^n_* is a right G_* module, and these structures are compatible in the obvious way. We denote this category $\text{grbimod}_{E^*-G_*}$.

We need a "tensor product" functor

$$\otimes_{F^*}\colon \text{grbimod}_{E^*-F_*} \times \text{grbimod}_{F^*-G_*} \longrightarrow \text{grbimod}_{E^*-G_*}. \tag{11.15}$$

Define $(B^*_* \otimes_{F^*} C^*_*)^*_*$ in $\text{grbimod}_{E^*-G_*}$ by

$$(B^*_* \otimes_{F^*} C^*_*)^n_k = (B^n_* \otimes_{F^*} G^*_k)_0, \tag{11.16}$$

the zero degree of the standard graded tensor product over $F^* = F_{-*}$, i.e.,

(11.17) $$(B^*_* \otimes_{F^*} C^*_*)^n_k = \sum_i B^n_i \otimes_R C^i_k/(br \otimes c - b \otimes rc), \qquad r \in F^*.$$

One is not allowed to call just anything a tensor product. Bob Thomason has justified calling this a tensor product by showing that it is a coend, as any good tensor product should be.

The *coend* of a functor [**ML**, p. 222]

(11.18) $$S: C^{op} \times C \longrightarrow D,$$

is an object, $d \in D$, and a dinatural transformation

(11.19) $$\eta: S(c, c) \longrightarrow d$$

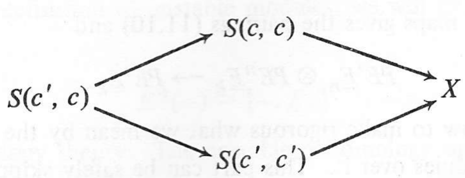
such that if $X \in D$ has a dinatural transformation

(11.20) $$\beta: S(c, c) \longrightarrow X,$$

then we have $\alpha: d \longrightarrow X$ with $\alpha\eta = \beta$.

To be dinatural means that for $\alpha: c \longrightarrow c'$, the following diagram commutes:

(11.21)



Let $C = F^*$, where, as a category, F^* has objects $i \in Z$ and morphisms

(11.22) $$F^{j-i} = \text{morph}(i, j).$$

$C^{op} = F_*$ with $F^{i-j} = F_{j-i} = \text{morph}(i, j)$. Define

(11.23) $$S(i, j) = B^*_i \otimes_R C^j_* \in \text{grbimod}_{E^*-G_*}.$$

The coend of this functor is just

(11.24) $$B^*_* \otimes_{F_*} C^*_*$$

as defined above.

Let

(11.25) $$B^*_* \in \text{grbimod}_{E^*-F_*}, \quad C^*_* \in \text{grbimod}_{F^*-G_*} \quad \text{and} \quad D^*_* \in \text{grbimod}_{E^*-G_*}.$$

Define the graded $E^* - F_*$ bimodule

(11.26) $$\text{Hom}_{R-G_*}\,{}^*_*(C^*_*, D^*_*)$$

by

(11.27) $$\text{Hom}_{R-G_*}\,{}^n_k(C^*_*, D^*_*) = \text{hom}_{R-G_*}(C^k_*, D^n_*).$$

The left E^* module structure is inherited from the left E^* module structure of D^*_*. The right F_* module structure is induced by the left F^* module structure on C^*_*.

The adjoint relation is

$$\text{hom}_{E^*-G_*}(B^*_* \otimes_{F^*} C^*_*, D^*_*) \simeq \text{hom}_{E^*-F_*}(B^*_*, \text{Hom}_{R-G_*}{}^*_*(C^*_*, D^*_*)). \tag{11.28}$$

Let $B^*_* \in \text{grbimod}_{E^*-E_*}$. A map

$$B^*_* \otimes_{E^*} B^*_* \longrightarrow B^*_* \tag{11.29}$$

puts a "*ring*" structure on B^*_*. The definition of *associative* is obvious since $\otimes_{E^*}$ is associative. A *unit* is a sequence of elements $1_n \in B^n_n$ which act as left and right units in the obvious fashion. When we say *ring* we mean an associative "ring" structure with unit. Observe that this is a sequence of maps

$$B^i_n \otimes B^n_k \longrightarrow B^i_k. \tag{11.30}$$

EXAMPLE 11.31. $\text{End}^*_*(M^*)$. Let M^* be a left E^* module. Let

$$\text{End}^n_k(M^*) \simeq \text{hom}_R(M^k, M^n).$$

The left E^* module structure of M^* turns $\text{End}^*_*(M^*)$ into an object of $\text{grbimod}_{E^*-E_*}$. The "ring" structure is just composition of maps and so it is clearly associative. The unit is the collection of identity maps $1_n \in \text{hom}_R(M^n, M^n)$.

More generally, we can define, for $M^*_* \in \text{grbimod}_{E^*-G_*}$,

$$\text{End}^n_k(M^*_*) = \text{hom}_{R-G_*}(M^k_*, M^n_*),$$

which is again a ring in $\text{grbimod}_{E^*-E_*}$. The first is a special case of this. □

EXAMPLE 11.32. B^*_* MODULES. A B^*_* *module*, $M^*_* \in \text{grbimod}_{E^*-G_*}$, for B^*_* a ring in $\text{grbimod}_{E^*-E_*}$, is a ring map

$$B^*_* \longrightarrow \text{End}^*_*(M^*_*),$$

which, by the adjoint relation, is the same as a definition given by an $E^* - G_*$ map,

$$B^*_* \otimes_{E^*} M^*_* \longrightarrow M^*_*,$$

with commuting diagram

$$\begin{array}{ccc} B^*_* \otimes_{E^*} B^*_* \otimes_{E^*} M^*_* & \xrightarrow{1 \otimes m} & B^*_* \otimes_{E^*} M^*_* \\ \downarrow{\scriptstyle m \otimes 1} & & \downarrow{\scriptstyle m} \\ B^*_* \otimes_{E^*} M^*_* & \xrightarrow{m} & M^*_* \end{array}$$

As above, we can specialize to the case of a left E^* module M^*. □

EXAMPLE 11.33. STABLE E^*E MODULES. Let E^*E be the stable cohomology operation ring for a generalized cohomology theory represented by the spectrum E. Define $\mathrm{St}^*_* \in \mathrm{grbimod}_{E^*-E_*}$ by

$$E^{n-k}E = \mathrm{St}^n_k.$$

This is not a very efficient way to say it, but a ring map $\mathrm{St}^*_* \longrightarrow \mathrm{End}^*_*(M^*)$ is the same as our usual concept of a *stable E^*E module structure on M^**.

There is a stable E^*E module structure induced on each $\Sigma^i M^*$, $i \in Z$, from

$$\mathrm{St}^*_* \simeq \mathrm{St}^{*-i}_{*-i} \longrightarrow \mathrm{End}^{*-i}_{*-i}(M^*) \simeq \mathrm{End}^*_*(\Sigma^i M^*). \quad \square$$

EXAMPLE 11.34. UNSTABLE E^*E MODULES. This is our motivating example. Let $E = \underline{E}_*$ be an Ω-spectrum with

$$E^*(\underline{E}_k \times \underline{E}_k) \simeq E^*\underline{E}_k \hat{\otimes} E^*\underline{E}_k$$

for every k. We have $PE^*\underline{E}_k \subset E^*\underline{E}_k$. Let $U^n_k = PE^n\underline{E}_k$. This is a ring in $\mathrm{grbimod}_{E^*-E_*}$. We define an *unstable E^*E module structure on M^** to be a ring map

$$U^*_* \longrightarrow \mathrm{End}^*_*(M^*).$$

The cohomology suspension $E^{n-k}E \longrightarrow PE^n\underline{E}_k$ gives a ring map $\mathrm{St}^*_* \longrightarrow U^*_*$. By composition, any unstable module has a stable structure on it. We say a stable E^*E module is unstable if the following diagram can be filled in:

$$\begin{array}{ccc} \mathrm{St}^*_* & \longrightarrow & \mathrm{End}^*_*(M^*_*) \\ \downarrow & \nearrow & \\ U^*_* & & \end{array}$$

A stable module may have many unstable structures, or none.

An unstable module structure on M^* can be lifted to an unstable structure on $\Sigma^i M^*$, $i \geqslant 0$, by using the cohomology suspension map $PE^*\underline{E}_* \longrightarrow PE^{*-i}\underline{E}_{*-i}$ to get

$$U^*_* \longrightarrow U^{*-i}_{*-i} \longrightarrow \mathrm{End}^{*-i}_{*-i}(M^*) \simeq \mathrm{End}^*_*(\Sigma^i M).$$

Extending the unstable structure on M^* to one on $\Sigma^{-1}M^*$ (or $\Sigma^{-i}M^*$) may not always be possible. This, of course, is one of the important points about unstable operations.

Every $\widetilde{E}^*X$, for X a space, is an unstable E^*E module, and $\widetilde{E}^*\Sigma^i X$ is an unstable module isomorphic to $\Sigma^i\widetilde{E}^*X$. If a space can be desuspended, say $X \simeq \Sigma Y$ $(Y = \Sigma^{-1}X)$, then

$$\widetilde{E}^*Y \simeq \Sigma^{-1}\widetilde{E}^*X \tag{11.35}$$

and we have an unstable structure on $\Sigma^{-1}\widetilde{E}^*X$. Therefore, if there is no compatible unstable structure on $\Sigma^{-1}\widetilde{E}^*X$, then X cannot be desuspended.

We have already noted that for $E = MU$ and BP,

$$E^*\underline{E}_k \simeq \hom_{E_*}(E_*\underline{E}_k, E_*) \tag{11.36}$$

and (11.6) holds. This is simply because $H_*\underline{E}_*$ has no torsion and the Atiyah-Hirzebruch spectral sequence collapses. Standard mod (p) homology also satisfies condition (11.6). The stable operations H^*H give the Steenrod algebra A_p. We can exhibit the striking differences between unstable BP and H operations and show the reasoning behind our philosophy that BP operations should contain a great deal of information. We have, for $n > 0$,

(11.37)
$$A_p \twoheadrightarrow PH^*\underline{K}_n \subset H^*\underline{K}_n \quad \text{primitively generated,}$$
$$BP^{*-n}BP \rightarrowtail PBP^*\underline{BP}_n \subset BP^*\underline{BP}_n \quad \text{not primitively generated.}$$

For mod (p) cohomology, the stable operations map onto the additive unstable operations and $H^*\underline{K}_n$ is primitively generated. Thus the only unstable operations are stable operations and cup products. The main unstable information is that some stable operations (for $p = 2$, Sq^i, $i > n$) are always trivial on n dimensional classes.

For BP, the stable operations inject into the additive unstable operations. As a bonus, $BP^*\underline{BP}_n$ is *not* primitively generated. So not only do we have a rich new collection of additive unstable operations, but there are serious new nonadditive operations not generated by cup products and the additive operations.

We will tend to ignore the nonadditive operations until useful applications of the additive ones have been found. Theoretically, though, it is clear how to deal with them, and computations like those in this section can be carried out.

Recalling that $H_*\underline{MU}_n$ is bipolynomial or exterior, we have $MU_*\underline{MU}_n$ is cofree and $MU^*\underline{MU}_*$ is either a completed polynomial algebra (n even) or a completed exterior algebra (n odd). We can see by duality that $MU^*\underline{MU}_n$ is not primitively generated.

Since there is never any torsion anywhere in BP (MU) we can feel free to study things rationally without loss of information. BP is just a bunch of Eilenberg-MacLane spaces rationally so we have

(11.38)
$$BP_{\mathbf{Q}}^{*-n}BP \simeq PBP_{\mathbf{Q}}^*\underline{BP}_n, \qquad n > 0,$$

with obvious modifications for $n \leqslant 0$. The isomorphism is given by cohomology suspension. This immediately implies

(11.39)
$$0 \longrightarrow BP^{*-n}BP \longrightarrow PBP^*\underline{BP}_n, \qquad n > 0,$$

with modifications for $n \leqslant 0$. From this it appears that the study of unstable operations for BP is just a problem of divisibility conditions for stable operations, and indeed, if we restrict our attention to k dimensional complexes this is the case, and the coker defined by

(11.40)
$$BP^{*-n}BP \longrightarrow PBP^*\underline{BP}_n \longrightarrow \text{coker} \longrightarrow 0$$

would be entirely torsion. However, as we let k go to ∞, we have an inverse limit with higher and higher torsion. (Think p-adics.) In other words we have completion problems, discussed more later, and unstable operations cannot be represented in terms of stable operations and divisibility problems, but they are truly different objects. Our main theorem is

THEOREM 11.41. *In* (11.40), *there is no torsion in the cokernel.* □

We defer the proof for a bit. We have:

$$\begin{array}{ccccc} BP_{\mathbf{Q}}^{*-n}BP & \supset & BP^{*-n}BP & \xrightarrow{\simeq} & \hom_{BP_*}(BP_*BP, BP_*) \\ \downarrow \simeq & & \cap & & \cap \\ PBP_{\mathbf{Q}}^{*}\underline{BP}_n & \supset & PBP^{*}\underline{BP}_n & \xrightarrow{\simeq} & \hom_{BP_*}(\mathbf{Q}BP_*\underline{BP}_n, BP_*) \end{array} \tag{11.42}$$

We now have two ways to describe our unstable operations. First, they are dual to $\mathbf{Q}BP_*\underline{BP}_n$, which we have an explicit basis for. Second, we know they are contained in the rational stable operations. For this to be a useful statement, we must know how they sit inside. Our first description can tell us. We have

$$BP_{\mathbf{Q}}^{*-n}BP \simeq \hom_{BP^*}(\mathbf{Q}BP_*\underline{BP}_n, BP_*^{\mathbf{Q}}) \tag{11.43}$$

and the unstable operations, $PBP^*\underline{BP}_n$, in

$$BP^{*-n}BP \subset PBP^*\underline{BP}_n \subset BP_{\mathbf{Q}}^{*-n}BP \tag{11.44}$$

are just those maps which send $\mathbf{Q}BP_*\underline{BP}_n$ to $BP_* \subset BP_*^{\mathbf{Q}}$. We need a more explicit description of the isomorphism (11.43).

We can think of $r \in BP^{k-n}BP$ as a map

$$r \in PBP^k\underline{BP}_n \subset BP^k\underline{BP}_n \simeq [\underline{BP}_n, \underline{BP}_k] \simeq \{\underline{BP}_n, \Sigma^k BP\}, \tag{11.45}$$

just by restricting to n dimensional classes (the cohomology suspension again). The map in (11.43) is induced as usual: r goes to

$$\hat{r} \in \hom_{BP_*}(\mathbf{Q}BP_*\underline{BP}_n, BP_*) \tag{11.46}$$

with

$$\hat{r}(y) = \langle r, y\rangle \quad \text{or} \quad \hat{r}(y) = \epsilon \circ s^\infty \circ r_*(y), \tag{11.47}$$

s^∞ the ∞ homology suspension and r_* the induced map (11.1).

Although our concern is mainly $\hat{r}$, evaluating r_* is an interesting problem in its own right. As mentioned already, it is enough to do this rationally.

Recall from §9 the notations $\circ$, $*$, $[a]$, $b = \Sigma_{i\geq 0} b_i$ and $x \in BP^2\mathbf{C}P^\infty$. The algebra structure on BP_*BP induces a coalgebra on BP^*BP with $r \longrightarrow \Sigma r' \hat{\otimes} r''$.

THEOREM 11.48. *Let* $r \in BP_{\mathbf{Q}}^{k-n}BP \subset PBP_{\mathbf{Q}}^k\underline{BP}_n$, *any* n *(also for MU). If* $r(x) = \Sigma_{i\geq 0} c_i x^{((k-n)/2)+i+1}$, $c_i \in BP_{2i}^{\mathbf{Q}}$, *then*

(i) $r_*(y * z) = r_*(y) * r_*(z)$, $\hat{r}(x * y) = 0$, (x, y) *in the augmentation ideal*;

(ii) $r_*(y \circ z) = \Sigma r'(y) \circ r''(z)$, $\hat{r}(y \circ z) = \Sigma \hat{r}'(y)\hat{r}''(z)$;

(iii) $r_*([a]) = [r(a)]$, $\hat{r}([a]) = r(a)$;

(iv) $r_*(b) = *_{i\geq 0}[c_i] \circ b^{\circ((k-n)/2)+i+1}$, $\hat{r}(b) = \Sigma_{k\geq 0} c_k = c$. □

This gives a complete evaluation of r_* for $r \in PBP_{\mathbf{Q}}^k\underline{BP}_n$ because we know that the $[a]$ and b's generate $BP_*\underline{BP}_n$ from §9. Since $BP^*\underline{BP}_n$ is primitively generated rationally, it is easy to extend this to all $r \in BP^k\underline{BP}_n$.

PROOF. (i) follows from the additivity of r and the fact that homology suspension is trivial on decomposables. (ii) and (iii) are elementary. To show (iv), we consider, rationally,

$$CP^\infty \xrightarrow{x} \underline{BP}_2 \xrightarrow{r} \underline{BP}_{2+k-n}.$$

Then

$$r_*(b) = r_*x_*(\beta) = r(x)_*(\beta) = \left(\sum c_i x^{((k-n)/2)+i+1}\right)_* \beta = \underset{i\geq 0}{*} [c_i] \circ b^{((k-n)/2)+i+1}.$$

From $\epsilon(b) = 1$, $s(x * y) = 0$, and $\epsilon(\eta_R(a)) = a$, we have

$$\hat{r}(b) = \epsilon \circ s^\infty \circ r_*(b) = \epsilon \circ s^\infty\left(\underset{i\geq 0}{*} [c_i] \circ b^{((k-n)/2)+i+1}\right)$$

$$= \epsilon\left(\sum_{i\geq 0} \eta_R(c_i) b^{((k-n)/2)+i+1} \circ b_1^{\circ\infty}\right) = \sum_{i\geq 0} c_i = c. \quad \square$$

We are now equipped to produce some honest unstable operations. If we are given

(11.49) $$r \in MU_{\mathrm{Q}}^{*-n}MU \simeq PMU_{\mathrm{Q}}^*\underline{MU}_n \simeq \hom_{MU_*}(QMU_*\underline{MU}_n, MU_*^{\mathrm{Q}})$$

we can evaluate it on $QMU_*\underline{MU}_n$, and if it lands in $MU_* \subset MU_*^{\mathrm{Q}}$ we have a legitimate unstable operation.

Several people have studied rational Adams operations for MU and BP, but generally they are not viewed as potential unstable operations [N, Ar$_2$, Sn].

Recall (1.53) that the MU_{Q} multiplicative operations are given by power series

(11.50) $$f(x) = x + \cdots$$

with coefficients in MU_*^{Q}. Let

(11.51) $$f(x) = [k](x)/k$$

define a rational multiplicative operation ψ^k. We know that for $x \in MU_{\mathrm{Q}}^2 CP^\infty$,

(11.52) $$\psi^k(x) = [k](x)/k.$$

THEOREM 11.53. *$\psi^k \in PMU^0\underline{MU}_0 \subset MU_{\mathrm{Q}}^0 MU$ and $k^i\psi^k \in PMU^{2i+\epsilon}\underline{MU}_{2i+\epsilon} \subset MU_{\mathrm{Q}}^0 MU$, $\epsilon = 0, 1, i \in Z$.* $\square$

REMARKS 11.54. Novikov [N] claims the first part for torsion free spaces in a footnote. The result is true for BP as well because the ψ^k exist there [Ar$_2$]. Note that the ψ^k imply torsion in the coker of $PBP^*\underline{BP}_* \rightarrowtail PBP^{*-2}\underline{BP}_{*-2}$.

PROOF. Recall (3.7) that mog $x = \log f^{-1}(x)$. We have $f^{-1}(x) = [1/k](kx)$ because

$$[1/k](k([k](x)/k) = [1/k]([k](x)) = x.$$

So

$$\begin{aligned}\operatorname{mog} x &= \log f^{-1}(x) = \log[1/k](kx) = \log \exp((1/k)\log(kx))\\ &= (1/k)\log(kx) = (1/k)\sum m_i(kx)^{i+1},\end{aligned}$$

and since

$$\sum_{i\geqslant 0} \psi_*^k(m_i)x^{i+1} = (1/k)\sum m_i(kx)^{i+1},$$

we have $\psi_*^k(m_i) = k^i m_i$ and it follows that, for $a \in MU^{-2i}$,

$$\psi^k(a) = k^i a.$$

We will prove only the $MU^0\underline{MU}_0$ part as it adequately demonstrates the techniques. Recall Theorem 9.13 that $MU_*\underline{MU}_*$ is generated by b's and $[MU^*]$. So $\mathbf{Q}MU_*\underline{MU}_0$ must be generated by elements of the form

$$[a] \circ b_1^{\circ j_1} \circ b_2^{\circ j_2} \cdots = [a] b^J$$

with $|a| = -2(j_1 + j_2 + \cdots)$, $[a] \in MU_0\underline{MU}_{|a|}$. We have $\hat{\psi}^k[a] = k^{-|a|/2}a$ and $\hat{\psi}^k(b) = [k](1)/k$. Since ψ^k is multiplicative we have

$$\begin{aligned}\hat{\psi}^k([a]b^J) &= \hat{\psi}^k([a])\hat{\psi}^k(b^J) = k^{-|a|/2}a\hat{\psi}(b^J)\\ &= k^{-|a|/2}a(\text{stuff})/k^{j_1+j_2+\cdots},\end{aligned}$$

which is integral! Likewise for the general case. □

Before we prove 11.41 we introduce the stable BP operations. Define monomials in the t's in BP_*BP by

(11.55) $$t^G = t_1^{g_1}t_2^{g_2} \cdots .$$

We define

(11.56) $$r_E \in BP^*BP$$

by the property

(11.57) $$\langle r_E, t^G\rangle = \delta_E^G.$$

Then $|r_E| = |E| = 2\Sigma_{i\geqslant 0}e_i(p^i - 1)$. Let R be the free $Z_{(p)}$ module on the r_E. R is a coalgebra with

(11.58) $$r_E \longrightarrow \sum_{E'+E''=E} r_{E'} \otimes r_{E''}$$

and

(11.59) $$BP^*BP \simeq BP^* \hat{\otimes} R.$$

An element $r \in BP^*BP$ is a possibly infinite sum

(11.60) $$r = \sum c_E r_E, \qquad c_E \in BP^*.$$

Our interest is in $BP^*_{\mathbf{Q}}BP$, which is not the same as tensoring BP^*BP with $\mathbf{Q}$, but is the same as taking sums (11.60) with the $c_E \in BP^*_{\mathbf{Q}}$. Define a *truly unstable operation* to be an element of $PBP^k\underline{BP}_n$ which is not in the image of $BP^{k-n}BP$, i.e., an additive unstable operation which is not a stable operation. We can rephrase 11.41.

THEOREM 11.61. *A truly unstable operation $r = \Sigma c_E r_E$ must be an infinite sum with unbounded negative powers of p in the coefficients c_E.* □

Define $a_i^E \in BP_{2i}$ by

$$r_E(x) = \sum_{i \geq 0} a_i^E x^{|E|/2+i+1}, \quad x \in BP^2\mathbf{CP}^\infty. \tag{11.62}$$

LEMMA 11.63. *Let $a^E = \Sigma_{i\geq 0} a_i^E$. Then*

$$b = c\left(\sum_{i\geq 0}{}^F t_i\right) = \sum_E a^E t^E. \quad \square$$

PROOF.

$$a^E = \langle r_E x, \beta\rangle = \left\langle r_E \otimes x, \sum_{j\geq 0} c\left(\sum_{i\geq 0}{}^F t_i\right)^j \otimes \beta_j\right\rangle \quad \text{by 3.13}$$

$$= \sum_{j\geq 0} \left\langle r_E, c\left(\sum_{i\geq 0}{}^F t_i\right)^j\right\rangle \langle x, \beta_j\rangle.$$

Since $\langle x, \beta_j\rangle = 0$ except when $j = 1$, we have

$$= \left\langle r_E, c\left(\sum_{i\geq 0}{}^F t_i\right)\right\rangle,$$

the required result. The first equality is just 3.14. □

Tables of some a_i^E for $p = 2$ have been computed and are presented at the end of this section. Thanks to Daciberg Goncalves and Don Davis for tremendous help.

COROLLARY 11.64. $\hat{r}_E(b) = a^E$. □

COROLLARY 11.65. $\hat{r}_{\Delta_i}(b_{(i)}) = a_0^{\Delta_i} = -1$. □

PROOF. Use induction and $\Sigma^F t_j c(t_i)^{p^j} = 1$. □

PROOF OF 11.61. Let $f = \Sigma d^E r_E \in PBP^*\underline{BP}_n \subset BP^{*-n}_{\mathbf{Q}}\underline{BP}$ not in $BP^{*-n}BP \subset PBP^*\underline{BP}_n$; then $0 \neq d^E \in BP^*_{\mathbf{Q}}/BP^*$ for some E. We assume that $p^k f = r \in BP^*BP$ for some minimal $k > 0$. We obtain a contradiction. This proves the result. Since we have multiplied f by p to get r, $\hat{r}$ must be trivial mod (p). By cohomology suspension, f is in all

$PBP^*\underline{BP}_i$, $i \leqslant n$, and the corresponding $\hat{r}$'s are also trivial modulo (p). Since we have chosen k minimal we can write

$$r = \sum c^E r_E + pr', \quad \text{some } r' \in BP^*BP$$

with $c^E \not\equiv 0$ (p) and some nontrivial c^E. The pr' will automatically have $\widehat{pr'} \equiv 0 \bmod (p)$. So since $\hat{r}$ is zero mod (p), we must have $r'' = \Sigma c^E r_E$ give $\hat{r}'' \equiv 0 \bmod (p)$. We show this does not happen, i.e., for any such r'' with $c^E \not\equiv 0 \bmod (p)$ we find that for large negative i, $\hat{r}''$ is nontrivial mod (p) on $QBP_*\underline{BP}_i$. Of course we need to find only one element x with $\hat{r}''(x) \not\equiv 0$ (p).

Order the sequences E by

$$E_1 < E_2 \quad \text{if } |E_1| < |E_2|,$$

$$E_1 < E_2 \quad \text{if } |E_1| = |E_2| \text{ and } l(E_1) > l(E_2),\ l(E) = \sum e_i,$$

$$E_1 < E_2 \quad \text{if } |E_1| = |E_2|,\ l(E_1) = l(E_2),$$

$$e_i^1 = e_i^2,\ i = 1, 2, \ldots, k-1, \text{ and } e_k^1 < e_k^2.$$

Pick the minimal E from the set of c^E. To construct our x we begin with $b^E = b_{(1)}^{\circ e_1} \circ b_{(2)}^{\circ e_2} \circ \cdots$. By our choice of E, 11.65 and 11.48,

$$\hat{r}''(b^E) = (-1)^{l(E)} c^E. \tag{11.66}$$

We started with a fixed $PBP^*\underline{BP}_n$ and we must produce an x in $PBP^*\underline{BP}_i$, $i \leqslant n$. This b^E may not meet this condition. Pick k and j very large so that $[v_k^{p^j}] \circ b^E \in PBP^*\underline{BP}_i$, $i \leqslant n$. Also choose k bigger than any v in c^E, and j bigger than $l(E)$ and $|c^E|$; then

$$\widehat{c^E r_E}([v_k^{p^j}] \circ b^E) \equiv \pm c^E v_k^{p^j} \quad (p). \tag{11.67}$$

For $\widehat{c^{E'} r_{E'}}$, $E' > E$, to give something nonzero we know that some of $r_{E'}$ must apply to $[v_k^{p^j}]$, by (11.66). If so, then it must apply to all p^j of the v_k's in order to be nonzero mod (p). This guarantees p^j v's smaller than v_k. However, by our choice of j, c^E in (11.67) has fewer such v's, so the term in (11.67) will never be canceled out. □

This gives some insight into the number and type of unstable operations. We can take a "basis" for $PBP^*\underline{BP}_*$ dual to either 9.20 or 9.21. For the element r corresponding to a basis element

$$[v^I] b^J, \tag{11.68}$$

the "lead" term will be $\pm r_E/_p l(I)$ for some E. The infinity of terms in the "tail" of r will be forced to be there in order to make $\hat{r}$ integral.

In practice it is a burden to handle the entire definition of unstable operations; a "local" condition does well enough. If we have a stable BP^*BP module M^*, then for any $x \in M^n$ we have a map

$$BP^{*-n}BP \longrightarrow M^* \tag{11.69}$$

which takes r to $r(x)$. If M^* has an unstable structure on it, then this map must factor through a BP^*BP module map

$$\begin{array}{ccc} BP^{*-n}BP & \longrightarrow & M^* \\ \big\downarrow & \nearrow & \\ PBP^*\underline{BP}_n & & \end{array} \tag{11.70}$$

This is just good old-fashioned algebraic topology.

The BP^*BP module structure of $PBP^*\underline{BP}_n$ is complicated. Basically there will be a bunch of generators $\iota_n, y_1, y_2, \ldots$, and a set of operations and relations

$$r_j\iota_n = \sum_i r_{ij}y_i. \tag{11.71}$$

Defining the map in (11.70) amounts to choosing values in M^* for the unstable operations

$$y_i \in PBP^*\underline{BP}_n \tag{11.72}$$

which are consistent with the relations (11.71). This is difficult to handle in practice, but it is always a straightforward calculation in primary operations.

We give a computational example. The result is well known and can be proven using standard unstable secondary operations. The hope, of course, is that the unstable primary BP operations can eventually be used to obtain results that would necessitate the use of standard higher order operations. To make our computations, we formalize the previous discussion in the equivalent form: Is M^n in the image of

$$\begin{aligned} M^* = \hom_{BP^*BP}(BP^*BP, M^*) &\longleftarrow \hom_{BP^*BP}(PBP^*\underline{BP}_n, M^*) \\ &\simeq \mathrm{Ext}^0_{BP^*BP}(PBP^*\underline{BP}_n, M^*)? \end{aligned} \tag{11.73}$$

We need to produce only one element x_n not in the image in order to show no unstable module structure exists. To show an unstable structure exists one must show that all of the local definitions patch together consistently.

For our example, consider the cofibration

$$RP^{15} \longrightarrow RP^{26} \longrightarrow RP^{26}_{16} = RP^{26}/RP^{15}. \tag{11.74}$$

No eleven-fold desuspension exists for RP^{26}_{16}. We prove this by showing there is no unstable module structure on

$$\Sigma^{-11}BP^*RP^{26}_{16} = M^*, \tag{11.75}$$

which is compatible with the given stable structure. We compute, for $p = 2$,

$$\hom_{BP^*BP}(PBP^*\underline{BP}_5, M^*). \tag{11.76}$$

and show that it is not onto M^5. It is enough to compute through internal degree 15 because $M^k = 0$, $k > 15$. Also $M^k = 0$, k even.

We describe a cofree BP_*BP resolution

$$0 \longrightarrow QBP_*\underline{BP}_5 \longrightarrow F_0 \longrightarrow F_1, \tag{11.77}$$

take the dual and apply hom to obtain

$$0 \longrightarrow \hom_{BP^*BP}(PBP^*\underline{BP}_5, M^*) \longrightarrow \hom_{BP^*BP}(F_0^*, M^*) \xrightarrow{r} \hom_{BP^*BP}(F_1^*, M^*). \tag{11.78}$$

To briefly describe how this goes (through deg 15),

$$F_0 \simeq \Sigma^5 BP_*BP \oplus \Sigma^{11} BP_*BP \oplus \Sigma^{15} BP_*BP \tag{11.79}$$

and

$$F_1 \simeq \Sigma^{11} BP_*BP \oplus \Sigma^{13} BP_*BP. \tag{11.80}$$

Now, (11.78) becomes

$$0 \longrightarrow \mathrm{Ker} \longrightarrow M^5 \oplus M^{11} \oplus M^{15} \xrightarrow{r} M^{11} \oplus M^{13} \tag{11.81}$$

where r is a matrix of stable operations. Interpreted as in (11.71), for every x_5 we must be able to choose an x_{11} and an x_{15} such that

$$0 = r_{11}x_5 + r_{12}x_{11} + r_{13}x_{15} \quad \text{and} \quad 0 = r_{21}x_5 + r_{22}x_{11} + r_{23}x_{15}. \tag{11.82}$$

This will not come out so clearly in the computation. What we want is to show that the composition of (11.81),

$$\begin{array}{c} 0 \longrightarrow \mathrm{Ker} \longrightarrow M^5 \oplus M^{11} \oplus M^{15} \\ \downarrow \\ M^5 \end{array} \tag{11.83}$$

is not surjective. We will never describe the matrix r in terms of the r_E, but we will keep our computations in homology, not cohomology.

From 9.21, or simple manipulations using 9.13 and 9.14, a basis for $QBP_*\underline{BP}_5$ is

(11.84)

deg	
5	$e_1 \circ b_1^{\circ 2}$
7	$e_1 \circ b_1 \circ b_2$
9	$e_1 \circ b_2^{\circ 2}$
11	$e_1 \circ b_1 \circ b_4,\ e_1 \circ [v_1] \circ b_1 \circ b_2^{\circ 2}$
13	$e_1 \circ b_2 \circ b_4,\ e_1 \circ [v_1] \circ b_2^{\circ 3}$
15	$e_1 \circ [v_1] \circ b_1 \circ b_2 \circ b_4,\ e_1 \circ [v_2] \circ b_1^{\circ 3} \circ b_2^{\circ 2}$.

The Boardman basis is preferred because it uses the smallest v's possible.

By homology suspension we consider this a subcomodule of $\Sigma^5 BP_* BP$. Recall that $[a]$ suspends to $\eta_R(a)$ and b_n suspends to b_{n-1}.

$$Q BP_* \underline{BP}_5 \subset \Sigma^5 BP_* BP$$

(11.85)

deg		
5	1	
7	b_1	
9	b_1^2	
11	b_3,	$b_1\eta_R(v_1)$
13	$b_1 b_3$,	$b_1^3\eta_R(v_1)$
15	$b_1 b_3 \eta_R(v_1)$,	$b_1^2\eta_R(v_2)$

From tables at the end of this section giving $\eta_R(v_1) = v_1 - 2b_1$ and $\eta_R(v_2) = v_2 + 5v_1^2 b_1 - 15 v_1 b_1^2 - 2b_3 + 14 b_1^3$ we can rewrite and slightly alter the basis to be

$$Q BP_* \underline{BP}_5 \subset \Sigma^5 BP_* BP$$

(11.86)

deg		
5	1	
7	b_1	
9	b_1^2	
11	b_3,	$2b_1^3$
13	$b_1 b_3$,	$v_1 b_1^3 - 2b_1^4$
15	$2b_1^2 b_3$,	$v_1 b_1^4 - 14 b_1^5$

To begin our resolution (11.77), (11.79), (11.80), in F_0 we give coefficients

	deg	$QBP_*\underline{BP}_5 \longrightarrow \Sigma^5 BP_*BP$	$\Sigma^{11}BP_*BP$	$\Sigma^{15}BP_*BP$
	5	$1 \longrightarrow 1_5$		
	7	$b_1 \longrightarrow b_1$		
	9	$b_1^2 \longrightarrow b_1^2$		
	11	$b_3 \longrightarrow b_3$		
(11.87)		$2b_1^3 \longrightarrow 2b_1^3$	1_{11}	
	13	$b_1b_3 \longrightarrow b_1b_3$	$3b_1$	
		$(v_1b_1^3 - 2b_1^4) \longrightarrow v_1b_1^3 - 2b_1^4$	$-3b_1$	
	15	$2b_1^2b_3 \longrightarrow 2b_1^2b_3$	$17b_1^2 - 2v_1b_1$	
		$(v_1b_1^4 - 14b_1^5) \longrightarrow (v_1b_1^4 - 14b_1^5)$	$-(15v_1b_1 + 36b_1^2)$	1_{15}

This requires some computations to verify as does

$$F_0 \xrightarrow{r} F_1$$

	deg		coefficients of 1_{11}	1_{13}
	11	$b_3 \longrightarrow$	0	
		$b_1^3 \longrightarrow$	1	
		$1_{11} \longrightarrow$	-2	
	13	$b_1^4 \longrightarrow$	$2v_1$	3
(11.88)		$b_1b_3 \longrightarrow$	$3v_1$	6
		$b_1 1_{11} \longrightarrow$	$-v_1$	-2
	15	$b_1^5 \longrightarrow$	$-10b_1^2 + 10v_1b_1$	$15b_1$
		$b_1^2b_3 \longrightarrow$	$-v_1^2 + 17v_1b_1 - 17b_1^2$	$-2v_1 + 34b_1$
		$b_1^2 1_{11} \longrightarrow$	$-2v_1b_1 + 2b_1^2$	$-4b_1$
		$1_{15} \longrightarrow$	$-17v_1^2 + 68v_1b_1 - 68b_1^2$	$-33v_1 + 66b_1$

Before we can compute (11.83) from (11.81) and (11.78), we must compute the groups M^* of (11.75).

We begin with the fibration

$$RP^\infty \longrightarrow CP^\infty \xrightarrow{2} CP^\infty. \tag{11.89}$$

The Atiyah-Hirzebruch spectral sequences all collapse and give an exact sequence, so

$$BP^*RP^\infty \simeq BP^*[[x]]/[2](x). \tag{11.90}$$

Furthermore, the spectral sequences collapse for everything in sight, giving surjections from the maps

$$RP^{2k} \longrightarrow RP^\infty, \quad RP^\infty_{2k} \longrightarrow CP^\infty_k, \quad RP^{2m}_{2k} \longrightarrow CP^n_k, \tag{11.91}$$

where $CP^n_k \simeq CP^n/CP^{k-1}$. For example,

$$BP^*RP^{2k} \simeq BP^*[[x]]/([2](x), x^{k+1}), \tag{11.92}$$

and

$$BP^*RP^{2n}_{2k} \tag{11.93}$$

has a $Z_{(2)}$ summand in degree $2k$ coming from $x^k \in BP^{2k}CP^\infty_{2k}$. In memory of CP^∞ we call the BP^* generators for (11.93), $x^k, x^{k+1}, \ldots, x^n$. Using the formula for $[2](x)$ at the end of §3 we can describe $BP^*RP^{26}_{16}$ explicitly from degree 16 on up.

(11.94) $BP^*RP^{26}_{16} = \Sigma^{11}M^*$

deg	groups	generators	relations
26	$Z/(2)$	x^{13}	$2x^{13} = 0$
24	$Z/(4)$	x^{12}	$2x^{12} = v_1x^{13}$
22	$Z/(8)$	x^{11}	$2x^{11} = v_1x^{12}$
20	$Z/(16), Z/(2)$	x^{10}, v_2x^{13}	$2x^{10} = -3v_1x^{11} + v_2x^{13}$
18	$Z/(32), Z/(4)$	x^9, v_2x^{12}	$2x^9 = -3v_1x^{10} - v_2x^{12}$
16	$Z_{(2)}, Z/(8)$	x^8, v_2x^{11}	

Returning to (11.81) we have

$$0 \longrightarrow \mathrm{Ker} \longrightarrow (\underset{y_1}{Z_{(2)}} \oplus \underset{y_2}{Z/(8)}) \oplus \underset{y_3}{Z/(8)} \oplus \underset{y_4}{Z/(2)} \xrightarrow{r=(r_1,r_2)} \underset{z_1}{Z/(8)} \oplus \underset{z_2}{Z/(4)}, \tag{11.95}$$

with new names for the generators. Since we obtain surjections from (11.91) we can compute our stable operations in $BP^*CP^{13}_8$. This is torsion free, so we can compute operations in here by duality in $BP_*CP^{13}_8$. These in turn can be computed using CP^∞. Our operation

r is given to us in terms of (11.88), and we must compute it on the y's in (11.95). For $r(x^n) = \Sigma a_i x^i$,

$$a_k = \langle rx^n, \beta_k \rangle = \epsilon \circ r_* \circ x_*^n(\beta_k), \tag{11.96}$$

which is convenient for us because we are given r in terms of r_*. The map x^n is just

$$CP^\infty \longrightarrow \bigwedge_n CP^\infty \longrightarrow \bigwedge_n \Sigma^2 BP \longrightarrow \Sigma^{2n} BP. \tag{11.97}$$

We will need the following computations which use the fact that $b_2 = -v_1 b_1 + 2b_1^2$ from the tables at the end of this section.

(11.98)

	modulo
$x_*^{13}\beta_{13} = 1$	(2)
$x_*^{11}\beta_{11} = 1$	(8)
$x_*^{11}\beta_{12} = 11b_1 = -b_1$	(4)
$x_*^{11}\beta_{13} = 11b_2 + \binom{11}{2}b_1^2 = v_1 b_1 + b_1^2$	(2)
$x_*^{8}\beta_{11} = 8b_3 + \binom{8}{3}b_1^3 + \binom{8}{1,1}b_1 b_2 = 0$	(8)
$x_*^{8}\beta_{12} = 2b_1^4$	(4)
$x_*^{8}\beta_{13} = 0$	(2)

We compute (11.95) from (11.96). We compute in $BP^*CP_8^{13}$ but because we reduce to $BP^*RP_{16}^{26}$, we can work modulo (2^i) as in the tables (11.98). Let

$$r_1(y_4) = a_0 x^{11} + a_1 x^{12} + a_2 x^{13}, \tag{11.99}$$

because $y_4 = x^{13}$, and we have $x_*^{13}\beta_{11} = x_*^{13}\beta_{12} = 0$, $r_1(y_4) = a_2 x^{13}$ and

$$a_2 = \langle r_1(y_4), \beta_{13} \rangle = \epsilon \circ r_{1*} x_*^{13}\beta_{13} = \epsilon \circ r_{1*}(1_{15}) = \tag{11.100}$$

from (11.88)

$$\epsilon(-17v_1^2 + 68v_1 b_1 - 68b_1^2) = -17v_1^2 \tag{11.101}$$

because $\epsilon(b) = 1$. Since $2x^{13} = 0$,

$$r_1(y_4) = v_1^2 x^{13} = 4x^{11} = 4z_1. \tag{11.102}$$

Computing $r_2(y_4) = 2z_2$ is about the same. So

$$r(y_4) = 4z_1 + 2z_2. \tag{11.103}$$

Note that for our purposes we need only the table of $\epsilon \circ r_*$ from (11.88), as

(11.104)

deg	$\epsilon \circ r_*$	coefficients of	mod
	$F_0 \longrightarrow 1_{11}$	1_{13}	
	$b_3 \longrightarrow 0$		(8)
11	$b_1^3 \longrightarrow 1$		
	$1_{11} \longrightarrow -2$		
13	$b_1^4 \longrightarrow 2v_1$	-1	(4)
	$b_1 b_3 \longrightarrow -v_1$	2	
	$b_1 1_{11} \longrightarrow -v_1$	2	
15	$b_1^5 \longrightarrow 0$	0	(2)
	$b_1^2 b_3 \longrightarrow v_1^2$	0	
	$b_1^2 1_{11} \longrightarrow 0$	0	
	$1_{15} \longrightarrow v_1^2$	v_1	

A similar calculation to that for $r(y_4)$ also gives

(11.105) $$r(y_3) = 4z_1 + 2z_2.$$

Computing $r(y_2)$ is slightly different because $y_2 = v_2 x^{11}$. We have

(11.106) $$\langle r(y_2), \beta_k \rangle = \epsilon \circ r_*(v_2 x^{11})_*(\beta_k) = \epsilon \circ r_* v_{2*} x_*^{11}(\beta_k)$$

where v_{2*} is just multiplication by $\eta_R(v_2)$. Recall that in (11.104) there is an absent 1_5, b_1, and b_1^2 going to zero. We compute

(11.107) $$r(y_2) = 0.$$

Computing $r(y_1)$ is fairly easy. We represent it as

(11.108) $$r(y_1) = a_0 x^8 + a_1 x^9 + a_2 x^{10} + a_3 x^{11} + a_4 x^{12} + a_5 x^{13}.$$

Then

(11.109) $$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} = \epsilon \circ r_* x_*^8 \begin{pmatrix} \beta_8 \\ \beta_9 \\ \beta_{10} \\ \beta_{11} \\ \beta_{12} \\ \beta_{13} \end{pmatrix} = \epsilon \circ r_* \begin{pmatrix} ? \\ ? \\ ? \\ 0 \\ 2b_1^4 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix}$$

and we have

$$r(y_1) = 2z_2. \tag{11.110}$$

We are almost done. We want to show the lack of surjectivity of (11.83), and it is now clear, from (11.103), (11.105), (11.107) and (11.110), put into (11.95), that y_1 cannot be in the image in (11.83).

We conclude this section with some formulas we have found useful. We use the formulas and tables found earlier in the paper, and

THEOREM 11.111. *Let* $[p](x) = \Sigma_{i\geq 0} a_i x^{i+1}$, *with* $b = \Sigma_{i\geq 0} b_i = c(\Sigma^{F}_{i\geq 0} t_i)$, *and* $a = \Sigma_{i\geq 0} a_i$, $a_i \in BP_{2i}$. *Then, with* $x(y) = \Sigma_{i\geq 0} x_i y^{i+1}$, $b(a) = \eta_R a(b)$, *or*

$$\sum_{i\geq 0} b_i a^{i+1} = \sum_{i\geq 0} \eta_R(a_i) b^{i+1}. \quad \square$$

PROOF. Just stabilize 9.12 for *BP*. □

This allows us to compute the right unit of v's in terms of b's and give the nongenerating b's in terms of the generating b's.

$p = 2$, $[2](x) = \Sigma_{i\geq 0} a_i x^{i+1}$. In BP_*BP

$$\begin{aligned}
\eta_R(a_0) &= 2,\\
\eta_R(a_1) &= -v_1 + 2b_1,\\
\eta_R(a_2) &= 2v_1^2 - 8v_1 b_1 + 8b_1^2,\\
\eta_R(a_3) &= -7v_2 - 8v_1^3 + 13v_1^2 b_1 + 9v_1 b_1^2 + 14b_3 - 34b_1^3,\\
\eta_R(a_4) &= 30v_1 v_2 + 26v_1^4 - 60v_2 b_1 - 58v_1^3 b_1 - 126v_1^2 b_1^2\\
&\quad - 60v_1 b_3 + 488v_1 b_1^3 - 424b_1^4 + 120b_1 b_3,\\
\eta_R(a_5) &= -111v_2 v_1^2 - 84v_1^5 + 444v_2 v_1 b_1 + 285v_1^4 b_1 - 444v_2 b_1^2 + 525v_1^3 b_1^2\\
&\quad + 222v_1^2 b_3 - 3714v_1^2 b_1^3 - 888v_1 b_1 b_3 + 6156v_1 b_1^4 + 888b_1^2 b_3 - 3528b_1^5,\\
\eta_R(a_6) &= 112v_2^2 + 502v_2 v_1^3 + 300v_1^6 - 1892v_1^2 v_2 b_1 - 1090v_1^5 b_1 + 2664v_1 v_2 b_1^2\\
&\quad - 1790v_1^4 b_1^2 - 448v_2 b_3 - 1004v_1^3 b_3 - 880v_2 b_1^3 + 17528v_1^3 b_1^3 + 3784v_1^2 b_3 b_1\\
&\quad - 39728v_1^2 b_1^4 - 5328v_1 b_1^2 b_3 + 39936v_1 b_1^5 + 448b_3^2 + 1760b_3 b_1^3 - 15072b_1^6.
\end{aligned} \tag{11.112}$$

In BP_*BP, $p = 2$,

$$
\begin{aligned}
\eta_R(2) &= 2,\\
\eta_R(v_1) &= v_1 - 2b_1,\\
b_2 &= -v_1b_1 + 2b_1^2,\\
\eta_R(v_2) &= v_2 + 5v_1^2b_1 - 15v_1b_1^2 - 2b_3 + 14b_1^3,\\
b_4 &= -v_1^3b_1 - 2v_2b_1 - 3v_1^2b_1^2 + 14v_1b_1^3 - v_1b_3 - 16b_1^4 + 6b_1b_3,\\
b_5 &= 4v_2v_1b_1 + 2v_1^4b_1 - 11v_2b_1^2 + v_1^3b_1^2 + v_1^2b_3\\
&\quad - 49v_1^2b_1^3 - 13v_1b_1b_3 + 121v_1b_1^4 + 28b_1^2b_3 - 98b_1^5,\\
b_6 &= -4v_1^2v_2b_1 - 2v_1^5b_1 + 27v_1v_2b_1^2 + 7v_1^4b_1^2 - 2v_2b_3\\
&\quad - 2v_1^3b_3 - 40v_2b_1^3 + 57v_1^3b_1^3 + 13v_1^2b_3b_1 - 321v_1^2b_1^4\\
&\quad - 60v_1b_1^2b_3 + 570v_1b_1^5 + 4b_3^2 + 80b_1^3b_3 - 368b_1^6.
\end{aligned}
\tag{11.113}
$$

$p = 2$, $b = c(\Sigma^F_{i\geq 0} t_i) = \Sigma_E a^E t^E$, $a^E = \Sigma_{i\geq 0} a_i^E$.

$$
\begin{aligned}
a^0 &= 1,\\
a^1 &= -1 + v_1 - v_1^2 + (2v_2 + 2v_1^3) - (3v_1^4 + 4v_1v_2) + (6v_1^2v_2 + 4v_1^5) + \cdots,\\
a^2 &= 2 - 5v_1 + 8v_1^2 - (17v_1^3 + 11v_2) + (37v_1v_2 + 34v_1^4) + \cdots,\\
a^3 &= -5 + 21v_1 - 49v_1^2 + (50v_2 + 118v_1^3) + \cdots,\\
a^{01} &= -1 + v_1 - v_1^2 + (2v_2 + 2v_1^3) + \cdots,\\
a^4 &= 14 - 84v_1 + 264v_1^2 + \cdots,\\
a^{11} &= 6 - 13v_1 + 21v_1^2 + \cdots,\\
a^5 &= -42 + 330v_1 + \cdots,\\
a^{21} &= -28 + 100v_1 + \cdots,\\
a^6 &= 132 + \cdots,\\
a^{31} &= 120 + \cdots,\\
a^{02} &= 4 + \cdots.
\end{aligned}
\tag{11.114}
$$

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