Calculating descent for 2-primary topological modular forms

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Abstract. We give an explicit presentation of the moduli stack of elliptic curves with full level three structures and a choice of a Weil pairing $\mathcal{M}(3)^{\zeta}$, together with a description of the action of $SL_2(\mathbb{Z}/3)$ (due to Charles Rezk). Then we proceed to compute the cohomology ring $H^*(SL_2(\mathbb{Z}/3), H^*(\mathcal{M}(3)^{\zeta}))$ with its full multiplicative structure including Massey products using several spectral sequences. The result is the E_2 page of a homotopy fixed point spectral sequence computing the homotopy groups of $Tmf[1/3]$.

1. Introduction

Topological modular forms exhibit some fascinating properties; for example, in [**Sto12**], the author showed that Tmf[1] is Anderson self dual (up to a shift), after inverting 2. The reason for concentrating on the 3-primary torsion in that paper was to avoid the technical difficulties that appear in the 2-primary calculations. The goal of the present work is precisely to address those algebraic technicalities.

1.1. Acknowledgements. Charles Rezk has generously shared his notes on the presentation described in Section 4, without which the present work would not have been possible. Most of the Massey product calculations are completely analogous to Tilman Bauer's in [**Bau08**]; Tilman also gets the credits for creating the spectral sequence latex packages (sseq and luasseq) that I used to create the figures for this paper.

2. Recalling elliptic curves

A curve over a base S is a map of schemes $p : E \to S$ which is flat, proper, has finite presentation and dimension one. An elliptic curve is a diagram

$$p : E \rightleftarrows S : e \tag{2.1}$$

where $p : E \to S$ is a curve of genus one whose geometric fibers are non-empty, connected, and smooth, and $e : S \to E$ is its section. An elliptic curve has a unique structure of an abelian group scheme with e as its identity [**KM85**, 2.1]. The object which classifies elliptic curves and isomorphisms between them is the moduli stack of elliptic curves, denoted[2] $\mathcal{M}^0$. Its compactification $\mathcal{M}$ classifies diagrams (2.1)

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[1] the non-connective and non-periodic version of topological modular forms

[2] as in [**DR73**].

where $p : E \to S$ is a curve of genus one whose geometric fibers can be smooth or have an isolated nodal singularity away from e [**DR73**, II.1.12].

Associated to an elliptic curve $p : E \to S$ is the sheaf $\omega_{E/S}$ of (translation-) invariant differentials, which can be described as the push-forward $p_*\Omega_{E/S}$ of the sheaf of relative differentials on E. Let $\mathcal{I}$ denote the ideal sheaf defining the identity section e; then by [**Har77**, II.8]

$$\omega_{E/S} = p_*\Omega_{E/S} = \mathcal{I}/\mathcal{I}^2. \tag{2.2}$$

The sheaf $\omega_{E/S}$ is invertible; if E is generalized (i.e. not necessarily smooth), $\omega_{E/S}$ can be defined as $\mathcal{I}/\mathcal{I}^2$, giving again an invertible line bundle. Consequently, the assignment

$$E/S \mapsto \omega_{E/S}$$

defines an invertible quasi-coherent sheaf on $\mathcal{M}$, which we denote by ω. The ring of (holomorphic) modular forms MF_* is defined to be the graded ring

$$H^0(\mathcal{M}, \omega^*) = \bigoplus_{n\geq 0} H^0(\mathcal{M}, \omega^{\otimes n})$$

Locally, a choice of an $\mathcal{O}_S$-basis for $\omega_{E/S}$ gives rise to a Weierstrass equation for E as follows. Let $U = \operatorname{Spec} R$ be an open subset of S on which $\omega_{E/S}$ is trivializable, with η as a generator. Note that η is unique up to multiplication by a unit $u \in R^\times$. For any $n \in \mathbb{Z}$, η^n generates $\mathcal{I}^n/\mathcal{I}^{n+1}$, and for $n > 0$, the sheaf $p_*\mathcal{I}^{-n}$ is locally free of rank n [**KM85**, 2.2.5].

The natural inclusion $\mathcal{O}_C \cong \mathcal{I}^0 \to \mathcal{I}^{-n}$ defines a generator 1 of $\mathcal{I}^{-n}$ for any $n \geq 0$. Let x be a generator of $\mathcal{I}^{-2}$ which reduces to η^{-2} in $\mathcal{I}^{-2}/\mathcal{I}^{-1}$, and let y be a generator of $\mathcal{I}^{-3}$ which reduces to η^{-3} in $\mathcal{I}^{-3}/\mathcal{I}^{-2}$. Then $\{1, x\}$ is a basis for $\mathcal{I}^{-2}$, and $\{1, x, y\}$ is a basis for $\mathcal{I}^{-3}$. Note x and y are uniquely determined up to a change of variables

$$\begin{aligned} x &\mapsto u^{-2}x + r, \\ y &\mapsto u^{-3}y + u^{-2}sx + t, \end{aligned} \tag{2.3}$$

where u is a unit, and r, s, t are arbitrary elements of R. Continuing in this fashion, we find that $p_*\mathcal{I}^{-4}$ is freely generated by $1, x, y, x^2$, and $p_*\mathcal{I}^{-5}$ by $1, x, y, x^2, xy$. Next, $p_*\mathcal{I}^{-6}$ is freely generated on either $1, x, y, x^2, xy, y^2$ or $1, x, y, x^2, xy, x^3$, where $y^2 - x^3$ is in fact an element of $\mathcal{I}^{-5}$, as x^3 and y^2 both reduce to η^{-6}. Therefore, a relation called a Weierstrass equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \tag{2.4}$$

must hold, for some $a_i \in R$. In fact, the map

$$\phi = [x, y, 1] : E_U \to \mathbb{P}^2_U$$

identifies E_U with the locus of vanishing of (2.4), with the identity section e mapping to the point at infinity $[0 : 1 : 0]$ in $\mathbb{P}^2_U$][**Sil86**, III.3],[**KM85**, 2.2.5]. The differential form η is expressed as

$$\eta = \frac{dx}{2y + a_1x + a_3} = \frac{dy}{3x^2 + 2a_2x + a_4 - a_1y}. \tag{2.5}$$

Conversely, any curve given by an equation (2.4) which is smooth or has at most a nodal singularity is a generalized elliptic curve. Consequently, the moduli stack of generalized elliptic curves $\mathcal{M}$ is represented by the Hopf algebroid (A, Γ), where $A = \mathbb{Z}[a_1, a_2, a_3, a_4, a_6]$ and $\Gamma = A[u^{\pm 1}, r, s, t]$, while the structure map $\psi : A \to \Gamma$

is deduced from the change of variables (2.3). The formulas can be found in [**Sil86**, Table 1.2] or [**Bau08**, Section 3].

This presentation eases the computation of the ring of modular forms; one has [**Bau08**]

$$MF_* = \mathbb{Z}[c_4, c_6, \Delta]/(12^3\Delta = c_4^3 - c_6^2),$$

where c_n is a global section of $\omega^{\otimes n}$, and Δ is the discriminant of the equation (2.4). The global sections of $\omega^{\otimes 12}$ c_4^3 and Δ do not simultaneously vanish (as that would indicate a cusp singularity), and they define a map

$$j = [c_4^3 : \Delta] : \mathcal{M} \to \mathbb{P}^1 \tag{2.6}$$

called the j-invariant (its target usually called the projective j-line), which classifies the line bundle $\omega^{\otimes 12}$ and which restricts to $j : \mathcal{M}^0 \to \mathbb{A}^1 \overset{i}{\hookrightarrow} \mathbb{P}^1$, where i includes the complement of the point $[1 : 0]$ in $\mathbb{P}^1$. The j invariant can be used to describe the compactification $\mathcal{M}$ of $\mathcal{M}^0$ as the normalization in the field of functions of $\mathcal{M}^0$ of the projective j-line [**DR73**].

The embedding $\phi : E_U \to \mathbb{P}^2_U$ also gives a geometric way to describe the group law on E_U by interpreting Abel's theorem [**KM85**, 2.1.2] as explained in [**Sil86**, III.2]. A line in $\mathbb{P}^2_U$ intersects E_U at exactly three points (counted with multipilicites) since the defining equation (2.4) has degree three. Then the sum $P + Q + R$ of three (not necessarily distinct) points of E is the identity if and only if they are collinear in $\mathbb{P}^2_U$.

3. Recalling level structures

Let n be a positive integer and let S be a scheme over $\mathbb{Z}[1/n]$; then multiplication by n on a smooth elliptic curve E/S is a finite map of degree n^2 whose kernel $E[n]$ is étale locally isomorphic to $(\mathbb{Z}/n)^2$. Specification of this isomorphism is called a (full) level n structure on E. For $i = 1, 2$, let $\big(E_i, \varphi_i : (\mathbb{Z}/n)^2 \to E_i\big)$ be two elliptic curves (both over S) with level n structures; an isomorphism

$$f : (E_1, \varphi_1) \to (E_2, \varphi_2)$$

is a commutative diagram

$$\begin{array}{ccc} (\mathbb{Z}/n)^2 & \xrightarrow{\varphi_1} & E_1 \\ \Big\| & & \big\downarrow \psi \\ (\mathbb{Z}/n)^2 & \xrightarrow{\varphi_2} & E_2 \end{array}$$

where ψ is an isomorphism of elliptic curves. We denote by $\mathcal{M}(n)^0$ the moduli stack classifying elliptic curves with level n structure and isomorphisms between them. In fact, $\mathcal{M}(n)^0$ is a scheme whenever $n \geq 3$ [**DR73**, IV.2.7]. Forgetting the level structure gives a covering map $f : \mathcal{M}(n)^0 \to \mathcal{M}^0[1/n]$, hence also a j-invariant $j : \mathcal{M}(n)^0 \to \mathbb{A}^1[1/n]$ by composition.

The finite group $E[n]$ of n-torsion points in E is equipped with a non-degenerate alternating form

$$e_n : E[n] \times E[n] \to \mu_n,$$

called the Weil pairing [**KM85**, 2.8] into the group of n-th roots of unity. While at first we have $\mathcal{M}(n)^0$ as a scheme over $\mathbb{Z}[\frac{1}{n}]$, the Weil pairing gives a map $\mathcal{M}(n)^0 \to \operatorname{Spec} \mathbb{Z}[\zeta_n, \frac{1}{n}]$ by sending (E, φ) to $e_n(\varphi(1, 0), \varphi(0, 1)) = \zeta_n$.

Now the compactification $\mathcal{M}(n)$ of $\mathcal{M}(n)^0$ can be described as the normalization in the field of functions of $\mathcal{M}(n)^0$ of the projective j-line over $\mathbb{Z}[\zeta_n]$ [**DR73**]. Deligne-Rapoport [**DR73**] develop insightful and important modular description of these compactifications using so-called Néron polygons, but for the purposes of this work we will stick to the approach via normalization.

The automorphism group $GL_2(\mathbb{Z}/n)$ of $(\mathbb{Z}/n)^2$ acts on the right on $\mathcal{M}(n)^0$ by precomposition; namely, $g \in GL_2(\mathbb{Z}/n)$ maps (E, φ) to $(E, \varphi \circ g)$. This action is in fact free and transitive, making the forgetful map $f : \mathcal{M}(n)^0 \to \mathcal{M}^0[1/n]$ a torsor for the group $GL_2(\mathbb{Z}/n)$. Moreover, the action extends over $\mathcal{M}(n)$, but the stabilizers of the cusps are non-trivial; they are conjugates of the subgroup

$$\pm U := \left\{ \pm \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\} \subset GL_2(\mathbb{Z}/n),$$

as described in [**DR73**, IV.5].

Since $\mathcal{M}(n)$ is a stack (scheme when $n > 2$) over $\mathbb{Z}[\zeta_n, \frac{1}{n}]$, after a finite étale extension, it splits as a disjoint union of stacks $\mathcal{M}(n)^\zeta$ indexed by the primitive n-th roots of unity. To be more precise, let k be a ring in which n is invertible and which contains a primitive n-th root of unity; then $k[\zeta_n] := k[x]/(x^2 + x + 1)$ splits as a product $k \times k$, and therefore any scheme or stack X over $k[\zeta_n]$ splits as a disjoint union indexed over $\mu_n^\times$. In particular, this happens for $\mathcal{M}(n)$ and we have a diagram

$$\begin{array}{ccccccc} \mathcal{M}(n)^\zeta & \longrightarrow & \coprod_{\mu_n^\times} \mathcal{M}(n)^\zeta & \longrightarrow & \mathcal{M}(n) & \longrightarrow & \mathcal{M} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec} k & \longrightarrow & \operatorname{Spec} k[\zeta_n] & \longrightarrow & \operatorname{Spec} \mathbb{Z}[\zeta_n] & \longrightarrow & \operatorname{Spec} \mathbb{Z}. \end{array}$$

The moduli stack $\mathcal{M}(n)^\zeta$ has action by the subgroup $SL_2(\mathbb{Z}/3)$, and in fact the composed map $\mathcal{M}(n)^\zeta \to \mathcal{M}$ is an $SL_2(\mathbb{Z}/3)$-torsor away from the cusps which have $\pm U \subset SL_2(\mathbb{Z}/n)$ as stabilizers.

4. Level-3-structures made explicit

In this section we describe an explicit presentation of the moduli stack $\mathcal{M}(3)$, due to Charles Rezk. From this point on, 3 will be assumed to be invertible everywhere.

Let E/S be a generalized elliptic curve (over a scheme on which 3 is invertible); by completing the cube in the Weierstrass equation (2.4), we get that locally E is isomorphic to a Weierstrass curve of the form

$$y^2 + a_1xy + a_3y = x^3 + a_4x + a_6, \tag{4.1}$$

with discriminant $\Delta = (a_1^3 - 27a_3)a_3^3$. The points of order three are the inflection points of E. Choose $P = (r, t)$ to be such a point; applying the transformation $(x, y) \mapsto (x + r, y + t)$ puts E in the form

$$y^2 + a_1xy + a_3y = x^3, \tag{4.2}$$

where now P has coordinates $(0, 0)$. The inversion map $[-1] : E \to E$ is given by $[-1](x, y) = (x, y - a_1x - a_3)$. Thus $[-1]P = (0, -a_3)$, and the tangent line to $[-1]P$ is $y = -a_1x - a_3$.

4.1. The nonsingular case. If the curve E is smooth, choose $Q = (e_2, e_3)$ to be another point of order three which is different from $\pm P$. There are exactly three points on E with x-coordinate equal to zero ($\pm P$ and the point at infinity), hence $e_2 = x(Q)$ is invertible. From (4.2) it follows then that $e_3 = y(Q)$ is also invertible. If $y = b_1 x + b_3$ is the tangent line to E at Q, we have (as Q is an inflection point)

$$x^3 - (x - e_2)^3 = (b_1 x + b_3)^2 + a_1 x(b_1 x + b_3) + a_3(b_1 x + b_3),$$

which yields

$$\begin{aligned} 3e_2 &= b_1^2 + a_1 b_1 \\ -3e_2^2 &= 2b_1 b_3 + a_1 b_3 + b_1 a_3 \\ e_2^3 &= b_3^2 + a_3 b_3, \end{aligned}$$

whence b_1, b_3, as well as $e_3 - b_3 = b_1 e_2$ must be invertible. In particular, the quotient e_3/b_3 cannot be 1. However,

$$\begin{aligned} \frac{e_3^3}{b_3^3} &= \frac{(b_1 e_2 + b_3^3)}{b_3^3} \\ &= \frac{(b_3^2 + a_3 b_3)b_1^3 - (2b_1 b_3 + a_1 b_3 + b_1 a_3)b_1^2 b_3 + (b_1^2 + a_1 b_1)b_1 b_3^2 + b_3^3}{b_3^3} = 1, \end{aligned}$$

hence $\frac{e_3}{b_3}$ must be a primitive third root of 1. Set $\zeta = e_3/b_3$, and denote $\gamma_1 = b_1$ and $\gamma_2 = a_1 + b_1$. We have the following formulas.

$$\begin{aligned} a_1 &= \gamma_2 - \gamma_1 \\ e_2 &= \frac{1}{3}\gamma_1\gamma_2 \\ b_3 &= -\frac{1}{9}(1 - \zeta^2)\gamma_1^2\gamma_2 \\ e_3 &= \frac{1}{9}(1 - \zeta)\gamma_1^2\gamma_2 \\ a_3 &= \frac{1}{9}(\zeta - 1)\gamma_1\gamma_2(\gamma_1 + \zeta^2\gamma_2) \\ a_1^3 - 27a_3 &= (\gamma_2 - \zeta\gamma_1)^3. \end{aligned}$$

4.2. A presentation. Let $\Gamma = \mathbb{Z}[1/3, \zeta][\gamma_1, \gamma_2]$ be the graded ring with γ_i in degree 1. The above discussion shows that the locus $\mathcal{M}^0(3)^\zeta$ of smooth curves in $\mathcal{M}(n)^\zeta$ is

$$\left(\operatorname{Spec}\Gamma[\Delta^{-1}]\right)//\mathbb{G}_m.$$

Consequently, the compactification $\mathcal{M}(3)^\zeta$ must be $\operatorname{Proj}\Gamma$.

4.3. The action of $GL_2(\mathbb{Z}/3)$. Fix an elliptic curve E and its Weierstrass equation adapted to the level structure (P, Q) as above, and think of a_1, b_1, ζ as functions of the level structure (P, Q). To determine the action of $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL_2(\mathbb{Z}/3)$ on $\mathcal{M}(3)$, we need to determine the Weierstrass equation associated to E with the level structure $(P, Q)A = (\alpha P + \gamma Q, \beta P + \delta Q)$ (giving $a_1((P, Q)A)$), the slope $b_1((P, Q)A)$ of the tangent line at $\beta P + \delta Q$, as well as $\zeta = \frac{y(\beta P + \delta Q)}{b_3((P,Q)A)}$.

Note that a_1 is in fact only a function of the first point of order three. We already saw that when P is at $(0,0)$, then $-P$ has coordinates $(0,-a_3)$ and a tangent line $y=-a_1x-a_3$. The transformation

$$\begin{aligned} x &\mapsto x \\ y &\mapsto y - a_1x - a_3 \end{aligned}$$

moves $-P$ to $(0,0)$, putting E in the form

$$y^2 - a_1xy - a_3 = x^3.$$

In particular $a_1(-P) = -a_1$.

Similarly, the transformation

$$\begin{aligned} x &\mapsto x + e_2 \\ y &\mapsto y + b_1x + e_3 \end{aligned}$$

moves Q to $(0,0)$ and gives that $a_1(Q) = a_1 + 2b_1$. This transformation also moves P to $(-e_2,-e_3)$, giving that

$$\begin{aligned} b_1(Q,P) &= -b_1, & b_3(Q,P) &= -e_3, \\ e_3(Q,P) &= -b_3, & \zeta(Q,P) &= \zeta^{-1}. \end{aligned}$$

The line through P and Q is $y = \frac{e_3}{e_2}x$, the other point of E which lies on this line is

$$R = \left(-\frac{a_3e_3}{e_2^2}, -\frac{a_3e_3^2}{e_2^3}\right),$$

and $R = -P-Q$. The tangent line at R is given by

$$y = -\zeta b_1 x - \frac{b_1^2}{9}((\zeta^2-1)a_1 + (\zeta-1)b_1).$$

Consequently,

$$\begin{aligned} b_1(Q,-P-Q) &= \zeta^2 b_1, & b_3(Q,-P-Q) &= -\frac{b_1^2}{9}((\zeta-1)a_1 + 2\zeta b_1), \\ e_3(Q,-P-Q) &= \frac{b_1^2}{9}((2\zeta+1)a_1 - 3\zeta^2 b_1), & \zeta(Q,-P-Q) &= \zeta. \end{aligned}$$

Putting all of the above together, we deduce the following result.

PROPOSITION 4.1. *The (left) action of $GL_2(\mathbb{Z}/3)$ on Γ is the ring action determined by*

$$\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix} : \begin{array}{l}\gamma_1 \mapsto -\gamma_1,\\ \gamma_2 \mapsto -\gamma_2,\end{array} \qquad \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} : \begin{array}{l}\gamma_1 \mapsto -\gamma_2,\\ \gamma_2 \mapsto \gamma_1,\end{array}$$

$$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} : \begin{array}{l}\gamma_1 \mapsto -\gamma_1,\\ \gamma_2 \mapsto \gamma_2,\end{array} \qquad \begin{pmatrix}0 & -1\\ 1 & -1\end{pmatrix} : \begin{array}{l}\gamma_1 \mapsto \zeta^2\gamma_1,\\ \gamma_2 \mapsto \gamma_2 - \zeta\gamma_1.\end{array}$$

The elements of $SL_2(\mathbb{Z}/3)$ preserve ζ, while the rest map ζ to $\zeta^{-1} = \zeta^2$.

Let

$$x = \left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right), \quad y = \left(\begin{smallmatrix}-1 & -1\\ -1 & 1\end{smallmatrix}\right), \quad z = \left(\begin{smallmatrix}0 & -1\\ 1 & -1\end{smallmatrix}\right)$$

be a choice of generators for $SL_2(\mathbb{Z}/3)$. Then Proposition 4.1 implies that $SL_2(\mathbb{Z}/3)$ acts on $\mathcal{M}(3)^\zeta = \operatorname{Proj}\Gamma$ by the map $\chi : SL_2(\mathbb{Z}/3) \to PGL_2(\mathbb{Z}[1/3,\zeta]) = \operatorname{Aut}(\operatorname{Proj}\Gamma)$ given by

$$x \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} -\zeta^2 & \zeta \\ \zeta & \zeta^2 \end{pmatrix}, \quad z \mapsto \begin{pmatrix} \zeta^2 & 0 \\ -\zeta & 1 \end{pmatrix}. \tag{4.3}$$

5. Serre Duality on $\mathcal{M}(3)^\zeta$

Since $\mathcal{M}(3)^\zeta = \operatorname{Proj}\Gamma$ is a projective line, it has Serre duality, and its dualizing sheaf is the invertible sheaf of differentials $\Omega = \Omega_{\mathcal{M}(3)^\zeta}$. Note that line bundles on $\mathcal{M}(3)^\zeta$ are in bijection with shifts of Γ as a module over itself; namely, for any integer k, $\mathcal{O}(k)$ denotes the line bundle corresponding to the graded module $\Gamma[k]$ which in degree t is the $(t+k)$-graded part of Γ. We have $\mathcal{O}(k)\otimes\mathcal{O}(n) = \mathcal{O}(k+n)$.

Now the differential form $\gamma_1 d\gamma_2 - \gamma_2 d\gamma_1$ is a nowhere vanishing differential form of degree two, hence it is a trivializing global section of $\mathcal{O}(2)\otimes\Omega$. We conclude that $\Omega \cong \mathcal{O}(-2)$.

On the other hand, the sheaf ω is a line bundle locally generated by the invariant differential $\eta = \frac{dx}{2y+a_1x+a_3}$ which is of degree 1, so $\omega \cong \mathcal{O}(1)$. Consequently, $\Omega \cong \omega^{-2}$.

The cohomology $H^*(\mathcal{M}(3)^\zeta, \Omega)$ is zero in degrees other than 1, and is $\mathbb{Z}[1/3,\zeta]$ in degree 1. The group $SL_2(\mathbb{Z}/3)$ acts on $H^1(\mathcal{M}(3)^\zeta,\Omega) =: \mathbb{Z}_\zeta$ via the determinant of the image of χ in $PGL_2(\mathbb{Z}[1/3,\zeta])$ (4.3). Hence x and y act trivially, and z acts as multiplication by ζ^2.

There is an $SL_2(\mathbb{Z}/3)$-equivariant Serre duality pairing [**Har77**, III.7.1]

$$H^0(\mathcal{M}(3)^\zeta, \omega^*) \otimes H^1(\mathcal{M}(3)^\zeta, \omega^{-*-2}) \to \mathbb{Z}_\zeta,$$

hence $H^1(\mathcal{M}(3)^\zeta, \omega^{-*-2}) \cong \operatorname{Hom}(\Gamma, \mathbb{Z}_\zeta)[-2] =: \Gamma_\zeta^\vee[-2]$. We will now proceed to compute the cohomology

$$H^*(SL_2(\mathbb{Z}/3), H^*(\mathcal{M}(3)^\zeta, \omega^*)) = H^*(SL_2(\mathbb{Z}/3), \Gamma \oplus \Gamma_\zeta^\vee[-2]).$$

6. Quaternion group cohomology

Having determined the action of $GL_2(\mathbb{Z}/3)$ on Γ, we will proceed to compute the cohomology ring $H^*(GL_2(\mathbb{Z}/3),\Gamma)$. We will make repeated use of Lyndon-Hochshield-Serre (LHSSS) and Bockstein spectral sequences (BSS), and we will keep track of Massey products, which in particular will be useful in identifying hidden extensions in the E_∞-pages of the various spectral sequences.

The quaternion group Q_8, which has a presentation

$$\langle x, y | xyx = y, x^2 = y^2, x^4 = 1\rangle$$

is a subgroup of $GL_2(\mathbb{Z}/3)$, and a Sylow 2-subgroup of $SL_2(\mathbb{Z}/3)$. As a preliminary calculation, we will determine $H^*(Q_8,\mathbb{F}_4)$, where Q_8 acts trivially on $\mathbb{F}_4$, and consequently the cohomology of Q_8 with trivial coefficients for any extension of $\mathbb{F}_4$.

The Lyndon-Hochschield-Serre spectral sequence for

$$1 \to C_2 \times C_2 \to Q_8 \to C_2 \to 1$$

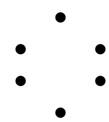
looks as

$$\mathbb{F}_4[a,b,c] = H^*(C_2, H^*(C_2\times C_2, \mathbb{F}_4)) \Rightarrow H^*(Q_8,\mathbb{F}_4),$$

where $a, b \in H^1(C_2 \times C_2, \mathbb{F}_4)$. Then $d_2(c) = a^2 + ab + b^2$, and $d_3(c^2) = a^2b + ab^2$. Consequently,

$$H^*(Q_8, \mathbb{F}_4) = \mathbb{F}_4[P][a,b]/(a^2 + ab + b^2, a^3, b^3, a^2b + ab^2),$$

where P is the class of c^4 [**?**, IV.2.10]. We can visualize this as the pattern



tensored with $\mathbb{F}_4[P]$, where each dot represents an F_4-generator, and they are arranged so that the cohomological degree is mapped along the vertical axis (not displayed).

The periodicity element P can be written as a Massey product

$$P = \langle a + b, ab, a + b, ab \rangle,$$

as $d_3(c^2) = a^2b + ab^2$, and P is the class of c^4. This is not a Massey product that we will use since it involves elements $(ab, a+b)$ which are not invariant under larger subgroups of $GL_2(\mathbb{Z}/3)$. However, a crucial Massey product is closely related to this one, and we describe it now.

Let Q_8 act trivially on $\bar{\gamma}$ in the polynomial ring $\Gamma_1 := \mathbb{F}_4[\bar{\gamma}]$. (We will see shortly how Γ_1 appears in some Bockstein spectral sequences.) Then

$$H^*(Q_8, \Gamma_1) = H^*(Q_8, \mathbb{F}_4) \otimes \Gamma_1.$$

Consider the elements

$$h_1 = (a + \zeta^2 b)\bar{\gamma}, h_2 = (a + \zeta b)\bar{\gamma}^2 \in H^1(Q_8, \Gamma_1)$$

(we will justify their appearance in 7 below). In $H^*(Q_8, \Gamma_1)$ we have a Massey product

$$P\bar{\gamma}^{12} = \left\langle \begin{pmatrix} h_2^2 & h_1 \end{pmatrix}, \begin{pmatrix} h_2 & h_1^2\bar{\gamma}^3 \\ h_1^2\bar{\gamma}^3 & h_2 \end{pmatrix}, \begin{pmatrix} h_2^2 & h_1 \\ h_1 & h_2^2 \end{pmatrix}, \begin{pmatrix} h_2 \\ h_1^2\bar{\gamma}^3 \end{pmatrix} \right\rangle;$$

following from the fact that $h_2^3 + h_1^3\bar{\gamma}^3 = (ab^2 + a^2b)\bar{\gamma}^6 = d_3(c^2\bar{\gamma}^6)$, and all the other products in the Massey product are represented by zero.

Further, we use naturality of Massey products to conclude that the same relation holds in $H^*(SL_2(\mathbb{Z}/3), \Gamma_1)$, where $z \cdot \bar{\gamma} = \zeta^2\bar{\gamma}$. We record this as

Lemma 6.1. *In $H^*(SL_2(\mathbb{Z}/3), \Gamma_1)$, there is a Massey product*

$$P\bar{\gamma}^{12} = \left\langle \begin{pmatrix} h_2^2 & h_1 \end{pmatrix}, \begin{pmatrix} h_2 & h_1^2\bar{\gamma}^3 \\ h_1^2\bar{\gamma}^3 & h_2 \end{pmatrix}, \begin{pmatrix} h_2^2 & h_1 \\ h_1 & h_2^2 \end{pmatrix}, \begin{pmatrix} h_2 \\ h_1^2\bar{\gamma}^3 \end{pmatrix} \right\rangle.$$

7. Using Bockstein spectral sequences

Before continuing, let us summarize the structure of the group $SL_2(\mathbb{Z}/3)$ and its action on Γ. The summary can also serve as guidelines for the method we will use to execute the computations.

The group $SL_2(\mathbb{Z}/3)$ has a presentation

$$SL_2(\mathbb{Z}/3) = \langle x, y, z | x^2 = y^2, x^4 = 1 = z^3, xyx = y, xz = zy^3, zyx = yz \rangle$$

such that the elements x and y generate a normal subgroup isomorphic to Q_8, and there is an exact sequence

$$1 \to Q_8 \to SL_2(\mathbb{Z}/3) \to C_3 \to 1.$$

This implies that if M is any $SL_2(\mathbb{Z}/3)$-module on which 3 is invertible, we have that

$$H^*(SL_2(\mathbb{Z}/3), M) = H^*(Q_8, M)^{C_3}.$$

To describe the action of $G = SL_2(\mathbb{Z}/3)$ on $\Gamma = \mathbb{Z}[1/3][\zeta][\gamma_1, \gamma_2]$, recall that the matrices corresponding to x, y, z are

$$x = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}, \quad z = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

respectively, and they act on Γ as follows

$$\begin{aligned} x :& \gamma_1 \mapsto -\gamma_2 & \qquad y :& \gamma_1 \mapsto -\zeta^2\gamma_1 + \zeta\gamma_2 \\ & \gamma_2 \mapsto \gamma_1 & & \gamma_2 \mapsto \zeta\gamma_1 + \zeta^2\gamma_2 \\ z :& \gamma_1 \mapsto \zeta^2\gamma_1 & & \\ & \gamma_2 \mapsto \gamma_2 - \zeta\gamma_1. & & \end{aligned}$$

This group action preserves the ideals $I_0 = (2)$ and $I_1 = (2, \gamma_1 + \gamma_2)$, and we will compute cohomology using the corresponding Bockstein spectral sequences.

First, $\Gamma_1 = \Gamma/I_1 = \mathbb{F}_4[\bar{\gamma}]$, where $\bar{\gamma}$ is the class of γ_i. The elements x and y of G act trivially on Γ_1, while z maps $\bar{\gamma}$ to $\zeta^2\bar{\gamma}$. We have, first of all

$$H^*(Q_8, \Gamma_1) = H^*(Q_8, \mathbb{Z}) \otimes \Gamma_1 = \mathbb{F}_4[\bar{\gamma}][P][a, b]/(a^2 + ab + b^2, a^2b + ab^2, a^3, b^3),$$

where P is the periodicity class in degree 4, and a and b are in degree 1. From the conjugation action of z on Q_8, i.e. the relations in G

$$zxz^{-1} = xy \qquad zyz^{-1} = x^3,$$

we get that z preserves the periodicity element P, and acts on a and b as

$$z : a \mapsto a + b, \qquad b \mapsto a.$$

In bidigree $(*, 0)$ (first is the cohomological, second is the internal grading), we have $\mathbb{F}_4[P][a, b]/(a^2 + ab + b^2, a^2b + ab^2, a^3, b^3)$, and the only C_3-invariants are the powers of P. In bidegree $(*, 2)$ we have $\bar{\gamma}\mathbb{F}_4[P][a, b]/(a^2 + ab + b^2, a^2b + ab^2, a^3, b^3)]$, and here $\bar{\gamma}P^k$ is acted on by multiplication by ζ^2. However, the elements $\bar{\gamma}(a + \zeta^2 b)P^k$ are invariant, as are $\bar{\gamma}(a^2 + \zeta^2 b^2)P^k$, for any $k \geq 0$. Similarly, in bidegree $(*, 4)$ the invariants are the elements $\bar{\gamma}^2(a + \zeta b)P^k$ as well as $\bar{\gamma}^2(a^2 + \zeta b^2)P^k$. Thus we obtain

$$\begin{aligned} & H^*(G, \Gamma_1) = H^*(Q_8, \Gamma_1)^{C_3} = \\ & \mathbb{F}_4[\bar{\gamma}^3, P] \left\langle 1, a^2b = ab^2, (a + \zeta^2 b)\bar{\gamma}, (a^2 + \zeta^2 b^2)\bar{\gamma}, (a + \zeta b)\bar{\gamma}^2, (a^2 + \zeta b^2)\bar{\gamma}^2 \right\rangle, \end{aligned}$$

with all the above relations and thus obtained multiplicative structure.

Denote suggestively

$$\begin{aligned} h_1 &= [(a+\zeta^2 b)\bar{\gamma}] \\ h_2 &= [(a+\zeta b)\bar{\gamma}^2] \\ e_0 &= [(a^2+\zeta^2 b^2)\bar{\gamma}] \\ e_3 &= [a^2 b] = [ab^2] \\ \bar{a}_3 &= [\bar{\gamma}^3]. \end{aligned}$$

Then $H^*(G,\Gamma_1) = \mathbb{F}_4[\bar{a}_3, P, h_1, h_2, e_3]/(\sim)$, the relations being described as

$$\begin{aligned} h_1^3 &= e_3\bar{a}_3 \\ h_2^3 &= h_1^3\bar{a}_3 \\ e_i^2 &= 0. \end{aligned}$$

Pictorially, we have the pattern in Figure 1, where the cohomological degree is along the vertical axis, while the horizontal axis depicts the "topological" degree $2t-s$. Each dot represents an $\mathbb{F}_4$, and multiplications by h_1 and h_2 are depicted as connecting line segments.

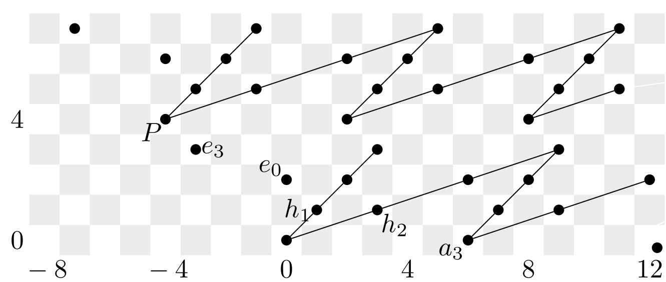


FIGURE 1. $H^*(SL_2(\mathbb{Z}/3),\Gamma_1)$

The relation $h_2^3 = a_3 h_1^3$ holds since

$$h_2^3 - a_3 h_1^3 = [(\zeta^2+\zeta)(a^2b+ab^2)\bar{\gamma}^6] = 2[e_3][\bar{\gamma}^6] = 0.$$

7.1. The a_1-Bockstein spectral sequence. We proceed to compute the a_1-BSS

$$H^*(G,\Gamma_1)[a_1] \Rightarrow H^*(G,\Gamma_0).$$

Recall that $a_1 = \gamma_1+\gamma_2$; to simplify the notation, $x \doteq y$ will mean that x and y are equal up to multiplication by a unit.

PROPOSITION 7.1. *The differentials in the a_1-BSS are determined by*

$$d_i(a_1) = 0 \qquad d_1(a_3) \doteq a_1h_2 \qquad d_1(e_0) \doteq a_1e_3 \qquad d_2(a_3^2) \doteq a_1^2h_1\bar{\gamma}^3.$$

PROOF. For $d_1(a_3)$ and $d_2(a_3^2)$, one checks that a_3 is not invariant in $\Gamma_0/(\gamma_1+\gamma_2)^2$ and a_3^2 is not invariant in $\Gamma_0/(\gamma_1+\gamma_2)^3$, so they need to support the specfied non-trivial differentials.

We can directly compute that $H^*(SL_2(\mathbb{Z}/3),(\Gamma_0)_1)$ (internal degree one) is $\mathbb{F}_4$ in cohomological degrees congruent to 1 modulo 4, and zero otherwise; the only differential making this work is $d_1(e_0) \doteq a_1e_3$. $\square$

The following are the resulting charts, in which a $\bullet$ denotes an element in the cohomology of Γ_1, a $\circ$ is an a_1-multiple of such an element, a $\square$ is an a_1^2-multiple of such an element, and for clarity we have omitted drawing all higher powers of a_1. The elements which support a differential or are hit by one are grayed out. The entire pattern is P-periodic and the result is also a_3^4-periodic.

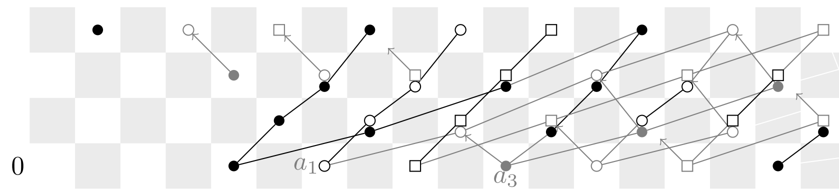


FIGURE 2. d_1 differential in the a_1-BSS

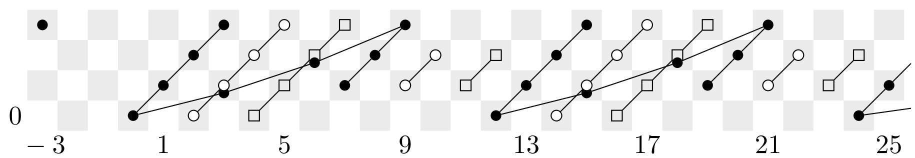


FIGURE 3. E_2-page of a_1-BSS

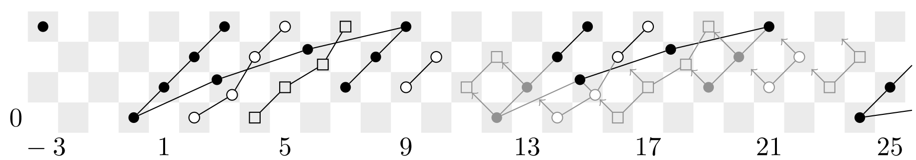


FIGURE 4. d_2 differential in the a_1-BSS

The resulting E_∞-page is in Figure 5, where we have stopped distinguishing between the elements which have different a_1-divisibility. The dotted lines denote hidden extensions which we prove in the following few results.

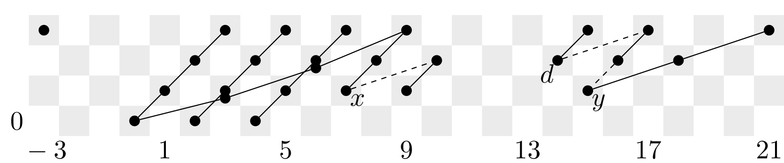


FIGURE 5. E_∞-page of a_1-BSS

LEMMA 7.2. *We have Massey products*

$$x := [a_3h_1] = \langle a_1, h_2, h_1\rangle$$
$$y := [a_3^2h_2] = \langle x, a_1^2, h_2\rangle = \langle a_1x, a_1, h_2\rangle.$$

Proof. Since $d_1(a_3) = a_1h_2$, and $h_2h_1h_1 = 0$, we conclude that $\langle a_1, h_2, h_1\rangle = [a_3h_1] = x$. For the next, we have that $d_2(a_3^2) = a_1^2$, $d_1(a_1a_3) = a_1^2h_2$, and $d_1(a_3) = a_1h_2$, hence

$$\langle x, a_1^2, h_2\rangle = [a_3^2h_2 + a_1a_3x] = [a_3^2h_2] = y, \text{ and}$$
$$\langle a_1x, a_1, h_2\rangle = [a_3^2h_2 + a_1a_3x] = y.$$

□

Corollary 7.3. *Let* $d = [a_3^2h_1^2]$. *We have the following multiplications in* $H^*(G,\Gamma_0)$

$$xh_2 = a_1xh_1, \qquad yh_1 = a_1d \qquad dh_2 = yh_1^2.$$

Proof. For the first equality, we note that xh_2 is represented by the class

$$(a^2 + ab + b^2)\gamma_1^2(\gamma_1 + \zeta\gamma_2)\gamma_1\gamma_2(\gamma_1 + \zeta^2\gamma_2)$$

and a_1xh_1 is represented by the class

$$\zeta^2(a^2 + \zeta b^2)\gamma_1^2(\gamma_1 + \gamma_2)\gamma_1\gamma_2(\gamma_1 + \zeta^2\gamma_2).$$

Hence the sum $xh_2+a_1xh_1$ is represented by the sum of the representatives, namely

$$\gamma_1^3\gamma_2(\gamma_1 + \zeta^2\gamma_2)(a + \zeta^2b)(\zeta^2a\gamma_1 + b\gamma_2).$$

But this element reduces to $\zeta a_3h_1h_2$, which is zero. Therefore $xh_2 = a_1xh_1$.

For the rest, we use Lemma 7.2, simple shuffling, and that $xh_2 = a_1xh_1$. We have

$$yh_1 = \langle a_1x, a_1, h_2\rangle h_1 = a_1x\langle a_1, h_2, h_1\rangle = a_1x[a_3h_1] = a_1d$$
$$yh_1^2 = \langle a_1x, a_1, h_2\rangle h_1^2 = a_1x\langle a_1, h_2, h_1^2\rangle = a_1x[a_3h_1^2] = [xh_2a_3h_1] = dh_2.$$

□

Proposition 7.4. *There is an extension* $h_1^4 = a_1^4P$

Proof. From Lemma 6.1, we get

$$\begin{aligned} a_1^4a_3^4P = a_1^4\bar\gamma^{12}P &= \left\langle \begin{pmatrix} h_2^2 & h_1\end{pmatrix}, \begin{pmatrix} h_2 & a_3h_1^2 \\ a_3h_1^2 & h_2\end{pmatrix}, \begin{pmatrix} h_2^2 & h_1 \\ h_1 & h_2^2\end{pmatrix}, \begin{pmatrix} h_2 \\ a_3h_1^2\end{pmatrix}\right\rangle a_1^4 \\ &\subseteq \left\langle \begin{pmatrix} h_2^2 & h_1\end{pmatrix}, \begin{pmatrix} 0 & a_1a_3h_1^2 \\ a_1a_3h_1^2 & 0\end{pmatrix}, \begin{pmatrix} 0 & a_1h_1 \\ a_1h_1 & 0\end{pmatrix}, \begin{pmatrix} h_2 \\ a_3h_1^2\end{pmatrix}\right\rangle a_1^2 \\ &= \langle h_1, a_1a_3h_1^2, a_1h_1, a_3h_1^2\rangle a_1^2 + \langle h_2^2, a_1a_3h_1^2, a_1h_1, h_2\rangle a_1^2 = L + R. \end{aligned}$$

By shuffling, we get

$$R \subseteq \langle a_1^2, h_2^2, a_1a_3h_1^2, a_1h_1\rangle h_2 = 0, \qquad L = h_1^4a_3^4.$$

But multiplication by a_3^4 is injective, so the result follows. □

7.2. The 2-Bockstein spectral sequence.

PROPOSITION 7.5. *In the 2-BSS we have the following differentials, which determine all the rest.*

$$d_1(a_1) \doteq 2h_1 \qquad d_1(x) = 2h_2^2 \qquad d_1(y) = 2d$$
$$d_2(a_1^2) \doteq 4h_2$$
$$d_3(e_3) = 8P.$$

PROOF. For the differentials on powers of a_1, we just check that a_1 is invariant mod 2 but not mod 4, and a_1^2 is invariant mod 4 but not mod 8.

The cohomology $H^*(SL_2(\mathbb{Z}/3), \mathbb{Z}) = \mathbb{Z}[P]/(8P)$, which gives that $d_3(e_3) = 8P$.

Since $d_1(a_1) = 2h_1$, and $d_1(h_1) = 0 = d_1(h_2)$ we get $d_1(x) = d_1(\langle a_1, h_2, h_1\rangle) = 2\langle h_1, h_2, h_1\rangle = 2h_2^2$.

$d_1(yh_1) = h_1 d_1(y) = d_1(a_1 d) = 2h_1 d$, so $d_1(y) = 2d$. □

The chart is displayed in Figure 6; the d_1 differentials are dotted, the d_2 differentials are dashed, and the d_3 differentials are the solid curved lines.

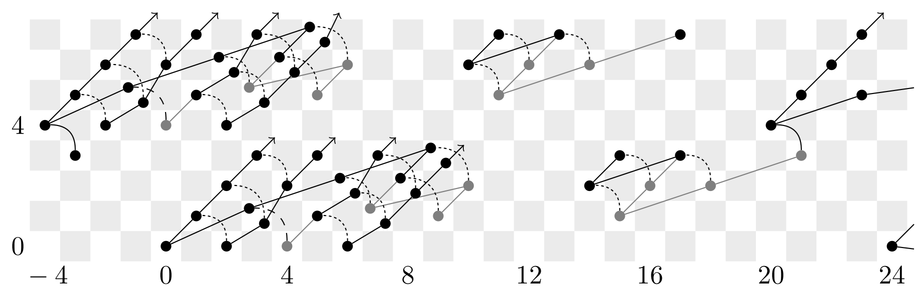


FIGURE 6. The mod 2-Bockstein spectral sequence

The resulting E_∞-page is in Figure 7; a bullet denotes an $\mathbb{F}_4$, and mod 2 extensions are depicted as circles around the bullet.

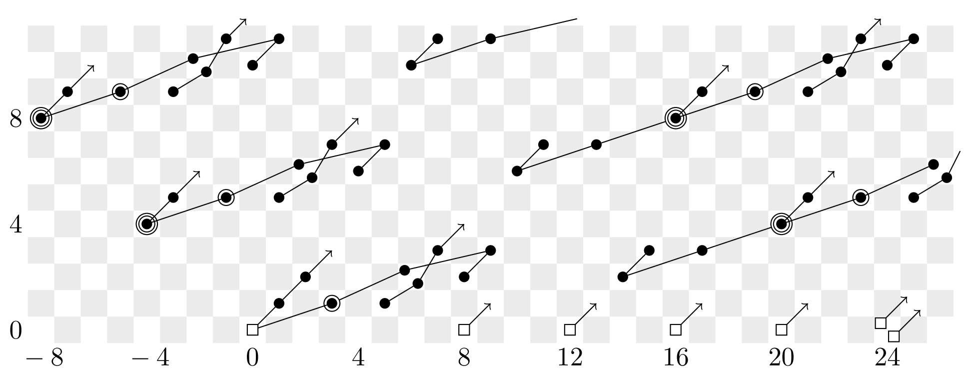


FIGURE 7. $H^*(G, \Gamma)$

8. The cohomology $H^*(SL_2(\mathbb{Z}/3), H^*(\mathcal{M}(3)^\zeta))$

Completely analogously to the above calculations, one can obtain the group cohomology of the twisted module $\Gamma_\zeta = \Gamma \otimes \mathbb{Z}_\zeta$, where $\mathbb{Z}_\zeta = H^1(\mathcal{M}(3)^\zeta, \omega^{-2})$ is the module on which Q_8 acts trivially, and the element $z \in SL_2(\mathbb{Z}/3)$ of order three acts as multiplication by ζ. The resulting pattern is displayed in Figure 8; again there are periodicity operators P of degree $(-4, 4)$ and a_3^4 of degree $(24, 0)$.

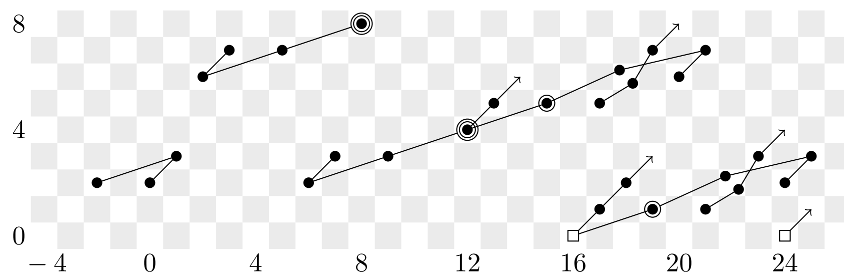


FIGURE 8. $H^*(SL_2(\mathbb{Z}/3), \Gamma_\zeta)$

To obtain the cohomology of the dual module Γ_ζ, we proceed as in Section 10 of [**Sto12**]. The result is the familiar pattern depicted in Figure 9.

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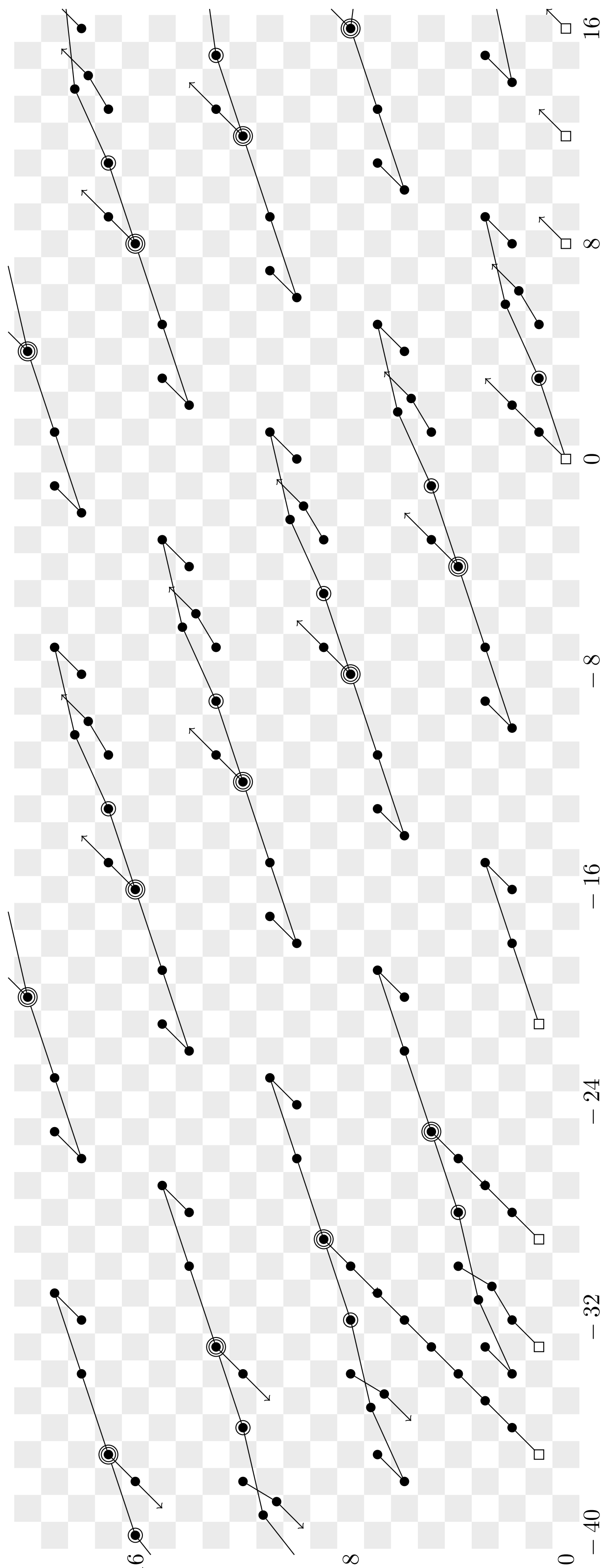
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