4. Permutation representations.

If G is a finite group and S a finite G-set we can consider the associated permutation representation V(S,F) of S over the commutative ring F. The assignment $S \longmapsto V(S,F)$ induces a ring homomorphism

$$h = h_F : A(G) \longrightarrow R(G;F)$$

of the Burnside ring into the representation ring. We shall describe some aspects of this homomorphism in particular when F is a field or the ring of integers Z. We describe the connection to the J-homomorphism of section 2 and to λ-rings.

4.1. p-adic completion.

Let p be a prime number and let G be a p-group. Let

$$A(G)^{\wedge}_{p} = \mathrm{inv}_n \lim A(G)/p^n A(G) \cong A(G) \otimes_Z Z_p$$

be the p-adic completion of A(G).

If $|G| = p^n$ and $m = q(1,p)$ we have seen in exercise 1.9.4 that $m^{n+1} \subset p\,A(G) \subset m$. Hence

Proposition 4.1.1.

If G is a p-group the p-adic and the m-adic topology on A(G) coincide.

Let now q be a prime different from p. Let $e: R(G,F_q) \rightarrow Z: x \mapsto \dim x$ be the augmentation and $I(G,F_p) = \mathrm{Kernel}\ e$ the augmentation ideal.

The ring $A(G)^{\wedge}_{p}$ is a local ring with maximal ideal $m^{\wedge}$, the completion of m.

We now consider the case $p \neq 2$. Since $A(G)[q^{-1}] \subset A(G)^{\wedge}_{p}$ we obtain from 2.1 the J-homomorphism

$$(4.1.2) \qquad J : R(G,F_q) \longrightarrow A(G)^{\wedge}_{p} .$$

We notice that for an F_pG-module V $eJ(V\text{-}\dim V) = 1$. Hence

$$(4.1.3) \qquad JI(G,F_q) \subset 1 + m^{\wedge}.$$

The set $1 + m^{\wedge} \subset A(G)^{\wedge}_{p}$ is compact and a topological group with respect to multiplication. A fundamental system of neighbourhoods of 1 is given by $(1+\hat{m}^{i})_{i \geq 1}$, or $(1+\hat{m}^{i}+p^{j}\hat{m})$. Since

$$J(p^{i}I(G,F_q)) \subset (1+\hat{m})^{p^{i}} \subset 1+\hat{m}^{i+1}$$

we see that $J : I(G,F_q) \longrightarrow 1+m^{\wedge}$ is p-adically continuous and therefore induces a continuous map

$$(4.1.4) \qquad J^{\wedge} : I(G,F_q)^{\wedge}_{p} \longrightarrow 1+m^{\wedge}$$

homomorphic from addition to multiplication.

4.2. Permutation representations over F_q.

We still assume that p is odd and consider the permutation representation map and its p-adic completion

$$(4.2.1) \qquad \begin{array}{l} h : A(G) \longrightarrow R(G,F_q) \\ h^{\wedge} : A(G)^{\wedge}_{p} \longrightarrow R(G,F_q)^{\wedge}_{p}. \end{array}$$

Since $h(m) \subset p\,R(G,F_q) + I(G,F_q)$ and because the p-adic and $I(G,F_q)$-adic topology on $R(G,F_q)$ coincide (see [6]) we obtain an induced continuous map between multiplicative topological groups

$$(4.2.2) \qquad h^{\wedge}:\ 1+m^{\wedge} \longrightarrow 1+I(G,F_q)^{\wedge}.$$

Definition 4.2.3.

We call the prime q p-generic if it generates a dense subgroup of the p-adic units (i. e. if q generates Z/p^2Z^*).

Theorem 4.2.4.

Let q be a p-generic prime. Then the composition

$$h^{\wedge}J^{\wedge}:\ I(G,F_q)^{\wedge} \longrightarrow 1+I(G,F_q)^{\wedge}$$

is an isomorphism.

In fact the proof will show that this is one of the isomorphisms which we had considered in the previous chapter on λ-rings, namely the map ϱ_q.

Proof.

In order to prove the equality $h^{\wedge}J^{\wedge} = \varrho_q$ we need only consider cyclic groups $G = Z/p^nZ$ because $J^{\wedge}$, $h^{\wedge}$ and ϱ_q are compatible with restriction to subgroups and elements in $R(G,F_q)^{\wedge}$ are detected by their restriction to cyclic subgroups.

We begin with the computation of ϱ_q for $G = Z/p^nZ$. The group algebra $F_qG = F_q[x]/(x^a-1)$, $a = p^n$, decomposes as $\bigoplus_{1\le t\le n} F_q[x]/\phi_t(x)$, where $\phi_t(x)$ is the p^t-th cyclotomic polynomial. If q is p-generic then $\phi_t(x)$ is irreducible. Hence the $F_q[x]/\phi_t(x) =: V_t$ are the irreducible

$F_q G$-modules in our case. By 3.12.2 we have the identity

$$\rho_q(V_t - \dim V_t)\ \Theta_q(\dim V_t) = \Theta_q(V_t).$$

Over a splitting field F of G the module V_t splits $V_t = \oplus_j V_t(j)$, where $V_t(j)$ is onedimensional and a generator of G acts as multiplication with u^j, where u is a primitive p^t-th root of unity and $j \in Z/p^t Z^*$. Since the Θ_q-operations are compatible with field extension we obtain from 3.7.2

$$\Theta_q(V_t) = \prod \Theta_q(V_t(j)) = \prod (1+V_t(j) + \dots + V_t(j)^{q-1}) .$$

It is enough, by naturality, to study this for t = n. We claim that in $R(G,F) \cong Z[y]/(y^a-1)$ $\Theta_q(V_n) = h(1+bG)$ where b satisfies $1+bp^n = q^a$. This means we have to check

$$\prod_j (1 + y^j + \dots + y^{j(q-1)}) = 1 + b(1 + y + \dots + y^{a-1}).$$

But this is true if we replace y by a-th roots of unity v and evaluation at such v determines elements of $Z[y]/(y^a-1)$. (This is essentially a computation with modular characters.) Now an easy checking of fixed point dimensions shows that $J(V_n) = 1 + bG$. This shows $hJ(V_t) = \Theta_q(V_t)$ and therefore $h^\wedge J^\wedge(V_t - \dim V_t) = \rho_q(V_t - \dim V_t)$. The equality $h^\wedge J^\wedge = \rho_q$ is now proved.

We now check that we are in a situation where 3.14.1 and 3.14.5 can be applied. To prove $\psi^k V = V$ for $(k,p) = 1$ and $F_q G$-modules V we again need only consider cyclic G and then this follows from the determination of the irreducible $F_q G$-modules above.

Remark 4.2.5.

If q is p-generic then the decomposition homomorphism

$$d : R(G,Q) \longrightarrow R(G,F_q)$$

(Serre [147] , 15.2) is an isomorphism.

4.3. Representations of 2-groups over F_3.

We now consider the analogue of 4.2 for 2-groups and restrict attention to representations over F_3. We first recall what the theory of oriented γ -rings tells us in this case.

In this section G shall be a 2-group. We have the following objects

$$R(G,F_3) \supset RO(G,F_3) \supset RSO(G,F_3) \supset ISO(G,F_3) .$$

Here $R(G,F_3)$ is the representation ring of F_3G-modules, RO the subring of those modules possessing a G-invariant quadratic form, RSO the subring of F_3G-modules on which each $g \in G$ acts with determinant one, and ISO is the augmentation ideal of zero-dimensional objects.

The ring $RSO(G,F_3)$ is an oriented λ-ring (3.10.2) and $ISO(G,F_3)$ is an oriented γ -ring. Let a roof denote 2-adic completion. We have from 3.14.10

Proposition 4.3.1.

The map

$$\varsigma_3^{or} : ISO(G,F_3)^{\wedge} \longrightarrow 1 + ISO(G,F_3)^{\wedge}$$

is an isomorphism.

In order to relate this isomorphism to the J-homomorphism and to permutation representations we compute the map for cyclic groups $G=Z/2^nZ$. We start with the representation ring.

We have a decomposition of the group ring

$$F_3G \cong \bigoplus_{1\leq t\leq n} F_3[x]/\phi_t(x)$$

where $\phi_t(x)$ is the 2^t-th cyclotomic polynomial. The ϕ_t are no longer irreducible for $t \geq 3$. If $K_t = F_3[u_t]$, where u_t is a primitive 2^t-th root of unity then $[K_t : F_3] = 2^{t-2}$, $t \geq 3$. Moreover $\phi_2(x) = x^2+1$ is irreducible and $K_2 = F_3 [u_t] = F_9$.

First assume $t \geq 3$. Let V_t be the F_3G-module K_t where a fixed generator $g \in G$ acts as multiplication with u_t. Then the dual module $V_t^* = \mathrm{Hom}(V_t, F_3)$ is K_t and g acting as u_t^{-1}. Moreover $F_3[x]/\phi_t(x) \cong V_t \oplus V_t^*$ and V_t is not isomorphic to V_t^*. The module V_t cannot carry a G-invariant quadratic form, because this would imply $V_t \cong V_t^*$. But

$$V_t \oplus V_t^* \longrightarrow F_3 : (x,y) \longmapsto \mathrm{Tr}(xy)$$

is a G-invariant, non-degenerate quadratic form (where $\mathrm{Tr} : K_t \longrightarrow F_3$ is the trace map).

If $t = 2$ let $V_t = F_3[u_2] = F_9$ with g acting as multiplication with u_2. Then the norm map $N : F_9 \longrightarrow F_3$ is a G-invariant quadratic form. The associated bilinear form is

$$b : F_9 \times F_9 \longrightarrow F_3 : (x,y) \longmapsto \varphi(x)y + x\,\varphi(y)$$

where φ is the Frobenius automorphism. The determinant of b is one.

Any G-invariant symmetric bilinear form must have determinant one in this case.

Finally there are two one dimensional representations, V_o the trivial representation, and $V_1 = F_3$ with g acting as multiplication with -1. They both carry quadratic forms $q : x \longmapsto x^2$ or $q^- : x \longmapsto -x^2$.

We now enter the computation of ρ_3^{or} for the elements $V_1 - \dim V_1$, $V_2 - \dim V_2$, $V_t + V_t^* - \dim(V_t + V_t^*)$. It is sufficient to compute Θ_3^{or} of the corresponding modules. Since character computations are easier, we compute for QG-module and then use the decomposition homomorphism. Let

$$W_t = \mathbb{Q}[x]/\phi_t(x), \quad t \geq 1$$

with g acting as multiplication with x. Let S_t be the homogeneous G-set with 2^t elements and $V(S_t)$ its permutation representation. Let a_t be the cardinality of K_t. Then we have

<u>Proposition 4.3.1.</u>

<u>For</u> $t \geq 3$:

$$\Theta_3^{or}(W_t) = V(S_1) - V(S_o) + 2^{-t}(a_t - 1)V(S_t).$$

<u>Moreover</u>

$$\Theta_3^{or}(W_2) = V(S_o) - V(S_1) + V(S_2)$$

$$\Theta_3^{or}(W_1 \oplus W_1) = V(S_o) - 2V(S_1) .$$

Proof.

Suppose $t \geqslant 3$. We compute the character of $\Theta_3^{or}(W_t)$. Over a splitting field W_t decomposes as $W_t = \oplus_j (W_t(j) + W_t(-j))$ where $W_t(j)$ is one-dimensional with g acting as multiplication with $(u_t)^j$ and $1 \leqslant j = 2k + 1 < 2^{t-1}$. From 3.10.12 we obtain

$$\Theta_3^{or}(W_t) = \pi_j (1 + W_t(j) \oplus W_t(-j))$$

with character value at g equal to

$$\pi_j (1 + u^j + u^{-j}), \qquad u = u_t.$$

This product is -1, as can be seen by using the identity

$$\pi_j (x + x^{-1} - (u^j + u^{-j})) = x^{-2^{t-2}} \phi_t(x)$$

and evaluating at x a cubic root of unity. The character value of $\Theta_3^{or}(W_t)$ at non-generators $x \neq 1$ of G is 1. The character value at 1 is a_t. It is an easy matter to check that the permutation representation of $S_1 - S_o + 2^{-t}(a_t-1)S_t$ has the same character.

Finally $\Theta_3^{or}(W_2) = 1+W_2$, $\Theta_3^{or}(W_1 \oplus W_1) = 1 + W_1 \oplus W_1$ and the assertion of the proposition is easily verified.

Connecting Θ_3^{or} with the quadratic J-homomorphism and permutation representations presents the difficulty that permutation representations do not generally preserve the orientation. We deal therefore with this problem first.

Let $A_o(G) \subset A(G)$ be the subring generated by finite G-sets S on which each $g \in G$ acts through even permutations.

If S is any finite G-set we can assign to it a homomorphism

$$s(S) : G \longrightarrow Z^* : g \longmapsto \text{signum}(l_g)$$

where $l_g : S \longrightarrow S$ is left translation by g. The assignment $S \longmapsto s(S)$ induces a homomorphism

$$s : A(G) \longrightarrow \text{Hom}(G,Z^*)$$

from the additive group of A(G) into the multiplicative group $\text{Hom}(G,Z^*)$. The kernel of s is $A_o(G)$. Let

$$j : \text{Hom}(G,Z^*) \longrightarrow A(G)$$

be given by

$$j(f) = G/H_f - |G/H_f| + 1$$

where H_f = kernel f. Then j maps into $A(G)^*$. Since $2A(G) \subset$ kernel s everything passes to the 2-adic completions. Let sign be the composition

$$(4.3.3) \qquad \text{sign} : A(G)^{\wedge} \xrightarrow[s]{} \text{Hom}(G/Z^*) \xrightarrow[j]{} \dot{A}(G) \subset A(G)^{\wedge}$$

Then $A(G)^{\wedge} \longrightarrow A(G)^{\wedge} : x \longmapsto x + \text{sign}(x) - 1$ has an image in $A_o(G)^{\wedge}$ and does not change the cardinality.

Let $QS(G,F_3)$ be the monoid of orientation preserving F_3G-modules with quadratic form under orthogonal sum. Denote $f : QS(G,F_3) \longrightarrow ISO(G,F_3)$ the map $(M,q) \longmapsto M - \dim M$.

We define a modified quadratic J-map

$$J' : QS(G,F_3) \longrightarrow A_o(G)^\wedge$$

by $J'(M,q) = (JQ(M,q) + \text{sign}\, JQ(M,q) - 1)_1$ where $(-)_1$ means that we divide the value in the bracket by its cardinality (which is a power of 3, hence invertible in $A_o(G)^\wedge$).

Theorem 4.3.4.

The following diagram is commutative

$$\begin{array}{ccc} QS(G,F_3) & \xrightarrow{\quad J' \quad} & A_o(G)^\wedge \\ \Big\downarrow f & & \Big\downarrow h \\ ISO(G,F_3) & \xrightarrow[\quad \rho_3^{or} \quad]{} & RSO(G,F_3)^\wedge \end{array} .$$

Proof.

It is sufficient to consider cyclic groups $G = Z/2^nZ$. In that case any (M,q) is orthogonal sum of forms carried by one of the modules $V_t + V_t^*$, $t \geq 3$, V_2, $V_1 \oplus V_1$. In the case of $V_t + V_t^*$ the form must be hyperbolic. From 2.3.4 one obtains $JQ(V_t \oplus V_t^*, q) = 1+2^{-t}(a_t-1)S_t$ (compare 4.3.2). Since sign $S_t = S_1 - 1$ we compute $J'(V_t \oplus V_t, q) = a_t^{-1}(S_1 - 1 + 2^{-t}(a_t-1)S_t)$ and with 4.3.2 we obtain the desired commutativity. The remaining cases give the following results:

$JQ(V_2,q) = 1-S_2$, $J'(V_2,q) = \frac{1}{3}(1-S_1+S_2)$

$JQ(V_1 \oplus V_1, q \oplus q) = JQ(V_1 \oplus V_1, q^- \oplus q^-) = 1-2S_1$

$J'(V_1 \oplus V_1, q \oplus q) = \frac{1}{3}(2S_1-1)$

$JQ(V_1 \oplus V_1, q \oplus q^-) = 1+S_1$, $J'(V_1 \oplus V_1, q \oplus q^-) = \frac{1}{3}(2S_1-1)$.

Again with 4.3.2 we obtain the desired commutativity.

4.4. Permutation representations over Q.

The previous investigations can be used to give a very round-about prove of

Theorem 4.4.1.

Let G be a p-group. Then

$$h_Q : A(G) \longrightarrow R(G,Q)$$

is surjective.

We make various remarks how this is related to the forgoing results. We have decomposition homomorphisms $d_q : R(G,Q) \longrightarrow R(G,F_q)$ and $d_3 : R(G,Q) \longrightarrow RO(G;F_3)$. If G is a p-group, $p \neq q$ and q is p-generic then d_q is an isomorphism. If G is a 2-group then d_3 is an isomorphism. In order to show that h_Q is surjective one can therefore try to show the same for h_{F_q} or h_{F_3}.

It is now easy to show that the cokernel of h_Q is annihilated by the order of the group G. This can be seen as follows. The characters in R(G,Q) are constant on conjugacy classes and the set of generators of a cyclic group. If $H < G$ is cyclic then $h(G/H)(g)$ is non-zero if and only if g is conjugate to an element in H and $h(G/H)(g) = |G/H^g|$ is divisible by $|NH/H|$. Hence any class function which is constant on generator sets of cyclic groups is a Z-linear combination of $|NH/H|^{-1} h(G/H)$, $H < G$ cyclic. As a consequence h_Q is surjective for a p-group if the p-adic completion is surjective. For $p \neq 2$ this follows immediately from 4.2.4. For p = 2 one deduces from 4.3.4 that

$A_o(G) \longrightarrow RSO(G)$ is surjective. But if V is any $Q[G]$-module let $D(V)$ be its determinant module. Then $D(V) \oplus 1$ is a permutation representation and $V \oplus D(V) \oplus 1$ is orientation preserving. Hence $V = V \oplus D(V) \oplus 1 - D(V) \oplus 1$ is in the image of d_Q.

4.5. Comments.

The material in this section is taken from Segal [146] . The presentation in 4.3 is unsatisfactory; I hope some reader can elaborate on it. There are important connections between the Burnside ring and integral permutation representations, see Oliver [121] , [122] and the references there to earlier work of Dress and Endo-Miyata. For 4.4.1 see also Ritter [133] .