6. Induction Theory.

In this section we present the formal theory of induced representations, restriction homomorphisms, transfer maps. This axiomatic theory was developed mainly by Green [88] and Dress [80], [81]. The basic axioms are abstract forms of the Frobenius reciprocity law and the Mackey double coset formula of ordinary representation theory. Later we shall apply the formalism to equivariant homology, cohomology, and topological transfer maps.

6.1. Mackey functors.

Let G be a finite group and let $G^\wedge$ or G-Set be category of finite G-sets and G-maps. Let Ab be the category abelian groups.
A bi-functor

$$M = (M^*, M_*) : G\text{-Set} \longrightarrow Ab$$

consists of a contravariant functor $M^* : G\text{-Set} \longrightarrow Ab$ and a covariant functor $M_* : G\text{-Set} \longrightarrow Ab$; the functors are assumed to coincide on objects. We write

$$M(S) = M_*(S) = M^*(S)$$

for a finite G-set S. If $f : S \longrightarrow T$ is a morphism we often use the notation

$$M_*(f) = f_*, \quad M^*(f) = f^*.$$

We use the topological notation: a lower star for covariant functors ("homology"). Dress unfortunately uses a different notation.

A bi-functor $M = (M^*, M_*)$ is called a Mackey functor if it has the following properties:

(6.1.1) For any pullback diagram in G-Set

$$\begin{array}{ccc} U & \xrightarrow{F} & S \\ {\scriptstyle H}\downarrow & & \downarrow{\scriptstyle h} \\ T & \xrightarrow{f} & V \end{array}$$

the diagram

$$\begin{array}{ccc} M(U) & \xrightarrow{F_*} & M(S) \\ {\scriptstyle H^*}\uparrow & & \uparrow{\scriptstyle h^*} \\ M(T) & \xrightarrow{f_*} & M(V) \end{array}$$

is commutative.

(6.1.2) The two embeddings $S \longrightarrow S + T \longleftarrow T$ into the disjoint union define an isomorphism

$$M^*(S+T) \longrightarrow M^*(S) \oplus M^*(T).$$

Let M and N be bi-functors. A natural transformation of bi-functors $X : M \longrightarrow N$ consists of a family of maps $X(S) : M(S) \longrightarrow N(S)$, indexed by the objects of G-Set, such that this family is a natural transformation $M_* \longrightarrow N_*$ and $M^* \longrightarrow N^*$.

Let M be a Mackey functor and S a G-set. Then

$$M_S : T \longrightarrow M(S \times T)$$

$$M_S^*(f) = M^*(id_S \times f), \quad M_{S*}(f) = M_*(id_S \times f)$$

defines a Mackey functor M_S, as one easily checks. The projection maps pr : S x T ⟶ T define natural transformation of bi-functors

$$\Theta^S : M \longrightarrow M_S \quad , \quad \Theta^S(T) = pr^*$$

$$\Theta_S : M_S \longrightarrow M \quad , \quad \Theta_S(T) = pr_* .$$

The relevant commutative diagrams follow from the functor properties of M and from 6.1.1.

The functor M is called S-<u>injective</u> (S-<u>projective</u>) if Θ^S (Θ_S) is split-injective (split-surjective) as a natural transformation of bi-functors.

<u>Proposition 6.1.3.</u>

<u>Let</u> M <u>be</u> <u>a</u> <u>Mackey</u> <u>functor</u>. <u>Then</u> <u>the</u> <u>following</u> <u>assertions</u> <u>are</u> <u>equivalent:</u>

i). M <u>is</u> S-<u>injective</u>.

ii) M <u>is</u> S-<u>projective</u>.

iii) M <u>is</u> <u>a</u> <u>direct</u> <u>summand</u> <u>of</u> M_S <u>as</u> <u>bi</u>-<u>functor</u>.

<u>Proof.</u>

i) ⇒ iii) By definition of S-injectivity.

iii) ⇒ i) The assumption of iii) is that we have natural transformation $\Theta : M \longrightarrow M_S$, $\Psi : M_S \longrightarrow M$ such that $\Psi\Theta = id$. We have to find a natural transformation $\Psi^S : M_S \longrightarrow M$ such that $\Psi^S\Theta^S = id$. For a G-set T we define $\Psi^S(T)$ by the following diagram

$$\begin{array}{ccccc} M(T) & \xrightarrow[\theta^S(T)]{pr^*} & M(S \times T) & \overset{{}^S(T)}{\dashrightarrow} & M(T) \\ \Big\downarrow{\scriptstyle \theta(T)} & & \Big\downarrow{\scriptstyle \theta(S\times T)} & & \Big\uparrow{\scriptstyle (T)} \\ M(S \times T) & \xrightarrow[pr^*]{} & M(S \times S \times T) & \xrightarrow[(d \times id)^*]{} & M(S \times T) \end{array}$$

where $d : S \longrightarrow S \times S$ is the diagonal. The left square is commutative by naturality. Using $(d \times id)^* pr^* = id$ and $\Psi(T)\theta(T) = id$ one proves $\Psi^S(T)\theta^S(T) = id$. Moreover Ψ^S is defined as a composition of three natural transformation of bi-functors hence itself such a natural transformation.

ii) <=> iii) is proved analogously.

Let S be a G-set. We let S^o be a point and $S^k = \prod_{i=o}^{k-1} S$. We denote $pr_i : S^{k+1} \longrightarrow S^k$ the projection which omits the i-th factor, $O \leq i \leq k$. If M is a Mackey functor we have two chain complexes

(6.1.4) $$O \longrightarrow M(S^o) \xrightarrow{d^o} M(S^1) \xrightarrow{d^1} M(S^2) \xrightarrow{d^2} \ldots$$

(6.1.5) $$O \longleftarrow M(S^o) \xleftarrow[d_o]{} M(S^1) \xleftarrow[d_1]{} M(S^2) \xleftarrow[d_2]{} \ldots$$

defined by $d^k = \sum_{i=o}^{k} (-1)^i \, pr_i^*$, $d_k = \sum_{i=o}^{k} (-1)^i \, pr_{i*}$.

Proposition 6.1.6.

Let M be a Mackey functor. Then

i) M_S is always S-injective and S-projective.

ii) If M is S-injective then the complexes 6.1.4 and 6.1.5 are exact.

Proof.

The splitting of $M_S \longrightarrow (M_S)_S$ appears in the proof of 6.1.3. Let Ψ be a splitting of Θ^S. Then a contracting homotopy of 6.1.4 is given by the maps

$$s^{k+1} := \Psi(S^k) : M(S \times S^k) \longrightarrow M(S^k)$$

A splitting of Θ_S gives a contracting homotopy for 6.1.5.

Remark.

Instead of using functors into Ab one can consider functors into the category of modules over a ring or into an abelian category. This remark also applies to subsequent developments.

It is often convenient to denote M(G/H) by M(H). If $H < K < G$ and $f : G/H \longrightarrow G/K$ the canonical map then

$$f \quad : M(K) = M(G/K) \longrightarrow M(G/H) = M(H)$$

is called restriction from K to H

$$res_H^K$$

and

$$f_* : M(H) = M(G/H) \longrightarrow M(G/K) = M(K)$$

is called induction from H to K

$$ind_H^K$$

The axioms for a Mackey functor essentially tell how res and ind behave under composition. This is the so called double coset formula which one can never remember and which is avoided by this axiomatic treatment. Let

$$\begin{array}{ccc} G/H \times G/K & \xrightarrow{P} & G/K \\ \Big\downarrow Q & & \Big\downarrow q \\ G/H & \xrightarrow{p} & G/G \end{array}$$

be the canonical pullback. The orbits $A_1, \ldots, A_r$ of $G/H \times G/K$ correspond to the double cosets $H \backslash G/K$. Let $P(i)$, $Q(i)$ be the restriction of P, Q to A_i. Then 6.1.1 and 6.1.2 say

(6.1.7) $$\operatorname{res}_H^G \operatorname{ind}_K^G = \sum_{i=1}^{r} P(i)_* \, Q(i)^*.$$

If A_i is the orbit through $(1,x)$ then via $A_i = G/G_{(1,x)}$

(6.1.8) $$Q(i)^* = \operatorname{res}^H_{H \cap x K x^{-1}}$$

and

(6.1.9) $$P(i)_* = \operatorname{ind}^K_{K \cap x^{-1} H x} \circ c(x)_*$$

where $c(x)$ is conjugation $g \longmapsto x^{-1} g x$. The double coset formulas 6.1.7 - 6.1.9 are sufficient to reconstruct the whole Mackey functor.

Similar remarks apply to the exact sequences 6.1.4 and 6.1.5. We spell out what the exactness of 6.1.4 at $M(S^1)$ means in terms of double cosets. Let $S = \coprod_{H \in F} G/H$, where F is a family of subgroups of G. Then $M(S) = \bigoplus_{H \in F} M(G/H)$. The image of $M(S^0)$ in $M(S)$ is equal to the difference kernel of the two projection maps $p_i^* : M(S) \longrightarrow M(S \times S)$ which are maps

$$\bigoplus_{H\in F} M(G/H) \longrightarrow \bigoplus_{(H,K)\in F\times F} M(G/H \times G/K)$$

Then $(x_H) \in \bigoplus_{H\in F} M(G/H)$ is in the kernel if and only if for each $x \in K$ and $(H,K) \in F \times F$ $\mathrm{res}(x_H) \in M(H \cap x\,Kx^{-1})$ is equal to $\mathrm{res} \circ c(x)x_K$, where again $c(x)$ is the map induced by the conjugation $x^{-1}Hx \cap K \longrightarrow xKx^{-1} \cap H$. It is seen that this difference kernel is actually an inverse limit.

6.2. Frobenius functors and Green functor.

Let M,N, and L be Mackey functors G-Set $\longrightarrow$ Ab. A pairing

$$M \times N \longrightarrow L$$

is a family of bilinear maps

$$M(S) \times N(S) \longrightarrow L(S) \; : \; (x,y) \longmapsto x\cdot y$$

indexed by the objects of G-Set, such that for any morphism $f : S \longrightarrow T$ the following holds

$$\begin{aligned} L^*f(x\cdot y) &= (M^*fx)\cdot(N^*fy), & x \in M(T), \; y \in N(T) \\ x\cdot(N_*fy) &= L_*f((M^*fx)\cdot y), & x \in M(T), \; y \in N(S) \\ (M_*fx)\cdot y &= L_*f(x\cdot(N^*fy)), & x \in M(S), \; y \in N(T). \end{aligned} \tag{6.2.1}$$

These formulas make sense if M,N, and L are just bi-functors. A bi-functor F together with a pairing $F \times F \longrightarrow F$ is called a Frobenius functor if $F(S) \times F(S) \longrightarrow F(S)$ makes $F(S)$ into an associative ring with unit and morphisms f^* preserve units.

A Green functor U : G-Set $\longrightarrow$ Ab is a Mackey functor U together with a pairing $U \times U \longrightarrow U$ making it into a Frobenius functor.

If U is a Green functor then a left U-module is a Mackey functor M together with a pairing U x M $\longrightarrow$ M such that via this pairing M(S) becomes a left U(S)-module (the unit $1_{U(S)} \in U(S)$ acting as identity).

Theorem 6.2.2.

Let U : G-Set $\longrightarrow$ Ab be a Green functor. Let S be a G-set. Then the following assertions are equivalent:

i) *The map* $f_* : U(S) \longrightarrow U(P)$ *is surjective* (P = Point).

ii) U *is* S-*injective*.

iii) *All* U-*modules are* S-*injective*.

Proof.

iii) $\Rightarrow$ ii), because U is a U-module.

ii) $\Rightarrow$ i), because by 6.1.3. U is S-projective; in particular $U_S(P) \longrightarrow U(P)$ is split surjective.

i) $\Rightarrow$ iii): Choose $x \in U(S)$ with $f_*(x) = 1$. Let M be a U-module. Define a natural transformation $\psi : M_S \longrightarrow M$ by

$$\psi(T) : M(S \times T) \longrightarrow M(T) : y \longmapsto q_*(p^*x \cdot y)$$

where $p : S \times T \longrightarrow S$ and $q : S \times T \longrightarrow T$ are the two projections. One checks that ψ is a natural transformation of Mackey functors. Moreover ψ is left inverse to $\theta^S : M \longrightarrow M_S$ because for $z \in M(T)$ one has by 6.2.1

$$\psi\theta^S(T)(z) = q_*(p^*x \cdot q^*y) = (q_*p^*x) \cdot y$$

and by 6.1.1. $q_*p^*x = g^*f_*x = g^*1 = 1$, where we have used the pullback diagram

$$\begin{array}{ccc} S \times T & \xrightarrow{\quad q \quad} & T \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g} \\ S & \xrightarrow[\quad f \quad]{} & P \end{array}$$

The universal example of a Green functor is the _Brunside ring functor_. We describe this aspect of the Burnside ring now. Let $A[S]$ be the Burnside ring of finite G-sets over S. If $f : S \longrightarrow T$ is a morphism then pullback along f defines a ring homomorphism $f^*: A[T] \longrightarrow A[S]$ and composition with f defines an additive map $f_* : A[S] \longrightarrow A[T]$. The ring structure on $A[S]$ defines the pairing $A \times A \longrightarrow A$. It is easily checked that these data make A into a Green functor. (Compare 5. where we have studied a slightly more general situation.)

Proposition 6.2.3.

Let M be a Mackey functor. Then M is in a canonical way a module over the Burnside ring functor.

Proof.

Given $f : T \longrightarrow S$ we consider the homomorphism $f_* f^* : M(S) \longrightarrow M(S)$. The assignment $(f,x) \longmapsto f_* f^* x$ is additive in f and induces therefore a bilinear map $A[S] \times M(S) \longrightarrow M(S)$. We leave it as an exercise to verify that this defines a pairing and makes M into an A-module.

Let U be a Green functor. The assignment $f : T \longrightarrow S \longmapsto f_* f^* 1_S$ induces a ring homomorphism

$$(6.2.4) \qquad h(S) = h : A[S] \longrightarrow U(S)$$

and the h(S) from a natural transformation of Green functors. This generalizes permutation representations.

We now discuss defect sets.

Proposition 6.2.4.

Let X *and* Y *be finite* G-*sets and let* U *be a Green functor. Then* U(X) ⟶ U(P) *and* U(Y) ⟶ U(P) *are surjective if and only if* U(X x Y) ⟶ U(P) *is surjective.* (P = Point.)

Proof.

If U(X x Y) ⟶ U(P) is surjective we see from the factorization U(X x Y) ⟶ U(X) ⟶ U(P) that U(X) ⟶ U(P) is surjective. If U(Y) ⟶ U(P) is surjective then U is Y-projective so that U(Y x X) ⟶ U(X) is surjective for any X.

Corollary 6.2.5.

There exists a unique minimal set D(U) *of conjugacy classes of subgroups of* G *such that the sum of the induction maps* U(H) ⟶ U(G), (H) $\in$ D(U) *is surjective.*

D(U) is called the *defect set* of the Green functor U. The famous induction theorem of Brauer is in this terminilogy the statement that the defect set of the complex representation ring are the groups S x P, P a p-group, S cyclic.

6.3. Hyperelementary induction.

An induction theorem for a given Mackey functor is a theorem which computes its defect base or gives at least some restrictions on the defect base. We shall present one general result of this nature.

We begin with a result about restriction and induction for the

Burnside ring. Let N be a family of subgroups of G which is closed with respect to subgroups and conjugation. Let p a prime and define

(6.3.1) $N^p = \{ H < G \mid \exists K \triangleleft H \text{ with } K \in N \text{ and } |H/K| \text{ a power of } p \}$.

Let an index (p) denote localization at the prime ideal (p). Let Ke(N) denote the kernel of the restriction maps $A(G)_{(p)} \longrightarrow \prod_{H \in N} A(H)_{(p)}$ and let $\mathrm{Im}(N^p)$ denote the image of the sum of the induction maps $\bigoplus_{H \in N^p} A(H)_{(p)} \longrightarrow A(G)_{(p)}$. Then we have

Proposition 6.3.2.

$\mathrm{Ke}(N) + \mathrm{Im}(N^p) = A(G)_{(p)}$.

Proof.

$\mathrm{Ke}(N) + \mathrm{Im}(N^p)$ is an ideal of $A(G)_{(p)}$ because Ke(N) certainly is an ideal as a kernel of a ring homomorphism and for any Frobenius functor the image of an induction map is an ideal (use 6.2.1). If this ideal is different from $A(G)_{(p)}$ then there exists a maximal ideal q of $A(G)_{(p)}$ with $\mathrm{Ke}(N) + \mathrm{Im}(N^p) \subset q$. This ideal q has the form q = q(L,p), see 5. 7. 2 . Since $\mathrm{Ke}(N) \subset q$ this ideal extends to $\prod_{H \in N} A(H)$ (use e. g. Atiyah - Mac Donald [11] , 5.10), i. e. we may assume q = q(L,p) with $L \in N$. By 5. 7. 1 q(L,p) = q(K,p) where $G/K \notin q$ and by 5. 7. 9 $K \in N^p$. Hence G/K is the image of 1 under the induction map $A(K) \longrightarrow A(G)$. But $G/K \notin q$ contradicts $G/K \in \mathrm{Im}(N^p) \subset q$. Hence a q with $\mathrm{Ke}(N) + \mathrm{Im}(N^p) \subset q$ cannot exist.

Let now U be a Green functor G-Set $\longrightarrow$ Ab. As usual we denote U(G/H) for the G-set G/H by U(H). Let N and N^p be as above.

Theorem 6.3.3.

Assume that any torsion element in U(G) is nilpotent. Assume that the restriction map

$$U(G) \otimes \mathbb{Q} \longrightarrow \prod_{H \in N} U(H) \otimes \mathbb{Q}$$

is injective. Then the induction map

$$\bigoplus_{H \in N^p} U(H)_{(p)} \longrightarrow U(G)_{(p)}$$

is surjective.

Proof.

The injectivity and nilpotency hypothesis of the theorem imply that any element in the kernel of $U(G)_{(p)} \longrightarrow \prod_{H \in N} U(H)_{(p)}$ is nilpotent. By 6.3.2 we find $x \in Ke(N)$, $y \in Im(N^p)$ with $x + y = 1$ $A(G)_{(p)}$. Now apply the natural transformation $h : A(G)_{(p)} \longrightarrow U(G)_{(p)}$ of 6.2.4. Then $h(x) + h(y) = 1 \in U(G)_{(p)}$ and $h(x)$, contained in the kernel of $U(G)_{(p)} \longrightarrow \prod_{H \in N} U(H)_{(p)}$, is nilpotent. Therefore $h(y) = 1 - h(x)$ is a unit. But $h(y)$ is in the image of $\bigoplus_{H \in N^p} U(H)_{(p)} \longrightarrow U(G)_{(p)}$, so that this image being an ideal must be all of $U(G)_{(p)}$.

If $N = C$ is family of cyclic subgroups of G, then N^p is the family of p-*hyperelementary* subgroups of G. A subgroup is called *hyperelementary* if it is p-hyperelementary for some prime p. Let Hy be the class of hyperelementary subgroups of G.

Corollary 6.3.4.

If U(G) is torsion free and $U(G) \longrightarrow \prod_{H \in C} U(H)$ *is injective then* U *satisfies hyperelementary induction, i. e. the induction map*

$$\bigoplus_{H \in Hy} U(H) \longrightarrow U(G)$$

is surjective.

A particular example where the hypothesis of 6.3.4 is fulfilled is the Green functor "rational representation ring". By 6.2.2. any module over this Green functor also satisfies hyperelementary induction.

6.4. Comments.

This section is based on Dress [80], [81]. We refer to these papers for further details, in particular for the connection with classical induction theorems. The reader should also study Dress [80], § 7 in order to see a general construction of Mackey functors which works in most of the algebraic applications. As a research problem I suggest that the reader takes the double coset formula of 5.12 and develops induction theory for compact Lie groups in analogy to the theory in this section. For applications of induction theory in topology see the next section (also for compact Lie groups).

6.5. Exercises.

1. Make multiplicative induction (5.12) as part of a Mackey functor.
2. Let $(p) \subset Z$ be a prime ideal. What is the defect set of the localized Burnside ring functor $A(G)_{q(H,p)}$?
3. Provide the details in the proof of 6.2.3.