5. The Burnside-Ring of a Compact Lie Group.

5.1. Euler Characteristic.

We collect the properties of the Euler-Characteristic that we shall need in the sequel and indicate proofs when appropriate references cannot be given.

Let R be a commutative ring and let A be an associative R-algebra with identity (e.g. A = R; A = R[G], G a finite group). In general, an Euler-Poincaré map is a map from a certain category of A-modules to an abelian group which is additive on certain exact sequences. We consider the following sufficiently general situation:

Let $Gr^R(A)$ be the abelian group (Grothendieck group) with generators [M] where M is a left A-module which is finitely generated and projective as an R-module, with relations [M] = [M'] + [M"] for each exact sequence $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ of such modules. Let Gr(A) be the Grothendieck group of finitely generated left A-modules and the analogous relations for exact sequences. A ring R is called regular if it is noetherian and every finitely generated R-module has a finite resolution by finitely generated projective R-modules.

Proposition 5.1.1.

Let R be a regular ring and A an R-algebra which is finitely generated and projective as an R-module. Then the forgetful map $Gr^R(A) \longrightarrow Gr(A)$ is an isomorphism.

Proof.

Swan-Evans [158], p. 2. (The symbol G_0 is used in [158] where we use Gr. Since we do not need G_1 and use G to denote groups we have chosen this non-standard notation.)

Remark.

In the case of the group ring $A = S[\pi]$, S a commutative ring, we denote $Gr^S(A)$ by $R(\pi,S)$. Tensor product over S induces a multiplication and $R(\pi,S)$ becomes a commutative ring the representation ring of π over S.

We call the assignment $M \mapsto [M] \in Gr^R(A)$ a universal Euler-Characteristic for the modules under consideration, because any map $M \mapsto e(M)$, $e(M) \in B$, B an abelian group, such that $e(M) = e(M') + e(M'')$ whenever $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$, is induced from a unique homomorphism $h : Gr^R(A) \longrightarrow B$, $e(M) = h[M]$. (Similar definition for Gr(A).) If $R = A$ is a field then $M \longmapsto \dim_R M \in Z$ is such a universal map, establishing $Gr(R) \cong Z$. If $R = A = Z$ then $M \mapsto \mathrm{rank}(M) = \dim_Q (M \otimes_Z Q) \in Z$ is a universal Euler-Characteristic. (by 5.1.1 $Gr^Z(Z) = Gr(Z)$).

If $M \; : 0 \longrightarrow M_o \longrightarrow M_1 \longrightarrow \dots \longrightarrow M_n \longrightarrow 0$ is a complex of A-modules which are finitely generated and projective as R-modules then we define

$$(5.1.2) \qquad \chi(M_\bullet) = \sum_{i=o}^{n} (-1)^i [M_i] \in Gr^R(A)$$

to be the Euler-Characteristic of the complex. We use the same terminology in case of Gr(A). If submodules of finitely A-modules are again finitely generated then for the homology groups $H_i(M_\bullet)$ of a complex

$$\chi(H_\bullet(M_\bullet)) := \quad (M \;) \quad Gr(A).$$

If $0 \longrightarrow M'_\bullet \longrightarrow M_\bullet \longrightarrow M''_\bullet \longrightarrow 0$ is an exact sequence of complexes then

$$(5.1.4) \qquad \chi(M_\bullet) = \chi(M'_\bullet) + \chi(M''_\bullet)$$

when everything is defined. If one works with $Gr^R(A)$ then one has to use <u>hereditary rings</u> , i.e. submodules of projective modules are projective (see Cartan-Eilenberg [45] , p. 14 for this notion). Examples are <u>Dedekind rings</u> R, i. e. integral domains in which all ideals are projective (see Swan-Evans [158] , p. 212 for various characterisations of Dedekind rings).

We now consider the special cases that are relevant for topology. Let (Y,A) be a pair of spaces such that the (singular) homology groups with integral coefficients $H_i(Y,A)$ are finitely generated and zero for large i. Then, by abuse of language, we define the <u>Euler-Characteristic</u> $\chi(Y,A)$ <u>of the pair</u> (Y,A) to be the integer

$$\chi(Y,A) = \sum_{i \geq 0} (-1)^i \operatorname{rank} H_i(Y,A) \tag{5.1.5.}$$

with the usual convention $\chi(Y) = \chi(Y,\emptyset)$. Standard properties are (see Dold [75] , p. 105):

<u>Proposition 5.1.6.</u>

(i) *If two of the numbers $\chi(Y)$, $\chi(A)$ and $\chi(Y,A)$ are defined then so is the third, and*

$$\chi(Y) = \chi(A) + \chi(Y,A).$$

(ii) *If $(Y;Y_1,Y_2)$ is an excisive triad and if two of the numbers $\chi(Y_1 \cup Y_2)$, $\chi(Y_1 \cap Y_2)$, $\chi(Y_1) + \chi(Y_2)$ are defined then so is the third, and*

$$\chi(Y_1) + \chi(Y_2) = \chi(Y_1 \cup Y_2) + \chi(Y_1 \cap Y_2).$$

(iii) *If (Y,A) is a relative CW-complex with Y-A containing many cells* then $\chi(Y,A)$ is defined and

$$\chi(Y,A) = \sum_{i \geq 0} (-1)^i n_i$$

where n_i *is the number of i-cells in* Y − A.

If F is a field we can consider

$$(5.1.7) \qquad \chi(Y,A;F) = \sum_{i \geq 0} (-1)^i \dim_F H_i(Y,A;F),$$

if this number is defined. Then 5.1.6 also holds with this type of Euler-Characteristic.

Proposition 5.1.8.

(i) *If* F *has characteristic zero then* $\chi(Y,A)$ *is defined if and only if* $\chi(Y,A;F)$ *is defined and* $\chi(Y,A) = \chi(Y,A;F)$.

(ii) *If* $\chi(Y,A)$ *is defined and* (Y,A) *has finitely generated integral homology then* $\chi(Y,A;F)$ *is defined for any field and* $\chi(Y,A) = \chi(Y,A;F)$.

Proof. This is a simple application of the universal coefficient formula. (See Dold [75], p. 156).

One can also define the Euler-Characteristic using various types of cohomology (singular-, Alexander-Spanier-, sheaf-, etc.) and use the universal coefficient formulas to see that homology and cohomology gives the same result under suitable finiteness conditions.

Proposition 5.1.9.

Let $p : E \longrightarrow B$ *be a Serre-fibration with typical fibre* F. *If* $\chi(B)$ *and* $\chi(F)$ *are defined and the local coefficient system* $(H_*(p^{-1}b;Q))$ *is trivial then* $\chi(E)$ *is defined and*

$$\chi(E) = \chi(F) \ \chi(B).$$

Proof.

Use the existence of the Serre spectral sequence; apply the Künneth-formula to the E_2-term; use 5.1.3 (see Spanier [152] , p. 481).

We actually need a more general result where fibrations are replaced by relative fibrations and the coefficient system may be non-trivial. This will be done in the next section when a suitable class of spaces with Euler-Characteristic (the Euclidean neighbourhoods retracts) has been described. A really general and satisfactory treatment of the Euler-Characteristic (and its generalization: the Lefschetz number) does not seem to exist.

5.2. Euclidean neighbourhood retracts.

We single out a convenient class of G-spaces X such that for all fixed point sets and other related spaces the Euler-Characteristic is defined.

Let G be compact Lie group. We define a G-ENR (Euclidean Neighbourhood Retract) to be a G-space X which is (G-homeomorphic to) a G-retract of some open G-subset in a G-module V.

Proposition 5.2.1.

If X is a G-ENR and i : X ⟶ W a G-embedding into a G-module W then iX is a G-retract of a neighbourhood.

Proof. As in Dold [75] , p. 81, using the Tietze-Gleason extension theorem (Bredon [37] , p. 36; Palais [124] , p. 19).

Proposition 5.2.2.

A differentiable G-manifold with a finite number of orbit types is a G-ENR.

Proof.

Embed the manifold differentiably into a G-module (Wasserman [165]) where it is a retract of a G-invariant neighbourhood.

If we have no group G acting we simply talk about ENR's. The following basic result of Borsuk shows that being an ENR is a local property. Recall that a space X is called locally contractible if every neighbourhood V of every point $x \in X$ contains a neighbourhood W of x such that $W \subset V$ is nullhomotopic fixing x. It is easy to see that an ENR is locally contractible (Dold [75], p. 81). A space is locally n-connected if every neighbourhood V of every point x contains a neighbourhood W such that any map $S^j \longrightarrow W$, $j \leq n$, is nullhomotopic in V.

Proposition 5.2.3.

If $X \subset \mathbb{R}^n$ is locally (n-1)-connected and locally compact then X is an ENR.

Proof.

Dold [75], IV 8.12, and 8.13 exercise 4.

Remarks 5.2.4.

A basic theorem of point set topology says that a separable metric space of (covering) dimension $\leq n$ can be embedded in $\mathbb{R}^{2n+1}$; see Hurewicz-Wallman [38] for the notion of dimension and this theorem. Hence a space is an ENR if and only if it is locally compact, separable metric, finite-dimensional and locally contractible. Using a local Hurewicz-theorem (Raußen [131]) one can express the local contractibility in terms of homology conditions.

Proposition 5.2.5.

Let X be a G-ENR. Then the orbit space X/G is an ENR.

Proof.

Let $X \xrightarrow{i} U \xrightarrow{r} X$ be a presentation of X as a neighbourhood retract (i.e. U open G-subset in a G-module, ri = id). We pass to orbit spaces. A retract of an ENR is an ENR. Hence we have to prove the Proposition for X a differentiable G-manifold (and then apply it to the manifold U). Let $p : X \longrightarrow X/G$ be the quotient map. Given $x \in V \subset X/G$, V open, $p^{-1}V$ contains a G-invariant tubular neighbourhood W of the orbit $p^{-1}x$. Hence pW is contractible. Therefore X/G is locally contractible. Moreover X/G is locally compact (Bredon [37], p. 38), separable metric (Palais [124], 1.1.12) and dim X/G $\leq$ dim X (use Hurewicz-Wallman [38]). Now apply 5.2.3, and 5.2.4.

Using 5.2.3 and the following result of Jaworowski we see that being a G-ENR is a local property too.

Proposition 5.2.6.

Let X be a G-space which is separable metric and finite-dimensional. Then X is a G-ENR if and only if X is locally compact, has a finite number of orbit types, and for every isotropy group H < G the fixed point set X^H is an ENR.

Proof.

Jaworowski [102].

Corollary 5.2.7.

If X is a G-ENR then $X_{(H)}$ is a G-ENR for every H < G.

Proposition 5.2.8.

If X is a compact ENR then the Euler-Characteristic $\chi(X)$ *is defined.*

Proof.

X is a retract of a space K which may be given as a finite union of cubes in a Euclidean space. Hence $H_i X$ is a direct summand in $H_i K$, which is finitely generated and zero for large i.

Proposition 5.2.9.

Let $E \longrightarrow B$ *be a fibre bundle with typical fibre F. If F and B are ENR's then E is an ENR.*

Proof.

Apply 5.2.3.

We now come to the generalization of 5.1.9.

Proposition 5.2.10.

Let $F : (X,A) \longrightarrow (Y,B)$ *be a continuous map between compact ENR's such that* $F(X \setminus A) = Y \setminus B$. *Suppose the induced map* $f : X \setminus A \longrightarrow Y \setminus B$ *is a fibration with typical fibre Z a compact ENR. Then*

$$\chi(X,A) = \chi(Z) \quad \chi(Y,B).$$

The Euler-Characteristic $\chi_c(X \setminus A)$ *of* $X \setminus A$ *computed with Alexander-Spanier cohomology with compact support and coefficients in a field exists and* $\chi(X,A) = \chi_c(X \setminus A)$.

Proof.

Since the integral homology groups are finitely generated, we can compute the Euler-Characteristic using any field of coefficients and homology or cohomology. We use cohomology with Z/2-coefficients. Since ENR's are locally contractible, 5.2.3, we can use singular or Alexander-Spanier cohomology (Spanier [152], 6.9.6.). Using Alexander-Spanier cohomology with compact support we have by Spanier [152], 6.6.11, that

$$H_c^i(X,A) = H_c^i(X \setminus A)$$

and similarly for (Y,B). The fibration $f : X \setminus A \longrightarrow Y \setminus B$ gives us a Leray spectral sequence with E_2-term

$$E_2^{p,q} = H_c^p(Y \setminus B;\ H_c^q(Z))$$

where the coefficients are $H_c^q(Z)$ considered as a local coefficient system on $Y \setminus B$ (Borel [25], XVI. 4.3; [27]). If this local coefficient system is trivial then our assertion follows as in 5.1.9. If it is non-trivial then the following ad hoc argument of Becker and Gottlieb reduces it to the case of a trivial coefficient system: Since $H_c^q(Z)$ is a finite group (Z/2 coefficients!) a finite covering of $Y \setminus B$ will make the coefficient system trivial. The relation $\chi(U') = N\ \chi(U)$ for a finite covering $U' \longrightarrow U$ of degree N (which will be proved in 5.3) and the result for trivial coefficients implies $\chi_c(X \setminus A) = \chi(Z)\ \chi_c(Y \setminus B)$.

Problem 5.2.11.

Give a satisfactory and general (not just for ENR's) proof for 5.2.10 and its generalization to Lefschetz numbers (compare Dold [77]).

Proposition 5.2.11.

Finite G-CW-complexes are G-ENR's.

Proof.

See Illman [100] for the notion of G-CW-complexes.

Use 5.2.3, 5.2.6.

5.3. Equivariant Euler-Characteristic.

If G is a compact Lie group and X is a G-space then the G-action on X induces a G-action on the cohomology groups $H^i(X;M)$ where M is an R-module. If G_o is the component of the identity of G then G_o acts trivially on $H^i(X;M)$ so that $H^i(X;M)$ becomes an $R[G/G_o]$-module. If $H^*(X;M) = (H^i(X;M))_{i\geq o}$ is R-finite, i. e. zero for large i and finitely generated as R-module, then we define the equivariant Euler-Characteristic of the G-space X to be the element

$$(5.3.1) \qquad \chi_G(X;R) = \sum_{i\geq o} (-1)^i H^i(X;R) \in Gr(R[G/G_o]).$$

If $R = \mathbb{C}$, the complex numbers, then $\chi_G(X;\mathbb{C}) \in R(G)$, where R(G) denotes the complex representation ring. We use similar definitions for pairs of G-spaces and homology. Actually for general spaces one has to specify the cohomology theory. For simplicity we make the following

Assumption 5.3.2:

X is a G-ENR. Cohomology is Alexander-Spanier cohomology with compact support (in this case isomorphic to sheaf- or presheaf cohomology with compact support; see Spanier [152], Chapter 6; Bredon [35], Chapter III).

Our task in this section is the computation of (5.3.1) in case R is the field of rational numbers. The computation will be in terms of non-equivariant Euler-Characteristics. The reader should convince himself that most of the results to follow are obvious if a finite group acts simplicially on a finite complex. In this case one can compute on the chain level.

Proposition 5.3.3.

Let G be a p-group acting freely on X. Suppose $H^*(X;F_p)$ is F_p-finite. Then $\chi(X/G;F_p)$ is defined and

$$\chi(X;F_p) = |G|\, \chi(X/G;F_p).$$

(*Recall 5.3.2 and that χ is defined using cohomology with compact support.*)

Proof.

If $H \triangleleft G$ then G/H acts freely on the G/H-ENR (by 5.2.5, 5.2.6) X/H. Hence using induction on the order of G it is sufficient to prove the Proposition for G = Z/p. We use the following fact:

(5.3.4) $$H^i(X;F_p) \cong H^i(X/G;A)$$

where A is the local coefficient system (= locally constant sheaf, Spanier [152], p. 360) with stalks $H^0(\pi^{-1}(x);F_p) \cong F_p[G]$, $\pi: X \to X/G$ the quotient map. In our case the group action on $H^i(X;F_p)$ corresponds via 5.3.4 to the group action on the coefficient system, which is a system of $F_p[G]$-modules (for a verification see Floyd [83] , III. 1). Since an $F_p[G]$-module always contains non-trivial G-fixed submodules if G is a p-group (e.g. by 1. 3.) we can find a filtration $A = A_1 \supset A_2 \supset \dots \supset A_k = 0$ of the coefficient system such that A_i/A_{i+1} is the constant system. The Cartan spectral sequence of a covering (Bredon [35] , p. 154) shows $H^i(X/G;F_p)$ to be finite dimensional. From the additivity of the Euler-Characteristic $\chi(X/G;A_i) = \chi(X/G;A_{i+1}) + \chi(X/G;A_i/A_{i+1})$ we obtain the result.

Proposition 5.3.4.

Let the finite group G act freely on X. Suppose $H^*(X;Z)$ is Z-finite. Then $\chi(X/G;Q)$ is defined and

$$\chi_G(X;Q) = \chi(X/G;Q)\cdot Q[G] \in R(G;Q) .$$

(*Here* $Q[G]$ *denotes the regular representation of* G *over* Q.)

Proof.

Two elements of R(G;Q) are equal if their characters are equal. Thus the assertion of the Proposition is equivalent to:

(5.3.5) $$\chi(X) = |G|\,\chi(X/G),$$

(5.3.6) $$\chi_G(X)(g) = 0 \qquad \text{for } g \neq 1.$$

(Note that $\chi_G(X)(g)$ is the Lefschetz-number

$$L(g,X) = \sum_{i\geq 0}(-1)^i \,(\mathrm{Trace}\,(g,H^i(X;Q))$$

for the action of g; and under reasonable circumstances the Lefschetz-number of a map without fixed points should be zero.)

We first prove 5.3.5 and 5.3.6 for cyclic groups. Since $H^*(X;Z)$ is finite the universal coefficient formula for cohomology with compact support (Spanier [152] , p. 338) shows

(5.3.7) $$\chi(X;Q) = \chi(X;F_p).$$

The Cartan spectral sequence of a covering shows that $H^*(X/G;Z)$ is Z-finite. Hence we obtain from 5.3.3 and 5.3.7, using induction on $|G|$, that 5.3.5 is true for cyclic G.

The existence of the transfer for finite groups implies the isomorphism (Bredon [37], III 7.2)

(5.3.8) $$H^i(X,Q)^G \cong H^i(X/G;Q) .$$

Since for any character ψ of G dim $\psi^G = |G|^{-1} \sum \psi(g)$ we obtain from 5.3.5 and 5.3.8

(5.3.9) $$\sum_{g\neq 1} \chi_G(X)(g) = 0 .$$

Using this we prove 5.3.6 for cyclic groups by induction over the group order: We start with

$$H^i(X,\mathbb{C}) \cong H^i(X/G;A)$$

where A again is the local coefficient system with typical stalk $\mathbb{C}[G]$. Let g be a generator of G. We decompose the coefficient system A according to the irreducible $\mathbb{C}[G]$-modules

$$A = \oplus A_j , \quad 0 \leq j < m = |G|$$

where g acts on A_j through multiplication with $\zeta^j = \exp(2\pi ij/m)$. The equalities

$$\mathrm{Tr}(g^k,H^i(X;\mathbb{C})) = \sum_j \mathrm{Tr}(g^k,H^i(X/G;A_j))$$

$$= \sum_j \zeta^{jk} \dim H^i(X/G;A_j)$$

yield for the Lefschetz-number

$$L(g^k,X) = \Sigma_j \ \zeta^{jk} \chi(X/G;A_j).$$

But $L(g^k,X) \in Z$ is obtained from $L(g,X)$ for $(k,m) = 1$ by applying a Galois automorphism of $Q(\zeta)$ over Q. Therefore $L(g^k,X) = L(g,X)$ for $(k,m) = 1$. From 5.3.9 we obtain

$$(5.3.10) \qquad 0 = \sum_{(k,m)=1} L(g^k,X) + \sum_{(k,m)\neq 1} L(g^k,X)$$

By the inductive assumption the second sum in 5.3.10 is zero, and since the summands of the first sum are all equal we see that $L(g,X) = 0$. This proves 5.3.6 in general. Again using 5.3.8 and 5.3.9 we obtain 5.3.5 for general G.

We have actually proved in 5.3.4 a special case of the Lefschetz fixed point theorem.

<u>Proposition 5.3.11.</u>

Let X be a compact G-ENR where G is a cyclic group with generator g. Then the Lefschetz number

$$L(g,X) = \Sigma_{i\geq 0}(-1)^i \text{ Trace } (g,H^i(X;Q))$$

is equal to the Euler-Characteristic $\chi(X^g)$.

<u>Proof.</u>

Let $X_1 = X^g$, $X_2,\ldots,X_r$ be the orbit bundles of X. Then $H^*(X_i,Z)$ (cohomology with compact support) is Z-finite and $L(g,X_j) = 0$ for $j > 1$ by 5.3.4. Hence

$$L(g,X) = \Sigma_j L(g,X_j) = L(g,X_1)$$

and clearly $L(g,X_1) = \chi(X^g)$.

<u>Corollary 5.3.12.</u>

Let G be a finite group and let X be a compact G-ENR. Then

$$\chi(X/G) = |G|^{-1} \sum_{g\in G} \chi(X^g).$$

<u>Proof.</u>

From $H^i(X/G) \cong H^i(X)^G$ and $\dim H^i(X)^G = |G|^{-1} \sum_{g\in G} \mathrm{Trace}\,(g,H^i(X))$ the result follows, using 5.3.11.

We can now compute the equivariant Euler-Characteristic $\chi_G(X)$.

<u>Theorem 5.3.13.</u>

Let G be a compact Lie group and X be a compact G-ENR. Then

$$\chi_G(X) = \sum_{(H)} \chi(X_{(H)}/G)\ \chi_G(G/H)$$

where the sum is taken over those isotropy types (H) of X such that NH/H is finite.

<u>Proof.</u>

By additivity of the Euler-Characteristic

$$\chi_G(X) = \sum_{(H)} \chi_G(X_{(H)}).$$

Thus we have to show: $\chi_G(X_{(H)}) = 0$ if NH/H is infinite and

(5.3.14) $$L(g,X_{(H)}) = \chi(X_{(H)}/G)\ L(g,G/H)$$

otherwise ($g \in G$). Let C be the closed subgroup of G generated by g.

Since $L(g,Y)$ only depends on the image of g in the group of components of C we can find an element $h \in C$ of finite order such that $L(g,Y)=L(h,Y)$ for all Y. We fix h with this property. Since X is a compact G-ENR we can find compact G-ENR's $Y \supset Z$ in X such that $Y \setminus Z = X_{(H)}$. The proof of 5.3.11 shows

$$L(h,X_{(H)}) = \chi(X_{(H)}{}^{h}).$$

Using the fibre bundle

$$G/H \longrightarrow X_{(H)} \longrightarrow X_{(H)}/G$$

and 5.2.10 we obtain

$$\chi(X_{(H)}{}^{h}) = \chi(X_{(H)}/G) \quad \chi(G/H^{h}).$$

Again by 5.3.11 $\chi(G/H^{h}) = L(g,G/H)$, so we see that 5.3.14 is true in general. But $\chi(G/H^{h}) = 0$ if NH/H is infinite because NH/H acts freely on G/H^{h}.

Remark 5.3.15.

If G is finite then $\chi_G(G/H)$ is just the permutation representation associated to the G-set G/H. In general $\chi_G(G/H) \in R(G/G_o;\mathbb{Q})$ where G_o is the component of the identity of G. We would like to see that this is actually a permutation representation.

Problem 5.3.16.

What are the most general assumptions on the spaces which imply the decomposition formula 5.3.13? A similar formula holds for the equivariant Lefschetz number of a G-map $f : X \longrightarrow X$ between compact G-ENR's. Also this should be generalized to more general spaces.

5.4. Universal Euler-Characteristic for G-spaces.

The classical computation of the Euler-Characteristic from a cell decomposition of a space indicates that suitable axioms (like 5.16 (i), (ii)) determine the Euler-Characteristic uniquely. This is carried out in Watts [166] . We present a similar argument for G-spaces without insisting on a minimal set of axioms.

An Euler Characteristic for finite G-CW-complexes consists of an abelian group A and map b which associates to each finite CW-complex X an element $b(X) \in A$ such that:

(i) If X and Y are G-homotopy-equivalent then $b(X) = b(Y)$.

(ii) If X and Y are subcomplexes of Z then

$b(X) + b(Y) = b(X \cup Y) + b(X \cap Y)$.

Given such an Euler-Characteristic b we show

Proposition 5.4.1.

Let X be a finite G-CW-complex. Then

$$b(X) = \Sigma_{(H)} \; n_H \, b(G/H)$$

where

$$n_H = \Sigma_{i \geq 0} \, (-1)^i \, n(H,i)$$

n(H,i) the number of i-cells of type (H), and the sum is taken over conjugacy classes of subgroups of G.

Proof.

Induction on the number of cells and dimension. Let $Z = X \cup (G/H \times e^n)$ be obtained from X by attaching an n-cell of type (H). Let $Y = G/H \times D^n(1/2)$

be the closed cell in $G/H \times e^n$ of radius 1/2. If we remove Y from Z then the resulting space is G-homotopy-equivalent to X. Therefore

$$b(Z) = b(X) + b(G/H \times D^n) - b(G/H \times S^{n-1}).$$

One shows by induction

$$b(G/H \times S^n) = 1 + (-1)^{n+1}\, b(G/H);$$

namely if D_+ and D_- are the upper and lower hemisphere of S^n respectively then

$$b(G/H \times S^n) = b(G/H \times D_+) + b(G/H \times D_-) - b(G/H \times S^{n-1})$$

$$= 2b(G/H) - (1+(-1)^n)\, b(G/H)$$

$$= 1 + (-1)^{n+1}\, b(G/H).$$

Put together we obtain

$$b(Z) = b(X) + (-1)^n\, b(G/H),$$

the induction step.

An Euler-Characteristic (U(G),u) for finite G-CW-complexes is called universal, if every Euler-Characteristic (A,b) as above is obtained from (U(G),u) by composing with a unique homomorphism $U(G) \longrightarrow A$. As usual for universal objects uniqueness up to isomorphism follows.

From 5.4.1 we obtain existence:

(5.4.2) U(G) free abelian group with basis

$$[G/H] \; , \; (H) \in C(G).$$

$$u(X) = \sum_{(H)} n_{(H)} [G/H] .$$

Instead of u(X) we also write [X] , in accordance with the notation [G/H] for the basis elements. We now aim at another characterisation of U(G) which is not based an CW-complex and which shows that b(X) in 5.4.1 is independent of the cell decomposition.

Proposition 5.4.3.

We have [X] = [Y] *in* U(G) *if and only if for all* $H < G$

$$\chi(X^H/NH) = \chi(Y^H/NH).$$

Proof.

Suppose [X] = [Y] . We consider the mapping

$$b_H : Z \longmapsto \chi(Z^H/NH)$$

from finite G-CW-complexes into Z. This mapping satisfies (i) and (ii) in the definition of an Euler-Characteristic for finite G-CW-complexes. From the universal property of U(G) we obtain $b_H(X) = b_H(Y)$. For the converse be have to show that the totallity of maps $b_H : U(G) \longrightarrow Z$ defines an injective map $U(G) \longrightarrow \pi_{(H)} Z$. Let $0 \neq x = \sum a_H [G/H] \in U(G)$. Let H be maximal such that $a_H \neq 0$. Then

$$b_H(x) = a_H \; \chi((G/H^H)/NH) = a_H \neq 0.$$

We now redefine the group U(G).

Definition and Proposition 5.4.4.

On the set of compact G-ENR introduce the equivalence relation: $X \sim Y \Leftrightarrow$ for all $H < G$ the equality $\chi(X^H/NH) = \chi(Y^H/NH)$ holds. Let U(G) be the set of equivalence classes and let $[X] \in U(G)$ be the class of X. Disjoint union induces on U(G) the structure of an abelian group. This group is free abelian with basis $[G/H]$, $H \in C(G)$. We have

$$(5.4.5) \qquad [X] = \Sigma_{(H)} \; \chi_c \, (X_{(H)}/G) \; [G/H] \, .$$

Proof.

We have to show that inverses exist for addition. Let K be a compact ENR with trivial G-action and $\chi(K) = -1$. Then $[X] + [K \times H] = 0$ in U(G) because $\chi(X^H) + \chi((K \times H)^H) = 0$ for all $H < G$. As in the proof of 5.4.3 one shows that the $[G/H]$ are linearly independent. We show that the $[G/H]$ span U(G) by proving 5.4.5. By additivity of the Euler-Characteristic we have

$$\chi(X^K/NK) = \Sigma_{(H)} \; \chi_c(X_{(H)}{}^K/NK) \, .$$

Now $X_{(H)} \longrightarrow X_{(H)}/G$ is a fibre bundle with fibre G/H and as G-space $X_{(H)}$ has the form $G/H \times_{NH} X_H$ (see Bredon , p. 88). Hence $X^K_{(H)}/NK \longrightarrow X_{(H)}/G$ is a fibre bundle with fibre $G/H^K/NK$. From we obtain

$$\chi_c(X_{(H)}{}^K/NK) = \chi((G/H^K)/NK) \;\; \chi_c(X_{(H)}/G) \, .$$

This shows that both sides of 5.4.5 describe the same element in U(G).

Definition and Proposition 5.4.6.

Cartesian product of representatives induces a multiplication on U(G). Addition and multiplication make U(G) into a commutative ring with identity. This ring is called the Euler-ring of the compact Lie group G.

Proof.

We need only show that multiplication is well-defined, i. e. we have to show that the numbers $\chi((X \times Y)^K/NK)$ can be computed from the $\chi(X^H/NH)$, $\chi(Y^H/NH)$ or, equivalently, from the $\chi_c(X_H/NH)$, $\chi_c(Y_H/NH)$. We begin with

$$\chi((X \times Y)^K/NK) = \Sigma_{(H)} \chi_c((X \times Y_{(H)})^K/NK).$$

The map

$$(X \times Y_{(H)})^K/NK \longrightarrow Y_{(H)}/G$$

is a fibre bundle with fibre

$$(X^K \times G/H^K)/NK.$$

Now we use the fact that G/H^K consists of a finite number of NK-orbits (Bredon [37] , p. 87), say

$$G/H^K = \Sigma_U NK/U$$

as NK-space. Using this information and 5.2.10 we obtain

$$\chi_c((X \times Y_{(H)})^K/NK) = \Sigma_U \chi_c(Y_{(H)}/G) \quad \chi(X^K/U).$$

Finally, using

$$\chi(X^K/U) = \Sigma_{(H)} \chi_c(X_{(H)}/G) \chi((G/H^K)/U),$$

we see that $\chi((X \times Y)^K/NK)$ can be computed from the $\chi_c(X_H/NH)$, $\chi_c(Y_H/NH)$.

We show in the next section that for finite G U(G) is the Burnside ring of G. For non-finite G U(G) contains nilpotent elements. In order to obtain the product structure one has to compute $[G/H]\ [G/K]$.

Proposition 5.4.7.

Suppose NH/H is not finite. Then $[G/H] \in U(G)$ is nilpotent.

Proof.

By the descending chain condition for subgroups of G the spaces G/H^k, $k \geq 1$, altogether only contain a finite number of isotropy groups. If $[G/H]^k = \sum_{(K)} a_K [G/K]$ with $a_K \neq 0$ and (K) maximal with this property then $[G/H]^{k+1}$ does not contain $[G/K]$ with a non-zero coefficients: Expanding $[G/H]^{k+1}$ then G/K could only occur from the expansion of $a_K\ [G/H]\ [G/K]$. But $(G/H \times G/K)_K = G/H^K \times NK/K$ and therefore $\chi_c((G/H \times G/K)_K/NK) = \chi(G/H^K) = 0$ because NH/H acts freely on G/H^K and $\chi(NH/H)$ is zero if NH/H is not finite (e.g. because a circle group acts freely on NH/H).

5.5. The Burnside ring of a compact Lie group.

Let G be a compact Lie group. On the set of compact G-ENR's consider the equivalence relation: $X \sim Y \Leftrightarrow$ for all $H < G$ the Euler-Characteristics $\chi(X^H)$ and $\chi(Y^H)$ are equal. Let A(G) be the set of equivalence classes and let $[X] \in A(G)$ be the class of X. Disjoint union and cartesian product induce composition laws addition and multiplication, respectively, on A(G). It is easy to see that A(G) with these composition laws is a commutative ring with identity. We call A(G) the Burnside ring of G. We will show in a moment that this definition is consistent with the earlier one of section 1. (for finite G).

Let $\phi(G)$ be the set of conjugacy classes (H) such that NH/H is finite.

Proposition 5.5.1.

Additively, A(G) is the free abelian group on $[G/H]$, $(H) \in \phi(G)$. For a compact G-ENR X we have the relation

$$[X] = \Sigma_{(H) \in \phi(G)} \chi(X_{(H)}/G)\,[G/H] .$$

The assignment $X \longmapsto \chi(X^H)$ induces a ring homomorphism $\varphi_H : A(G) \longrightarrow Z$.

Proof. (Compare 5.4.4).

The last assertion is obvious from the definition. The $[G/H]$, $(H) \in \phi(G)$, are linearly independent: Given $x = \Sigma\, a_H\,[G/H] \in A(G)$. Choose (H) maximal such that $a_H \neq 0$. Then

$$\varphi_H\, x = a_H\, \chi(G/H^H) = a_H\, |NH/H| \neq 0$$

and therefore $x \neq 0$.

Given a comapct G-ENR X. Then

$$\chi(X^K) = \Sigma_{(H)}\, \chi({X_{(H)}}^K) = \Sigma\, \chi(G/H^K)\, \chi(X_{(H)}/G).$$

The summands with NH/H not finite vanish, because NH/H then acts freely on G/H^K so that $\chi(G/H^K) = 0$. This computation shows that $[X]$ and $\Sigma\, \chi(X_{(H)}/G)\,[G/H]$ have the same image under φ_K, $(K) \in \phi(G)$, hence are equal in A(G).

The map

$$v : U(G) \longrightarrow A(G) : [X] \longrightarrow [X]$$

is a well-defined ring homomorphism. By 5.5.1 and 5.4.4 it is surjective, and bijective for finite G. In particular we have

Proposition 5.5.2.

For finite G the rings U(G), A(G), and the Burnside ring of finite G-sets are canonically isomorphic.

Proposition 5.5.3.

The kernel of v : U(G) ⟶ A(G) is the nilradical (= set of nilpotent elements) of U(G).

Proof.

Since the φ_H : A(G) ⟶ Z detect the elements of A(G) the ring A(G) cannot have nilpotent elements (different from zero). Now use 5.4.4, 5.4.7, and 5.5.1.

Remark 5.5.4.

The previous Proposition implies in particular that U(G) and A(G) have the same prime ideal spectrum.

Remark 5.5.5.

In contrast to the situation in section 1 with our new definition of A(G) also the negatives of all elements are represented by geometric objects.

We now give some immediate applications of the geometric definition of A(G).

Recall that we have in 5.3 associated with every compact G-ENR X the equivariant Euler-Characteristic

$$\chi_G(X) = \sum_{i \geq 0} (-1)^i H^i(X;Q) \in R(G;Q).$$

Proposition 5.5.6.

The assignment $X \longmapsto \chi_G(X)$ induces a ring homomorphism

$$\chi_G : A(G) \longrightarrow R(G;Q).$$

Proof. In order to show χ_G is well-defined we have to show that the character $\chi_G(X)$ can be computed from Euler-Characteristics of fixed point sets. But this is the content of 5.3.11, and the same Proposition shows that χ_G respects addition and multiplication.

Remark 5.5.7.

The homomorphism χ_G generalizes the permutation representation of finite G-sets.

We have mentioned in 1.5 the construction of units of A(G) using representations. We can now make this precise.

The homomorphisms $\varphi_H : A(G) \longrightarrow Z$ combine to an injective (by definition A(G)) ring homomorphism

(5.5.8) $$\varphi : A(G) \longrightarrow \prod_{(H)} Z$$

where the product is taken over the set C(G) of conjugacy classes of closed subgroups of G. We use φ to identify elements of A(G) with functions $C(G) \longrightarrow Z$ (see 5.6 for an elaboration of this point of view).

Proposition 5.5.9.

Let V be a real representation of G. Then $u(V) : (H) \longmapsto (-1)^{\dim V^H}$ is a unit of A(G). The assignment $V \longmapsto u(V)$ induces a homomorphism

$$u : R(G;\mathbb{R}) \longrightarrow A(G)^*.$$

(*Here* R(G;ℝ) *is the real representation ring of* G, *also denoted* RO(G).)

Proof.

Let S(V) be the unit sphere in V. Then

$$\chi(S(V)^H) = 1 - (-1)^{\dim V^H}.$$

Hence $1 - [SV] \in A(G)$ represents the function u.

Proposition 5.5.10.

The multiplication table of the $[G/H] \in A(G)$ *has non-negative* coefficient, *i.e. if*

$$[G/H]\ [G/K] = \Sigma_{(L)}\ n_L\ [G/L]$$

then $n_L \geq 0$.

Proof.

We have $n_L = \chi((G/H \times G/K)_{(L)}/G)$.

Moreover

$$\begin{aligned}(G/H \times G/K_{(L)}/G &\cong (G/H \times G/K)_L/NL \\ &\subset (G/H \times G/K)^L/NL\ .\end{aligned}$$

But by Bredon [37] , II. 5.7, the space G/H^L consists of finitely many NL/L-orbits. Since NL/L is finite the set $(G/H \times G/K)_{(L)}/G$ is finite and its Euler-Characteristic therefore non-negative.

5.6. The space of subgroups.

We recall some notions from point set topology. Let E be a metric space with bounded metric d. Let F(E) be the set of non-empty subsets of E.

On F(E) one has the Hausdorff metric h defined by

$$h(A,B) = \max(r(A,B),\ r(B,A))$$

with
$$r(A,B) = \sup(d(x,B) \mid x \in A).$$

If E is complete then F(E) is complete. If E is compact then F(E) is compact.

The convergence of a sequence X_i to the limit X can be expressed as follows: For any $\varepsilon > 0$ there exists n_o such that for $n > n_o$:

(a) for $x \in X_n$ there exists $y \in X$ with $d(x,y) < \varepsilon$.

(b) for $x \in X$ there exists $y \in X_n$ with $d(x,y) < \varepsilon$.

If Y_n is the closure of $\bigcup_{p \geq o} X_{n+p}$ then X is the intersection of the Y_n.

We want to use this metric on the set S(G) of closed subgroups of the compact Lie group G.

Proposition 5.6.1.

(i) S(G) is a closed (hence compact) subset of F(G).

(ii) The action $G \times S(G) \longrightarrow S(G) : (g,H) \longmapsto gHg^{-1}$ is continuous. The quotient space C(G) is a countable, hence a totally disconnected, compact Hausdorff space.

(iii) $\Phi(G) \subset C(G)$ is a closed subspace.

Proof.

(i) We start with a bi-invariant metric d on G. Let $X = \lim H_i$, $H_i \in S(G)$. Given $x,y \in X$, $\varepsilon > 0$, choose n_o such that for $n > n_o$ there exists x_n, $y_n \in H_n$ with $d(x,x_n) < \varepsilon/2$, $d(y,y_n) < \varepsilon/2$. Then $d(xy^{-1}, x_n y_n^{-1}) < \varepsilon$. If $xy^{-1} \notin X$ then $X \cup \{xy^{-1}\}$ would satisfy conditions (a), (b) above, a contradiction.

(ii) Let $\lim g_i = g$ in G and $\lim H_i = H$ in S(G). Using $d(g_n x_n g_n^{-1}, gxg^{-1}) \leq 2d(g,g_n) + d(x,x_n)$, which follows from the triangle inequality and bi-invariance, one shows that gHg^{-1} is precisely the set of points satisfying (a) and (b) above for the sequence $g_n H_n g_n^{-1}$. The space C(G) is countable: see Palais [124], 1.7.27.

(iii) We show that $S_o(G) = \{H \mid NH/H \text{ finite}\}$ is closed in S(G). Let $H = \lim H_i$, $H_i \in S(G)$. By a theorem of Montgomery and Zippin (Bredon [37], p. 87) there exists an $\varepsilon > 0$ such that any subgroup in the ε-neighbourhood of H is conjugate to a subgroup of H. Hence the H_i are eventually conjugate to subgroup of H. But if $K \in S_o(G)$ and $K < H$ then $H \in S_o(G)$; this follows e.g. from Bredon [37], II. 5.7, because G/H^K consists of finitely many NK/K-orbits hence is a finite set with free NH/H-action.

We now show that convergence in S(G) and C(G) is equivalent in the following sense.

<u>Proposition 5.6.2.</u>

<u>Let</u> (H) = lim (H_i) <u>in</u> C(G). <u>There exists an</u> n_o <u>and</u> $K_n \in S(G)$, $n \geq n_o$, <u>such that</u> $(K_n) = (H_n)$, $K_n < H$, $\lim K_n = H$.

<u>Proof.</u>

By the theorem of Montgomery and Zippin (Bredon [37], II. 5.6) we can find for each $\varepsilon > 0$ an integer $n_o(\varepsilon)$ such that for $n > n_o(\varepsilon)$ there exists an u_n with $d(u_n, 1) < \varepsilon$ and $u_n H_n u_n^{-1} < H$. Therefore we can find a sequence $g_n \in G$ converging to 1 such that for almost all n $g_n H_n g_n^{-1} < H$.

In view of the preceding Proposition it is interesting to know which compact Lie groups G are limits of a sequence of proper subgroups.

Proposition 5.6.3.

G *is a limit of proper subgroups if and only if* G *is not semi-simple*.

Proof.

Suppose $G = \lim H_n$, $H_n \neq G$. Let G^o be the component of 1 of G and put $K_n = G^o \cap H_n$. Then $\lim K_n = G^o$ so that without loss of generality we can assume that G is connected. By passing to a subsequence we can assume that the components H_n^o converge to H and therefore must have eventually the same dimension as H. But then the H_n^o are conjugate to H and by conjugating the whole sequence we arrive at the situation: $G = \lim L_n$, $L_n^o =: L$ for all n, $L \neq G$. Since $L \triangleleft L_n$ we must have $L \triangleleft G$ and G/L is the limit of finite subgroups L_n/L. We now invoke the theorem of Jordan (Wolf [163]) which says that there exists an integer j such that any finite subgroup of G/L has a normal abelian subgroup of index less than j. Choose such a large abelian normal subgroup A_n in L_n/L. The limit A of the A_n is then an abelian normal subgroup of index less than j in G/L. Since G/L is connected we must have G/L = A a torus and therefore G is not semi-simple.

Conversely if G is not semi-simple we can find a normal subgroup L of G^o such that G^o/L is a non-trivial torus (Hochschild [97] , XIII Theorem 1.3). By Lie algebra considerations (e.g. Helgason [96] , II. Proposition 6.6) the group L is a characteristic subgroup of G^o and therefore a normal subgroup of G. Therefore G/L =: P is a finite extension of a torus

$$1 \longrightarrow T \longrightarrow P \longrightarrow F \longrightarrow 1,$$

T a torus, F finite. If we show that P is a limit of proper subgroups then G is a limit of proper subgroups. We shall show in section 5.10 what the finite subgroups of P are, in particular we shall see that P is a limit of finite subgroups.

Proposition 5.6.4.

If X is a compact G-ENR then the mapping $C(G) \longrightarrow Z : (H) \longmapsto \chi(X^H)$ is continuous (Z carries the discrete topology).

Proof.

Let $(H) = \lim (H(i))$. By 5.6.2 we can assume $H(i) < H$ and $H = \lim H(i)$. We can and do assume $H = G$ (otherwise consider the H-space X). We choose a bi-invariant metric on X. Put $\varepsilon = \min h(K,G)$ where (K) runs through the finite set of orbit-types of X unequal to G. Since $(L) < (K)$ implies $h(L,G) \geqslant h(K,G)$ we have: $h(L,G) < \varepsilon$ implies $(L) \not< (K)$ for all isotropy types of X except possibly (G). Thus if $h(H(i),G) < \varepsilon$ then $X^{H(i)} = \cup X^{H(i)}_{(K)} = X^{H(i)}_{(G)} = X^G$.

5.7. The prime ideal spectrum of A(G).

Recall the ring homomorphisms $\varphi_H : A(G) \longrightarrow Z$ (see 5.5). If $(p) \subset Z$ is a prime ideal then

$$q(H,p) := \varphi_H^{-1}(p) \subset A(G)$$

is a prime ideal of A(G). We show that all prime ideals of A(G) arise in this way.

Proposition 5.7.1.

Given $H \triangleleft K < G$. Assume that K/H is an extension of a torus by a finite p-group (K/H a torus if p = o). Then $q(H,p) = q(K,P)$.

Proof.

For a certain L we have $H \triangleleft L \triangleleft K$, L/H is a torus, and K/L a finite p-group. Let X be a compact G-ENR. The group K/L acts on M^L with fixed point set M^K. Hence $\chi(M^K) \equiv \chi(M^L) \bmod p$ and $\chi(M^L) = \chi(M^H)$ by an easy application of Theorem 5.3.

Theorem 5.7.2.

Every prime ideal q of A(G) has the form q(H,p) for a suitable (H) $\in \Phi(G)$. Given q there exists a unique (K) $\in \Phi(G)$ with $q = q(K,p)$ and $\varphi_K(G/K) \not\equiv 0(p)$ where p is the characteristic of A(G)/q.

Proof.

We closely follow Dress [79] ! Let

$$T(q) = \{ (H) \in \Phi(G) \mid [G/H] \notin q \} .$$

Then T(q) is not empty because (G) $\in \Phi(G)$ and $[G/G] = 1 \notin q$. Let (H) be minimal in T(q); this exists because compact Lie groups satisfy the descending condition. We claim that for any $x \in A(G)$ we have a relation of the type

$$[G/H]\, x = \varphi_H(x)\, [G/H] + \sum a_K\, [G/K] \tag{5.7.3}$$

where the sum is over $(K) < (H)$, $(K) \neq (H)$. To see this we take $x = [X]$ look at the orbits of $G/H \times X$ and see from 5.5.1 that a relation must hold as claimed with some constant c instead of $\varphi_H(x)$: We then determine c if we apply φ_H to both sides of this equation. (This uses $\varphi_H(G/H) \neq 0$, i.e. (H) $\in \Phi(G)$.) But 5.7.3 implies $[G/H]\, x \equiv \varphi_H(x)[G/H]$ mod q (by minimality of G/H) and dividing by $[G/H] \notin q$ we get $x \equiv \varphi_H(x) \bmod q$ or $q = q(H,p)$ with p the characteristic of A(G)/q.

If K is any subgroup of G with $q = q(K,p)$ and $\varphi_K(G/K) \not\equiv 0 \bmod p$ for $p = \operatorname{char} A(G)/q$ then for an (H) as in the beginning of the proof $\varphi_K(G/K) \equiv \varphi_H(G/K) \not\equiv 0 \bmod p$. In particular G/H^K is not empty; and similarly G/K^H is not empty. This can only happen if $(H) = (K)$.

Proposition 5.7.4.

Every homomorphism $f : A(G) \longrightarrow R$ *into an integral domain* R *has the form* $f(x) = \varphi_K(x)\cdot 1$ *for a suitable* $K < G$.

Proof.

The kernel of f is a prime ideal $q(K,p)$. Therefore $f : A(G) \longrightarrow A(G)/q(K,p) \longrightarrow R$ must be the map $x \longmapsto \varphi_K(x)\cdot 1$, because there is a unique isomorphism $A(G)/q(K,p) \cong Z/(p)$.

Proposition 5.7.5.

(i) *If* $q(K,o) = q(L,o)$ *and* $(K) \in \Phi$ *then (up to conjugation)* $L \lhd K$ *and* K/L *is a torus*.

(ii) *Given* $L < G$ *there exists* $K \in \Phi$ *such that* $L \lhd K$ *and* K/L *is a torus. Moreover we have in this case* $\varphi_L = \varphi_K$.

Proof.

(i) Since $q(K,o) = q(L,o)$ by 5.7.2 $\varphi_K = \varphi_L$. From $\chi(G/K^L) = \varphi_L(G/K) = \varphi_K(G/K) = |NK/K| \neq 0$, we see that G/K^L is non-empty and hence $(L) < (K)$. We take $L < K$. Let T be a maximal torus in NL/L and let P be its inverse image in NL. By 5.7.1 $q(P,o) = q(L,o)$. We show $(P) \in \Phi$; then by 5.7.2 $(P) = (K)$. Assume $(P) \notin \Phi$. Then NP/P contains a non-trivial maximal torus S. We let Q be its inverse image in NP. We claim that L is still normal in Q. Let $q \in Q$ induce the conjugation automorphism c_q on P. Since Q/P is a torus, c_q is homotopic to an inner automorphism, hence (e.g. by Conner-Floyd [47], 38.1) an inner automorphism itself and preserves the normal subgroup L. From the exact sequence

$$0 \longrightarrow P/L \longrightarrow Q/L \longrightarrow S \longrightarrow 0$$

and $P/L = T$ we conclude that Q/L is a torus and hence T is not a maximal torus.

(ii) Use the proof of (i) and 5.7.1.

As a corollary of 5.6.4 and 5.7.5 we obtain

Corollary 5.7.6.

Let $C(\Phi(G),Z)$ be the ring of continuous (= locally constant, in this case) functions. Then

(5.7.7) $$\varphi : A(G) \longrightarrow C(\Phi(G),Z)$$

$\varphi(x) : (H) \longmapsto \varphi_H(x)$, is defined and an injective ring homomorphism.

The possible equalities $q(H,p) = q(K,p)$ are not so easy to describe. We show that in a certain sense 5.7.1 is the only reason for such equalities. Given $K < G$. If NK/K is not finite or $|NK/K| \equiv 0 \bmod p$ we find a subgroup $K \triangleleft P$ with $q(K,p) = q(P,p)$ as follows: Either by the procedure in the proof of 5.7.5 we let P be the inverse image in NK of a maximal torus in NK/K or we let P be the inverse image in NK of a Sylow p-group of NK/K. Then $(P) \in \Phi$ but it may happen that $|NP/P| \equiv 0$ mod p. In this case we can iterate the procedure. Either we arrive after a finite number of steps at a group Q with $|NQ/Q| \not\equiv 0 \bmod p$, or we get a sequence

$$P_0 = K \triangleleft P_1 \triangleleft P_2 \triangleleft P_3 \triangleleft \dots$$

of groups with $q(P_i,p) = q(P_{i-1},p)$ and $|NP_i/P_i| \equiv 0 \bmod p$ for $i \geq 1$. Let in this case Q be the closure in G of $\cup P_i$ (this is the limit in the space of subgroups, see 5.6). By continuity 5.6.4 we still have

$q(Q,p) = q(K,p)$. Now again we can apply our construction to Q if $|NQ/Q| \equiv 0 \bmod p$. Sooner or later we arrive at the defining group L of the prime ideal with $|NL/L| \not\equiv 0 \bmod p$.

That an infinite chain as above can actually occur is shown by the group $G = O(2)$. The groups in ϕ are $O(2)$, $SO(2)$ and the dihedral groups D_m. We have $ND_m = D_{2m}$. Hence

$$q(D_m,2) = q(D_n,2) \qquad \text{if} \qquad n = 2^j m.$$

For finite G the situation is more tractable.

Proposition 5.7.8.

Suppose $q(H,p) = q(K,p)$, $H \in \phi$, $K \in \phi$, $|NH/H| \not\equiv 0 \bmod p$, $|K/K_o| \not\equiv 0(p)$ *where* K_o *is the component of the identity in* K, *and* $p \neq 0$. *Then up to conjugation* $K \triangleleft H$ *and* H/K *is a finite* p-*group*.

Proof.

Choose P such that $NK > P > K$ and P/K a Sylow p-group of NK/K. We claim that $NP < NK$. Take $a \in NP$ and let K^a be the a-conjugate of K. Then $K/(K \cap K^a) < P/K^a$, hence $K/(K \cap K^a)$ is a finite p-group. On the other hand K, K^a, and P have the same component K_o of the identity, hence $K/(K \cap K^a)$ is a quotient of K/K_o which has order prime to p by assumption. Therefore $K = K \cap K^a = K^a$ and $a \in NK$. But then $|NP/P| \not\equiv 0 \bmod p$, because P/K was a Sylow p-group of NK/K. Now 5.7.1 and 5.7.2 imply $(P) = (H)$ and hence the assertion.

In particular if G is finite and $|NH/H| \not\equiv 0 \bmod p$ then there exists a unique smallest normal subgroup H_p of H such that H/H_p is a p-group and we have (with these notations)

Proposition 5.7.9.

$q(H,p) = q(K,p)$ *if and only if* $(H_p) \leq (K) \leq (H)$.

We shall see later that the cokernel of 5.7.7 is a torsion group of bounded exponent. We now make some remarks on the topology of Spec A(G), the prime ideal spectrum of A(G) with the Zariski topology.

Proposition 5.7.10.

The map

$$q : \phi(G) \times \operatorname{Spec} Z \longrightarrow \operatorname{Spec} A(G)$$

$$(H),\ (p) \longmapsto q(H,p)$$

is continuous, closed and surjective.

Proof.

An element $x \in C(\phi(G),Z) =: C$, being a locally constant function, is an integral linear combination of idempotent functions. Therefore this ring is integral over any subring. By an elementary result of commutative algebra (Atiyah-Mac Donald [11] , p. 67, Exercise 1) the mapping

$$\operatorname{Spec} \varphi : \operatorname{Spec} C \longrightarrow \operatorname{Spec} A(G)$$

is closed (and surjective by 5.7.2). Hence the Proposition follows from the next Lemma.

Lemma 5.7.11.

Let X be a compact, totally disconnected space. Then

$$(x,(p)) \longmapsto \{f \mid f(x) \in (p)\}$$

<u>defines a homeomorphism</u>

$$F : X \times \text{Spex } Z \longrightarrow \text{Spec } C(X,Z).$$

<u>Proof.</u>

We ask the reader to recall the topology on Spec (Bourbaki [33] , Ch. II). Certainly $\{f \mid f(x) \in (p)\}$ is a prime ideal in $C(X,Z)$ for any x and (p), so that F is defined. To define an inverse, let $k : Z \to C(X,Z)$ take n to the constant function $k_n : x \longmapsto n$. This induces a continuous map $k^* : \text{Spec } C(X,Z) \longrightarrow \text{Spec } Z$. Given $b \in \text{Spec } C(X,Z)$, let p be the element generating $k^* b$. Then we claim that $P = \bigcap_{f \in b} f^{-1}(p)$ consists of a single element of X. For if $p \neq 0$ and P is empty, then for each $x \in X$ there is a function $g_x \in b$ with $g_x(x) \notin (p)$. Since $k_p \in b$, for each $x \in X$ there is an $f_x \in b$ with $f_x(x) = 1$, i.e. the sets $f_x^{-1}(1)$ form a closed-open cover of X. Choose a finite subcover

$$U_i = f_{x_i}^{-1}(1), \qquad 1 \leq i \leq n.$$

Then one shows by induction on i that the characteristic function $K(V_i)$ of $V_i = U_i \cup \ldots \cup U_i$ is in b and in particular $k_1 \in b$, a contradiction. For $p = 0$, the same type of argument shows that k_m with $m = \text{l.c.m.}\ (g_{x_i}(x_i))$ is in b, contradicting $k^* b = (o)$. But if $x,y \in P$, choose $f \in b$ with $f(x) \in (p)$, and choose a closed-open U with $x \in U$, $y \in U$. Then setting

$$f_1 = f\,K(U) + (1-K(U))$$

$$f_2 = f(1-k(U)) + K(U)$$

we have $f_1 f_2 = f \in b$. Since $f_2(x) = 1$, $f_2 \notin b$, hence $f_1 \in b$, but $f_1(y) = 1$, hence $y \notin P$. Now we have a map $d : \text{Spec } C(X,Z) \longrightarrow X$ taking b to the unique element P, and the maps F and $d \times k^*$ are clearly inverse.

For the continuity of $d \times k^*$ we need only show d continuous. But for a closed -open $V \subset X$, $d^{-1}(V) = \{ b \mid K(V) \notin b \}$, which is open, while such V form a base of the topology of X.

It remains to be seen that F itself is continuous. But if $U = \{ b \mid f \notin b \}$ is a basic open set for some $f \in C(X,Z)$, and $q \in U$, then writing q as $F(x,(p))$ we have $f(x) = m \notin (p)$, and $V = f^{-1}(m)$ is closed-open in X containing x. Thus $q \in F(V \times \{(p) \mid m \notin (p)\}) \subset U$.

5.8. Relations between Euler-Characteristics.

We have described the Burnside ring of finite G-sets using congruences among fixed point sets (see 1.3). We generalize this description to compact Lie groups. The geometric interpretation of the Burnside ring then shows that we obtain a complete set of congruences that hold among the Euler characteristics of fixed point sets. We have already used the classical relations:

(5.8.1) $\chi(X) \equiv \chi(X^P) \bmod p$, P a p-group

(5.8.2) $\chi(X) = \chi(X^T)$, T a torus.

Using 5.8.2 we have shown in 5.7 that is suffices to consider subgroups H with finite index in their normalizer. Therefore we pose the problem: Describe the image of

$$\varphi : A(G) \longrightarrow C(\phi(G),Z) =: C$$

The next Proposition shows that this can be done by using congruences.

Proposition 5.8.3.

C is a free abelian group with basis $x_H = |NH/H|^{-1} \varphi(G/H)$, $(H) \in O(G)$.

Proof.

A priori the x_H are only contained in $C \otimes Q$. But since NH/H acts freely on every fixed point set G/H^K, $(K) \in \phi(G)$, we see that the numbers $\chi(G/H^K)$ are divisible by $|NH/H|$, and therefore $x_H \in C$. The elements x_H are linearly independent over Z because the G/H are. We have to show that each $x \in C$ is an integral linear combination of the x_H. Since x is continuous it attains only a finite number of values. Let $(H_1), \dots, (H_k)$ be the maximal elements of $\phi(G)$ such that $x(H_i) \neq 0$. Consider $x - \sum_{1 \leq i \leq k} x(H_i) x_{H_i} =: y \in C$. If $y(K) \neq 0$ then (K) is strictly smaller than one of the (H_i). Induction, using the descending chain condition for subgroups, gives the result.

Now let X be a compact G-ENR. For $(H) \in \phi(G)$ we consider the NH/H-space X^H. Since NH/H is a finite group we obtain as in 1.3

$$\sum_{n \in NH/H} \chi_{NH/H}(X^H)(n) \equiv 0 \mod |NH/H|$$

and this congruence can be rewritten in the form, using 5.3.,

$$(5.8.4) \qquad \sum_{(K)} n(H,K)\, \chi(X^K) \equiv 0 \mod |NH/H|,$$

where the sum is taken over conjugacy classes (K) of $K < G$ such that $K \triangleright H$ and K/H is cyclic; the n(H,K) are integers such that $n(H,H) = 1$.

Proposition 5.8.5.

The congruences 5.8.4 *are a complete set of congruences for the image of* $\varphi : A(G) \longrightarrow C$, i.e. $z \in C$ *is contained in* $\varphi A(G)$ *if and only if for all* $(H) \in \phi(G)$

$$\Sigma_{(K)}\ n(H,K)\ z(H) \equiv 0 \bmod |NH/H|,$$

with the summation convention as for 5.8.4.

Proof.

Write z according to 5.8.3 as integral linear combination $z = \Sigma\, n_K\, x_K$ and suppose that z satisfies the congruences. If we can show that n_K is divisible by $|NK/K|$ then $z \in \varphi\, A(G)$. Choose (H) maximal with $n_{(H)} \neq 0$. Consider the congruence belonging to H. The only term which is non-zero is $n(H,H)\ z(H) = n_H$ which has to be zero mod $|NH/H|$. Therefore $n_H\, x_H \in \varphi\, A(G)$. Apply the same argument to $z - n_H\, x_H$ etc. Induction on the "length" of z in terms of the x_K gives that $z \in \varphi\, A(G)$.

Proposition 5.8.5 tells you which congruences hold among the Euler-Characteristics of fixed point sets X^H if X is a compact G-ENR. One would like to know the most general class of spaces for which such congruences hold. We must ensure that the results of 5.3 are applicable: The equivariant Euler-Characteristics $\chi_{NH/H}(X^H)$ should be defined and the decomposition formula 5.3 should hold.

Remark 5.8.6.

A different proof for 5.8.5 in the more general context of certain modules over A(G) was given in tom Dieck - Petrie [69] .

Remark 5.8.7.

As in 1.2.4 one shows that $\varphi : A(G) \longrightarrow C$ can be recovered from the ring structure of A(G): namely φ is the inclusion of A(G) into the integral closure in its total quotient ring.

5.9. Finiteness theorems.

We collect some finiteness theorems for compact Lie groups.

Proposition 5.9.1.

Let M be a compact differentiable G-manifold. Then M has only a finite number of orbit-types.

Proof.

Induct over dim M. An equivariant tubular neighbourhood U of an orbit $X \subset M$ is a G-vector bundle hence has only isotropy groups appearing on X or on the unit sphere of a fibre. By induction U has finite orbit type. (See Palais [124], 1.7.25 for more details.)

Proposition 5.9.2.

Let G be a compact Lie group. There are only a finite number of conjugacy classes of subgroups which are normalizers of connected subgroups.

Proof.

(Bredon [26], VII Lemma 3.2) Let L be the Lie algebra of G, E its exterior algebra, and P(E) the projective space of E. If h is a linear subspace of L with basis $h_1, \dots, h_k$ then $h_1 \wedge \dots \wedge h_k$ determines a point ph of P(E) which is independent of the choice of the basis. The adjoint action of G on L induces an action of G on P(E). A subgroup N of G leaves h invariant if and only if ph is fixed under N. If H is a subgroup with Lie algebra h then:

$$gHg^{-1} = H \iff ad(g)h = h \iff g(ph) = ph .$$

Thus $NH = G_{ph}$. Now apply 5.9.1.

Proposition 5.9.3.

A compact Lie group G contains only a finite number of conjugacy classes (K) where K is the centralizer of a closed subgroup.

Proof.

Let G act on M = G via conjugation. If $H < G$ then M^H is the centralizer ZH. Apply 5.9.1.

We now come to a classical theorem of Jordan. Let $\mathfrak{S}(G)$ be the set of finite subgroups of G.

Theorem 5.9.4.

There exists an integer j, depending only on the dimension and the number of components of G, with the following properties: For each $H \in \mathfrak{S}(G)$ there exists an abelian normal subgroup A_H of H such that $|H/A_H| < j$. Moreover the A_H can be chosen such that $H < K$ implies $A_H < A_K$.

Proof.

(Boothby and Wang [24] . Wolf [163] .) Given integers k and d there are only a finite number of groups G with $|G/G_o| = k$ and dim G = d, up to isomorphism (see 5.9.5). These groups can therefore be embedded into a fixed O(n). Hence it suffices to prove the theorem for G = O(n). A simply proof may be found for instance in Wolf [169] , p. 100 - 103.

Theorem 5.9.5.

There exist only a finite number of non-isomorphic compact Lie groups of a given dimension and number of components.

Proof.

This depends on various classical results. We only describe the ingredients.

We begin with connected groups G. Then G is of the form

$$G = (T \times H)/D$$

where T is a torus, H is compact semi-simple, D is a finite central subgroup of T x H such that $D \cap T$ and $D \cap H$ are trivial (Hochschild [97] XIII Theorem 1.3). Therefore the projection of D into H is injective with image contained in the center ZH of H. This center ZH is finite by a theorem of Weyl (Helgason [96] , II. 6.9.). Hence given T and H there only a finite number of G's. By the classification theorem for semi-simple groups there are only a finite number of H's (Bourbaki [34]). This establishes the theorem for connected groups.

For the general case one has to study finite extensions

$$1 \longrightarrow G_o \longrightarrow G \longrightarrow E \longrightarrow 1$$

where G_o is connected and E is finite. By the general theory of group extensions and the finiteness of the cohomology of finite groups (Mac Lane [112] , IV) one sees that the following has to be proved: There are only a finite number of conjugacy classes of homomorphisms $E \longrightarrow Aut(G_o)/In(G_o)$ into the group of automorphisms modulo inner automorphisms. In case G_o is a torus the required finiteness follows from the Jordan-Zassenhaus theorem (Curtis-Reiner [48] , §79) and the general case is easily reduced to this case.

<u>Theorem 5.9.6.</u>

<u>Let G be a compact connected Lie group. Then there exist only finitely many conjugacy classes of connected subgroups of maximal rank.</u>

Proof.

Borel - de Siebenthal [23] .

We now consider solvable groups. A compact Lie group is called solvable if it is an extension of a torus by a finite solvable group. The derived group $G^{(1)}$ of G is the closure of the subgroup generated by commutators. We put inductively $G^{(n)} = (G^{(n-1)})^{(1)}$. A group H is called perfect if $H = H^{(1)}$. If $1 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 1$ is an exact sequence of compact Lie groups, then B is solvable if and only if A and C' are solvable. A compact Lie group G is solvable if and only if there exists an integer n such that $G^{(n)} = \{1\}$. We list the following elementary facts.

Proposition 5.9.7.

a) Any subgroup H of G has a unique minimal normal subgroup H_a such that H/H_a is solvable.

b) For each H there exists an integer n such that $H^{(n)} = H_a$.

c) H_a is a perfect characteristic subgroup of H.

d) $H = H_a$ if and only if H is perfect.

e) $(H) = (K) \Rightarrow (H_a) = (K_a)$.

f) $K \triangleleft H$, H/K solvable $\Rightarrow K_a = H_a$.

Proof.

a): If $K \triangleleft H$, $L \triangleleft H$ and H/K, H/L are solvable then $K \cap L \triangleleft H$ and $H/K \cap L$ is solvable. By the descending chain condition for subgroups there is a minimal group as stated. b), c) and d): Since $H/H^{(1)}$ is abelian, by induction $H/H^{(k)}$ is solvable, hence $H^{(k)} > H_a$ for all k, and $H^{(k)}/H_a$ is solvable. If $H^{(k)} \neq H_a$ then $H^{(k)}$ has a non-trivial abelian quotient, hence $H^{(k)} \neq H^{(k+1)}$. By the descending chain condition there is an n such that $H^{(n)} = H^{(n+1)}$ and for this n necessarily $H^{(n)} = H_a$ and $H^{(n)}$ is perfect. The $H^{(n)}$ are characteristic subgroups.

e) and f) are obvious.

Theorem 5.9.8.

Let G be a compact Lie group. There exists an integer n such that for all $H < G$ we have $H^{(n)} = H_a$.

Proof.

Note that $H^{(n)} = H_a$ if and only if $(H/H_a)^{(n)}$ is the trivial group. Therefore we consider pairs H,K such that $H \trianglelefteq K < G$ and K/H is solvable. We show that there is an integer n such that for all such pairs $(K/H)^{(n)}$ is the trivial group. Let us call the smallest integer k such that $L^{(k)} = 1$ for a solvable group L the length $l(L)$ of L.

Take a pair K,H as above. Since K/H is solvable we have an exact sequence

$$1 \longrightarrow T \longrightarrow K/H \longrightarrow F \longrightarrow 1$$

where T is a torus and F is finite solvable. We have

$$l(K/H) \leq l(T) + l(T) = 1 + l(F).$$

So we need only show that the length of finite solvable subquotients is bounded. Let generally K_o denote the component of 1 of K. Then $K/H \longrightarrow F$ induces a surjection $p : K/K_o \longrightarrow F$. We show in a moment that there exists an integer $b(G)$ such that for any $K < G$ there exists an abelian normal subgroup A_K of K/K_o such that $|K/K_o : A_K| < b(G)$. Let be F_o be a pA_K. Then F/F_o has order less than $b(G)$. But $l(F) \leq l(F_o) + l(F/F_o) = 1 + l(F/F_o)$ because F_o is abelian. But $l(F/F_o)$ is bounded because only a finite number of groups occur.

The existence of the integer b(G) is proved by induction over dim G and $|G : G_o|$. Given G, the bound exists for the finite subgroups of G by Theorem 5.9.4. Let K be a subgroup of positive dimension. Consider

$$K_o < K < NK < NK_o.$$

Then K/K_o is a finite subgroup of $NK_o/K_o =: U$, and $\dim U < \dim G$. By 5.9.2. only a finite number of U occur up to isomorphism. This gives by induction the required finiteness.

We put WH = NH/H.

<u>Theorem 5.9.9.</u>

<u>There exists an integer</u> b <u>such that for each closed subgroup</u> H <u>of</u> G <u>the index</u> $|WH : (WH)_o|$ <u>is less than</u> b.

<u>Proof.</u>

The proof proceeds in three steps: We first reduce to the case that WH is finite; then we reduce to the case that H is finite; and finally we show that for finite H with finite WH the order of WH is uniformly bounded.

The group Aut H/In H of automorphism modulo inner automorphisms is discrete. Conjugation induces an injective homomorphism

$$NH/ZH \cdot H \longrightarrow \mathrm{Aut}\ H/\mathrm{In}\ H$$

where ZH is the centralizer of H. Hence NH/ZH·H being compact and discrete is finite. Hence

Lemma 5.9.10.

WH is finite if and only if $ZH/ZH \cap H$ is finite.

Lemma 5.9.11.

For any $H < G$ the group $ZH \cdot H$ has finite index in its normalizer.

This follows from the previous Lemma and the relations $Z(ZH \cdot H) < ZH < ZH \cdot H$.

If $n \in G$ normalizes H then also ZH and hence $ZH \cdot H$. We therefore have

$$NH/ZH \cdot H < N(ZH \cdot H)/ZH \cdot H \ .$$

Using Lemma 5.9.11 and the existence of an upper bound for the set

$$F(G) := \{ |WH| \mid H < G, \ WH \ \text{finite} \}$$

we obtain

Lemma 5.9.12.

There exists an integer c such that for all $H < G$ we have $|NH/ZH \cdot H| < c$.

Now we obtain the first reduction of our problem. From the exact sequence

$$1 \longrightarrow ZH/ZH \cap H \longrightarrow WH \longrightarrow NH/ZH \cdot H \longrightarrow 1$$

we see that $WH/(WH)_o \longrightarrow NH/ZH \cdot H$ has the kernel which is a quotient of $ZH/(ZH)_o$. Now Proposition 5.9.3 and Lemma 5.9.12 show that

$$\{ |WH/(WH)_o| \mid H < G \}$$

is bounded.

We show by induction over $|G/G_o|$ and dim G that F(G) has an upper bound $a = a(G/G_o, \dim G)$. For finite G we can take $a = |G|$. Suppose that an upper bound $a(K/K_o, \dim K)$ is given for all K with $\dim K < \dim G$. Let $T(G) = \{H < G \mid WH \text{ finite}\}$. Suppose $H \in T(G)$ is not finite. We consider the projection

$$p : NH_o \longrightarrow NH_o/H_o =: U .$$

Let V be the normalizer of H/H_o in U. Then $WH = V/(H/H_o)$ and therefore $H/H_o \in T(U)$. Since $\dim U < \dim G$ we obtain by induction hypothesis $|WH| \leq a(U/U_o, \dim U)$. We show that the possible values for $|U/U_o|$ are finite: This follows from 5.9.2. Hence for a given G the possible $|U/U_o|$ are bounded, say $|U/U_o| \leq m(G)$. We have

$$|U/U_o| \leq |G/G_o| \, m(G_o) .$$

By the classification theory of compact connected Lie groups there are only a finite number in each dimension. Hence there exists a bound for $|U/U_o|$ depending only on $|G/G_o|$ and dim G. This proves the induction step as far as the non-finite H in T(G) are concerned.

For the remaining case we use 5.9.4. and 5.9.6.

If $H \in T(G)$ is finite then also $K = NH$ is finite and by Lemma 5.9.11 $K \in T(G)$. We choose $j = j(|G/G_o|, \dim G)$ and A_H, A_K according to 5.9.4. We have

$$|K/H| \leq |K/A_K| \cdot |A_K/H \cap A_K| \leq j |A_K/H \cap A_K| .$$

Hence it suffices to find a bound for the $|A_K/H \cap A_K|$. Consider the exact sequence $1 \longrightarrow A_H \longrightarrow H \longrightarrow S \longrightarrow 1$. The conjugation c(a) with

$a \in A_K$ is trivial on A_H, because $A_K > A_H$, and hence $c(a)$ induces an automorphism of S. Since $|S| \leq j$ this automorphism has order at most $J = j!$, i. e. $c(a^r)$ is the identity on S and A_H for a suitable $r \leq J$. The group of such automorphisms modulo the subgroups of inner automorphisms by elements of A_H is isomorphic to $H^1(S;A_H)$, with S acting on A_H by conjugation. Since this group is annihilated by $|S|$ we see that $c(a^s)$ is an inner automorphism by an element of A_H for a suitable $s \leq J|S| \leq jJ$. In other words: $a^s h^{-1} \in ZH$. Hence it is sufficient to find a bound for the order of

$$A_K \cap ZH/H \cap A_K \cap ZH \ .$$

Let $U_1 = A_K \cap ZH$. By Borel-Serre [28] , Théorème 1, U_1 is contained in the normalizer NT of a maximal torus of G. Put $U = U_1 \cap T$. Then $|U_1/U| \leq |G/G_o||wG_o|$ where wG_o denotes the Weyl group of G_o. We estimate the order of U. Since U is abelian we have $U < ZU$. Moreover $H < ZU$ by definition of ZH. Since U is contained in the center $C = CZU$ of ZU. The inclusion $H < ZU$ implies $C < NH$. Hence C is finite.

We proceed to show that for the order of a finite center $C(G)$ of G there exists a bound depending only on $|G/G_o|$ and dim G. We let G/G_o act by conjugation on $C(G_o)$ Then $C(G) \cap G_o$ is the fixed point set of this action. We have $C(G_o) = A \times T_1$, where A is a finite abelian group and T_1 is a torus. The group A is the center of a semisimple group and therefore $|A|$ is bounded by a constant c depending only on dim G. The exact cohomology sequence associated to the universal covering

$$0 \longrightarrow \pi_1 T_1 \longrightarrow V \longrightarrow T_1 \longrightarrow 0$$

shows, that the fixed point set of the action of G/G_o on $T_1 = C(G_o)_o$ is isomorphic to $H^1(G/G_o, \pi_1 T_1)$, hence its order is bounded by a constant d depending only on $|G/G_o|$ and the rank of T_1. Hence

$$|C(G)| \leq |G/G_o| \, cd \; .$$

Finally we show that for the possible groups ZU the order $|ZU/(ZU)_o|$ is bounded. U is contained in a maximal torus of G. Therefore ZU is a subgroup of maximal rank and $(ZU)_o$ a connected subgroup of maximal rank. By Theorem 5.9.6 there exist only finitely many conjugacy classes of connected subgroups of maximal rank. We have

$$|ZU/(ZU)_o| \leq |G/G_o| \; |N_o(ZU)_o/(ZU)_o| \; .$$

There are only finitely many possibilities for normalizers $N_o(ZU)_o$ in G_o of $(ZU)_o$.

This finishes the proof of Theorem 5.9.9. .

The last Theorem together with Proposition 5.8.3 gives the following result.

<u>Proposition 5.9.13.</u>

Let n be the least common multiple of the numbers $|NH/H|$ where $(H) \in \Phi(G)$. Then the cokernel of $A(G) \longrightarrow C(\Phi(G), Z)$ is annihilated by n.

5.10. Finite extensions of the torus.

We have seen earlier that the appearance of infinitely many elements in $\Phi(G)$ is connected with subgroups of G which are not semi-simple. The typical situation is given, when G itself is an extension of a torus T by a finite group F

$$(5.10.1) \qquad a \longrightarrow T \longrightarrow G \xrightarrow{p} F \longrightarrow 1.$$

In particular if we are given a homomorphism $h = F \rightarrow Aut(T) = GL(n,Z)$, $n = \dim T$, we can form the semi-direct product of T with F and h as twisting, call this G_h. Note that h is an integral representation of F. It would be interesting to know what the Burnside ring $A(G_h)$ can say about the integral representation (or vice versa). We are going to make a few elementary remarks concerning the Burnside ring A(G) for groups G as in 5.10.1.

Given G as in 5.10.1 let $h : F \longrightarrow Aut\ (T)$ be the homomorphism induced by conjugation. We call a pair (F',T') with $F' < F$, $T' < T$ and T' invariant under F' admissible, and call $H < G$ an (F',T')-subgroup if $p(H) = F'$ and $H \cap T = T'$.

Let $\sigma \in H^2(F,T)$ be the class given by 5.10.1. We have maps

$$k_* : H^2(F',T) \longrightarrow H^2(F',T/T')$$

$$i^* : H^2(F,T) \longrightarrow H^2(F',T) \quad .$$

Elementary diagram chasing then tells us

Proposition 5.10.2.

An (F',T')-subgroup exists in G if and only if $\sigma \in Ker(k_* i^*)$.

Now choose any section $s : F \longrightarrow G$ to p and parametrize G as $F \times T$ via $g \longmapsto (pg, spg^{-1} \cdot g)$. The multiplication in G takes the form

$$(5.10.3) \qquad (f,t)(f',t') = (ff',,(t)f' + t' + \mu(f,f'))$$

where $(t)f = g^{-1}tg$ for $g \in p^{-1}(f)$ and

$$\mu(f,f') = s(ff')^{-1}\, s(f)s(f') .$$

We always assume $s(1) = 1$ from now on.

Proposition 5.10.3.

If H *is an* (F',T')*-subgroup of* G, *and* s *is a section with* $s(F') \subset H$, *then a 1-1 correspondence between the* (F',T')*-subgroups of* G *and the crossed homomorphisms* $\alpha : F' \longrightarrow T/T'$ *is established by associating to* H' *the crossed homomorphism*

$$\alpha(f') = k(s(f')^{-1}h(f'))$$

for $h(f')$ *any element of* $H' \cap p^{-1}(f')$.

We leave the proof as an exercise. We denote the group described in 5.10.3 by (F',T',α). If G is a semi-direct product then $\sigma = 0$ and (F',T')-subgroups always exist; in this case it is advisable to choose s as a homomorphism.

We now describe the effect of conjugation. For conjugation by elements of T, note that in our parametrization

$$(1,t)(f',t')(1,t)^{-1} = (f',t' + (t)f' - t) .$$

Thus denoting by $d_t : F' \longrightarrow T$ the principal crossed homomorphism

$d_t(f') = (t)f'-t$, the result of conjugating (F',T',α) by $(1,t)$ is (F',T',α') with $\alpha'(f') = \alpha(f') + k(d_t(f'))$.

Proposition 5.10.4.

Given a choice of H and s as in 5.10.3. There is a 1-1 correspondence between classes of (F',T')-subgroups under conjugation by elements of T and the elements of $H^1(F',T/T')$.

Proposition 5.10.5.

If $H < G$ *is an* (F',T')*-subgroup then*

$$NH\cap T/H\cap T \;=\; \mathrm{Fix}(F',T/T') \;.$$

Proofs are again left as exercises.

Proposition 5.10.6.

If H is an (F',T')-subgroup then the following are equivalent:

i) $H \in \Phi(G)$.

ii) $\mathrm{Fix}(F',T/T')$ *is finite*

iii) T' *contains the zero-component of* $\mathrm{Fix}(F',T)$.

Proof.

The equivalence i) $\Leftrightarrow$ ii) follows immediately from 5.10.5. The equivalence i) $\Leftrightarrow$ iii) is elementary Lie group theory and will be left to the reader.

From 5.10.2. and 5.10.6 one obtains

Proposition 5.10.7.

$\Phi(G)$ *is infinite if and only if the action of F on T is non-trivial.*

This can be used to give an analogous result for an arbitrary compact Lie group.

Proposition 5.10.8.

Let G be a compact Lie group. Then $\Phi(G)$ is finite if and only if the action of the Weyl group on the maximal torus is trivial.

Proof.

If the action is trivial then G_o can have no semi-simple component. Hence G is of type 5.10.1 and 5.10.6 says that $\Phi(G)$ is finite.

Now assume that in

$$0 \longrightarrow T \longrightarrow NT \longrightarrow WT \longrightarrow 1 ,$$

T a maximal torus, the action of the Weyl group WT on T is non-trivial. By 5.10.7 $\Phi(NT)$ is infinite. We show that an infinite number of elements of $\Phi(NT)$ are contained in $\Phi(G)$. We know that $NT = \lim H_i$, $H_i \neq NT$. By continuity our assertion follows with the help of the next Lemma.

Lemma 5.10.9.

Let $H < K < G$. Then $(H) \in \Phi(G)$ if and only if $(H) \in \Phi(K)$ and G/K^H is finite.

Proof.

If $(H) \in \Phi(G)$ then, of course, $(H) \in \Phi(K)$ and G/K^H is finite because it consists of a finite number of NH/H-orbits. For the other direction, note that $H < N_K H < N_G H$ yields a fibre bundle $N_K H/H \longrightarrow N_G H/H \longrightarrow N_G H/N_K H$. But the inclusion $N_G H \longrightarrow G$ induces an injective map

$$N_G H/N_K H = N_G H/N_G H \cap K \longrightarrow G/K^H .$$

Thus if $(H) \in \Phi(K)$ and G/K^H is finite, both base and fibre are finite.

We now report briefly about cyclic extensions of a torus (see Gordon [87]).

Proposition 5.10.10.

If G is an extension of T by F and F' < F is cyclic, then any two (F',T') subgroups of G are conjugate under an element of T.

Proof.

If F' is cyclic and Fix(F',T/T') is finite, then $H^1(F',T/T') = 0$. Now use 5.10.4.

If f is cyclic of order n with generator f and M is any F-module then

$$H^2(F,M) \cong \mathrm{Fix}(F,M)/N^*M$$

where $N^*M = \sum_{i=0}^{n-1} (m)f^i$. Since for an r-torus $M = T^r$ we have $H^2(F,T^r) \cong H^3(F,Z^r)$ this group is finite. Thus N^*T^r contains the zero-component of $\mathrm{Fix}(F,T^r)$. On the other hand, if $\varphi : I \longrightarrow T^r$ is any path from 0 to t, then $\sum_{i=0}^{n-1} (\varphi)f^i$ is a path in N^*T^r from 0 to $\sum_{i=0}^{n-1} (t)f^i$, so that N^*T^r is connected. Hence for any torus T^r, N^*T^r is precisely $\mathrm{Fix}(F,T^r)_o$.

The isomorphism $H^2(F,T) \cong \mathrm{Fix}(F,T)/N^*T$ means that the extension G is characterized by a component of Fix(F,T). Now note that it is no essential restriction to assume $N^*T = 0$. For if L is any compact Lie group and $(ZL)_o$ the zero-component of its center, then $L \longrightarrow L/(ZL)_o$ induces an isomorphism of rings

$$A(L/(ZL)_o) \cong A(L) \ .$$

Now choose any element $s(f) \in p^{-1}(f)$ and construct a section s by putting $s(f^i) = s(f^i)$, $0 \leq i < n$. Then since $s(f)^{-1}s(f)^n s(f) = s(f)^n$, $\tau = s(f)^n \in \mathrm{Fix}(F,T)$, and τ is the image of $[G] \in H^2(F,T)$ in $\mathrm{Fix}(F,T) = \mathrm{Fix}(F,T)/N^{*}T$. If now (F',T') is an admissible pair then there exists an (F',T') subgroup H with NH/H finite if and only if both the zero-component and the τ-component of $\mathrm{Fix}(F',T')$ are in T'.

Suppose $\tau \in \mathrm{Fix}(F,T)$, latter being discrete, and let T' be the (finite) subgroup generated by τ. Then T/T' inherits an F-operation. With these notations one has

Theorem 5.10.11.

If G is the extension of T by F defined by τ, and G' is the semi-direct product of T/T' and F in the action above, then $A(G) \cong A(G')$.

Proof.

There exists a map $t : G \longrightarrow G'$ making the following diagram commutative

$$\begin{array}{ccccc} T & \longrightarrow & G & \longrightarrow & F \\ \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle id} \\ T/T' & \longrightarrow & G' & \longrightarrow & F \end{array} .$$

By the analysis of (F',T')-subgroups of G given above it is seen that t induces the required isomorphism.

5.11. Idempotent elements.

In section 1.4 we have described the idempotents of A(G) for finite G. We generalize this to compact Lie group, using results of 5.9. and 5.6.

Let S = S(G) be the space of closed subgroups of G and cS the quotient space under the conjugation action (see 5.6). Let $H^{(1)}$ be the commutator subgroup of H and H_a the smallest normal subgroup of H such that H/H_a is solvable (see 5.9.8). Let P be the space of perfect subgroups in S

Proposition 5.11.1.

The maps $H \longmapsto H^{(1)}$ *and* $H \longmapsto H_a$ *are continuous maps* $S \longrightarrow S$. *The space* P *is closed in* S.

Proof.

In view of the compactness of S and 5.9.8 we need only show that $H \longmapsto H^{(1)}$ is continuous. Let $H_1, H_2, \dots$ be a sequence of subgroups converging to H. Without loss of genrality we can assume that the H_i are conjugate to subgroups of H. Moreover by 5.6.2 we can find a sequence $g_i \in G$ converging to 1 such that $K_i = g_i H_i g_i^{-1}$ is contained in H. We show that lim $K_i^{(1)}$ exists and is equal to $H^{(1)}$. Fix $\varepsilon > 0$ and choose n such that in the Hausdorff metric $d(K_i,H) < \varepsilon$ for $i \geq n$. Let c^kK be the closed subspace of a group K consisting of elements which are product of a most k commutators. Then $d(K_i,H) < \varepsilon$ implies $d(c^kK_i, c^kH) < 4k\varepsilon$ Choose k such that $d(c^kH, H^{(1)}) < \varepsilon$. Then for $i \geq n$ we have $d(c^kK_i, H^{(1)}) < (4k+1)\varepsilon$ and a fortiori $d(K_i^{(1)}, H^{(1)}) < (4k+1)\varepsilon$.

As a corollary we obtain

Proposition 5.11.2.

Given a perfect subgroup H *of* G. *Then* $\{K \mid K_a = H\}$ *and* $\{K \mid K_a \sim H\}$ *are closed subsets of* S.

In 5.7 we obtained the closed quotient map

$$q : S \times \text{Spec } Z \longrightarrow \text{Spec}A(G) : (H),(p) \longmapsto q(H,p) .$$

Let r be the composition

$$S \times \text{Spec } Z \xrightarrow[pr]{} S \xrightarrow[a]{} P \xrightarrow[c]{} cP$$

where pr is the projection, a the map $aH = H_a$, and c the map $cH = (H)$ into the space cP of conjugacy classes of perfect subgroups. Then r is continuous by 5.11.1.

Proposition 5.11.3.

The map r *factors over* q *inducing a continuous surjective map* $s : \text{Spec } A(G) \longrightarrow cP$.

Proof.

Suppose $q(H,p_1) = q(K,p_2)$. Since p is the residue characteristic of $q(H,p)$ we must have $p_1 = p_2$. Put $p = p_1$. Let (H^*) be the unique conjugacy class such that $q(H,p) = q(H^*,p)$ and NH^*/H^* is finite (see 5.7.2). By 5.7 we can find a countable transfinite sequence $H \triangleleft H_1 \triangleleft H_2 \ldots H_\lambda \sim H^*$ such that H_{i+1}/H_i is solvable and H_j is the limit of the preceeding subgroups if j is a limit ordinal. It follows from Proposition 5.11.1 that $H_a = (H_\lambda)_a$.

The space cP being a countable compact metric space is totally disconnected. Hence we get a unique continuous map e which makes the following diagram commutative

$$\begin{array}{ccc} & \text{Spec } A(G) & \\ {}^{s}\swarrow & & \searrow^{\pi} \\ cP & \xrightarrow[e]{\hspace{4em}} & \pi\,\text{Spec } A(G) \end{array}$$

Here π is the projection onto the space of components.

<u>Theorem 5.11.4.</u>

<u>The map</u> e <u>is a homomorphism</u>.

<u>Proof.</u>

π Spec A(G) is a quotient of a quasi compact space hence quasi compact. The space cP is a Hausdorff space. We therefore need only show that e is bijective. We already know that e is surjective.

Given two components B and C of Spec A(G). Choose elements $q(H,p) \in B$, $q(K,l) \in C$. Assume that $e(B) = e(C)$, hence

$$(H_a) = sq(H,p) = sq(K,l) = (K_a).$$

Since H/H_a is solvable we can find a finite chain of subgroups

$$H = H_1 \triangleright H_2 \triangleright \dots \triangleright H_k = H_a$$

such that H_i/H_{i+1} is a torus or finite cyclic of prime order. By 5.7.1 $q(H_i,p_i) = q(H_{i+1},p_i)$ for a suitable prime. If $\bar{q}(H,p)$ denotes the closure of the point $q(H,p)$ we have

$$q(H,p) \in \bar{q}(H_1,o), \ \bar{q}(H_i,o) \cap \bar{q}(H_{i+1},o) \neq \emptyset ,$$

and therefore $q(H_a,o) \in B$. Similarly $q(K_a,o) \in C$ and therefore $B = C$.

We now show how Theorem 5.11.4 leads to a description of idempotent elements.

Let U be an open and closed subset of Spec A(G). Then U is a union of components and projects into an open and closed subset of cP called s(U). Let e(U) be the idempotent element of A(G) which corresponds to U (Bourbaki [33] , II. 4.3, Proposition 15). Let $S(U) = \{H < G \mid \varphi_H e=1\}$

Proposition 5.11.5.

$H \in S(U) \iff (H_a) \in S(U)$.

Proof.

Since e(U) is idempotent $\varphi_H(e(U))$ is 0 or 1. We have to recall how to pass from U to e(U). Let Z be the complement of U in Spec A(G). Then

$$Z = V(A(G)e(U)) = \{q \in \text{Spec } A(G) \mid q \supset A(G)e(U)\} .$$

Moreover $e(Z) = 1-e(U)$. Suppose $\varphi_H e(U) = 1$, then $\varphi_H(e(Z)) = 0$, so $\varphi_H A(G)e(Z) = (0)$, which means $q(H,o) \supset A(G)e(Z)$, $q(H,o) \in V(A(G)e(Z))=U$ and therefore $(H_a) \in s(U)$.

Conversely, if $(H_a) \in s(U)$, then $q(H,o) \in U$, $\varphi_H A(G)e(Z) = (0)$, $\varphi_H e(U) = 1$.

The idempotent is indecomposable if and only if U is a component. If the perfect subgroup H of G is not a limit of perfect subgroups then

$\{q(K,p) \mid (K_a) = (H)\} := U(H)$ is a component and H yields an indecomposable idempotent $e_H := e(U(H))$.

We are now going to show that the topological considerations above are necessary in that usually an infinite number of conjugacy classes of perfect subgroups exists. Let $1 \longrightarrow T \longrightarrow G \longrightarrow F \longrightarrow 1$ be an exact sequence where T is a torus and F a finite group. Conjugation in G induces a homomorphism $\varphi : F \longrightarrow \mathrm{Aut}(T)$ which we also interprete as action of F on T (compare section 5.10.) Let F_U be the kernel of φ .

Proposition 5.11.6.

Let G be a finite extension of a torus as above. Then the number of conjugacy classes of perfect subgroups of G is finite if and only if F/F_U is solvable. If F/F_U is solvable then the set of perfect subgroups is finite.

Proof.

A quotient of a perfect group is perfect. Let F/F_u be solvable. Let $H < G$ be perfect. Then the image under $G \longrightarrow F \longrightarrow F/F_u$ is perfect hence trivial. Therefore H is an extension $1 \longrightarrow H \cap T \longrightarrow H \longrightarrow P \longrightarrow 1$ with $P < F_u$ perfect and trivial action of P on $H \cap T$ and T. Let K be the pre-image of P under $p : G \longrightarrow F$. Then $H \triangleleft K$ since T is contained in the center of K. The group $K/H = T/H \cap T$ is solvable. Hence $H = K_a$. There a perfect group comes via the map $K \longmapsto K_a$ from a finite set of subgroups.

Now let us assume that F/F_u is not solvable. Let $P < F/F_u$ be a non-trivial perfect subgroup. Let H be the pre-image of P under $G \longrightarrow F/F_u$ and $Q < F$ be its group of components. Let T_o be the component of 1 in the fixed point set of the Q action on T. Since $Q > F_u$, $Q \neq F_u$, we have $T_o \neq T$. The group T_o is contained in the center of H and $H \longrightarrow H/T_o$ induces an injective ring homomorphism $A(H/T_o) \longrightarrow A(H)$. If $A(H/T_o)$ has an infinite number of idempotents then A(H) has an infinite number of idempotents, hence an infinite number of conjugacy classes of perfect subgroups. The action of Q on H/T_o has zero-dimensional fixed point set. Hence we have reduced the problem to the case $T_o = \{1\}$. But then a subgroup L of H which projects onto P under $H \longrightarrow Q \longrightarrow P$ has finite index in its normalizer. Let L be such a group and consider its derived group $L^{(1)}$. Then $L^{(1)}$ also projects onto P because P is perfect. Therefore $NL^{(1)}/L^{(1)}$ is finite and $L/L^{(1)} < NL^{(1)}/L^{(1)}$. But we have shown in 5.9.4 that there exists a number b such that for any $L < H$ with finite index in its normalizer $|NL/L| < b$. Together with 5.9.8 we see that there is an integer n such that L/L_a is finite of order less than b^n. Hence if there exists an infinite set of subgroups of H which projects onto P and which contains groups of arbitrary large order then the set of conjugacy classes of perfect subgroups is infinite. But infinite sequence of subgroups of the required sort is easily constructed, using the techniques of 5.10.

5.12. Functorial properties.

If X is a G-space and $H < G$ then X can be considered as H-space. This induces the forgetful functor r_H^G : G-Top $\longrightarrow$ H-Top from the category of G-spaces to the category of H-spaces. This functor has a left adjoint, called <u>extension</u> from H-spaces to G-spaces. On objects it is defined by

$$e_H^G(X) = Gx_H X$$

for an H-space X. The adjointness means that for H-spaces X and G-spaces Y we have a natural bijection

$$\mathrm{Map}_G(Gx_H X, Y) \cong \mathrm{Map}_H(X, r_H^G Y) \quad ,$$

where Map_G is the set of G-maps. If $f : X \longrightarrow Y$ is an H-map then $f' : Gx_H X \longrightarrow Y : (g,x) \longmapsto gf(x)$ is the adjoint G-map.

<u>Proposition 5.12.1.</u>

The assignment $X \longmapsto Gx_H X$ *induces an additive homomorphism*

$$e_H^G : A(H) \longrightarrow A(G).$$

(*X a compact* H-ENR.)

<u>Proof.</u>

Given $K < G$, then $Gx_H X)^K \neq \emptyset \Rightarrow G/H^K \neq \emptyset \Rightarrow (K) < (H)$. Assume $K < H$. We have to show that $\chi((Gx_H X)^K)$ can be computed from Euler-Characteristics of fixed point sets X^L. The set G/H^K is finite (if $K \in \Phi(G)$). The fibre of $(Gx_H X)^K \longrightarrow G/H^K$ over gH is homeomorphic to $X^{gKg^{-1} \cap H}$. Hence

$$\chi((Gx_H X)^K) = \sum_{gH \in G/H^K} \chi(X^{gKg^{-1} \cap H}) \quad .$$

If $f : H \longrightarrow K$ is a continuous homomorphism between compact Lie groups then a K-space X can be considered via f as an H-space. This induces a ring homomorphism

$$A(f) = f^{*} : A(K) \longrightarrow A(H)$$

and A(-) becomes a contravariant functor from compact Lie groups to commutative rings. If $f : H \subset K$ then f is called restriction, also denoted r_H^K.

We want to investigate the various interrelations between the e_H^G and r_H^G. We need a slightly more general map then the e_H^G. This is done best by redefining e_H^G and r_H^G using a more general concept than the Burnside ring.

Let S be a closed differentiable G-manifold and let a(S) be the set of differentiable G-maps $M \longrightarrow S$ which are proper submersions. On a (S) we induce the following equivalence relation : $p : M \longrightarrow S$ equivalent to $q : N \longrightarrow S$ if and only if for all $s \in S$ and all $H < G_s$ the equality

$$\chi(p^{-1}(s)^H) = \chi(q^{-1}(s)^H)$$

holds. Disjoint union (addition) and fibre product over S (multiplication) makes the set of equivalence classes into a commutative ring with identity, denotes A [S] . If S is a point this is the Burnside ring; hence we call A [S] the Burnside ring of G-manifolds over S. We are going to describe the functorial properties of this ring.

Let $f : T \longrightarrow S$ be a differentiable G-map. Let $p : M \longrightarrow S$ be a submersion as above. Then in the pull-back diagram

$$\begin{array}{ccc} N & \xrightarrow{F} & M \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ T & \xrightarrow{f} & S \end{array}$$

the map q is a proper submersion and defines an element in $A[T]$. The assignment $p \longmapsto q$ induces a ring homomorphism $f^{*} : A[S] \longrightarrow A[T]$.

We also have covariant maps. Let $f : T \longrightarrow S$ be a submersion. Then composition with f induces an additive (but not multiplicative) map $f_{*} : A[T] \longrightarrow A[S]$. These maps have the following properties.

Proposition 5.12.2.

i) f^{*} is a homomorphism of rings. We have $(id)^{*} = id$ and $(fg)^{*} = g^{*}f^{*}$.

ii) For any submersion $f : T \longrightarrow S$ the map f_{*} is well-defined and additive. We have $(id)_{*} = id$ and $(fg)_{*} = f_{*}g_{*}$.

iii) For $a \in A[S]$ and $b \in A[T]$ we have

$$af_{*}(b) = f_{*}(f^{*}(a)b).$$

iv) Let

$$\begin{array}{ccc} T' & \xrightarrow{F} & S' \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle p} \\ T & \xrightarrow{f} & S \end{array}$$

be a pull-back diagram with f and hence F a submersion. Then $p^{*}f_{*} = F_{*}P^{*}$.

v) If $f_o, f_1 : T \longrightarrow S$ are G-homotopic then $f_o^{*} = f_1^{*}$.

The proofs are straightforward and left to the reader. The connection with material at the beginning of this section if obtained using a canonical isomorphism $A[G/H] \cong A(H) : p : M \longrightarrow G/H \longmapsto p^{-1}(H/H)$.

Proposition 5.12.2 iv) generalizes the main property of Mackey functors in the sense of Dress [80] to compact Lie groups. But in the case of non-finite Lie groups there exists a double coset formula which is a less formal generalization of the Mackey axiom and is more accessible to computation. We are going to describe this formula.

We consider a pull-back diagram

$$\begin{array}{ccc} S & \xrightarrow{\ \bar{h}\ } & G/L \\ {\scriptstyle \bar{k}}\downarrow & & \downarrow{\scriptstyle k} \\ G/K & \xrightarrow[\ h\ ]{} & G/P \end{array} .$$

The problem is to compute $k^* h_*$. We use a decomposition of S into homogeneous spaces but slightly more refined than the decomposition in the Burnside ring. As in Section 5.5 we have the decomposition $S = \bigcup S_{(H)}$ into the subspaces of a given orbit type. We let $S_{(H),b}$ be the invers image in $S_{(H)}$ of the connected components of $S_{(H)}/G$. So the index b distinguishes the components. Then we still have a decomposition

$$S = \sum n_{(H),b} [M_{(H),b}]$$

in A(G) with $n_{(H),b} := \chi_c(S_{(H),b}/G) \in Z$ and $M_{(H),b}$ an orbit in $S_{(H),b}$. We let $k_{(H),b} : M_{(H),b} \longrightarrow G/K$ and $h_{(H),b}$ G/L be the maps which are compositions of the inclusion $M_{(H),b} \subset S$ with the maps $\bar{k}$ and $\bar{h}$ respectively. Then we claim

Theorem 5.12.3.

We have the equality of maps

$$k^{*}h_{*} = \sum_{(H),b} n_{(H),b}(h_{(H),b})_{*}(k_{(H),b})^{*} .$$

Proof.

Given an element x in $A[G/K]$ represented by $f : M \longrightarrow G/K$. Then $k^{*}h_{*}x$ is represented by $\bar{h}F$ in the pull-back diagram below (where the squares and hence also the rectangle are pull-backs).

$$\begin{array}{ccccc}
\bar{M} & \xrightarrow{F} & S & \xrightarrow{h} & G/L \\
\downarrow & & \downarrow{\scriptstyle \bar{k}} & & \downarrow{\scriptstyle k} \\
M & \xrightarrow[f]{} & G/K & \xrightarrow[h]{} & G/P
\end{array} .$$

Since pull-backs are transitive the pull-back of $f : M \longrightarrow G/K$ along $k_{(H),b}$ is the fibre of $F : \bar{M} \longrightarrow S$ over $M_{(H),b}$, say $F_{(H),b} : \bar{M}_{(H),b} \rightarrow M_{(H),b}$ and this represents $k^{*}_{(H),b}x$. Hence $(h_{(H),b})_{*}(k_{(H),b})^{*}x$ is represented by the composition

$$h_{(H),b}F_{(H),b} : \bar{M}_{(H),b} \longrightarrow M_{(H),b} \longrightarrow G/L.$$

So we have to show that the following two elements are equal in $A[G/L]$, namely $[\bar{h}F]$ and $\sum n_{(H),b}[h_{(H),b}F_{(H),b}]$. This means by definition of $A[G/L]$ that we have to show: For each $U < L$ the U-fixed points of the fibres over the coset L/L of G/L have the same Euler-characteristic.

The fibre of $\bar{h}F$ is the fibre of hf over k(L/L), considered as

L-manifold. Since we are now dealing with G-spaces over G/L the whole situation can be reconstructed from the fibres over L/L, which we denote by an index zero, using canonical G-diffeomorphisms like $G \times_L \overline{M}^o = \overline{M}$. We have for $V < L$

$$M_{(H),b} = G \times_L M^o_{(H),b}, \quad S_{(V)} = G \times_L S^o_{(V)}, \quad S_{(V),b} = G \times_L S^o_{(V),b}$$

using the identification $S_{(V)}/G = S^o_{(V)}/L$.

Let $F : \overline{M}^o \longrightarrow S^o$ be the restriction of the map $F : \overline{M} \longrightarrow S$. As in Section 5.5 we have

(5.12.4) $$\chi((\overline{M}^o)^U) = \sum_{V,b} \chi_c((F^{-1}S^o_{(V),b})^U) .$$

The map

$$F^{-1}(S^o_{(V),b}) \longrightarrow S^o_{(V),b} \longrightarrow S^o_{(V),b}/L$$

is a fibre bundle with the fibre $F^{-1}(M^o_{(V),b})$ such that the U-fixed points again yield a fibration with typical fibre $F^{-1}(M^o_{(V),b})^U$. Then the ((V),b)-summand in (5.12.4) is by Proposition equal to

$$\chi_c(F^{-1}(M^o_{(V),b})^U) \; \chi_c(S^o_{(V),b}/L)$$

$$= \chi_c(F^{-1}(M^o_{(V),b})^U) \; \chi_c(S_{(V),b}/G)$$

$$= \chi_c(F^{-1}(M^o_{(V),b})^U) n_{(V),b}$$

and this was to be shown.

5.13. Multiplicative induction and symmetric powers.

Let K be a subgroup of finite index in G. Let $\mathrm{Hom}_K(G,X)$, for a K-space X, be the space of K-maps $G \longrightarrow X$ with G-action induced by right translation on G'. The functor $X \longmapsto \mathrm{Hom}_K(G,X)$ from K-spaces to G-spaces is right adjoint to the restriction functor and preserves in particular products. Explicitely, we have a natural bijection

$$\mathrm{Map}_G(Y,\mathrm{Hom}_K(G,X)) \cong \mathrm{Map}_K(Y,X),$$

where Y is any G-space. Given $f : Y \longrightarrow \mathrm{Hom}_K(G,X)$ in the set on the left we compose with the K-map $\mathrm{Hom}_K(G,X) \longrightarrow X : f \longmapsto f(1)$ to obtain the corresponding element in the set on the right side. We have chosen K to be of finite index in G in order to avoid some technical problems: In our case $\mathrm{Hom}_K(G,X)$ as a topological space is simply the product $\prod_{y\in G/K} X$ of $|G/K|$ copies of X.

Proposition 5.13.1.

The assignment $X \longmapsto \mathrm{Hom}_K(G,X)$ induces a map $A(K) \longrightarrow A(G)$ which, in general, is not additive but preserves products (X a compact G-ENR).

Proof.

Given $H < G$ we have to compute $\chi(\mathrm{Hom}_K(G,X)^H)$. Since K has finite index in G the space G/H is K-homeomorphic to a finite disjoint union $\coprod_i K/K(i)$ of homogeneous spaces. The equalities

$$\begin{aligned}\mathrm{Hom}_K(G,X)^H &= \mathrm{Hom}_G(G/H,\ \mathrm{Hom}_K(G,X))\\ &= \mathrm{Hom}_K(G/H,X)\\ &= \mathrm{Hom}_K(\coprod_i K/K(i),X)\\ &= \prod_i \mathrm{Hom}_K(K/K(i),X)\\ &= \prod_i X^{K(i)}\end{aligned}$$

show that the Euler-Characteristics in question can be computed from Euler-Characteristics of fixed point sets X^L, $L < K$.

We call $X \longmapsto \mathrm{Hom}_K(G,X)$ and the map induced on the Burnside ring <u>multiplicative induction</u>.

<u>Proposition 5.13.2.</u>

<u>Let</u> L <u>be</u> <u>a</u> <u>finite</u> <u>normal</u> <u>subgroup</u> <u>of</u> G. <u>The</u> <u>assignment</u> $X \longmapsto X/L$ <u>induces</u> <u>a</u> <u>map</u> $A(G) \longrightarrow A(G/L)$. (X <u>a</u> <u>compact</u> G-ENR.)

<u>Proof.</u>

Given $H < G/L$ we have to show that $\chi(X/L^H)$ is determined by Euler-Characteristics of fixed point sets of X. Let P be the inverse image of H in G. Let $B = p^{-1}(X/L^H)$ where $p : X \longrightarrow X/L$ is the quotient map. We consider X and B as P-spaces. An Orbit of X isomorphic to P/U is contained in B if and only if P = LU. Hence B is a union of orbit bundles. From Proposition [B] = [B'] in A(P) where $B' \subset X'$ has a similar meaning as B. Now

$$\chi(X/L^H) = \chi(B/L) = |L|^{-1} \sum_{g \in L} \chi(B^g).$$

Here we have used 5.3.12. Hence $\chi(X/L^H)$ can be computed from Euler-Characteristics as we wanted. We still have to show that X/L is a G/L-ENR. By 5.2.6 if suffices to see that all $M/L^H = B/L$ are ENR. But B is an ENR by 5.2.6 and hence B/L an ENR by 5.2.5.

We now discuss symmetric powers. Let S_r be the symmetric group on r symbols. If X is a G-space then the diagonal action of G on X^r and the permutation action of S_r commute, so we can view X^r as $(S_r \times G)$-space. If M is an S_r-space with trivial G-action then $M \times X^r$ is an $(S_r \times G)$-space. Dividing out the S_r-action yields the G-space $(M \times X^r)/S_r$.

Proposition 5.13.3.

The assignment $(M,X) \longmapsto (M \times X^r)/S_r$ induces a map

$$A(S_r) \times A(G) \longrightarrow A(G).$$

(M,X compact G-ENR's.)

Proof.

We begin by showing that $X \longmapsto X^r$ induces a map $w : A(G) \longrightarrow A(S_r \times G)$. The standard embedding $S_{r-1} \subset S_r$ gives $S_{r-1} \times G$ as a subgroup of finite index in $S_r \times G$. Viewing X as an $(S_{r-1} \times G)$-space via the projection $S_{r-1} \times G \longrightarrow G$ then the $(S_r \times G)$-space X^r is obtained from X using the multiplicative induction corresponding to $S_{r-1} \times G < S_r \times G$. Therefore w is well-defined by 5.13.1. Now consider the following composition of maps

$$A(S_r) \times A(G) \xrightarrow{p \times w} A(S_r \times G) \times A(S_r \times G)$$

$$\xrightarrow{m} A(S_r \times G) \xrightarrow{q} A(G)$$

where w is as above, p is induced by the projection $S_r \times G \longrightarrow S_r$, m is ring-multiplication, and q is the quotient map of 5.13.2. We check that on representatives the above composition is $(M,X) \longmapsto (M \times X^r)/S_r$.

Let $\pi < S_r$ be a subgroup. Then X^r/π is the π-symmetric power, a G-space if X is a G-space. Note that $(S_r/\pi \times X^r)/S_r = X^r/\pi$. Hence we have

Corollary 5.13.4.

$X \longmapsto X^r/\pi$ induces a map $A(G) \longrightarrow A(G)$.

We are going to analyse the formal properties of the map 5.13.3. We write this map

$$A(S_r) \times A(G) \longrightarrow A(G) : (x,y) \longmapsto x\cdot y . \tag{5.13.5}$$

We recall some constructions with the symmetric group. Let X,Y be S_r-, S_t-spaces, respectively. We write

$$X\cdot Y = S_{r+t} \times_{S_r \times S_t} (X \times Y) \tag{5.13.6}$$

using the standard embedding $S_r \times S_t \subset S_{r+t}$.

Let $S_r \int G$ be the <u>wreath-product</u> of G with S_r. This is the set $S_r \times G^r$ with group-law

$$(s;g_1,\dots,g_r)(t;h_1,\dots,h_r) = (st;g_1 h_{s^{-1}(1)},\dots,g_r h_{s^{-1}(r)})$$

If M is a G-space then M^r becomes an $S_r \int G$-space with action

$$(s;g_1,\dots,g_r)(m_1,\dots,m_r) = (g_1 m_{s^{-1}(1)},\dots,g_r m_{s^{-1}(r)}).$$

We consider $S_r \int S_t$ as a subgroup of S_{rt}: If $M = S_t$ as S_t-space then $S_r \int S_t$ acts as a group of permutations on M^r; now identify M^r in a sensible way with $\{1,2,\dots,rt\}$. (The conjugacy class of $S_r \int S_t$ in S_{rt} is then uniquely determined.) Let X,Y be S_r-, S_t-spaces respectively. We write

$$X * Y = S_{rt} \times_{(S_r \int S_t)} (X \times Y^r). \tag{5.13.7}$$

<u>Proposition 5.13.8.</u>

<u>The constructions</u> $(X,Y) \longmapsto X\cdot Y$ <u>and</u> $(X,Y) \longmapsto X * Y$ <u>induce maps</u>

$$A(S_r) \times A(S_t) \longrightarrow A(S_{r+t}) \ : \ (x,y) \longmapsto x \cdot y$$

$$A(S_r) \times A(S_t) \longrightarrow A(S_{rt}) \ : \ (x,y) \longmapsto x * y$$

respectively. The graded additive group

$$A = \bigoplus_{r \geq 0} A(S_r)$$

becomes a graded ring with multiplication $\circ$. Moreover one has

$$(a+b) * c = a * c + b * c$$

$$(a \circ b) * c = (a * c) \circ (b * c)$$

$$(a * b) * c = a * (b * c)$$

$$b * 1 = b.$$

Here $1 \in B(S_o) = Z$.

Proof. The formal algebraic properties of these constructions follow by considering representatives once we have shown that there are well defined induced maps $\circ$ and $*$.

We factorise the required map $\circ$ as

$$A(S_r) \times A(S_t) \xrightarrow[p_1^* \times p_2^*]{} A(S_r \times S_t) \times A(S_r \times S_t)$$

$$\longrightarrow A(S_r \times S_t) \longrightarrow A(S_{r+t})$$

where p_1, p_2 are the projections, the second map is the multiplication in the ring $A(S_r \times S_t)$ and the third map is the extension homomorphism 5.12.1.

Similarly we factorise the map $*$ as

$$A(S_r) \times A(S_t) \xrightarrow[p_1^* \times w]{} A(S_r \int S_t) \times A(S_r \int S_t)$$

$$\longrightarrow A(S_r \int S_t) \longrightarrow A(S_{rt})$$

where p is the projection $S_r \int S_t \longrightarrow S_r$ and where w is induced by $Y \longmapsto Y^r$ (this well-defined!); the second map is again multiplication and the third extension.

We return to the map 5.13.5 which, obviously, is additive in the first variable, so that we obtain an action $A \times A(G) \longrightarrow A(G)$. Moreover the constructions of 5.13.8 have the following properties.

Proposition 5.13.9.
For $a_1, a_2 \in A$ and $b \in A(G)$

$$(a_1 \circ a_2)\cdot b = (a_1 \circ b)(a_2 \circ b)$$

$$(a_1 * a_2)\cdot b = a_1\cdot(a_2\cdot b) \quad .$$

The interpretation of these formulas is this: $a \in A$ induces an operation $b \longmapsto a\cdot b$ on A(G). Addition and multiplication in A corresponds to pointwise addition and multiplication of operation. Finally $*$ is composition of operations. Hence A is a ring of operations. The operations have some obvious naturality properties which we do not write down. The proof of the identities is given by looking at representatives.

5.14. An example: The group SO(3).

Using Wolf [169] , 2.6., one can see that SO(3) has the following conjugacy classes of subgroups:

$SO(3)$	
$S^1 \cong SO(1)$	maximal torus
$S \cong NS^1 \cong O(1)$	normalizer of S^1
$I = A_5$	icosahedral group
$O = S_4$	octahedral group
$T = A_4$	tetrahedral group
$D_n,\ n \geq 2$	dihedral group of order 2n
$Z/n,\ n \geq 1,$	cyclic group.

One has $ND_n = D_{2n}$, $n = 2$; $ND_2 = S_4$, $NA_4 = S_4$, $NS_4 = S_4$, $NA_5 = A_5$, $NO(1) = O(1)$. The cyclic groups do not have finite index in their normalizer.

The ring A(SO(3)) is the set of functions $z \in C(\Phi, Z)$ such that

i) $z(H)$ arbitrary for $H = SO(3), A_5, S_4$, NT.

ii) $z(D_n) \equiv z(D_{2n}) \bmod 2,\ n \neq 2$

iii) $z(A_4) \equiv z(S_4) \bmod 2$

iv) $z(S) \equiv z(S^1) \bmod 2$

v) $z(D_2) + 2z(A_4) + 3z(D_4) \equiv 0 \bmod 6.$

The continuity of z means $\lim_j z(D_{2^j n}) = z(S)$.

If H is a subgroup of SO(3) we denote for simplicity with the same symbol the element [G/H] of A(SO(3)). We give the multiplication table of the elements H. We put (k,n) for the greatest common divisor and let $d_{(k,n)} = 1$ if $(k,n) = k$ and $d_{(k,n)} = 0$ otherwise.

$(S)^2 = S + D_2$ $\qquad$ $S S^1 = S^1$

$S \cdot D_k = D_k + 2d_{(2,k)} D_2$ $\qquad$ $SI = D_5 + D_3 + D_2$

$S \cdot T = D_2$ $\qquad$ $SO = D_4 + D_3 + D_2$

$(S^1)^2 = 2S^1$ $\qquad$ $S^1 H = 0$ for $H \neq S, S^1$.

$$(D_k)^2 = 2D_k + 4d_{(2,k)} D_2$$

$$D_k D_n = 2D_{(k,n)} + 4d_{(2,(k,n))} D_2$$

$$D_k O = 2d_{(4,k)} D_4 + 2d_{(3,k)} D_3 + 2d_{(2,k)} (2-d_{(4,k)}) D_2$$

$$D_k T = 2d_{(2,k)} D_2$$

$$I^2 = I + T + D_5 + D_3$$

$$IT = 2T$$

$$IO = T + 2D_3 + D_2$$

$$O^2 = O + D_4 + D_3 + D_2$$

$$OT = T + D_2$$

$$T^2 = 2T$$

$$D_k I = 2d_{(5,k)} D_5 + 2d_{(3,k)} D_3 + 2d_{(2,k)} D_2$$

The ring A(SO(3)) contains the following idempotent elements

$$x = I - T - D_5 - D_3$$

$$y = S + O - D_4 - D_3$$

$x+y$, $1-x$, $1-y$, $1-x-y$.

5.15. Comments.

The general theory of the Burnside ring of a compact Lie group is based on the authors papers [64], [65], [66]. As far as the equivariant Euler characteristic is concerned there has been a parallel development in the cohomology of groups, see K. Brown [39], [40], [41]. We have been guided in 5.3 by Brown [39]. For 5.3.3 see Floyd ([83], III §3). For 5.3.4 see [39].

It would be desirable to give a unified treatment of the Burnside ring and results in Brown [39] . Also Bass [16] is relevant. The universal ring for Euler-Characteristic in 5.4 has been introduced by Oliver [118] and has also been used by Becker-Gottlieb 5.5.10 and 5.14 is due to Schwänzl [140] , 5.7 is an extension of work of Dress [79] . For general compact groups see Gordon [86] . It would be interesting to find a more general class of G-spaces which satisfy the relations between Euler-Characteristics 5.8.5; suitable finiteness conditions for cohomology should suffice. For 5.9.8 see Zassenhaus [171] and Raghunatan [130] . The results of 5.10 are based on the thesis of Gordon [86] ; see also Gordon [87] . The reader can see that a purely algebraic definition of the Burnside ring for finite torus extensions can be given. This algebraic definition is then also applicable to other arithmetic situations, e.g. representations over p-adic integers. If G acts on a disk D such that all D^H are either empty or contractible then D represents an idempotent in A(G). Oliver [118] has shown that essentially all idempotents of A(G) arise in this way. For 5.13 I could make us of an unpublished manuscript of Rymer [137] . For operations in the Burnside ring see also Siebeneicher [149] .

5.16. Exercises.

1. Compute the ring U(G) of 5.4 for G = SO(3).
2. Given a natural number $n \geq 2$. Can U(G) contain elements x such that $x^{n-1} \neq 0$, but $x^n = 0$?
3. Show that A(SO(3)) has three idecomposable idempotent elements.
4. Compute the units of A(SO(3)) and compare with the units obtainable from 1.5.3.
5. If G is cyclic then permutation representations given an isomorphism $A(G) \cong R(G;Q)$.

6. Use 5.13 to define a λ-ring structure on A(G) by symmetric powers and show that the isomorphism of exercise 5 is compatible with λ-operations (but don't take exterior powers!).

7. Let S be any subring of the rationals. Determine the idempotent elements of $A(G) \otimes_Z S$, in particular for $S = Z_{(p)}$.

8. Let S_G be the homotopy category of pointed G-CW-complexes. Consider the Grothendieck group $K(S_G)$ of this category: The universal abelian group $S_G \longrightarrow K(S_G) : X \longmapsto [X]$, where each cofibration sequence $A \longrightarrow X \longrightarrow X/A$ gives rise to a relation $[X] = [A] + [X/A]$. Show that smashed-product $(X,Y) \mapsto X \wedge Y$ makes $K(S_G)$ into a commutative ring. Show that $K(S_G) \cong U(G)$.