2. The J-homomorphism and quadratic forms.

Having defined the Burnside ring of finite G-sets in the previous chapter we go on to study finite G-sets which arise from G-modules over finite fields and G-invariant quadratic forms on such modules. This will later be used to study permutation representations. In this chapter G will always denote a finite group.

2.1. The J-homomorphism.

We consider torsion G-modules M, i. e. finite abelian groups M together with a left G-action by group automorphisms. Forgetting the group structure on M yields a finite G-set and therefore an element J(M) in the Burnside ring A(G). Since $\varphi_H J(M) = |M^H|$ we have

(2.1.1) $$J(M \oplus N) = J(M)\, J(N)$$

for two torsion G-modules M and N. But J(M) is in general not a unit in A(G) so that J does not immediately extend to a homomorphism from a suitable Grothendieck group. On the category of torsion modules with torsion prime to $|G|$ taking H-fixed points is an exact functor so that $J(M) = J(N)\, J(P)$ for an exact sequence $0 \longrightarrow P \longrightarrow M \longrightarrow N \longrightarrow 0$ of such modules.

Proposition 2.1.2.

Let M be a torsion G-module with $q = |M|$ prime to $|G|$. Then $J(M) \in A(G)[q^{-1}]$ (i.e. q made invertible) is a unit.

Proof.

Using φ of 1.2.2 we see that $\varphi\, J(M)$ is certainly a unit in $\prod Z[q^{-1}]$. We have to show: the inverse is contained in $A(G)[q^{-1}]$. Note that by

1.2.3 the cokernel of $\varphi[q^{-1}]$ is a finite group because q is prime to $|G|$. The next algebraic lemma implies the result.

Lemma 2.1.3.

Let R be a subring of the commutative ring S. Assume that $R \subset S$ is an integral extension (e.g. S/R is a finite group). Then $R^* = R \cap S^*$.

Proof.

Clearly $R^* \subset S^*$. Given $x \in R \cap S^*$. Suppose $y \in S$ satisfies $xy = 1$. Since $S \supset R$ is integral we have $y^n + a_1 y^{n-1} + \ldots + a_n = 0$ for suitable $a_i \in R$. Multiplying this equation by x^{n-1} we obtain $y + a_1 + \ldots + a_n x^{n-1} = 0$, hence $y \in R$.

Let $T_q(G)$ be the Grothendieck group with respect to exact sequences of q-torsion G-modules. Let R(G;F) be the Grothendieck group of finitely generated FG-modules, F a field. Then 2.1.2 implies

Proposition 2.1.4.

Let q be prime to $|G|$. The assignment $M \mapsto J(M)$ induces a homomorphism $J : T_q(G) \longrightarrow A(G)[q^{-1}]^*$. If F is a finite field of characteristic q then we obtain a homomorphism $J : R(G;F) \longrightarrow A(G)[q^{-1}]^*$.

We call this homomorphism the *J-homomorphism*. The relation to the J-homomorphism of algebraic topology will become clear later.

2.2. Quadratic forms on torsion groups. Gauß sums.

Let M be a finite abelian group.

Definition 2.2.1.

A *quadratic form on* M is a map $q : M \longrightarrow Q/Z$ such that

i) q is quadratic, i. e. $q(am) = a^2 q(m)$ for $a \in Z$ and $m \in M$.

ii) the map $b : M \times M \longrightarrow Q/Z$,

$b(m,n) = q(m+n) - q(m) - q(n)$ is biadditive.

If moreover M is a ZG-module we call q G-<u>invariant</u> if for $g \in G$ and $m \in M$ the relation $q(gm) = q(m)$ holds. The form is called <u>non-degenerate</u> if $b^* : M \longrightarrow \mathrm{Hom}(M, Q/Z) : m \longmapsto b(m,-)$ is an isomorphism.

We shall only consider non-degenerate forms. Let $e : Q/Z \longrightarrow \mathbb{C}^*$ be the standard character $e(x \bmod Z) = \exp(2\pi i x)$.

<u>Definition 2.2.2.</u>

Let (M,q) be a quadratic torsion form. The <u>associated</u> (<u>quadratic</u>) <u>Gauß sum</u> is

$$G(M,q) = \sum_{m \in M} eq(m).$$

(We use the letter G despite of its use for groups.)

We now list some formal properties of Gauß sums. If (M_1,q_1) and (M_2,q_2) are quadratic torsion forms we have the <u>orthogonal sum</u>

$$(M_1,q_1) \perp (M_2,q_2) =: (M,q)$$

which is $(M_1 \oplus M_2, q)$ with

$$q(m_1,m_2) = q_1(m_1) + q_2(m_2).$$

Obviously one has

$$G(M,q) = G(M_1,q_1)\, G(M_2,q_2). \tag{2.2.3}$$

Definition 2.2.4.

A quadratic torsion form (M,q) is called split or metabolic if there exists a subgroup $N \subset M$ such that for all $n \in N$ $q(n) = 0$ and moreover $N^{\perp} := \{n \mid b(n,N) = 0\}$ equals N. We then call N a metabolizer of (M,q).

Proposition 2.2.5.

If (M,q) is split with metabolizer N then $G(M,q) = |N|$.

Proof.

Since q is non-degenerate the map

$$M \longrightarrow Hom(M,Q/Z) \longrightarrow Hom(N,Q/Z)$$

is surjective with kernel $N^{\perp}$. By assumption $N = N^{\perp}$. The induced map $\bar{b} : N \times M/N \longrightarrow Q/Z$ is non-degenerate. Therefore $|N| = |M/N|$, $|M| = |N|^2$. For $m \in M$ we have

$$\sum_{n \in N} eq(m+n) = eq(m) \sum_{n \in N} eb(m,n) .$$

If $m \notin N$ then $n \longmapsto eb(m,n)$ is a non-trivial character of N. The sum above is therefore zero in this case. There remains the sum for m = 0 which is equal to $|N|$.

If (M,q) is torsion form, then $(M,q) \perp (M,-q)$ is always split, with metabolizer the diagonal of $M \oplus M$. On the set $KQ^{+}(Q,Z)$ of isomorphism classes of quadratic torsion forms one has the relation of Witt equivalence: $(M_1,q_1) \sim (M_2,q_2)$ if and only if there exist split forms (V_i,r_i) such that $(M_1,q_1) \perp (V_1,q_1) \cong (M_2,q_2) \perp (V_2,q_2)$. The set of Witt equivalence classes WQ(Q/Z) becomes an abelian group, the group structure being induced from orthogonal sum. From 2.2.5 we see that the assignment $(M,q) \longmapsto G(M,q)/\sqrt{|M|}$ induces a homomorphism

$$(2.2.6) \qquad \gamma : WQ(Q/Z) \longrightarrow \mathbb{C}^* .$$

In particular we have

$$(2.2.7) \qquad |G(M,q)|^2 = |M|$$

for any torsion form and $\gamma(M,q)$ is a root of unity.

For the convenience of the reader we now collect the relevant material about Witt groups. The general reference will be Milnor-Husenmoller [117] . Let W(R) be the Witt ring of symmetric inner product spaces ([117] , p. 14) and WQ(R) the Witt algebra of quadratic forms ([117] , p. 112) for a commutative ring R. If we assign to a quadratic form its associated bilinear form we obtain a homomorphism

$$a : WQ(R) \longrightarrow W(R)$$

which is an isomorphism if 2 is a unit in R. There is an exact sequence ([117] , p. 90)

$$(2.2.8) \qquad 0 \longrightarrow W(Z) \longrightarrow W(Q) \longrightarrow W(Q/Z) \longrightarrow 0$$

where W(Q/Z) is the Witt group of symmetric bilinear forms on torsion groups. Moreover

$$W(Q/Z) \cong \bigoplus_p W(Z[p^{-1}]/Z)$$

because a torsion form is uniquely the orthogonal sum of its restrictions to the p-primary parts. Moreover one has an isomorphism

(2.2.9) $$W(F_p) \cong W(Z[p^{-1}]/Z)$$

viewing a form over the ring $F_p = Z/pZ$ as a torsion form. The ring $W(F_p)$ is computed in [117], p. 87. One has $W(Z) = Z$ by the signature homomorphism and the signature splits $W(Z) \longrightarrow W(Q)$.

In the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & WQ(Z) & \longrightarrow & WQ(Q) & \longrightarrow & WQ(Q/Z) & \longrightarrow & 0 \\ & & \downarrow a(Z) & & \downarrow a(Q) & & \downarrow a(Q/Z) & & \\ 0 & \longrightarrow & W(Z) & \longrightarrow & W(Q) & \longrightarrow & W(Q/Z) & \longrightarrow & 0 \end{array}$$

the map $a(Q)$ is an isomorphism and so is $a(Z[p^{-1}]/Z)$ for p odd. The map $a(Z)$ is injective with cokernel of order 8 ([117], p.24). The map

$$WQ(Z[2^{-1}]/Z) \longrightarrow W(Z[2^{-1}]/Z)$$

is surjective and the source is isomorphic to $Z/8Z \times Z/2Z$. A torsion form of order 8 in the Witt group is

$$q : Z/2Z \longrightarrow Q/Z$$

$$q(0) = 0, \quad q(1) = \frac{1}{4} .$$

The value $\gamma(Z/2Z, q)$ of 2.2.6 is in this case $\frac{1}{\sqrt{2}}(1+i)$, a primitive 8-th root unity.

From the quoted results one sees already that $\gamma(M,q)$ has order 2^i,

$0 \leq i \leq 3$. For the actual computation of γ see [117], Appendix 4, or Lang [108], IV §3.

We now study more closely the case of quadratic forms on F_p-modules (alias torsion form). We assume that p is an odd prime.

If (M,q) is given then for $a \in F_p$, $a \neq 0$

(2.2.10) $$q^{-1}(a^2b) = aq^{-1}(b)$$

and the sets $q^{-1}(a^2b)$ and $q^{-1}(b)$ have the same cardinality. Therefore

(2.2.11) $$G(M,q) = \sum_{b \bmod p} q^{-1}(b) \exp(2\pi ib/p)$$

$$= P + Q\alpha + N\beta$$

where

$Q = q^{-1}(b)$ for any non-zero square b in F_p

$N = q^{-1}(c)$ for any non-square c in F_p

$P = q^{-1}(0)$

and

$$-1-\beta = \alpha = \sum \exp(2\pi ib/p)$$

summed over the non-zero squares in F_p. We write 2.2.11 as

(2.2.12) $$G(M,q) = P - N + (Q-N)\alpha ,$$

and we are going to compute P - N and Q - N.

We use the following notations:

$$1 + 2\alpha = g = \sum_{a \bmod p} \exp(2\pi i a^2/p)$$

$(\frac{x}{p})$ Legendre symbol

$D(q) \in F_p/F_p^2$ determinant of the form q.

Proposition 2.2.13.

Let (M,q) *be a form with* $|M| = p^n$. *Then*

$$G(M,q) = (\frac{D(q)}{p})\, g^n .$$

Proof.

Both expressions behave multiplicatively with respect to orthogonal sum. A form over F_p, p odd, is an orthogonal sum of one-dimensional forms. Therefore it suffices to consider the case n = 1. But then the equality is a simple calculation (see Lang [108], QS 1 on p. 85).

From 2.2.12 and 2.2.13 we obtain

$$P - N + (Q-N)(\tfrac{1}{2}(g-1)) = (\frac{D(q)}{p})\,g^n \qquad (2.2.14)$$

where P also denotes the cardinality of the set P, etc. We now use the fact that the absolute value of g is $\sqrt{p}$. Comparing coefficients gives

Proposition 2.2.15.

If $n = 2k$, *then* $Q - N = 0$ *and* $P - N = (\frac{D(q)}{p})g^{2k}$.

If $n = 2k+1$, *then* $2(P-N) = Q-N$ *and* $P-N = (\frac{D(q)}{p})g^{2k}$.

Remark.

Using $P + \frac{1}{2}(p-1)Q + \frac{1}{2}(p-1)N = p^n$ and 2.2.15 one can solve for P,Q, and N thus obtaining the number of solutions of $q(x) = b$.

Finally we recall the elementary computation (Lang [108] , p. 77)

$$g^2 = (\frac{-1}{p})p. \tag{2.2.16}$$

2.3. The quadratic J-homomorphism.

We use equivariant Gauß sums to describe certain refinements of the construction in 2.1.

Let M be a ZG-module which is finite as an abelian group and let (M,q) be a G-invariant quadratic form on M as in 2.2. Since $q:M \longrightarrow Q/Z$ is G-invariant the sets

$$q^{-1}(x), \quad x \in Q/Z$$

are finite G-sets. We consider the equivariant Gauß sum

$$G(M,q) = \sum_{x \in Q/Z} q^{-1}(x)e(x) . \tag{2.3.1}$$

(This is essentially a finite sum). We think of G(M,q) as an element in

$$A(G)[\zeta] = A(G) \otimes_Z Z[\zeta] \subset A(G) \times_Z \mathbb{C}$$

where ζ is a root of unity that generates eqM. For an orthogonal sum we have

(2.3.2) $$G((M_1,q_1) \perp (M_2,q_2)) = G(M_1,q_1)G(M_2,q_2)$$

If we forget the G-action, i. e. put $|q^{-1}(x)| \in Z$ in 2.3.1, then we obtain the Gauß sum G(M,q) of 2.2. Since $b^*: M \longrightarrow Hom(M,Q/Z)$ is an ZG-isomorphism by assumption, q induces on each fixed point set M^H a quadratic form called q^H. Therefore

(2.3.3) $$G(M,q)^H = G(M^H,q^H)$$

with the obvious meanings of the symbols.

As in 2.2.12 we can write

$$G(M,q) = P - N + (Q-N)\alpha$$

where now P,N, and Q are finite G-sets. Here again we work with F_pG-modules, p odd, for simplicity. We describe these G-sets through its fixed point numbers, using 2.2.13. We obtain

Proposition 2.3.4.

Let p be an odd prime and q a G-invariant quadratic form on the F_pG-module M. Then the elements P - N and Q - N of the Burnside ring A(G) have the following fixed point functions:

$$P - N : (H) \longmapsto \left(\frac{D(q^H)}{p}\right) p_*^{\left[\frac{1}{2}\dim M^H\right]}$$

$$Q - N : (H) \longmapsto (1-(-1)^{\dim M^H})\left(\frac{D(q^H)}{p}\right) p_*^{\left[\frac{1}{2}\dim M^H\right]}$$

with $p_* = \left(\frac{-1}{p}\right)p$.

Here $[x]$ is the greatest integer m such that $m \leq x$ and dim is the dimension as F_p vector space. (If $M^H = \{o\}$, then $P = 1$, $Q = N = o$.)

This proposition shows that the equivariant Gauß sum of (M,q) only depends on the underlying F_pG-module and the determinant function, i.e. the determinants $D(q^H)$ of all fixed point forms. If $KQ(G;F_p)$ denotes the Grothendieck group of quadratic forms on F_pG-module (with orthogonal sum as addition) we consider the quotient group which only records the isomorphism type of the underlying module and the determinant. We denote this group $RO'(G,F_p)$. We have natural homomorphisms

(2.3.5)
$$r : RO'(G,F_p) \longrightarrow RO(G,F_p)$$
$$d : RO'(G,F_p) \longrightarrow \prod_{(H)} Z^*.$$

Here r associates to the class of (M,q) the underlying F_pG-module M and $RO(G,F_p)$ is simply the image of r in the representation ring $R(G,F_p)$. Hence r is surjective by definition, Moreover d associates to (M,q) the function $(H) \longmapsto \left(\frac{D(q^H)}{p}\right) \in Z^* = \{+1,-1\}$. The homomorphism

$$(r,d) : RO'(G,F_p) \longrightarrow RO(G,F_p) \times \prod_{(H)} Z^*$$

is injective, by definition. Hence additively the torsion of $RO'(G,F_p)$ contains only elements of order two and the torsion subgroup is mapped injectively under d.

The assignment

$$(M,q) \longmapsto P - N$$

induces a well-defined map

$$(2.3.6) \qquad JQ : RO'(G,F_p) \longrightarrow A(G)\,[p^{-1}]$$

which is not homomorphic from addition to multiplication. We call JQ the <u>quadratic</u> J-<u>homomorphism</u>.

2.4. Comments.

The construction in 2.1 and 2.3 are taken from Segal [146]. For the localization sequence for Witt groups see Pardon [125], and, in the equivariant case, Dress [81]. The use of equivariant Witt groups in topology is explained in Alexander-Conner-Hamrick [3], where the reader will find many computations. For quadratic forms on torsion see e. g. Wall [164], Brumfiel -Morgan [44], and Alexander-Hamrick-Vick [4]. For 2.2.15 and the remark following it see Siegel [150] p. 344. Proposition 2.3.4 is related to recent work of Tornehave [160] (see Madsen [113]).

2.5. Exercises.

1. Let n be a natural number. Let S be a finite G-set. Let n^S be the function $(H) \longmapsto n^{|S^H|}$. Show that $n^S \in A(G)$.

2. It is seen from 2.3.4 that JQ is not additive. Verify the following formula for the deviation from additivity

$$JQ((M_1,q_1) \perp (M_2,q_2)) = d(M_1,M_2)JQ(M_1,q_1)JQ(M_2,q_2)$$

where

$$d(M_1,M_2) = (1+(p_* -1)\tfrac{1}{4}\ (d(M_1)-1)\ (d(M_2)-1))$$

with $d(M) : (H) \longmapsto (-1)^{\dim M^H}$. (Compare 1.5.3)

3. Let F be a field of characteristic not 2 and let G be a group of order prime to char(F). Show that any G-invariant quadratic form over F is an orthogonal sum of indecomposable quadratic modules (M,q). If (M,q) is indecomposable then either M is irreducible and isomorphic to its dual M^* or $M = N \oplus N^*$, $N \neq N^*$, N irreducible, and q is hyperbolic.

4. Extend 2.3.4 to general quadratic forms on torsion groups.

5. Since the signature of $x \in WQ(Z)$ is divisible by 8 the signature homomorphism $WQ(Q) \longrightarrow Z/8Z$ factors over WQ(Q/Z). Compute it! (Compare the formula of Milgram in Milnor-Husemoller [117], p. 127.)