# Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

766



Tammo tom Dieck


# Transformation Groups and Representation Theory


Springer-Verlag
Berlin Heidelberg New York 1979

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AMS Subject Classifications (1970): 20C10, 20C15, 20D15, 55-02, 55B25, 55E50, 57-02, 57D85, 57E15, 57E25

ISBN 3-540-09720-1 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-09720-1 Springer-Verlag New York Heidelberg Berlin

Library of Congress Cataloging in Publication Data
Dieck, Tammo tom.
Transformation groups and representation theory.
(Lecture notes in mathematics; 766)
Bibliography: p.
Includes index.
1. Topological transformation groups. 2. Representations of groups. I. Title. II. Series:
Lecture notes in mathematics (Berlin); 766.
QA3.L28 no. 766 [QA613.7] 510'.8s [514'.223] 79-24606
ISBN 0-387-09720-1

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Printed in Germany

Printing and binding: Beltz Offsetdruck, Hemsbach/Bergstr.
2141/3140-543210

# Preface

These are extended lecture notes for a course on transformation groups which I gave at the Mathematical Institute at Göttingen during the summer term 1978.

The purpose of these notes is to give an introduction to that part of the theory of transformation groups which centers around the Burnside ring and the topology of group representations. It is assumed that the reader is acquainted with the basic material in algebraic topology, representation theory, and transformation groups. Nevertheless we have presented some elementary topics in detail.

Section 11 contains joint work with Henning Hauschild.

My thanks are due to Christian Okonek who read part of the manuscript and made many valuable suggestions and to Margret Rose Schneider who typed the manuscript.

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