The Grothendieck Construction and Gradings for Enriched Categories

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July 1, 2009

Abstract

The Grothendieck construction is a process to form a single category from a diagram of small categories. In this paper, we extend the definition of the Grothendieck construction to diagrams of small categories enriched over a symmetric monoidal category satisfying certain conditions. Symmetric monoidal categories satisfying the conditions in this paper include the category of k-modules over a commutative ring k, the category of chain complexes, the category of simplicial sets, the category of topological spaces, and the category of modern spectra. In particular, we obtain a generalization of the orbit category construction in [CM06]. We also extend the notion of graded categories and show that the Grothendieck construction takes values in the category of graded categories. Our definition of graded category does not require any coproduct decompositions and generalizes k-linear graded categories indexed by small categories defined in [Low08].

There are two popular ways to construct functors from the category of graded categories to the category of oplax functors. One of them is the smash product construction defined and studied in [CM06, Asaa, Asab] for k-linear categories and the other one is the fiber functor. We construct extensions of these functors for enriched categories and show that they are "right adjoint" to the Grothendieck construction in suitable senses.

As a byproduct, we obtain a new short description of small enriched categories.

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1 Introduction

1.1 The Grothendieck Construction

Given a diagram of small categories

$$X : I \longrightarrow \mathbf{Categories}$$

indexed by a small category I, there is a way to form a single category $\mathrm{Gr}(X)$, called the Grothendieck construction on X. The construction first appeared in §8 of Exposé VI in [SGA71]. It can be used to prove an equivalence of categories of prestacks and fibered categories

$$\mathrm{Gr} : \mathbf{Prestacks}(I) \longleftrightarrow \mathbf{Fibered}(I) : \Gamma, \tag{1}$$

where $\mathbf{Prestacks}(I)$ is the category of lax presheaves (contravariant lax functors satisfying certain conditions)

$$X : I^{op} \longrightarrow \mathbf{Categories}$$

and $\mathbf{Fibered}(I)$ is the category of fibered categories

$$\pi : E \longrightarrow I$$

over I, which is a full subcategory of the category of prefibered categories $\mathbf{Prefibered}(I)$ over I. See [Hol, Vis], for example.

It was Quillen who first realized the usefulness of (pre)fibered categories in homotopy theory. In particular, he proved famous Theorem A and B for the classifying spaces of small categories in [Qui73]. The Grothendieck construction was used implicitly in the proofs of these theorems. Subsequently, the classifying space of the Grothendieck construction of a diagram of categories was studied by Thomason [Tho79] who found a description in terms of the homotopy colimit construction of Bousfield and Kan [BK72]

$$B\,\mathrm{Gr}(X) \simeq \mathop{\mathrm{hocolim}}_{I} B \circ X, \tag{2}$$

where

$$B : \mathbf{Categories} \longrightarrow \mathbf{Spaces}$$

is the classifying space functor described, for example, in [Seg68].

Since then the Grothendieck construction has been one of the most indispensable tools in homotopy theory of classifying spaces, as is exposed by Dwyer [DH01]. The work of Quillen [Qui78] suggests the usefulness of the classifying space techniques in combinatorics in which posets are one of the central objects of study. When the indexing category I is a poset and the functor X takes values in the category of posets, the Grothendieck construction of X is called the poset limit of X. Thomason's homotopy colimit description (2) has been proved to be useful in topological combinatorics. See [WZŽ99], for example.

When G is a group regarded as a category with a single object, giving a functor

$$X : G \longrightarrow \mathbf{Categories}$$

is equivalent to giving a category X equipped with a left action of G. The Grothendieck construction has been used implicitly in this context. One of the most fundamental examples is the semidirect product of groups.

The translation groupoid $G \ltimes X$ for an action of a topological group G on a space X used in the study of orbifolds [Moe02] is another example. When X takes values in posets, Borcherds [Bor98] called the Grothendieck construction the homotopy quotient of X by G. This terminology is based on Thomason's homotopy equivalence (2)

$$B\operatorname{Gr}(X) \simeq \underset{G}{\operatorname{hocolim}} BX = EG \times_G BX$$

and the fact that the Borel construction $EG \times_G BX$ is regarded as a "homotopy theoretic quotient" of BX under the action of G.

Algebraists have been studying group actions on k-linear categories

$$X : G \longrightarrow k\textbf{-Categories}$$

and several "quotient category" constructions are known in relation to covering theory of k-linear categories. It turns out that some of them [CM06, Asaa, Asab] are equivalent to a k-linear version of the Grothendieck construction. More general diagrams of k-linear categories are considered by Gerstenhaber and Schack in their study of deformation theory. In particular, the Grothendieck construction was used in [GS83a] in a process of assembling a diagram A of k-linear category into an algebra $A!$.

Thomason's theorem also suggests the Grothendieck construction can be regarded as a kind of colimit construction. It is also stated in Thomason's paper that the construction can be naturally extended to oplax functors. (See Definition 2.42 for oplax functors.) In fact, category theorists studied the Grothendieck construction as a model of 2-colimits in the bicategory of small categories. It was proved by J. Gray [Gra69] (also stated in [Tho79]) that the Grothendieck construction regarded as a functor

$$\operatorname{Gr} : \overleftarrow{\textbf{Oplax}}(I, \textbf{Categories}) \longrightarrow \textbf{Categories}$$

is left adjoint to the diagonal (constant) functor

$$\Delta : \textbf{Categories} \longrightarrow \overleftarrow{\textbf{Oplax}}(I, \textbf{Categories}),$$

where $\overleftarrow{\textbf{Oplax}}(I, \textbf{Categories})$ is the 2-category of oplax functors.

The original motivation for the Grothendieck construction in [SGA71] and the equivalence (1) suggest, however, we should regard Gr as a functor

$$\operatorname{Gr} : \overleftarrow{\textbf{Oplax}}(I, \textbf{Categories}) \longrightarrow \overleftarrow{\textbf{Categories}} \downarrow I,$$

where $\overleftarrow{\textbf{Categories}} \downarrow I$ is a 2-category of comma categories over I whose morphisms are relaxed by taking "left natural transformations" into account. See §2.4 for precise definitions.

On the other hand, the orbit category construction in [CM06, Asaa] gives rise to a functor

$$\operatorname{Gr} : \overleftarrow{\textbf{Funct}}(G, k\textbf{-Categories}) \longrightarrow k\textbf{-}\overleftarrow{\textbf{Categories}}_G,$$

where $k\textbf{-}\overleftarrow{\textbf{Categories}}_G$ is the 2-category of G-graded k-linear categories. In a recent paper [Asab] Asashiba proved that this functor induces an equivalence of 2-categories

$$\operatorname{Gr} : \overleftarrow{\textbf{Funct}}(G, k\textbf{-Categories}) \longleftrightarrow k\textbf{-}\overleftarrow{\textbf{Categories}}_G : \Gamma. \qquad (3)$$

As is suggested by the similarity between this equivalence and (1), we should regard the orbit category construction as a k-linear version of the Grothendieck construction.

This ubiquity of the Grothendieck construction suggests us to work in a more general framework. For example, when we study the derived category of a k-linear category equipped with a group action, it would be useful if we have a general theory of the Grothendieck construction for dg categories, i.e. categories enriched over the category of chain complexes, which can be regarded as a model for enhanced triangulated categories according to Bondal and Kapranov [BK90]. Another important model of enhanced triangulated categories is the notion of stable quasicategory (stable $(\infty, 1)$-category) by Lurie [Lur], which is closely related to stable simplicial categories appeared in a work of Toën and Vezzosi [TV04], i.e. categories enriched over the category of simplicial sets.

1.2 Gradings of Categories

In order to fully understand the meaning of the above similarities and attack the problem of extending the Grothendieck construction to general enriched categories, we should find a unified way to handle the comma category $\overleftarrow{\mathbf{Categories}} \downarrow I$ and the category $k\text{-}\overleftarrow{\mathbf{Categories}}_G$ of G-graded k-linear categories. It is immediate to extend the definition of the Grothendieck construction to diagrams of enriched categories, as we will see in §3.1. The problem is to find the right co-domain category of the Grothendieck construction for general enriched categories.

The notion of group graded k-linear categories has been used as a natural "many objectification" of that of group graded k-algebras. They are often defined in terms of coproduct decompositions, i.e. a k-linear category A graded by a group G is a k-linear category whose module $A(x, y)$ of morphisms has a coproduct decomposition

$$A(x, y) = \bigoplus_{g \in G} A^g(x, y)$$

for each pair of objects x, y satisfying certain compatibility conditions.

For a (non-enriched) small category X, we may also define a G-grading as a coproduct decomposition

$$\mathrm{Mor}_X(x, y) = \coprod_{g \in G} \mathrm{Mor}_X^g(x, y)$$

of the set of morphisms for each x, y, which satisfies the analogous compatibility conditions for group graded k-linear categories. This coproduct approach can be extended to categories graded by a small category, including k-linear categories. See, for example, [Low08].

Notice, however, that in the case of group graded small (non-enriched) categories, a G-grading on a category X can be defined simply as a functor

$$p : X \longrightarrow G.$$

All the necessary compatibility conditions are encoded into the functoriality of p. And this definition of grading is close to the co-domain in (1). It is desirable to redefine graded categories without referring to coproduct decompositions in order to find a correct co-domain of the Grothendieck construction, although the idea of describing a G-graded category as a k-linear functor from A to $k[G]$ fails immediately.

In fact such an approach has already appeared in a classical work on group graded algebras by Cohen and Montgomery [CM84]. When translated into the language of comodules, their observation can be stated as follows.

Lemma 1.1. *Let A be an algebra over a commutative ring k and G be a group. Then there is a one-to-one correspondence between gradings of A by G and comodule algebra structure on A over $k[G]$.*

We pursue this observation and define graded k-linear categories as follows.

Definition 1.2. Let I be a small category. An I-grading on a k-linear category A is a structure of comodule category on A over the coalgebra category $I \otimes k$ generated by I.

Undefined terminologies appearing in the above "definition" will be explained in §2. This definition extends immediately to categories enriched over more general symmetric monoidal categories. A precise definition and basic properties of graded categories will be given in §3.2 in the general context of categories enriched over a symmetric monoidal category.

It turns out this comodule approach of Cohen and Montgomery is appropriate for generalizing the Grothendieck construction and gradings for enriched categories. The Grothendieck construction should be regarded as a process of forming a comodule category from a diagram of categories.

As a byproduct, we also obtain a simple characterization of categories enriched over a symmetric monoidal category in terms of comodules.

Definition 1.3. Let $\mathbf{V}$ be a symmetric monoidal category satisfying a certain mild condition and S be a set. A category enriched over $\mathbf{V}$ with the set of objects S is a monoid object in the monoidal category of S-S-bicomodules.

The definition of S-S-bicomodule and the monoidal structure on the category of S-S-bicomodules will be given in §A.1, including related definitions.

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1.3 Aim and Scope

The aim of this article, therefore, is the following:

- We define a grading of an enriched category by a small category I in terms of comodule structures over a coalgebra category and investigate basic properties of the 2-category of I-graded categories.

- We extend the definition of the Grothendieck construction to enriched categories as a 2-functor

$$\mathrm{Gr} : \overleftarrow{\mathbf{Oplax}}(I, \boldsymbol{V}\text{-}\mathbf{Categories}) \longrightarrow \boldsymbol{V}\text{-}\overleftarrow{\mathbf{Categories}}_I,$$

where $\overleftarrow{\mathbf{Oplax}}(I, \boldsymbol{V}\text{-}\mathbf{Categories})$ is the 2-category of oplax functors and left transformations from I to $\boldsymbol{V}$-enriched categories and $\boldsymbol{V}\text{-}\overleftarrow{\mathbf{Categories}}_I$ is the 2-category of left I-graded categories.

- We show that Gr has a right adjoint

$$\overleftarrow{\Gamma} : \boldsymbol{V}\text{-}\overleftarrow{\mathbf{Categories}}_I \longrightarrow \overleftarrow{\mathbf{Oplax}}(I, \boldsymbol{V}\text{-}\mathbf{Categories}).$$

We also study right versions. In particular we define 2-functors

$$\begin{aligned} \mathrm{Gr} &: \overrightarrow{\mathbf{Lax}}(I^{op}, \boldsymbol{V}\text{-}\mathbf{Categories}) \longrightarrow \boldsymbol{V}\text{-}\overrightarrow{\mathbf{Categories}}_I \\ \overrightarrow{\Gamma} &: \boldsymbol{V}\text{-}\overrightarrow{\mathbf{Categories}}_I \longrightarrow \overrightarrow{\mathbf{Lax}}(I^{op}, \boldsymbol{V}\text{-}\mathbf{Categories}). \end{aligned}$$

- We extend the notion of (pre)fibered and (pre)cofibered categories by Grothendieck and graded (pre)fibered categories by Lowen [Low08] to enriched graded (pre)fibered and (pre)cofibered categories. We define 2-functors

$$\begin{aligned} \Gamma_{\mathrm{cof}} &: \mathbf{Precofibered}(I) \longrightarrow \boldsymbol{V}\text{-}\overleftarrow{\mathbf{Categories}}_I \\ \Gamma_{\mathrm{fib}} &: \mathbf{Prefibered}(I) \longrightarrow \boldsymbol{V}\text{-}\overrightarrow{\mathbf{Categories}}_I \end{aligned}$$

and investigate their relations to $\overleftarrow{\Gamma}$ and $\overrightarrow{\Gamma}$.

1.4 Organization

The paper is written as follows:

- We set up a categorical framework in §2. After a brief summary on monoidal categories in §2.1, we define and study comonoids and comodules over them in §2.2. We describe the standard definition of enriched categories in §2.3. 2-categorical notions used in this paper are recalled in §2.4. We introduce notions of coalgebra categories and comodule categories over a coalgebra category and investigate their properties in §2.5.

- The Grothendieck construction for diagrams of enriched categories is defined in §3. The construction is given in §3.1. We introduce the 2-category of graded categories in §3.2 and the Grothendieck construction is extended into a 2-functor taking values in the 2-category of graded categories in §3.3.

- §4 is devoted to definitions of taking "fibers" of graded enriched categories. We define three ways to take fibers over an object in the grading category in §4.1. The notions of prefibered, precofibered, fibered, and cofibered categories are extended to enriched categories in §4.2.

- In §5, we prove several properties of the Grothendieck construction. We show that the comma category constructions defined in the previous section can be regarded as an extension of the smash product construction and prove that it is right adjoint to the Grothendieck construction. A main result is Theorem 5.1. We also study relations between $\overleftarrow{\Gamma}$ and Γ_{cof} in §5.2.

- We include four appendices. We define categories enriched over a symmetric monoidal category in terms of comodules in §A.1. §A.2 is an enriched version of a well-known fact that the Grothendieck construction can be regarded as a 2-colimit. We specialize the contents of §3 to the case of categories enriched over a symmetric monoidal category whose unit object is terminal and the tensor product is given by the product in §A.3. Constructions corresponding to those in §5 are described in §A.4.

Acknowledgements

2 Categorical Preliminaries

2.1 Monoidal Categories

Throughout this paper, we fix a symmetric monoidal category V and work in V. Before we begin our discussion, let us briefly summarize basic definitions and properties of monoidal categories. A convenient summary is the appendix of [Pfe], for example.

Definition 2.1. Let V be a category. A monoidal structure on V is a collection of the following data:

(1) an object 1 of V,

($\bullet \otimes \bullet$) a (covariant) functor

$$\otimes : V \times V \longrightarrow V,$$

($\bullet \otimes \bullet \otimes \bullet$) a natural isomorphism

$$a_{A,B,C} : A \otimes (B \otimes C) \longrightarrow (A \otimes B) \otimes C$$

called the associator,

($1 \otimes \bullet$) a natural isomorphism

$$\ell_A : 1 \otimes A \longrightarrow A,$$

($\bullet \otimes 1$) a natural isomorphism

$$r_A : A \otimes 1 \longrightarrow A.$$

They are required to satisfy the following three conditions:

($\bullet \otimes \bullet \otimes \bullet \otimes \bullet$) For any objects A, B, C, D in V, the following diagram is commutative

($\bullet \otimes \bullet \otimes 1$) For any objects A, B in V, the following diagram is commutative

($1 \otimes 1$) $\ell_1 = r_1 : 1 \otimes 1 \longrightarrow 1$.

Remark 2.2. The commutativity of the following diagrams follows from the above axioms, according to [Kel64].

(1 ⊗ • ⊗ •) For any objects A, B in $\boldsymbol{V}$, the following diagram is commutative

$$
\begin{array}{ccc}
1 \otimes (A \otimes B) & \xrightarrow{\ \ell\ } & A \otimes B \\
\Big\downarrow{\scriptstyle a} & \nearrow{\scriptstyle \ell} & \\
(1 \otimes A) \otimes B. &
\end{array}
$$

(• ⊗ 1 ⊗ •) For any objects A, B in $\boldsymbol{V}$, the following diagram is commutative

$$
\begin{array}{ccc}
A \otimes (1 \otimes B) & \xrightarrow{\ 1 \otimes \ell\ } & A \otimes B \\
\Big\downarrow{\scriptstyle a} & \nearrow{\scriptstyle r \otimes 1} & \\
(A \otimes 1) \otimes B. &
\end{array}
$$

Our primary examples are the category **Sets** of sets, the category k-**Mod** of k-modules over a commutative ring k, the category $C(k)$ of chain complexes over k, the category **Spaces** of topological spaces, the category **Sets**$^{\Delta^{op}}$ of simplicial sets, and the category **Categories** of small categories. We may also consider one of models of modern category **Spectra** of spectra (in the sense of algebraic topology) [EKMM97, HSS00, MM02]. All these categories have coproducts.

Definition 2.3. We say a monoidal category $(\boldsymbol{V}, \otimes, 1)$ is a monoidal category with coproducts if $\boldsymbol{V}$ is closed under coproducts and $\otimes$ distributes with respect to coproducts.

k-**Mod** and $C(k)$ are different from **Sets**, **Spaces**, and **Categories**.

Definition 2.4. A monoidal category $(\boldsymbol{V}, \otimes, 1)$ is an additive monoidal category if

1. $\boldsymbol{V}$ has a structure of additive category, i.e. it has a 0-object, has finite coproducts, the set of morphisms $\mathrm{Mor}_{\boldsymbol{V}}(A, B)$ from A to B has a structure of Abelian group, and the composition of morphisms is bilinear.

2. $\boldsymbol{V}$ is a monoidal category with finite coproducts,

3. the monoidal structure

$$
\otimes : \mathrm{Mor}_{\boldsymbol{V}}(A, B) \times \mathrm{Mor}_{\boldsymbol{V}}(C, D) \longrightarrow \mathrm{Mor}_{\boldsymbol{V}}(A \otimes C, B \otimes D)
$$

is bilinear.

An additive monoidal category $\boldsymbol{V}$ is called an Abelian monoidal category if it is an Abelian category.

Example 2.5. k-**Mod** and $C(k)$ are Abelian monoidal categories. □

The remaining examples but **Spectra** have the following structure.

Definition 2.6. We say a monoidal category is of product type if the unit object 1 is terminal and $\otimes$ is given by the categorical product.

Example 2.7. **Sets**, **Spaces**, **Sets**$^{\Delta^{op}}$, and **Categories** are of product type. □

The category **Spectra** of spectra is neither Abelian nor of product type. But it can be fit into our work by replacing isomorphisms by weak equivalences. See Remark 2.22.

All of our monoidal categories are symmetric.

Definition 2.8. Let V be a monoidal category. A switching operation on V is a natural transformation

$$t_{A,B} : A \otimes B \longrightarrow B \otimes A,$$

i.e. the diagram

$$
\begin{array}{ccc}
A \otimes B & \xrightarrow{t_{A,B}} & B \otimes A \\
{\scriptstyle f \otimes g} \downarrow & & \downarrow {\scriptstyle g \otimes f} \\
C \otimes D & \xrightarrow{t_{C,D}} & D \otimes C
\end{array}
$$

is commutative for any morphisms f and g.

Definition 2.9. A monoidal category V with a switching operation t is called a symmetric monoidal category if

1. $t_{A,B} \circ t_{B,A} = 1$ for any objects A, B.

2. The following diagram is commutative for any triple of objects A, B, C:

We often have a left adjoint to the "underlying set" functor.

Definition 2.10. Let $(V, \otimes, 1)$ be a monoidal category. The functor

$$\mathrm{Mor}_V(1, -) : V \longrightarrow \mathbf{Sets}$$

is called the "underlying set" functor.

For example, the forgetful functor from k-**Mod** to **Sets** can be expressed as $\mathrm{Mor}_{k\text{-}\mathbf{Mod}}(k, -)$ and k is the unit object in k-**Mod**. In this case we have a left adjoint

$$(-) \otimes k : \mathbf{Sets} \longrightarrow k\text{-}\mathbf{Mod}$$

which assigns to each set S the free module generated by S.

We require this property as a fundamental assumption on our symmetric monoidal category V.

Assumption 2.11. In the rest of this paper, we assume V is a symmetric monoidal category with coproducts satisfying the following conditions.

1. V is closed under finite limits.

2. The "underlying set" functor

$$\mathrm{Mor}_V(1, -) : V \longrightarrow \mathbf{Sets}$$

has a left adjoint

$$(-) \otimes 1 : \mathbf{Sets} \longrightarrow V.$$

Example 2.12. The categories **Sets**, **Sets**$^{\Delta^{op}}$, **Spaces**, k-**Mod**, $C(k)$, **Categories** and **Spectra** all satisfy the above conditions. $\square$

A monoidal category can be regarded as a 2-category with a single object. When we talk about functors between monoidal categories, we need a notion of lax monoidal functor.

Definition 2.13. Let $(\boldsymbol{C}, \otimes, 1_{\boldsymbol{C}})$ and $(\boldsymbol{D}, \otimes, 1_{\boldsymbol{D}})$ be monoidal categories.

1. A lax monoidal functor is a triple (F, μ, η), where

   - $F : \boldsymbol{C} \to \boldsymbol{D}$ is a functor,
   - $\mu_{x,y} : F(x) \otimes F(y) \to F(x \otimes y)$ is a natural transformation,
   - $\eta : 1_{\boldsymbol{D}} \to F(1_{\boldsymbol{C}})$ is a morphism in $\boldsymbol{D}$,

   which make the following diagrams commutative:

   (a)

$$
\begin{array}{ccc}
 & (F(x) \otimes F(y)) \otimes F(z) & \\
\swarrow{\scriptstyle \mu \otimes 1} & & \searrow{\scriptstyle a} \\
F(x \otimes y) \otimes F(z) & & F(x) \otimes (F(y) \otimes F(z)) \\
\downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle 1 \otimes \mu} \\
F((x \otimes y) \otimes z) & & F(x) \otimes F(y \otimes z) \\
\searrow{\scriptstyle F(a)} & & \swarrow{\scriptstyle \mu} \\
 & F(x \otimes (y \otimes z)) &
\end{array}
$$

   (b)

$$
\begin{array}{ccc}
1_{\boldsymbol{D}} \otimes F(x) & \longrightarrow & F(1_{\boldsymbol{C}}) \otimes F(x) \\
\downarrow & & \downarrow \\
F(x) & \longleftarrow & F(1_{\boldsymbol{C}} \otimes x)
\end{array}
$$

   (c)

$$
\begin{array}{ccc}
F(x) \otimes 1_{\boldsymbol{D}} & \longrightarrow & F(x) \otimes F(1_{\boldsymbol{C}}) \\
\downarrow & & \downarrow \\
F(x) & \longleftarrow & F(x \otimes 1_{\boldsymbol{C}})
\end{array}
$$

2. An oplax monoidal functor is a triple (F, μ, η), where

   (a) $F : \boldsymbol{C} \to \boldsymbol{D}$ is a functor;
   (b) $\mu_{x,y} : F(x \otimes y) \to F(x) \otimes F(y)$ is a natural transformation;
   (c) $\eta : F(1_{\boldsymbol{C}}) \to 1_{\boldsymbol{D}}$ is a morphism in $\boldsymbol{D}$,

   which make the following diagrams are commutative:

(a)

$$(F(x) \otimes F(y)) \otimes F(z)$$

$$\mu \otimes 1 \qquad\qquad a$$

$$F(x \otimes y) \otimes F(z) \qquad\qquad F(x) \otimes (F(y) \otimes F(z))$$

$$\mu \uparrow \qquad\qquad \uparrow 1 \otimes \mu$$

$$F((x \otimes y) \otimes z) \qquad\qquad F(x) \otimes F(y \otimes z)$$

$$F(a) \qquad\qquad \mu$$

$$F(x \otimes (y \otimes z))$$

(b)

$$1_{\boldsymbol{D}} \otimes F(x) \longleftarrow F(1_{\boldsymbol{C}}) \otimes F(x)$$

$$\downarrow \qquad\qquad \uparrow$$

$$F(x) \longleftarrow F(1_{\boldsymbol{C}} \otimes x)$$

(c)

$$F(x) \otimes 1_{\boldsymbol{D}} \longleftarrow F(x) \otimes F(1_{\boldsymbol{C}})$$

$$\downarrow \qquad\qquad \uparrow$$

$$F(x) \longleftarrow F(x \otimes 1_{\boldsymbol{C}})$$

2.2 Comonoids, Comodules, and Coproduct Decompositions

In this section, we introduce and study the notion of comonoids in a symmetric monoidal category and comodules over them, which play central roles in this paper.

Let $\boldsymbol{V}$ be a symmetric monoidal category satisfying the conditions in Assumption 2.11.

Definition 2.14. A comonoid object in $\boldsymbol{V}$ is an object C equipped with morphisms

$$\begin{aligned} \Delta &: & C &\longrightarrow C \otimes C \\ \varepsilon &: & C &\longrightarrow 1 \end{aligned}$$

making the following diagrams commutative

$$C$$

$$\cong \swarrow \quad \downarrow \Delta \quad \searrow \cong$$

$$1 \otimes C \xleftarrow{\;\varepsilon \otimes 1\;} C \otimes C \xrightarrow{\;1 \otimes \varepsilon\;} C \otimes 1$$

$$C \xrightarrow{\;\Delta\;} C \otimes C$$

$$\Delta \downarrow \qquad\qquad \downarrow \Delta \otimes 1$$

$$\qquad\qquad (C \otimes C) \otimes C$$

$$\qquad\qquad \uparrow a$$

$$C \otimes C \xrightarrow{\;1 \otimes \Delta\;} C \otimes (C \otimes C),$$

10

where a is the associator in $\boldsymbol{V}$.

The morphisms Δ and ε are called the coproduct and the counit of C, respectively.

Morphisms of comonoids are defined in an obvious way. The category of comonoids in $\boldsymbol{V}$ is denoted by $\mathbf{Comonoids}(\boldsymbol{V})$.

Example 2.15. When 1 is terminal and $\otimes$ is the product, i.e. when $\boldsymbol{V}$ is of product type (Definition 2.6), any object in $\boldsymbol{V}$ has a canonical comonoid structure. $\qquad\square$

A typical example of such a monoidal category is the category of sets. Under our assumption on $\boldsymbol{V}$ (Assumption 2.11), we can compare the category of sets and $\boldsymbol{V}$ by the "free object" functor

$$(-) \otimes 1 : \mathbf{Sets} \longrightarrow \boldsymbol{V},$$

which is left adjoint to the underlying set functor.

Lemma 2.16. $(-) \otimes 1$ *is an oplax monoidal functor.*

Proof. We need to define a natural transformation

$$\theta : ((-) \times (-)) \otimes 1 \Longrightarrow ((-) \otimes 1) \otimes ((-) \otimes 1).$$

For sets S, T, consider the composition

$$
\begin{array}{rl}
S \times T & \longrightarrow \quad \mathrm{Mor}_{\boldsymbol{V}}(1, S \otimes 1) \times \mathrm{Mor}_{\boldsymbol{V}}(1, T \otimes 1) \\
& \xrightarrow{\otimes} \quad \mathrm{Mor}_{\boldsymbol{V}}(1 \otimes 1, (S \otimes 1) \otimes (T \otimes 1)) \\
& \xrightarrow{\cong} \quad \mathrm{Mor}_{\boldsymbol{V}}(1, (S \otimes 1) \otimes (T \otimes 1)).
\end{array}
$$

By taking the adjoint, we obtain

$$\theta_{S,T} : (S \times T) \otimes 1 \longrightarrow (S \otimes 1) \otimes (T \otimes 1).$$

By taking the left adjoint to the map

$$1_1 : \{*\} \longrightarrow \mathrm{Mor}_{\boldsymbol{V}}(1, 1)$$

representing the identity morphism on the unit object, we obtain

$$\eta : \{*\} \otimes 1 \longrightarrow 1.$$

It is straightforward to check that θ and η satisfy the condition for an oplax monoidal functor. $\qquad\square$

Example 2.17. For any set S, $S \otimes 1$ has a comonoid structure defined as follows. We have

$$\theta : (S \times S) \otimes 1 \longrightarrow (S \otimes 1) \otimes (S \otimes 1)$$

by the above Lemma. By composing with

$$\Delta \otimes 1 : S \otimes 1 \longrightarrow (S \times S) \otimes 1,$$

we obtain a coproduct

$$\Delta : S \otimes 1 \longrightarrow (S \otimes 1) \otimes (S \otimes 1).$$

The counit is defined by

$$\varepsilon : S \otimes 1 \longrightarrow \{*\} \otimes 1 \xrightarrow{\eta} 1.$$

$\qquad\square$

Definition 2.18. Let C be a comonoid in $\boldsymbol{V}$. A right coaction of C on an object M in $\boldsymbol{V}$ is a morphism

$$\mu : M \longrightarrow M \otimes C$$

making the following diagrams commutative

$$
\begin{array}{ccc}
& & M \\
& {\scriptstyle\cong}\swarrow & \downarrow{\scriptstyle\mu} \\
M\otimes 1 & \xleftarrow{\;1\otimes\varepsilon,\;} & M\otimes C
\end{array}
$$

$$
\begin{array}{ccc}
M & \xrightarrow{\;\mu\;} & M\otimes C \\
\downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle 1\otimes\Delta} \\
& & M\otimes(C\otimes C) \\
& & \downarrow{\scriptstyle a} \\
M\otimes C & \xrightarrow{\;\mu\otimes 1\;} & (M\otimes C)\otimes C.
\end{array}
$$

An object M equipped with a right coaction of C is called a right C-comodule. Morphisms of right comodules are defined in an obvious way.

Left coactions are defined analogously. An object M equipped with both right coaction μ^R and left coaction μ^L is called a C-C-bimodule if the following diagram is commutative.

$$
\begin{array}{ccc}
M & \xrightarrow{\qquad\mu^L\qquad} & C\otimes M \\
\downarrow{\scriptstyle\mu^R} & & \downarrow{\scriptstyle 1\otimes\mu^L} \\
M\otimes C & & C\otimes(M\otimes C) \\
& {\scriptstyle\mu^L\otimes 1}\searrow \quad \swarrow{\scriptstyle a} & \\
& (C\otimes M)\otimes C &
\end{array}
$$

The categories of right C-comodules, left C-comodules, and C-C-bimodules are denoted by **Comod**-C, C-**Comod**, and C-**Comod**-C, respectively.

Example 2.19. Suppose $\boldsymbol{V}$ is of product type. By Example 2.15, any object C can be regarded as a comonoid and any morphism

$$
f : M \longrightarrow C
$$

is a morphism of comonoids. Define

$$
\mu : M \xrightarrow{\;\Delta\;} M\otimes M \xrightarrow{\;1\otimes f\;} M\otimes C,
$$

then μ is a coaction of C on M.

Conversely any coaction μ is determined by the composition

$$
\pi : M \xrightarrow{\;\mu\;} M\otimes C \xrightarrow{\;\mathrm{pr}_2\;} C,
$$

since the composition of μ and the first projection is always the identity morphism by the counit condition. $\quad\square$

The following is a direct extension of an observation (Lemma 1.1) by Cohen and Montgomery in [CM84].

Lemma 2.20. *Suppose $\boldsymbol{V} = k$-**Mod** for a commutative ring k. Let M be an object of $\boldsymbol{V}$ and S be a set. Then there is a one-to-one correspondence between right $S\otimes 1$-comodule structures on M and coproduct decompositions on M indexed by S*

$$
M \cong \bigoplus_{s\in S} M^s.
$$

Proof. Suppose we have a coproduct decomposition

$$M \cong \bigoplus_{s \in S} M^s.$$

Define

$$\mu : M \longrightarrow M \otimes (S \otimes 1)$$

on each component M^s by the composition

$$M^s \overset{\cong}{\longleftarrow} M^s \otimes 1 \overset{1 \otimes s}{\longrightarrow} M^s \otimes (S \otimes 1).$$

The coassociativity follows from the commutativity of the following diagram

The counitality is obvious.

Conversely suppose we have a comodule structure

$$\mu : M \longrightarrow M \otimes (S \otimes 1).$$

For each $s \in S$, define a morphism

$$p_s : M \longrightarrow M$$

by the composition

By the coassociativity of μ and the fact that the coproduct on $S \otimes 1$ is given by the diagonal on S, we have

$$p_s \circ p_t = \begin{cases} p_s, & s = t \\ 0, & s \neq t. \end{cases}$$

By the definition of p_s and the counitality of μ,

$$\sum_{s \in S} p_s = 1_M$$

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and we have a coproduct decomposition

$$M \cong \bigoplus_{s \in S} \operatorname{Im} p_s.$$

□

The above discussion is based on the fact that k-**Mod** is Abelian. Even when V is not Abelian, we often obtain a coproduct decomposition from a comodule structure. Suppose V is a product type monoidal category. In such a monoidal category, a coaction

$$\mu : M \longrightarrow M \otimes (S \otimes 1)$$

defines and is defined by a morphism

$$\pi : M \xrightarrow{\mu} M \otimes (S \otimes 1) \longrightarrow S \otimes 1$$

by Example 2.19.

Example 2.21. Let V be the category **Spaces** of topological spaces. This is a product type monoidal category and thus any object C is a comonoid and a coaction of C on another object M determines and is determined by a morphism

$$p : M \longrightarrow C$$

When C has a discrete topology, i.e. $C = S \otimes 1$ for a set S, such a continuous map induces a coproduct decomposition

$$M \cong \coprod_{s \in S} M^s$$

by $M^s = p^{-1}(s)$. We have analogous decompositions when V is the category of sets, of simplicial sets, or of small categories. □

Remark 2.22. When V is one of models of symmetric monoidal category of spectra, we do not have such a correspondence between comodule structures and coproduct decompositions. However, we can obtain analogous coproduct decomposition from a comodule structure by replacing isomorphisms by weak equivalences.

The following operation plays an important role.

Definition 2.23. Let C be a comonoid in V and M and N be a right and a left C-comodules. Define the cotensor product $M \square_C N$ of M and N over C by the following equalizer diagram

$$M \square_C N \longrightarrow M \otimes N \underset{1_M \otimes \mu_N}{\overset{\mu_M \otimes 1_N}{\rightrightarrows}} M \otimes C \otimes N.$$

We need the following condition on C.

Definition 2.24. An object C in V is called flat if $C \otimes (-)$ preserves equalizers.

Remark 2.25. Since V is symmetric, if C is flat, $(-) \otimes C$ also preserves equalizers.

The condition that $C \otimes (-)$ preserves equalizers is satisfied when V is **Sets**, **Spaces**, **Sets**$^{\Delta^{op}}$ for any C and when V is k-**Mod** and C is flat. In particular, when $V = k$-**Mod** and $C = S \otimes 1$, the free k-module generated by a set S, the condition is satisfied.

Lemma 2.26. *Let C be a flat comonoid object in V. For C-C-bicomodules M and N, $M \square_C N$ has a structure of C-C-bicomodule under which the canonical morphism*

$$M \square_C N \longrightarrow M \otimes N$$

*is a morphism of bicomodules. Furthermore $\square_C$ defines a monoidal structure on C-**Comod**-C. The unit object is C.*

Proof. For C-C-bicomodules

$$
\begin{aligned}
\mu_M^L &: M \longrightarrow C \otimes M, \\
\mu_M^R &: M \longrightarrow M \otimes C, \\
\mu_N^L &: N \longrightarrow C \otimes N, \\
\mu_N^R &: N \longrightarrow N \otimes C,
\end{aligned}
$$

we need to define a bimodule structure on $M\square_C N$.

Since $C \otimes (-)$ preserves equalizers, we can define a left coaction

$$
\mu_M^L \square_C 1 : M\square_C N \longrightarrow C \otimes (M\square_C N)
$$

by the following diagram

The right comodule structure is defined analogously.

The associator of $\otimes$ defines an associator for $\square_C$ and $(C\text{-}\mathbf{Comod}\text{-}C, \square_C, C)$ becomes a monoidal category. $\square$

We also need to compare comodules over different comonoids.

Definition 2.27. Let C and D be comonoids in $\boldsymbol{V}$ and M and N be right comodules over C and D, respectively. A morphism of right comodules from M to N

$$
f : M \longrightarrow N
$$

is a pair $f = (f_0, f_1)$ morphisms

$$
\begin{aligned}
f_0 &: C \longrightarrow D \\
f_1 &: M \longrightarrow M,
\end{aligned}
$$

where f_0 is a morphism of comonoids and f_1 makes the following diagram commutative

Morphisms of left comodules and bicomodules are defined analogously.

2.3   Enriched Categories

In this section, we collect the standard definitions concerning categories enriched over a symmetric monoidal category. We can use the language developed in §2.2 to define (small) enriched categories and related notions in a very compact way, as is shown in §A.1. We have chosen to use the traditional definitions for the convenience of the reader. Our reference is Kelly's book [Kel82]. The reader is encouraged to compare definitions in this section and the corresponding definitions in §A.1.

Definition 2.28. A category enriched over $\boldsymbol{V}$, or simply a $\boldsymbol{V}$-category A consists of

- a class of objects A_0;

- for two objects a, b in A, an object $A(a, b)$ in $\boldsymbol{V}$;

- for three objects a, b, c in A, a morphism

$$\circ : A(b, c) \otimes A(a, b) \longrightarrow A(a, c)$$

in V;

- for an object a in A, a morphism in V

$$1_a : 1 \longrightarrow A(a, a)$$

satisfying the following conditions:

1. for any objects a, b, c, d, the following diagram is commutative

$$
\begin{array}{ccc}
(A(c,d) \otimes A(b,c)) \otimes A(a,b) & \overset{a}{\longrightarrow} & A(c,d) \otimes (A(b,c) \otimes A(a,b)) \\
{\scriptstyle \circ \otimes 1} \Big\downarrow & & \Big\downarrow {\scriptstyle 1 \otimes \circ} \\
A(b,d) \otimes A(a,b) & & A(c,d) \otimes A(a,c) \\
& \overset{\circ}{\searrow} \qquad \swarrow {\scriptstyle \circ} & \\
& A(a,d) &
\end{array}
$$

2. for any objects a, b, the following diagram is commutative

$$
\begin{array}{ccccc}
A(b,b) \otimes A(a,b) & \overset{\circ}{\longrightarrow} & A(a,b) & \overset{\circ}{\longleftarrow} & A(a,b) \otimes A(a,a) \\
\Big\uparrow & \nearrow & & \nwarrow & \Big\uparrow \\
1 \otimes A(a,b) & & & & A(a,b) \otimes 1
\end{array}
$$

where 1 is the unit object in V.

Any enriched category has its underlying category.

Definition 2.29. Let A be a V-category. Define an ordinary category $\underline{A}$ as follows. Objects are the same as objects in A. The set of morphisms from $a \in A_0$ to $b \in A_0$ is defined by

$$\mathrm{Mor}_{\underline{A}}(a, b) = \mathrm{Mor}_V(1, A(a, b)).$$

The composition is given by

$$
\begin{aligned}
\mathrm{Mor}_{\underline{A}}(b, c) \times \mathrm{Mor}_{\underline{A}}(a, b) \quad &= \quad \mathrm{Mor}_V(1, A(b, c)) \times \mathrm{Mor}_V(1, A(a, b)) \\
&\longrightarrow \quad \mathrm{Mor}_V(1 \otimes 1, A(b, c) \otimes A(a, b)) \\
&\overset{\cong}{\longleftarrow} \quad \mathrm{Mor}_V(1, A(b, c) \otimes A(a, b)) \\
&\longrightarrow \quad \mathrm{Mor}_V(1, A(a, c)) \\
&= \quad \mathrm{Mor}_{\underline{A}}(a, c).
\end{aligned}
$$

When it is obvious from the context, a V-category and its underlying category are denoted by the same symbol.

Definition 2.30. Let A be a V-category. For a morphism in the underlying category $f \in \mathrm{Mor}_{\underline{A}}(a, b)$, define morphisms in V

$$
\begin{aligned}
f_* &: \quad A(c, a) \longrightarrow A(c, b), \\
f^* &: \quad A(b, c) \longrightarrow A(a, c)
\end{aligned}
$$

by the following compositions

$$
\begin{aligned}
f_* &: \quad A(c, a) \cong 1 \otimes A(c, a) \overset{f \otimes 1}{\longrightarrow} A(a, b) \otimes A(c, a) \overset{\circ}{\longrightarrow} A(c, b), \\
f^* &: \quad A(b, c) \cong A(b, c) \otimes 1 \overset{1 \otimes f}{\longrightarrow} A(b, c) \otimes A(a, b) \overset{\circ}{\longrightarrow} A(a, c).
\end{aligned}
$$

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The following is the standard description of V-functors.

Definition 2.31. Let A and B be small V-categories. A V-functor f from A to B consists of a map

$$f_0 : A_0 \longrightarrow B_0$$

and a morphism

$$f_1 : A(a,b) \longrightarrow B(f(a), f(b))$$

in V for each pair a, b of objects in A, satisfying the following conditions:

1. the following diagram is commutative

$$
\begin{array}{ccc}
A(b,c) \otimes A(a,b) & \longrightarrow & A(a,c) \\
\downarrow & & \downarrow \\
B(f(b), f(c)) \otimes B(f(a), f(b)) & \longrightarrow & B(f(a), f(c)),
\end{array}
$$

2. and the following diagram is commutative

$$
\begin{array}{ccc}
1 & \longrightarrow & A(a,a) \\
& \searrow & \downarrow \\
& & B(f(a), f(a)).
\end{array}
$$

Definition 2.32. The category of small V-categories and V-functors is denoted by V-**Categories**.

The monoidal structure on V-**Categories** (Lemma A.10) can be described as follows.

Lemma 2.33. *For small V-categories A and B, define a V-category $A \otimes B$ by*

$$(A \otimes B)_0 = A_0 \times B_0$$

and

$$(A \otimes B)((a,b),(a',b')) = A(a,a') \otimes B(b,b').$$

Define a V-category 1 with a single object $*$ by $1(*,*) = 1$.

If V is symmetric monoidal, $(V\text{-}\mathbf{Categories}, \otimes, 1)$ forms a symmetric monoidal category whose associator is defined by that of V.

Example 2.34. When V is the category of k-modules for a commutative ring k, V-categories are called k-linear categories and the category of small k-linear categories is denoted by k-**Categories**. When V is the category of (unbounded) chain complexes over a fixed commutative ring k, V-categories are called dg (differential graded) categories over k.

When V is the category of simplicial sets, topological spaces, or spectra (in the sense of algebraic topology), V-categories are called simplicial categories, topological categories, or spectral categories. □

Example 2.35. Consider the case when V is the category **Categories** of small categories. **Categories**-enriched categories are usually called (strict) 2-categories. **Categories**-functors are called 2-functors. See §2.4 for more details on 2-categories. □

The following is the standard definition of V-natural transformation.

Definition 2.36. Let

$$f, g : A \longrightarrow B$$

be V-functors. A V-natural transformation φ from f to g, denoted by

$$\varphi : f \Longrightarrow g$$

consists of a family of morphisms in $\boldsymbol{V}$

$$\varphi(a) : 1 \longrightarrow B(f(a), g(a))$$

indexed by objects in A making the following diagram commutative for any pair of objects in A:

The composition of $\boldsymbol{V}$-natural transformations is defined in an obvious way.

Definition 2.37. Let

$$f, g, h : A \longrightarrow B$$

be $\boldsymbol{V}$-functors and

$$f \overset{\varphi}{\Longrightarrow} g \overset{\psi}{\Longrightarrow} h$$

be $\boldsymbol{V}$-natural transformations. The composition

$$\psi \circ \varphi : f \Longrightarrow h$$

is defined by

$$(\psi \circ \varphi)(a) : 1 \overset{\cong}{\longrightarrow} 1 \otimes 1 \overset{\psi(a) \otimes \varphi(a)}{\longrightarrow} B(g(a), h(a)) \otimes B(f(a), g(a)) \overset{\circ}{\longrightarrow} B(f(a), h(a)).$$

2.4   The Language of 2-Categories

We need the language of 2-categories in order to fully understand the role and meanings of the Grothendieck construction. Our reference is [Str72].

Definition 2.38. A (strict) 2-category $\boldsymbol{C}$ is a category enriched over the category of small categories. In other words, for each pair of objects x and y in $\boldsymbol{C}$, $\boldsymbol{C}(x, y)$ is a category. Objects in $\boldsymbol{C}(x, y)$ are called 1-morphisms and morphisms in $\boldsymbol{C}(x, y)$ are called 2-morphisms.

The category obtained from a 2-category $\boldsymbol{C}$ by forgetting 2-morphisms is denoted by $\mathrm{sk}_1 \boldsymbol{C}$.

Example 2.39. $\boldsymbol{V}$-categories, $\boldsymbol{V}$-functors, and $\boldsymbol{V}$-natural transformations form a 2-category $\boldsymbol{V}$-**Categories**. ☐

Example 2.40. Any category can be regarded as a 2-category whose 2-morphisms are identities. ☐

Example 2.41. Let B be a $\boldsymbol{V}$-category. Define a 2-category $\boldsymbol{V}$-$\overleftarrow{\textbf{Categories}} \downarrow B$ as follows. Objects are $\boldsymbol{V}$-functors

$$\pi : E \longrightarrow B.$$

A left morphism from $\pi : E \to B$ to $\pi' : E' \to B$ is a pair (F, φ) of a $\boldsymbol{V}$-functor

$$F : E \longrightarrow E'$$

and a $\boldsymbol{V}$-natural transformation

$$\varphi : \pi' \circ F \Longrightarrow \pi.$$

For left morphisms
$$(F, \varphi), (G, \psi) : \pi \longrightarrow \pi',$$
a 2-morphism from (F, φ) to (G, ψ) is a $\boldsymbol{V}$-natural transformation
$$\xi : F \Longrightarrow G$$
making the following diagram commutative

$$
\begin{array}{ccc}
\pi' \circ F & \xrightarrow{\ \pi' \circ \xi\ } & \pi' \circ G \\
& \varphi \searrow \quad \swarrow \psi & \\
& \pi &
\end{array}
$$

By reversing the direction of φ in left morphisms, we obtain right morphisms and another 2-category $\overrightarrow{\boldsymbol{V}\text{-}\mathbf{Categories}} \downarrow B$ with the same objects. $\qquad\square$

We can define and discuss "functors" between 2-categories. One of them is 2-functors in Example 2.35. However, we often encounter "functors up to natural transformations". The notion of lax functor was introduced by Street in [Str72] in order to describe such "functors". For the Grothendieck construction, we need oplax functors from ordinary categories to 2-categories.

Definition 2.42. Let I be a category and $\boldsymbol{C}$ be a 2-category. An oplax functor from I to $\boldsymbol{C}$ consists of the following:

- a morphism of quivers
$$F : I \longrightarrow \mathrm{sk}_1 \, \boldsymbol{C},$$

- for each object $i \in I_0$, a 2-morphism
$$\eta_i : F(1_i) \Longrightarrow 1_{F(i)},$$

- for each pair of composable morphisms $i \xrightarrow{u} i' \xrightarrow{u'} i''$, a 2-morphism
$$\theta_{u',u} : F(u' \circ u) \Longrightarrow F(u') \circ F(u),$$

satisfying the following conditions:

1. For any morphism $u : i \to j$ in I, the following diagram of 2-morphisms are commutative

$$
\begin{array}{ccc}
F(u) \circ 1_{F(i)} & \xLongleftarrow{\quad} & F(u) \circ F(1_i) \\
\big\| & & \big\Uparrow \\
F(u) & =\!=\!=\!= & F(u \circ 1_i)
\end{array}
$$

$$
\begin{array}{ccc}
F(1_j) \circ F(u) & \Longrightarrow & 1_{F(j)} \circ F(u) \\
\big\Uparrow & & \big\| \\
F(1_j \circ u) & =\!=\!=\!= & F(u)
\end{array}
$$

2. For each sequence $a \xrightarrow{u} b \xrightarrow{v} c \xrightarrow{w} d$ in I, the following diagram of natural transformations is commutative

$$
\begin{array}{ccc}
F(w \circ v \circ u) & \Longrightarrow & F(w \circ v) \circ F(u) \\
\big\Downarrow & & \big\Downarrow \\
F(w) \circ F(v \circ u) & \Longrightarrow & F(w) \circ F(v) \circ F(u).
\end{array}
$$

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Remark 2.43. There seem to be confusions on these terminologies. Our oplax functors are called lax functors by Goerss and Jardine [GJ99]. Lax functors in the sense of Street are required to "compose" functors

$$\theta_{u',u} : F(u') \circ F(u) \Longrightarrow F(u' \circ u),$$

with respect to compositions of morphisms in I. We follow the original terminology in [Str72]. A precise definition of lax functor can be found in this paper of Street's.

Definition 2.44. Let I and C be as above. Let

$$X, Y : I \longrightarrow C$$

be oplax functors. A left transformation from X to Y consists of

- a family of 1-morphisms

$$F(i) : X(i) \longrightarrow Y(i)$$

  indexed by objects $i \in I_0$,

- a family of 2-morphisms $\varphi(u)$ indexed by morphisms in I with

$$
\begin{array}{ccc}
X(i) & \xrightarrow{\ F(i)\ } & Y(i) \\
\scriptstyle X(u) \downarrow & \overset{\varphi(u)}{\swarrow\!\!\!\!\diagup} & \downarrow \scriptstyle Y(u) \\
X(j) & \xrightarrow[\ F(j)\ ]{} & Y(j)
\end{array}
$$

  if $u : i \to j$ in I,

satisfying the following conditions:

1. For any object $i \in I_0$, the following diagram is commutative

$$
\begin{array}{ccc}
Y(1_i) \circ F(i) & \Longrightarrow & F(i) \circ X(1_i) \\
\big\| & & \big\| \\
1_{Y(i)} \circ F(i) & = & F(i) \circ 1_{X(i)}
\end{array}
$$

2. For composable morphisms $i \xrightarrow{u} j \xrightarrow{v} k$ in I, the following diagram is commutative

$$
\begin{array}{ccccc}
Y(v \circ u) \circ F(k) & \Longrightarrow & Y(u) \circ Y(v) \circ F(k) & \Longrightarrow & Y(u) \circ F(j) \circ X(v) \\
\big\| & & & & \big\Downarrow \\
F(k) \circ X(v \circ u) & & = \!\!\!=\!\!\!=\!\!\!=\!\!\!=\!\!\!=\!\!\!=\!\!\!\Longrightarrow & & F(k) \circ X(v) \circ X(u).
\end{array}
$$

We denote

$$(F, \varphi) : X \longrightarrow Y.$$

For left transformations

$$X \xrightarrow{(F,\varphi)} Y \xrightarrow{(G,\psi)} Z,$$

the composition $(G \circ F, \psi \circ \varphi) : X \to Z$ is defined by

$$(G \circ F)(i) = G(i) \circ F(i)$$

for $i \in I_0$ and, for $u : i \to j$ in I,

$$(\psi \circ \varphi)(u) : Z(u) \circ (G \circ F)(i) \Longrightarrow (G \circ F)(j) \circ X(u)$$

is defined by the composition

$$
\begin{array}{rcl}
Z(u) \circ (G \circ F)(i) & = & Z(u) \circ G(i) \circ F(i) \\
& \Longrightarrow & G(j) \circ Y(u) \circ F(i) \\
& \Longrightarrow & G(j) \circ F(j) \circ X(u) \\
& = & (G \circ F)(j) \circ X(u).
\end{array}
$$

The reader might have noticed that the natural transformation $\varphi(u)$ in the above definition goes in a wrong direction with respect to F. But it is in the right direction with respect to u.

Definition 2.45. Let X and Y be as above. A right transformation from X to Y consists of a family of functors

$$F(i) : X(i) \longrightarrow Y(i)$$

indexed by $i \in I_0$ and a family of 2-morphisms $\varphi(u)$

$$
\begin{array}{ccc}
X(i) & \xrightarrow{\ F(i)\ } & Y(i) \\
{\scriptstyle X(u)}\big\downarrow & {\scriptstyle \varphi(u)} \nearrow\!\!\!\!\!\nearrow & \big\downarrow {\scriptstyle Y(u)} \\
X(j) & \xrightarrow[\ F(j)\ ]{} & Y(j)
\end{array}
$$

in C satisfying conditions analogous to those of left transformations.

We may dualize and define left and right transformations for lax functors.

Definition 2.46. Let I and C be as above. Let

$$X, Y : I \longrightarrow C$$

be lax functors. A left transformation of lax functors from X to Y consists of

- a family of 1-morphisms
  $$F(i) : X(i) \longrightarrow Y(i)$$
  indexed by $i \in I_0$,

- a family of 2-morphisms
  $$\varphi(u) : Y(u) \circ F(i) \Longrightarrow F(j) \circ X(u)$$
  for $u : i \to j$,

satisfying the following conditions:

1. For any object $i \in I_0$, the following diagram is commutative

$$
\begin{array}{ccc}
Y(1_i) \circ F(i) & \Longrightarrow & F(i) \circ X(1_i) \\
\big\Updownarrow & & \big\Uparrow \\
1_{Y(i)} \circ F(i) & =\!=\!= & F(i) \circ 1_{X(i)}
\end{array}
$$

2. For composable morphisms $i \xrightarrow{\ u\ } j \xrightarrow{\ v\ } k$ in I, the following diagram is commutative

$$
\begin{array}{ccccc}
Y(v) \circ F(j) \circ X(u) & \Longleftarrow & Y(v) \circ Y(u) \circ F(i) & \Longrightarrow & Y(v \circ u) \circ F(i) \\
\big\Updownarrow & & & & \big\Updownarrow \\
F(k) \circ X(v) \circ X(u) & & =\!=\!=\!=\!=\!=\!=\!=\!=\!=\!\Longrightarrow & & F(k) \circ X(v \circ u).
\end{array}
$$

Right transformations of lax functors are defined by reversing the direction of $\varphi(u)$ and by changing the conditions accordingly.

We have 2-morphisms.

Definition 2.47. Let

$$X, Y : I \longrightarrow C$$

be oplax functors and

$$(F, \varphi), (G, \psi) : X \longrightarrow Y$$

be left transformations of oplax functors. A morphism of left transformations from (F, φ) to (G, ψ) is a collection of 2-morphisms in $\mathbf{C}$

$$\theta(i) : F(i) \Longrightarrow G(i)$$

indexed by $i \in I_0$ making the following diagram commutative

$$
\begin{array}{ccc}
Y(u) \circ F(i) & \overset{\theta(i)}{\Longrightarrow} & Y(u) \circ G(i) \\
\varphi(u) \big\Downarrow & & \big\Downarrow \psi(u) \\
F(j) \circ X(u) & \underset{\theta(j)}{\Longrightarrow} & G(j) \circ X(u)
\end{array}
$$

for each morphism $u : i \to j$ in I.

The composition of morphisms of left transformations is defined by the composition of 2-morphisms in $\mathbf{C}$. Morphisms between right transformations are defined by reversing arrows appropriately. We also have lax versions of left and right transformations.

Oplax functors and lax functors form 2-categories but there are two variations.

Definition 2.48. Let I be a small category and $\mathbf{C}$ be a 2-category.

The 2-category consisting of oplax functors from I to $\mathbf{C}$, left transformations, and morphisms of left transformations is denoted by $\overleftarrow{\mathbf{Oplax}}(I, \mathbf{C})$. By using right transformations instead of left transformations, we also obtain a 2-category, which is denoted by $\overrightarrow{\mathbf{Oplax}}(I, \mathbf{C})$.

Lax functors, left transformations, morphisms of left transformations form a 2-category and it is denoted by $\overleftarrow{\mathbf{Lax}}(I, \mathbf{C})$. The 2-category obtained by replacing left transformations by right transformations is denoted by $\overrightarrow{\mathbf{Lax}}(I, \mathbf{C})$.

By restricting objects to strict functors, we obtain the full 2-subcategories $\overleftarrow{\mathbf{Funct}}(I, \mathbf{C})$ and $\overrightarrow{\mathbf{Funct}}(I, \mathbf{C})$ whose 1-morphisms are left and right transformations, respectively.

We have the following diagonal functor.

Definition 2.49. Define a functor

$$\Delta : \mathbf{C} \longrightarrow \overleftarrow{\mathbf{Oplax}}(I, \mathbf{C})$$

by

$$\Delta(X)(i) = X$$

on objects.

It is useful to have an explicit description of a morphism of oplax functors from F to $\Delta(A)$.

Lemma 2.50. *Let A be an object of a 2-category $\mathbf{C}$ and $X : I \to \mathbf{C}$ be an oplax functor. A morphism of oplax functors*

$$(F, \varphi) : X \longrightarrow \Delta(A)$$

consists of

- *a family of 1-morphisms*

$$F(i) : X(i) \to A$$

  indexed by objects in I, and

- *a family of 2-isomorphisms*

$$\varphi(u) : F(i) \Longrightarrow F(j) \circ X(u)$$

  indexed by morphisms in I,

satisfying the following conditions:

1. *For each object $i \in I_0$,*

$$\eta_i \circ \varphi(1_i) = 1_{F(i)}.$$

2. The following diagram of 2-morphisms is commutative

$$
\begin{array}{ccc}
F(k) & \Longrightarrow & F(k) \circ X(v) \\
\big\| & & \big\| \\
F(k) \circ X(v \circ u) & \Longrightarrow & F(k) \circ X(v) \circ X(u).
\end{array}
$$

Proof. A direct translation of definition. □

Example 2.51. Let G be a group. Consider the case $C = k\text{-}\mathbf{Categories}$. A strict functor

$$X : G \longrightarrow k\text{-}\mathbf{Categories}$$

is given by a k-linear category X equipped with an action of G. For a k-linear category Y, a morphism of oplax functors

$$F : X \longrightarrow \Delta(Y)$$

is given by a functor

$$F(*) : X(*) = X \longrightarrow \Delta(Y)(*) = Y$$

and a a family of natural transformations

$$\varphi(\alpha) : F(*) \Longrightarrow F(*) \circ X(\alpha)$$

indexed by $\alpha \in G$ making the following diagram commutative

$$
\begin{array}{ccc}
F(*) & \Longrightarrow & F(*) \circ X(\beta) \\
\big\| & & \big\| \\
F(*) \circ X(\alpha\beta) & \Longrightarrow & F(*) \circ X(\alpha) \circ X(\beta).
\end{array}
$$

This is nothing but the definition of right G-invariant functor in [Asaa]. □

2.5 Comodule Categories

We have seen in §2.2 that we often obtain a coproduct decomposition from a comodule structure, extending the idea of a characterization of group graded algebras by Cohen and Montgomery [CM84].

The notion of group graded algebras has been extended to group graded k-linear categories and to k-linear categories graded by a small category. See [CM06, Asaa, Low08], for example. In order to extend their definitions to $\boldsymbol{V}$-categories graded by a small category I, we introduce and investigate "many objectifications" of comodules.

Recall from Lemma 2.33 that the category of $\boldsymbol{V}$-categories has a symmetric monoidal structure.

Definition 2.52. A coalgebra $\boldsymbol{V}$-category is a comonoid object in the monoidal category of $\boldsymbol{V}$-categories.

Remark 2.53. In other words, a coalgebra $\boldsymbol{V}$-category is a $\boldsymbol{V}$-category C equipped with a family of morphisms

$$\Delta_{a,b} : C(a,b) \longrightarrow C(a,b) \otimes C(a,b)$$
$$\varepsilon_{a,b} : C(a,b) \longrightarrow 1$$

indexed by pairs of objects $a, b \in C_0$ satisfying the following conditions:

1. $\Delta_{a,b}$ is compatible with compositions, i.e. the following diagram is commutative

$$
\begin{array}{ccc}
C(b,c) \otimes C(a,b) & \xrightarrow{\quad\circ\quad} & C(a,c) \\
{\scriptstyle \Delta_{b,c} \otimes \Delta_{a,b}}\big\downarrow & & \big\downarrow{\scriptstyle \Delta_{a,c}} \\
(C(b,c) \otimes C(b,c)) \otimes (C(a,b) \otimes C(a,b)) & & \\
{\scriptstyle T}\big\downarrow & & \\
(C(b,c) \otimes C(a,b)) \otimes (C(b,c) \otimes C(a,b)) & \xrightarrow{\circ \otimes \circ} & C(a,c) \otimes C(a,c),
\end{array}
$$

where T is an appropriate composition of associators and a symmetry operator. (Recall that we assume V is symmetric monoidal.)

2. $\Delta_{a,a}$ preserves identity morphisms, i.e. the following diagram is commutative for each object $a \in C_0$

$$
\begin{array}{ccc}
1 & \xleftarrow{\;\cong\;} & 1 \otimes 1 \\
\downarrow{\scriptstyle 1_a} & & \downarrow{\scriptstyle 1_a \otimes 1_a} \\
C(a,a) & \xrightarrow{\;\Delta_{a,a}\;} & C(a,a) \otimes C(a,a),
\end{array}
$$

3. $\varepsilon_{a,b}$ is a counit for $\Delta_{a,b}$, i.e. the following diagram is commutative

$$
\begin{array}{ccc}
& C(a,b) & \\
{\scriptstyle\cong}\swarrow & \downarrow{\scriptstyle \Delta_{a,b}} & \searrow{\scriptstyle\cong} \\
1 \otimes C(a,b) \xleftarrow{\varepsilon_{a,b} \otimes 1} & C(a,b) \otimes C(a,b) & \xrightarrow{1 \otimes \varepsilon_{a,b}} C(a,b) \otimes 1
\end{array}
$$

4. $\Delta_{a,b}$ is coassociative, i.e. the following diagram is commutative

$$
\begin{array}{ccc}
C(a,b) & \xrightarrow{\;\Delta_{a,b}\;} & C(a,b) \otimes C(a,b) \\
\downarrow{\scriptstyle \Delta_{a,b}} & & \downarrow{\scriptstyle 1 \otimes \Delta_{a,b}} \\
& & C(a,b) \otimes (C(a,b) \otimes C(a,b)) \\
& & \downarrow{\scriptstyle\cong} \\
C(a,b) \otimes C(a,b) & \xrightarrow{\;\Delta_{a,b} \otimes 1\;} & (C(a,b) \otimes C(a,b)) \otimes C(a,b).
\end{array}
$$

Another way of saying this is that a coalgebra V-category is a category enriched over the category of comonoid objects in V.

Example 2.54. Let I be a small category. Then we have a V-category $I \otimes 1$ by Lemma A.13. Each $(I \otimes 1)(i,j)$ has a structure of comonoid by Example 2.17. The comonoid structure is compatible with the compositions of morphisms and preserves identities. Thus $I \otimes 1$ is a coalgebra category. $\square$

Example 2.55. When V is of product type, any object C has a canonical comonoid structure by Example 2.15. The assumption that the unit object 1 is terminal guarantees the existence of counit morphisms

$$\varepsilon_{a,b} : C(a,b) \longrightarrow 1$$

and we have a coalgebra structure on C. $\square$

Definition 2.56. Let C be a coalgebra V-category. When a V-category X is equipped with a right comodule structure over C

$$\mu : X \longrightarrow X \otimes C,$$

it is called a right comodule category over C. Left comodule categories are defined analogously.

It is convenient to have a more concrete description.

Lemma 2.57. *Let C be a coalgebra V-category and X be a V-category. A right comodule structure on X consists of*

- *a map*
$$p : X_0 \longrightarrow C_0$$
between objects, and

- *a collection of morphisms*

$$\mu_{x,y} : X(x,y) \longrightarrow X(x,y) \otimes C(p(x),p(y))$$

indexed by pairs of objects in X

satisfying the following conditions:

1. *The following diagram is commutative*

$$
\begin{array}{ccc}
X(y,z) \otimes X(x,y) & \xrightarrow{\;\;\circ\;\;} & X(x,z) \\
\Big\downarrow{\scriptstyle \mu_{y,z}\otimes\mu_{x,y}} & & \Big\downarrow{\scriptstyle \mu_{x,z}} \\
X(y,z) \otimes (C(p(y),p(z))) \otimes (X(x,y) \otimes C(p(x),p(y))) & & \\
\Big\downarrow{\scriptstyle T} & & \\
(X(y,z) \otimes X(x,y)) \otimes (C(p(y),p(z)) \otimes C(p(x),p(y))) & \xrightarrow{\;\circ\otimes\circ\;} & X(x,z) \otimes C(p(x),p(z)),
\end{array}
$$

where T is an appropriate composition of associators and a symmetry operator.

2. *The following diagram is commutative for each object $x \in X_0$*

$$
\begin{array}{ccc}
1 & \xleftarrow{\;\cong\;} & 1 \otimes 1 \\
\Big\downarrow{\scriptstyle 1_x} & & \Big\downarrow{\scriptstyle 1_x \otimes 1_p(x)} \\
X(x,x) & \xrightarrow{\;\mu_{x,x}\;} & X(x,x) \otimes C(p(x),p(x)).
\end{array}
$$

3. *The following diagram is commutative*

$$
\begin{array}{ccc}
 & & X(x,y) \\
 & {\scriptstyle\cong}\nearrow & \Big\downarrow{\scriptstyle \mu_{x,y}} \\
X(x,y) \otimes 1 & \xleftarrow{\;1\otimes\varepsilon_{x,y}\cdot\;} & X(x,y) \otimes C(p(x),p(y))
\end{array}
$$

4. *$\mu_{x,y}$ is coassociative in the sense that the following diagram is commutative*

$$
\begin{array}{ccc}
X(x,y) & \xrightarrow{\;\mu_{x,y}\;} & X(x,y) \otimes C(p(x),p(y)) \\
\Big\downarrow{\scriptstyle \mu_{x,y}} & & \Big\downarrow{\scriptstyle \mu_{x,y}\otimes 1} \\
 & & (X(x,y) \otimes C(p(x),p(y))) \otimes C(p(x),p(y) \\
 & & \Big\downarrow{\scriptstyle \cong} \\
X(x,y) \otimes C(p(x),p(y)) & \xrightarrow{\;1\otimes\Delta_{x,y}\;} & X(x,y) \otimes (C(p(x),p(y)) \otimes C(p(x),p(y))).
\end{array}
$$

Proof. A functor $\mu : X \to X \otimes C$ defines a map

$$\mu : X_0 \longrightarrow X_0 \times C_0.$$

Define $p = \mathrm{pr}_2 \circ \mu : X_0 \to C_0$. Then the counit condition implies

$$\mu(x) = (x, p(x))$$

for $x \in C_0$ and for each pair of objects $x, y \in X_0$ we obtain

$$\mu_{x,y} : X(x,y) \longrightarrow X(x,y) \otimes C(p(x),p(y)).$$

It is easy to verify that the conditions on μ for comodule structure corresponds to conditions in this Lemma. $\quad\square$

Example 2.58. Suppose V is of product type. By Example 2.55, any V-category has a canonical coalgebra structure and any V-functor

$$p : X \longrightarrow C$$

is compatible with coproducts and counits. The composition

$$X(x,y) \xrightarrow{\Delta} X(x,y) \otimes X(x,y) \xrightarrow{1 \times p} X(x,y) \otimes C(p(x), p(y))$$

makes X into a comodule over C. $\qquad\qquad\square$

Note that right comodules over a comonoid object in a symmetric monoidal category, in general, form a category in an obvious way (Definition 2.27). Namely by requiring a morphism to make the diagram commutative

$$
\begin{array}{ccc}
M & \longrightarrow & N \\
\downarrow & & \downarrow \\
M \otimes C & \longrightarrow & N \otimes C.
\end{array}
$$

In the case of comodule categories, we should relax the commutativity of this diagram in the following way.

Definition 2.59. Let C be a coalgebra V-category and (X, μ) and (X', μ') be right comodule categories over C. A left morphism of right comodule categories from (X, μ) to (X', μ') is a pair (F, φ) of a V-functor

$$F : X \longrightarrow X'$$

and a V-natural transformation

$$
\begin{array}{ccc}
X & \xrightarrow{\ F\ } & X' \\
\mu \downarrow & \overset{\varphi}{\nearrow} & \downarrow \mu' \\
X \otimes C & \xrightarrow[F \otimes 1]{} & X' \otimes C.
\end{array}
$$

The composition of left morphisms are defined by the following diagram

$$
\begin{array}{ccccc}
X & \xrightarrow{\ F\ } & X' & \xrightarrow{\ F'\ } & X'' \\
\mu \downarrow & \nearrow & \mu' \downarrow & \nearrow & \downarrow \mu'' \\
X \otimes C & \xrightarrow[F \otimes 1]{} & X' \otimes C & \xrightarrow[F' \otimes 1]{} & X'' \otimes C.
\end{array}
$$

For left morphisms

$$(F, \varphi), (G, \psi) : (X, \mu) \longrightarrow (X', \mu'),$$

a 2-morphism from (F, φ) to (G, ψ) is a V-natural transformation

$$\xi : F \Longrightarrow G$$

making the following diagram commutative

$$
\begin{array}{ccc}
\mu' \circ F & \overset{\varphi}{\Longrightarrow} & (F \otimes 1) \circ \mu \\
{\scriptstyle \mu' \circ \xi} \big\Downarrow & & \big\Downarrow {\scriptstyle (\xi \otimes 1) \circ \mu} \\
\mu' \circ G & \overset{\psi}{\Longrightarrow} & (G \otimes 1) \circ \mu
\end{array}
$$

The composition of 2-morphisms are given by the composition of V-natural transformations.

Right morphisms for right comodules are defined analogously by reversing the direction of $\Rightarrow$. For left comodules, we also define left and right morphisms analogously.

The 2-categories of right comodule categories over C and left morphisms, of right comodules categories and right morphisms, of left comodule categories and left morphisms, and of left comodule categories and right morphisms are denoted by $\overleftarrow{\textbf{Comod}}\text{-}C$, $\overrightarrow{\textbf{Comod}}\text{-}C$, $C\text{-}\overrightarrow{\textbf{Comod}}$, and $C\text{-}\overleftarrow{\textbf{Comod}}$, respectively. 2-categories of bicomodule categories $C\text{-}\overleftarrow{\textbf{Comod}}\text{-}C$ and $C\text{-}\overrightarrow{\textbf{Comod}}\text{-}C$ are defined similarly.

Remark 2.60. The above definition can be generalized to comodules over a comonoid object in a symmetric monoidal 2-category.

Example 2.61. Suppose V is of product type. Recall from Example 2.58 that any V-category C can be regarded as a coalgebra category and any functor

$$p : X \longrightarrow C$$

defines a right comodule structure

$$X \xrightarrow{\Delta} X \otimes X \xrightarrow{1 \otimes p} X \otimes C$$

on X. It is easy to see that this correspondence defines a 2-functor

$$V\text{-}\overleftarrow{\mathbf{Categories}} \downarrow C \longrightarrow \overleftarrow{\mathbf{Comod}}\text{-}C,$$

where the structure of 2-category on $V\text{-}\overleftarrow{\mathbf{Categories}} \downarrow C$ is defined in Example 2.41. Since $\otimes$ is the direct product, we can recover p from $(1 \otimes p) \circ \Delta$. However, there are differences in 1-morphisms. For example, a morphism

$$(F, \varphi) : (X, \mu) \longrightarrow (X', \mu')$$

of right C-comodule categories is given by a V-functor

$$F : X \longrightarrow X'$$

and a family of morphisms

$$\varphi(x) : (F(x), p(F(x))) \longrightarrow (F(x), p(x))$$

in $X' \otimes C$. Since $\otimes$ is the product in V, $\varphi(x)$ is of the form

$$\varphi(x) = \varphi_1(x) \otimes \varphi_2(x) \in (X \otimes C)((F(x), p(F(x))), (F(x), p(x))) = X(F(x), F(x)) \otimes C(p(F(x)), p(x)).$$

The 1-morphisms coming from $V\text{-}\overleftarrow{\mathbf{Categories}} \downarrow C$ are exactly those morphisms having natural transformations whose first component is the identity.

Thus we can regard $V\text{-}\overleftarrow{\mathbf{Categories}} \downarrow C$ as a 2-subcategory of $\overleftarrow{\mathbf{Comod}}\text{-}C$. $\qquad \square$

When V is Abelian, we cannot expect such a simple characterization of comodule categories as above. We will see in §3.2 that, in this case, the notion of comodules over a free V-category $I \otimes 1$ generated by a small category I corresponds to the notion of graded categories.

3 The Grothendieck Construction

In this section, we define the Grothendieck construction as a 2-functor from the category of oplax functors from I to the category of comodules over $I \otimes 1$. The construction works for lax functors by reversing arrows appropriately.

3.1 The Grothendieck Construction for Oplax and Lax Functors

The following is our definition of the Grothendieck construction. Recall that V is a symmetric monoidal category satisfying Assumption 2.11. In particular it is closed under arbitrary coproducts.

Definition 3.1. Let I be a small category. For an oplax functor

$$X : I \longrightarrow V\text{-}\mathbf{Categories},$$

define a V-category $\mathrm{Gr}(X)$ as follows: Objects are given by

$$\mathrm{Gr}(X)_0 = \coprod_{i \in I_0} X(i)_0 \times \{i\}.$$

For $(x, i), (y, j) \in \mathrm{Gr}(X)_0$, define

$$\mathrm{Gr}(X)((x, i), (y, j)) = \bigoplus_{u: i \to j} X(j)(X(u)(x), y).$$

The composition

$$\circ : \mathrm{Gr}(X)((y, j), (z, k)) \otimes \mathrm{Gr}(X)((x, i), (y, j)) \longrightarrow \mathrm{Gr}(X)((x, i), (z, k))$$

is given, on each component, by

$$
\begin{aligned}
X(k)(X(v)(y), z) \otimes X(j)(X(u)(x), y) &\xrightarrow{1 \otimes X(v)} X(k)(X(v)(y), z) \otimes X(k)(X(v)(X(u)(x)), X(v)(y)) \\
&\xrightarrow{\theta^*_{v,u}} X(k)(X(v)(y), z) \otimes X(k)(X(v \circ u)(x)), X(v)(y)) \\
&\xrightarrow{\circ} X(k)(X(v \circ u)(x), z).
\end{aligned}
$$

One of the simplest examples is the semidirect product construction for groups.

Example 3.2. Suppose a group G acts on another group H. We regard G as a category with a single object $*$. Then the action defines a functor

$$H : G \longrightarrow \mathbf{Groups} \subset \mathbf{Categories}$$

by

$$H(*) = H$$

and

$$H(g) = g\cdot : H \longrightarrow H.$$

The Grothendieck construction of this functor is nothing but the semidirect product. □

This example can be extended as follows.

Example 3.3. Let A be a $\boldsymbol{V}$-category. Suppose a group G acts on A from the left via $\boldsymbol{V}$-functors. We regard it as a functor

$$A : G \longrightarrow \boldsymbol{V}\text{-}\mathbf{Categories}.$$

The Grothendieck construction of A is called the orbit category by Cibils and Marcos in [CM06] when $\boldsymbol{V}$ is the category of k-modules.

By definition, objects of $\mathrm{Gr}(A)$ can be identified with objects in A

$$\mathrm{Gr}(A)_0 = A_0 \times \{*\} \cong A_0.$$

For $x, y \in A_0$, morphisms are given by

$$\mathrm{Gr}(A)(x, y) = \bigoplus_{g \in G} A(gx, y)$$

and compositions are given, on each component, by

$$A(gy, z) \otimes A(hx, y) \xrightarrow{1 \otimes A(g)} A(gy, z) \otimes A(ghx, gy) \xrightarrow{\circ} A(ghx, z).$$

□

It is not difficult to define Gr as a 2-functor

$$\mathrm{Gr} : \overleftarrow{\mathbf{Oplax}}(I, \boldsymbol{V}\text{-}\mathbf{Categories}) \longrightarrow \boldsymbol{V}\text{-}\mathbf{Categories}.$$

Definition 3.4. For a left transformation of oplax functors
$$(F, \varphi) : X \longrightarrow Y,$$
define
$$\mathrm{Gr}(F, \varphi) : \mathrm{Gr}(X) \longrightarrow \mathrm{Gr}(Y)$$
by
$$\mathrm{Gr}(F, \varphi)(x, i) = (F(i)(x), i)$$
for objects and
$$\mathrm{Gr}(F, \varphi) : \mathrm{Gr}(X)((x, i), (y, j)) \longrightarrow \mathrm{Gr}(Y)((F(i)(x), i), (F(j)(y), j))$$
by the composition
$$X(j)(X(u)(x), y) \xrightarrow{F(j)} Y(j)(F(j)(X(u)(x), F(j)(y))) \xrightarrow{\varphi(u)^*} Y(j)(Y(u)(F(i)(x)), F(j)(y))$$
on each component.

Definition 3.5. For a 2-morphism
$$\theta : (F, \varphi) \Longrightarrow (G, \psi)$$
in $\overleftarrow{\mathbf{Oplax}}(I, \mathbf{V\text{-}Categories})$, define a 2-morphism
$$\mathrm{Gr}(\theta) : \mathrm{Gr}(F, \varphi) \Longrightarrow \mathrm{Gr}(G, \psi)$$
in $\mathbf{V\text{-}Categories}$ by
$$\mathrm{Gr}(\theta)(x, i) = (\theta(i)(x), 1_i) : \mathrm{Gr}(F, \varphi)(x, i) = (F(i)(x), i) \longrightarrow (G(i)(x), i) = \mathrm{Gr}(G, \psi)(x, i).$$

Proposition 3.6. *The above constructions define a 2-functor*
$$\mathrm{Gr} : \overleftarrow{\mathbf{Oplax}}(I, \mathbf{V\text{-}Categories}) \longrightarrow \mathbf{V\text{-}Categories}.$$

We may dualize the above construction to obtain
$$\mathrm{Gr} : \overrightarrow{\mathbf{Lax}}(I^{op}, \mathbf{V\text{-}Categories}) \longrightarrow \mathbf{V\text{-}Categories}.$$

We briefly describe the construction.

Definition 3.7. For a lax functor
$$X : I^{op} \longrightarrow \mathbf{V\text{-}Categories},$$
define a $\mathbf{V}$-category $\mathrm{Gr}(X)$ by
$$\mathrm{Gr}(X)_0 = \coprod_{i \in I_0} \{i\} \times X(i)_0$$
and
$$\mathrm{Gr}(X)((i, x), (j, y)) = \bigoplus_{u : i \to j} X(i)(x, X(u)(y)).$$
The composition is given on each component by
$$X(j)(y, X(v)(z)) \otimes X(i)(x, X(u)(y)) \xrightarrow{X(u) \otimes 1} X(i)(X(u)(y), X(u) \circ X(v)(z)) \otimes X(i)(x, X(u)(y))$$
$$\xrightarrow{\theta_*} X(i)(X(u)(y), X(v \circ u)(z)) \otimes X(i)(x, X(u)(y))$$
$$\xrightarrow{\circ} X(i)(x, X(v \circ u)(z)).$$

For a right transformation (F, φ) of lax functors

$$
\begin{array}{ccc}
X(j) & \xrightarrow{F(j)} & Y(j) \\
{\scriptstyle X(u)}\downarrow & {\scriptstyle \varphi(u)} \nearrow & \downarrow {\scriptstyle Y(u)} \\
X(i) & \xrightarrow[F(i)]{} & Y(i),
\end{array}
$$

define
$$\mathrm{Gr}(F, \varphi) : \mathrm{Gr}(X) \longrightarrow \mathrm{Gr}(Y)$$
by
$$\mathrm{Gr}(F, \varphi)(i, x) = (i, F(i)(x))$$
for objects and
$$\mathrm{Gr}(F, \varphi) : \mathrm{Gr}(X)((i, x), (j, y)) \longrightarrow \mathrm{Gr}(Y)((i, F(i)(x)), (j, F(j)(y)))$$
by the composition
$$X(i)(x, X(u)(y)) \xrightarrow{F(i)} Y(i)(F(i)(x), F(i)(X(u)(y))) \xrightarrow{\varphi(u)_*} Y(i)(F(i)(x), Y(u)(F(j)(y))).$$

Proposition 3.8. *The above constructions define a 2-functor*

$$\mathrm{Gr} : \overrightarrow{\mathbf{Lax}}(I^{op}, \boldsymbol{V}\text{-}\mathbf{Categories}) \longrightarrow \boldsymbol{V}\text{-}\mathbf{Categories}.$$

The above propositions say the Grothendieck construction is a process to glue $\boldsymbol{V}$-categories in a given diagram of $\boldsymbol{V}$-categories together to form a single $\boldsymbol{V}$-category. We show in §A.2 that this Gr is left adjoint to the diagonal functor and $\mathrm{Gr}(X)$ can be regarded as a 2-colimit of X. If we regard the Grothendieck construction in this way, there is no hope to recover the original diagram. Note that the Grothendieck construction is defined by coproducts and $\mathrm{Gr}(X)$ is a $\boldsymbol{V}$-category whose morphism objects have coproduct decompositions. It turns out that these coproduct decompositions allow us to construct a diagram of $\boldsymbol{V}$-categories which is very closely related to the original diagram X.

We use the notion of graded category, which is the subject of the next section §3.2, in order to make the above statement precise.

3.2 Graded Categories

Given a group G, the notion of G-graded category was introduced and has been studied for k-linear categories by extending the notion of G-graded algebras. More generally, k-linear categories graded by a small category have been defined and used in the deformation theory of k-linear prestacks [Low08, LdB]. In these approaches graded categories are defined by using coproducts.

We have seen in §2.2 that coproduct decompositions are intimately related to comodule structures over a "free coalgebra". This observation allows us to define graded categories without coproduct decompositions. This viewpoint also suggests us to extend the notion of morphisms by incorporating appropriate natural transformations. Our definition is also suggested by the definition of degree preserving functor by Asashiba [Asab]. Note that morphisms between graded categories defined in [Low08] correspond to strictly degree preserving functors in [Asab] and our category of I-graded $\boldsymbol{V}$-categories is larger than Lowen's.

The following is our definition of $\boldsymbol{V}$-categories graded by a small category I.

Definition 3.9. Let I be a small category and X be a $\boldsymbol{V}$-category. An I-grading on X is a right comodule structure

$$\mu : X \longrightarrow X \otimes (I \otimes 1)$$

on X over the coalgebra category $I \otimes 1$.

Remark 3.10. We may also use left comodules. Since $I \otimes 1$ is a cocommutative coalgebra category, it does not make any essential difference.

A more concrete description of I-grading can be obtained in many cases.

Lemma 3.11. *Suppose $\boldsymbol{V}$ is the category k-$\mathbf{Mod}$ for a commutative ring k. Then an I-grading on a $\boldsymbol{V}$-category X consists of a map*

$$p : X_0 \longrightarrow I_0$$

and a family of coproduct decompositions

$$X(x, y) \xrightarrow{\cong} \bigoplus_{u : p(x) \to p(y) \ in \ I} X^u(x, y)$$

indexed by pairs of objects in X satisfying the following conditions:

1. *For any object $x \in X_0$, we have the following diagram*

$$
\begin{array}{ccc}
1 & \xrightarrow{\;1_x\;} & X(x,x) \\
& & \Big\uparrow \\
& & X^{1_{p(x)}}(x,x)
\end{array}
$$

2. *For objects $x, y, z \in X_0$, the following diagram is commutative*

$$
\begin{array}{ccc}
X(y,z) \otimes X(x,y) & \xrightarrow{\;\circ\;} & X(x,z) \\
\Big\uparrow & & \Big\uparrow \\
X^u(y,z) \otimes X^v(x,y) & \dashrightarrow & X^{uv}(x,z).
\end{array}
$$

Proof. This is a direct translation of Lemma 2.20 and Lemma 2.57. □

Example 3.12. Let G be a group and consider the case $\boldsymbol{V} = k\text{-}\mathbf{Mod}$. The unit object of $k\text{-}\mathbf{Mod}$ is the ground ring k and the k-linear category $G \otimes k$ is nothing but the group algebra $k[G]$ over k regarded as a k-linear category with a single object.

Since $G \otimes 1 = k[G]$ has the only object $*$, a G-grading on a k-linear category A is given by a coproduct decomposition

$$
A(x,y) = \bigoplus_{g \in G} A^g(x,y)
$$

satisfying the condition that the functor p induces

$$
\circ : A^g(y,z) \otimes A^h(x,y) \longrightarrow A^{gh}(x,z).
$$

and

$$
\begin{array}{ccc}
1 & \xrightarrow{\;1_x\;} & A(x,x) \\
& & \Big\uparrow \\
& & A^1(x,x).
\end{array}
$$

This is the definition of a G-graded k-linear category in [CM06, Asaa, Asab]. □

Even if $\boldsymbol{V}$ is not Abelian, we often obtain an analogous coproduct decomposition.

Example 3.13. Let I be a small category. Consider the case that $\boldsymbol{V}$ is the category of sets. Suppose a small category X has an I-grading. As we have seen in Example 2.61, it is determined by a functor

$$
p : X \longrightarrow I.
$$

We have a map of morphism sets

$$
p_{x,y} : \mathrm{Mor}_X(x,y) \longrightarrow \mathrm{Mor}_I(p(x), p(y))
$$

for each pair of objects $x, y \in X_0$. Denote

$$
\mathrm{Mor}_X^u(x,y) = p_{x,y}^{-1}\{u\}
$$

and we have a coproduct decomposition

$$
\mathrm{Mor}_X(x,y) = \coprod_{u : p(x) \to p(y)} \mathrm{Mor}_X^u(x,y).
$$

The composition of morphisms in X induces a map

$$\mathrm{Mor}_X^u(y,z) \times \mathrm{Mor}_X^v(x,y) \longrightarrow \mathrm{Mor}^{uv}(x,z).$$

Conversely, we can recover p from the above coproduct decomposition by using the following composition

$$\mathrm{Mor}_X(x,y) = X(x,y) \cong \coprod_{u:p(x)\to p(y)} X^u(x,y) \longrightarrow \coprod_{u:p(x)\to p(y)} \{u\} = \mathrm{Mor}_I(p(x),p(y)).$$

Note that this argument works without a change when V is the category of topological spaces, simplicial sets, and small categories. □

Example 3.14. Suppose V is a stable model category and X is an I-graded V-category. By Remark 2.22, we have a coproduct decomposition of each $X(x,y)$ up to weak equivalences. In particular, we have

$$X(x,y) \simeq \bigvee_{u:p(x)\top(y)} X^u(x,y)$$

when X is an I-graded spectral category in our sense. See [Tab, BM] for spectral categories. □

We have also seen in Example 2.61 that, when V is of product type, the comma 2-category $V\text{-}\overleftarrow{\mathbf{Categories}} \downarrow$ $(I \otimes 1)$ can be regarded as a 2-subcategory of the 2-category $\overleftarrow{\mathbf{Comod}}\text{-}(I \otimes 1)$ of comodules over $I \otimes 1$. Thus we would like to define a 2-category of I-graded categories which generalizes the 2-category of the comma category $V\text{-}\overleftarrow{\mathbf{Categories}} \downarrow (I \otimes 1)$ when V is of product type and the 2-category of G-graded categories defined by Asashiba [Asab]. Note that, although we have defined a 2-category of comodule categories in Definition 2.59, 1-morphisms in this 2-category is less restrictive than 1-morphisms of G-graded categories.

Example 3.15. Let G be a group and

$$\begin{aligned} \mu &: A \longrightarrow A \otimes k[G] \\ \mu' &: B \longrightarrow B \otimes k[G] \end{aligned}$$

be G-graded k-linear categories regarded as comodules over $k[G]$. Let

$$(F, \varphi) : (A, \mu) \longrightarrow (B, \mu')$$

be a left morphism of $k[G]$-comodules. F is a k-linear functor

$$F : A \longrightarrow B$$

and φ is a k-linear natural transformation

Note that μ and μ' are identity on objects and, for each object $a \in A_0$, $\varphi(a)$ can be regarded as an element

$$\varphi(a) \in (B \otimes k[G])(\mu'(F(a)), (1 \otimes F)(\mu(a))) = B(F(a), F(a)) \otimes k[G].$$

The condition that φ is a k-linear natural transformation implies that the following diagram is commutative

Suppose $\varphi(a)$ is of the form

$$\varphi(a) = 1_{F(a)} \otimes \varphi_2(a)$$

for some $\varphi_2(a) \in G$ and suppose $f \in A^h(a, a')$ and $F(f) \in B^{h'}(F(a), F(a'))$. Then we have

$$\begin{aligned} \mu(f) &= f \otimes h \\ \mu'(F(f)) &= F(f) \otimes h' \end{aligned}$$

and the commutativity of the diagram implies

$$F(f) \otimes (h\varphi_2(a)) = F(f) \otimes (\varphi_2(a')h').$$

In other words, F restricts to

$$F : A^{\varphi_2(a')g}(a, a') \longrightarrow B^{g\varphi_2(a)}(F(a), F(a')).$$

This is the definition of degree preserving functor in Definition 4.1(2) in [Asab]. $\qquad\square$

Definition 3.16. Let I be a small category. A left morphism

$$(F, \varphi) : (X, \mu) \longrightarrow (X', \mu')$$

in $\overleftarrow{\textbf{Comod-}}(I \otimes 1)$ is called degree-preserving if there exists a family of morphisms in I

$$\varphi_2(x) \in \mathrm{Mor}_I(p'(F(x)), p(x)) \subset \mathrm{Mor}_{\underline{I \otimes 1}}(p'(F(x)), p(x)) = \mathrm{Mor}_{\textbf{\textit{V}}}(1, (I \otimes 1)(p'(F(x)), p(x)))$$

with

$$\varphi(x) = 1 \otimes \varphi_2(x).$$

We define degree preserving right morphisms in the same way.

Remark 3.17. We required that $\varphi_2(x)$ is homogeneous in the sense that it belongs to I not in $I \otimes 1$.

Definition 3.18. Let I be a small category. The subcategories of the 2-categories $\overleftarrow{\textbf{Comod-}}(I \otimes 1)$ and $\overrightarrow{\textbf{Comod-}}(I \otimes 1)$ consisting of degree preserving (left or right) morphisms are denoted by $\textbf{\textit{V}}\text{-}\overleftarrow{\textbf{Categories}}_I$ and $\textbf{\textit{V}}\text{-}\overrightarrow{\textbf{Categories}}_I$ and called the 2-categories of left or right I-graded $\textbf{\textit{V}}$-categories, respectively.

Remark 3.19. The definition of morphisms of graded categories in [Low08] requires φ to be the identity.

3.3 The Grothendieck Construction as a Graded Category

In this section, we extend the Grothendieck construction for oplax functors to a 2-functor

$$\mathrm{Gr} : \overleftarrow{\textbf{Oplax}}(I, \textbf{\textit{V}}\text{-}\textbf{Categories}) \longrightarrow \textbf{\textit{V}}\text{-}\overleftarrow{\textbf{Categories}}_I.$$

Let us begin with objects.

Definition 3.20. For an oplax functor

$$X : I \longrightarrow \textbf{\textit{V}}\text{-}\textbf{Categories},$$

define a $\textbf{\textit{V}}$-functor

$$\mu_X : \mathrm{Gr}(X) \longrightarrow \mathrm{Gr}(X) \otimes (I \otimes 1)$$

as follows. On objects,

$$\mu_X : \mathrm{Gr}(X)_0 = \coprod_{i \in I_0} X(i)_0 \times \{i\} \longrightarrow \mathrm{Gr}(X)_0 \otimes I_0$$

is defined by

$$\mu_X(x, i) = (x, i, i).$$

Define, on each component of morphisms, by

$$\begin{CD} \mathrm{Gr}(X)((x,i),(y,j)) @>{\mu_X}>> (\mathrm{Gr}(X) \otimes (I \otimes 1))((x,i),(y,j)) \\ @AAA @AAA \\ X(j)(X(u)(x),y) @>{\cong}>> X(j)(X(u)(x),y) \otimes (1 \otimes 1) @>{1 \otimes u}>> X(j)(X(u)(x),y) \otimes (u \otimes 1). \end{CD}$$

Let
$$(F, \varphi) : X \longrightarrow X'$$
be a left morphism of oplax functors. Recall that in Definition 3.4, we defined a $\boldsymbol{V}$-functor
$$\mathrm{Gr}(F, \varphi) : \mathrm{Gr}(X) \longrightarrow \mathrm{Gr}(X').$$

In order to obtain a left morphism in $\overleftarrow{\textbf{Comod-}}(I \otimes 1)$, we need a $\boldsymbol{V}$-natural transformation
$$\mathrm{Gr}(\varphi) : \mu_{X'} \circ \mathrm{Gr}(F, \varphi) \Longrightarrow (\mathrm{Gr}(F, \varphi) \otimes 1) \circ \mu_X.$$

Definition 3.21. For a left morphism of oplax functors
$$(F, \varphi) : X \longrightarrow X',$$
define a $\boldsymbol{V}$-natural transformation
$$\mathrm{Gr}(\varphi) : \mu_{X'} \circ \mathrm{Gr}(F) \Longrightarrow (\mathrm{Gr}(F) \otimes 1) \circ \mu_X,$$
namely a morphism

$$\mathrm{Gr}(\varphi)(x, i) : 1 \rightarrow (\mathrm{Gr}(X') \otimes (I \otimes 1))((\mu_{X'} \circ \mathrm{Gr}(F))(x, i), (\mathrm{Gr}(F) \otimes 1) \circ \mu_X(x, i))$$
$$= X'(i)(F(x), F(x)) \otimes (\mathrm{Mor}_I(i, i) \otimes 1)$$

by the identity morphism
$$\mathrm{Gr}(\varphi)(x, i) = 1_{F(x)} \otimes 1_i.$$

Lemma 3.22. *The pair $(\mathrm{Gr}(F, \varphi), \mathrm{Gr}(\varphi))$ defines a left morphism*
$$(\mathrm{Gr}(F, \varphi), \mathrm{Gr}(\varphi)) : (\mathrm{Gr}(X), \mu_X) \longrightarrow (\mathrm{Gr}(X'), \mu_{X'})$$
in $\boldsymbol{V}\text{-}\overleftarrow{\textbf{Categories}}_I$.

Proof. It is easy to check that $(\mathrm{Gr}(F, \varphi), \mathrm{Gr}(\varphi))$ is a left morphism in $\overleftarrow{\textbf{Comod-}}(I \otimes 1)$.
  For an object (x, i) in $\mathrm{Gr}(X)$, define
$$\mathrm{Gr}(\varphi)_2(x, i) = 1_i : 1 \longrightarrow (I \otimes 1)(i, i).$$
Then $\mathrm{Gr}(\varphi)(x, i) = 1_{F(x)} \otimes \mathrm{Gr}(\varphi)_2(x, i)$ and we have a degree preserving left morphism. $\square$

Let
$$(F, \varphi), (G, \psi) : X \longrightarrow X'$$
be morphisms of oplax functors. For a 2-morphism
$$\theta : (F, \varphi) \Longrightarrow (G, \psi)$$
in $\overleftarrow{\textbf{Oplax}}(I, \boldsymbol{V}\text{-}\textbf{Categories})$, we have defined a $\boldsymbol{V}$-natural transformation
$$\mathrm{Gr}(\theta) : \mathrm{Gr}(F, \varphi) \Longrightarrow \mathrm{Gr}(G, \psi)$$
in Definition 3.5.

Lemma 3.23. *The above constructions define a functor*
$$\mathrm{Gr} : \overleftarrow{\textbf{Oplax}}(I, \boldsymbol{V}\text{-}\textbf{Categories})(X, X') \longrightarrow (\boldsymbol{V}\text{-}\overleftarrow{\textbf{Categories}}_I)(\mathrm{Gr}(X), \mathrm{Gr}(X')).$$

Proof. We need to check the following:

1. For a 2-morphism θ in $\overleftarrow{\mathbf{Oplax}}(I, \boldsymbol{V}\text{-}\mathbf{Categories})$, $\mathrm{Gr}(\theta)$ is a 2-morphism in $(I \otimes 1)\text{-}\mathbf{Comod}$, i.e. it makes the following diagram commutative

$$
\begin{array}{ccc}
\mu_{X'} \circ \mathrm{Gr}(F, \varphi) & \xRightarrow{\mathrm{Gr}(\varphi)} & (F \otimes 1) \circ \mu_X \\
{\scriptstyle \mu_{X'} \circ \mathrm{Gr}(\theta)} \big\Downarrow & & \big\Downarrow {\scriptstyle (\mathrm{Gr}(\theta) \otimes 1) \circ \mu_X} \\
\mu_{X'} \circ \mathrm{Gr}(G, \psi) & \xRightarrow{\mathrm{Gr}(\psi)} & (G \otimes 1) \circ \mu.
\end{array}
$$

2. $\mathrm{Gr}(1_{(F,\varphi)}) = 1_{(\mathrm{Gr}(F,\varphi), \mathrm{Gr}(\varphi))}$ for a 1-morphism (F, φ) in $\overleftarrow{\mathbf{Oplax}}(I, \boldsymbol{V}\text{-}\mathbf{Categories})$.

3. For 2-morphisms

$$
(F_0, \varphi_0) \xRightarrow{\theta_1} (F_1, \varphi_1) \xRightarrow{\theta_2} (F_2, \varphi_2),
$$

we have

$$
\mathrm{Gr}(\theta_2 \circ \theta_1) = \mathrm{Gr}(\theta_2) \circ \mathrm{Gr}(\theta_1).
$$

The first and second parts are obvious from the definition of $\mathrm{Gr}(\theta)$. For the third part, recall that both compositions $\theta_2 \circ \theta_1$ and $\mathrm{Gr}(\theta_2) \circ \mathrm{Gr}(\theta_1)$ are compositions of natural transformations. The equality follows from the definition of the composition of natural transformations. $\qquad\square$

Proposition 3.24. *The above constructions make the Grothendieck construction into a 2-functor*

$$
\mathrm{Gr} : \overleftarrow{\mathbf{Oplax}}(I, \boldsymbol{V}\text{-}\mathbf{Categories}) \longrightarrow \boldsymbol{V}\text{-}\overleftarrow{\mathbf{Categories}}_I \subset \overleftarrow{\mathbf{Comod}}\text{-}(I \otimes 1).
$$

Proof. It remains to prove the following:

1. The following diagram is commutative

$$
\begin{array}{ccc}
\overleftarrow{\mathbf{Oplax}}(I, \boldsymbol{V}\text{-}\mathbf{Categories})(X', X'') \times \overleftarrow{\mathbf{Oplax}}(I, \boldsymbol{V}\text{-}\mathbf{Categories})(X, X') & \xrightarrow{\ \circ\ } & \overleftarrow{\mathbf{Oplax}}(I, \boldsymbol{V}\text{-}\mathbf{Categories})(X, X'') \\
{\scriptstyle \mathrm{Gr} \times \mathrm{Gr}} \Big\downarrow & & \Big\downarrow {\scriptstyle \mathrm{Gr}} \\
(\boldsymbol{V}\text{-}\overleftarrow{\mathbf{Categories}}_I)(\mathrm{Gr}(X'), \mathrm{Gr}(X'')) \times (\boldsymbol{V}\text{-}\overleftarrow{\mathbf{Categories}}_I)(\mathrm{Gr}(X), \mathrm{Gr}(X')) & \xrightarrow{\ \circ\ } & (\boldsymbol{V}\text{-}\overleftarrow{\mathbf{Categories}}_I)(\mathrm{Gr}(X), \mathrm{Gr}(X'')).
\end{array}
$$

2. The following diagram is commutative

$$
\begin{array}{ccc}
1 & \longrightarrow & \overleftarrow{\mathbf{Oplax}}(I, \boldsymbol{V}\text{-}\mathbf{Categories})(X, X) \\
& \searrow & \big\downarrow {\scriptstyle \mathrm{Gr}} \\
& & (\boldsymbol{V}\text{-}\overleftarrow{\mathbf{Categories}}_I)(\mathrm{Gr}(X), \mathrm{Gr}(X)),
\end{array}
$$

where 1 in the diagram is the trivial category.

For morphisms of oplax functors

$$
X \xrightarrow{(F, \varphi)} X' \xrightarrow{(F', \varphi')} X'',
$$

the composition $(F', \varphi') \circ (F, \varphi)$ is given by

$$
(F', \varphi') \circ (F, \varphi) = (F' \circ F, \varphi' \circ \varphi)
$$

and

$$
\mathrm{Gr}((F', \varphi') \circ (F, \varphi)) = (\mathrm{Gr}(F' \circ F, \varphi' \circ \varphi), \mathrm{Gr}(\varphi' \circ \varphi)).
$$

For $(x, i) \in \mathrm{Gr}(X)_0$,

$$
\mathrm{Gr}(F' \circ F, \varphi' \circ \varphi)(x, i) = ((F' \circ F)(i)(x), i) = (F'(i)(F(i)(x)), i) = \mathrm{Gr}(F')(\mathrm{Gr}(F)(x, i)).
$$

For morphisms in $\mathrm{Gr}(X)$, the compatibility of the compositions and Gr follows from the definition of morphisms of oplax functors.

The second part is obvious. $\qquad\square$

We also obtain a 2-functor

$$\mathrm{Gr} : \overrightarrow{\mathbf{Lax}}(I^{op}, \boldsymbol{V}\text{-}\mathbf{Categories}) \longrightarrow \boldsymbol{V}\text{-}\overrightarrow{\mathbf{Categories}}_I.$$

The details are omitted.

4   Comma Categories and the Smash Product Construction

Given a functor

$$p : E \longrightarrow I$$

of small categories, the inverse image $p^{-1}(x)$ of $x \in I_0$ is a natural candidate for the fiber over x. There are two more ways to take fibers, which should be regarded as "homotopy fibers", i.e. comma categories $p \downarrow x$ and $x \downarrow p$, thanks to the work of Quillen [Qui73].

In this section, we extend definitions of these fibers to enriched contexts.

4.1   Fibers of Gradings

As we have seen in §3.2, a correct enriched analogue of a functor

$$p : E \longrightarrow I$$

is a grading

$$\mu : E \longrightarrow E \otimes (I \otimes 1).$$

In this section, we define fibers of a grading.

Definition 4.1. Let I be a small category and

$$\mu : E \longrightarrow E \otimes (I \otimes 1)$$

be an I-grading. For each object $i \in I_0$, define a $\boldsymbol{V}$-category $E|_i$ by

$$(E|_i)_0 = \{e \in E_0 \mid \mu(e) = (e, i)\}$$

and, for $e, e' \in (E|_i)_0$, $(E|_i)(e, e')$ is defined by the following pullback diagram

$$
\begin{array}{ccc}
(E|_i)(e, e') & \longrightarrow & E(e, e') \\
\downarrow & & \downarrow {\scriptstyle p} \\
E(e, e') \otimes 1 & \xrightarrow{1 \otimes 1_i} & E(e, e') \otimes (I(i, i) \otimes 1).
\end{array}
$$

The composition is defined by the following diagram

$$
\begin{array}{ccccc}
(E|_i)(e', e'') \otimes (E|_i)(e, e') & \longrightarrow & & & E(e', e'') \otimes E(e, e') \\
& \searrow & & & \\
\downarrow & & (E|_i)(e, e'') \longrightarrow E(e, e'') & & \downarrow \\
& & \downarrow \qquad\qquad \downarrow & & \\
& & E(e, e'') \otimes 1 \longrightarrow E(e, e'') \otimes I(i, i) \otimes 1 & & \\
& \nearrow & & \nwarrow & \\
E(e', e'') \otimes 1 \otimes E(e, e') \otimes 1 & \longrightarrow & & & E(e', e'') \otimes I(i, i) \otimes 1 \otimes E(e, e') \otimes I(i, i) \otimes 1.
\end{array}
$$

The following is an analogue of $p \downarrow i$.

Definition 4.2. Let

$$\mu : E \longrightarrow E \otimes (I \otimes 1)$$

be an I-grading and define

$$p = \mathrm{pr}_2 \circ \mu_0 : E_0 \longrightarrow E_0 \times I_0 \longrightarrow I_0.$$

For each object $i \in I_0$, define a $\boldsymbol{V}$-category $\mu \downarrow i$ as follows. Objects are defined by

$$(\mu \downarrow i)_0 = \coprod_{e \in E_0} \{e\} \times \mathrm{Mor}_I(p(e), i).$$

For $(e, u), (e', u') \in (\mu \downarrow i)_0$, define an object $(\mu \downarrow i)((e, u), (e', u'))$ in $\boldsymbol{V}$ by the following pullback diagram in $\boldsymbol{V}$

In $\mu \downarrow i$, the identity morphisms are defined by

The composition

$$(\mu \downarrow i)((e', u'), (e'', u'')) \otimes (\mu \downarrow i)((e, u), (e', u')) \longrightarrow (\mu \downarrow i)((e, u), (e'', u''))$$

is given by the following diagram

37

$$
\begin{array}{ccc}
(\mu \downarrow i)((e',u'),(e'',u'')) \otimes (\mu \downarrow i)((e,u),(e',u')) & \longrightarrow & E(e',e'') \otimes E(e,e')
\end{array}
$$

The following is a commutative diagram:

- Top-left: $(\mu \downarrow i)((e',u'),(e'',u'')) \otimes (\mu \downarrow i)((e,u),(e',u'))$
- Top-right: $E(e',e'') \otimes E(e,e')$
- $(\mu \downarrow i)((e',u'),(e'',u'')) \otimes (E \otimes (I \otimes 1))((e,p(e)),(e',p(e')))$
- $(E \otimes (I \otimes 1))((e',p(e')),(e'',p(e''))) \otimes (E \otimes (I \otimes 1))((e,p(e)),(e',p(e')))$
- Arrow labeled $\mu \otimes \mu$
- $E(e,e'')$
- $(E(e',e'') \otimes 1) \otimes (\mu \downarrow i)((e,u),(e',u'))$
- $(E(e',e'') \otimes 1) \otimes (E \otimes (I \otimes 1))((e,p(e)),(e',p(e')))$
- Arrow labeled $(1 \otimes u') \otimes 1$
- $(E(e',e'') \otimes (I \otimes 1)(p(e'),i)) \otimes (E \otimes (I \otimes 1))((e,p(e)),(e',p(e')))$
- Arrow labeled $u''_* \otimes 1$
- $E(e,e'') \otimes (I \otimes 1)(p(e),p(e''))$
- Arrow labeled μ
- $(E(e',e'') \otimes 1) \otimes (E(e,e') \otimes 1)$
- Arrow labeled $1 \otimes (1 \otimes u)$
- $(E(e',e'') \otimes 1) \otimes (E(e,e') \otimes (I \otimes 1)(p(e),i))$
- Arrow labeled $1 \otimes u'_*$
- $E(e,e'') \otimes 1$
- Arrow labeled $1 \otimes u$
- $E(e,e'') \otimes (I \otimes 1)(p(e),i).$
- Arrow labeled $1 \otimes u''_*$
- Arrows labeled $\circ$

Definition 4.3. Let

$$\mu : E \longrightarrow E \otimes (I \otimes 1)$$

be an I-grading and define

$$p = \mathrm{pr}_2 \circ \mu : E_0 \longrightarrow E_0 \times I_0 \longrightarrow I_0.$$

For each $i \in I_0$, define a $\boldsymbol{V}$-category $i \downarrow \mu$ as follows. Objects are defined by

$$(i \downarrow \mu)_0 = \coprod_{e \in E_0} \mathrm{Mor}_I(i, p(e)) \times \{e\}.$$

Define $(i \downarrow \mu)((e, u), (e', u'))$ by the following pullback diagram

$$
\begin{array}{ccc}
(i \downarrow \mu)((u, e), (u', e')) & \longrightarrow & E(e, e') \\
\Big\downarrow & & \Big\downarrow \mu \\
& & (E \otimes (I \otimes 1))((e, p(e)), (e', p(e'))) \\
& & \Big\| \\
& & E(e, e') \otimes (I \otimes 1)(p(e), p(e')) \\
& & \Big\downarrow 1 \otimes u^* \\
E(e, e') \otimes 1 & \xrightarrow{\ 1 \otimes u'_*\ } & E(e, e') \otimes (I \otimes 1)(i, p(e')).
\end{array}
$$

The identities and compositions are defined analogously to the case of $\mu \downarrow i$.

It follows from the fact that μ is a functor satisfying coassociativity and counitality that $\mu \downarrow i$ is a $\boldsymbol{V}$-category for each $i \in I_0$.

Remark 4.4. $(\mu \downarrow i)((e, u), (e', u'))$ can be also defined by an equalizer

$$(\mu \downarrow i)((e, u), (e', u')) \longrightarrow E(e, e') \underset{u}{\overset{u_* \circ \mu}{\rightrightarrows}} E(e, e') \otimes (I \otimes 1)(p(e), i)$$

These "homotopy fibers" define functors on I.

Definition 4.5. Let E be a right I-graded category. Then define

$$\overleftarrow{\Gamma}(\mu) : I \longrightarrow \boldsymbol{V}\text{-}\mathbf{Categories}$$

as follows. For $i \in I_0$

$$\overleftarrow{\Gamma}(\mu)(i) = \mu \downarrow i.$$

For a morphism $u : i \to i'$ in I, define a $\boldsymbol{V}$-functor

$$\overleftarrow{\Gamma}(\mu)(u) : \overleftarrow{\Gamma}(\mu)(i) \longrightarrow \overleftarrow{\Gamma}(\mu)(i')$$

as follows. For an object (e, v) in $\overleftarrow{\Gamma}(\mu)(i) = \mu \downarrow i$, define

$$\overleftarrow{\Gamma}(\mu)(u)(e, v) = (e, u \circ v).$$

The morphism

$$\overleftarrow{\Gamma}(\mu)(u) : \overleftarrow{\Gamma}(i)((e, v), (e', v')) \longrightarrow \overleftarrow{\Gamma}(\mu)(i')((e, u \circ v), (e', u \circ v'))$$

is defined by the following commutative diagram

$$
\begin{array}{ccc}
\overleftarrow{\Gamma}(\mu)(i)((e,v),(e',v')) & \longrightarrow & E(e,e') \\
\downarrow & & \downarrow \mu \\
& & E(e,e') \otimes (I \otimes 1)(p(e),p(e')) \\
& & \downarrow 1\otimes v'_* \\
E(e,e') \otimes 1 \xrightarrow{\ 1\otimes v\ } E(e,e') \otimes (I \otimes 1)(p(e),i) \\
\end{array}
$$

with diagonal arrows labeled $1\otimes (u\circ v')_*$, $1\otimes u_*$, $1\otimes (u\circ v)$ to the object

$$E(e,e') \otimes (I \otimes 1)(p(e),i').$$

Dually, for an I-graded category

$$\mu : E \longrightarrow E \otimes (I \otimes 1),$$

define

$$\overrightarrow{\Gamma}(\mu) : I^{op} \longrightarrow \textbf{\textit{V}-Categories}$$

by

$$\overrightarrow{\Gamma}(\mu)(i) = i \downarrow \mu$$

on objects and

$$\overrightarrow{\Gamma}(\mu)(u)(e,v) = (e, v \circ u).$$

for a morphism $u : i' \to i$.

Lemma 4.6. $\overleftarrow{\Gamma}(\mu)$ and $\overrightarrow{\Gamma}(\mu)$ are functors.

Proof. Let us check $\overleftarrow{\Gamma}(\mu)$ is a functor. The case of $\overrightarrow{\Gamma}(\mu)$ is analogous. For an identity morphism

$$1_i : i \longrightarrow i$$

in I, the morphism

$$\overleftarrow{\Gamma}(\mu)(1_i) : \overleftarrow{\Gamma}(\mu)(i)((e,v),(e',v')) \longrightarrow \overleftarrow{\Gamma}(\mu)(i)((e,v),(e',v'))$$

is defined by the following diagram

$$
\begin{array}{ccc}
\overleftarrow{\Gamma}(\mu)(i)((e,v),(e',v')) & \longrightarrow & E(e,e') \\
\downarrow & & \downarrow \mu \\
& & E(e,e') \otimes (I \otimes 1)(p(e),p(e')) \\
& & \downarrow 1\otimes v'_* \\
E(e,e') \otimes 1 \xrightarrow{\ 1\otimes v\ } E(e,e') \otimes (I \otimes 1)(p(e),i) \\
\end{array}
$$

with diagonal arrows labeled $1\otimes (1_i\circ v')_*$, $1\otimes (1_i)_* = 1$, $1\otimes (1_i\circ v)$ to the object

$$E(e,e') \otimes (I \otimes 1)(p(e),i).$$

It follows that $\overleftarrow{\Gamma}(\mu)(1_i)$ is the identity morphism.

For a composable morphisms

$$i_0 \xrightarrow{\ u_1\ } i_1 \xrightarrow{\ u_2\ } i_2$$

in I, the composition

$$\overleftarrow{\Gamma}(\mu)(u_2) \circ \overleftarrow{\Gamma}(\mu)(u_1) : \overleftarrow{\Gamma}(\mu)(i_0)((i,v),(i',v')) \longrightarrow \overleftarrow{\Gamma}(\mu)(i_2)((e,u_2\circ u_1\circ v),(e',u_2\circ u_1\circ v'))$$

satisfies the universality of the pullback and we have

$$\overleftarrow{\Gamma}(\mu)(u_2) \circ \overleftarrow{\Gamma}(\mu)(u_1) = \overleftarrow{\Gamma}(\mu)(u_2 \circ u_1).$$

$\square$

Example 4.7. Consider the case I is a group G and $\boldsymbol{V} = k\text{-}\mathbf{Mod}$. For a G-graded category

$$\mu : A \longrightarrow A \otimes k[G]$$

we obtain a functor

$$\overleftarrow{\Gamma}(\mu) : G \longrightarrow k\text{-}\mathbf{Categories}.$$

Since G has a single object $*$, $\overleftarrow{\Gamma}(\mu)$ is determined by the k-linear category $\overleftarrow{\Gamma}(\mu)(*)$ and an action of G on it.

The k-linear category $\overleftarrow{\Gamma}(\mu)(*)$ has objects

$$\overleftarrow{\Gamma}(\mu)(*)_0 = \{(x, g) \mid x \in A_0, g \in \mathrm{Mor}_G(*, *)\} = A_0 \times G.$$

The k-module of morphisms $\overleftarrow{\Gamma}(\mu)(*)((x, g), (y, h))$ is given by the pullback diagram

$$
\begin{array}{ccc}
\overleftarrow{\Gamma}(\mu)(*)((x,g),(y,h)) & \longrightarrow & A(x,y) \\
\downarrow & & \downarrow{\scriptstyle \mu} \\
& & A(x,y) \otimes k[G] \\
& & \downarrow{\scriptstyle h_*} \\
A(x,y) \otimes 1 & \xrightarrow{1 \otimes g} & A(x,y) \otimes k[G].
\end{array}
$$

Since μ is G-grading, we have a coproduct decomposition

$$A(x,y) = \bigoplus_{g \in G} A^g(x,y)$$

and, for $f \in A^g(x,y)$, $\mu(f)$ is given by

$$\mu(f) = f \otimes g.$$

Thus we have an identification

$$\overleftarrow{\Gamma}(\mu)(*)((x,g),(y,h)) \cong A^{h^{-1}g}(x,y)$$

for $g, h \in G$. The is the smash product construction in [CM06, Asaa, Asab]. $\square$

Let us extend $\overleftarrow{\Gamma}$ and $\overrightarrow{\Gamma}$ as 2-functors

$$
\begin{aligned}
\overleftarrow{\Gamma} &: \boldsymbol{V}\text{-}\overleftarrow{\mathbf{Categories}}_I \longrightarrow \overleftarrow{\mathbf{Oplax}}(I, \boldsymbol{V}\text{-}\mathbf{Categories}) \\
\overrightarrow{\Gamma} &: \boldsymbol{V}\text{-}\overrightarrow{\mathbf{Categories}}_I \longrightarrow \overrightarrow{\mathbf{Lax}}(I^{op}, \boldsymbol{V}\text{-}\mathbf{Categories}).
\end{aligned}
$$

Definition 4.8. Let

$$(F, \varphi) : (E, \mu) \longrightarrow (E', \mu').$$

be a morphism in $\boldsymbol{V}\text{-}\overleftarrow{\mathbf{Categories}}_I$, i.e. a degree-preserving left morphism in $\overleftarrow{\mathbf{Comod}}\text{-}(I \otimes 1)$. φ can be written as

$$\varphi(e) = 1 \otimes \varphi_2(e)$$

for $\varphi_2(e) \in \mathrm{Mor}_I(p'(F(e)), p(e))$.

Define a left morphism of oplax functors

$$(\Gamma(F, \varphi), \Gamma(\varphi)) : \overleftarrow{\Gamma}(\mu) \longrightarrow \overleftarrow{\Gamma}(\mu')$$

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as follows. For each $i \in I_0$, define a $\boldsymbol{V}$-functor
$$\overleftarrow{\Gamma}(F, \varphi)(i) : \overleftarrow{\Gamma}(\mu)(i) \longrightarrow \overleftarrow{\Gamma}(\mu')(i)$$
by
$$\overleftarrow{\Gamma}(F, \varphi)(i)(e, v) = (F(e), v \circ \varphi_2(e))$$
for an object $(e, v) \in \overleftarrow{\Gamma}(\mu)(i)_0$ and
$$\overleftarrow{\Gamma}(F, \varphi)(i) : \overleftarrow{\Gamma}(\mu)(i)((e, v), (e', v')) \longrightarrow \overleftarrow{\Gamma}(\mu')(i)((F(e), v \circ \varphi_2(e)), (F(e'), v' \circ \varphi_2(e')))$$
is defined by the commutativity of the following diagram

where the commutativity of the top right hexagon follows from the naturality of φ.

For each $u : i \to i'$ in I and $(e, v) \in \overleftarrow{\Gamma}(\mu)(i)_0$, define
$$\overleftarrow{\Gamma}(\varphi)(u)(e, v) : 1 \longrightarrow \overleftarrow{\Gamma}(\mu')(i')((\overleftarrow{\Gamma}(\mu')(u) \circ \overleftarrow{\Gamma}(F, \varphi))(e, v), (\overleftarrow{\Gamma}(F, \varphi)(i') \circ \overleftarrow{\Gamma}(\mu)(u))(e, v))$$
$$= \overleftarrow{\Gamma}(\mu')(i')((F(e), u \circ (v \circ \varphi_2(e))), (F(e), (u \circ v) \circ \varphi_2(e)))$$
to be the identity.

Definition 4.9. Let
$$(F, \varphi), (G, \psi) : (E, \mu) \longrightarrow (E', \mu')$$
be left morphisms of I-graded $\boldsymbol{V}$-categories. For a 2-morphism
$$\xi : (F, \varphi) \Longrightarrow (G, \psi)$$
in $\boldsymbol{V}\text{-}\overleftarrow{\mathbf{Categories}}_I$, define a 2-morphism in $\overleftarrow{\mathbf{Oplax}}(I, \boldsymbol{V}\text{-}\mathbf{Categories})$
$$\overleftarrow{\Gamma}(\xi) : (\overleftarrow{\Gamma}(F, \varphi), \overleftarrow{\Gamma}(\varphi)) \Longrightarrow (\overleftarrow{\Gamma}(G, \psi), \overleftarrow{\Gamma}(\psi)),$$
i.e. a $\boldsymbol{V}$-natural transformation
$$\overleftarrow{\Gamma}(\xi) : \overleftarrow{\Gamma}(F, \varphi) \Longrightarrow \overleftarrow{\Gamma}(G, \psi)$$
by the following diagram

Lemma 4.10. *The above constructions define a functor*

$$\overleftarrow{\Gamma} : \boldsymbol{V}\text{-}\overleftarrow{\mathbf{Categories}}_I((\mu), (\mu')) \longrightarrow \overleftarrow{\mathbf{Oplax}}(I, \boldsymbol{V}\text{-}\mathbf{Categories})(\overleftarrow{\Gamma}(\mu), \overleftarrow{\Gamma}(\mu')).$$

Proof. By the universality of pullback, the identity natural transformation induces the identity. For composable 2-morphisms

$$(F_0, \varphi_0) \overset{\xi_1}{\Longrightarrow} (F_1, \varphi_1) \overset{\xi_2}{\Longrightarrow} (F_2, \varphi_2),$$

consider the following diagram

The commutativity of the diagram and the universality of the pullback implies that

$$(\Gamma(\xi_2) \circ \Gamma(\xi_1))(e, v) = \Gamma(\xi_2)(e, v) \circ \Gamma(\xi_1)(e, v).$$

$\square$

It is not hard to see that $\overleftarrow{\Gamma}$ preserves the identity left morphisms and compositions of left morphisms.

Proposition 4.11. *We obtain a 2-functor*

$$\overleftarrow{\Gamma} : \boldsymbol{V}\text{-}\overleftarrow{\mathbf{Categories}}_I \longrightarrow \overleftarrow{\mathbf{Oplax}}(I, \boldsymbol{V}\text{-}\mathbf{Categories}).$$

Proof. Strict 2-categories are categories enriched over **Categories**. By definition of enriched functors (Definition 2.31), it suffices to prove that the following diagrams are commutative in **Categories**.

The commutativity of the first diagram is obvious. It is tedious but straightforward to check the commutativity of the second diagram. $\square$

Dually we have

Proposition 4.12. *We obtain a 2-functor*

$$\overrightarrow{\Gamma} : \boldsymbol{V}\text{-}\overrightarrow{\mathbf{Categories}}_I \longrightarrow \overrightarrow{\mathbf{Lax}}(I^{op}, \boldsymbol{V}\text{-}\mathbf{Categories}).$$

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4.2   Fibered and Cofibered Categories

We have seen in §4.1 that there are three ways to take a fiber over an object $i \in I_0$ for an I-graded category

$$\mu : E \longrightarrow E \otimes (I \otimes 1).$$

We have also seen that two of them, $\mu \downarrow i$ and $i \downarrow \mu$ can be extended to 2-functors

$$\overleftarrow{\Gamma} \quad : \quad \textbf{\textit{V}-\overleftarrow{Categories}}_I \longrightarrow \overleftarrow{\textbf{Oplax}}(I, \textbf{\textit{V}-Categories}),$$
$$\overrightarrow{\Gamma} \quad : \quad \textbf{\textit{V}-\overrightarrow{Categories}}_I \longrightarrow \overrightarrow{\textbf{Lax}}(I^{op}, \textbf{\textit{V}-Categories}).$$

In the case of non-enriched categories, we need the notions of fibered and cofibered categories introduced by Grothendieck in order to extend the remaining construction $E|_i$ to a 2-functor. A definition of fibered I-graded category is introduced for k-linear categories by Lowen [Low08] recently. We reformulate Lowen's definition in order to incorporate it into our definition of graded categories.

The idea of Grothendieck is to compare $\Gamma_{\mathrm{cof}}(\mu)(i)$ and $E|_i$ for a given I-graded category

$$\mu : E \longrightarrow E \otimes (I \otimes 1).$$

Definition 4.13. Let $\mu : E \to E \otimes (I \otimes 1)$ be an I-graded category. Define

$$i_i : E|_i \longrightarrow \mu \downarrow i$$

as follows. For $e \in (E|_i)_0$,
$$i_i(e) = (e, 1_i).$$

For $e, e' \in (E|_i)_0$,
$$i_i : (E|_i)(e, e') \longrightarrow (\mu \downarrow i)((e, 1_i), (e', 1_i))$$

is defined by the following diagram

Similarly we define

$$j_i : E|_i \longrightarrow i \downarrow \mu$$

by

$$j_i(e) = (e, 1_i)$$

on objects.

Lemma 4.14. *For each $i \in I_0$,*

$$i_i \quad : \quad E|_i \longrightarrow \mu \downarrow i,$$
$$j_i \quad : \quad E|_i \longrightarrow i \downarrow \mu$$

are $\textbf{V}$-functors.

We define prefibered and precofibered categories in terms of these functors.

Definition 4.15. When i_i has a left adjoint

$$s_i : \mu \downarrow i \longrightarrow E|_i$$

for each $i \in I_0$, μ is called precofibered. The collection $\{s_i\}_{i \in I_0}$ is called a precofibered structure on μ.
  When j_i has a right adjoint

$$t_i : i \downarrow \mu \longrightarrow E|_i$$

for each $i \in I_0$, μ is called prefibered. The collection $\{t_i\}_{i \in I_0}$ is called a prefibered structure on μ.

  We obtain a lax functor from a prefibered category.

Definition 4.16. Let

$$\mu : E \longrightarrow E \otimes (I \otimes 1)$$

be a prefibered category. For $i \in I_0$, define

$$\Gamma_{\mathrm{fib}}(\mu)(i) = E|_i.$$

For a morphism $u : i \to i'$ in I, define

$$\Gamma_{\mathrm{fib}}(\mu)(u) : E|_{i'} \longrightarrow E|_i$$

by the composition

$$E|_{i'} \xrightarrow{j_{i'}} i' \downarrow \mu \xrightarrow{\overrightarrow{\Gamma}(\mu)(u)} i \downarrow \mu \xrightarrow{t_i} E|_i.$$

  Dually, for a precofibered category

$$\mu : E \longrightarrow E \otimes (I \otimes 1),$$

define

$$\Gamma_{\mathrm{cof}}(\mu)(i) = E|_i$$

for each $i \in I_0$ and define

$$\Gamma_{\mathrm{cof}}(\mu)(u) : E|_i \longrightarrow E|_{i'}$$

for each $u : i \to i'$ by the composition

$$E|_i \xrightarrow{i_i} \mu \downarrow i \xrightarrow{\overleftarrow{\Gamma}(\mu)(u)} \mu \downarrow i' \xrightarrow{s_{i'}} E|_{i'}.$$

Lemma 4.17. *When μ is prefibered*

$$\Gamma_{\mathrm{fib}}(\mu) : I^{op} \longrightarrow \boldsymbol{V}\text{-}\mathbf{Categories}$$

is a lax functor. When μ is precofibered

$$\Gamma_{\mathrm{cof}}(\mu) : I \longrightarrow \boldsymbol{V}\text{-}\mathbf{Categories}$$

is an oplax functor.

Proof. For a composable morphisms $i \xrightarrow{u} i' \xrightarrow{u'} i''$, we have the following diagram

By definition, we have a natural transformation

$$\varepsilon_{i'} : j_{i'} \circ t_{i'} \Longrightarrow 1_{i' \downarrow \mu}$$

for each $i \in I_0$. Define

$$\theta_{u',u} : \Gamma_{\mathrm{fib}}(\mu)(u) \circ \Gamma_{\mathrm{fib}}(\mu)(u') \Longrightarrow \Gamma_{\mathrm{fib}}(\mu)(u' \circ u)$$

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by using $\varepsilon_{i'}$

$$\Gamma_{\mathrm{fib}}(\mu)(u) \circ \Gamma_{\mathrm{fib}}(\mu)(u') = (t_i \circ \overrightarrow{\Gamma}(\mu)(u) \circ j_{i'}) \circ (t_{i'} \circ \overrightarrow{\Gamma}(\mu)(u) \circ j_{i''})$$
$$\Longrightarrow t_i \circ \overrightarrow{\Gamma}(\mu)(u) \circ \overrightarrow{\Gamma}(\mu)(u') \circ j_{i''} = t_i \circ \overrightarrow{\Gamma}(\mu)(u' \circ u) \circ j_{i''} = \Gamma_{\mathrm{fib}}(\mu)(u' \circ u).$$

Then we obtain a lax functor.

Similarly, if μ is precofibered, define

$$\theta_{u',u} : \Gamma_{\mathrm{cof}}(\mu)(u' \circ u) \Longrightarrow \Gamma_{\mathrm{cof}}(\mu)(u') \circ \Gamma_{\mathrm{cof}}(\mu)(u)$$

by using

$$\eta_{i'} : 1_{E|_{i'}} \longrightarrow i_{i'} \circ s_{i'}.$$

And we obtain an oplax functor. $\qquad \square$

Definition 4.18. We say a prefibered I-graded category $\mu : E \to E \otimes (I \otimes 1)$ is fibered if the above natural transformations $\varepsilon_{i'}$, hence $\theta_{u',u}$, are isomorphisms. A precofibered category is called cofibered if $\eta_{i'}$ are all natural isomorphisms.

We would like to define morphisms of prefibered and precofibered categories corresponding to morphisms of lax and oplax functors under Γ_{fib} and Γ_{cof}.

Definition 4.19. Let

$$\begin{aligned}
\mu &: & E &\longrightarrow E \otimes (I \otimes 1) \\
\mu' &: & E' &\longrightarrow E' \otimes (I \otimes 1)
\end{aligned}$$

be prefibered graded categories over I. A morphism of prefibered graded categories from μ to μ' is a right morphism

$$(F, \varphi) : (E, \mu) \longrightarrow (E', \mu')$$

of I-graded categories. 2-morphisms are also 2-morphisms in $\boldsymbol{V\text{-}\overrightarrow{\mathbf{Categories}}}_I$.

Definition 4.20. Let

$$(F, \varphi) : (E, \mu) \longrightarrow (E', \mu')$$

be a morphism of prefibered I-graded categories. Define a right morphism of lax functors

$$(\Gamma_{\mathrm{fib}}(F, \varphi), \Gamma_{\mathrm{fib}}(\varphi)) : \Gamma_{\mathrm{fib}}(\mu) \longrightarrow \Gamma_{\mathrm{fib}}(\mu')$$

as follows. For an object $i \in I_0$, we need to define a functor

$$\Gamma_{\mathrm{fib}}(F, \varphi)(i) : E|_i \longrightarrow E'|_i.$$

For $e \in (E|_i)_0$, we have

$$\varphi(e) : i = p(e) \longrightarrow p'(F(e)).$$

In other words, $(F(e), \varphi(e)) \in (i \downarrow \mu')_0$. By applying

$$t_i' : i \downarrow \mu' \longrightarrow E'|_i$$

we obtain

$$\Gamma_{\mathrm{fib}}(F, \varphi)(i)(e) = t_i'(F(e), \varphi(e)) \in (E'|_i)_0.$$

For a pair of objects $e, e' \in (E|_i)_0$, define

$$(E|_i)(e, e') \longrightarrow (E'|_i)(\Gamma_{\mathrm{fib}}(F, \varphi)(i)(e), \Gamma_{\mathrm{fib}}(F, \varphi)(i)(e'))$$

by the composition

$$(E|_i)(e, e') \xrightarrow{\bar{F}} (i \downarrow E')((F(e), \varphi(e)), (F(e'), \varphi(e'))) \xrightarrow{t_i'} (E'|_i)(t_i'(F(e), \varphi(e)), t_i'(F(e'), \varphi(e'))),$$

where $\tilde{F}$ is defined by the following diagram

$$
\begin{array}{ccc}
E(e,e) & & \\
& & \\
& (i \downarrow E')((F(e),\varphi(e)),(F(e'),\varphi(e'))) \xrightarrow{\quad} E'(F(e),F(e')) \\
& \downarrow & \downarrow \\
& E'(F(e),F(e')) \xrightarrow{\quad} E'(F(e),F(e')) \otimes (I \otimes 1)(i, p'(F(e'))).
\end{array}
$$

The commutativity of the outside square comes from the naturality of φ.

We define $\Gamma_{\mathrm{fib}}(\varphi)$ for

$$
\begin{array}{ccc}
E|_{i'} & \xrightarrow{\Gamma_{\mathrm{fib}}(F,\varphi)(i')} & E'|_{i'} \\
\Gamma_{\mathrm{fib}}(\mu)(u) \downarrow & \overset{\Gamma_{\mathrm{fib}}(\varphi)}{\Longrightarrow} & \downarrow \Gamma_{\mathrm{fib}}(\mu)(u) \\
E|_{i} & \xrightarrow[\Gamma_{\mathrm{fib}}(F,\varphi)(i)]{} & E'|_{i}
\end{array}
$$

as follows. The adjunction ε_i induces

$$ F(t_i(e,u)) \longrightarrow F(e), $$

which in turn induces

$$ j_i(F(t_i(e,u))) \longrightarrow (F(e),\varphi(e)) $$

or

$$ F(t_i(e,u)) \longrightarrow t_i(F(e),\varphi(e)) $$

and we obtain a morphism

$$ (F(t_i(e,u)),\varphi(t_i(e,u))) \longrightarrow (t_i'(F(e),\varphi(e)),u) $$

in $i \downarrow \mu'$. By applying t_i', we obtain $\Gamma_{\mathrm{fib}}(\varphi)$.

For 2-morphisms we define as follows.

Definition 4.21. Let

$$ (F,\varphi),(G,\psi) : (E,\mu) \longrightarrow (E',\mu') $$

be right morphisms of prefibered I-graded categories. For a 2-morphism

$$ \xi : (F,\varphi) \Longrightarrow (G,\psi), $$

define a morphism of right transformations

$$ \Gamma_{\mathrm{fib}}(\xi) : (\Gamma_{\mathrm{fib}}(F,\varphi),\Gamma_{\mathrm{fib}}(\varphi)) \Longrightarrow (\Gamma_{\mathrm{fib}}(G,\psi),\Gamma_{\mathrm{fib}}(\psi)) $$

as follows. For $i \in X_0$, we need to define a $\boldsymbol{V}$-natural transformation

$$ \Gamma_{\mathrm{fib}}(\xi)(i) : \Gamma_{\mathrm{fib}}(F,\varphi)(i) \Longrightarrow \Gamma_{\mathrm{fib}}(G,\psi)(i). $$

For each object e in $E|_i$, we have

$$ \xi(e) : F(e) \longrightarrow G(e). $$

We also have a commutative diagram

$$
\begin{array}{ccc}
x = p(e) & \xrightarrow{\varphi(e)} & p'(F(e)) \\
\| & & \downarrow p'(\xi(e)) \\
x = p(e) & \xrightarrow[\psi(e)]{} & p'(G(e))
\end{array}
$$

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and we have a morphism
$$\xi(e) : (F(e), \varphi(e)) \longrightarrow (G(e), \psi(e))$$

in $i \downarrow \mu'$ and we obtain
$$\Gamma_{\mathrm{fib}}(\xi)(i)(e) \in E'(t'_i(F(e), \varphi(e)), t'_i(G(e), \psi(e))) = (E'|_i)(\Gamma_{\mathrm{fib}}(F, \varphi)(i)(e), \Gamma_{\mathrm{fib}}(G, \psi)(i)(e)).$$

It is not difficult to check that $\Gamma_{\mathrm{fib}}(\xi)$ satisfies the condition for a morphism of right transformation.

The above definitions can be dualized and we obtain a 2-functor Γ_{cof}.

Proposition 4.22. *We have 2-functors*

$$\Gamma_{\mathrm{fib}} \quad : \quad \mathbf{Prefibered}(I) \longrightarrow \overrightarrow{\mathbf{Lax}}(I^{op}, \boldsymbol{V}\text{-}\mathbf{Categories}),$$
$$\Gamma_{\mathrm{cof}} \quad : \quad \mathbf{Precofibered}(I) \longrightarrow \overleftarrow{\mathbf{Oplax}}(I, \boldsymbol{V}\text{-}\mathbf{Categories}),$$

whose behaviors on objects are given by $\Gamma_{\mathrm{fib}}(\mu)$ and $\Gamma_{\mathrm{cof}}(\mu)$, respectively.

5 The Grothendieck Construction as Left Adjoints

In §3.3, we have seen that the Grothendieck construction defines a 2-functor

$$\mathrm{Gr} : \overleftarrow{\mathbf{Oplax}}(I, \boldsymbol{V}\text{-}\mathbf{Categories}) \longrightarrow \boldsymbol{V}\text{-}\overleftarrow{\mathbf{Categories}}_I \subset \overleftarrow{\mathbf{Comod}}\text{-}(I \otimes 1).$$

We have constructed 2-functors

$$\overleftarrow{\Gamma} : \boldsymbol{V}\text{-}\overleftarrow{\mathbf{Categories}}_I \longrightarrow \overleftarrow{\mathbf{Funct}}(I, \boldsymbol{V}\text{-}\mathbf{Categories}) \subset \overleftarrow{\mathbf{Oplax}}(I, \boldsymbol{V}\text{-}\mathbf{Categories})$$

and

$$\Gamma_{\mathrm{cof}} : \mathbf{Cofibered}(I) \longrightarrow \overleftarrow{\mathbf{Oplax}}(I, \boldsymbol{V}\text{-}\mathbf{Categories})$$

in §4.

The aim of this section is to show that Gr is left adjoint to $\overleftarrow{\Gamma}$. We also show that a restriction of Gr to a certain 2-subcategory is left adjoint to Γ_{cof} in a weak sense. We, of course, have "lax versions" of these adjunctions.

By composing the forgetful functor, we obtain

$$\mathrm{Gr} : \overleftarrow{\mathbf{Oplax}}(I, \boldsymbol{V}\text{-}\mathbf{Categories}) \longrightarrow \boldsymbol{V}\text{-}\mathbf{Categories},$$

which is the 2-functor Gr defined in §3.1. We show that this 2-functor also has a right adjoint in §A.2.

5.1 The Grothendieck Construction and Graded Categories

In this section, we prove that $\overleftarrow{\Gamma}$ is right adjoint to Gr.

Theorem 5.1. *For an oplax functor $X : I \to \boldsymbol{V}\text{-}\mathbf{Categories}$ and an I-graded category $\mu : E \to E \otimes (I \otimes 1)$, we have an isomorphism of categories*

$$(\boldsymbol{V}\text{-}\overleftarrow{\mathbf{Categories}}_I)((\mathrm{Gr}(X), \mu_X), (E, \mu)) \cong \overleftarrow{\mathbf{Oplax}}(I, \boldsymbol{V}\text{-}\mathbf{Categories})(X, \overleftarrow{\Gamma}(\mu)).$$

We need to define morphisms

$$\eta_X \quad : \quad X \longrightarrow \overleftarrow{\Gamma}(\mu_X),$$
$$\varepsilon_\mu \quad : \quad \mathrm{Gr}(\overleftarrow{\Gamma}(\mu)) \longrightarrow \mu,$$

in $\overleftarrow{\mathbf{Oplax}}(I, \boldsymbol{V}\text{-}\mathbf{Categories})$ and $\boldsymbol{V}\text{-}\overleftarrow{\mathbf{Categories}}_I$, respectively.

Let us first define η_X. We need to define a family of functors

$$\eta_X(i) : X(i) \longrightarrow \overleftarrow{\Gamma}(\mu_X)(i)$$

indexed by $i \in I_0$ and a family of natural transformations

$$\eta_X(u) : \overleftarrow{\Gamma}(\mu_X)(u) \circ \eta_X(i) \Longrightarrow \eta_X(j) \circ X(u)$$

indexed by morphisms $u : i \to j$ in I.

Objects of $\overleftarrow{\Gamma}(\mu_X)(i)$ are given by

$$\overleftarrow{\Gamma}(\mu_X)(i)_0 = \coprod_{(x,j) \in \mathrm{Gr}(X)_0} \{(x,j)\} \times \mathrm{Mor}_I(j,i)$$

and we have a canonical inclusion

$$\eta_X(i) : X(i)_0 \hookrightarrow X(i)_0 \times \{i\} \times \{1_i\} \subset \overleftarrow{\Gamma}(\mu_X)(i)_0.$$

For $x, x' \in X(i)_0$,

$$\overleftarrow{\Gamma}(\mu_X)(i)(\eta_X(i)(x), \eta_X(i)(x')) = \overleftarrow{\Gamma}(\mu_X)((x,i,1_i),(x',i,1_i))$$

is defined by the following pullback diagram

and we have

$$\overleftarrow{\Gamma}(\mu_X)((x,i,1_i),(x',i,1_i)) \cong X(i)(x,x') \times \{1_i\}.$$

This identification defines a functor

$$\eta_X(i) : X(i) \longrightarrow \overleftarrow{\Gamma}(\mu_X)(i)$$

by the canonical inclusion.

For a morphism $u : i \to j$ and an object $x \in X(i)$, we have

$$\begin{aligned}
\overleftarrow{\Gamma}(\mu_X)(u) \circ \eta_X(i)(x) &= (x,i,u) \\
\eta_X(j) \circ X(u)(x) &= (X(u)(x),j,1_j).
\end{aligned}$$

Define a morphism in $\boldsymbol{V}$

$$\eta_X(u)(x) : 1 \longrightarrow \overleftarrow{\Gamma}(\mu_X)(j)(\overleftarrow{\Gamma}(\mu_X)(u) \circ \eta_X(i)(x), \eta_X(j) \circ X(u)(x)) = \overleftarrow{\Gamma}(\mu_X)(j)((x,i,u),(X(u)(x),j,1_j))$$

by the following diagram

Lemma 5.2. *The above construction defines a morphism of oplax functors*

$$\eta_X : X \longrightarrow \overleftarrow{\Gamma}(\mu_X).$$

Proof. We need to check the following diagrams are commutative.

$$
\begin{array}{ccc}
\overleftarrow{\Gamma}(\mu_X)(1_i) \circ \eta_X(i) & \Longrightarrow & \eta_X(i) \circ X(1_i) \\
\big\Downarrow & & \big\Downarrow \\
1_{\overleftarrow{\Gamma}(\mu_X)(i)} \circ \eta_X(i) & = & \eta_X(i) \circ 1_{X(i)}
\end{array}
$$

$$
\begin{array}{ccccc}
\overleftarrow{\Gamma}(\mu_X)(v \circ u) \circ \eta_X(k) & \Longrightarrow & \overleftarrow{\Gamma}(\mu_X)(u) \circ \overleftarrow{\Gamma}(\mu_X)(v) \circ \eta_X(k) & \Longrightarrow & \overleftarrow{\Gamma}(\mu_X)(u) \circ \eta_X(j) \circ X(v) \\
\big\Downarrow & & & & \big\Downarrow \\
\eta_X(k) \circ X(v \circ u) & = & \!\longrightarrow & \eta_X(k) \circ X(v) \circ X(u).
\end{array}
$$

The commutativity of these diagrams follows from the commutativity of diagrams of 2-morphisms in Definition 2.42. The details are omitted. $\square$

Let us define ε_μ for an I-graded category (E, μ). We need to define a $\boldsymbol{V}$-functor

$$\varepsilon'_\mu : \mathrm{Gr}(\overleftarrow{\Gamma}(\mu)) \longrightarrow E$$

and a $\boldsymbol{V}$-natural transformation

$$
\begin{array}{ccc}
\mathrm{Gr}(\overleftarrow{\Gamma}(\mu)) & \xrightarrow{\mu_{\overleftarrow{\Gamma}(\mu)}} & \mathrm{Gr}(\overleftarrow{\Gamma}(\mu)) \otimes (I \otimes 1) \\
{\scriptstyle \varepsilon'_\mu}\big\downarrow & {\scriptstyle \varepsilon''_\mu}\!\!\!\nearrow & \big\downarrow {\scriptstyle \varepsilon'_\mu \otimes 1} \\
E & \xrightarrow{\quad \mu \quad} & E \otimes (I \otimes 1).
\end{array}
$$

Let us denote the map induced by μ on objects by

$$\mu_0 = 1_{E_0} \times p : E_0 \longrightarrow E_0 \times I_0.$$

ε'_μ is defines as follows. Objects of $\mathrm{Gr}(\overleftarrow{\Gamma}(\mu))$ are given by

$$\mathrm{Gr}(\overleftarrow{\Gamma}(\mu))_0 = \coprod_{i \in I_0} \overleftarrow{\Gamma}(\mu)(i)_0 \times \{i\} = \coprod_{i \in I_0} \left\{ (e, p(e) \xrightarrow{u} i, i) \;\middle|\; e \in E_0, u \in I_1 \right\}.$$

For objects $(e, u, i), (e', u', i')$ in $\mathrm{Gr}(\overleftarrow{\Gamma}(\mu))$,

$$\mathrm{Gr}(\overleftarrow{\Gamma}(\mu))((e, u, i), (e', u', i')) = \bigoplus_{v : i \to i'} \overleftarrow{\Gamma}(\mu)(i')(\overleftarrow{\Gamma}(\mu)(v)(e, u), (e'u'))$$

and each component $\overleftarrow{\Gamma}(\mu)(i')((e, v \circ u), (e', u'))$ is defined by the following pullback diagram

$$
\begin{array}{ccc}
\overleftarrow{\Gamma}(\mu)(i')((e, v \circ u), (e', u')) & \longrightarrow & E(e, e') \\
\big\downarrow & & \big\downarrow {\scriptstyle \mu} \\
& & E(e, e') \otimes (I \otimes 1)(p(e), p(e')) \\
\big\downarrow & & \big\downarrow {\scriptstyle u'_*} \\
E(e, e') \otimes 1 & \xrightarrow{1 \otimes (v \circ u)} & E(e, e') \otimes (I \otimes 1)(p(e), i').
\end{array}
$$

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ε'_μ is defined by the top morphism in the above diagram.

For an object $(x, u, i) \in \mathrm{Gr}(\overleftarrow{\Gamma}(\mu))$, define

$$\varepsilon''_\mu(x, u, i) : 1 \to (I \otimes 1)(p \circ \varepsilon'_\mu(x, u, i), p_{\overleftarrow{\Gamma}(\mu)}(x, u, i)) = (I \otimes 1)(p(e), i)$$

by

$$\varepsilon''_\mu(x, u, i) = u.$$

It is tedious but elementary to check that ε''_μ is a $\boldsymbol{V}$-natural transformation and we obtain a morphism $\varepsilon_\mu = (\varepsilon'_\mu, \varepsilon''_\mu)$ in $\boldsymbol{V}\text{-}\mathbf{Categories}_I$.

Lemma 5.3. *The pair $\varepsilon_\mu = (\varepsilon'_\mu, \varepsilon''_\mu)$ is a morphism in $\boldsymbol{V}\text{-}\overleftarrow{\mathbf{Categories}}_I$.*

Proof of Theorem 5.1. It remains to check the following diagrams are commutative

For an oplax functor X, consider the composition

$$\mathrm{Gr}(X) \xrightarrow{\mathrm{Gr}(\eta_X)} \mathrm{Gr}(\overleftarrow{\Gamma}(\mu_X)) \xrightarrow{\varepsilon_{\mathrm{Gr}(X)}} \mathrm{Gr}(X).$$

For objects, we have

$$(x, i) \longmapsto (x, 1_i, i) \longmapsto (x, i).$$

For objects $(x, i), (x', i')$ in $\mathrm{Gr}(X)$, we have

$$\begin{aligned}
\mathrm{Gr}(\overleftarrow{\Gamma}(\mu_X))(\eta_X(x, i), \eta_X(x', i')) &= \bigoplus_{u : i \to i'} \overleftarrow{\Gamma}(\mu_X)(i')(\overleftarrow{\Gamma}(\mu_X)(u)(x, 1_i), (x', 1_{i'})) \\
&= \bigoplus_{u : i \to i'} \overleftarrow{\Gamma}(\mu_X)(i')((x, u), (x', 1_{i'}))
\end{aligned}$$

and each component $\overleftarrow{\Gamma}(\mu_X)(i')((x, u), (x', 1_{i'}))$ can be identified with $X(i')(X(u)(x), x')$. It follows that the composition

$$\mathrm{Gr}(X)((x, i), (x', i')) \longrightarrow \mathrm{Gr}(\overleftarrow{\Gamma}(\mu_X))(\eta_X(x, i), \eta_X(x', i')) \longrightarrow \mathrm{Gr}(X)((x, i), (x, i'))$$

is the identity.

For an I-graded category $\mu : E \to E \otimes (I \otimes 1)$, consider the composition

$$\overleftarrow{\Gamma}(\mu) \xrightarrow{\eta_{\overleftarrow{\Gamma}(\mu)}} \overleftarrow{\Gamma}(\mathrm{Gr}(\overleftarrow{\Gamma}(\mu))) \xrightarrow{\overleftarrow{\Gamma}(\varepsilon_\mu)} \overleftarrow{\Gamma}(\mu)$$

of morphisms of oplax functors. Note that $\overleftarrow{\Gamma}$ takes values in strict functors and $\eta_{\overleftarrow{\Gamma}(\mu)}$ and $\overleftarrow{\Gamma}(\varepsilon_\mu)$ are ordinary natural transformations. Thus it suffices to consider the composition

$$\overleftarrow{\Gamma}(\mu)(i) \xrightarrow{\eta_{\overleftarrow{\Gamma}(\mu)}} \overleftarrow{\Gamma}(\mathrm{Gr}(\overleftarrow{\Gamma}(\mu)))(i) \xrightarrow{\overleftarrow{\Gamma}(\varepsilon_\mu)} \overleftarrow{\Gamma}(\mu)(i)$$

of $\boldsymbol{V}$-functors for each object $i \in I_0$.

Objects of $\overleftarrow{\Gamma}(\mathrm{Gr}(\overleftarrow{\Gamma}(\mu)))(i)$ are

$$\overleftarrow{\Gamma}(\mathrm{Gr}(\overleftarrow{\Gamma}(\mu)))(i)_0 = \coprod_{(e, p(e) \xrightarrow{u} i, i) \in \mathrm{Gr}(\overleftarrow{\Gamma}(\mu))_0} \{(e, p(e) \xrightarrow{u} i, i)\} \times \mathrm{Mor}_I(i, i)$$

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and, for $(e, u) \in \overleftarrow{\Gamma}(\mu)(i)_0$,

$$
\begin{aligned}
(\overleftarrow{\Gamma}(\varepsilon_\mu) \circ \eta_{\overleftarrow{\Gamma}(\mu)})(e, u) &= \overleftarrow{\Gamma}(\varepsilon_\mu)((e, u, i), 1_i) \\
&= (\varepsilon'_\mu(e, u, i), 1_i \circ \varepsilon''_\mu(e, u, i)) \\
&= (e, u).
\end{aligned}
$$

The composition

$$
\overleftarrow{\Gamma}(\mu)(i)((e, u), (e', u')) \overset{\eta_{\overleftarrow{\Gamma}(\mu)}}{\longrightarrow} \overleftarrow{\Gamma}(\mathrm{Gr}(\overleftarrow{\Gamma}(\mu)))(i)((e, u, i), (e', u', i)) \overset{\overleftarrow{\Gamma}(\varepsilon_\mu)}{\longrightarrow} \overleftarrow{\Gamma}(\mu)(i)((e, u), (e', u'))
$$

is easily seen to be the identity, since η is given by the canonical inclusion and ε' is given by the projection. □

5.2 The Smash Product Construction for Precofibered and Prefibered Categories

In this section, we study the relations between $\overleftarrow{\Gamma}$ and Γ_{cof} and $\overrightarrow{\Gamma}$ and Γ_{fib} for precofibered and prefibered categories, respectively. We concentrate on the case of precofibered categories. The case of prefibered categories is analogous and is left to the reader.

Lemma 5.4. *Let* $\mu : E \to E \otimes (I \otimes 1)$ *be a precofibered I-graded category. The family of functors*

$$
i_i : \Gamma_{\mathrm{cof}}(\mu)(i) \longrightarrow \overleftarrow{\Gamma}(\mu)(i)
$$

indexed by $i \in I_0$ defines a left transformation of oplax functors

$$
i : \Gamma_{\mathrm{cof}}(\mu) \longrightarrow \overleftarrow{\Gamma}(\mu).
$$

Proof. For $u : i \to i'$ in I, we have the following diagram

Since $s_{i'}$ is left adjoint to $i_{i'}$, we have a $\boldsymbol{V}$-natural transformation

$$
1_{\overleftarrow{\Gamma}(\mu)(i')} \Longrightarrow i_{i'} \circ s_{i'},
$$

which induces a left transformation. □

We only need the fact that $s_{i'}$ is left adjoint to $i_{i'}$ in the above proof and thus a similar argument implies that we obtain a right transformation from $\{s_i\}$.

Lemma 5.5. *The functors*

$$
s_i : \overleftarrow{\Gamma}(\mu)(i) \longrightarrow \Gamma_{\mathrm{cof}}(\mu)(i)
$$

defines a right transformation of oplax functors from $\overleftarrow{\Gamma}(\mu)$ to $\Gamma_{\mathrm{cof}}(\mu)$.

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When the natural transformations $\varepsilon_i : s_i \circ i_i \Rightarrow 1_{\Gamma_{\mathrm{cof}}(\mu)(i)}$ are natural isomorphisms, we obtain a left transformation.

Corollary 5.6. *When μ is precofibered and $\{\varepsilon_i\}$ are natural isomorphisms, we obtain a left transformation*

$$s : \overleftarrow{\Gamma}(\mu) \longrightarrow \Gamma_{\mathrm{cof}}(\mu).$$

Corollary 5.7. *Let μ be a precofibered I-graded $\boldsymbol{V}$-category. If the natural transformations*

$$\begin{aligned}
\varepsilon_i &: s_i \circ i_i \Longrightarrow 1_{\Gamma_{\mathrm{cof}}(\mu)(i)} \\
\eta_i &: 1_{\overleftarrow{\Gamma}(\mu)(i)} \Longrightarrow i_i \circ s_i
\end{aligned}$$

*are natural isomorphisms, for any oplax functor $X : I \to \boldsymbol{V}$-**Categories**, the above left transformations induce an equivalence of categories*

$$i : \overleftarrow{\mathbf{Oplax}}(I, \boldsymbol{V}\text{-}\mathbf{Categories})(X, \overleftarrow{\Gamma}(\mu)) \longleftrightarrow \overleftarrow{\mathbf{Oplax}}(I, \boldsymbol{V}\text{-}\mathbf{Categories})(X, \Gamma_{\mathrm{cof}}(\mu)) : s.$$

Corollary 5.8. *Under the assumptions as above, we have an equivalence of categories*

$$\boldsymbol{V}\text{-}\overleftarrow{\mathbf{Categories}}_I(\mathrm{Gr}(X), \mu) \simeq \overleftarrow{\mathbf{Oplax}}(I, \boldsymbol{V}\text{-}\mathbf{Categories})(X, \Gamma_{\mathrm{cof}}(\mu)).$$

In other words, under these assumptions, Γ_{cof} can be regarded as a right adjoint functor to Gr in a weak sense.

A Appendices

A.1 Enriched Categories by Comodules

Given a set S, the category of quivers with the set of vertices is the comma category

$$\mathbf{Quivers}(S) = \mathbf{Sets} \downarrow (S \times S).$$

This category can be made into a monoidal category under the pullback

It is a well-known fact that a category X with the set of objects S is a monoid object in this monoidal category. See [DS04], for example. This characterization can be generalized to a definition of small enriched categories by using the notion of bicomodules, which is the subject of this appendix. For the standard definition of enriched categories, see §2.3.

Our starting point is to regard C-C-bimodules as "quivers enriched over $\boldsymbol{V}$ with vertices given by C".

Definition A.1. When $C = S \otimes 1$ for a set S, C-C-bicomodules are called $\boldsymbol{V}$-quivers with the set of vertices S. The category of $\boldsymbol{V}$-quivers with the set of vertices S is denoted by $\boldsymbol{V}$-$\mathbf{Quivers}(S)$.

Lemma 2.26 allows us to define enriched categories without referring to each object.

Definition A.2. Let C be a flat comonoid object in $\boldsymbol{V}$. Then a monoid object in the monoidal category $(C\text{-}\mathbf{Comod}\text{-}C, \square_C, C)$ is called a category object in $\boldsymbol{V}$ with objects C.

For a category object M in $\mathbf{V}$ with objects C, the right and left coactions are denoted by

$$
\begin{aligned}
s &: M \longrightarrow M \otimes C \\
t &: M \longrightarrow C \otimes M
\end{aligned}
$$

and called the source and the target.

When $C = S \otimes 1$ for a set S, a category object in $\mathbf{V}$ with objects $S \otimes 1$ is called a category enriched over $\mathbf{V}$, or simply $\mathbf{V}$-category, with the set of objects S.

We use the following convention for simplicity.

Convention A.3. For a $\mathbf{V}$-category X, the set of objects is denoted by X_0. As an object of $\mathbf{V}$, X is denoted by X_1.

We also use traditional notations described in §2.3, when we have a coproduct decomposition

$$
X_1 = \bigoplus_{x,y \in X_0} X(x,y).
$$

Example A.4. Let $\mathbf{V} = k\text{-}\mathbf{Mod}$. The canonical natural transformation

$$
\theta_{S,T} : (S \times T) \otimes 1 \longrightarrow (S \otimes 1) \otimes (T \otimes 1)
$$

is an isomorphism for any S and T.

A category A enriched over $k\text{-}\mathbf{Mod}$ with the set of objects A_0 is a k-module A_1 equipped with a bicomodule structure

$$
\begin{aligned}
s &: A_1 \longrightarrow A_1 \otimes (A_0 \otimes 1), \\
t &: A_1 \longrightarrow (A_0 \otimes 1) \otimes A_1
\end{aligned}
$$

and a monoid structure

$$
\circ : A_1 \square_{A_0 \otimes 1} A_1 \longrightarrow A_1.
$$

Lemma 2.20 implies that we have a coproduct decomposition

$$
A_1 \cong \bigoplus_{(a,b) \in A_0 \times A_0} A(a,b)
$$

and

$$
A_1 \square_{A_0 \otimes 1} A_1 = \bigoplus_{a,b,c \in A_0} A(b,c) \otimes A(a,b).
$$

The monoid structure $\circ$ induces

$$
\circ : A(b,c) \otimes A(a,b) \longrightarrow A(a,c).
$$

The unit

$$
\iota : A_0 \otimes 1 \longrightarrow A_1
$$

induces a morphism

$$
1 \xrightarrow{s} \{s\} \otimes 1 \xrightarrow{\iota} A(a,a)
$$

serving as identity morphisms. And we obtain the standard definition of k-linear category.

Note that we the inclusion

$$
A_1 \square_{A_0 \otimes 1} A_1 \hookrightarrow A_1 \otimes A_1
$$

has a canonical retraction

$$
r : A_1 \otimes A_1 \longrightarrow A_1 \square_{A_0 \otimes 1} A_1
$$

and the composition

$$
A_1 \otimes A_1 \xrightarrow{r} A_1 \square_{A_0 \otimes 1} A_1 \xrightarrow{\circ} A_1
$$

makes A_1 into an algebra (possibly without a unit). This is the algebra associated with a k-linear category A, which is often denoted by $\Lambda(A)$. See [GS83b, CR05], for example. $\qquad\square$

Example A.5. Let Q be a quiver with the set of vertices Q_0, i.e. a diagram

$$Q_1 \underset{t}{\overset{s}{\rightrightarrows}} Q_0.$$

We obtain a $(Q_0 \otimes 1)$-$(Q_0 \otimes 1)$-bimodule structure on $Q_1 \otimes 1$ by

We obtain a functor

$$(-) \otimes 1 : \mathbf{Quivers}(Q_0) \longrightarrow (Q_0 \otimes 1)\text{-}\mathbf{Comod}\text{-}(Q_0 \otimes 1) = \boldsymbol{V}\text{-}\mathbf{Quivers}(Q_0).$$

$\square$

Example A.6. Let $\boldsymbol{V}$ be the category $\mathbf{Spaces}$ of topological spaces. As we have seen in Example 2.21, any object C is a comonoid and a coaction of C on another object M is determined by a morphism.

$$\pi : M \longrightarrow C.$$

Thus the category C-$\mathbf{Comod}$-C of bicomodules over C can be identified with the comma category $\mathbf{Spaces} \downarrow C \otimes C$, since the monoidal structure is given by the product. The corresponding monoidal structure on $\mathbf{Spaces} \downarrow C \otimes C$ is the monoidal structure described at the beginning of this section. A monoid object M in this monoidal category is, therefore, a topological category with the space of objects C.

When C has a discrete topology, i.e. $C = S \otimes 1$ for a set S, we have a coproduct decomposition

$$M \cong \coprod_{(s,t) \in S} M(s,t)$$

as we have seen in Example 2.21 and we obtain the standard definition of a category enriched over $\mathbf{Spaces}$ with the set of objects S. $\square$

Definition A.7. Let A and B be $\boldsymbol{V}$-categories. A $\boldsymbol{V}$-functor from A to B is a morphism of bicomodules

$$f = (f_0, f_1) : A \longrightarrow B$$

making the following diagrams commutative

The category of $\boldsymbol{V}$-categories and $\boldsymbol{V}$-functors is denoted by $\boldsymbol{V}$-$\mathbf{Categories}$.

A monoidal structure on $\boldsymbol{V}$-$\mathbf{Categories}$ is induced from the following monoidal structure on the category of bicomodules.

Definition A.8. For bicomodules M and N over C and D, respectively, define left and right coactions of $C \otimes D$ on $M \otimes N$ by

$$M \otimes N \xrightarrow{\mu_M^R \otimes \mu_N^R} (M \otimes C) \otimes (N \otimes D) \cong (M \otimes N) \otimes (C \otimes D),$$

$$M \otimes N \xrightarrow{\mu_M^L \otimes \mu_N^L} (C \otimes M) \otimes (D \otimes N) \cong (C \otimes D) \otimes (M \otimes N).$$

Lemma A.9. *The above operation makes the category* **Bicomodules**(V) *of bicomodules in* V *into a symmetric monoidal category. The unit is given by*

$$1 \longrightarrow 1 \otimes 1.$$

For V-categories A and B, we would like to define a morphism

$$(A \otimes A) \square_{(A_0 \otimes 1) \otimes (B_0 \otimes 1)} (B \otimes B) \longrightarrow A \otimes B.$$

We have the following diagram

$$
\begin{array}{ccccc}
(A \otimes B)\square_{(A_0 \otimes 1) \otimes (B_0 \otimes 1)}(A \otimes B) & \longrightarrow & (A \otimes B) \otimes (A \otimes B) & \underset{t}{\overset{s}{\rightrightarrows}} & (A \otimes B) \otimes (A_0 \otimes 1) \otimes (B_0 \otimes 1) \otimes (A \otimes B) \\
& & \downarrow & & \downarrow \\
(A \square_{A_0 \otimes 1} A) \otimes (B \square_{B_0 \otimes 1} B) & \longrightarrow & (A \otimes A) \otimes (B \otimes B) & \underset{t}{\overset{s}{\rightrightarrows}} & (A \otimes (A_0 \otimes 1) \otimes A) \otimes (B \otimes (B_0 \otimes 1) \otimes B).
\end{array}
$$

$$(4)$$

Lemma A.10. *Suppose the bottom row in the diagram (4) is an equalizer. Then the resulting morphism*

$$(A \otimes B)\square_{(A_0 \otimes 1) \times (B_0 \otimes 1)}(A \otimes B) \longrightarrow (A\square_{A_0 \otimes 1} A) \otimes (B\square_{B_0 \otimes 1} B) \xrightarrow{\circ \otimes \circ} A \otimes B.$$

defines a structure of monoid on $A \otimes B$.

If the bottom row in (4) is an equalizer for any V-categories A and B, then we obtain a symmetric monoidal structure on V-**Categories**.

Remark A.11. The condition for the above Lemma is satisfied when V is the category of k-modules, chain complexes, topological spaces, simplicial sets, and small categories.

We also have 2-morphisms in V-**Categories**.

Definition A.12. Let A and B be V-categories and

$$f, g : A \longrightarrow B$$

be V-functors. A V-natural transformation from f to g

$$\varphi : f \Longrightarrow g$$

is a morphism

$$\varphi : A_0 \otimes 1 \longrightarrow B_1$$

satisfying the following conditions:

1. The following diagram is commutative

$$
\begin{array}{ccccc}
(A_0 \otimes 1) \otimes (A_0 \otimes 1) & \xleftarrow{\Delta} & A_0 \otimes 1 & \xrightarrow{\Delta} & (A_0 \otimes 1) \otimes (A_0 \otimes 1) \\
{\scriptstyle \varphi \otimes f_0 \otimes 1}\downarrow & & {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle \varphi \otimes g_0 \otimes 1} \\
B_1 \otimes (B_0 \otimes 1) & \xleftarrow{s} & B_1 & \xrightarrow{t} & B_1 \otimes (B_0 \otimes 1).
\end{array}
$$

2. The following diagram is commutative

$$
\begin{array}{ccc}
& A_1 & \\
{}^{s}\swarrow & & \searrow^{T \circ t} \\
A_1 \square_{A_0 \otimes 1}(A_0 \otimes 1) & & (A_0 \otimes 1)\square_{A_0 \otimes 1}A_1 \\
{}_{g \otimes \varphi}\downarrow & & \downarrow^{\varphi \otimes f} \\
B_1 \square_{B_0 \otimes 1}B_1 & & B_1 \square_{B_0 \otimes 1}B_1 \\
{}_{\circ}\searrow & & \swarrow_{\circ} \\
& B_1 &
\end{array}
$$

We have seen in Lemma A.5 that the "free object functor"

$$(-) \otimes 1 : \mathbf{Sets} \longrightarrow \boldsymbol{V}$$

can be extended to

$$(-) \otimes 1 : \mathbf{Quivers} \longrightarrow \boldsymbol{V}\text{-}\mathbf{Quivers}.$$

It induces a functor

$$(-) \otimes 1 : \mathbf{Categories} \longrightarrow \boldsymbol{V}\text{-}\mathbf{Categories}.$$

Lemma A.13. *For a small category I with the set of objects I_0 and the set of morphisms I_1, define a $\boldsymbol{V}$-category $I \otimes 1$ by $(I \otimes 1)_0 = I_0$ and $(I \otimes 1)_1 = I_1 \otimes 1$. The structure morphisms are induced by those of I.*
Then we obtain a 2-functor

$$(-) \otimes 1 : \mathbf{Categories} \longrightarrow \boldsymbol{V}\text{-}\mathbf{Categories}.$$

A.2 The Grothendieck Construction as a Left Adjoint to the Diagonal Functor

Recall from Definition 2.49 that we have the diagonal functor

$$\Delta : \boldsymbol{C} \longrightarrow \overleftarrow{\mathbf{Oplax}}(I, \boldsymbol{C}).$$

The following adjunction is well-known for non-enriched categories. For example, it can be found in Thomason's [Tho79]. According to Thomason, it is originally due to J. Gray [Gra69]. For the sake of completeness, we give a proof of an enriched version of this adjunction.

Theorem A.14. *For any oplax functor $X : I \to \boldsymbol{V}$-**Categories** and a $\boldsymbol{V}$-category A, we have the following natural isomorphism of categories*

$$\boldsymbol{V}\text{-}\mathbf{Categories}(\mathrm{Gr}(X), A) \cong \overleftarrow{\mathbf{Oplax}}(I, \boldsymbol{V}\text{-}\mathbf{Categories})(X, \Delta(A)).$$

Proof. We need to define morphisms

$$
\begin{array}{rcl}
\eta_X & : & X \longrightarrow \Delta(\mathrm{Gr}(X)) \\
\varepsilon_A & : & \mathrm{Gr}(\Delta(A)) \longrightarrow A
\end{array}
$$

in $\overleftarrow{\mathbf{Oplax}}(I, \boldsymbol{V}\text{-}\mathbf{Categories})$ and $\boldsymbol{V}$-**Categories**, respectively.
For each $i \in I_0$,

$$\eta_X(i) : X(i) \hookrightarrow \Delta(\mathrm{Gr}(X))(i) = \mathrm{Gr}(X).$$

is given by the canonical inclusion. For a morphism $u : i \to j$ in I, we need to define a $\boldsymbol{V}$-natural transformation

$$\Delta(\mathrm{Gr}(X))(u) \circ \eta_X(i) \Longrightarrow \eta_X(j) \circ X(u).$$

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It is given, for $x \in X(i)_0$, by the composition

$$1 \stackrel{1_{X(u)(x)}}{\longrightarrow} X(j)(X(u)(x), X(u)(x)) \hookrightarrow \mathrm{Gr}(X)((x,i),(X(u)(x),j)) = \mathrm{Gr}(X)(\eta_X(i)(x), \eta_X(j)(X(u)(x))).$$

It is left to the reader to check that these family of morphisms form a V-natural transformation.

The category $\mathrm{Gr}(\Delta(A))$ has objects

$$\mathrm{Gr}(\Delta(A))_0 = \coprod_{i \in I_0} \Delta(A)(i)_0 \times \{i\} = A_0 \times I_0.$$

For $(a,i), (a',i') \in \mathrm{Gr}(\Delta(A))_0$, we obtain a morphism

$$\mathrm{Gr}(\Delta(A))((a,i),(a',i')) = \bigoplus_{u:i \to i'} A(a,a') \longrightarrow A(a,a') \otimes (I \otimes 1)(i,i').$$

or

$$\mathrm{Gr}(\Delta(A)) \longrightarrow A \otimes (I \otimes 1).$$

The counit ε of $I \otimes 1$ induces

$$\varepsilon_A : \mathrm{Gr}(\Delta(A)) \longrightarrow A \otimes (I \otimes 1) \stackrel{1 \otimes \varepsilon}{\longrightarrow} A \otimes 1 \cong A.$$

It is elementary to check that η_X and ε_A give us adjunctions we want. The proof is omitted. $\quad\square$

Example A.15. Let G be a group and consider the case $A = k\text{-}\mathbf{Mod}$. $\Delta(A)$ is $k\text{-}\mathbf{Mod}$ equipped with the trivial G-action. Thus $\overleftarrow{\mathbf{Oplax}}(G, k\text{-}\mathbf{Categories})(X, \Delta(A))$ is the category of right G-invariant functors from X to $k\text{-}\mathbf{Mod}$. (See Example 2.51 for right G-invariant functors).

On the other hand, the category $k\text{-}\mathbf{Categories}(\mathrm{Gr}(X), k\text{-}\mathbf{Mod})$ is the category of representations of the Grothendieck construction of X. Cibils and Marcos [CM06] regard $\mathrm{Gr}(X)$ as a version of orbit category. Thus we can identify the category of right G-invariant functors with the category of representations of the Cibils-Marcos orbit category. This is observed by Asashiba and stated as Theorem 6.2 in [Asaa]. $\quad\square$

A.3   The Grothendieck Construction over Product Type Monoidal Categories

In this and next sections, we specialize the constructions in this paper to the case V is a product type symmetric monoidal category. In this case, we do not need to use comodules.

Throughout this section, V is a product type symmetric monoidal category. Let us modify the Grothendieck construction as a 2-functor

$$\mathrm{Gr} : \overleftarrow{\mathbf{Oplax}}(I, V\text{-}\mathbf{Categories}) \longrightarrow V\text{-}\overleftarrow{\mathbf{Categories}} \downarrow I \otimes 1.$$

Definition A.16. For an oplax functor

$$X : I \longrightarrow V\text{-}\mathbf{Categories},$$

define

$$p_X : \mathrm{Gr}(X) \longrightarrow I \otimes 1$$

by

$$p_X(x, i) = i$$

for objects and

$$p_X : \mathrm{Gr}(X)((x,i),(y,j)) = \bigoplus_{u:i \to j} X(j)(X(u)(x), y) \longrightarrow (I \otimes 1)(i,j)$$

is defined by

$$X(j)(X(u)(x), y) \longrightarrow 1 \stackrel{\eta}{\longrightarrow} \{u\} \otimes 1 \hookrightarrow \mathrm{Mor}_I(i,j) \otimes 1 = (I \otimes 1)(i,j)$$

on each component. Recall that we assume that 1 is a terminal object in V. The morphism η is one of structure morphisms of the lax monoidal functor $(-) \otimes 1$.

Definition A.17. For a morphism of oplax functors

$$(F, \varphi) : X \longrightarrow Y,$$

define

$$\mathrm{Gr}(F, \varphi) : \mathrm{Gr}(X) \longrightarrow \mathrm{Gr}(Y)$$

by Definition 3.4.

Define a natural transformation

$$\mathrm{Gr}(\varphi) : p_Y \circ \mathrm{Gr}(F, \varphi) \Longrightarrow p_X$$

by the identity

$$\mathrm{Gr}(\varphi)(i) : p_Y \circ \mathrm{Gr}(F, \varphi)(x, i) = i \xrightarrow{1_i} i = p_X(x, i).$$

Lemma A.18. *The pair* $(\mathrm{Gr}(F, \varphi), \mathrm{Gr}(\varphi))$ *is a 1-morphism in* $\boldsymbol{V}\text{-}\overleftarrow{\mathbf{Categories}} \downarrow I \otimes 1$.

Proof. Obvious from the definition. $\qquad\square$

Definition A.19. For a 2-morphism

$$\theta : (F, \varphi) \Longrightarrow (G, \psi)$$

in $\overleftarrow{\mathbf{Oplax}}(I, \boldsymbol{V}\text{-}\mathbf{Categories})$, define

$$\mathrm{Gr}(\theta) : (\mathrm{Gr}(F, \varphi), \mathrm{Gr}(\varphi)) \Longrightarrow (\mathrm{Gr}(G, \psi), \mathrm{Gr}(\psi))$$

in $\boldsymbol{V}\text{-}\overleftarrow{\mathbf{Categories}} \downarrow I$ by

$$\mathrm{Gr}(\theta)(x, i) = (\theta(i)(x), 1_i) : \mathrm{Gr}(F, \varphi)(x, i) = (F(i)(x), i) \longrightarrow (G(i)(x), i) = \mathrm{Gr}(G, \psi)(x, i).$$

This makes the following diagram of natural transformations commutative

since all morphisms in the diagram are identity. And we obtain a 2-morphism

$$\mathrm{Gr}(\theta) : (\mathrm{Gr}(F, \varphi), \mathrm{Gr}(\varphi)) \Longrightarrow (\mathrm{Gr}(G, \psi), \mathrm{Gr}(\psi)).$$

Lemma 3.23 can be translated as follows.

Lemma A.20. *The above constructions define a functor*

$$\mathrm{Gr} : \overleftarrow{\mathbf{Oplax}}(I, \boldsymbol{V}\text{-}\mathbf{Categories})(X, X') \longrightarrow (\boldsymbol{V}\text{-}\overleftarrow{\mathbf{Categories}} \downarrow I \otimes 1)(\mathrm{Gr}(X), \mathrm{Gr}(X')).$$

And Proposition 3.24 becomes the following form.

Proposition A.21. *The Grothendieck construction defines a functor of* $\mathbf{Categories}$*-enriched categories, i.e. a 2-functor*

$$\mathrm{Gr} : \overleftarrow{\mathbf{Oplax}}(I, \boldsymbol{V}\text{-}\mathbf{Categories}) \longrightarrow \boldsymbol{V}\text{-}\overleftarrow{\mathbf{Categories}} \downarrow I \otimes 1.$$

A.4 Comma Categories for Enriched Categories

When $\boldsymbol{V}$ is of product type, we can regard the Grothendieck construction as

$$\mathrm{Gr} : \overleftarrow{\mathbf{Oplax}}(I, \boldsymbol{V}\text{-}\mathbf{Categories}) \longrightarrow \boldsymbol{V}\text{-}\overleftarrow{\mathbf{Categories}} \downarrow (I \otimes 1).$$

In order to define a right adjoint to the Grothendieck construction of this form, we need comma categories for enriched categories. There seems to be a general theory of comma objects in enriched categories. (See, for example, [Woo78].) We choose, however, a more down-to-earth approach, which is enough for our purposes.

Definition A.22. Let V be a monoidal category closed under pullbacks and

$$p : E \longrightarrow B$$

be a V-functor. For an object $x \in B_0$, define a V-category $p \downarrow x$ as follows. Objects are given by

$$(p \downarrow x)_0 = \coprod_{e \in E_0} \{e\} \times \mathrm{Mor}_{\underline{B}}(p(e), x).$$

For a pair $(e, f), (e', f') \in (p \downarrow x)_0$, define an object $(p \downarrow x)((e, f), (e', f'))$ in V by the following pullback diagram in V

The composition

$$(p \downarrow x)((e', f'), (e'', f'')) \otimes (p \downarrow x)((e, f), (e', f')) \longrightarrow (p \downarrow x)((e, f), (e'', f''))$$

is defined by the commutativity of the following diagram

The identity morphisms are defined by

If we restrict our attention to 1-morphisms in V-$\overleftarrow{\mathbf{Categories}}_I$, i.e. degree preserving morphisms, we have an alternative construction for the smash product.

Definition A.23. Let I be a small category and

$$p : X \longrightarrow I \otimes 1$$

a V-functor. Define a functor

$$\overleftarrow{\Gamma}(p) : I \longrightarrow V\text{-}\mathbf{Categories}$$

as follows. For an object $i \in I_0$,

$$\overleftarrow{\Gamma}(p)(i) = p \downarrow i.$$

For a morphism $u : i \to j$ in I, define a V-functor

$$\overleftarrow{\Gamma}(u) = p \downarrow u : p \downarrow i \longrightarrow p \downarrow j$$

as follows. For an object (x, v) in $p \downarrow i$, define

$$(p \downarrow u)(x, f) = (x, u \circ f).$$

For morphisms,

$$p \downarrow u : (p \downarrow i)((x, f), (x', f')) \longrightarrow (p \downarrow j)((x, u \circ f), (x, u \circ f'))$$

is defined by the following commutative diagram

Remark A.24. When p is a graded category defined by a coproduct decompositions, we can define $\overleftarrow{\Gamma}$ without assuming that V is closed under pullbacks.

For an object $i \in I_0$, the definition of $\overleftarrow{\Gamma}(p)(i)_0$ is the same. For $(x, f), (x', f') \in \overleftarrow{\Gamma}(i)_0$, define an object $\overleftarrow{\Gamma}(p)(i)((x, f), (x', f'))$ in V by

$$\overleftarrow{\Gamma}(p)(i)((x, f), (x', f')) = \bigoplus_{u : f' \circ u = f} X^u(x, x').$$

The composition

$$\overleftarrow{\Gamma}(p)(i)((x', f'), (x'', f'')) \otimes \overleftarrow{\Gamma}(p)(i)((x, f), (x', f')) \longrightarrow \overleftarrow{\Gamma}(p)(i)((x, f), (x'', f''))$$

is given by the composition

$$X^{u'}(x', x'') \otimes X^u(x, x') \longrightarrow X^{u' \circ u}(x, x'')$$

on each component.

For a morphism $u : i \to j$ in I, define a V-functor

$$\overleftarrow{\Gamma}(p)(u) : \overleftarrow{\Gamma}(p)(i) \longrightarrow \overleftarrow{\Gamma}(p)(j)$$

as follows. For an object (x, f) in $\overleftarrow{\Gamma}(p)(i)$, define

$$\overleftarrow{\Gamma}(p)(u)(x, f) = (x, u \circ f).$$

For morphisms, define

$$\overleftarrow{\Gamma}(p)(u) : \overleftarrow{\Gamma}(p)(i)((x, f), (x', f')) \longrightarrow \overleftarrow{\Gamma}(p)(j)((x, u \circ f), (x, u \circ f'))$$

by the identity

$$X^v(x, x') \longrightarrow X^v(x, x')$$

on each component.

It is straightforward to check that the above construction is compatible with the compositions and the identities.

Lemma A.25. *The above construction defines a functor*

$$\overleftarrow{\Gamma}(p) = p \downarrow (-) : I \longrightarrow \boldsymbol{V}\text{-}\mathbf{Categories}.$$

We would like to extend Γ to a 2-functor

$$\overleftarrow{\Gamma} : \boldsymbol{V}\text{-}\overleftarrow{\mathbf{Categories}} \downarrow (I \otimes 1) \longrightarrow \overleftarrow{\mathbf{Oplax}}(I, \boldsymbol{V}\text{-}\mathbf{Categories}).$$

Definition A.26. Let

$$
\begin{aligned}
p &: X \longrightarrow I \otimes 1 \\
p' &: X' \longrightarrow I \otimes 1
\end{aligned}
$$

be objects in $\boldsymbol{V}\text{-}\overleftarrow{\mathbf{Categories}} \downarrow (I \otimes 1)$. For a morphism

$$(F, \varphi) : p \longrightarrow p',$$

define a morphism of oplax functors

$$(\overleftarrow{\Gamma}(F, \varphi), \Gamma(\varphi)) : \overleftarrow{\Gamma}(p) \longrightarrow \overleftarrow{\Gamma}(p')$$

as follows. For $i \in I_0$, a $\boldsymbol{V}$-functor

$$\overleftarrow{\Gamma}(F, \varphi)(i) : \overleftarrow{\Gamma}(p)(i) = p \downarrow i \longrightarrow p' \downarrow i = \overleftarrow{\Gamma}(p')(i)$$

is defined on objects by

$$\overleftarrow{\Gamma}(F, \varphi)(i)(x, f) = (F(x), f \circ \varphi(x))$$

and on morphisms by the commutativity of the following diagram

For each $u : i \to j$ in I, define a $\boldsymbol{V}$-natural transformation

$$\overleftarrow{\Gamma}(\varphi)(u) : \overleftarrow{\Gamma}(p')(u) \circ \overleftarrow{\Gamma}(F, \varphi)(i) \Longrightarrow \overleftarrow{\Gamma}(F, \varphi)(j) \circ \overleftarrow{\Gamma}(p)(u)$$

to be the identity.

It is straightforward to check that the pair $(\overleftarrow{\Gamma}(F, \varphi), \overleftarrow{\Gamma}(\varphi))$ is a morphism of oplax functors. This correspondence extends as follows.

Lemma A.27. *The above construction Γ defines a functor*

$$\overleftarrow{\Gamma} : (\boldsymbol{V}\text{-}\overleftarrow{\mathbf{Categories}} \downarrow I \otimes 1)(p, p') \longrightarrow \overleftarrow{\mathbf{Oplax}}(I, \boldsymbol{V}\text{-}\mathbf{Categories})(\Gamma(p), \Gamma(p')).$$

Proof. This follows from Lemma 4.10. □

Proposition 4.11 specializes to the following.

Proposition A.28. $\overleftarrow{\Gamma}$ *defines a 2-functor*

$$\overleftarrow{\Gamma} : \boldsymbol{V}\text{-}\overleftarrow{\mathbf{Categories}} \downarrow (I \otimes 1) \longrightarrow \overleftarrow{\mathbf{Oplax}}(I, \boldsymbol{V}\text{-}\mathbf{Categories}).$$

We have seen in Example 2.61 that $\boldsymbol{V}\text{-}\overleftarrow{\mathbf{Categories}}_I$ can be identified with $\boldsymbol{V}\text{-}\overleftarrow{\mathbf{Categories}} \downarrow (I \otimes 1)$. By translating $\overleftarrow{\Gamma}$ for comma categories into $\overleftarrow{\Gamma}$ for comodules, we obtain the following corollary to Theorem 5.1.

Theorem A.29. *Let* $\boldsymbol{V}$ *be a product type symmetric monoidal category. Then the 2-functor*

$$\overleftarrow{\Gamma} : \boldsymbol{V}\text{-}\overleftarrow{\mathbf{Categories}} \downarrow (I \otimes 1) \longrightarrow \overleftarrow{\mathbf{Oplax}}(I, \boldsymbol{V}\text{-}\mathbf{Categories})$$

is right adjoint to

$$\mathrm{Gr} : \overleftarrow{\mathbf{Oplax}}(I, \boldsymbol{V}\text{-}\mathbf{Categories}) \longrightarrow \boldsymbol{V}\text{-}\overleftarrow{\mathbf{Categories}} \downarrow (I \otimes 1).$$

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