Operads, Algebras and Modules in Model Categories and Motives

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1. Introduction

This thesis consists of two parts. In the first part we develop a general machinery to study operads, algebras and modules in symmetric monoidal model categories. In particular we obtain a well behaved theory of E_∞-algebras and modules over them, where E_∞-algebras are an appropriate substitute of commutative algebras in model categories. This theory gives a derived functor formalism for commutative algebras and modules over them in any nice geometric situation, for example for categories of sheaves on manifolds or, as we show in the second part of the thesis, for triangulated categories of motivic sheaves on schemes. As our main application of this theory we construct a so called limit motive functor, which is a motivic analogue and generalization of the limit Hodge structures considered by Schmid, Steenbrink et.al. and can also be viewed as a refinement of the vanishing cycle functor. As a corollary one can obtain motivic tangential base point functors for triangulated categories of Tate motives on rational curves. This answers a question of Deligne asked in [Del2].

We start with a brief historical review. Recently important new applications of model categories appeared, for example in the work of Voevodsky and others on the $\mathbb{A}^1$-local stable homotopy category of schemes. But also for certain questions in homological algebra model categories became quite useful, for example when one deals with unbounded complexes in abelian categories. In topology, mainly in the stable homotopy category, one is used to deal with objects having additional structures, for example modules over ring spectra. The work of [EKMM] made it possible to handle commutativity appropriately, namely the special properties of the linear isometries operad lead to a strictly associative and commutative tensor product for modules over E_∞-ring spectra. As a consequence many constructions in topology became more elegant or even possible at all (see [EKM]). Moreover the category of E_∞-algebras could be examined with homotopical methods because this category carries a model structure. In [KM] a parallel theory in algebra was developed (see [May]).

Parallel to the achievements in topology the abstract model category theory was further developed (see [Hov1] for a good introduction to model categories, see also [DHK]). Categories of algebras and of modules over algebras in monoidal model categories have been considered ([SS], [Hov2]). Also localization techniques for model categories have become important, because they yield many new useful model structures (for example the categories of spectra of [Hov3]). The most general statement for the existence of localizations is given in [Hir].

In all these situations it is as in topology desirable to be able to work in the commutative world, i.e. with commutative algebras and modules over them. Since a reasonable model structure for commutative algebras in a given symmetric monoidal model category is quite unlikely to exist the need for a theory of E_∞-algebras arises. Also for the category of modules over an E_∞-algebra a symmetric monoidal structure is important. One of the aims of this paper is to give adequate answers to these requirements.

E_∞-algebras are algebras over particular operads. Many other interesting operads appeared in various areas of mathematics, starting from the early application for recognition principles of iterated loop spaces (which was the reason to introduce operads), later for example to handle homotopy Lie algebras which are necessary for general deformation theory, the operads appearing in two dimensional conformal quantum field theory or the operad of moduli spaces of stable curves in algebraic geometry. In many cases the necessary operads are only well defined up to quasi isomorphism or another sort of weak equivalence (as is the case for example for E_∞-algebras), therefore a good homotopy theory of operads is desireable. A related question is then the invariance (up to homotopy) of the categories of algebras over weakly equivalent operads and also of modules over weakly equivalent algebras. We will also give adequate solutions to these questions. This part of the paper was motivated by and owes many ideas to [Hin1] and [Hin2].

So in the first half of Part I we develop the theory of operads, algebras and modules in the general situation of a cofibrantly generated symmetric monoidal model category satisfying some technical conditions which are usually fulfilled. Our first aim is to provide these categories with model structures. It turns out that in general we cannot quite get model structures in the case of operads and algebras, but a slightly weaker structure which we call a J-semi model structure. A version of this structure already appeared in [Hov2]. To the knowledge of the author no restrictions arise in the applications when using J-semi model structures instead of model structures. The J-semi model structures are necessary since the free operad and algebra functors are not linear (even not polynomial). These structures appear in two versions, an absolute one and a version relative to a base category.

We have two possible conditions for an operad or an algebra to give model structures on the associated categories of algebras or modules, the first one is being cofibrant (which is in some sense the best condition), and the second one being cofibrant in an underlying model category.

In the second half of Part I we demonstrate that the theory of $\mathbb{S}$-modules of [EKMM] and [KM] can also be developed in our context if the given symmetric monoidal model category $\mathcal{C}$ either receives a symmetric monoidal left Quillen functor from **SSet** (i.e. is simplicial) or from $Comp_{\geq 0}(\mathbf{Ab})$. The linear isometries operad $\mathcal{L}$ gives via one of these functors an E_∞-operad in $\mathcal{C}$ with the same special properties responsible for the good behavior of the theories of [EKMM] and [KM]. These theories do not yield honest units for the symmetric monoidal category of modules over $\mathcal{L}$-algebras, and we have to deal with the same problem. In the topological theory of [EKMM] it is possible to get rid of this problem, in the algebraic or simplicial one it is not. Nevertheless it turns out that the properties the unit satisfies are good enough to deal with operads, algebras and modules in the category of modules over a cofibrant $\mathcal{L}$-algebra. This seems to be a little counterproductive, but we need this to prove quite strong results on the behavior of algebras and modules with respect to base change and projection morphisms. These results are even new for the cases treated in [EKMM] and [KM].

In a remark we show that one can always define a product on the homotopy category of modules over an $\mathcal{O}$-algebra for an arbitrary E_∞-operad $\mathcal{O}$ without relying on the special properties of the linear isometries operad, but we do not construct associativity and commutativity isomorphisms in this situation! In the case when $\mathbb{S}$-modules are available this product structure is naturally isomorphic to the one defined using $\mathbb{S}$-modules.

Certainly this general theory will have many applications, for example the ones we give in the second part of this thesis or to develop the theory of schemes in symmetric monoidal cofibrantly generated model categories (see [TV]).

Part II of the thesis is concerned with the applications of the general theory of Part I to $\mathbb{A}^1$-local homotopy categories of schemes and of sheaves with transfers introduced by Vladimir Voevodsky. Our main application will be the construction of what we call *limit motives*. This construction has predecessors in the world of Hodge structures, the so called limit Hodge structures, and for special cases in other realization categories, for example the l-adic one, as introduced by Deligne in [Del]. He considers sheaves on a pointed curve and defines a functor which associates to such a sheaf another sheaf on the pointed tangent space at the point missing on the curve. This functor computes the local monodromy around the point. Deligne also describes a more general geometric situation of a smooth variety and normal crossing divisors on it for which he conjectures the existence of a local monodromy functor which associates to a sheaf on the open variety a sheaf on the product of the pointed normal bundles of the divisor over the intersection of the divisors. We will define such a functor for this situation over a general base for some class of triangulated categories of motivic sheaves. We will compare this construction with the classical ones in a forthcoming paper.

The first section of Part II briefly sketches in a topological context the way we construct the local monodromy functor. The construction makes use of a general principle which enables one to identify a certain subcategory of (some sort of) sheaves on a scheme X over a base S consisting of generalized unipotent objects relative to S with the category of modules over the relative cohomology algebra of X on S. The abstract version of this principle is given in the second section.

We then introduce in a uniform way the $\mathbb{A}^1$-local model categories we consider. We use cd-model structures throughout, which are finitely generated model structures using the special properties of the Nisnevich or cdh-topology. There are two types of these model categories. The first one is based on simplicial sheaves on some site of schemes. The corresponding model categories will give $\mathbb{A}^1$-local homotopy categories of schemes, for example the stable motivic homotopy category. The second sort of model categories involve complexes of sheaves with transfers. They give triangulated categories of motives or motivic sheaves. We compare these categories over a field of characteristic 0 to the categories constructed in [Vo3] and give some properties of the behaviour of their $\mathbf{T}$-stabilizations.

The construction of the local monodromy functor producing limit motives works in enrichments of the stable motivic homotopy categories (i.e. in modules over algebras in there). We restricted to this case because for the triangulated categories of motives we do not know the gluing exact triangles. Working with modules over the motivic Eilenberg Mac Lane spectrum gives a substitute for the triangulated categories of motives in some interesting cases.

Finally we sketch the proofs of the statements about the behaviour of the local monodromy functor with respect to composition.

I would like to thank Bertrand Toen for many useful discussions on the subject. My special thanks are to Prof. Dr. G. Harder who supported my work and drew my attention to many interesting questions.

Part I

2. Preliminaries

We first review some standard arguments from model category theory which we will use throughout the paper (see for the first part e.g. the introduction to [Hov2]).

Let $\mathcal{C}$ be a cocomplete category. For a pushout diagram in $\mathcal{C}$

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \uparrow{\scriptstyle\varphi} & & \uparrow \\ K & \xrightarrow{g} & L \end{array}$$

we call f the pushout of g by φ, and we call B the pushout of A by g with attaching map φ. If we say that B is a pushout of A by g and g is an object of $\mathcal{C}$ rather than a map then we mean that $B = g$ and A need not be defined in this case (we need this convention to handle pathological cases in the statements describing pushouts of operads and algebras over operads in $\mathcal{C}$ correctly).

Let I be a set of maps in $\mathcal{C}$. Let I-inj denote the class of maps in $\mathcal{C}$ which have the right lifting property with respect to I, I-cof the class of maps in $\mathcal{C}$ which have the left lifting property with respect to I-inj and I-cell the class of maps which are transfinite compositions of pushouts of maps from I. Note that I-cell $\subset$ I-cof and that I-inj and I-cof are closed under retracts.

Let us suppose now that the domains of the maps in I are small relative to I-cell. Then by the small object argument there exists a functorial factorization of every map in $\mathcal{C}$ into a map from I-cell followed by a map from I-inj. Moreover every map in I-cof is a retract of a map in I-cell such that the retract induces an isomorphism on the domains of the two maps. Also the domains of the maps in I are small relative to I-cof.

Now let $\mathcal{C}$ be equipped with a symmetric monoidal structure such that the product $\otimes: \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves colimits (e.g. if the monoidal structure is closed). We denote the pushout product of maps $f: A \to B$ and $g: C \to D$,

$$A \otimes D \sqcup_{A \otimes C} B \otimes C \to B \otimes D\ ,$$

by $f \Box g$.

For ordinals ν and λ we use the convention that the well-ordering on the product ordinal $\nu \times \lambda$ is such that the elements in ν have higher significance. We will need the

Lemma 2.1. *Let $f: A_0 \to A_\mu = \mathrm{colim}_{i<\mu} A_i$ and $g: B_0 \to B_\lambda = \mathrm{colim}_{i<\lambda} B_i$ be a μ- and a λ-sequence such that the transition maps $A_i \to A_{i+1}$ and $B_i \to B_{i+1}$ are pushouts by maps $\varphi_i: K_i \to K_{i+1}$ and $\psi_i: L_i \to L_{i+1}$. Then the pushout product $f \Box g$ is a $(\mu \times \lambda)$-sequence $M_0 \to M_{\mu\times\lambda} = \mathrm{colim}_{i<\mu\times\lambda} M_i$ such that the transition maps $M_{(i,j)} \to M_{(i,j+1)}$ are pushouts by the maps $\varphi_i \Box \psi_j$.*

Proof. For any $(i, j) \in \mu \times \lambda$ define $M_{(i,j)}$ to be the colimit of the diagram

$$\begin{array}{ccccccccc} A_\mu \otimes B_0 & & & & A_{i+1} \otimes B_j & & & & A_i \otimes B_\lambda \ . \\ & \nwarrow & & \nearrow & & \nwarrow & & \nearrow & \\ & & A_{i+1} \otimes B_0 & & & & A_i \otimes B_j & & \end{array}$$

Clearly $M_{(0,0)} = A_\mu \otimes B_0 \sqcup_{A_0 \otimes B_0} A_0 \otimes B_\lambda$ is the domain of $f \Box g$. We have obvious transition maps $M_{(i,j)} \to M_{(i,j)+1}$ induced by the maps $A_i \to A_{i+1}$ and $A_{i+1} \to A_{i+2}$. For fixed $i \in \mu$ both $\operatorname{colim}_{j<\lambda} M_{(i,j)}$ and $M_{(i+1,0)}$ are canonically isomorphic to the pushout of the diagram $A_\mu \otimes B_0 \leftarrow A_{i+1} \otimes B_0 \to A_{i+1} \otimes B_\lambda$. From this and the fact that $\otimes$ commutes with colimits it follows that the assignment $(i, j) \mapsto M_{(i,j)}$ is a $(\mu \times \lambda)$-sequence. It also follows that the limit of this sequence is $A_\mu \otimes B_\lambda$ and the map from $M_{(0,0)}$ to this limit is $f \Box g$. We have to show that a transition map $M_{(i,j)} \to M_{(i,j+1)}$ is a pushout by $\varphi_i \Box \psi_j$. To do this one shows that in the obvious diagram

$$\begin{array}{ccc} K_{i+1} \otimes L_j \sqcup_{K_i \otimes L_j} K_i \otimes L_{j+1} & \longrightarrow & K_{i+1} \otimes L_{j+1} \\ \downarrow & & \downarrow \\ A_{i+1} \otimes B_j \sqcup_{A_i \otimes B_j} A_i \otimes B_{j+1} & \longrightarrow & A_{i+1} \otimes B_{j+1} \\ \downarrow & & \downarrow \\ M_{(i,j)} & \longrightarrow & M_{(i,j+1)} \end{array}$$

the upper and the lower square are pushout squares. $\Box$

The pushout product is associative. For maps $f_i : A_i \to B_i$, $i = 1, \ldots, n$, in $\mathcal{C}$ giving a map from the domain of $g := \Box_{i=1}^n f_i$ to an object $X \in \mathcal{C}$ is the same as to give maps φ_j from the

$$S_j := (\bigotimes_{i=1}^{j-1} B_i) \otimes A_j \otimes \bigotimes_{i=j+1}^{n} B_i$$

to X for $j = 1, \ldots, n$ such that φ_j and $\varphi_{j'}$ $(j' > j)$ coincide on

$$I_{j,j'} := (\bigotimes_{i=1}^{j-1} B_i) \otimes A_j \otimes (\bigotimes_{i=j+1}^{j'-1} B_i) \otimes A_{j'} \otimes \bigotimes_{i=j'+1}^{n} B_i$$

after the obvious compositions. We call the S_j the *summands* of the domain of g and the $I_{j,j'}$ the *intersections* of these summands. Sometimes some of the f_i will coincide. Then there is an action of a product of symmetric groups on g, and the quotient of a summand with respect to the induced action of the stabilizer of this summand will also be called a summand (and similarly for the intersections).

For the rest of the paper we fix a cofibrantly generated symmetric monoidal model category $\mathcal{C}$ with generating cofibrations I and generating trivial cofibrations J. For simplicity we assume that the domains of I and J are small relative to the whole category $\mathcal{C}$. The interested reader may weaken this hypothesis appropriately in the statements below.

For a monad $\mathbb{T}$ in $\mathcal{C}$ we write $\mathcal{C}[\mathbb{T}]$ for the category of $\mathbb{T}$-algebras in $\mathcal{C}$. The following theorem summarizes the general method to equip categories of objects in $\mathcal{C}$ with "additional structure" with model structures (e.g. as in [Hov2, Theorem 2.1]).

Theorem 2.2. *Let $\mathbb{T}$ be a monad in $\mathcal{C}$, assume that $\mathcal{C}[\mathbb{T}]$ has coequalizers and suppose that every map in $\mathbb{T}J$-cell, where the cell complex is built in $\mathbb{C}[\mathbb{T}]$, is a weak equivalence in $\mathcal{C}$. Then there is a cofibrantly generated model structure on $\mathcal{C}[\mathbb{T}]$, where a map is a weak equivalence or fibration if and only if it is a weak equivalence or fibration in $\mathcal{C}$.*

Proof. We apply [Hov1, Theorem 2.1.19] with generating cofibrations $\mathbb{T}I$, generating trivial cofibrations $\mathbb{T}J$ and weak equivalences the maps which are weak equivalences in $\mathcal{C}$.

By [McL, VI.2, Ex 2], $\mathcal{C}[\mathbb{T}]$ is complete and by [BW, 9.3 Theorem 2] cocomplete. Property 1 of [Hov1, Theorem 2.1.19] is clear, properties 2 and 3 follow by adjunction from our smallness assumptions on the domains of I and J. Since each element of J is in I-cof, hence a retract of a map in I-cell, each element of $\mathbb{T}J$ is in $\mathbb{T}I$-cof, hence together with our assumption we see that property 4 is fulfilled. By adjunction $\mathbb{T}I$-inj (resp. $\mathbb{T}J$-inj) is the class of maps in $\mathcal{C}[\mathbb{T}]$ which are trivial fibrations (resp. fibrations) in $\mathcal{C}$. Hence property 5 and the second alternative of 6 are fulfilled. □

In most of the cases we are interested in the hypothesis of this theorem that every map in $\mathbb{T}J$-cell is a weak equivalence won't be fulfilled. The reason is that we are considering monads which are not linear. The method to circumvent this problem was found by Hovey in [Hov2, Theorem 3.3]. He considers categories which are not quite model categories. We will call them semi model categories.

Definition 2.3. *(I) A* J-semi model category *over $\mathcal{C}$ is a left adjunction $F: \mathcal{C} \to \mathcal{D}$ and subcategories of weak equivalences, fibrations and cofibrations in $\mathcal{D}$ such that the following axioms are fulfilled:*

1. *The adjoint of F preserves fibrations and trivial fibrations.*
2. *$\mathcal{D}$ is bicomplete and the two out of three and retract axioms hold in $\mathcal{D}$.*
3. *Cofibrations in $\mathcal{D}$ have the left lifting property with respect to trivial fibrations, and trivial cofibrations whose domain becomes cofibrant in $\mathcal{C}$ have the left lifting property with respect to fibrations.*
4. *Every map in $\mathcal{D}$ can be functorially factored into a cofibration followed by a trivial fibration, and every map in $\mathcal{D}$ whose domain becomes cofibrant in $\mathcal{C}$ can be functorially factored into a trivial cofibration followed by a fibration.*
5. *Cofibrations in $\mathcal{D}$ whose domain becomes cofibrant in $\mathcal{C}$ become cofibrations in $\mathcal{C}$, and the initial object in $\mathcal{D}$ is mapped to a cofibrant object in $\mathcal{C}$.*
6. *Fibrations and trivial fibrations are closed under pullback.*

We say that $\mathcal{D}$ is cofibrantly generated *if there are sets of morphisms I and J in $\mathcal{D}$ such that I*-inj *is the class of trivial fibrations and J*-inj *the class of fibrations in $\mathcal{D}$ and if the domains of I are small relative to I*-cell *and the domains of J are small relative to maps from J*-cell *whose domain becomes cofibrant in $\mathcal{C}$.*

$\mathcal{D}$ is called left proper (relative to $\mathcal{C}$) *if pushouts by cofibrations preserve weak equivalences whose domain and codomain become cofibrant in $\mathcal{C}$ (hence all objects*

which appear become cofibrant in $\mathcal{C}$). $\mathcal{D}$ is called right proper *if pullbacks by fibrations preserve weak equivalences.*

(II) A category $\mathcal{D}$ is called a J-semi model category if conditions (2) to (4) and (6) of Definition 2.3 are fulfilled where the condition of becoming cofibrant in $\mathcal{C}$ is replaced by the condition of being cofibrant.

The same is valid for the definition of being cofibrantly generated and of being right proper.

(Note that the only reasonable property to require in a definition for a J-semi model category to be left proper, namely that weak equivalences between cofibrant objects are preserved by pushouts by cofibrations, is automatically fulfilled as is explained below when we consider homotopy pushouts.)

Alternative: One can weaken the definition of a J-semi model category (resp. of a J-semi model category over $\mathcal{C}$) slightly by only requiring that a factorization of a map in $\mathcal{D}$ into a cofibration followed by a trivial fibration should exist if the domain of this map is cofibrant (resp. becomes cofibrant in $\mathcal{D}$). We then include into the definition of cofibrant generation that the cofibrations are all of I-cof. Using this definition all statements from section 3 on remain true if one does not impose any further smallness assumptions on the domains of I and J as we did at the beginning. This follows in each of the cases from the fact that the domains of I and J are small relative to I-cof.

Of course a J-semi model category over $\mathcal{C}$ is a J-semi model category. There is also the notion of an I-semi (and also (I, J)-semi) model category (over $\mathcal{C}$), where the parts of properties 3 and 4 concerning cofibrations are restricted to maps whose domain is cofibrant (becomes cofibrant in $\mathcal{C}$).

We summarize the main properties of a J-semi model category $\mathcal{D}$ (relative to $\mathcal{C}$) (compare also [Hov2, p. 14]):

By the factorization property and the retract argument it follows that a map is a cofibration if and only if it has the left lifting property with respect to the trivial fibrations. Similarly a map is a trivial fibration if and only if it has the right lifting property with respect to the cofibrations. These two statements remain true under the alternative definition if $\mathcal{D}$ is cofibrantly generated.

A map in $\mathcal{D}$ whose domain is cofibrant (becomes cofibrant in $\mathcal{C}$) is a trivial cofibration if and only if it has the left lifting property with respect to the fibrations, and a map whose domain is cofibrant (becomes cofibrant in $\mathcal{C}$) is a fibration if and only if it has the right lifting property with respect to the trivial cofibrations whose domains are cofibrant (become cofibrant in $\mathcal{C}$).

Pushouts preserve cofibrations (also under the alternative definition if $\mathcal{D}$ is cofibrantly generated). Trivial cofibrations with cofibrant domain (whose domain becomes cofibrant in $\mathcal{C}$) are preserved under pushouts by maps with cofibrant codomain (whose codomain becomes cofibrant in $\mathcal{C}$).

In the relative case the functor F preserves cofibrations (also in the alternative definition if $\mathcal{D}$ is cofibrantly generated), and trivial cofibrations with cofibrant domain.

Ken Brown's Lemma ([Hov1, Lemma 1.1.12]) remains true, and its dual version has to be modified to the following statement: Let $\mathcal{D}$ be a J-semi model category (over $\mathcal{C}$) and $\mathcal{D}'$ be a category with a subcategory of weak equivalences which satisfies the two out of three property. Suppose $F: \mathcal{D} \to \mathcal{D}'$ is a functor which takes trivial fibrations between fibrant objects with cofibrant domain (whose domain becomes cofibrant in $\mathcal{C}$) to weak equivalences. Then F takes all weak equivalences between fibrant objects with cofibrant domain (whose domain becomes cofibrant in $\mathcal{C}$) to weak equivalences.

We define cylinder and path objects and the various versions of homotopy as in [Hov1, Definition 1.2.4]. Cylinder and path objects exist for cofibrant objects (for objects which become cofibrant in $\mathcal{C}$).

We give the J-semi version of [Hov1, Proposition 1.2.5]:

Proposition 2.4. *Let $\mathcal{D}$ be a J-semi model category (over $\mathcal{C}$) and let $f, g: B \to X$ be two maps in $\mathcal{D}$.*

(1) *If $f \overset{l}{\sim} g$ and $h: X \to Y$, then $hf \overset{l}{\sim} hg$. Dually, if $f \overset{r}{\sim} g$ and $h: A \to B$, then $fh \overset{r}{\sim} gh$.*
(2) *Let $h: A \to B$ and suppose A and B are cofibrant (become cofibrant in $\mathcal{C}$) and X is fibrant. Then $f \overset{l}{\sim} g$ implies $fh \overset{l}{\sim} gh$. Dually, let $h: X \to Y$. Suppse X and Y are cofibrant (become cofibrant in $\mathcal{C}$) and B is cofibrant. Then $f \overset{r}{\sim} g$ implies $hf \overset{r}{\sim} hg$.*
(3) *If B is cofibrant, then left homotopy is an equivalence relation on* $\mathrm{Hom}(B, X)$.
(4) *If B is cofibrant and X is cofibrant (becomes cofibrant in $\mathcal{C}$), then $f \overset{l}{\sim} g$ implies $f \overset{r}{\sim} g$. Dually, if X is fibrant and B is cofibrant (becomes cofibrant in $\mathcal{C}$), then $f \overset{r}{\sim} g$ implies $f \overset{l}{\sim} g$.*
(5) *If B is cofibrant and $h: X \to Y$ is a trivial fibration or weak equivalence between fibrant objects with X cofibrant (such that X becomes cofibrant in $\mathcal{C}$), then h induces an isomorphism*

$$\mathrm{Hom}(B, X)/\overset{l}{\sim} \xrightarrow{\cong} \mathrm{Hom}(B, Y)/\overset{l}{\sim} .$$

Dually, suppose X is fibrant and cofibrant (becomes cofibrant in $\mathcal{C}$) and $h: A \to B$ is a trivial cofibration with A cofibrant (such that A becomes cofibrant in $\mathcal{C}$) or a weak equivalence between cofibrant objects, then h induces an isomorphism

$$\mathrm{Hom}(B, X)/\overset{r}{\sim} \xrightarrow{\cong} \mathrm{Hom}(A, X)/\overset{r}{\sim} .$$

This Proposition is also true for the alternative definition of a J-semi model category (over $\mathcal{C}$). We changed the order between 4 and 5, because it is a priori not clear that right homotopy is an equivalence relation (under suitable condition), this follows only after comparison with the left homotopy relation.

As in [Hov1, Corollary 1.2.6 and 1.2.7] it follows that if B is cofibrant and X is fibrant and cofibrant (becomes cofibrant in $\mathcal{C}$), then left and right homotopy coincide and are equivalence relations on $\mathrm{Hom}(B, X)$ and the homotopy relation on $\mathcal{D}_{cf}$ is an equivalence relation and compatible with composition. The statement of [Hov1, Proposition 1.2.8] that a map in $\mathcal{D}_{cf}$ is a weak equivalence if and only if it is a homotopy equicalence is proved exactly in the same way. The same holds for the fact that $\mathrm{Ho}\,\mathcal{D}_{cf}$ is naturally isomorphic to $\mathcal{D}_{cf}/\sim$ ([Hov1, Corollary 1.2.9]). Finally the existence of the cofibrant and fibrant replacement functor RQ implies that the map $\mathrm{Ho}\,\mathcal{D}_{cf} \to \mathrm{Ho}\,\mathcal{D}$ is an equivalence.

Definition 2.5. *A functor $L : \mathcal{D} \to \mathcal{D}'$ between J-semi model categories is a* left Quillen functor *if it has a right adjoint and if the right adjoint preserves fibrations and trivial fibrations.*

Of course in the relative situation F is a left Quillen functor. We show that a left Quillen functor induces an adjunction between the homotopy categories (also when we use the alternative definition). L preserves (trivial) cofibrations between cofibrant objects, hence by Ken Brown's Lemma it preserves weak equivalences between cofibrant objects. This induces a functor $\mathrm{Ho}\,\mathcal{D} \to \mathrm{Ho}\,\mathcal{D}'$. By the dual version of Ken Brown's Lemma the adjoint of L preserves weak equivalences between fibrant and cofibrant objects which gives a functor $\mathrm{Ho}\,\mathcal{D}' \to \mathrm{Ho}\,\mathcal{D}$. One easily checks that L preserves cylinder objects on cofibrant objects and that the adjoint of L preserves path objects on fibrant objects. As in Lemma [Hov1, Lemma 1.3.10] it follows that on the derived functors between $\mathrm{Ho}\,\mathcal{D}$ and $\mathrm{Ho}\,\mathcal{D}'$ there is induced a natural derived adjunction.

Next we are going to consider Reedy model structures and homotopy function complexes. We have the analogue of [Hov1, Theorem 5.1.3]:

Proposition 2.6. *Let $\mathcal{D}$ be a J-semi model category and $\mathcal{B}$ be a direct category. Then the diagram category $\mathcal{D}^{\mathcal{B}}$ is a J-semi model category with objectwise weak equivalences and fibrations and where a map $A \to B$ is a cofibrations if and only if the maps $A_i \sqcup_{L_i A} L_i B \to B_i$ are cofibrations for all $i \in \mathcal{B}$.*

Proof. As in [Hov1, Proposition 5.1.4] one shows that cofibrations have the left lifting property with respect to trivial fibrations. Then it follows that if $A \to B$ is a map in $\mathcal{D}^{\mathcal{B}}$ with A cofibrant such that the maps $A_i \sqcup_{L_i A} L_i B \to B_i$ are (trivial) cofibrations then the map $\mathrm{colim} A \to \mathrm{colim} B$ is a (trivial) cofibration in $\mathcal{D}$. So a good trivial cofibration (definition as in the proof of [Hov1, Theorem 5.1.3]) with cofibrant domain is a trivial cofibration and trivial cofibrations with cofibrant domain have the left lifting property with respect to fibrations. We then can construct functorial factorizations into a good trivial cofibration followed by a fibration for maps with cofibrant domain as in the proof of [Hov1, Theorem 5.1.3]) and also the factorization into a cofibration followed by a trivial fibration (for the alternative definition for maps with cofibrant domain). It follows that a trivial cofibration with cofibrant domain is a good trivial cofibration. All other properties are immediate. □

Similarly but easier we have that for an inverse category $\mathcal{B}$ the diagram category $\mathcal{D}^{\mathcal{B}}$ is a J-semi model category.

We can combine both results as in [Hov1, Theorem 5.2.5] to get

Proposition 2.7. *Let $\mathcal{D}$ be a J-semi model category and $\mathcal{B}$ a Reedy category. Then $\mathcal{D}^{\mathcal{B}}$ is a J-semi model category where a map $f : A \to B$ is a weak equivalence if and only if it is objectwise a weak equivalence, a cofibration if and only if the maps $A_i \sqcup_{L_i A} L_i B \to B_i$ are cofibrations and a fibration if and only if the maps $A_i \to B_i \times_{M_i B} M_i A$ are fibrations.*

It is easily checked that cosimplicial and simplicial frames (see [Hov1, Definition 5.2.7]) exist on cofibrant objects. In the following we denote by $A^\bullet$ and $A_\bullet$ functorial cosimplicial and simplicial frames on cofibrant $A \in \mathcal{D}$. We are going to equip the category $\mathcal{D}_{cf}$ with a strict 2-category structure $\mathcal{D}_{cf}^{\leq 2}$ with underlying 1-category $\mathcal{D}_{cf}$ and with associated homotopy category $\operatorname{Ho}\mathcal{D}_{cf}$. Let $A, B \in \mathcal{D}_{cf}$. As in [Hov1, Proposition 5.4.7] there are weak equivalences

$$\operatorname{Hom}_{\mathcal{D}}(A^\bullet, B) \to \operatorname{diag}(\operatorname{Hom}_{\mathcal{D}}(A^\bullet, B_\bullet)) \leftarrow \operatorname{Hom}_{\mathcal{D}}(A, B_\bullet)$$

in **SSet** which are isomorphisms in degree 0, and we define the morphism category $\operatorname{Hom}_{\mathcal{D}_{cf}^{\leq 2}}(A, B)$ to be the groupoid associated to one of these simplicial sets. By the groupoid associated to a $K \in \mathbf{SSet}$ we mean the groupoid with set of objects $K[0]$ and set of morphisms $\operatorname{Hom}(x, y)$ for $x, y \in K[0]$ the homotopy classes of paths from x to y in the topological realization of K. We have to give composition functors

$$\operatorname{Hom}_{\mathcal{D}_{cf}^{\leq 2}}(A, B) \times \operatorname{Hom}_{\mathcal{D}_{cf}^{\leq 2}}(B, C) \to Hom_{\mathcal{D}_{cf}^{\leq 2}}(A, C) \ .$$

These are the normal composition on objects and are induced on the morphisms by the map of simplicial sets

$$\operatorname{Hom}_{\mathcal{D}}(A^\bullet, B) \times \operatorname{Hom}_{\mathcal{D}}(B, C_\bullet) \to \operatorname{diag}(\operatorname{Hom}_{\mathcal{D}}(A^\bullet, C_\bullet)) \ .$$

In the following we write $\underset{0}{\circ}$ for the composition of 2-morphisms over objects and $\underset{1}{\circ}$ for the composition of 2-morphisms over 1-morphisms. We claim that for $A, B, C \in \mathcal{D}_{cf}$, morphisms $f, g : \ A \to B$, $f', g' : \ B \to C$ and 2-morphisms $\varphi : \ f \to g$, $\psi : \ f' \to g'$ we have

$$\psi \underset{0}{\circ} \varphi = (\operatorname{Id}_{f'} \underset{0}{\circ} \varphi) \underset{1}{\circ} (\psi \underset{0}{\circ} \operatorname{Id}_g) = (\psi \underset{0}{\circ} \operatorname{Id}_f) \underset{1}{\circ} (\operatorname{Id}_{g'} \underset{0}{\circ} \varphi) \ .$$

This follows from the corresponding equation of homotopy classes of paths in $\operatorname{Hom}_{\mathcal{D}}(A^\bullet, B) \times \operatorname{Hom}_{\mathcal{D}}(B, C_\bullet)$. Moreover for a 1-morphism $f'' : \ C \to D$ we have $(\operatorname{Id}_{f''} \underset{0}{\circ} \psi) \underset{0}{\circ} \operatorname{Id}_f = \operatorname{Id}_{f''} \underset{0}{\circ} (\psi \underset{0}{\circ} \operatorname{Id}_f)$, and the assignments $\operatorname{Hom}_{\mathcal{D}_{cf}^{\leq 2}}(B, C) \to \operatorname{Hom}_{\mathcal{D}_{cf}^{\leq 2}}(B, D)$, $a \mapsto \operatorname{Id}_{f''} \underset{0}{\circ} a$, and $\operatorname{Hom}_{\mathcal{D}_{cf}^{\leq 2}}(B, C) \to \operatorname{Hom}_{\mathcal{D}_{cf}^{\leq 2}}(A, C)$, $a \mapsto a \underset{0}{\circ} \operatorname{Id}_f$, are functors. From these three properties it follows that $\underset{0}{\circ}$ is associative and that $\underset{0}{\circ}$ and $\underset{1}{\circ}$ are compatible. Hence $\mathcal{D}_{cf}^{\leq 2}$ is a strict 2-category. We set $\operatorname{Ho}^{\leq 2}\mathcal{D} := \mathcal{D}_{cf}^{\leq 2}$. One can show that this 2-category is weakly equivalent to the 2-truncation of the 1-Segal category (see [Hi-Si]) associated to $\mathcal{D}$.

Let $\llcorner$ be the category whose diagrams (i.e. functors into another category) are the “lower left triangles”, and $\square$ the category whose diagrams are the commutative squares like the square at the beginning of this section. There is an obvious inclusion functor $\llcorner \to \square$. For a category $\mathcal{D}$ denote by $\mathcal{D}^{\llcorner}$ (resp. $\mathcal{D}^{\square}$) the category of $\llcorner$-diagrams (resp. of $\square$-diagrams) in $\mathcal{D}$. There is a restriction functor $r : \ \mathcal{D}^{\square} \to \mathcal{D}^{\llcorner}$.

Let $\mathcal{D}$ be a J-semi model category. Then there is a canonical way to define a *homotopy pushout functor*

$$\sqcup : \ (\operatorname{Ho}\mathcal{D})^{\llcorner} \to (\operatorname{Ho}\mathcal{D})^{\square}$$

which sends a triangle
$$\begin{array}{ccc} B & & \\ \uparrow & & \\ A & \longrightarrow & C \end{array}$$
to the square
$$\begin{array}{ccc} B & \longrightarrow & B \sqcup_A C \\ \uparrow & & \uparrow \\ A & \longrightarrow & C \end{array}$$
, together with a natural isomorphism from $r \circ \sqcup$ to the identity. This is done by lifting a triangle to a triangle in $\mathcal{D}$ where all objects are cofibrant and at least one map is a cofibration. Then by the cube lemma ([Hov1, Lemma 5.2.6]), which is also valid for J-semi model categories, the pushout does not depend on the choices and indeed yields a well-defined square in $\operatorname{Ho}\mathcal{D}$. We call a square in $\operatorname{Ho}\mathcal{D}$ a *homotopy pushout square* if it is in the essential image of the functor $\sqcup$. A *homotopy pushout square* in $\mathcal{D}$ is defined to be any commutative square weakly equivalent to a pushout square

$$\begin{array}{ccc} B & \longrightarrow & D \\ {\scriptstyle f}\uparrow & & \uparrow \\ A & \xrightarrow{g} & C \end{array}$$

where all objects are cofibrant and f or g is a cofibration.

Taking A to be an initial object in $\operatorname{Ho}\mathcal{D}$ (i.e. the image of an initial object in $\mathcal{D}$) the product $\sqcup_A$ gives the categorical coproduct on $\operatorname{Ho}\mathcal{D}$. For general A the homotopy pushout need not be a categorical pushout in $\operatorname{Ho}\mathcal{D}$.

A commutative square

$$\begin{array}{ccc} B & \xrightarrow{g'} & D \\ {\scriptstyle f}\uparrow & \searrow\!\!\searrow^{\varphi} & \uparrow{\scriptstyle f'} \\ A & \xrightarrow{g} & C \end{array}$$

in $\operatorname{Ho}^{\leq 2}\mathcal{D}$ is called a homotopy pushout square if it is equivalent to the image of a homotopy pushout square in $\mathcal{D}$. Note that a homotopy pushout square in $\operatorname{Ho}^{\leq 2}\mathcal{D}$ need not be a categorical homotopy pushout.

Note that it follows that for any $T \in \operatorname{Ho}\mathcal{D}$ and homotopy pushout square as above the map

$$\operatorname{Hom}(B \sqcup_A C, T) \to \operatorname{Hom}(B, T) \times_{\operatorname{Hom}(A,T)} \operatorname{Hom}(C, T) ,$$

where all homomorphism sets are in $\operatorname{Ho}\mathcal{D}$, is always surjective.

There is a dual *homotopy pullback functor* $\times$ and the dual notion of a homotopy pullback square in both $\operatorname{Ho}\mathcal{D}$ and $\operatorname{Ho}^{\leq 2}\mathcal{D}$.

For any homotopy pushout square $\triangle_1$

$$\begin{array}{ccc} B & \longrightarrow & D \\ \uparrow & & \uparrow \\ A & \longrightarrow & C \end{array}$$

in $\operatorname{Ho}\mathcal{D}$ and object $T \in \operatorname{Ho}\mathcal{D}$ the square induced on homotopy function complexes

$$\begin{array}{ccc} \operatorname{map}(B, T) & \longleftarrow & \operatorname{map}(D, T) \\ \downarrow & & \downarrow \\ \operatorname{map}(A, T) & \longleftarrow & \operatorname{map}(C, T) \end{array}$$

in Ho **SSet** is a homotopy pullback square. If we have another homotopy pullback square $\triangle_2$

$$\begin{array}{ccc} B' & \longrightarrow & D' \\ \uparrow & & \uparrow \\ A' & \longrightarrow & C' \end{array}$$

in $\mathrm{Ho}\,\mathcal{D}$ and a map $\triangle_1 \to \triangle_2$ such that the maps $A \to A'$, $B \to B'$ and $C \to C'$ are isomorphisms then the map $D \to D'$ is also an isomorphism, since an analogous statement is valid for the diagrams of homotopy function complexes.

We remark that it is possible to lift any commutative square

$$\begin{array}{ccc} B & \longrightarrow & D \\ \uparrow & & \uparrow \\ A & \longrightarrow & C \end{array}$$

in $\mathrm{Ho}\,\mathcal{D}$ to a *commutative* square in $\mathcal{D}_{cf}$ by first lifting it to a square in $\mathcal{D}_{cf}$ with cofibrations as morphisms starting at A with a homotopy between the two compositions and then replacing D by a path object on D.

It follows that for any such square in $\mathrm{Ho}\,\mathcal{D}$ there is a map $B \sqcup_A C \to D$ compatible with the squares. Hence such a square is a homotopy pushout if and only if the induced squares on homotopy function complexes are homotopy pushouts for all $T \in \mathrm{Ho}\,\mathbf{SSet}$.

For a cofibrant object $A \in \mathcal{D}$ the category $A \downarrow \mathcal{D}$ of objects under A is again a J-semi model category. The 2-functor

$$\mathcal{D} \to \mathbf{Cat}\,,$$

$$A \mapsto \mathrm{Ho}\,((QA) \downarrow \mathcal{D})$$

where $QA \to A$ is a cofibrant replacement, descents to a 2-functor

$$\mathrm{Ho}^{\leq 2}\mathcal{D} \to \mathbf{Cat}\,,$$

$$A \mapsto D(A \downarrow \mathcal{D})$$

such that the image functors f_* of all maps f in $\mathrm{Ho}^{\leq 2}\mathcal{D}$ have right adjoints f^*. The functor f_* preserves homotopy pushout squares, and the functor f^* preserves homotopy pullback and homotopy pushout squares. For $f : 0 \to A$ the map from an initial object to an object in $\mathrm{Ho}^{\leq 2}\mathcal{D}$ the functor $f^* : D(A \downarrow \mathcal{D}) \to \mathrm{Ho}\,\mathcal{D}$ factors through $A \downarrow \mathrm{Ho}\,\mathcal{D}$ and the map from A to the image of the initial object in $D(A \downarrow \mathcal{D})$ is an isomorphism.

Consider a commutative square

$$\begin{array}{ccc} B & \xrightarrow{g'} & D \\ {\scriptstyle f}\uparrow & \overset{\varphi}{\searrow\!\!\searrow} & \uparrow{\scriptstyle f'} \\ A & \xrightarrow{g} & C \end{array}$$

in $\mathrm{Ho}^{\leq 2}\mathcal{D}$. Let $E \in D(B \downarrow \mathcal{D})$. There is a base change morphism

$$g_* f^* E \to {f'}^* g'_* E$$

adjoint to the natural map $f^*E \to f^*g'^*g'_*M \overset{\varphi}{\cong} g^*f'^*g'_*M$. This base change morphism applied to diagrams

$$\begin{array}{ccc} B & \longrightarrow & C \\ \uparrow & \Searrow & \uparrow \\ A & \xrightarrow{\mathrm{Id}} & A \end{array}$$

enables one to construct a 2-functor

$$(A \downarrow \mathrm{Ho}^{\leq 2}\mathcal{D}) \to D(A \downarrow \mathcal{D})$$

which gives an equivalence after 1-truncation of the left hand side.

Remark 2.8. *The above construction should generalize to give functors between (weak)* $(n+1)$*-categories*

$$\mathrm{Ho}^{\leq n+1}\mathcal{D} \to n - \mathbf{Cat}$$
$$A \mapsto D^{\leq n}(A \downarrow \mathcal{D}) \ ,$$

where $\mathrm{Ho}^{\leq n+1}\mathcal{D}$ *is the* $(n+1)$*-truncation of the* 1*-Segal category associated to* $\mathcal{D}$*,* $n - \mathbf{Cat}$ *is the* $(n+1)$*-category of* n*-categories and* $D^{\leq n}(A \downarrow \mathcal{D}) := \mathrm{Ho}^{\leq n}(QA \downarrow \mathcal{D})$ *for* $QA \to A$ *a cofibrant replacement.*

There are dual constructions for objects over an object in $\mathcal{D}$.

The following theorem is the main source to obtain J-semi model categories.

Theorem 2.9. *Let* $\mathbb{T}$ *be a monad in* $\mathcal{C}$ *and assume that* $\mathcal{C}[\mathbb{T}]$ *has coequalizers. Suppose that every map in* $\mathbb{T}J$-cell *whose domain is cofibrant in* $\mathcal{C}$ *is a weak equivalence in* $\mathcal{C}$ *and every map in* $\mathbb{T}I$-cell *whose domain is cofibrant in* $\mathcal{C}$ *is a cofibration in* $\mathcal{C}$ *(here in both cases the cell complexes are built in* $\mathcal{C}[\mathbb{T}]$*). Assume furthermore that the initial object in* $\mathcal{C}[\mathbb{T}]$ *is cofibrant in* $\mathcal{C}$*. Then there is a cofibrantly generated* J*-semi model structure on* $\mathcal{C}[\mathbb{T}]$ *over* $\mathcal{C}$*, where a map is a weak equivalence or fibration if and only if it is a weak equivalence or fibration in* $\mathcal{C}$*.*

Proof. We define the weak equivalences (resp. fibrations) as the maps in $\mathcal{C}[\mathbb{T}]$ which are weak equivalences (resp. fibrations) as maps in $\mathcal{C}$. By adjointness the fibrations are $\mathbb{T}J$-inj and the trivial fibrations are $\mathbb{T}I$-inj. We define the class of cofibrations to be $\mathbb{T}I$-cof. Since the adjoint of $\mathbb{T}$ is the forgetful functor property 1 of Definition 2.3 is clear.

The bicompleteness of $\mathcal{C}[\mathbb{T}]$ follows as in the proof of Theorem 2.2. The 2-out-of-3 and retract axioms for the weak equivalences and the fibrations hold in $\mathcal{C}[\mathbb{T}]$ since they hold in $\mathcal{C}$, the retract axiom for the cofibrations holds because $\mathbb{T}I$-cof is closed under retracts. So property 2 is fulfilled.

The first half of property 3 is true by the definition of the cofibrations. By our smallness assumptions we have functorial factorizations of maps into a cofibration followed by a trivial fibration and into a map from $\mathbb{T}J$-cell followed by a fibration. We claim that a map f in $\mathbb{T}J$-cell whose domain is cofibrant in $\mathcal{C}$ is a trivial cofibration. f is a weak equivalence by assumption. Factor f as $p \circ i$ into a cofibration followed by a trivial fibration. Since f has the left lifting property with respect to p, f is a retract of i by the retract argument, hence also a cofibration. Hence we have shown property 4.

Now let f be a trivial cofibration whose domain is cofibrant in $\mathcal{C}$. We can factor f as $p \circ i$ with $i \in \mathbb{T}J$-cell and p a fibration. p is a trivial fibration by the 2-out-of-3 property, hence f has the left lifting property with respect to p, so f is a retract of i and has therefore the left lifting property with respect to fibrations. This is the second half of property 3. Property 5 immediately follows from the assumptions, and property 6 is true since limits in $\mathcal{C}[\mathbb{T}]$ are computed in $\mathcal{C}$. □

Alternative: Assume that $\mathcal{C}[\mathbb{T}]$ has coequalizers, that sequential colimits in $\mathcal{C}[\mathbb{T}]$ are computed in $\mathcal{C}$ and that the pushout of an object in $\mathcal{C}[\mathbb{T}]$ which is cofibrant in $\mathcal{C}$ by a map from $\mathbb{T}I$ (resp. from $\mathbb{T}J$) is a cofibration (resp. weak equivalence) as a map in $\mathcal{C}$. Then the same conclusion holds as in the Theorem above. Moreover the conclusion also holds for the alternative definition of J-semi model category without the smallness assumptions on the domains of I and J which we made at the beginning of this section.

Example 2.10. *Let* $\mathrm{Ass}(\mathcal{C})$ *be the category of associative unital algebras in* $\mathcal{C}$*. Then* $\mathrm{Ass}(\mathcal{C})$ *is a* J*-semi model category over* $\mathcal{C}$ *(see* [Hov2, Theorem 3.3]*).*

Will will need the

Lemma 2.11. *Let* R *be a ring with unit in* $\mathcal{C}$*,* i *a map in* $(I \otimes R)$-cof *(taken in* R*–Mod*$_r$*) and* j *a map in* R*–Mod which is a (trivial) cofibration in* $\mathcal{C}$*. Then* $i \Box_R j$ *is a (trivial) cofibration in* $\mathcal{C}$*. If* i *is in* $(J \otimes R)$-cof*, then* $i \Box_R j$ *is a trivial cofibration in* $\mathcal{C}$*.*

Proof. This follows either by [Hov1, Lemma 4.2.4] applied to the adjunction of two variables $R\text{–}Mod_r \times R\text{–}Mod \to \mathcal{C}$, $(M, N) \mapsto M \otimes_R N$, or by Lemma 2.1. □

3. Operads

For a group G write $\mathcal{C}[G]$ for the category of objects in $\mathcal{C}$ together with a right G-action. This is the same as $\mathbb{1}[G]\text{–}Mod_r$, where $\mathbb{1}[G]$ is the group ring of G in $\mathcal{C}$. Let $\mathcal{C}^{\mathbb{N}}$ be the category of sequences in $\mathcal{C}$ and $\mathcal{C}^{\Sigma}$ the category of symmetric sequences, i.e. $\mathcal{C}^{\Sigma} = \coprod_{n \in \mathbb{N}} \mathcal{C}[\Sigma_n]$. Finally let $\mathcal{C}^{\mathbb{N},\bullet}$ (resp. $\mathcal{C}^{\Sigma,\bullet}$) be the category of objects X from $\mathcal{C}^{\mathbb{N}}$ (resp. from $\mathcal{C}^{\Sigma}$) together with a map $\mathbb{1} \to X(1)$.

Proposition 3.1. *For any group* G *the category* $\mathcal{C}[G]$ *has a natural structure of cofibrantly generated model category with generating cofibrations* $I[G]$ *and generating trivial cofibrations* $J[G]$*.*

Proof. Easy from [Hov1, Theorem 2.1.19]. □

Hence there are also canonical model structures on $\mathcal{C}^{\mathbb{N}}$, $\mathcal{C}^{\mathbb{N},\bullet}$, $\mathcal{C}^{\Sigma}$ and $\mathcal{C}^{\Sigma,\bullet}$. For each such model category $\mathcal{C}^{\sim}$ and $n \in \mathbb{N}$ there is a left Quillen functor $i_n : \mathcal{C} \to \mathcal{C}^{\sim}$ adjoint to the forgetful functor at the n-th place. $\mathcal{C}^{\sim}$ is cofibrantly generated with generating cofibrations $\coprod_{i \in \mathbb{N}} i_n(I)$ and generating trivial cofibrations $\coprod_{i \in \mathbb{N}} i_n(J)$. We denote them by $\mathbb{N}I$, $\mathbb{N}^{\bullet}I$, ΣI and $\Sigma^{\bullet} I$ respectively (similarly for J).

Note that a map of groups $\varphi : H \to G$ induces a left Quillen functor $\mathcal{C}[H] \to \mathcal{C}[G]$. If φ is injective the right adjoint to this functor preserves (trivial) cofibrations.

Let $\mathrm{Op}(\mathcal{C})$ be the category of operads in $\mathcal{C}$, where an operad in $\mathcal{C}$ is defined as in [KM, Definition 1.1]. Let $F : \mathcal{C}^{\mathbb{N}} \to \mathrm{Op}(\mathcal{C})$ be the functor which assigns to a sequence X the free operad FX on X. This functor naturally factors through $\mathcal{C}^{\mathbb{N},\bullet}$, $\mathcal{C}^{\Sigma}$ and $\mathcal{C}^{\Sigma,\bullet}$, and the functors starting from one of these categories going to $\mathrm{Op}(\mathcal{C})$

are also denoted by F. The right adjoints of F, i.e. the forgetful functors, map $\mathcal{O}$ to $\mathcal{O}^\sharp$.

For any object $A \in \mathcal{C}$ there is the endomorphism operad $\mathrm{End}^{\mathrm{Op}}(A)$ given by $\mathrm{End}^{\mathrm{Op}}(A)(n) = \underline{\mathrm{Hom}}(A^{\otimes n}, A)$.

We come to the main result of this section:

Theorem 3.2. *The category* $\mathrm{Op}(\mathcal{C})$ *is a cofibrantly generated J-semi model category over $\mathcal{C}^{\Sigma,\bullet}$ with generating cofibrations $F(\mathbb{N}I)$ and generating trivial cofibrations $F(\mathbb{N}J)$. If $\mathcal{C}$ is left proper (resp. right proper), then* $\mathrm{Op}(\mathcal{C})$ *is left proper relative to $\mathcal{C}^{\Sigma,\bullet}$ (resp. right proper).*

We first give an explicit description of free operads and pushouts by free operad maps, which will be needed for the proof of this Theorem.

Definition 3.3.
1. *An n-*tree *is a finite connected directed graph T such that any vertex of T has ≤ 1 ingoing arrows, the outgoing arrows of each vertex v of T are numbered by* $1, \ldots, \mathrm{val}(v)$*, where* $\mathrm{val}(v)$ *is the number of these arrows, and there are n arrows which do not end at any vertex, which are called* tails *and which are numbered by $1, \ldots, n$. By definition the empty tree has one tail, so it is a 1-tree.*
2. *A* doubly colored *n-tree is an n-tree together with a decomposition of the set of vertices into* old *and* new *vertices.*
3. *A* proper *doubly colored n-tree is a doubly colored n-tree such that every arrow starting from an old vertex is either a tail or goes to a new vertex.*

We denote the set of n-trees by $\mathcal{T}(n)$, the set of doubly colored n-trees by $\mathcal{T}_{\mathrm{dc}}(n)$ and the set of proper doubly colored n-trees by $\mathcal{T}_{\mathrm{dc}}^{p}(n)$. Set $\mathcal{T} := \coprod_{n\in\mathbb{N}} \mathcal{T}(n)$ and $\mathcal{T}_{\mathrm{dc}}^{(p)} := \coprod_{n\in\mathbb{N}} \mathcal{T}_{\mathrm{dc}}^{(p)}(n)$.

The n-trees will describe the n-ary operations of free operads, and indeed $\mathcal{T}(\bullet)$ is endowed with a natural operad structure in **Set**. Let $n, m_1, \ldots, m_n \in \mathbb{N}$, $m := \sum_{i=1}^{n} m_i$ and $T \in \mathcal{T}(n)$, $T_i \in \mathcal{T}(m_i)$, $i = 1, \ldots, n$. Then the corresponding structure map γ of this operad sends $(T, T_1, \ldots, T_n)$ to the tree which one obtains from T by glueing the root of T_i to the i-th tail of T for every $i = 1, \ldots, n$. The previously j-th tail of T_i gets the label $j + \sum_{k=1}^{i-1} m_k$. The free right action of Σ_n on $\mathcal{T}(n)$ (which is also defined on $\mathcal{T}_{\mathrm{dc}}^{(p)}(n)$) is such that $\sigma \in \Sigma$ sends a tree $T \in \mathcal{T}(n)$ to the tree obtained from T by changing the label i of a tail of T into $\sigma^{-1}(i)$. So $\gamma(T, T_1, \ldots, T_n)^{\sigma(m_{\sigma(1)}, \ldots, m_{\sigma(n)})} = \gamma(T^\sigma, T_{\sigma(1)}, \ldots, T_{\sigma(n)})$, where $\sigma(m_1, \ldots, m_n)$ permutes blocks of lenth m_i in $1, \ldots, m$ as σ permutes $1, \ldots, n$.

Note that an n-tree has a natural embedding into the plane and this embedding is equivalent to the numbering of the arrows. It follows that there exists a *canonical* labelling of the tails of an n-tree, namely the one which labels the tails succesively from the left to the right in the planar embedding of the tree.

For T an element of $\mathcal{T}$ or $\mathcal{T}_{\mathrm{dc}}^{(p)}$ let $V(T)$ denote the set of vertices of T (this is defined up to unique isomorphism, since our trees do not have automorphisms) and let $u(T)$ be the number of vertices of T of valency 1 and $U(T)$ be the set of vertices of T of valency 1. For $T \in \mathcal{T}_{\mathrm{dc}}^{(p)}$ write $V_{\mathrm{old}}(T)$ (resp. $V_{\mathrm{new}}(T)$) for the set of old (resp. new) vertices of T and $U_{\mathrm{old}}(T)$ (resp. $U_{\mathrm{new}}(T)$) for the set of old (resp. new) vertices in $U(T)$ and $u_{\mathrm{old}}(T)$ (resp. $u_{\mathrm{new}}(T)$) for their number.

Proposition 3.4. (1) *The free operad FX on $X \in \mathcal{C}^{\mathbb{N}}$ is given by*

$$(FX)(n) = \coprod_{T \in \mathcal{T}(n)} \bigotimes_{v \in V(T)} X(\mathrm{val}(v)) \; .$$

(2) *The free operad FX on $X \in \mathcal{C}^{\mathbb{N},\bullet}$ is given by an ω-sequence*

$$FX = \mathrm{colim}_{i<\omega} F_i X$$

in $\mathcal{C}^{\mathbb{N}}$, where $(F_iX)_n$ is a pushout of $(F_{i-1}X)_n$ by the map

$$\coprod_{\substack{T \in \mathcal{T}(n) \\ u(T) = i}} \left(\bigotimes_{v \in V(T) \setminus U(T)} X(\mathrm{val}(v)) \right) \otimes e^{\square(U(T))} \; ,$$

where e is the unit map $\mathbb{1} \to X(1)$.

(3) *The free operad on $X \in \mathcal{C}^{\Sigma}$ is given by*

$$(FX)(n) = \left(\coprod_{T \in \mathcal{T}(n)} \bigotimes_{v \in V(T)} X(\mathrm{val}(v)) \right) / \sim \; ,$$

where the equivalence relation $\sim$ identifies for every isomorphism of directed graphs $\varphi : T \to T'$, $T, T' \in \mathcal{T}(n)$, which respects the numbering of the tails but not necessarily of the arrows, the summands $\bigotimes_{v \in V(T)} X(\mathrm{val}(v))$ and $\bigotimes_{v \in V(T')} X(\mathrm{val}(v))$ by the map $\bigotimes_{v \in V(T)} \sigma_v$, where $\sigma_v : X(\mathrm{val}(v)) \to X(\mathrm{val}(\varphi(v))) = X(\mathrm{val}(v))$ is the action of the element $\sigma_v \in \Sigma_{\mathrm{val}(v)}$ such that φ maps the i-th arrow of v to the $\sigma_v(i)$-th arrow of $\varphi(v)$.

(4) *The free operad FX on $X \in \mathcal{C}^{\Sigma,\bullet}$ is given by an ω-sequence*

$$FX = \mathrm{colim}_{i<\omega} F_i X$$

in $\mathcal{C}^{\mathbb{N}}$, where $(F_iX)_n$ is a pushout of $(F_{i-1}X)_n$ by the map

$$\left(\coprod_{T \in \mathcal{T}(n), u(T) = i} \left(\bigotimes_{v \in V(T) \setminus U(T)} X(\mathrm{val}(v)) \right) \otimes e^{\square(U(T))} \right) / \sim \; ,$$

where e is as in 2 and the equivalence relation $\sim$ is like in 3.

In cases 2 and 4 the attaching map is induced from the operation of removing a vertex of valency 1 from a tree. Note that the morphism in 4 and the attaching morphism respects the equivalence relation. The Σ_n-actions are induced from the Σ_n-action on $\mathcal{T}(n)$.

Proof. We claim that in all four cases the functors F define a monad the algebras of which are the operads in $\mathcal{C}$. So we have to define in all four cases maps $m : FFX \to FX$ and $e : X \to FX$ satisfying the axioms for a monad. We will restrict ourselves to case i) and leave the other cases to the interested reader.

The domain of the map $m(n)$ is a coproduct over all $T \in \mathcal{T}(n)$, $T_v \in \mathcal{T}(\mathrm{val}(v))$ for all $v \in V(T)$ of the

$$\bigotimes_{v \in V(T), w \in V(T_v)} X(\mathrm{val}(w)) \; ,$$

and the map m sends such an entry via the identity to the entry associated to the tree in $\mathcal{T}(n)$ obtained by replacing every vertex v of T by the tree T_v in such a way that the numbering of the arrows starting at v and the numbering of the tails of T_v correspond. The map e sends $X(n)$ to the summand $X(n)$ in FX which belongs to the tree with one vertex and n tails such that the labelling of the arrows coincides with the labelling of the tails (which are of course all arrows in this case) (i.e. the labelling of the tails is the canonical one). It is clear that m is associative and e is a two-sided unit. To see that an F-algebra is the same as an operad one proceeds as follows: Let X be an F-algebra. Let $\mathcal{O}(n) := X(n)$. The structure maps of the operad structure we will define on $\mathcal{O}$ are obtained from the algebra map by restricting it to the summands belonging to trees where every arrow starting at the root goes to a vertex which has only tails as outgoing vertices and where the labelling of the tails is the canonical one. The unit in $\mathcal{O}(1)$ corresponds to the empty tree. The right action of a $\sigma \in \Sigma_n$ on $\mathcal{O}(n)$ is given by the algebra map restricted to the tree with one vertex and n tails such that the i-th arrow simultaneously is the $\sigma^{-1}(i)$-th tail. That 1 acts as the identity is the unit property of X, and the associativity of the action follows from the associativity of X. It is easy to see that the associativity and symmetry properties of $\mathcal{O}$ also follow from the associativity of X. The unit properties follow from the behaviour of the empty tree.

On the other hand let $\mathcal{O}$ be an operad. We define an F-algebra structure on $X := \mathcal{O}^\sharp$: Let $T \in \mathcal{T}(n)$ be a tree with canonical labelling of the tails. Then it is clear how to define a map from the summand in FX corresponding to T to $X(n)$ by iterated application of the structure maps of $\mathcal{O}$ (the unit of $\mathcal{O}$ is needed to get the map for the empty tree). The map on the summand corresponding to T^σ for $\sigma \in \Sigma_n$ is the map for T followed by the action of σ on $X(n) = \mathcal{O}(n)$. One then can check that the associativity, symmetry and unit properties of the structure maps of $\mathcal{O}$ imply that we get indeed an F-algebra with structure map $FX \to X$ just described. □

For describing pushouts by free operad maps we need an operation which changes a new vertex in a tree in $\mathcal{T}^p_{\mathrm{dc}}(n)$ into an old vertex and gives again a tree in $\mathcal{T}^p_{\mathrm{dc}}(n)$. This is given by first making the new vertex into an old vertex to get an element of $\mathcal{T}_{\mathrm{dc}}(n)$ and then removing all arrows joining only old vertices and identifying the old vertices which have been joined. The numbering of the arrows of the new tree is most easily described by noting that this numbering corresponds to a planar embedding of the tree and the operation of removing the arrows and identifying the vertices can canonically be done in the plane. For $T \in \mathcal{T}^p_{\mathrm{dc}}$ and $v \in V_{\mathrm{new}}(T)$ denote by $\mathrm{ch}_T(v) \in \mathcal{T}^p_{\mathrm{dc}}$ the tree obtained by changing the new vertex v in T into an old vertex. Note that for $\mathcal{O} \in \mathrm{Op}(\mathcal{C})$ there is a concatenation map

$$\mathrm{conc}^{\mathcal{O}}_T(v): \ \mathcal{O}(\mathrm{val}(v)) \otimes \bigotimes_{v' \in V_{\mathrm{old}}(V)} \mathcal{O}(\mathrm{val}(v')) \longrightarrow \bigotimes_{v' \in V_{\mathrm{old}}(\mathrm{ch}_T(v))} \mathcal{O}(\mathrm{val}(v'))$$

induced by applying the operad maps of $\mathcal{O}$.

Proposition 3.5. *Let $\mathcal{O} \in \mathrm{Op}(\mathcal{C})$ and $f: A \to B$ and $\varphi: A \to \mathcal{O}^\sharp$ be maps in $\mathcal{C}^{\mathbb{N}}$. Then the pushout $\mathcal{O}'$ of $\mathcal{O}$ by Ff with attaching map the adjoint of φ is given by an $\omega \times (\omega+1)$-sequence $\mathcal{O}' = \mathrm{colim}_{(i,j)<\omega\times(\omega+1)} \mathcal{O}_{(i,j)}$ in $\mathcal{C}^{\mathbb{N}}$, where for $j < \omega$ $\mathcal{O}_{(i,j)}(n)$ is a pushout of $\mathcal{O}_{(i,j)-1}(n)$ in $\mathcal{C}$ by the quotient of the map*

$$\coprod \left(\bigotimes_{v \in V_{\mathrm{old}}(T) \setminus U_{\mathrm{old}}(T)} \mathcal{O}(\mathrm{val}(v)) \right) \otimes e^{\square(U_{\mathrm{old}}(T))} \square \mathop{\square}_{v \in V_{\mathrm{new}}(T)} f(\mathrm{val}(v)) \ ,$$

where the coproduct is over all $T \in \mathcal{T}^p_{\mathrm{dc}}(n)$ *with* $\sharp V_{\mathrm{new}}(T) = i$ *and* $u_{\mathrm{old}}(T) = j$*, with respect to the equivalence relation which identifies for every isomorphism of doubly colored directed graphs* $\varphi : T \to T'$*,* $T, T' \in \mathcal{T}^p_{\mathrm{dc}}$*, which respects the labeling of the tails and of the arrows starting at new vertices, the summands corresponding to* T *and* T' *via a map analogous to the map in Proposition 3.4.3. Here* e *is the unit* $\mathbb{1} \to \mathcal{O}(1)$ *and the attaching map is the following: The domain of the above map is obtained by glueing* $i + j$ *objects together, hence we have to give* $i + j$ *maps compatible with glueing. The first* i *maps are induced by removing one of the vertices in* $U_{\mathrm{old}}(T)$ *from* T*, and the other* j *maps are induced by changing one of the vertices in* $V_{\mathrm{new}}(T)$ *into an old vertex and applying the maps* $\mathrm{conc}^{\mathcal{O}}_v(T)$*,* $v \in V_{\mathrm{new}}(T)$*. (Note that for* $n = 1$ *the operad* $\mathcal{O}$ *appears in the second step of the limit, in all other cases in the first.) The* Σ_n*-actions are induced from the ones on* $\mathcal{T}^p_{\mathrm{dc}}(n)$*.*

There are similar descriptions of pushouts of $\mathcal{O}$ *by free operad maps on maps from* $\mathcal{C}^{\mathbb{N},\bullet}$*,* $\mathcal{C}^{\Sigma}$ *and* $\mathcal{C}^{\Sigma,\bullet}$*.*

Proof. Let $\widetilde{\mathcal{O}}(n)$ be the colimits described in the Proposition. First of all we check that this is well defined, i.e. that firstly the $i + j$ maps we have described glue together. This is the case because the processes of removing old vertices of valency 1 and/or changing a new vertex into an old one and concatenating commute with each other. Secondly this map factors through the quotient described in the Proposition because of the symmetry properties of $\mathcal{O}$ and because of the fact that in previous steps quotients with respect to analogous equivalence relations have been taken.

Next we have to equip $\widetilde{\mathcal{O}} \in \mathcal{C}^{\mathbb{N}}$ with an operad structure. The unit is the one coming from $\mathcal{O}$. We define the structure map $\gamma : \widetilde{\mathcal{O}}(n) \otimes \widetilde{\mathcal{O}}(m_1) \otimes \cdots \otimes \widetilde{\mathcal{O}}(m_n) \to \widetilde{\mathcal{O}}(m)$ $(m = \sum_{i=1}^n m_i)$ in the following way: For $T \in \mathcal{T}^p_{\mathrm{dc}}(n)$ let

$$S(T) := \left(\bigotimes_{v \in V_{\mathrm{old}}(T)} \mathcal{O}(\mathrm{val}(v)) \right) \otimes \bigotimes_{v \in V_{\mathrm{new}}(T)} B(\mathrm{val}(v)) \; .$$

First one defines for trees $T \in \mathcal{T}^p_{\mathrm{dc}}(n)$, $T_i \in \mathcal{T}^p_{\mathrm{dc}}(m_i)$, $i = 1, \ldots, n$, a map

$$\xi_{(T, T_1, \ldots, T_n)} : \; S(T) \otimes S(T_1) \otimes \cdots \otimes S(T_n) \to \widetilde{\mathcal{O}}(m) \; .$$

Therefore one glues the tree T_i to the tail of T with label i and concatenates such that one gets a tree $\widetilde{T} \in \mathcal{T}^p_{\mathrm{dc}}(m)$. Then by applying structure maps of $\mathcal{O}$ one gets a map $S(T) \otimes S(T_1) \otimes \cdots \otimes S(T_n) \to S(\widetilde{T})$ and composes this with the canonical map $S(\widetilde{T}) \to \widetilde{\mathcal{O}}(m)$.

Let $m_0 := n$. Suppose we have already defined for a $0 \leq k \leq n$ and for all trees $T_i \in \mathcal{T}^p_{\mathrm{dc}}(m_i)$, $i = k, \ldots, n$, a map $(\bigotimes_{i=0}^{k-1} \widetilde{\mathcal{O}}(m_i)) \otimes S(T_k) \otimes \cdots \otimes S(T_n) \to \widetilde{\mathcal{O}}(m)$. From this data one then obtains the same data for $k+1$ instead of k as follows: Let $T_i \in \mathcal{T}^p_{\mathrm{dc}}(m_i)$, $i = k+1, \ldots, n$. One defines the map $\varphi : \; (\bigotimes_{i=0}^{k} \widetilde{\mathcal{O}}(m_i)) \otimes S(T_{k+1}) \otimes \cdots \otimes S(T_n) \to \widetilde{\mathcal{O}}(m)$ by transfinite induction on the terms of the $\omega \times (\omega+1)$-sequence defining $\widetilde{\mathcal{O}}(m_k)$: So let $(\bigotimes_{i=0}^{k-1} \widetilde{\mathcal{O}}(m_i)) \otimes \mathcal{O}_{(i,j)}(m_k) \otimes S(T_k) \otimes \cdots \otimes S(T_n) \to \widetilde{\mathcal{O}}(m)$ be already defined and let ψ be the map by which $\mathcal{O}_{(i,j+1)}(m_k)$ is a pushout of $\mathcal{O}_{(i,j)}(m_k)$. We define φ on $(\bigotimes_{i=0}^{k-1} \widetilde{\mathcal{O}}(m_i)) \otimes \mathcal{O}_{(i,j+1)}(m_k) \otimes S(T_k) \otimes \cdots \otimes S(T_n)$ by using the data described above for k to get the map after taking the appropriate quotient on the codomain of ψ. One has to check the compatibility of this map with the given map via the attaching map. To do this for one of the $i+j$ summands of the domain of ψ one uses the fact that the same kind of compatibility is valid in $\widetilde{\mathcal{O}}(m)$. Finally when arriving at $k = n$ we get the desired structure map.

The associativity of the structure maps follows by proving the corresponding statement for the $\xi_{(T,T_1,\dots,T_n)}$. This one gets by first glueing trees without concatenating and then observing that the concatenation processes at different places commute. The symmetry properties follow in the same way as for free operads, the unit properties are forced by the fact that in the ψ's the pushout product over the unit maps is taken. Hence $\widetilde{\mathcal{O}}$ is an operad. It receives canonical compatible maps in $\mathrm{Op}(\mathcal{C})$ from $\mathcal{O}$ and FB.

In the end we have to show that our operad $\widetilde{\mathcal{O}}$ indeed satisfies the universal property of the pushout by Ff. We need to show that a map $g: \mathcal{O} \to \mathcal{O}''$ in $\mathrm{Op}(\mathcal{C})$ together with a map $h: B \to (\mathcal{O}'')^\sharp$ compatible with the attaching map is the same as a map $g': \widetilde{\mathcal{O}} \to \mathcal{O}''$. To get g' from g and h one first defines for any $T \in \mathcal{T}^p_{\mathrm{dc}}(n)$ a map $S(T) \to \mathcal{O}''$ using the structure maps of $\mathcal{O}''$. Then one checks that these maps indeed glue together to a g'. To get g and h from g' one composes g' with $\mathcal{O} \to \widetilde{\mathcal{O}}$ and $B \to FB \to \widetilde{\mathcal{O}}$. These processes are invers to each other. □

Lemma 3.6. *Let $\mathcal{O}$, f, φ and $\mathcal{O}'$ be as in Proposition 3.5, assume that $\mathcal{O} \in \mathrm{Op}(\mathcal{C})$ is cofibrant as object in $\mathcal{C}^{\Sigma,\bullet}$ and that f is a (trivial) cofibration. Then the pushout $\mathcal{O} \to \mathcal{O}'$ is a (trivial) cofibration in $\mathcal{C}^{\Sigma,\bullet}$. There is an analogous statement for f a (trivial) cofibration in $\mathcal{C}^{\mathbb{N},\bullet}$, $\mathcal{C}^{\Sigma}$ and $\mathcal{C}^{\Sigma,\bullet}$.*

The proof is given after the next Lemma. In this Lemma we use the fact that if we have a G-action on an object L and a Σ_n-action on M, then there is a canonical action of the wreath product $\Sigma_n \ltimes G^n$ on $M \otimes L^{\otimes n}$.

Lemma 3.7. *Let $n_1, \dots, n_k \in \mathbb{N}_{>0}$, let $G_1, \dots, G_k$ be groups and g_i be a cofibration in $\mathcal{C}[G_i]$, $i = 1, \dots, k$. Let f be a cofibration in $\mathcal{C}[\prod_{i=1}^k \Sigma_{n_i}]$. Then the map*

$$h := f \Box \mathop{\Box}_{i=1}^k g_i^{\Box n_i}$$

is a cofibration in $\mathcal{C}[(\prod_{i=1}^k \Sigma_{n_i}) \ltimes (\prod_{i=1}^k G^{n_i})]$. If f or one of the g_i is trivial, so is h.

Proof. We restrict to the case $k = 1$, the general case is done in the same way. Set $n := n_1$, $G := G_1$ and $g := g_1$. We can assume that $g \in I[G]$-cell and $f \in I[\Sigma_n]$-cell (or $f \in J[G]$-cell or $g \in J[\Sigma_n]$-cell). Let $g: L_0 \to \mathrm{colim}_{i<\mu} L_i$ and $f: M_0 \to \mathrm{colim}_{i<\lambda} M_i$ such that $L_i \to L_{i+1}$ is a pushout by $\psi_i \in I[G]$ and $M_i \to M_{i+1}$ is a pushout by $\varphi_i \in I[\Sigma_n]$. Then by Lemma 2.1 $f \Box g^{\Box n}$ is a $\lambda \times \mu^n$-sequence, and the transition maps are pushouts by the $\varphi_i \Box \psi_{i_1} \Box \cdots \Box \psi_{i_n}$, $i < \lambda$; $i_1, \dots, i_n < \mu$. We can modify this sequence to make it invariant under the Σ_n-action: Let S be the set of unordered sequences of length n with entries in μ, and for $s \in S$ let j_s be the set of ordered sequences of length n with entries in μ which map to s. Let $s, s' \in S$. In the following let us view s and s' as monotonly increasing sequences of length n. We say that $s < s'$ if there is a $1 \le i < n$ such that $s(j) = s'(j)$ for $i < j$ and $s(i) < s'(i)$. With this order S is well-ordered. Now $g^{\Box n}$ is an S-sequence with s-th transition map $\psi'_s := \coprod_{w \in j_s} \psi_{w(1)} \Box \cdots \Box \psi_{w(n)}$, so $f \Box g^{\Box n}$ is the corresponding $\lambda \times S$-sequence with transition maps the $\varphi_i \Box \psi'_s$, $i < \lambda$, $s \in S$. Note that on these maps there is a $\Sigma_n \ltimes G^n$-action. Now to prove our claim it suffices to show that every $\varphi_i \Box \psi'_s$ is a (trivial) cofibration in $\mathcal{C}[\Sigma_n \ltimes G^n]$, which can easily be seen by noting that every φ_i and ψ_i is of the form $h[G]$ for $h \in I$ (or $h \in J$). $\square$

Proof of Lemma 3.6. Let $\sim$ be the equivalence relation on $\mathcal{T}^p_{\text{dc}}$ which identifies T and T' in $\mathcal{T}^p_{\text{dc}}$ if there is an isomorphism of directed graphs $T \to T'$ which respects the labeling of the arrows starting at new vertices. Let C be an equivalence class of $\sim$ in $\mathcal{T}^p_{\text{dc}}(n)$. The Σ_n-action on $\mathcal{T}^p_{\text{dc}}(n)$ restricts to a Σ_n-action on C. We have to show that the part of the map in Proposition 3.5 given as the appropriate quotient of

$$\coprod_{T \in C} \left(\bigotimes_{v \in V_{\text{old}}(T) \setminus U_{\text{old}}(T)} \mathcal{O}(\text{val}(v)) \right) \otimes e^{\square(U_{\text{old}}(T))} \square \mathop{\square}_{v \in V_{\text{new}}(T)} f(\text{val}(v)) \quad (*)$$

is a (trivial) cofibration in $\mathcal{C}[\Sigma_n]$. Let Γ be a doubly colored directed graph, where the arrows starting at new vertices are labelled, isomorphic to the objects of the same type underlying the objects from C. Set

$$\varphi := \left(\bigotimes_{v \in V_{\text{old}}(\Gamma) \setminus U_{\text{old}}(\Gamma)} \mathcal{O}(\text{val}(v)) \right) \otimes e^{\square(U_{\text{old}}(\Gamma))} \square \mathop{\square}_{v \in V_{\text{new}}(\Gamma)} f(\text{val}(v)) \; .$$

On φ there is an action of $\text{Aut}(\Gamma)$. Let t be the set of tails of Γ. There is an action of $\text{Aut}(\Gamma)$ on t. It is easily seen that the quotient of the map (*) we are considering is isomorphic to $\varphi \times_{\text{Aut}(\Gamma)} \Sigma_t$. Hence we are finished if we show that φ is a (trivial) cofibration in $\mathcal{C}[\text{Aut}(\Gamma)]$. This is done by induction on the depth of Γ. Let $\Gamma_1, \ldots, \Gamma_k$ be the different isomorphism types of doubly colored directed graphs, such that the arrows starting at new vertices are labelled, sitting at the initial vertex of Γ with multiplicities $n_1, \ldots, n_k$ and set $G_i := \text{Aut}(\Gamma_i)$, $i = 1, \ldots, k$. Then, if the initial vertex of Γ is old, $\text{Aut}(\Gamma) = (\prod_{i=1}^k \Sigma_{n_i}) \ltimes (\prod_{i=1}^k G_i^{n_i})$, otherwise $\text{Aut}(\Gamma) = \prod_{i=1}^k G_i^{n_i}$, and the map φ is given like the map h in Lemma 3.7. Now the claim follows from Lemma 3.7 and the induction hypothesis. $\square$

Proof of Theorem 3.2. We apply Theorem 2.9 to the monad $\mathbb{T}$ which maps X to $(FX)^\sharp$. It is known that $\text{Op}(\mathcal{C})$ is cocomplete. Since filtered colimits in $\text{Op}(\mathcal{C})$ are computed in $\mathcal{C}^{\mathbb{N}}$, it follows from Lemma 3.6 that those maps from FI-cell (resp. FJ-cell) whose domain is cofibrant in $\mathcal{C}^{\Sigma,\bullet}$ are cofibrations (resp. trivial cofibrations) in $\mathcal{C}^{\Sigma,\bullet}$.

It is clear that $\text{Op}(\mathcal{C})$ is right proper if $\mathcal{C}$ is. If $\mathcal{C}$ is left proper, then $\mathcal{C}^{\Sigma,\bullet}$ is left proper, and the pushout in $\text{Op}(\mathcal{C})$ by a cofibration whose domain is cofibrant in $\mathcal{C}^{\Sigma,\bullet}$ is a retract of a transfinite composition of pushouts by cofibrations in $\mathcal{C}^{\Sigma,\bullet}$, hence weak equivalences are preserved by these pushouts. $\square$

Remark 3.8. *Let R be a commutative ring with unit and $\mathcal{C}$ be the symmetric monoidal model category of unbounded chain complexes of R-modules with the projective model structure. Here the generating trivial cofibrations are all maps $0 \to D^n R$, $n \in \mathbb{Z}$. The $D^n R$ are clearly null-homotopic. From this it follows that for a generating trivial cofibration f in $\mathcal{C}^{\mathbb{N}}$ the codomain of the maps in Proposition 3.5 along which the pushouts are taken (these maps have domain 0, so the pushouts are trival) are also null-homotopic by the homotopy which is on the summand corresponding to a tree $T \in \mathcal{T}^p_{\text{dc}}$ the sum over the homotopies from above over all new* *vertices of T (this homotopy factors through the quotient which is taken). Hence the conditions of Theorem 2.2 are fulfilled, so we get a model structure on* $\mathrm{Op}(\mathcal{C})$ *which is the same as the one provided by* [Hin1, Theorem 6.1.1].

Remark 3.9. *One can use exactly the same methods as above to give the category of colored operads in $\mathcal{C}$ for any set of labels the structure of a J-semi model category. In the case of unbounded complexes over a commutative unital ring as above this J-semi model structure is again a model structure.*

4. Algebras

For an operad $\mathcal{O} \in \mathrm{Op}(\mathcal{C})$ let us denote by $\mathrm{Alg}(\mathcal{O})$ the category of algebras over $\mathcal{O}$. Let $F_{\mathcal{O}} : \mathcal{C} \to \mathrm{Alg}(\mathcal{O})$ be the free algebra functor which is given by

$$F_{\mathcal{O}}(X) = \coprod_{n \geq 0} \mathcal{O}(n) \otimes_{\Sigma_n} X^{\otimes n} \, .$$

The right adjoint of $F_{\mathcal{O}}$ maps A to $A^{\sharp}$.

Remark 4.1. *An $\mathcal{O}$-algebra structure on an object $A \in \mathcal{C}$ is the same as to give a map of operads $\mathcal{O} \to \mathrm{End}^{\mathrm{Op}}(A)$.*

Lemma 4.2. *Let I be a small category and let $D : I \to \mathrm{Op}(\mathcal{C})$, $i \mapsto \mathcal{O}_i$, be a functor. Set $\mathcal{O} := \mathrm{colim}_{i \in I} \mathcal{O}_i$ and let $A, B \in \mathcal{C}$. Then the following is valid.*

1. *To give an $\mathcal{O}$-algebra structure on A is the same as to give $\mathcal{O}_i$-algebra structures on A compatible with all transition maps in D.*
2. *Assume that A and B have $\mathcal{O}$-algebra structures and let $f : A \to B$ be a map in $\mathcal{C}$. Then f is a map of $\mathcal{O}$-algebras if and only if it is a map of $\mathcal{O}_i$-algebras for all $i \in D$.*

Proof. The first part follows from the Remark above.

Let f be compatible with all $\mathcal{O}_i$-algebra structures. Then it can be checked directly that f is also compatible with the algebra structure on $\mathcal{O}' := \coprod_{i \in D} \mathcal{O}_i$. But since the maps $\mathcal{O}'(n) \to \mathcal{O}(n)$ are coequalizers in $\mathcal{C}$ the claim follows. □

The first main result of this section is

Theorem 4.3. *Let $\mathcal{O} \in \mathrm{Op}(\mathcal{C})$ be cofibrant. Then the category* $\mathrm{Alg}(\mathcal{O})$ *is a cofibrantly generated J-semi model category over $\mathcal{C}$ with generating cofibrations $F_{\mathcal{O}} I$ and generating trivial cofibrations $F_{\mathcal{O}} J$. If $\mathcal{C}$ is left proper (resp. right proper), then* $\mathrm{Alg}(\mathcal{O})$ *is left proper relative to $\mathcal{C}$ (resp. right proper). If the monoid axiom holds in $\mathcal{C}$, then* $\mathrm{Alg}(\mathcal{O})$ *is a cofibrantly generated model category.*

We want to describe pushouts by free algebra maps. The following definition has its origin in [Hin2, Definitions 3.3.1 and 3.3.2].

Definition 4.4.
1. *A* doubly colored am-tree *is the same as a doubly colored n-tree except that instead of the labeling of the tails every tail is marked by either a or m.*
2. *A* proper *doubly colored* am*-tree is a doubly colored* am*-tree such that every arrow starting from an old vertex is either a tail or goes to a new vertex and every vertex with only tails as outgoing arrows is new and at least one of the outgoing tails is marked by m.*

Note that in particular a proper doubly colored am-tree has no vertices of valency 0.

Let $\mathcal{T}_{\mathrm{am}}$ be the set of isomorphism classes of doubly colored am-trees and $\mathcal{T}^p_{\mathrm{am}}$ the set of isomorphism classes of proper doubly colored am-trees. For $T \in \mathcal{T}_{\mathrm{am}}$ let $a(T)$ be the set of tails of T marked by a and $m(T)$ the set of tails of T marked by m.

Let $T \in \mathcal{T}^p_{\mathrm{am}}$. Similarly as in the case of operads there is the operation of changing a new vertex v of T into an old vertex and also of changing a tail marked by m into a tail marked by a. Denote the resulting trees in $\mathcal{T}^p_{\mathrm{am}}$ by $\mathrm{ch}_T(v)$ for $v \in V_{\mathrm{new}}(T)$ and by $\mathrm{ch}_T(t)$ for $t \in m(T)$. For $\mathcal{O} \in \mathrm{Op}(\mathcal{C})$ and $A \in \mathrm{Alg}(\mathcal{O})$ there is as in the operad case a concatenation map

$$\mathrm{conc}^{\mathcal{O}}_T(v):\ \mathcal{O}(\mathrm{val}(v)) \otimes \bigotimes_{v' \in V_{\mathrm{old}}(T)} \mathcal{O}(\mathrm{val}(v')) \otimes A^{\otimes(a(T))} \longrightarrow \bigotimes_{v' \in V_{\mathrm{old}}(\mathrm{ch}_T(v))} \mathcal{O}(\mathrm{val}(v')) \otimes A^{\otimes(a(\mathrm{ch}_T(v)))}$$

induced by the operad maps of $\mathcal{O}$ and the structure maps of A. There is also a concatenation map

$$\mathrm{conc}^{\mathcal{O},A}_T(t):\ A \otimes \bigotimes_{v \in V(T)} \mathcal{O}(\mathrm{val}(v)) \otimes A^{\otimes(a(T))} \longrightarrow \bigotimes_{v \in V(\mathrm{ch}_T(t))} \mathcal{O}(\mathrm{val}(v)) \otimes A^{\otimes(a(\mathrm{ch}_T(t)))}$$

induced by the structure maps of the algebra A.

Proposition 4.5. *Let $\mathcal{O} \in \mathrm{Op}(\mathcal{C})$ and $f:\ X \to Y$ and $\varphi:\ X \to \mathcal{O}^\sharp$ be maps in $\mathcal{C}^{\mathbb{N}}$. Let $\mathcal{O}'$ be the pushout of $\mathcal{O}$ by Ff with attaching map the adjoint of φ. Let A be an $\mathcal{O}'$-algebra and let $g:\ M \to N$ and $\psi:\ M \to A^\sharp$ be maps in $\mathcal{C}$. Let B be the pushout of A as $\mathcal{O}$-algebra by $F_{\mathcal{O}}(g)$ with attaching map the adjoint of ψ and B' the pushout of A as $\mathcal{O}'$-algebra by $F_{\mathcal{O}'}(g)$. Then the canonical map $h:\ B \to B'$ is given by an $\omega \times \omega \times (\omega+1)$-sequence $B' = \mathrm{colim}_{(i,j,k)} B_{(i,j,k)}$, where for (i,j,k) a successor $B_{(i,j,k)}$ is a pushout of $B_{(i,j,k)-1}$ by the quotient of the map*

$$\coprod \left(\bigotimes_{v \in V_{\mathrm{old}}(T) \setminus U_{\mathrm{old}}(T)} \mathcal{O}(\mathrm{val}(v)) \right) \otimes A^{\otimes(a(T))} \otimes e^{\square(U_{\mathrm{old}}(T))} \square g^{\square(m(T))} \square \mathop{\square}_{v \in V_{\mathrm{new}}(T)} f(\mathrm{val}(v)),$$

where the coproduct is over all $T \in \mathcal{T}^p_{\mathrm{am}}$ with $\sharp V_{\mathrm{new}}(T) = i$, $\sharp m(T) = j$ and $u_{\mathrm{old}}(T) = k$, with respect to the equivalence relation which identifies for every isomorphism of directed graphs $\varphi:\ T \to T'$, $T, T' \in \mathcal{T}^p_{\mathrm{am}}$, which respects the labeling of the tails and of the arrows which start at new vertices, the summands corresponding to T and T' by a map which is described on the $\otimes$-part of the summands involving vertices from $V_{\mathrm{old}}(T) \setminus U_{\mathrm{old}}(T)$ as in Proposition 3.4.3 and on the other parts by the identification of the indexing sets via φ. The attaching map is induced on the different parts of the domain of the above map by either the operation of removing a vertex of valency one, by changing a new vertex into an old vertex or by changing a tail labelled by m into a tail labelled by a and then by applying either a unit map, a map $\mathrm{conc}_T(v)$ or a map $\mathrm{conc}_T(t)$.

Proof. We have to do the same steps as in the proof of Proposition 3.5. Let C be the colimit described in the Proposition. The attaching maps are again well-defined because the various concatenation processes commute with each other and because of the symmetry properties of $\mathcal{O}$ and the equivalence relations appearing in previous steps.

We equip C with an $\mathcal{O}'$-algebra structure: Let us define the structure map $\mathcal{O}'(n) \otimes C^{\otimes n} \to C$. For $T \in \mathcal{T}^p_{\mathrm{dc}}(n)$ let $S(T)$ be as in the proof of Proposition 3.5. For $T \in \mathcal{T}^p_{\mathrm{am}}$ let

$$S^a(T) := \left(\bigotimes_{v \in V_{\mathrm{old}}(T)} \mathcal{O}(\mathrm{val}(v)) \right) \otimes A^{\otimes(a(T))} \otimes N^{\otimes(m(T))} \otimes \bigotimes_{v \in V_{\mathrm{new}}(T)} Y(\mathrm{val}(v)) \,.$$

Let $T \in \mathcal{T}^p_{\mathrm{dc}}(n)$ and $T_i \in \mathcal{T}^p_{\mathrm{am}}$, $i = 1, \dots, n$. We obtain a tree $\widetilde{T} \in \mathcal{T}^p_{\mathrm{am}}$ by glueing T_i to the tail of T labelled by i and then concatenating. By applying operad and algebra structure maps we get a map $S(T) \otimes S^a(T_1) \otimes \cdots \otimes S^a(T_n) \to S^a(\widetilde{T})$. It is then possible by similar considerations as in the proof of Proposition 3.5 to get from these maps the desired structure map of C. It is easy to see that these structure maps are associative and symmetric. Hence C is an $\mathcal{O}'$-algebra which receives an $\mathcal{O}$-algebra map from B and $\mathcal{O}'$-algebra maps from A and $F_{\mathcal{O}'}(N)$ which are compatible with each other in the obvious way.

We have to check that for an $\mathcal{O}'$-algebra D a map $c : C \to D$ is the same as a map of $\mathcal{O}'$-algebras $a : A \to D$ and a map $n : N \to A^\sharp$ which are compatible with each other. We get the maps a and n from c by the obvious compositions. Given a and n we first obtain a map of $\mathcal{O}$-algebras $B \to D$. Moreover for any $T \in \mathcal{T}^p_{\mathrm{am}}$ there is a map $S^a(T) \to D$ by applying the $\mathcal{O}'$-algebra structure maps of D. It is then easy to check that these maps glue together to give the map c. These processes are invers to each other. □

Lemma 4.6. *Let the notation be as in the Proposition above. If $\mathcal{O}$ is cofibrant as an object in $\mathcal{C}^{\Sigma,\bullet}$, A is cofibrant as an object in $\mathcal{C}$, f is a cofibration in $\mathcal{C}^{\mathbb{N}}$ and g is a cofibration in $\mathcal{C}$ then the map $h : B \to B'$ is a cofibration in $\mathcal{C}$. If f or g is a trivial cofibration then so is h. If f or g is a trivial cofibration and A is arbitrary, then h lies in $(\mathcal{C} \otimes J)$-*cof*, hence is a weak equivalence if the monoid axiom holds in $\mathcal{C}$.*

Proof. Let $\sim$ be the equivalence relation on $\mathcal{T}^p_{\mathrm{am}}$ which identifies T and T' in $\mathcal{T}^p_{\mathrm{am}}$ if there is an isomorphism of directed graphs $T \to T'$ which respects the labeling of the tails and of the arrows starting at new vertices. Let C be an equivalence class of $\sim$ in $\mathcal{T}^p_{\mathrm{am}}$. We have to show that the appropriate quotient of the map

$$\coprod_{T \in C} \left(\bigotimes_{v \in V_{\mathrm{old}}(T) \setminus U_{\mathrm{old}}(T)} \mathcal{O}(\mathrm{val}(v)) \right) \otimes A^{\otimes(a(T))} \otimes e^{\square(U_{\mathrm{old}}(T))} \square g^{\square(m(T))} \square \mathop{\square}_{v \in V_{\mathrm{new}}(T)} f(\mathrm{val}(v))$$

is a (trivial) cofibration in $\mathcal{C}$ (or lies in $(\mathcal{C} \otimes J)$-cof under the assumptions of the last statement). This is done as in the proof of Lemma 3.6 by induction on the depth of the trees in C. This time instead of using Lemma 3.7 it is sufficient to use Lemma 2.11 applied to rings of the form $\mathbb{1}[\prod_{i=1}^k \Sigma_{n_i}]$. □

Proof of Theorem 4.3. We apply Theorem 2.9 to the monad $\mathbb{T}_{\mathcal{O}}$ which maps X to $(F_{\mathcal{O}} X)^\sharp$. It is known that $\mathrm{Alg}(\mathcal{O})$ is cocomplete. Since filtered colimits in $\mathrm{Alg}(\mathcal{O})$ are computed in $\mathcal{C}$ we are reduced to show that the pushout of an $\mathcal{O}$-algebra A which is cofibrant as an object in $\mathcal{C}$ by a map in $F_{\mathcal{O}} I$ (resp. in $F_{\mathcal{O}} J$) is a cofibration (resp. trivial cofibration) in $\mathcal{C}$. Since $\mathcal{O}$ is a retract of a cell operad (i.e. a cell complex in $\mathrm{Op}(\mathcal{C})$) such a pushout is a retract of a pushout of the same kind with the additional hypothesis that $\mathcal{O}$ is a cell operad. So let $\mathcal{O}$ be a cell operad. Then the pushout in question is a transfinite composition of maps h as in Proposition 4.5, hence by Lemma 4.6 it is a (trivial) cofibration.

It is clear that $\mathrm{Alg}(\mathcal{O})$ is right proper if $\mathcal{C}$ is. The pushout in $\mathrm{Alg}(\mathcal{O})$ by a cofibration whose domain is cofibrant in $\mathcal{C}$ is a retract of a transfinite composition of pushouts by cofibrations in $\mathcal{C}$, hence if $\mathcal{C}$ is left proper weak equivalences are preserved by these pushouts, so $\mathrm{Alg}(\mathcal{C})$ is also left proper.

The last statement follows again from Lemma 4.6. □

The second result concerning algebras is

Theorem 4.7. *Let $\mathcal{O}$ be an operad in $\mathcal{C}$ which is cofibrant as an object in $\mathcal{C}^\Sigma$. Then* $\mathrm{Alg}(\mathcal{O})$ *is a cofibrantly generated J-semi model category with generating cofibrations $F_\mathcal{O}I$ and generating trivial cofibrations $F_\mathcal{O}J$. If $\mathcal{C}$ is right proper, so is* $\mathrm{Alg}(\mathcal{O})$.

The next result enables one to control pushouts of cofibrant algebras by free algebra maps.

For an ordinal λ denote by S_λ the set of all maps $f:\ \lambda \to \frac{1}{2}\mathbb{N}$ such that $f(i)$ is $\neq 0$ only for finitely many $i<\lambda$, if $f(i)\notin\mathbb{N}$ then $i>0$ and $f(i')=0$ for all $i'<i$ and if λ is a successor then $f(\lambda-1)=0$. For $f,f'\in S_\lambda$ say that $f<f'$ if there is an $i<\lambda$ such that $f(i')=f'(i')$ for all $i'>i$ and $f(i)<f'(i)$. With this ordering S_λ is well-ordered. For $i<\lambda$ denote by f_i the element of S_λ with $f_i(i)=\frac{1}{2}$ and $f_i(i')=0$ for $i'\neq i$. Set $S_{\lambda,+}:=S_\lambda\sqcup\{*\}$, where $*$ is by definition smaller than any other element in $S_{\lambda,+}$. Note that $f\in S_{\lambda,+}$ is a successor if and only if $f\neq *$ and $f(\lambda)\subset\mathbb{N}$. For $f\in S_{\lambda,+}$ a successor let $|f|:=\sum_{i<\lambda}f(i)\in\mathbb{N}$ and $\Sigma_f:=\prod_{i<\lambda}\Sigma_{f(i)}$.

Proposition 4.8. *Let $\mathcal{O}\in Op(\mathcal{C})$ and $A=\mathrm{colim}_{i<\lambda}A_i$ be a $F_\mathcal{O}(\mathrm{Mor}(\mathcal{C}))$*-cell *$\mathcal{O}$-algebra ($\mathrm{Mor}(\mathcal{C})$ is the class of all morphisms in $\mathcal{C}$) with $A_0=\mathcal{O}(0)$, where the transition maps $A_i\to A_{i+1}$ are pushouts of free $\mathcal{O}$-algebra maps on maps $g_i:\ K_i\to L_i$ in $\mathcal{C}$ by maps adjoint to $\varphi_i:\ K_i\to A_i^\sharp$. Then A is a transfinite composition $A=\mathrm{colim}_{f\in S_{\lambda,+}}A_f$ in $\mathcal{C}$ such that*

1. *$A_*=0$ and $A_{f_i}=A_i$ for $i<\lambda$,*
2. *for $f\in S_\lambda$ such that for an $i_0<\lambda$ we have $f(i_0)\notin\mathbb{N}$, there is for all $m\in\mathbb{N}$, successors $l\in S_{\lambda,+}$ with $l<f$ and $n:=m+|l|$ a map*

$$\Psi_{f,m,l}:\ \mathcal{O}(n)\otimes_{(\Sigma_m\times\Sigma_l)}\left(A_{i_0}^{\otimes m}\otimes\bigotimes_{i<\lambda}L_i^{\otimes l(i)}\right)\to A_f$$

   compatible with the structure map $\mathcal{O}(n)\otimes_{\Sigma_n}A^{\otimes n}\to A$. By applying permutations to $\mathcal{O}(n)$ and the big bracket there are similar maps for other orders of the factors in the big bracket. These maps satisfy the following conditions:
   (a) *They are compatible with the maps $L_i\to A_{i_0}$ for $i<i_0$. Moreover, if we replace a factor L_{i_0} by K_{i_0} we can either go to L_{i_0} or to A_{i_0} and apply suitable maps Ψ. Then the two compositions coincide.*
   (b) *They are associative in the following sense: Let $f_1,\dots,f_k\in S_\lambda$ be limit elements with $f_i<f$, $i=1,\dots,k$, and let for each f_i be given*

m_i, l_i and n_i satisfying the same conditions as m, l and n for f. Let D_i be the domain of Ψ_{f_i,m_i,l_i}. Then the two possible ways to get from

$$\mathcal{O}(n) \otimes \left(\bigotimes_{i=1}^{k} D_i \right) \otimes A_{i_0}^{\otimes m} \otimes \bigotimes_{i<\lambda} L_i^{\otimes l(i)}$$

to A_f given by either applying the Ψ_{f_i,m_i,l_i} and then $\Psi_{f,m+k,l}$ or by applying the obvious operad structure maps and a suitable permutation of $\Psi_{f,m+\sum_{i=1}^{k} m_i,\, l+\sum_{i=1}^{k} l_i}$ coincide.

(3) *For any successor $f \in S_{\lambda,+}$ the map $A_{f-1} \to A_f$ is a pushout by*

$$\mathcal{O}(|f|) \otimes_{\Sigma_f} \square_{i<\lambda}\, g_i^{\square f(i)} ,$$

where the attaching maps on the various parts of the domain of this map are induced from the maps in (2) (see below).

Proof. The whole Proposition is shown by induction on λ, so suppose that it is true for ordinals smaller than λ. We construct the map in 2, prove its properties and define the attaching map in 3 by transfinite induction: Suppose $f \in S_{\lambda,+}$ is a successor, that $A_{f'}$ is defined for $f' < f$ and that the map in 2 is defined for all limit elements $\tilde{f} \in S_\lambda$ with $\tilde{f} < f$. Let $i_0 \in \lambda$ with $f(i_0) > 0$ and let f' coincide with f except that $f'(i_0) = f(i_0) - 1$. The attaching map on the summand

$$S := \mathcal{O}(|f|) \otimes_{\Sigma_{f'}} \left(\left(\bigotimes_{i<i_0} L_i^{\otimes f(i)} \right) \otimes K_{i_0} \otimes L_{i_0}^{\otimes (f(i_0)-1)} \otimes \bigotimes_{i_0<i<\lambda} L_i^{\otimes f(i)} \right)$$

of the domain of

$$\mathcal{O}(|f|) \otimes_{\Sigma_f} \square_{i<\lambda}\, g_i^{\square f(i)}$$

is given as follows: Let $\tilde{f}, l \in S_\lambda$ be defined by $\tilde{f}(i_0) = f(i_0) - \frac{1}{2}$, $l(i_0) = f(i_0) - 1$, $\tilde{f}(i) = l(i) = 0$ for $i < i_0$ and $\tilde{f}(i) = l(i) = f(i)$ for $i > i_0$. Let $m := 1 + \sum_{i<i_0} f(i)$. There is a canonical map

$$S \to \mathcal{O}(|f|) \otimes_{\Sigma_{f'}} \left(A_{i_0}^{\otimes (m-1)} \otimes A_{i_0} \otimes L_{i_0}^{\otimes (f(i_0)-1)} \otimes \bigotimes_{i_0<i<\lambda} L_i^{\otimes f(i)} \right)$$

whose codomain maps naturally to the domain of $\Psi_{\tilde{f},m,l}$. So we get maps $S \to A_{\tilde{f}} \to A_{f-1}$ the composition of which is the attaching map on the summand S. These maps glue together for various summands S: There are two cases to distinguish. In the first one the intersection of two summands contains K_{i_0} twice. Then the two maps on this intersection coincide because of the symmetric group invariance. In the second case the intersection I contains $K_{i_0'}$ and K_{i_0} with $i_0' < i_0$. Let $\tilde{f}$ be as above and $\tilde{f}'$ be similarly defined for i_0'. Now the two properties 2(a) of the maps Ψ state that both maps $I \to A_f$ are equal the map induced by first mapping both $K_{i_0'}$ and K_{i_0} to A_{i_0} and then applying a suitable map Ψ.

Now suppose $f \in S_\lambda$ is a limit element with $f(i_0) \notin \mathbb{N}$ for some $i_0 < \lambda$. Define A_f as the colimit of the preceeding $A_{f'}$, $f' < f$. Let m, l and n be as in 2. We define $\Psi_{f,m,l}$ by induction on m and on S_{i_0} using the fact that $A_{i_0} = \operatorname{colim}_{f' \in S_{i_0}} A_{f'}$ by induction hypothesis for the induction on λ. For abbreviation set $\mathcal{L} := \bigotimes_{i<\lambda} L_i^{\otimes l(i)}$. Let $f' \in S_{i_0}$ be a successor and let a map

$$\psi_{f'-1} :\ \mathcal{O}(n) \otimes \left(A_{i_0}^{\otimes (m-1)} \otimes A_{f'-1} \otimes \mathcal{L} \right) \to A_f$$

be already defined. $A_{f'}$ is a pushout of $A_{f'-1}$ by

$$\varphi:\ \mathcal{O}(|f'|)\otimes_{\Sigma_{f'}}\square_{i<i_0} g_i^{\square f'(i)}\ .$$

Let $C := \bigotimes_{i<i_0} L_i^{\otimes f'(i)}$. Then the codomain of φ is $\mathcal{O}(|f'|)\otimes_{\Sigma_{f'}} C$. Moreover by induction hypothesis for the m-induction there is a map

$$\mathcal{O}(n+|f'|-1)\otimes\left(A_{i_0}^{\otimes(m-1)}\otimes C\otimes\mathcal{L}\right)\to A_f\ ,$$

hence by plugging in $\mathcal{O}(|f'|)$ into the m-th place of $\mathcal{O}(n)$ we get a map

$$\mathcal{O}(n)\otimes\left(A_{i_0}^{\otimes(m-1)}\otimes\mathcal{O}(|f'|)\otimes C\otimes\mathcal{L}\right)\to A_f\ .$$

This map and $\psi_{f'-1}$ glue together to a map $\psi_{f'}$: We have to show that they coincide after composition on domains of the form

$$\mathcal{O}(n)\otimes A_{i_0}^{\otimes(m-1)}\otimes\mathcal{O}(|f'|)\otimes_{\Sigma_{f''}} S'\otimes\mathcal{L}$$

for $\mathcal{O}(|f'|)\otimes_{\Sigma_{f''}} S'$ a summand of the domain of φ containing $K_{i_0'}$ for some $i_0'<i_0$ (the definition of f'' is similar to the one of f'). To do this we can restrict for every A_{i_0} to objects $\mathcal{O}(|f_i'|)\otimes_{\Sigma_{f_i'}} C_i$, $i=1,\dots,m-1$, for C_i of the same shape as C and $f_i'\in S_{i_0,+}$ successors. Then the two possible ways to get from

$$\mathcal{O}(n)\otimes\left(\bigotimes_{i=1}^{m-1}\mathcal{O}(|f_i'|)\otimes_{\Sigma_{f_i'}} C_i\right)\otimes\mathcal{O}(|f'|)\otimes_{\Sigma_{f''}} S'\otimes\mathcal{L}$$

to A_f can be compared by mapping $K_{i_0'}$ to $A_{i_0'}$, unwrapping the definitions of A_f and $\Psi_{f'-1}$ and using associativity of $\mathcal{O}$. We arrive at a map $\mathcal{O}(n)\otimes A_{i_0}^{\otimes m}\otimes\mathcal{L}\to A_f$. That it factors through the $(\Sigma_m\times\Sigma_l)$-quotient follows after replacing $A_{i_0}^{\otimes m}$ by $\left(\bigoplus_{i=1}^k\mathcal{O}(|f_i'|)\otimes_{\Sigma_{f_i'}} C_i\right)^{\otimes m}$ (the C_i and f_i' as above) in the domain of this map since then the $(\Sigma_m\times\Sigma_l)$-relation is obviously also valid in A_f.

Both properties 2(a) and (b) follow easily by the technique of restricting any appearing A_i by a factor $\mathcal{O}(|f'|)\otimes_{\Sigma_{f'}} C$.

Now using the maps Ψ and property 2(b) we can equip $\widetilde{A} := \operatorname{colim}_{f\in S_{\lambda,+}}$ with an $\mathcal{O}$-algebra structure (to do this accurately we have to enlarge λ a bit and the corresponding sequence by trivial pushouts).

We are left to prove the universal property for $\widetilde{A}$ by transfinite induction on λ. So let it be true for ordinals less than λ. If λ is a limit ordinal or the successor of a limit ordinal there is nothing to show. Let $\lambda=\alpha+2$, let B be an $\mathcal{O}$-algebra and $A_\alpha\to B$ a map in $\mathrm{Alg}(\mathcal{O})$ and $L_\alpha\to B^\sharp$ a map in $\mathcal{C}$ such that these two maps are compatible via the attaching map. We define maps $A_f\to B$ by transfinite induction on $S_{\lambda,+}$, starting with the given map on $A_{f_\alpha}=A_\alpha$. So let $f_\alpha<f<\lambda$ be a successor. Since for any $i\le\alpha$ there is a map $L_i\to B^\sharp$ we have a natural map

$$\mathcal{O}(|f|)\otimes_{\Sigma_f}\bigotimes_{i<\lambda} L_i^{\otimes f(i)}\to B$$

using the algebra structure maps of B. We have to show that this is compatible via the attaching map from the domain D of $\mathcal{O}(|f|)\otimes_{\Sigma_f}\square_{i<\lambda} g_i^{\square f(i)}$ to A_{f-1} with the map $A_{f-1}\to B$ coming from the induction hypothesis. We check this again on a summand S of D containing some K_{i_0}. The attaching map on S is induced from

$\Psi_{\tilde{f},m,l}$ as above. The canonical map from the domain of $\Psi_{\tilde{f},m,l}$ to B is compatible with $A_{\tilde{f}} \to B$ (as one checks again by replacing any A_{i_0} by essentially products of L_i's as above), which together with the fact that $L_{i_0} \to B$ and $A_{i_0+1} \to B$ coincide on K_{i_0} implies the compatibility. By construction and the definition of the algebra structure on $A_{\alpha+1}$ the map $A_{\alpha+1} \to B$ just defined is an $\mathcal{O}$-algebra map.

If we have on the other hand a map of $\mathcal{O}$-algebras $A_{\alpha+1} \to B$ we can restrict it to get compatible maps $A_\alpha \to B$ and $L_\alpha \to B^\sharp$. These two assignments are inverse to each other. □

Proof of Theorem 4.7. Let $\mathcal{O} \in \mathrm{Op}(\mathcal{C})$ be cofibrant in $\mathcal{C}^\Sigma$. We have to show that the pushout of an $\mathcal{O}$-algebra such that the map from the initial $\mathcal{O}$-algebra to A is in $F_\mathcal{O}I$-cof by a map from $F_\mathcal{O}I$ (resp. $F_\mathcal{O}J$) is a cofibration (resp. trivial cofibration) in $\mathcal{C}$. We can assume that A is a $F_\mathcal{O}I$-cell $\mathcal{O}$-algebra, since in the general situation all maps we look at are retracts of corresponding maps in this situation. But if A is a cell $\mathcal{O}$-algebra our claim immediately follows from Proposition 4.8 and Lemma 2.11. □

5. Module structures

In this section we want to show that if $\mathcal{C}$ is simplicial $\mathrm{Alg}(\mathcal{O})$ is also a simplicial J-semi model category in the cases when the assumptions of Proposition 4.3 or Proposition 4.7 are fulfilled. Also $\mathrm{Op}(\mathcal{C})$ is simplicial if $\mathcal{C}$ is.

Definition 5.1. *Let $\mathcal{D}$ and $\mathcal{E}$ be J-semi model categories (maybe over $\mathcal{C}$) and let $\mathcal{S}$ be a model category. Then a* Quillen bifunctor $\mathcal{D} \times \mathcal{S} \to \mathcal{E}$ *is an adjunction of two variables $\mathcal{D} \times \mathcal{S} \to \mathcal{E}$ such that for any cofibration $g: K \to L$ in $\mathcal{S}$ and fibration $p: Y \to Z$ in $\mathcal{E}$, the induced map*

$$\mathrm{Hom}_{r,\square}(g,p):\ \mathrm{Hom}_r(L,Y) \to \mathrm{Hom}_r(L,Z) \times_{\mathrm{Hom}_r(K,Z)} \mathrm{Hom}_r(K,Y)$$

is a fibration in $\mathcal{D}$ which is trivial if g or p is.

(See also [Hov1, Lemma 4.2.2]*.)*

It follows that for f a cofibration in $\mathcal{D}$ and g a cofibration in $\mathcal{S}$ both of which have cofibrant domains the pushout $f\square g$ is a cofibration in $\mathcal{E}$ which is trivial if f or g is.

Definition 5.2. *Let $\mathcal{D}$ be a J-semi model category (maybe over $\mathcal{C}$) and let $\mathcal{S}$ be a symmetric monoidal model category. Then a* Quillen $\mathcal{S}$-module structure *on $\mathcal{D}$ is a $\mathcal{S}$-module structure on $\mathcal{D}$ such that the action map $\otimes:\ \mathcal{D} \times \mathcal{S} \to \mathcal{D}$ is a Quillen bifunctor and the map $X \otimes (QS) \to X \otimes S \cong X$ is a weak equivalence for all cofibrant $X \in \mathcal{D}$, where $QS \to S$ is a cofibrant replacement.*

If $\mathcal{D}$ has a Quillen $\mathcal{S}$-module structure we say that $\mathcal{D}$ is an $\mathcal{S}$-module.

Let now $\mathcal{S}$ be a symmetric monoidal model category where the tensor product is the categorical product on $\mathcal{S}$, so let us denote this by $\times$ (e.g. $\mathcal{S} = \mathbf{SSet}$). Let be given a symmetric monoidal left Quillen functor $\mathcal{S} \to \mathcal{C}$.

Proposition 5.3. *Let the situation be as above and assume that either $\mathbb{1}$ is cofibrant in $\mathcal{S}$ or that $\mathcal{C}$ is left proper and the maps in I have cofibrant domains. Let $\mathcal{O}$ be an operad in $\mathcal{C}$ which is either cofibrant in $Op(\mathcal{C})$ or cofibrant as an object in $\mathcal{C}^\Sigma$. Then the J-semi model category (in the first case over $\mathcal{C}$) $\mathrm{Alg}(\mathcal{O})$ is naturally an $\mathcal{S}$-module and the functor $\mathcal{C} \to \mathrm{Alg}(\mathcal{C})$ is an $\mathcal{S}$-module homomorphism.*

Proof. Let $A \in \mathcal{C}$ and $K \in \mathcal{S}$. We denote by A^K the homomorphism object $\underline{\mathrm{Hom}}(K, A) \in \mathcal{C}$. There is a map of operads

$$\mathrm{End}^{\mathrm{Op}}(A) \to \mathrm{End}^{\mathrm{Op}}(A^K) ,$$

which is described as follows: We give the maps

$$\underline{\mathrm{Hom}}(A^{\otimes n}, A) \to \underline{\mathrm{Hom}}((A^K)^{\otimes n}, A^K)$$

on T-valued points ($T \in \mathcal{C}$): A map $T \otimes A^{\otimes n} \to A$ is sent to the composition

$$T \otimes (A^K)^{\otimes n} \to T \otimes (A^{\otimes n})^{K^n} \to T \otimes (A^{\otimes n})^K \to A^K ,$$

where the second map is induced by the diagonal $K \to K^n$.

Hence for objects $K \in \mathcal{S}$ and $A \in \mathrm{Alg}(\mathcal{O})$ the object $(A^\sharp)^K$ has a natural structure of $\mathcal{O}$-algebra given by the composition $\mathcal{O} \to \mathrm{End}^{\mathrm{Op}}(A) \to \mathrm{End}^{\mathrm{Op}}(A^K)$. We denote this $\mathcal{O}$-algebra by A^K.

For a fixed $K \in \mathcal{S}$ the functor $\mathrm{Alg}(\mathcal{O}) \to \mathrm{Alg}(\mathcal{O})$, $A \mapsto A^K$, has a left adjoint $A \mapsto A \otimes K$, which is given for a free $\mathcal{O}$-algebra $F_{\mathcal{O}}(X)$, $X \in \mathcal{C}$, by $F_{\mathcal{O}}(X) \otimes K = F_{\mathcal{O}}(X \otimes K)$ and which is defined in general by be requirement that $\_ \otimes K$ respects coequalizers (note that every $\mathcal{O}$-algebra is a coequalizer of a diagram where only free $\mathcal{O}$-algebras appear). So we have a functor $\mathrm{Alg}(\mathcal{O}) \times \mathcal{S} \to \mathrm{Alg}(\mathcal{O})$.

Let now $B \in \mathrm{Alg}(\mathcal{O})$ be fixed. By a similar argument as above the functor $\mathcal{S}^{\mathrm{op}} \to \mathrm{Alg}(\mathcal{O})$, $K \mapsto B^K$, has a left adjoint $A \mapsto \underline{\mathrm{Hom}}^{\mathcal{S}}(A, B)$, which sends a free $\mathcal{O}$-algebra $F_{\mathcal{O}}(X)$, $X \in \mathcal{C}$, to the image of $\underline{\mathrm{Hom}}(X, B^\sharp)$ in $\mathcal{S}$.

One checks that the functor $\mathrm{Alg}(\mathcal{O}) \times \mathcal{S} \to \mathrm{Alg}(\mathcal{O})$ we constructed defines an action of $\mathcal{S}$ on $\mathrm{Alg}(\mathcal{O})$.

It remains to show that this functor is a Quillen bifunctor and that the unit property is fulfilled. So let $g: K \to L$ be a cofibration in $\mathcal{S}$ and $p: Y \to Z$ a fibration in $\mathrm{Alg}(\mathcal{O})$. We have to show that $\mathrm{Hom}_{\Box,r}(g, p)$ is a fibration in $\mathrm{Alg}(\mathcal{O})$, i.e. lies in $F_{\mathcal{O}}J$-inj. By adjointness this means that p has the right lifting property with respect to the maps $(F_{\mathcal{O}}f)\Box g = F_{\mathcal{O}}(f \Box g)$ for all $f \in J$, which is by adjointness the case because $f \Box g$ is a trivial cofibration. When p or f is trivial we want to show that $\mathrm{Hom}_{\Box,r}(g, p)$ lies in $F_{\mathcal{O}}I$-inj, so p should have the right lifting property with respect to the maps $F_{\mathcal{O}}(f \Box g)$ for all $f \in I$, which is again the case by adjointness.

If $\mathbb{1}$ is cofibrant in $\mathcal{S}$ we are ready. In the other case the unit property follows by transfinite induction from the explicit description of algebra pushouts, and hence the structure of cell algebras, given in Proposition 4.5 and the structure of cell algebras given in Proposition 4.8. □

In a similar manner one shows

Proposition 5.4. *Let the situation be as before Proposition 5.3 and assume that either $\mathbb{1}$ is cofibrant in $\mathcal{S}$ or that $\mathcal{C}$ is left proper and the maps in I have cofibrant domains. Then $\mathrm{Op}(\mathcal{C})$ is naturally an $\mathcal{S}$-module and the functor $\mathcal{C} \to \mathrm{Op}(\mathcal{C})$ is an $\mathcal{S}$-module homomorphism.*

6. Modules

Let $\mathcal{O} \in \mathrm{Op}(\mathcal{C})$ and $A \in \mathrm{Alg}(\mathcal{O})$. We denote the category of A-modules by $(\mathcal{O}, A)$–*Mod*, or A–*Mod* if no confusion is likely. Let $F_{(\mathcal{O},A)} : \mathcal{C} \to A\text{–}Mod$ (or F_A for short) be the free A-module functor. It is given by $M \mapsto U_{\mathcal{O}}(A) \otimes M$, where $U_{\mathcal{O}}(A)$ is the universal enveloping algebra of the $\mathcal{O}$-algebra A. Recall that $\mathrm{Ass}(\mathcal{C})$ denotes the category of associative unital algebras in $\mathcal{C}$, and let F_{Ass} be the free associative algebra functor $\mathcal{C} \to \mathrm{Ass}(\mathcal{C})$.

The main result of this section is

Theorem 6.1. *Let $\mathcal{O} \in \mathrm{Op}(\mathcal{C})$ and $A \in \mathrm{Alg}(\mathcal{O})$. Let one of the following two conditions be satisfied:*

1. *$\mathcal{O}$ is cofibrant as an object in $\mathcal{C}^{\Sigma}$ and A is a cofibrant $\mathcal{O}$-algebra.*
2. *$\mathcal{O}$ is cofibrant in $\mathrm{Op}(\mathcal{C})$ and A is cofibrant as an object in $\mathcal{C}$.*

Then there is cofibrantly generated model structure on A–Mod with generating cofibrations $F_A I$ and generating trivial cofibrations $F_A J$. There is a right $\mathcal{C}$-module structure on A–Mod.

This theorem will follow from the fact that in each of the two cases the enveloping algebra $U_{\mathcal{O}}(A)$ is cofibrant in $\mathcal{C}$, since A–*Mod* is canonically equivalent to $U_{\mathcal{O}}(A)$–*Mod*.

Note that there is a canonical surjection from the tensor algebra to the universal enveloping algebra

$$T_{\mathcal{O}}(A) := \coprod_{n \in \mathbb{N}} \mathcal{O}(n+1) \otimes_{\Sigma_n} A^{\otimes n} \to U_{\mathcal{O}}(A) \ .$$

Proposition 6.2. *Let $\mathcal{O} \in \mathrm{Op}(\mathcal{C})$ and $f : X \to Y$ and $\varphi : X \to \mathcal{O}^{\sharp}$ be maps in $\mathcal{C}^{\mathbb{N}}$. Let $\mathcal{O}'$ be the pushout of $\mathcal{O}$ by f with attaching map the adjoint of φ. Let A be an $\mathcal{O}'$-algebra. Then $U_{\mathcal{O}'}(A)$ is a pushout of $U_{\mathcal{O}}(A)$ in $\mathrm{Ass}(\mathcal{C})$ by the map $F_{\mathrm{Ass}}(\coprod_{n\in\mathbb{N}} f(n+1) \otimes A^{\otimes n})$ with attaching map the adjoint to the composition $\coprod_{n\in\mathbb{N}} X(n+1) \otimes A^{\otimes n} \to \coprod_{n\in\mathbb{N}} \mathcal{O}(n+1) \otimes_{\Sigma_n} A^{\otimes n} \to U_{\mathcal{O}}(A)$.*

Proof. (Compare to [Hin1, 6.8.1. Lemma.]) A $(\mathcal{O}', A)$-module structure on a $(\mathcal{O}, A)$-module M is given by maps $Y(n+1) \otimes A^{\otimes n} \otimes M \to M$ for $n \in \mathbb{N}$ such that the compositions with $f(n+1) \otimes A^{\otimes n} \otimes M$ equals the composition $X(n+1) \otimes A^{\otimes n} \otimes M \to \mathcal{O}(n+1) \otimes A^{\otimes n} \otimes M \to M$. The same statement is true for a module structure under the described pushout algebra on a $U_{\mathcal{O}}(A)$-module. □

Corollary 6.3. *Let $\mathcal{O} \in \mathrm{Op}(\mathcal{C})$ be cofibrant and let A be an $\mathcal{O}$-algebra which is cofibrant as an abject in $\mathcal{C}$. Then $U_{\mathcal{O}}(A)$ is cofibrant in $\mathrm{Ass}(\mathcal{C})$, in particular is cofibrant as an object in $\mathcal{C}$.*

Hence the second part of Theorem 6.1 is proven.

Corollary 6.4. *Let $\mathcal{C}$ be left proper, let $\mathcal{O} \in \mathrm{Op}(\mathcal{C})$ be cofibrant and let $A \to A'$ be a weak equivalence between $\mathcal{O}$-algebras both of which are cofibrant as objects in $\mathcal{C}$. Then the map $U_{\mathcal{O}}(A) \to U_{\mathcal{O}}(A')$ is a weak equivalence.*

We have an analogous result to Proposition 4.8 for the enveloping algebra of a cell algebra.

Proposition 6.5. *Let $\mathcal{O} \in Op(\mathcal{C})$ and $A = \mathrm{colim}_{i<\lambda} A_i$ be a $F_{\mathcal{O}}(\mathrm{Mor}(\mathcal{C}))$-cell $\mathcal{O}$-algebra with $A_0 = \mathcal{O}(0)$, where the transition maps $A_i \to A_{i+1}$ are pushouts of free $\mathcal{O}$-algebra maps on maps $g_i : K_i \to L_i$ in $\mathcal{C}$ by maps adjoint to $\varphi_i : K_i \to A_i^\sharp$. Then $U := U_{\mathcal{O}}(A)$ is a transfinite composition $U = \mathrm{colim}_{f\in S_{\lambda,+}} U_f$ in $\mathcal{C}$ such that*

1. *$U_* = 0$ and $U_{f_i} = U_{\mathcal{O}}(A_i)$ for $i < \lambda$,*
2. *for $f \in S_\lambda$ such that for an $i_0 < \lambda$ we have $f(i_0) \notin \mathbb{N}$, there is for all $m \in \mathbb{N}$, successors $l \in S_{\lambda,+}$ with $l < f$ and $n := m + |l|$ a map*

$$\mathcal{O}(n+1) \otimes_{(\Sigma_m \times \Sigma_l)} \left(A_{i_0}^{\otimes m} \otimes \bigotimes_{i<\lambda} L_i^{\otimes l(i)} \right) \to U_f$$

   compatible with the map $\mathcal{O}(n+1) \otimes_{\Sigma_n} A^{\otimes n} \to U$ and

3. *for any successor $f \in S_{\lambda,+}$ the map $U_{f-1} \to U_f$ is a pushout by*

$$\mathcal{O}(|f|+1) \otimes_{\Sigma_f} \square_{i<\lambda}\, g_i^{\square f(i)}\ ,$$

   where the attaching maps on the various parts of the domain of this map are induced from the maps in (2).

Proof. This Proposition is proven in essentially the same way as Proposition 4.8 except that this time we have to define associative algebra structures on the U_{f_i} and to verify the universal property stating the equivalence of module categories. For the associative algebra structure one uses the same formulas as for the tensor algebra and checks that they are compatible with the attaching maps. For the universal property one uses the fact that an A-module M is given by maps

$$\mathcal{O}(|f|+1) \otimes_{\Sigma_f} \left(\bigotimes_{i<\lambda} L_i^{\otimes f(i)} \right) \otimes M \to M$$

which are compatible in various ways the explicit formulation of which we leave to the reader. □

Corollary 6.6. *For $\mathcal{O}$ an operad in $\mathcal{C}$ which is cofibrant in $\mathcal{C}^\Sigma$ and A a cofibrant $\mathcal{O}$-algebra the enveloping algebra $U_{\mathcal{O}}(A)$ is cofibrant as an object in $\mathcal{C}$.*

Hence also the first part of Theorem 6.1 is proven.

Corollary 6.7. *Let $\mathcal{C}$ be left proper, let $f : \mathcal{O} \to \mathcal{O}'$ be a weak equivalence between operads in $\mathcal{C}$ both of which are cofibrant as objects in $\mathcal{C}^\Sigma$ and let A be a cofibrant $\mathcal{O}$-algebra. Let A' be the pushforward of A with respect to f. Then the induced maps $A \to A'$ and $U_{\mathcal{O}}(A) \to U_{\mathcal{O}'}(A')$ are weak equivalences.*

Definition 6.8. *Let $\mathcal{C}$ be left proper and let $\mathbb{1}$ and the domains of the maps in I be cofibrant in $\mathcal{C}$.*

1. *For $\mathcal{O} \in \mathrm{Op}(\mathcal{C})$ define the* derived category of $\mathcal{O}$-algebras $D\mathrm{Alg}(\mathcal{O})$ *to be* $\mathrm{Ho}\,\mathrm{Alg}(Q\mathcal{O})$, *where $Q\mathcal{O} \to \mathcal{O}$ is a cofibrant replacement in* $\mathrm{Op}(\mathcal{C})$. *Define the* derived 2-category of $\mathcal{O}$-algebras $D^{\leq 2}\mathrm{Alg}(\mathcal{O})$ *to be* $\mathrm{Ho}^{\leq 2}\mathrm{Alg}(Q\mathcal{O})$.
2. *For $\mathcal{O} \in \mathrm{Op}(\mathcal{C})$ and $A \in \mathrm{Alg}(\mathcal{O})$ define the* derived category of A-modules *$D(A\text{–}Mod)$ to be* $\mathrm{Ho}\,(QA\text{–}Mod)$, *where $QA \to A$ is a cofibrant replacement of A in* $\mathrm{Alg}(Q\mathcal{O})$ *with $Q\mathcal{O} \to \mathcal{O}$ a cofibrant replacement in* $\mathrm{Op}(\mathcal{C})$.

Note that these definitions do not depend (up to equivalence up to unique isomorphism or up to equivalence up to isomorphism, which is itself defined up to unique isomorphism in the case of $D^{\leq 2}\mathrm{Alg}(\mathcal{O})$) on the choices by Corollary 6.7 and [Hov2, Theorem 2.4], that if $\mathcal{O} \in \mathrm{Op}(\mathcal{C})$ is cofibrant in $\mathcal{C}^{\Sigma}$ there is a canonical equivalence $D\mathrm{Alg}(\mathcal{O}) \sim \mathrm{Ho\,Alg}(\mathcal{O})$ and that for a cofibrant $\mathcal{O} \in \mathrm{Op}(\mathcal{C})$ and $A \in \mathrm{Alg}(\mathcal{O})$ which is cofibrant in $\mathcal{C}$ there is a canonical equivalence $D(A\text{–}Mod) \sim \mathrm{Ho}\,(A\text{–}Mod)$.

7. Functoriality

In this section let $\mathcal{C}$ be left proper and let $\mathbb{1}$ and the domains of the maps in I be cofibrant in $\mathcal{C}$.

Proposition 7.1. (1) *There is a well defined* 2*-functor*

$$\mathrm{Ho}^{\leq 2}\mathrm{Op}(\mathcal{C}) \to \mathbf{Cat}\ ,$$

$$\mathcal{O} \mapsto D\mathrm{Alg}(\mathcal{O})$$

such that for any cofibrant operad $\mathcal{O}$ in $\mathcal{C}$ there is a canonical equivalence $D\mathrm{Alg}(\mathcal{O}) \sim \mathrm{Ho\,Alg}(\mathcal{O})$ and every functor in the image of this 2*-functor has a right adjoint.*

(2) *For $\mathcal{O} \in \mathrm{Op}(\mathcal{C})$ there is a well defined* 2*-functor*

$$D^{\leq 2}\mathrm{Alg}(\mathcal{O}) \to \mathbf{Cat}\ ,$$

$$A \mapsto D(A\text{–}Mod)$$

such that for any cofibrant $A \in \mathrm{Alg}(Q\mathcal{O})$ ($Q\mathcal{O} \to \mathcal{O}$ a cofibrant replacement) there is a canonical equivalence $D(A\text{–}Mod) \sim \mathrm{Ho}\,(A\text{–}Mod)$ and every functor in the image of this 2*-functor has a right adjoint.*

Remark 7.2. *The* 2*-functor in the second part of the Proposition should be well defined for an object $\mathcal{O} \in \mathrm{Ho}^{\leq 3}\mathrm{Op}(\mathcal{C})$ and should depend on $\mathcal{O}$ functorially.*

Proof. We prove the first part of the Proposition, the second one is similar. Let $\mathcal{O}, \mathcal{O}' \in \mathrm{Op}(\mathcal{C})_{cf}$, $f, g \in \mathrm{Hom}(\mathcal{O}, \mathcal{O}')$ and φ a 2-morphism from f to g in $\mathrm{Ho}^{\leq 2}\mathrm{Op}(\mathcal{C})$. First of all it is clear that the pushforward functor $f_* :\ \mathrm{Alg}(\mathcal{O}) \to \mathrm{Alg}(\mathcal{O}')$ is a left Quillen functor between J-semi model categories by the definition of the J-semi model structures. We have to show that φ induces a natural isomorphism between f_* and g_* on the level of homotopy categories. So let $\mathcal{O}^{\bullet}$ be a cosimplicial frame on $\mathcal{O}$. φ can be represented by a chain of 1-simplices in $\mathrm{Hom}(\mathcal{O}^{\bullet}, \mathcal{O}')$, and a homotopy between two representing chains by a chain of 2-simplices. So we can assume that φ is a 1-simplex, i.e. $\varphi \in \mathrm{Hom}(\mathcal{O}^1, \mathcal{O}')$. We have maps $\mathcal{O} \sqcup \mathcal{O} \xrightarrow{i_0 \sqcup i_1} \mathcal{O}^1 \xrightarrow{p} \mathcal{O}$, and $\mathrm{Ho\,Alg}(\mathcal{O}^1) \to \mathrm{Ho\,Alg}(\mathcal{O})$ is an equivalence. Hence for $A \in \mathrm{Ho\,Alg}(\mathcal{O})$ there is a unique isomorphism $\varphi'(A) :\ i_{0*}(A) \to i_{1*}(A)$ with $p_*(\varphi'(A)) = \mathrm{Id}$. Then the $\varphi(\varphi'(A))$ define a natural isomorphism between $(\varphi \circ i_0)_*$ and $(\varphi \circ i_1)_*$. Now if we have a homotopy $\Phi \in \mathrm{Hom}(\mathcal{O}^2, \mathcal{O}')$, the three natural transformations which are defined by the three 1-simplices of Φ are compatible, since on a given object they are the images in $\mathrm{Ho\,Alg}(\mathcal{O}')$ of three compatible isomorphisms between the three possible images of A in $\mathrm{Ho\,Alg}(\mathcal{O}^2)$. □

Let $f :\ \mathcal{O} \to \mathcal{O}'$ be a map of operads in $\mathcal{C}$ and let $A \in D^{\leq 2}\mathrm{Alg}(\mathcal{O})$. Then there is an adjunction

$$D(A\text{–}Mod) \rightleftarrows D(f_*A\text{–}Mod)\ .$$

It follows that for $B \in D^{\leq 2}\mathrm{Alg}(\mathcal{O}')$ there is also an adjunction

$$D(f^*B\text{–}Mod) \rightleftarrows D(B\text{–}Mod) \ .$$

Of course for A and B as above and a map $f_*A \to B$ there is a similar adjunction.

Now let $\mathcal{D}$ be a second left proper symmetric monoidal cofibrantly generated model category with suitable smallness assumptions on the domains of the generating cofibrations and trivial cofibrations (depending on which definition of J-semi model category one takes) and with a cofibrant unit. Let $L: \mathcal{C} \to \mathcal{D}$ be a symmetric monoidal left Quillen functor with right adjoint R. For objects $X, Y \in \mathcal{D}$ there is always a natural map

$$R(X) \otimes R(Y) \to R(X \otimes Y)$$

adjoint to the map

$$F(R(X) \otimes R(Y)) \cong FR(X) \otimes FR(Y) \to X \otimes Y$$

which respects the associativity and commutativity isomorphisms (so R is a *pseudo symmetric monoidal* functor). It follows that L can be lifted to preserve operad, algebra and module structures.

Hence there is induced a pair of adjoint functors

$$\mathrm{Op}(\mathcal{C}) \underset{R_{\mathrm{Op}}}{\overset{L_{\mathrm{Op}}}{\rightleftarrows}} \mathrm{Op}(\mathcal{D}) \ ,$$

which is a Quillen adjunction between J-semi model categories by the definition of the model structures.

For $\mathcal{O} \in \mathrm{Op}(\mathcal{C})$ there is induced a pair of adjoint functors

$$\mathrm{Alg}(\mathcal{O}) \underset{R_{\mathcal{O}}}{\overset{L_{\mathcal{O}}}{\rightleftarrows}} \mathrm{Alg}(L_{\mathrm{Op}}(\mathcal{O})) \ ,$$

which is a Quillen adjunction between J-semi model categories in the cases where $\mathcal{O}$ is either cofibrant in $\mathrm{Op}(\mathcal{C})$ or cofibrant as an object in $\mathcal{C}^{\Sigma}$.

So for $\mathcal{O} \in \mathrm{Op}(\mathcal{C})$, $\mathcal{O}' \in \mathrm{Op}(\mathcal{D})$ and $f: L_{\mathrm{Op}}(\mathcal{O}) \to \mathcal{O}'$ a map there are induced adjunctions

$$D\mathrm{Alg}(\mathcal{O}) \rightleftarrows D\mathrm{Alg}(\mathcal{O}') \quad \text{and}$$

$$D^{\leq 2}\mathrm{Alg}(\mathcal{O}) \overset{\Psi}{\rightleftarrows} D^{\leq 2}\mathrm{Alg}(\mathcal{O}') \ .$$

Now let $A \in D^{\leq 2}\mathrm{Alg}(\mathcal{O})$, $B \in D^{\leq 2}\mathrm{Alg}(\mathcal{O}')$ and $g: \Psi(A) \to B$ be a map. Then there is induced an adjunction

$$D(A\text{–}Mod) \rightleftarrows D(B\text{–}Mod) \ .$$

All the adjunctions are compatible (in an appropriate weak categorical sense) with compositions of the maps which induce these adjunctions.

8. E_∞-Algebras

Let $\mathcal{N}$ be the operad in $\mathcal{C}$ whose algebras are just the commutative unital algebras in $\mathcal{C}$, i.e. $\mathcal{N}(n) = \mathbb{1}$ for $n \in \mathbb{N}$, and let $\mathcal{P}$ be the operad whose algebras are objects in $\mathcal{C}$ pointed by $\mathbb{1}$, i.e. $\mathcal{P}(n) = \mathbb{1}$ for $n = 0, 1$, $\mathcal{P}(n) = 0$ otherwise. There is an obvious map $\mathcal{P} \to \mathcal{N}$.

Definition 8.1. (1) *An E_∞-*operad *in $\mathcal{C}$ is an operad $\mathcal{O}$ in $\mathcal{C}$ which is cofibrant as an object in $\mathcal{C}^\Sigma$ together with a map $\mathcal{O} \to \mathcal{N}$ which is a weak equivalence.*

(2) *A* pointed *E_∞-operad in $\mathcal{C}$ is an E_∞-operad $\mathcal{O}$ in $\mathcal{C}$ together with a map $\mathcal{P} \to \mathcal{O}$ such that the composition with the map $\mathcal{O} \to \mathcal{N}$ is the canonical map $\mathcal{P} \to \mathcal{N}$.*

(3) *A* unital *E_∞-operad in $\mathcal{C}$ is a pointed E_∞-operad in $\mathcal{C}$ such that the map $\mathcal{P}(0) \to \mathcal{O}(0)$ is an isomorphism (this is the same as an E_∞-operad $\mathcal{O}$ in $\mathcal{C}$ such that the map $\mathcal{O}(0) \to \mathcal{N}(0)$ is an isomorphism).*

The unit $\mathbb{1}$ is an $\mathcal{N}$-algebra, hence it is an algebra for any E_∞-operad. Let $\mathcal{O}$ be a pointed E_∞-operad. Then an $\mathcal{O}$-algebra A is naturally pointed, i.e. there is a canonical map $\mathbb{1} \to A$, but note that this need not be a map of algebras. If it is, we say that A is a *unital* $\mathcal{O}$-algebra. Let us denote the category of unital $\mathcal{O}$-algebras by $\mathrm{Alg}^u(\mathcal{O})$. This is just the category of objects in $\mathrm{Alg}(\mathcal{O})$ under $\mathbb{1}$. If $\mathcal{O}$ is unital, then every $\mathcal{O}$-algebra is unital.

We first want to show that under suitable conditions unital E_∞-operads always exist.

For $\mathcal{O} \in \mathrm{Op}(\mathcal{C})$ let us denote by $\mathcal{O}_{\leq 1}$ the operad with $\mathcal{O}_{\leq 1}(0) = \mathcal{O}(0)$, $\mathcal{O}_{\leq 1}(1) = \mathbb{1}$ and $\mathcal{O}_{\leq 1}(n) = 0$ for $n > 1$. There is a canonical map $\mathcal{O}_{\leq 1} \to \mathcal{O}$ in $\mathrm{Op}(\mathcal{C})$. If $\mathcal{O}$ is an E_∞-operad there is also a map $\mathcal{O}_{\leq 1} \to \mathcal{P}$ in $\mathrm{Op}(\mathcal{C})$, and we denote by $\widetilde{\mathcal{O}}$ the pushout of $\mathcal{O}$ with respect to this map.

Lemma 8.2. *Let $\mathcal{O}$ be an E_∞-operad which admits a pointing.*

(1) *Then there is a canonical equivalence $\mathrm{Alg}^u(\mathcal{O}) \sim \mathrm{Alg}(\widetilde{\mathcal{O}})$, in particular an $\mathcal{O}$-algebra is unital if and only if it comes from an $\widetilde{\mathcal{O}}$-algebra.*

(2) *Assume that $\mathcal{C}$ is left proper, that $\mathbb{1}$ is cofibrant in $\mathcal{C}$ and that $\mathcal{O}$ is cofibrant in $\mathrm{Op}(\mathcal{C})$. Then $\widetilde{\mathcal{O}}$ is a unital E_∞-operad in $\mathcal{C}$.*

Proof. By Lemma 4.2(1) an $\widetilde{\mathcal{O}}$-algebra A is the same as an $\mathcal{O}$-algebra A together with a map $\mathbb{1} \to A$ such that the structure map $\mathcal{O}(0) \to A$ is the composition $\mathcal{O}(0) \to \mathbb{1} \to A$. Hence a unital $\mathcal{O}$-algebra comes from an $\widetilde{\mathcal{O}}$-algebra. On the other hand if A is an $\widetilde{\mathcal{O}}$-algebra we have to show that the induced pointing $\mathbb{1} \to A$ is a map of algebras. This follows easily from the fact that the map $\mathcal{O}(0)$ has a right inverse (a pointing of $\mathcal{O}$). For the first part of the Lemma it remains to prove that an $\mathcal{O}$-algebra morphism between $\widetilde{\mathcal{O}}$-algebras is in fact an $\widetilde{\mathcal{O}}$-algebra morphism, which follows from Lemma 4.2(2).

$\widetilde{\mathcal{O}}$ is unital by the last part of Lemma 8.3 and cofibrant as object in $\mathcal{C}^\Sigma$ by Corollary 8.4. We have to show that $\widetilde{\mathcal{O}} \to \mathcal{N}$ is a weak equivalence. Consider the commutative square

$$\begin{array}{ccc} \mathcal{O}(0) & \longrightarrow & F_{\mathcal{O}}(\mathbb{1}) \\ \downarrow & & \downarrow \\ \mathbb{1} & \longrightarrow & F_{\widetilde{\mathcal{O}}}(\mathbb{1}) \end{array}$$

of $\mathcal{O}$-algebras and let P be the pushout of the left upper triangle of the square. We want to show that the canonical map $P \to F_{\widetilde{\mathcal{O}}}(\mathbb{1})$ is an isomorphism. By the first part of the Lemma P is an $\widetilde{\mathcal{O}}$-algebra. Now again by the first part of the Lemma it is easily seen that P has the same universal property as $\widetilde{\mathcal{O}}$-algebra as $F_{\widetilde{\mathcal{O}}}(\mathbb{1})$.

So the above square is a pushout square in $\mathrm{Alg}(\mathcal{O})$, and hence by left properness of $\mathrm{Alg}(\mathcal{O})$ over $\mathcal{C}$ (Theorem 4.3) the right vertical arrow is a weak equivalence. This implies that $\mathcal{O} \to \widetilde{\mathcal{O}}$ is a weak equivalence. □

Let us call a vertex $v \in V(T)$ of a tree $T \in \mathcal{T}$ a *no-tail* vertex if one cannot reach a tail from v. Let us call T *0-special* if the only no-tail vertices of T are vertices of valency 0. A proper 0-special doubly colored tree is a proper doubly colored tree which is 0-special such that any vertex of valency 0 is old. Let $\widetilde{\mathcal{T}}^p_{\mathrm{dc}}(n)$ be the set of isomorphism classes of such trees with n tails.

Lemma 8.3. *Let $\mathcal{O} = \mathrm{colim}_{i<\lambda}\mathcal{O}_i$ be an operad in $\mathcal{C}$ such that the transition maps $\mathcal{O}_i \to \mathcal{O}_{i+1}$ are pushouts of free operad maps on maps $g_i : K_i \to L_i$ in $\mathcal{C}^{\mathbb{N}}$ and such that $\mathcal{O}_0$ is the initial operad. Let $E \in \mathcal{C}$ and let $\mathcal{O}(0) \to E$ be a morphism in $\mathcal{C}$. Let $\widetilde{E}$ be the operad with $\widetilde{E}(0) = E$, $\widetilde{E}(1) = \mathbb{1}$ and $\widetilde{E}(n) = 0$ for $n > 1$. Let the squares*

$$\begin{array}{ccc} \mathcal{O}_{i,\leq 1} & \longrightarrow & \mathcal{O}_i \\ \downarrow & & \downarrow \\ \widetilde{E} & \longrightarrow & \widetilde{\mathcal{O}}_i \end{array}$$

be pushout squares in $\mathrm{Op}(\mathcal{C})$, where either $i < \lambda$ or i is the blanket. Then $\widetilde{\mathcal{O}} = \mathrm{colim}_{i<\lambda}\widetilde{\mathcal{O}}_i$, and every map $\widetilde{\mathcal{O}}_i \to \widetilde{\mathcal{O}}_{i+1}$ is an $\omega \times (\omega+1)$-sequence in $\mathcal{C}^{\mathbb{N}}$ as in Proposition 3.5, where for $j < \omega$ $\mathcal{O}_{(i,j)}(n)$ is a pushout of $\mathcal{O}_{(i,j)-1}(n)$ in $\mathcal{C}$ by the quotient of the map

$$\coprod \left(\bigotimes_{v \in V_{\mathrm{old}}(T)\setminus U_{\mathrm{old}}(T)} \widetilde{\mathcal{O}}_i(\mathrm{val}(v)) \right) \otimes e^{\square(U_{\mathrm{old}}(T))} \square \mathop{\square}_{v \in V_{\mathrm{new}}(T)} g_i(\mathrm{val}(v)) \ ,$$

where the coproduct is over all $T \in \widetilde{\mathcal{T}}^p_{\mathrm{dc}}(n)$ with $\sharp V_{\mathrm{new}}(T) = i$ and $u_{\mathrm{old}}(T) = j$, with respect to an equivalence relation analogous to the one in Proposition 3.5. In particular we have $\widetilde{\mathcal{O}}_i(0) = E$ for all $i < \lambda$ or i the blanket.

Corollary 8.4. *Let the notation be as in the Lemma above and assume that the maps g_i are cofibrations in $\mathcal{C}^{\mathbb{N}}$ and that E is cofibrant in $\mathcal{C}$. Then the operad $\widetilde{\mathcal{O}}$ is cofibrant in $\mathcal{C}^{\Sigma,\bullet}$.*

Proof. The proof is along the same lines as the proof of Lemma 3.6. □

For the rest of this section let us fix a pointed E_∞-operad $\mathcal{O}$ in $\mathcal{C}$.

Lemma 8.5. *If $\mathbb{1}$ is cofibrant in $\mathcal{C}$ and $\mathcal{O}$ is cofibrant in* $\mathrm{Op}(\mathcal{C})$ *there is a J-semi model structure on* $\mathrm{Alg}^u(\mathcal{O})$ *over $\mathcal{C}$.*

Proof. In any J-semi model category $\mathcal{D}$ over $\mathcal{C}$ the category of objects under an object from $\mathcal{D}$ which becomes cofibrant in $\mathcal{C}$ is again a J-semi model category over $\mathcal{C}$. □

Lemma 8.6. *Assume that $\mathcal{C}$ is left proper and that the domains of the maps in I are cofibrant. Let $A \in$* $\mathrm{Alg}(\mathcal{O})$ *be cofibrant. Then the canonical map of A-modules $U_{\mathcal{O}}(A) \to A$ adjoint to the pointing $\mathbb{1} \to A$ is a weak equivalence.*

Proof. We can assume that A is a cell $\mathcal{O}$-algebra. It is easy to see that the map $U_{\mathcal{O}}(A) \to A$ is compatible with the descriptions of A and $U_{\mathcal{O}}(A)$ in Proposition 4.8 and Proposition 6.5 as transfinite compositions, and the map ψ from the map of part (3) of Proposition 6.5 to the map of part (3) of Proposition 4.8 is induced by the map $\mathcal{O}(|f|+1) = \mathcal{O}(|f|+1)\otimes\mathbb{1}^{\otimes|f|}\otimes\mathbb{1} \to \mathcal{O}(|f|+1)\otimes\mathcal{O}(1)^{\otimes|f|}\otimes\mathcal{O}(0) \to \mathcal{O}(|f|)$, which itself is induced by the unit, the pointing and a structure map of $\mathcal{O}$. Since $\mathcal{O}$ is an E_∞-operad this is a weak equivalence, hence since the domains of the maps in I are cofibrant ψ is a weak equivalence. Now the claim follows by transfinite induction and left properness of $\mathcal{C}$. □

Corollary 8.7. *Assume that $\mathcal{C}$ is left proper, that the domains of the maps in I are cofibrant and that $\mathcal{O}$ is cofibrant in* $\mathrm{Op}(\mathcal{C})$*. Let $A \in$* $\mathrm{Alg}(\mathcal{O})$ *be cofibrant as object in $\mathcal{C}$. Then the canonical map of A-modules $U_{\mathcal{O}}(A) \to A$ adjoint to the pointing $\mathbb{1} \to A$ is a weak equivalence.*

Proof. Let $QA \to A$ be a cofibrant replacement. Then in the commutative square

$$\begin{array}{ccc} U_{\mathcal{O}}(QA) & \longrightarrow & U_{\mathcal{O}}(A) \\ \downarrow & & \downarrow \\ QA & \longrightarrow & A \end{array}$$

the horizonrtal maps are weak equivalences (the upper one by Corollary 6.4) and the left vertical arrow is a weak equivalence by the Lemma above, hence the right vertical map is also a weak equivalence. □

Corollary 8.8. *Assume that $\mathcal{C}$ is left proper and that the domains of the maps in I are cofibrant. Let $A \to A'$ be a weak equivalence between cofibrant $\mathcal{O}$-algebras. Then the map $U_{\mathcal{O}}(A) \to U_{\mathcal{O}}(A')$ is also a weak equivalence.*

Proof. This follows immediately from Lemma 8.6. □

This Corollary has the consequence that under the assumptions of the Corollary there is a canonical equivalence $D(A\text{–}Mod) \sim \mathrm{Ho}\, A\text{–}Mod$ for a cofibrant $\mathcal{O}$-algebra A.

9. $\mathbb{S}$-Modules and Algebras

In this section we generalize the theories developed in [EKMM] and [KM].

Definition 9.1. *(I) A* symmetric monoidal category with pseudo-unit *is a category $\mathcal{D}$ together with*

- *a functor* $\_ \boxtimes \_ : \mathcal{D} \times \mathcal{D} \to \mathcal{D}$,
- *natural isomorphisms* $(X \boxtimes Y) \boxtimes Z \to X \boxtimes (Y \boxtimes Z)$ *and* $X \boxtimes Y \to Y \boxtimes X$ *which satisfy the usual equations and*
- *an object* $\mathbb{1} \in \mathcal{D}$ *with morphisms* $\mathbb{1} \boxtimes X \to X$ *(and hence morphisms* $X \boxtimes \mathbb{1} \to X$ *induced by the symmetry isomorphisms) such that the diagram*

$$\begin{array}{ccc} \mathbb{1} \boxtimes (X \boxtimes Y) & \longrightarrow & X \boxtimes Y \\ \downarrow & \nearrow & \\ (\mathbb{1} \boxtimes X) \boxtimes Y & & \end{array}$$

  commutes and such that the two possible maps $\mathbb{1} \boxtimes \mathbb{1} \to \mathbb{1}$ *agree.*

(II) A symmetric monoidal functor *between symmetric monoidal categories with pseudo-unit $\mathcal{D}$ and $\mathcal{D}'$ is a functor $F : \mathcal{D} \to \mathcal{D}'$ together with natural isomorphisms $F(X) \boxtimes F(Y) \to F(X \boxtimes Y)$ compatible with the associativity and commutativity isomorphisms and with a map $F(\mathbb{1}_{\mathcal{D}}) \to \mathbb{1}_{\mathcal{D}'}$ compatible with the unit maps.*

Definition 9.2. *(I) A* symmetric monoidal model category with weak unit *is a model category $\mathcal{D}$ which is a symmetric monoidal category with pseudo-unit such that the functor $\mathcal{D} \times \mathcal{D} \to \mathcal{D}$ has the structure of a Quillen bifunctor (*[Hov1, p. 108]*) and such that the composition $Q\mathbb{1} \boxtimes X \to \mathbb{1} \boxtimes X \to X$ is a weak equivalence for all cofibrant $X \in \mathcal{D}$, where $Q\mathbb{1} \to \mathbb{1}$ is a cofibrant replacement.*

(II) A symmetric monoidal Quillen functor *between symmetric monoidal model categories with weak unit $\mathcal{D}$ and $\mathcal{D}'$ is a left Quillen functor $\mathcal{D} \to \mathcal{D}'$ which is a symmetric monoidal functor between symmetric monoidal categories with pseudo-unit such that the composition $F(Q\mathbb{1}_{\mathcal{D}}) \to F(\mathbb{1}_{\mathcal{D}}) \to \mathbb{1}_{\mathcal{D}'}$ is a weak equivalence.*

The homotopy category of a symmetric monoidal model category with weak unit is a closed symmetric monoidal category.

Let us assume now that $\mathcal{C}$ is either simplicial (i.e. there is a symmetric monoidal left Quillen functor $\mathbf{SSet} \to \mathcal{C}$) or that there is a symmetric monoidal left Quillen functor $Comp_{\geq 0}(\mathbf{Ab}) \to \mathcal{C}$, where $Comp_{\geq 0}(\mathbf{Ab})$ is endowed with the projective model structure. In both cases we denote by $\mathcal{L}$ the image of the linear isometries operad in $\mathrm{Op}(\mathcal{C})$ via either the simplicial complex functor or the simplicial complex functor followed by the normalized chain complex functor. Clearly $\mathcal{L}$ is a unital E_∞-operad. Let $\mathbb{S} := \mathcal{L}(1)$. $\mathbb{S}$ is a ring with unit in $\mathcal{C}$ which is cofibrant as an object in $\mathcal{C}$.

As in [EKMM] or [KM] we define a tensor product on $\mathbb{S}$–*Mod* by

$$M \boxtimes N := \mathcal{L}(2) \otimes_{\mathbb{S} \otimes \mathbb{S}} M \otimes N \ .$$

[KM, Theorem V.1.5] and [KM, Lemma V.1.6] also work in our context, hence $\mathbb{S}$–*Mod* is a symmetric monoidal category with pseudo-unit.

There is an internal Hom in $\mathbb{S}$–*Mod* given by

$$\underline{\mathrm{Hom}}^{\boxtimes}(M,N) := \underline{\mathrm{Hom}}_{\mathbb{S}}(\mathcal{L}(2)\otimes_{\mathbb{S}} M, N)\ ,$$

where, when forming $\mathcal{L}(2)\otimes_{\mathbb{S}} M$, $\mathbb{S}$ acts on $\mathcal{L}(2)$ through $\mathbb{S}=\mathbb{1}\otimes\mathbb{S}\to\mathbb{S}\otimes\mathbb{S}$, when forming $\underline{\mathrm{Hom}}_{\mathbb{S}}$, $\mathbb{S}$ acts on $\mathcal{L}(2)\otimes_{\mathbb{S}} M$ via its left action on $\mathcal{L}(2)$ and the left action of $\mathbb{S}$ on $\underline{\mathrm{Hom}}^{\boxtimes}(M,N)$ is induced through the right action of $\mathbb{S}$ on $\mathcal{L}(2)$ through $\mathbb{S}=\mathbb{S}\otimes\mathbb{1}\to\mathbb{S}\otimes\mathbb{S}$.

There is an augmentation $\mathbb{S}\to\mathbb{1}$ which is a map of algebras with unit.

Proposition 9.3. *The category $\mathbb{S}$–Mod is a cofibrantly generated symmetric monoidal model category with weak unit with generating cofibrations $\mathbb{S}\otimes I$ and generating trivial cofibrations $\mathbb{S}\otimes J$. The functor $\mathcal{C}\to\mathbb{S}$–Mod, $X\mapsto\mathbb{S}\otimes X$, is a Quillen equivalence, and its left inverse, the functor $\mathbb{S}$–Mod $\to\mathcal{C}$, $M\mapsto M\otimes_{\mathbb{S}}\mathbb{1}$, is a symmetric monoidal Quillen equivalence. Moreover there is a closed action of $\mathcal{C}$ on $\mathbb{S}$–Mod.*

Proof. That R–*Mod* is a cofibrantly generated model category together with a closed action of $\mathcal{C}$ on it is true for any associative unital ring R in $\mathcal{C}$ which is cofibrant as an object in $\mathcal{C}$.

Let f and g be cofibrations in $\mathcal{C}$. The $\boxtimes$-pushout product of $\mathbb{S}\otimes f$ and $\mathbb{S}\otimes g$ is isomorphic to $\mathcal{L}(2)\otimes(f\Box g)$. As a left $\mathbb{S}$-module $\mathcal{L}(2)$ is (non canonically) isomorphic to $\mathbb{S}$, hence $\mathcal{L}(2)\otimes(f\Box g)$ is a cofibration $\mathbb{S}$–*Mod*, and it is trivial if one of f or g is trivial. To show that for a cofibrant $\mathbb{S}$-module M the map $Q\mathbb{1}\boxtimes M\to M$ is a weak equivalence we can assume that M is a cell $\mathbb{S}$-module and we can take $Q\mathbb{1}=\mathbb{S}$. Then M is a transfinite composition where the transition maps are pushouts of maps $f:\ \mathbb{S}\otimes K\to\mathbb{S}\otimes L$, where $K\to L$ is a cofibration in $\mathcal{C}$ with K cofibrant. But the composition $\mathbb{S}\boxtimes\mathbb{S}\to\mathbb{1}\boxtimes\mathbb{S}\to\mathbb{S}$ is a weak equivalence between cofibrant objects in $\mathbb{S}$–*Mod*, hence the composition $\mathbb{S}\boxtimes f\to\mathbb{1}\boxtimes f\to f$ is a weak equivalence between cofibrations in $\mathbb{S}$–*Mod*. So by transfinite induction the composition $\mathbb{S}\boxtimes M\to\mathbb{1}\boxtimes M\to M$ is a weak equivalence between cofibrant objects in $\mathbb{S}$–*Mod*. □

Note that in the simplicial case $\mathbb{1}\boxtimes\mathbb{S}$ is cofibrant in $\mathcal{C}$, hence for cofibrant M both maps $\mathbb{S}\boxtimes M\to\mathbb{1}\boxtimes M\to M$ are weak equivalences.

Let $\mathbb{S}$–*Mod*^{u} be the category of unital $\mathbb{S}$-modules, i.e. the objects in $\mathbb{S}$–*Mod* under $\mathbb{1}\in\mathbb{S}$–*Mod*. For $M\in\mathbb{S}$–*Mod*^{u} and $N\in\mathbb{S}$–*Mod* there are the products $M\lhd N$ and $N\rhd M$, and for $M,N\in\mathbb{S}$–*Mod*^{u} there is the product $M\boxdot N$. These products are defined as in [KM, Definition V.2.1] and [KM, Definition V.2.6].

$\mathbb{S}$–*Mod*^{u} is a symmetric monoidal category with $\boxdot$ as tensor product.

Analogous to [KM, Theorem V.3.1] and [KM, Theorem V.3.3] we have

Proposition 9.4.
- $\mathrm{Alg}(\mathcal{L})$ *is naturally equivalent to the category of commutative rings with unit in $\mathbb{S}$–Mod^u. Hence for $A,B\in\mathrm{Alg}(\mathcal{L})$ there is a natural isomorphism $A\sqcup B\cong A\boxdot B$.*
- *For $A\in\mathrm{Alg}(\mathcal{L})$ an A-module M is the same as an $\mathbb{S}$-module M together with a map $A\lhd M\to M$ satisfying the usual identities.*

Definition 9.5. *For $A\in\mathrm{Alg}(\mathcal{L})$ let* $\mathrm{Comm}(A)$ *be the category of commutative unital A-algebras in $\mathbb{S}$–Mod^u, i.e. the objects in* $\mathrm{Alg}(\mathcal{L})$ *under A. In particular* *we have* $\mathrm{Alg}(\mathcal{L}) \sim \mathrm{Comm}(\mathbb{1}_{\mathbb{S}}) =: \mathrm{Comm}_{\mathcal{C}}$*, where we denote by* $\mathbb{1}_{\mathbb{S}}$ *the algebra* $\mathbb{1}$ *in* $\mathbb{S}$*–Mod*^{u}*. Finally set* $D\mathrm{Comm}_{\mathcal{C}} := \mathrm{Ho}\,\mathrm{Comm}_{\mathcal{C}}$ *and* $D^{\leq 2}\mathrm{Comm}_{\mathcal{C}} := \mathrm{Ho}^{\leq 2}\mathrm{Comm}_{\mathcal{C}}$*.*

For the rest of the section let us make the following

Assumption 9.6. *The model category* $\mathcal{C}$ *is left proper and* $\mathbb{1}$ *and the domains of the maps in* I *are cofibrant in* $\mathcal{C}$*.*

Corollary 9.7.
- *Comm*$_{\mathcal{C}}$ *is a cofibrantly generated* J*-semi model category.*
- *For any cofibrant* $A \in \mathrm{Comm}_{\mathcal{C}}$ *the category* $\mathrm{Comm}(A)$ *is also a cofibrantly generated* J*-semi model category.*
- *If* $A \to B$ *is a weak equivalence between cofibrant* $A, B \in \mathrm{Comm}_{\mathcal{C}}$*, then the induced functor* $\mathrm{Comm}(A) \to \mathrm{Comm}(B)$ *is a Quillen equivalence.*

Proof. Follows from Theorem 4.7. □

Definition 9.8. *For* $A \in \mathrm{Comm}_{\mathcal{C}}$ *let* $D\mathrm{Comm}(A)$ *be* $\mathrm{Ho}\,\mathrm{Comm}(QA)$ *for* $QA \to A$ *a cofibrant replacement in* $\mathrm{Comm}_{\mathcal{C}}$*, and let* $D^{\leq 2}\mathrm{Comm}(A) := \mathrm{Ho}^{\leq 2}\mathrm{Comm}(QA)$*. The 2-functor* $\mathrm{Comm}_{\mathcal{C}} \to \mathbf{Cat}$*,* $A \mapsto D\mathrm{Comm}(A)$*, descents to a 2-functor* $D^{\leq 2}\mathrm{Comm}_{\mathcal{C}} \to \mathbf{Cat}$*,* $A \mapsto D\mathrm{Comm}(A)$*.*

Let $A \in \mathrm{Comm}_{\mathcal{C}}$ and $M, N \in A$*–Mod*. As in [KM, Definition V.5.1] or [KM, Remark V.5.2] we define the tensor product $M \boxtimes_A N$ as the coequalizer in the diagram

$$(M \rhd A) \boxtimes N \cong M \boxtimes (A \lhd N) \underset{\mathrm{Id}\boxtimes m}{\overset{m\boxtimes \mathrm{Id}}{\rightrightarrows}} M \boxtimes N \longrightarrow M \boxtimes_A N \quad \text{or}$$

$$M \boxtimes A \boxtimes N \underset{\mathrm{Id}\boxtimes m}{\overset{m\boxtimes \mathrm{Id}}{\rightrightarrows}} M \boxtimes N \longrightarrow M \boxtimes_A N \ .$$

With this product the category A*–Mod* has the structure of a symmetric monoidal category with pseudo-unit, where the pseudo-unit is A. As for $\mathbb{S}$-modules one can define products $\lhd_A$, $\rhd_A$ and $\boxdot_A$. There is also an analogue of Proposition 9.4 for A-algebras and modules over A-algebras.

The free A-module functor $\mathbb{S}$*–Mod* $\to A$*–Mod* is given by $M \mapsto A \lhd M$. More generally for $A \to B$ a map in $\mathrm{Comm}_{\mathcal{C}}$ the pushforward of modules is given by $M \mapsto B \lhd_A M$. In particular there is a canonical isomorphism of A-modules $U_{\mathcal{L}}(A) \cong A \lhd \mathbb{S}$.

Lemma 9.9. *Let* $A \to B$ *be a map in Comm*$_{\mathcal{C}}$*, let* $M, N \in A$*–Mod and* $P \in B$*–Mod. Then there are canonical ismorphisms*

$$M \boxtimes_A P \cong (B \lhd_A M) \boxtimes_B P \quad \textit{and}$$

$$(B \lhd_A M) \boxtimes_B (B \lhd_A N) \cong B \lhd_A (M \boxtimes_A N) \ .$$

Proof. Similar to the proof of [KM, Proposition V.5.8]. □

For $M, N \in A$*–Mod* define the internel Hom $\underline{\mathrm{Hom}}^{\boxtimes}_A(M, N)$ in A*–Mod* as the equalizer

$$\underline{\mathrm{Hom}}^{\boxtimes}_A(M, N) \longrightarrow \underline{\mathrm{Hom}}^{\boxtimes}(M, N) \rightrightarrows \underline{\mathrm{Hom}}^{\boxtimes}(A \lhd M, N)$$

like in [KM, Definition V.6.1].

Proposition 9.10. *For a cofibrant $A \in \mathrm{Comm}_{\mathcal{C}}$ the category A–Mod is a cofibrantly generated symmetric monoidal model category with weak unit with generating cofibrations $A \lhd (\mathbb{S} \otimes I)$ and generating trivial cofibrations $A \lhd (\mathbb{S} \otimes J)$. If $f : A \to B$ is a map in $\mathrm{Comm}_{\mathcal{C}}$ between cofibrant algebras the pushforward $f_* : A$–Mod $\to B$–Mod is a symmetric monoidal Quillen functor which is a Quillen equivalence if f is a weak equivalence.*

Proof. A–*Mod* is a cofibrantly generated model category by Theorem 6.1(1). Let f and g be cofibrations in $\mathcal{C}$. By Lemma 9.9 the $\boxtimes_A$-pushout product of the maps $A \lhd (\mathbb{S} \otimes f)$ and $A \lhd (\mathbb{S} \otimes g)$ is given by $A \lhd (\mathcal{L}(2) \otimes (f \square g))$, hence since $\mathcal{L}(2) \cong \mathbb{S}$ as $\mathbb{S}$-modules this is a cofibration in A–*Mod*, and it is trivial if one of f or g is trivial.

Note that $A \lhd \mathbb{S}$ is cofibrant in A–*Mod* and that the map $A \lhd \mathbb{S} \cong U_{\mathcal{L}}(A) \to A$ is a weak equivalence by Lemma 8.6. So we have to show that for cofibrant $M \in A$–*Mod* the map $(A \lhd \mathbb{S}) \boxtimes_A M \to M$ is a weak equivalence, which follows from the fact that the maps of the form $(A \lhd \mathbb{S}) \boxtimes_A (A \lhd (\mathbb{S} \otimes f)) \to A \lhd (\mathbb{S} \otimes f)$ for cofibrations $f \in \mathcal{C}$ with cofibrant domain are weak equivalences between cofibrations in A–*Mod*. The first part of the last statement follows from Lemmas 9.9 and 8.6, and the second part by Corollary 8.8. □

Definition 9.11. *For any algebra $A \in \mathrm{Comm}_{\mathcal{C}}$ set $D(A\text{–}Mod) := \mathrm{Ho}\,(QA\text{–}Mod)$ and $D^{\leq 2}(A\text{–}Mod) := \mathrm{Ho}^{\leq 2}(QA\text{–}Mod)$ for $QA \to A$ a cofibrant replacement.*

$D(A\text{–}Mod)$ is a closed symmetric monoidal category with tensor product $\otimes_A$ induced by $\boxtimes_A$. The assignment $A \mapsto D(A\text{–}Mod)$ factors through a 2-functor $D^{\leq 2}\mathrm{Comm}_{\mathcal{C}} \to \mathbf{Cat}^{\mathrm{sm}}$, $A \mapsto D(A\text{–}Mod)$, where $\mathbf{Cat}^{\mathrm{sm}}$ is the 2-category of symmetric monoidal categories, such that the image functors of all maps in $D^{\leq 2}\mathrm{Comm}_{\mathcal{C}}$ have right adjoints.

We finally consider base change and projection morphisms. Let

$$\begin{array}{ccc} B & \xrightarrow{g'} & B' \\ \uparrow f & \overset{\varphi}{\searrow\!\!\searrow} & \uparrow f' \\ A & \xrightarrow{g} & A' \end{array}$$

be a commutative square in $D^{\leq 2}\mathrm{Comm}_{\mathcal{C}}$. Let $M \in D(B\text{–}Mod)$. Then we have a base change morphism

$$g_* f^* M \to f'^* g'_* M$$

defined to be the adjoint of the natural map $f^* M \to f^* g'^* g'_* M \overset{\varphi}{\cong} g^* f'^* g'_* M$ or equivalently of the map $f'_* g_* f^* M \overset{\varphi}{\cong} g'_* f_* f^* M \to g'_* M$.

The base change morphism is natural with respect to composition of commutative squares.

The following statement is trivial in the context of usual commutative algebras, but is a rather strong structure result in our context.

Proposition 9.12. *Let the notation be as above. If the square is a homotopy pushout, then the base change morphism $g_* f^* M \to f'^* g'_* M$ is an isomorphism.*

The proof will be given in the next section.

Let $A \to B$ be a map in $D^{\leq 2}\mathrm{Comm}_{\mathcal{C}}$. Let $M \in D(A\text{–}Mod)$ and $N \in D(B\text{–}Mod)$. There is a projection morphism

$$M \otimes_A f^* N \to f^*(f_* M \otimes_B N)$$

adjoint to the natural map $f_*(M \otimes_A f^* N) = f_* M \otimes_B f_* f^* N \to f_* M \otimes_B N$. Note that for B-modules M', N' there is a natural map $f^* M' \otimes_A f^* N' \to f^*(M' \otimes_B N')$, and the projection morphism is equivalently described as the composition $M \otimes_A f^* N \to f^* f_* M \otimes_A f^* N \to f^*(f_* M \otimes_B N)$.

Proposition 9.13. *Let the notation be as above. Then the projection morphism $M \otimes_A f^* N \to f^*(f_* M \otimes_B N)$ is an isomorphism.*

We give the proof in the next section.

Let a square in $D^{\leq 2}\mathrm{Comm}_{\mathcal{C}}$ be given as above and let $M \in D(B\text{–}Mod)$, $N \in D(A'\text{–}Mod)$ and $P \in D(A\text{–}Mod)$. Set $M' := f^* M$, $N' := g^* N$, $\widetilde{M} := g'_* M$, $\widetilde{N} := f'_* N$ and $\widetilde{P} := g'_* f_* P \cong f'_* g_* P$.

Lemma 9.14. *Let the notation be as above. Then the diagram*

$$\begin{array}{ccccc}
(M' \otimes_A P) \otimes_A N' & \longrightarrow & g^*(g_*(M' \otimes_A P) \otimes_{A'} N) & \longrightarrow & g^* f'^*((\widetilde{M} \otimes_{B'} \widetilde{P}) \otimes_{B'} \widetilde{N}) \\
\uparrow & & & & \uparrow \\
M' \otimes_A (P \otimes_A N') & \longrightarrow & f^*(M \otimes_B f_*(P \otimes_A N')) & \longrightarrow & f^* g'^*(\widetilde{M} \otimes_{B'} (\widetilde{P} \otimes_{B'} \widetilde{N}))
\end{array} ,$$

where in the first two horizontal maps the projection morphism is applied and in the second two an adjunction and the base change morphism, commutes.

Proof. Let $F := g^* f'^* \overset{\varphi}{\cong} f^* g'^*$. One checks that both compositions are equal to the composition $M' \otimes_A P \otimes_A N' \to F^* \widetilde{M} \otimes_A F^* \widetilde{P} \otimes_A F^* \widetilde{N} \to F^*(\widetilde{M} \otimes_{B'} \widetilde{P} \otimes_{B'} \widetilde{N})$, where the first arrow is a tensor product of obvious objectwise morphisms. □

Let $A \in D^{\leq 2}\mathrm{Comm}_{\mathcal{C}}$. We can use the two Propositions above to give the natural functor $M : D\mathrm{Comm}(A) \to D(A\text{–}Mod)$ a symmetric monoidal structure with respect to the coproduct on $D\mathrm{Comm}(A)$ and the tensor product $\otimes_A$ on $D(A\text{–}Mod)$: We use the fact that $D\mathrm{Comm}(A)$ is equivalent to the 1-truncation of $A \downarrow D^{\leq 2}\mathrm{Comm}_{\mathcal{C}}$. So let $B \leftarrow A \to C$ be a triangle in $D^{\leq 2}\mathrm{Comm}_{\mathcal{C}}$ and complete it by a homotopy pushout to a square with upper right corner $B \sqcup_A C$. First we apply the base change isomorphism to the unit $\mathbb{1}_B$ in $D(B\text{–}Mod)$, which says that there is a natural isomorphism

$$(C \to B \sqcup_A C)^*(\mathbb{1}_{B \sqcup_A C}) \cong (A \to C)_*(M(B)) \; .$$

Applying $(A \to C)^*$ to the left hand side of this isomorphism we get $M(B \sqcup_A C)$, applying this map to the right hand side we get $M(B) \otimes_A M(C)$ by the projection formula. This establishes the isomorphism $M(B) \otimes_A M(C) \cong M(B \sqcup_A C)$. That this isomorphism respects the commutativity isomorphisms follows from Lemma 9.14 with $P = \mathbb{1}_A$. That it respects the associativity isomorphisms for objects $f : A \to B$, $h : A \to C$ and $g : A \to A'$ in $A \downarrow \mathrm{Ho}^{\leq 2}\mathcal{D}$ also follows from Lemma 9.14 with $M = \mathbb{1}_B$, $N = \mathbb{1}_{A'}$ and $P = h^* \mathbb{1}_C$ and a diagram chase.

10. Proofs

In this section we give the proofs of Propositions 9.12 and 9.13. Assume throughout that Assumption 9.6 is fulfilled.

We need the concept of operads in A–*Mod* for $A \in \mathrm{Comm}_{\mathcal{C}}$. We also give the definition of a pointed operad, because it is needed in the Remark. In the context of symmetric monoidal categories with pseudo-unit a pointed operad is not just an operad $\mathcal{O}$ together with a pointing of $\mathcal{O}(0)$, the domains of the structure maps also have to be adjusted (see below).

So let us fix $A \in \mathrm{Comm}_{\mathcal{C}}$. Let A–*Mod*^{u} be the category of pointed A-modules, i.e. the category of objects in A–*Mod* under A. For M a pointed or unpointed A-module and N a pointed or unpointed A-module let $M \circledast N$ be either $M \boxtimes_A N$, $M \lhd_A N$, $M \rhd_A N$ or $M \boxdot_A N$, depending on whether M and N are unpointed, M is pointed and N is unpointed, M is unpointed and N is pointed or M and N are pointed. $M \circledast N$ is an object in A–*Mod* unless both M and N are pointed in which case it is an object in A–*Mod*^{u}. Note that for $M_1, \dots, M_n$ A-modules each of them either pointed or unpointed the product $M_1 \circledast \cdots \circledast M_n$ is well defined, despite the fact that for different bracketings of this expression the symbols for which $\circledast$ actually stands can be different.

Definition 10.1. *An* operad $\mathcal{O}$ *in A–Mod is an object $\mathcal{O}(n) \in (A\text{–}Mod)[\Sigma_n]$ for each $n \in \mathbb{N}$, where $\mathcal{O}(1)$ is pointed, together with maps*

$$\mathcal{O}(m) \circledast \mathcal{O}(n_1) \circledast \cdots \circledast \mathcal{O}(n_m) \to \mathcal{O}(n) \ ,$$

where $m, n_1 \dots, n_m \in \mathbb{N}$ and $n = \sum_{i=1}^m n_i$, such that the usual diagrams for these structure maps commute. A pointed operad *in A–Mod is the same as above with the exception that $\mathcal{O}(0)$ is also pointed.*

Let $\mathrm{Op}(A\text{–}Mod)$ be the category of operads in A–*Mod* and $\mathrm{Op}^p(A\text{–}Mod)$ the category of pointed operads in A–*Mod*. A pointed operad $\mathcal{O}$ in A–*Mod* is called *unital* if the pointing $A \to \mathcal{O}(0)$ is an isomorphism. Let $\mathrm{Op}^u(A\text{–}Mod)$ be the category of unital operads in A–*Mod*.

Let $(A\text{–}Mod)^{\Sigma,\bullet\bullet}$ be the category of collections of objects $\mathcal{O}(n) \in (A\text{–}Mod)[\Sigma_n]$, which are pointed for $n = 0, 1$ and unpointed otherwise. As for ordinary operads we have free (pointed) operad functors F starting from the categories $(A\text{–}Mod)^{\mathbb{N}}$, $(A\text{–}Mod)^{\Sigma}$, $(A\text{–}Mod)^{\Sigma,\bullet\bullet}$ in the pointed case and various other pointed versions of these categories to $\mathrm{Op}(A\text{–}Mod)$ or $Op^p(A\text{–}Mod)$. Note that if A is cofibrant all these source categories of the functors F are model categories.

Theorem 10.2. *Let A be cofibrant in $\mathrm{Comm}_{\mathcal{C}}$. Then the category $\mathrm{Op}(A\text{–}Mod)$ (resp. $\mathrm{Op}^p(A\text{–}Mod)$) has the structure of a cofibrantly generated J-semi model category over $(A\text{–}Mod)^{\Sigma,\bullet}$ (resp. over $(A\text{–}Mod)^{\Sigma,\bullet\bullet}$) with generating cofibrations $F(\mathbb{N}(F_A I))$ and generating trivial cofibrations $F(\mathbb{N}(F_A J))$. If $\mathcal{C}$ is left proper, then $\mathrm{Op}(A\text{–}Mod)$ (resp. $\mathrm{Op}^p(A\text{–}Mod)$) is left proper relative to $(A\text{–}Mod)^{\Sigma,\bullet}$ (resp. relative to $(A\text{–}Mod)^{\Sigma,\bullet\bullet}$). If $\mathcal{C}$ is right proper, so are $\mathrm{Op}(A\text{–}Mod)$ and $\mathrm{Op}^p(A\text{–}Mod)$.*

Let f be a map in A–*Mod* or A–*Mod*^{u} and let g be a map in A–*Mod* or A–*Mod*^{u}. Let $f \Box_* g$ be the pushout product of f and g with respect to the product $\circledast$. $f \Box_* g$ is a map in A–*Mod* unless both f and g are maps in A–*Mod*^{u} in which case $f \Box_* g$ is a map in A–*Mod*^{u}.

Note that if A is cofibrant the category A–*Mod*^{u} has a natural model structure as category of objects under A in the model category A–*Mod*. Note however that A–*Mod*^{u} is *not* symmetric monoidal (with potential tensor product $\boxdot_A$), since this product is not closed.

Lemma 10.3. *Let A be cofibrant in* $\mathrm{Comm}_{\mathcal{C}}$*, let f be a cofibration in A–Mod or A–Mod^u, let g be a cofibration in A–Mod or A–Mod^u let M be cofibrant in A–Mod or A–Mod^u and let N be cofibrant in A–Mod or A–Mod^u. Then*

- *the pushout product $f\Box_* g$ is a cofibration in A–Mod or A–Mod^u which is trivial if f or g is,*
- *the product $M \circledast f$ is a cofibration in A–Mod or A–Mod^u which is trivial if f is and*
- *the product $M \circledast N$ is cofibrant in A–Mod or A–Mod^u.*

There is also a version of this statement when the map or object in A–Mod has a right action of a discrete group G and the other map or object is in A–Mod^u (resp. when both maps or objects are in A–Mod and have actions of discrete groups G and G'). The resulting map or object is then a cofibration or cofibrant object in $(A$–Mod$)[G]$ (resp. $(A$–Mod$)[G \times G']$).

Note that in a symmetric monoidal category cases 2 and 3 would be special cases of case 1.

Proof. It suffices to show this for relative cell complexes f and g and cell complexes M and N, for which it follows for the first case by writing the pushout product of a λ-sequence and a μ-sequence as a $\lambda \times \mu$-sequence. Let $M \in A$–*Mod*^{u}. Then if $A \to M$ is a λ-sequence, M itself is a $(1 + \lambda)$-sequence in A–*Mod*. One concludes now by writing the products in cases 2 and 3 again as appropriate sequences. The cases with group actions work in the same way. $\square$

We remark now that there are versions of Proposition 3.4 and Proposition 3.5 for $\mathrm{Op}^p(A$–*Mod*$)$ where all tensor products are repaced by $\circledast$-products and all pushout products by the $\circledast$-pushout product $\Box_*$. There is also a version of Lemma 3.6, from which Theorem 10.2 follows in the same way as Theorem 3.2.

Definition 10.4. *Let $\mathcal{O} \in \mathrm{Op}(A$–Mod$)$ (resp. $\mathcal{O} \in \mathrm{Op}^p(A$–Mod$)$).*

(1) *An* $\mathcal{O}$-algebra *is an object $B \in A$–Mod (resp. $B \in A$–Mod^u) together with maps*
$$\mathcal{O}(n) \circledast B^{\circledast n} \to A$$
satisfying the usual identities. The category of $\mathcal{O}$-algebras is denoted by $\mathrm{Alg}(\mathcal{O})$.

(2) *Let $B \in \mathrm{Alg}(\mathcal{O})$. A* B-module *is an object $M \in A$–Mod together with maps*
$$\mathcal{O}(n+1) \circledast B^{\circledast n} \circledast M \to M$$
satisfying the usual identities. The category of B-modules is denoted by B–Mod.

Let $\mathcal{O} \in \mathrm{Op}^{(p)}(A$–*Mod*$)$. The free $\mathcal{O}$-algebra functor $F_{\mathcal{O}} :\ A$–*Mod* $\to \mathrm{Alg}(\mathcal{O})$ is given by
$$F_{\mathcal{O}}(M) = \coprod_{n \geq 0} \mathcal{O}(n) \circledast_{\Sigma_n} M^{\boxtimes_A n}\ .$$

In the pointed case $F_{\mathcal{O}}$ factors through A–Mod^u.

As in section 4 one shows the

Theorem 10.5. *Let A be cofibrant in* $\mathrm{Comm}_{\mathcal{C}}$ *and let* $\mathcal{O} \in \mathrm{Op}(A\text{–}Mod)$ *(resp.* $\mathcal{O} \in \mathrm{Op}^p(A\text{–}Mod)$*).*

(1) *If $\mathcal{O}$ is cofibrant the category* $\mathrm{Alg}(\mathcal{O})$ *is a cofibrantly generated J-semi model category over A–Mod with generating cofibrations $F_{\mathcal{O}}F_AI$ and generating trivial cofibrations $F_{\mathcal{O}}F_AJ$. If $\mathcal{C}$ is left proper (resp. right proper), then* $\mathrm{Alg}(\mathcal{O})$ *is left proper relative to A–Mod (resp. right proper).*
(2) *Let $\mathcal{O}$ be cofibrant as an object in $(A\text{–}Mod)^{\Sigma,\bullet}$ (resp. in $(A\text{–}Mod)^{\Sigma,\bullet\bullet}$). Then* $\mathrm{Alg}(\mathcal{O})$ *is a cofibrantly generated J-semi model category with generating cofibrations $F_{\mathcal{O}}F_AI$ and generating trivial cofibrations $F_{\mathcal{O}}F_AJ$. If $\mathcal{C}$ is right proper, so is* $\mathrm{Alg}(\mathcal{O})$.

Let $\mathcal{N}_A \in \mathrm{Op}(A\text{–}Mod)$ (resp. $\mathcal{N}_A^u \in \mathrm{Op}^u(A\text{–}Mod)$) be the operad with $\mathcal{N}_A(n) = A$ (resp. $\mathcal{N}_A^u(n) = A$) for $n \in \mathbb{N}$ and the natural structure maps. Note that both categories $\mathrm{Alg}^{(u)}(\mathcal{N}_A)$ are *not* equivalent to the category $\mathrm{Comm}(A)$ but there are functors

$$C_{\mathcal{N}_A}^{(u)} : \mathrm{Alg}(\mathcal{N}_A^{(u)}) \to \mathrm{Comm}(A) ,$$

which are defined to be the left adjoints of the pullback functors $\mathrm{Comm}(A) \to \mathrm{Alg}(\mathcal{N}_A^{(u)})$. These adjoints exist since they exist on free algebras and every algebra is a coequalizer of two maps between free algebras (as is always the case for algebras over a monad).

Let $\mathcal{O} \in \mathrm{Op}^{(p)}(A\text{–}Mod)$ and $B \in \mathrm{Alg}(\mathcal{O})$. As for ordinary algebras one defines the universal enveloping algebra $U_{\mathcal{O}}(B)$ as the quotient of the tensor algebra

$$\coprod_{n\geq 0} \mathcal{O}(n+1) \circledast_{\Sigma_n} B^{\circledast n}$$

by the usual relations. $U_{\mathcal{O}}(B)$ is an associative unital algebra in A–Mod, hence it is an A_∞-algebra in $\mathcal{C}$ (i.e. an algebra over the operad $\mathcal{L}$ considered as a *non*-Σ operad), which also has a universal enveloping algebra $U_{\mathcal{L}}(U_{\mathcal{O}}(B)) \in \mathrm{Ass}(\mathcal{C})$. One has canonical equivalences

$$B\text{–}Mod \sim U_{\mathcal{O}}(B)\text{–}Mod \sim U_{\mathcal{L}}(U_{\mathcal{O}}(B))\text{–}Mod .$$

Let $F_B : A\text{–}Mod \to B\text{–}Mod$ be the free B-module functor.

As in section 6 one shows the

Theorem 10.6. *Let A be cofibrant in* $\mathrm{Comm}_{\mathcal{C}}$*, let* $\mathcal{O} \in \mathrm{Op}(A\text{–}Mod)$ *(resp.* $\mathcal{O} \in \mathrm{Op}^p(A\text{–}Mod)$*) and* $B \in \mathrm{Alg}(\mathcal{O})$*. Let one of the following two conditions be satisfied:*

(1) *$\mathcal{O}$ is cofibrant as an object in $(A\text{–}Mod)^{\Sigma,\bullet}$ (resp. in $(A\text{–}Mod)^{\Sigma,\bullet\bullet}$) and B is a cofibrant $\mathcal{O}$-algebra.*
(2) *$\mathcal{O}$ is cofibrant in* $\mathrm{Op}(A\text{–}Mod)$ *(resp.* $\mathrm{Op}^p(A\text{–}Mod)$*) and A is cofibrant as an object in A–Mod (resp. in A–Mod^u).*

Then there is cofibrantly generated model structure on B–Mod with generating cofibrations F_BF_AI and generating trivial cofibrations F_BF_AJ.

Definition 10.7. *An* E_∞-operad *(resp. pointed* E_∞*-operad) in* A*–Mod is an object* $\mathcal{O} \in \mathrm{Op}(A\text{–}Mod)$ *(resp.* $\mathcal{O} \in \mathrm{Op}^p(A\text{–}Mod)$*) which is cofibrant as an object in* $(A\text{–}Mod)^{\Sigma,\bullet}$ *(resp. in* $(A\text{–}Mod)^{\Sigma,\bullet\bullet}$*) together with a map* $\mathcal{O} \to \mathcal{N}_A$ *which is a weak equivalence. A pointed* E_∞*-operad* $\mathcal{O}$ *is called* unital *if it is unital as an object in* $\mathrm{Op}^p(A\text{–}Mod)$.

For $\mathcal{O}$ a pointed E_∞-operad in A–*Mod* let us define the operad $\widetilde{\mathcal{O}}$ in the same way as in section 8. Then we have analogues of Lemmas 8.2 and 8.3 and Corollary 8.4. So we are able to construct a unital E_∞-operad in A–*Mod* by first taking a cofibrant resolution $\mathcal{O} \to \mathcal{N}_A$ in $\mathrm{Op}(A\text{–}Mod)$ and then forming $\widetilde{\mathcal{O}}$. This will be relevant in the Remark.

Let $B \in \mathrm{Alg}(\mathcal{O})$ be cofibrant. As in Lemma 8.6 one can show that the map $U_{\mathcal{O}}(B) \to B$ adjoint to the pointing $A \to B$ is a weak equivalence.

For the rest of this section let us fix an unpointed E_∞-operad $\mathcal{O}$ in A–*Mod* (we could also take a pointed one). Let π be the map $\mathcal{O} \to \mathcal{N}_A$.

Lemma 10.8. *Let* A *be cofibrant in* $\mathrm{Comm}_{\mathcal{C}}$*. Then the composition*

$$\mathrm{Alg}(\mathcal{O}) \xrightarrow{\pi_*} \mathrm{Alg}(\mathcal{N}_A) \xrightarrow{C_{\mathcal{N}_A}} \mathrm{Comm}(A)$$

is a Quillen equivalence.

Proof. This follows from the fact that for a cofibrant A-module M the map

$$\mathcal{O}(n) \circledast_{\Sigma_n} M^{\boxtimes_A n} \to M^{\boxtimes_A n}/\Sigma_n$$

is a weak equivalence. □

Lemma 10.9. *Let* A *be cofibrant in* $\mathrm{Comm}_{\mathcal{C}}$ *and let* $B \in \mathrm{Alg}(\mathcal{O})$ *be cofibrant. Then the functor*

$$B\text{–}Mod \to (C_{\mathcal{N}_A} \circ \pi)_*(B)\text{–}Mod$$

is a Quillen equivalence.

Proof. This follows from the fact that the map $U_{\mathcal{L}}(U_{\mathcal{O}}(B)) \to U_{\mathcal{L}}((C_{\mathcal{N}_A} \circ \pi)_*(B))$ is a weak equivalence, which follows itself from the description of these algebras in terms of transfinite compositions as in Propositions 4.8 and 6.5. □

Lemma 10.10. *Let* A *be cofibrant in* $\mathrm{Comm}_{\mathcal{C}}$ *and let* $B \in \mathrm{Alg}(\mathcal{O})$ *be cofibrant. Then* $U_{\mathcal{O}}(B)$ *is cofibrant as object in* A*–Mod*^{u}.

Proof. Follows by the description of $U_{\mathcal{O}}(B)$ as in Proposition 6.5. □

Corollary 10.11. *Let* A *be cofibrant in* $\mathrm{Comm}_{\mathcal{C}}$*. Then for cofibrant* $B \in \mathrm{Alg}(\mathcal{O})$ *and cofibrant* $M \in B$*–Mod the underlying* A*-module* M *is cofibrant in* A*–Mod.*

Proof. Follows from Lemmas 10.10 and 10.3 and transfinite induction. □

Lemma 10.12. *Let* μ *and* λ *be ordinals and let* $S_{\mu,+}$ *and* $S_{\lambda,+}$ *be as in Proposition 4.8. Then there is a (necessarily unique) isomorphism*

$$\varphi : \; S_{\lambda+\mu,+} \cong S_{\mu,+} \times S_{\lambda,+}$$

of well-ordered sets.

Proof. There is a natural inclusion $S_{\lambda,+} \hookrightarrow S_{\lambda+\mu,+}$, and φ maps its image to $\{*\} \times S_{\lambda,+}$ in the natural way. Now let $f \in S_{\lambda+\mu}$ with $f(i) \notin \mathbb{N}$ for some $i \in \mu$. There is a segment $M_f \subset S_{\lambda+\mu}$ starting at f which is isomorphic to $S_{\lambda,+}$ as a well-ordered set. Via this identification S_λ corresponds to all $f' \in S_{\lambda+\mu}$ with $f'|_\mu = f|_\mu + \frac{1}{2}$. Then φ maps M_f to $\{f|_\mu\} \times S_{\lambda,+}$ if $i > 0$ and to $\{f|_\mu - \frac{1}{2}\} \times S_{\lambda,+}$ if $i = 0$. It is easy to see that this way φ is well-defined, bijective and order-preserving. $\square$

Remark 10.13. *If $f \in S_{\lambda,+}$ and $g \in S_{\mu,+}$ are successors, then φ maps $(f \sqcup g) - 1$ to $(g - 1, f - 1)$.*

Proof of Proposition 9.12. By Lemmas 10.8 and 10.9 we can work in $\mathrm{Alg}(\mathcal{O})$. So let $B, C \in \mathrm{Alg}(\mathcal{O})$ be cofibrant. Let us denote the coproduct in $\mathrm{Alg}(\mathcal{O})$ by $\sqcup_A$. We have to prove the base change isomorphism for the diagram

$$\begin{array}{ccc} B & \xrightarrow{g'} & B \sqcup_A C \\ \uparrow{\scriptstyle f} & & \uparrow{\scriptstyle f'} \\ A & \xrightarrow{g} & C \end{array} \quad .$$

Let $M \in B\text{–}Mod$ be cofibrant. Then f^*M is cofibrant in $A\text{–}Mod$ by Corollary 10.11. Hence the base change morphism is represented by the morphism of $U_\mathcal{O}(C)$-modules $U_\mathcal{O}(C) \lhd_A M \to U_\mathcal{O}(B \sqcup_A C) \lhd_{U_\mathcal{O}(B)} M$ which is adjoint to the map $M \cong A \lhd_A M \to U_\mathcal{O}(B \sqcup_A C) \lhd_{U_\mathcal{O}(B)} M$. We can assume that M is a cell module. Then by transfinite induction we are reduced to the following statement: Let $K \in A\text{–}Mod$ be cofibrant. Then the map $U_\mathcal{O}(C) \lhd_A (U_\mathcal{O}(B) \lhd_A K) \to U_\mathcal{O}(B \sqcup_A C) \lhd_A K$ is a weak equivalence. By Lemma 10.3 this follows if we show that the map of B-modules $\psi : U_\mathcal{O}(B) \boxdot_A U_\mathcal{O}(C) \to U_\mathcal{O}(B \sqcup C)$ (where we exchanged the roles of B and C) is a weak equivalence. It suffices to prove this for cell algebras B and C. So let $B = \mathrm{colim}_{i<\lambda} B_i$, where the transition maps are given by pushouts by cofibrations $g_i : K_i \to L_i$ in $A\text{–}Mod$ with cofibrant domain as in Proposition 4.8. Similarly let $C = \mathrm{colim}_{i<\mu} C_i$, where the transition maps are given by pushouts by cofibrations $h_i : M_i \to N_i$ in $A\text{–}Mod$ with cofibrant domain. Then the map $0 \to U_\mathcal{O}(B \sqcup_A C)$ is described as in Proposition 6.5 by a $S_{\lambda+\mu,+}$-sequence (1). Since the maps $0 \to U_\mathcal{O}(B)$ resp. $0 \to U_\mathcal{O}(C)$ are $S_{\lambda,+}$- resp. $S_{\mu,+}$-sequences, the map $0 \to U_\mathcal{O}(B) \boxdot_A U_\mathcal{O}(C)$ is a $S_{\mu,+} \times S_{\lambda,+}$-sequence (2) by Lemma 2.1 (this also holds in the case of a symmetric monoidal category with pseudo-unit). Let $\alpha : S_{\mu,+} \times S_{\lambda,+} \to S_{\lambda+\mu,+}$ be the isomorphism of well-ordered sets of Lemma 10.12. Let $f \in S_{\lambda,+}$ and $f' \in S_{\mu,+}$ be successors. Then α identifies $(f \sqcup f') - 1$ and $(f' - 1, f - 1)$, and the relevant pushouts in the sequences (1) and (2) are by maps

$$\mathcal{O}(|f \sqcup f'| + 1) \circledast_{\Sigma_{f \sqcup f'}} \square_{*i<\lambda} g_i^{\square_* f(i)} \square_* \square_{*i<\mu} h_i^{\square_* f'(i)} \quad \text{and}$$

$$\mathcal{O}(|f| + 1) \circledast \mathcal{O}(|f'| + 1) \circledast_{\Sigma_f \times \Sigma_{f'}} \square_{*i<\lambda} g_i^{\square_* f(i)} \square_* \square_{*i<\mu} h_i^{\square_* f'(i)} \quad .$$

It is easy to see by transfinite induction that the map ψ is compatible with sequences (1) and (2) via the identification α on the indexing sets and with the above pushouts by the map induced by the tensor multiplication map $\mathcal{O}(|f| + 1) \circledast \mathcal{O}(|f'| + 1) \to \mathcal{O}(|f \cup f'| + 1)$ which inserts the second object into the last slot of the first object. This map is a weak equivalence because $\mathcal{O}$ is an E_∞-operad, hence the claim follows by transfinite induction. $\square$

Proof of Proposition 9.13. By Lemmas 10.8 and 10.9 we can assume that we have a cofibrant $\widetilde{B} \in \mathrm{Alg}(\mathcal{O})$, a cofibrant $\widetilde{N} \in \widetilde{B}$–*Mod* and a cofibrant $M \in A$–*Mod* and prove the projection isomorphism for M and the image N of $\widetilde{N}$ in B–*Mod*, where B is the image of $\widetilde{B}$ in $\mathrm{Comm}(A)$. Since $\widetilde{N}$ is cofibrant as A-module by Corollary 10.11 the projection morphism is represented by the composition

$$M \boxtimes_A \widetilde{N} \to M \boxtimes_A N \cong (B \lhd_A M) \boxtimes_B N \ ,$$

where the isomorphism at the second place is from Lemma 9.9. So we have to show that the first map is a weak equivalence. We can assume that $\widetilde{N}$ is a cell module. Then by transfinite induction one is left to show that for a cofibrant A-module K the map $M \boxtimes_A (U_{\mathcal{O}}(\widetilde{B}) \lhd_A K) \to M \boxtimes_A (B \lhd_A K)$ is a weak equivalence. But this map is the map from the free $\widetilde{B}$-module on $M \boxtimes_A K$ to the free B-module on $M \boxtimes_A K$, which is a weak equivalence by Lemma 10.9. Hence we are finished. □

11. Remark

Assume that Assumption 9.6 is fulfilled.

In this section we give an alternative definition of a product on the derived category of modules over an algebra in $D^{\leq 2}\mathrm{Comm}_{\mathcal{C}} := D^{\leq 2}\mathrm{Alg}(\mathcal{N})$ without using the special properties of the linear isometries operad. Unfortunately it seems to be rather ugly (or difficult) to construct associativity and commutativity isomorphisms, and we did not try hard to do this! Note that $D^{\leq 2}\mathrm{Comm}_{\mathcal{C}}$ is the same up to canonical equivalence as the category denoted with the same symbol in section 9. If $\mathcal{O}$ is a unital E_∞-operad and $A \in D^{\leq 2}\mathrm{Comm}_{\mathcal{C}}$, then there is a representative $\widetilde{A} \in \mathrm{Ho}^{\leq 2}\mathrm{Alg}(\mathcal{O})$ which is well defined up to an isomorphism which itself is well defined up to a unique 2-isomorphism. There is a similar statement for a lift of A into $\mathrm{Alg}(\mathcal{O})$.

Let us first treat the case where $\mathcal{C}$ is simplicial, since it is a bit nicer. Let $\mathcal{O}$ be a pointed E_∞-operad in **SSet** and denote by $\mathcal{O}$ also its image in $\mathrm{Op}(\mathcal{C})$. In **SSet** the diagonal $\triangle : \ \mathcal{O} \to \mathcal{O} \times \mathcal{O}$ is a map of operads, hence we also have a map of operads $\mathcal{O} \to \mathcal{O} \otimes \mathcal{O}$ in $\mathrm{Op}(\mathcal{C})$.

We will define a tensor product on $\mathrm{Ho}\, A$–*Mod* for a cofibrant $\mathcal{O}$-algebra A.

First note that for $\mathcal{O}$-algebras A and B the tensor product $A \otimes B$ is a $\mathcal{O} \otimes \mathcal{O}$-algebra, hence also a $\mathcal{O}$-algebra via $\triangle$. Also for an A-module M and a B-module N the tensor product $M \otimes N$ has a natural structure of an $A \otimes B$-module. If A, B are unital there are induced maps in $\mathrm{Alg}^u(\mathcal{O})$ $A = A \otimes \mathbb{1} \to A \otimes B$ and $B = \mathbb{1} \otimes B \to A \otimes B$.

Proposition 11.1. *Assume that $\mathcal{O}$ is either unital or cofibrant in* $\mathrm{Op}(\mathcal{C})$*. Let $A, B \in \mathrm{Alg}^u(\mathcal{O})$ be cofibrant. Then the canonical map $A \sqcup B \to A \otimes B$ in $\mathrm{Alg}^u(\mathcal{O})$ induced by the maps $A \to A \otimes B$ and $B \to A \otimes B$ is a weak equivalence.*

Proof. This proof is very similar to a part of the proof of Proposition 9.12. By Lemma 8.2 we are reduced to the case where $\mathcal{O}$ is unital. It suffices to prove the claim for cell algebras A and B. So let $A = \mathrm{colim}_{i<\lambda} A_i$, where the transition maps are given by pushouts by maps $g_i : K_i \to L_i$ as in Proposition 4.8. Similarly let $B = \mathrm{colim}_{i<\mu} B_i$, where the transition maps are given by pushouts by maps

$h_i : M_i \to N_i$. Then the map $0 \to A \sqcup B$ is described by Proposition 4.8 by a $S_{\lambda+\mu,+}$-sequence (1). Since the maps $0 \to A$ resp. $0 \to B$ are $S_{\lambda,+}$- resp. $S_{\mu,+}$-sequences, the map $0 \to A \otimes B$ is a $S_{\mu,+} \times S_{\lambda,+}$-sequence (2). Let $\alpha : S_{\lambda+\mu,+} \to S_{\lambda,+} \times S_{\mu,+}$ be the isomorphism of well-ordered sets of Lemma 10.12. Let $f \in S_{\lambda,+}$ and $f' \in S_{\mu,+}$ be successors. Then α identifies $(f \sqcup f') - 1$ and $(f' - 1, f - 1)$. The relevant pushouts in the sequences (1) and (2) are by maps

$$\mathcal{O}(|f \sqcup f'|) \otimes_{\Sigma_{f \sqcup f'}} \square_{i<\lambda} g_i^{\square f(i)} \square \square_{i<\mu} h_i^{\square f'(i)} \text{ and}$$

$$\mathcal{O}(|f|) \otimes \mathcal{O}(|f'|) \otimes_{\Sigma_f \times \Sigma_{f'}} \square_{i<\lambda} g_i^{\square f(i)} \square \square_{i<\mu} h_i^{\square f'(i)} ,$$

and again one shows by transfinite induction that the map $\psi : A \sqcup B \to A \otimes B$ is compatible with sequences (1) and (2) via the identification α on the indexing sets and with the above pushouts by the map induced by

$$\mathcal{O}(|f| + |f'|) \xrightarrow{\triangle} \mathcal{O}(|f| + |f'|) \otimes \mathcal{O}(|f| + |f'|) \xrightarrow{\beta \otimes \gamma} \mathcal{O}(|f|) \otimes \mathcal{O}(|f'|) ,$$

where β inserts the pointing $\mathbb{1} \to \mathcal{O}(0)$ into the last $|f'|$ slots of $\mathcal{O}(|f| + |f'|)$ and γ inserts the pointing into the first $|f|$ slots. This map is a weak equivalence since $\mathcal{O}$ is an E_∞-operad, so our claim follows by transfinite induction and the assumptions. $\square$

Assume that $\mathcal{O}$ is either unital or cofibrant in $\mathrm{Op}(\mathcal{C})$. For any cofibrant $\mathcal{O}$-algebra A let Q_A denote a cofibrant replacement functor in A–*Mod*. Let $A \in \mathrm{Alg}^u(\mathcal{O})$ be cofibrant. Then the map $A \sqcup A \to A \otimes A$ is a weak equivalence. Now define a functor

$$T : A\text{–}Mod \times A\text{–}Mod \to A\text{–}Mod \text{ by}$$

$$T(M, N) := (A \sqcup A \to A)_*(Q_{(A \sqcup A)}(Q_A M \otimes Q_A N)) .$$

It is clear that T descents to a functor

$$T : D(A\text{–}Mod) \times D(A\text{–}Mod) \to D(A\text{–}Mod) .$$

We will see that this functor is naturally isomorphic to the tensor product defined in section 9.

Now we skip the restriction of $\mathcal{C}$ being simplicial. Let $\mathcal{O}$ be a unital E_∞-operad in $\mathcal{C}$ which always exists by Lemma 8.2. Then the operad $\mathcal{O} \otimes \mathcal{O}$ is also a unital E_∞-operad. Let $A, B \in \mathrm{Alg}(\mathcal{O} \otimes \mathcal{O})$. Let $\pi_1 : \mathcal{O} \otimes \mathcal{O} \to \mathcal{O} \otimes \mathcal{N} \cong \mathcal{O}$ and $\pi_2 : \mathcal{O} \otimes \mathcal{O} \to \mathcal{N} \otimes \mathcal{O} \cong \mathcal{O}$ be the two projections and define $A_i := \pi_{i,*} A$, $B_i := \pi_{i,*} B$, $i = 1, 2$. Note that π_1 and π_2 are weak equivalences. There are maps

$$A_1 \otimes \mathbb{1} \to A_1 \otimes B_2 \text{ and}$$

$$\mathbb{1} \otimes B_2 \to A_1 \otimes B_2$$

of $\mathcal{O} \otimes \mathcal{O}$-algebras and natural isomorphisms of $\mathcal{O} \otimes \mathcal{O}$-algebras $A_1 \otimes \mathbb{1} \cong \pi_1^* A_1$ and $\mathbb{1} \otimes B_2 \cong \pi_2^* B_2$, which are on the underlying objects in $\mathcal{C}$ the isomorphisms $A_1 \otimes \mathbb{1} \cong A_1$ and $\mathbb{1} \otimes B_2 \cong B_2$. Using the adjunction units $A \to \pi_1^* A_1$ and $B \to \pi_2^* B_2$ we finally get maps $A \to A_1 \otimes B_2$ and $B \to A_1 \otimes B_2$, hence a map

$$A \sqcup B \to A_1 \otimes B_2$$

of $\mathcal{O} \otimes \mathcal{O}$-algebras.

Proposition 11.2. *Let $A, B \in \mathrm{Alg}(\mathcal{O} \otimes \mathcal{O})$ be cofibrant. Then the map $A \sqcup B \to A_1 \otimes B_2$ constructed above is a weak equivalence.*

Proof. The proof of this Proposition is exactly the same as the one for Proposition 11.1, except that this time the relevant pushouts in the sequences (1) and (2) are by maps

$$(\mathcal{O}(|f \sqcup f'|) \otimes \mathcal{O}(|f \sqcup f'|)) \otimes_{\Sigma_{f \sqcup f'}} \square_{i<\lambda} g_i^{\square f(i)} \square \square_{i<\mu} h_i^{\square f'(i)} \quad \text{and}$$

$$\mathcal{O}(|f|) \otimes \mathcal{O}(|f'|) \otimes_{\Sigma_f \times \Sigma_{f'}} \square_{i<\lambda} g_i^{\square f(i)} \square \square_{i<\mu} h_i^{\square f'(i)} \ .$$

The map $A \sqcup B \to A_1 \otimes B_2$ is again compatible with these pushouts by the map induced by

$$\mathcal{O}(|f| + |f'|) \otimes \mathcal{O}(|f| + |f'|) \xrightarrow{\beta \otimes \gamma} \mathcal{O}(|f|) \otimes \mathcal{O}(|f'|) \ ,$$

where β inserts the pointing $\mathbb{1} \to \mathcal{O}(0)$ into the last $|f'|$ slots of $\mathcal{O}(|f| + |f'|)$ and γ inserts the pointing into the first $|f|$ slots. This map is again a weak equivalence since $\mathcal{O}$ is an E_∞-operad, so we are done. $\square$

Let $D\mathrm{Comm}_{\mathcal{C}} := D\mathrm{Alg}(\mathcal{N})$.

Corollary 11.3. *The natural functor* $M :\ D\mathrm{Comm}_{\mathcal{C}} \to \mathrm{Ho}\,\mathcal{C}$ *has a natural symmetric monoidal structure with respect to the coproduct on* $D\mathrm{Comm}_{\mathcal{C}}$ *and the tensor product on* $\mathrm{Ho}\,\mathcal{C}$*.*

If $\mathbb{S}$-modules are available in $\mathcal{C}$ it is clear that this symmetric monoidal structure is naturally isomorphic to the one constructed at the end of section 9.

Let now $A \in \mathrm{Alg}(\mathcal{O} \otimes \mathcal{O})$ be cofibrant. Note that for $M, N \in A\text{–}Mod$ the tensor product $\pi_{1,*}M \otimes \pi_{2,*}N$ is an $A_1 \otimes A_2$-module, hence also an $A \sqcup A$-module. Consider the functor

$$T :\ A\text{–}Mod \times A\text{–}Mod \to A\text{–}Mod \ ,$$
$$(M, N) \mapsto (A \sqcup A \to A)_*(Q_{A \sqcup A}(\pi_{1,*}(Q_A M) \otimes \pi_{2,*}(Q_A N))) \ .$$

It is again clear that T descents to a functor

$$T :\ D(A\text{–}Mod) \times D(A\text{–}Mod) \to D(A\text{–}Mod) \ .$$

To see that this functor is isomorphic to the previous functor T in the simplicial case one takes the previous $\mathcal{O}$ to be $\mathcal{O} \otimes \mathcal{O}$ and looks at the map of $\mathcal{O} \otimes \mathcal{O}$-algebras (obtained via the diagonal) $A \otimes A \to (A_1 \otimes \mathbb{1}) \otimes (\mathbb{1} \otimes A_2)$. The last algebra is isomorphic to the $\mathcal{O} \otimes \mathcal{O}$-algebra $A_1 \otimes A_2$. Hence for A-modules M and N we get a map of $A \otimes A$-modules $M \otimes N \to M_1 \otimes N_2$ which is a weak equivalence. From this one gets the natural isomorphism we wanted to construct.

It remains to show that in the cases $\mathcal{C}$ receives a symmetric monoidal left Quillen functor from **SSet** or $Comp_{\geq 0}(\mathbf{Ab})$ the functor T is isomorphic to the tensor product $\otimes_A$ defined in section 9.

To do this let $\mathcal{O}$ be a unital E_∞-operad in $\mathbb{S}\text{–}Mod = \mathbb{1}_{\mathbb{S}}\text{–}Mod$ and let $\overline{\mathcal{O}} := \mathcal{O} \otimes_{\mathbb{S}} \mathbb{1}$ be its image in $\mathrm{Op}(\mathcal{C})$. The operad $\mathcal{O} \circledast \mathcal{O}$ (which is defined componentwise) is also a unital E_∞-operad whose image in $\mathrm{Op}(\mathcal{C})$ is $\overline{\mathcal{O}} \otimes \overline{\mathcal{O}}$. Then by the above procedure one can define a tensor product on $\mathrm{Ho}\,(A\text{–}Mod)$ for a cofibrant $\mathcal{O} \circledast \mathcal{O}$-algebra A, and it is easy to see that this coincides (after the appropriate identifications) with the product T defined above on $\mathrm{Ho}\,(\overline{A}\text{–}Mod)$ ($\overline{A}$ is the image of A in $\mathrm{Alg}(\overline{\mathcal{O}} \otimes \overline{\mathcal{O}})$) on the one hand and with the product $\boxtimes_{A'}$ on $\mathrm{Ho}\,(A'\text{–}Mod)$, where A' is the image of A in $\mathrm{Comm}_{\mathcal{C}}$, on the other hand.

Part II

12. A toy model

In the second part of this thesis we want to give applications of the general theory of the first section. Our main application is the construction of what we call *limit motives*. They are a motivic analogue of limit Hodge structures considered by Schmid, Steenbrink, Varchenko, et. al. The basic idea is the following: Let $D := \{z \in \mathbb{C} \mid |z| < 1\}$ and $D^* := D \setminus \{0\}$. Consider a proper family of complex algebraic manifolds $f : X \to D$ such that $f^* : X^* := f^{-1}(D^*) \to D^*$ is smooth and $Y := f^{-1}(0)$ is a divisor with normal crossings in X. Let $X_t := f^{-1}(t)$ for $t \in D^*$. Then the $H^n(X_t, \mathbb{C})$ are part of pure Hodge structures and there is a way to put a mixed Hodge structure on $\lim_{t\to 0} H^n(X, \mathbb{C})$ depending on the direction in which t moves to 0 such that the weight filtration is given in terms of the monodromy action around 0. The considerations in [Del] suggest that these limit Hodge structures are fibers of a unipotent variation of mixed Hodge structures on $\mathbb{C}^*$, the pointed tangent space of D at 0.

Let now C be a smooth curve over a field $k \subset \mathbb{C}$ and $x_0 \in C(k)$. Let $C^\circ := C \setminus \{x_0\}$. We are going to propose a construction which associates to any motivic sheaf $\mathcal{F}$ on C° a unipotent motivic sheaf $\widetilde{\mathcal{F}}$ on T°_{C,x_0}, the pointed tangent space of x_0 in C. Let $f : X \to C$ be a proper morphism such that $f^\circ : X^\circ := f^{-1}(C^\circ) \to C^\circ$ is smooth and $Y := f^{-1}(x_0)$ is a divisor with normal crossings in X. Let $\mathcal{F} := \mathbb{R}f_*\mathbb{Z}$ as a motivic sheaf on C°, i.e. $\mathcal{F}$ is an object in a certain triangulated category possessing suitable Hodge realizations. Then the Hodge realizations of $\widetilde{\mathcal{F}}_t$, $t \in T^\circ_{C,x_0}(\mathbb{C})$, should give the limit Hodge structures of the $H^n(X(\mathbb{C})_x, \mathbb{C})$, $x \in C^\circ(\mathbb{C})$. We will examine this and further questions in a forthcoming paper.

We now sketch the method of the construction of the functor $\mathcal{F} \mapsto \widetilde{\mathcal{F}}$ in a toy model. Let S be a 2-dimensional real manifold, $x \in S$ and $S^\circ = S \setminus \{x\}$. Let D be a small disc around x in S and $D^\circ = D \setminus \{x\}$. For a manifold M let $DM(M)$ be the derived category of the category of sheaves of abelian groups on M. This is the homotopy category of a symmetric monoidal model category satisfying our assumptions of Part I. Let $UDM(M)$ be the smallest triangulated subcategory of $DM(M)$ containing the constant sheaf $\mathbb{Z}$ and closed under arbitrary sums. The objects in $UDM(M)$ should be thought of as generalized unipotent objects. Clearly we can identify $UDM(D^\circ)$ and $UDM(T^\circ_{S,x})$ via some choice of inclusion $D \hookrightarrow T_{S,x}$ inducing the identity on the tangent spaces at x and 0.

Let $\mathcal{F} \in DM(S^\circ)$ such that $\mathcal{F}|_{D^\circ} \in UDM(D^\circ)$. Then we get a sheaf $\widetilde{\mathcal{F}}$ as the image of $\mathcal{F}|_{D^\circ}$ in $UDM(T^\circ_{S,x})$. Unfortunately this assignment is not algebraic if we replace S by a complex smooth algebraic curve. What we can do is the following: Let $i : \{x_0\} \to S$ be the closed and $j : S^\circ \to S$ the open inclusion. Let $p : T^\circ_{S,x_0} \to \{x_0\}$ be the projection. Then we have two main observations:

There is a canonical isomorphism $i^*\mathbb{R}j_*\mathbb{Z} \cong \mathbb{R}p_*\mathbb{Z}$ in $DM(\{x_0\}) = DComp(\mathbf{Ab})$. This isomorphism can also be constructed as an isomorphism in $D\mathrm{Comm}_{Comp(\mathbf{Ab})}$ (for the definition of $D\mathrm{Comm}_{Comp(\mathbf{Ab})}$ see 9.5). This isomorphism is not enough to get an equivalence of the derived categories of modules over these algebras. But fortunately there is also a chain of weak isomorphisms in $D^{\leq 2}\mathrm{Comm}_{Comp(\mathbf{Ab})}$

connecting $i^*\mathbb{R}j_*\mathbb{Z}$ and $\mathbb{R}p_*\mathbb{Z}$, which is sufficient to get a canonical equivalence $D(i^*\mathbb{R}j_*\mathbb{Z}\text{–}Mod) \sim D(\mathbb{R}p_*\mathbb{Z}\text{–}Mod)$. (for the definition of the module categories see 9.11). The motivic version of these statements is Proposition 15.19.

The second observation is that $D(\mathbb{R}p_*\mathbb{Z}\text{–}Mod)$ is equivalent to the full subcategory $UDM(T^\circ_{S,x_0})$ of $DM(T^\circ_{S,x_0})$. This is motivated by the fact that the Ext groups in both categories between the tensor unit and itself are the same. An abstract version of this statement is Theorem 13.4 of the next section. The motivic versions are Corollaries 15.12 and 15.14.

Now we can start with any $\mathcal{F} \in DM(S^\circ)$ and form $i^*\mathbb{R}j_*\mathcal{F}$ as a module over $i^*\mathbb{R}j_*\mathbb{Z}$. Sending $i^*\mathbb{R}j_*\mathcal{F}$ further along the two equivalences

$$D(i^*\mathbb{R}j_*\mathbb{Z}\text{–}Mod) \sim D(\mathbb{R}p_*\mathbb{Z}\text{–}Mod) \sim UDM(T^\circ_{S,x_0})$$

defines indeed a natural functor

$$DM(S^\circ) \to DM(T^\circ_{S,x_0})$$

which now has a motivic analogue as we shall see.

The existence of this motivic local monodromy functor will in particular give tangential basepoint functors in the motivic setting, for example it will be possible to define the motivic fundamental group of $\mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$ with base point $\vec{01}$ as was done in [Del] for the realization categories. Then it is not hard to construct motivic polylogarithm sheaves such that all conditions of [BD] are satisfied for the motivic proof of the weak Zagier Conjecture. We will come back to this application in a future paper.

13. Unipotent objects as a module category

The equivalence $D(\mathbb{R}p_*\mathbb{Z}\text{–}Mod) \sim UDM(T^\circ_{S,x_0})$ mentioned in the introduction also has a relative version. In the applications we will need such a relative version in the following situation: Let $f : X \to S$ be a morphism of schemes. In the next sections we will consider triangulated categories of motivic sheaves $DM(S)$ and $DM(X)$. There is a functor $f^* : DM(S) \to DM(X)$ which is the functor induced on the homotopy categories of a symmetric monoidal left Quillen functor between symmetric monoidal model categories. The statement we are going to formulate is that under some conditions it is possible to describe the full subcategory of $DM(X)$ consisting of objects which are unipotent relative to S (which means that they are successive (possibly infinitely many) extensions of objects coming from $DM(S)$ via f^*, see below) by the category of modules over the relative cohomology algebra $\mathbb{R}f_*\mathbb{1}$.

In the following we will replace the notation $\mathbb{R}f_*$ etc. by f_* for the derived functors. If f_* is taken for the model categories or the homotopy categories will depend on whether the object f_* is applied to is an element of the model or the homotopy category.

We examine the question above in the following general situation: Let $f^* : \mathcal{C} \to \mathcal{D}$ be a symmetric monoidal Quillen functor between cofibrantly generated symmetric monoidal model categories satisfying Assumption 9.6. Let f_* be the right adjoint of f^*. Let $\mathbb{1}_{\mathcal{D}} \to R\mathbb{1}_{\mathcal{D}}$ be a fibrant replacement of the initial object in $\mathrm{Comm}_{\mathcal{D}}$ and set $A := Qf_*(R\mathbb{1}_{\mathcal{D}}) \in \mathrm{Comm}_{\mathcal{C}}$. We then get an adjunction

$$D(A\text{–}Mod) \underset{\tilde{f}_*}{\overset{\tilde{f}^*}{\rightleftarrows}} \mathrm{Ho}\,\mathcal{D} \ ,$$

where $\tilde{f}^*$ first maps to $D(f^*A\text{–}Mod)$ and then to $\mathcal{D}$ via pushforward along the natural map $f^*A \to \mathbb{1}_{\mathcal{D}}$.

Our aim is to formulate conditions under which $\tilde{f}^*$ is an equivalence onto its image. If this is the case the objects in the image of $\tilde{f}^*$ can be viewed as generalized unipotent objects with respect to $\mathcal{C}$, i.e. they are constructed by some iterated homotopy colimits of objects coming from $\mathcal{C}$, see below.

13.1. **Subcategories generated by homotopy colimits.** By a *homotopy colimit* in $\mathrm{Ho}\,\mathcal{C}$ we mean the image in $\mathrm{Ho}\,\mathcal{C}$ of a homotopy colimit over a diagram $D:\ I \to \mathcal{C}_c$ (see [Hir, Definition 20.1.2] for homotopy colimits). By the homotopy colimit of a diagram $D : I \to \mathcal{C}$ we mean the homotopy colimit of the diagram QD. For an ordinal λ a homotopy λ-sequence is a homotopy colimit of a diagram $D:\ \lambda \to \mathcal{C}$ such that for any limit ordinal $\nu < \lambda$ the map from the homotopy colimit of $D|_\nu$ to $D(\nu)$ is a weak equivalence. We call a full subcategory $\mathcal{C}'$ of $\mathcal{C}$ *saturated* if $\mathcal{C}'$ is equal to its essential image.

Definition 13.1. *Let C be a class of objects in $\mathrm{Ho}\,\mathcal{C}$. By the full subcategory of $\mathrm{Ho}\,\mathcal{C}$ C-generated by homotopy colimits we mean the smallest saturated full subcategory $\langle C\rangle_{\mathrm{Ho}\,\mathcal{C}}$ of $\mathrm{Ho}\,\mathcal{C}$ which contains C and is closed under homotopy colimits, i.e. contains all homotopy colimits whose terms map to $\langle C\rangle_{\mathrm{Ho}\,\mathcal{C}}$.*

13.2. **The result.** Recall that a model category $\mathcal{C}$ is called *stable* if the suspension functor on $\mathrm{Ho}\,\mathcal{C}$ is an equivalence (see [Hov1, Definition 7.1.1]). In this case $\mathrm{Ho}\,\mathcal{C}$ is a (classical) triangulated category.

Lemma 13.2. *Let $\mathcal{C}$ be a stable model category and*

$$\begin{array}{ccc} B & \xrightarrow{\tilde{g}} & D \\ \uparrow{\scriptstyle f} & & \uparrow{\scriptstyle \tilde{f}} \\ A & \xrightarrow{g} & C \end{array}$$

be a homotopy pushout square in $\mathrm{Ho}\,\mathcal{C}$. Then there is an exact triangle in $\mathrm{Ho}\,\mathcal{C}$ of the form

$$A \xrightarrow{(f,-g)} B \oplus C \xrightarrow{\tilde{g}\oplus\tilde{f}} D \longrightarrow A[1] \ .$$

There is a dual statement for homotopy pullback squares.

Proof. We have to check that the diagram

$$\begin{array}{ccc} 0 & \longrightarrow & D \\ \uparrow & & \uparrow{\scriptstyle \tilde{g}\oplus\tilde{f}} \\ A & \xrightarrow{(f,-g)} & B \oplus C \end{array}$$

is a homotopy pushout square in $\mathrm{Ho}\,\mathcal{C}$. Therefore it is sufficient to check that for any $T \in \mathrm{Ho}\,\mathcal{C}$ the induced square on mapping complexes

$$\begin{array}{ccc} \mathrm{pt} & \longleftarrow & \mathrm{map}(D,T) \\ \downarrow & & \downarrow \\ \mathrm{map}(A,T) & \longleftarrow & \mathrm{map}(B,T) \times \mathrm{map}(C,T) \end{array}$$

is a homotopy pullback square in $\mathrm{Ho}\,\mathbf{SSet}$. Since $\mathcal{C}$ is stable we know that $\mathrm{map}(A,T)$ has the homotopy type of a loop space. So we are reduced to the following situation: Let X, Y and Y' be topological spaces and suppose that X is pointed. Let $f:\ Y \to \Omega X$ and $f':\ Y' \to \Omega X$ be maps. The homotopy fiber product of $Y \to \Omega X \leftarrow Y'$ is given by the space E consisting of triples (y, y', φ), where $y \in Y$, $y' \in Y'$ and φ is a homotopy from $f(y)$ to $f'(y')$. Let f^{-1} be the map which is the composition of f and the automorphism of ΩX which sends a path to its invers. Then the homotopy fiber product of $\mathrm{pt} \to \Omega X \overset{(f^{-1},f')}{\longleftarrow} Y \times Y'$ is given by the space F consisting of triples (y, y', φ), where $y \in Y$, $y' \in Y'$ and φ is a path from $f(y)^{-1} * f'$ to the identity. Clearly there is a natural homotopy equivalence $E \sim F$ giving the square above in $\mathrm{Ho}\,\mathbf{SSet}$ after the identifications $\Omega X = \mathrm{map}(A,T)$, $Y = \mathrm{map}(B,T)$ and $Y' = \mathrm{map}(C,T)$. This proves our claim. □

Remark 13.3. *If we allow for one of the summands $\tilde{g}$ or $\tilde{f}$ in the map $\tilde{g} \oplus \tilde{f}$ appearing in the Lemma to be arbitrary then there is an easy proof of the statement involving only arguments in the triangulated category* $\mathrm{Ho}\,\mathcal{C}$.

The main theorem of this section is

Theorem 13.4. *Let $\mathcal{C}$, $\mathcal{D}$, etc. be as in the first paragraph of this section. Assume that the following conditions are fulfilled:*

1. *For $M \in \mathrm{Ho}\,\mathcal{C}$ the image of a domain or codomain of a generating cofibration of $\mathcal{C}$ the projection morphism $M \otimes A^\sharp \to f_*(f^*M)$ is an isomorphism.*
2. *The functor $f_*:\ \mathrm{Ho}\,\mathcal{D} \to \mathrm{Ho}\,\mathcal{C}$ commutes with homotopy λ-sequences for all ordinals λ.*
3. *$\mathcal{C}$ (and hence also $\mathcal{D}$) is a stable model category.*

Then the symmetric monoidal functor $\tilde{f}^*$ is an equivalence onto its image and its essential image is the full subcategory of $\mathcal{D}$ $f^*(D)$-generated by homotopy colimits, where D is the set of all domains and codomains of the generating cofibrations of $\mathcal{C}$. This subcategory is a $\otimes$-subcategory.

Proof. For the equivalence it is sufficient to show that the unit $\mathrm{Id}_{\mathrm{Ho}\,\mathcal{C}} \to \tilde{f}_* \circ \tilde{f}^*$ of the adjunction is an isomorphism. Let I be the generating cofibrations of $\mathcal{C}$. Let $X:\ \lambda \to \mathcal{C}$ be an $A \lhd (\mathbb{S} \otimes I)$-cell complex in A–*Mod* (so X_0 is the inital object), such that X_{i+1} is a pushout of X_i by a map $A \lhd (\mathbb{S} \otimes f_i):\ A \lhd (\mathbb{S} \otimes A_i) \to A \lhd (\mathbb{S} \otimes B_i)$ with $f_i \in I$. Let $\pi:\ f^*A \to R\mathbb{1}_{\mathcal{D}}$ be the natural map in $\mathrm{Comm}_{\mathcal{D}}$. We show by induction on $i \in \lambda$ that the map $X_i \to f_*(R(\pi_* f^* X_i))$ is a weak equivalence. For i a limit ordinal this follows from our second condition. Let us prove it for $i+1$: Since $\mathcal{C}$ is stable Lemma 13.2 gives an exact triangle

$$\triangle_1:\ A \lhd (\mathbb{S} \otimes A_i) \to A \lhd (\mathbb{S} \otimes B_i) \oplus X_i \to X_{i+1} \to (A \lhd (\mathbb{S} \otimes A_i))[1]$$

in the triangulated category $D(A\text{--}Mod)$ which $\pi_* \circ f^*$ maps to the exact triangle

$$\triangle_2 : \ f^*(A_i) \to f^*(B_i) \oplus \tilde{f}^*(X_i) \to \tilde{f}^*(X_{i+1}) \to f^*(A_i)[1]$$

in $\mathrm{Ho}\,\mathcal{D}$. Now again by stability the unit gives a map of exact triangles $\triangle_1 \to \tilde{f}_*(\triangle_2)$, which is by the projection ismorphism (the condition 1) and by induction hypothesis an isomorphism on the first two terms, hence it is an isomorphism on the third.

The essential image of $\tilde{f}^*$ clearly is contained in the full subcategory of $\mathrm{Ho}\,\mathcal{D}$ generated by colimits in $f^*(\mathrm{Ho}\,\mathcal{C})$. Let $D : \ I \to (R\mathbb{1}_{\mathcal{D}}\text{--}Mod)_{cf}$ be a small diagram which maps on the homotopy level to the essential image of $\tilde{f}^*$. Then $f_*(D)$ is a diagram in $A\text{--}Mod$, and the natural map $\pi_* f^*(Qf_*(D)) \to D$ is an objectwise weak equivalence by the first part of the Theorem. The claim now follows since $\pi_* f^*$ preserves homotopy colimits. □

14. Examples

Our main examples will be for the $\mathbb{A}^1$-local homotopy categories of spaces or of motives over some base scheme, which have all been introduced by Vladimir Voevodsky. The homotopy categories of spaces will be modeled on the category of sheaves of simplicial sets on some site of schemes, the homotopy categories of motives on the category of complexes of sheaves of abelian groups with transfers on sites as above. The meaning of the various expressions will be explained later. What is important is that there will be Quillen functors on the corresponding model categories from spaces to motives and for a change of base schemes. In this section we will give a general scheme which will set all these model structures and functorialities on a common footing.

14.1. **Basic example.** We start with a closed symmetric monoidal bicomplete category $\mathcal{S}$ and a symmetric monoidal complete Grothendieck abelian category $\mathcal{A}$ (which hence is also cocomplete) together with a symmetric monoidal left adjoint $l : \ \mathcal{S} \to \mathcal{A}$. For example in the case of the toy model of section 12 $\mathcal{S}$ is the category of sheaves of sets on a manifold M, $\mathcal{A}$ the category of sheaves of abelian groups on M and l the functor which sends a sheaf of sets F to the sheaf of abelian groups freely generated by F. Let $Ch(\mathcal{A})$ denote either $Comp(\mathcal{A})$ or $Comp_{\geq 0}(\mathcal{A})$. We have an induced pseudo symmetric monoidal functor $L : \ \triangle^{\mathrm{op}}\mathcal{S} \to Ch(\mathcal{A})$ which is the composition of $\triangle^{\mathrm{op}} l$ and the associated normalized complex functor. Let R denote the right adjoint of L. Let $\mathcal{W} \subset \triangle^{\mathrm{op}}\mathcal{S}$ be a subcategory such that $\mathcal{W}$ as weak equivalences and the monomorphisms as cofibrations are part of a left proper model structure on $\triangle^{\mathrm{op}}\mathcal{S}$. We call this model structure the *injective* model structure. On $Comp_{\geq 0}(\mathcal{A})$ and $Comp(\mathcal{A})$ there is also an injective proper model structure as explained in [Hov4], and the natural embedding $Comp_{\geq 0}(\mathcal{A}) \to Comp(\mathcal{A})$ is a left Quillen functor. We suppose that $L(\mathcal{W})$ consists of quasi isomorphisms, that a map f in $Comp_{\geq 0}(\mathcal{A})$ is a quasi isomorphism if and only if $R(f) \in \mathcal{W}$ and that for a trivial cofibration f in **SSet** and a monomorphism g in $\mathcal{S}$ we have $f \Box g \in \mathcal{W}$.

Let I^s (respectively I^a) be the set of generating cofibrations of **SSet** (respectively of $Ch(\mathbf{Ab})$) and J^s (respectively J^a) the set of generating trivial cofibrations of **SSet** (respectively of $Ch(\mathbf{Ab})$).

Let $\mathcal{M}$ be a set of monomorphisms in $\mathcal{S}$ such that $l(\mathcal{M})$ also consists of monomorphisms. Set $I^s_{\mathcal{M}} := I^s\Box\mathcal{M}$, $J^s_{\mathcal{M}} := J^s\Box\mathcal{M}$, $I^a_{\mathcal{M}} := I^a\Box\mathcal{M}$ and $J^a_{\mathcal{M}} := J^a\Box\mathcal{M}$. Here one should have in mind the situation of our toy model from section 12 where $\mathcal{M}$ will be the set of maps $[V] \to [U]$ for open inclusions $V \subset U \subset X$ ($[U]$ denotes the sheaf represented by U). We formulate the following conditions for $\mathcal{M}$:

- C0 $\mathbb{1} \in \mathcal{S}$ appears as a codomain of a map of $\mathcal{M}$.
- C1 For any domain or codomain X of a map of $\mathcal{M}$ the map $\emptyset \to X$ is also contained in $\mathcal{M}$.
- C2 Let $f \in J^s_{\mathcal{M}}$-inj. Then $f \in \mathcal{W}$ if and only if for any codomain X of a map of $\mathcal{M}$ the map $\mathrm{Hom}(X, f)$ is a weak equivalence in **SSet**.
- C3 Any codomain X of a map of $\mathcal{M}$ is finite, i.e. the functor $\mathrm{Hom}(X, \_)$ commutes with sequential colimits.
- C4 For $f, g \in \mathcal{M}$ the pushout product $f\Box g$ is also contained in $\mathcal{M}$.

Then we have

Proposition 14.1. *If $\mathcal{M}$ satisfies $C1 - C3$ the sets $I^s_{\mathcal{M}}$ and $J^s_{\mathcal{M}}$ (respectively $I^a_{\mathcal{M}}$ and $J^a_{\mathcal{M}}$) form a set of generating cofibrations and generating trivial cofibrations for a left proper model structure on $\triangle^{\mathrm{op}}\mathcal{S}$ (respectively a proper model structure on $Ch(\mathcal{A})$) such that L is a left Quillen functor. The domains and codomains of the generating (trivial) cofibrations of these model structures and, if C0 is fulfilled, the units of the symmetric monoidal structures are cofibrant and finite. If moreover C4 is valid the model structures are symmetric monoidal and L is a pseudo symmetric monoidal Quillen functor.*

Proof. First we verify the conditions of [Hov1, Theorem 2.1.19] for the two categories in question. Clearly the domains and codomains of the generating (trivial) cofibrations are finite, hence properties 2 and 3 are fulfilled. 1 and 4 are true by our assumptions and by the injective model structures. By C1 a map $f \in J^s_{\mathcal{M}}$-inj lies in $I^s_{\mathcal{M}}$-inj if and only if it has the right lifting property with respect to all maps $(\partial\Delta^n \hookrightarrow \Delta^n) \times X$ for every codomain X of a map of $\mathcal{M}$. Hence condition 5 and the second alternative of condition 6 follow from C2. These conditions for $Ch(\mathcal{A})$ follow then by adjunction applying in the case of $Comp(\mathcal{A})$ appropriate shifts of complexes. If C4 is fulfilled the model structures are symmetric monoidal by [Hov1, Corollary 4.2.5]. Left properness follows from left properness for the injective model structures. The remaining statements of the Proposition are clear. $\square$

Remark 14.2. *If we enlarge the set of monomorphisms $\mathcal{M}$ by maps whose domain and codomain appears already as a codomain of a map in $\mathcal{M}$ the conditions $C0 - C3$ are still fulfilled.*

14.2. **cd-structures.** We give examples of the above situation. $\mathcal{S}$ will always be the category of sheaves of sets on a small site coming from a complete, bounded and regular cd-structure P on a category C (see [Vo1, Section 2]). If we take as set of monomorphisms $\mathcal{M}_P$ all maps $\rho(A) \to \rho(X)$ for squares in P as in [Vo1, Def. 2.1] (here $\rho(X)$ is the sheaf associated to the presheaf represented by X) together will all maps $\emptyset \to \rho(X)$ for $X \in C$ the conditions for $\mathcal{M}$ are fulfilled except possibly C4. Proposition 14.1 yields the model structure of [Vo1, Theorem 4.5]. But also model structures with enlarged set of monomorphisms are interesting, for example to ensure C4.

Let S be a separated Noetherian scheme, Sch/S the category of separated Noetherian schemes of finite type over S, Sm/S the full subcategory of smooth schemes over S and $Prop/S$ the full subcategory of proper schemes over S. Let C be one of these categories and P one of the complete cd-structures of [Vo1, Lemma 2.2] on C. P is also bounded and regular by the results of [Vo1]. In the following $\mathcal{S}$ will always be the category of sheaves on C for the topology generated by P. We call this topology t_P.

For the cd-structures P which are either contained in the upper (respectively the lower) cd-structure $\mathcal{M}_P$ also fulfills C4, since the domain of a pushout product of maps of $\mathcal{M}_P$ is again an open (respectively a closed) subscheme of the codomain by the definition of the topology t_P. For the combined cd-structures we have to enlarge $\mathcal{M}_P$ in the following way: First we build the union over all maps $\square_{i=1}^n f_i$, $f_i \in \mathcal{M}_P$ to get the set M, and then we adjoin all maps $\emptyset \to X$ to M for all domains X of maps of M arriving at a set $\mathcal{M}_{P,\square}$. We set $\mathcal{M}_{P,\square} := \mathcal{M}_P$ if P is contained in the upper or lower cd-structure.

Lemma 14.3. *$\mathcal{M}_{P,\square}$ satisfies condition C2.*

Proof. The "only if" part is clear. Let $f: A \to B$ be a map in $J_{\mathcal{M}_{P,\square}}$-inj $\cap \mathcal{W}$. The general domain of a map in M (see above) is the domain D of a map $(Z \subset X)\square(U \subset Y)$ for $Z \subset X$ closed and $U \subset Y$ open, X, Y of finite type over S. We know that $\mathrm{Hom}(D, f)$ is a fibration, so to show that it is a weak equivalence we can choose a point $y \in \mathrm{Hom}(D, B)$ and show that $F := \mathrm{Hom}(D, f)^{-1}(y)$ is contractible. We have $\mathrm{Hom}(D, A) = \mathrm{Hom}(X \times_S U, A) \times_{\mathrm{Hom}(Z \times_S U, A)} \mathrm{Hom}(Z \times_S Y, A)$ (similar for $\mathrm{Hom}(D, B)$). Let y_1, y_2 and $\tilde{y}$ be the images of y in $\mathrm{Hom}(X \times_S U, B)$, $\mathrm{Hom}(Z \times_S Y, B)$ and $\mathrm{Hom}(Z \times_S U, B)$ and F_1, F_2 and $\widetilde{F}$ be the corresponding fibers with respect to the obvious maps. Then $F = F_1 \times_{\widetilde{F}} F_2$, and the maps $F_i \to \widetilde{F}$, $i = 1, 2$, are fibrations by the following Lemma. Hence F is contractible as well. □

Lemma 14.4. *Let $\mathcal{M}$ be a set of injections in C, let $\varphi: A \to B$ be a map in $\mathcal{M}$ and let $f: X \to Y$ be a map in $J_{\mathcal{M}}$-inj. Let $y \in \mathrm{Hom}(B, Y)$ and y' be the image in $\mathrm{Hom}(A, Y)$. Then the map of fibers $\mathrm{Hom}(B, f)^{-1}(y) \to \mathrm{Hom}(A, f)^{-1}(y')$ is a fibration.*

Proof. It follows from the definition of $J_{\mathcal{M}}$ that the map in question has the right lifting property with respect to the maps $\Lambda_k^n \subset \Delta^n$. □

So $\mathcal{M}_{P,\square}$ is a set of monomorphisms in $\triangle^{\mathrm{op}}\mathcal{S}$ satisfying C0-C4.

14.3. **Sheaves with transfers.** We now have to explain what $\mathcal{A}$ will be. We denote by $Cor(C)$ (respectively $Cor_{equi}(C)$) the category with the same objects as C and with homomorphism groups $\mathrm{Hom}_{Cor(C)}([X], [Y]) = c(X, Y) := c(X \times_S Y/X, 0)$ (respectively $\mathrm{Hom}_{Cor_{equi}(C)}([X], [Y]) = c_{equi}(X, Y) := c_{equi}(X \times_S Y/X, 0)$) (for notation see [SV1, after Lemma 3.3.9]).

There is a functor $C \to Cor_{(equi)}(C)$, which sends X to $[X]$ and a morphism $f: X \to Y$ to the cycle associated to the closed subscheme $X_{\mathrm{red}} \subset X \times_S Y$, the graph of f. A *presheaf with (equidimensional) transfers* on $Cor_{(equi)}(C)$ is an additive contravariant functor from $Cor_{(equi)}(C)$ to the category of abelian groups. It is called a t_P-sheaf with transfers if the composite with $C \to Cor_{(equi)}(C)$ is a

t_P-sheaf. The category of presheaves with transfers (respectively of t_P-sheaves with transfers) is denoted by $PreShv(Cor_{(equi)}(C))$ (respectively $Shv_P(Cor_{(equi)}(C))$).

Assumption 14.5. *We have to make the following restrictions to ensure that there is an exact associated sheaf functor: We use either the Nisnevich topology with sheaves with equidimensional transfers on Sch/S or Sm/S or the cdh-topology with sheaves with all transfers on Sch/S.*

Assuming this assumption $PreShv(Cor_{(equi)}(C))$ and $Shv_P(Cor_{(equi)}(C))$ are Grothendieck abelian categories and there is an exact associated sheaf functor

$$PreShv(Cor_{(equi)}(C)) \to Shv_P(Cor_{(equi)}(C))$$

which commutes with the functor which forgets the transfers. Write $\mathbb{Z}_{tr}(X)$ for the corresponding t_P-sheaf with transfers associated to the presheaf with transfers represented by $[X]$. Note that the map $\mathbb{Z}_{tr}(X_{\text{red}}) \to \mathbb{Z}_{tr}(X)$ is an isomorphism.

In the following we sometimes abbreviate $\times_S$ by $\times$.

14.4. **The tensor structure for sheaves with transfers.**

Proposition 14.6. *$Cor_{(equi)}(C)$ is a symmetric monoidal additive category where the tensor product on objects is given by $[X] \otimes [Y] = [X \times_S Y]$.*

Proof. In the following we write $c(X,Y)$ for either $c(X,Y)$ or $c_{equi}(X,Y)$ (similarly for $c(X/S,0)$). Let $X, Y, X', Y' \in C$. We define a bilinear exterior product map

$$\_ \boxtimes \_ : \ c(X,Y) \times c(X',Y') \to c(X \times_S X', Y \times_S Y')$$

in the following way: Let $W \in c(X,Y)$ and $W' \in c(X',Y')$. Let $Z' := cycl(X \times X' \to X')(W') \in c(X \times X' \times Y'/X \times X', 0)$ and $Z := cycl(X \times X' \times Y' \to X)(W) \in c(X \times Y \times X' \times Y'/X \times X' \times Y', 0)$ (see [SV1, p. 29] for the definition of *cycl*). Then the correspondence homomorphism from [SV1, 3.7] yields a cycle $W \boxtimes W' := Cor(Z, Z') \in c(X \times Y \times X' \times Y'/X \times X', 0)$, which defines the desired map.

Claim 1: The diagram

$$\begin{array}{ccc} c(X,Y) \otimes c(X',Y') & \longrightarrow & c(X \times_S X', Y \times_S Y') \\ \downarrow & & \downarrow \\ c(X',Y') \otimes c(X,Y) & \longrightarrow & c(X' \times_S X, Y' \times_S Y) \end{array} ,$$

where the vertical maps are (induced from) natural commutativity morphisms, commutes.

Proof. Let $W = \sum_{i=1}^n n_i W_i$ and $W' = \sum_{i=1}^m m_i W_i'$, where the W_i and W_i' are integral schemes. Let $\eta_1, \ldots, \eta_r$ be the generic points of $X \times_S X'$. The $W_i \times_S W_j'$ are naturally schemes over $X \times_S X'$, and we denote by $(W_i \times_S W_j')_{\eta_k}$ their pullbacks to the η_k. We show that

$$W \boxtimes W' = \sum_{i,j,k} n_i m_j \, cycl_{\eta_k \times_S Y \times_S Y'}((W_i \times_S W_j')_{\eta_k}) \in Cycl(X \times Y \times X' \times Y') ,$$

from which the claim follows because of the symmetry of the expression on the right hand side (here $cycl_{\_}(\_)$ is defined as in [SV1, 2.3]). First note that $cycl(X \times X' \to X')(W') = \sum_{i,k} m_i \, cycl_{\eta_k \times_S Y'}((X \times_S W_i')_{\eta_k})$, because every η_k lies over a generic point of X' and so the pullback to a blow-up of X' appearing in the definition of *cycl* does not matter in this case. Let τ be a generic point of $\tau' := (X \times_S W_i')_{\eta_k}$ appearing in there with multiplicity l_τ and T_τ be the closure of τ in $X \times X' \times Y'$. Then by definition of *Cor* the product $W \boxtimes W'$ is the sum over i, k and all such τ of the $m_i l_\tau \cdot cycl(T_\tau \hookrightarrow X \times X' \times Y')(Z) = m_i l_\tau \cdot cycl(T_\tau \to X)(W)$. As above we have $cycl(T_\tau \to X)(W) = \sum_j n_j \, cycl_{\tau \times_S Y}(\tau \times_X W_j)$. Moreover by comparing dimensions of vector spaces over $\kappa(\eta_k)$ we have $\sum_\tau l_\tau \cdot cycl_{\eta_k \times_S Y \times_S Y'}(\tau \times_X W_j) = cycl_{\eta_k \times_S Y \times_S Y'}(\tau' \times_X W_j)$, where the sum is over all generic points τ of τ', so because $\tau' \times_X W_j = (W_j \times_S W_i')_{\eta_k}$ we are finished. □

Claim 2: Let $X, Y, Z, W \in Cor_{(equi)}(C)$ and $f \in c(X, Y)$ and $g \in c(Y, Z)$. Then

$$(g \boxtimes \mathrm{Id}_W) \circ (f \boxtimes \mathrm{Id}_W) = (g \circ f) \boxtimes \mathrm{Id}_W \,.$$

Proof. First observe that $f \boxtimes \mathrm{Id}_W = i_*(cycl(X \times W \to X)(f))$, where $i: \; X \times Y \times W \hookrightarrow X \times Y \times W \times W$ is identity times diagonal. Let $g_X := cycl(X \times Y \to Y)(g)$. Then by [SV1, Proposition 3.6.2] $(g \circ f) \boxtimes \mathrm{Id}_W$ is the pushforward of $h := cycl(X \times W \to X)(Cor(g_X, f))$ with respect to the natural map $\varphi: \; X \times Y \times Z \times W \to X \times Z \times W \times W$. Let $f_W := cycl(X \times W \to X)(f)$ and $g_{XW} := cycl(X \times Y \times W \to Y)(g)$. By [SV1, Theorem 3.7.3] we have $h = Cor(g_{XW}, f_W)$, so one needs to show that $\varphi_*(Cor(g_{XW}, f_W)) = (g \boxtimes \mathrm{Id}_W) \circ (f \boxtimes \mathrm{Id}_W)$, which we leave to the reader since it is straightforward. □

Claim 3: Let $X, Y, X', Y' \in Cor_{(equi)}(C)$ and $f \in c(X, Y)$, $g \in c(X', Y')$. Then we have

$$(f \boxtimes \mathrm{Id}_{Y'}) \circ (\mathrm{Id}_X \boxtimes g) = f \boxtimes g \,.$$

Proof. Straightforward. □

Now the three claims immediately imply that the exterior product $\boxtimes$ together with the obvious associativity and commutativity morphisms define a symmetric monoidal structure on $SmCor(S)$. □

From now on we assume that the conditions of Assumption 14.5 are fulfilled.

Definition 14.7. *Let $F, G, H \in Shv_P(Cor_{(equi)}(C))$. A* bilinear map $F \times G \to H$ *is a bilinear map for F, G and H considered as presheaves on C such that the induced bilinear maps $F(U) \times G(V) \to F(U \times V) \times G(U \times V) \to H(U \times V)$ are functorial in U and V for all maps from $Cor_{(equi)}(C)$. Denote by $\mathrm{Bil}(F \times G, H)$ the group of bilinear maps from $F \times G$ to H.*

Remark. This is the same as giving a system of bilinear maps as in [SV2, Lemma 2.1].

Lemma 14.8. *For $X, Y \in C$ there is a bilinear map*

$$b_{X,Y}: \; \mathbb{Z}_{tr}(X) \times \mathbb{Z}_{tr}(Y) \to \mathbb{Z}_{tr}(X \times_S Y)$$

which is universal for bilinear maps $\mathbb{Z}_{tr}(X) \times \mathbb{Z}_{tr}(Y) \to H$, $H \in Shv_{Nis}(SmCor(S))$.

Proof. For $U \in C$ the bilinear map $c(U,X) \times c(U,Y) \to c(U, X \times Y)$ is given as the composition $c(U,X) \times c(U,Y) \to c(U \times U, X \times Y) \to c(U, X \times Y)$, where the first arrow is the exterior product map and the second one is composition with the diagonal $U \to U \times U$. The corresponding bilinear maps $\mathbb{Z}_{tr}(X)(U) \times \mathbb{Z}_{tr}(Y)(V) \to \mathbb{Z}_{tr}(X \times Y)(U \times V)$ are the exterior product maps, which are functorial for morphisms from $Cor_{(equi)}(C)$ by Proposition 14.6. This defines $b_{X,Y}$.

Now let $\varphi : \mathbb{Z}_{tr}(X) \times \mathbb{Z}_{tr}(Y) \to H$, $H \in Shv_P(Cor_{(equi)}(C))$, be a bilinear map and p_X and p_Y be the projections from $X \times Y$ to X and Y. Let ψ be the map $\mathbb{Z}_{tr}(X \times Y) \to H$ corresponding to $h := \varphi(p_X, p_Y) \in H(X \times Y)$. Let $(f,g) \in c(U,X) \times c(U,Y)$. Clearly $(\psi \circ b_{X,Y})(f,g) = (U \to X \times Y)^*(h)$, which is by the functoriality of φ with respect to morphisms from $Cor_{(equi)}(C)$ also equal to $\varphi(f,g)$, so $\varphi = \psi \circ b_{X,Y}$. Moreover ψ is uniquely determined by this equality. □

Lemma 14.9. *Let I and J be small categories and $F : I \to Shv_P(Cor_{(equi)}(C))$, $G : J \to Shv_P(Cor_{(equi)}(C))$ be diagrams. Then for $H \in Shv_P(Cor_{(equi)}(C))$ we have*

$$\mathrm{Bil}((\mathrm{colim} F) \times (\mathrm{colim} G), H) = \lim_{(i,j) \in I \times J} \mathrm{Bil}(F_i \times G_j, H) .$$

Proof. Straightforward. □

Let the notation be as in the Lemma. It follows that if we have universal bilinear maps $F_i \times G_j \to H_{(i,j)}$ then the natural bilinear map $(\mathrm{colim} F) \times (\mathrm{colim} G) \to \mathrm{colim}_{(i,j)} H_{(i,j)}$ is also universal. Since every sheaf is the colimit of sheaves of the form $\mathbb{Z}_{tr}(X)$ for $X \in C$ and because of Lemma 14.8 we can make the

Definition 14.10. *For $F, G \in Shv_P(Cor_{(equi)}(C))$ a* tensor product *for F and G is a universal bilinear map $F \times G \to F \otimes G$. This exists, is unique up to unique isomorphism and commutes with colimits.*

One easily shows that this tensor product defines a symmetric monoidal structure on $Shv_P(Cor_{(equi)}(C))$. The embedding $Cor_{(equi)}(C) \hookrightarrow Shv_P(Cor_{(equi)}(C))$, $[X] \mapsto \mathbb{Z}_{tr}(X)$, is symmetric monoidal.

14.5. **Spaces and sheaves with transfers.** We still work in the situation of Assumption 14.5 for the abelian side. So let $\mathcal{A}$ be either $Shv_{Nis}(Cor_{equi}(Sm/S))$ or $Shv_{cdh}(Cor(Sch/S))$ and $\mathcal{S}$ either $Shv_{Nis}(Sm/S)$ or $Shv_{cdh}(Sch/S)$. The functor $C \to Cor_{(equi)}(C)$ extends to a left adjoint $l : \mathcal{S} \to \mathcal{A}$ by requiring that it commutes with colimits. One easily checks that $\mathcal{S}$, $\mathcal{A}$ and l satisfy the conditions of section 14.1. Let $\mathcal{M}_{P,\square}$ be as in section 14.2.

Lemma 14.11. *$l(\mathcal{M}_{P,\square})$ consists of monomorphisms.*

Proof. A map in $l(\mathcal{M}_{P,\square})$ for which we have to prove something is of the form $\mathbb{Z}_{tr}((Z \subset X)\square(U \subset Y))$ for $Z \subset X$ closed and $U \subset Y$ open. We prove that the sequence

$$0 \to \mathbb{Z}_{tr}(Z \times U) \to \mathbb{Z}_{tr}(X \times U) \oplus \mathbb{Z}_{tr}(Z \times Y) \xrightarrow{\varphi} \mathbb{Z}_{tr}(X \times Y)$$

is exact as a sequence of presheaves. Let $V \in C$. Let $\alpha = \alpha_1 \oplus \alpha_2 \in \ker(\varphi(V))$. Then the α_i, $i = 1, 2$, consist of integral subschemes supported on $V \times Z \times U$, and by the next Lemma the corresponding cycle on $V \times Z \times U$ belongs to $c_{(equi)}(V \times Z \times U/V, 0)$. □

Lemma 14.12. *Let $X \in Sch/S$ and $Y \subset X$ locally closed. Let $\mathcal{Z} = \sum m_i z_i \in Cycl(Y)$ (see* [SV1, Section 2.3]*) be a cycle such that the closures of the z_i in Y are proper over S. We denote by $\mathcal{Z}$ also the image of $\mathcal{Z}$ in $Cycl(X)$.*

(1) *If $\mathcal{Z}$ belongs to $Cycl(X/S,r)$ (respectively to one of the other cycle subgroups of $Cycl(X)$ defined in* [SV1, Definition 3.1.3]*), then $\mathcal{Z}$ also belongs to $Cycl(Y/S,r)$ (respectively to the corresponding cycle subgroup of $Cycl(Y)$).*
(2) *If $\mathcal{Z}$ belongs to $c(X/S,r)$ (respectively to $c_{equi}(X/S,r)$), then $\mathcal{Z}$ also belongs to $c(Y/S,r)$ (respectively to $c_{equi}(Y/S,r)$).*

Proof. The first point follows from the fact that the pullback of $\mathcal{Z}$ along fat points does not depend on whether we consider $\mathcal{Z}$ as a cycle on Y or on X because of the properness assumption. The other cycle subgroups of [SV1, Definition 3.1.3] also do not depend on whether we consider $\mathcal{Z}$ on Y or X. For the second point we have to check that the equivalent conditions of [SV1, Lemma 3.3.9] are fulfilled for $\mathcal{Z}$ on Y if they are fulfilled for $\mathcal{Z}$ on X. This is the case since the pullback of $\mathcal{Z}$ along a map $T \to S$ does again not depend on whether we consider $\mathcal{Z}$ on Y or X and because $Cycl(Y/S,r)_{\mathbb{Q}} \cap Cycl(X/S,r) = Cycl(Y/S,r)$. □

Hence we have all conditions satisfied for $\mathcal{M}_{P,\square}$, so by Proposition 14.1 we get symmetric monoidal model structures on $\triangle^{\mathrm{op}}\mathcal{S}$ and $Ch(\mathcal{A})$ together with a pseudo symmetric monoidal Quillen functor $\triangle^{\mathrm{op}}\mathcal{S} \to Ch(\mathcal{A})$.

14.6. $\mathbb{A}^1$**-localizations.** From now on we suppose that C is either Sch/S or Sm/S.

Lemma 14.13. *The model structures on $\triangle^{\mathrm{op}}\mathcal{S}$ and $Ch(\mathcal{A})$ are cellular.*

Proof. Since the model structures are finitely generated it suffices to check condition 3 of [Hov3, Definition A.1], which is immediate. □

Hence by [Hir, Theorem 4.1.1] we can take the left Bousfield localization of $\triangle^{\mathrm{op}}\mathcal{S}$ (respectively $Ch(\mathcal{A})$) with respect to maps $\Delta^n \times (\mathbb{A}^1_X \to X)$, $n \in \mathbb{N}$ (respectively $S^n\mathbb{Z}_{tr}(\mathbb{A}^1_X \to X)$, $n \in \mathbb{Z}$), where X runs through a set of representatives of isomorphism classes of C. We denote the corresponding model category by $Spc(S)$ (respectively $M^{\mathit{eff}}(S)$). These are symmetric monoidal model categories. Let $\mathcal{H}(S) := \mathrm{Ho}\, Spc(S)$ and $DM^{\mathit{eff}}_{(\geq 0)}(S) := \mathrm{Ho}\, M^{\mathit{eff}}(S)$ (where (≥ 0) refers to which of $Comp(\mathcal{A})$ or $Comp_{\geq 0}(\mathcal{A})$ we have taken for $Ch(\mathcal{A})$). Note that for $C = Sm/S$ and P the upper cd-structure $\mathcal{H}(S)$ is the motivic homotopy category defined in [MV]. Let $Spc_\bullet(S)$ be the pointed version of $Spc(S)$ provided by [Hov1, Proposition 1.1.8] and set $\mathcal{H}_\bullet(S) := \mathrm{Ho}\, Spc_\bullet(S)$.

We have symmetric monoidal left adjoints

$$\mathcal{H}(S) \to \mathcal{H}_\bullet(S) \to DM^{\mathit{eff}}_{\geq 0}(S) \to DM^{\mathit{eff}}(S)\ .$$

14.7. **T-stabilizations.** Let $\mathbf{T}$ be a cofibrant object in $Spc_\bullet(S)$ weakly equivalent to $(\mathbb{P}^1, \infty)$. Denote by $Sp^{\Sigma}_{\mathbf{T}}(S)$ (respectively $M_{(\geq 0)}(S)$) the symmetric monoidal category of symmetric $\mathbf{T}$-spectra in $Spc_\bullet(S)$ (respectively of symmetric $L(\mathbf{T})$-spectra in $M^{\mathit{eff}}_{(\geq 0)}(S)$) provided by [Hov3, Theorem 7.11]. Note that the functor $M_{\geq 0}(S) \to M(S)$ is a Quillen equivalence. We denote the corresponding homotopy categories by $S\mathcal{H}(S)$ and $DM(S)$.

Very often it is possible to compare symmetric and non-symmetric spectra for these categories (i.e. to construct equivalences for the homotopy categories), but up to now not in full generality.

14.8. **Functoriality.** The model categories $Spc(S)$, $Spc_\bullet(S)$, $M^{eff}_{(\geq 0)}(S)$, $Sp^\Sigma_{\mathbf{T}}(S)$ and $M_{(\geq 0)}(S)$ depend functorially on S, i.e. they define left Quillen presheaves on the category of separated Noetherian schemes (see [Hi-Si, Section 17]). The natural Quillen functors between these model categories extend to morphisms of left Quillen presheaves.

Let $\mathcal{C}(S)$ be one of these model categories and let $f : X \to S$ be an object of the underlying site $C = C(S)$. We have functors $M_S : C(S) \to \mathcal{C}(S)$ and $M_X : C(X) \to \mathcal{C}(X)$. In this situation f^* is also a right Quillen functor with left adjoint $f_!$ which sends $M_X(Y)$ to $M_S(Y)$ for $Y \in C(X)$.

Proposition 14.14. *Let $f : X \to S$ be an object in $C(S)$ and let $A, B \in \mathrm{Ho}\,\mathcal{C}(S)$ and $C \in \mathrm{Ho}\,\mathcal{C}(X)$. Then we have:*

(1) *There is a canonical isomorphism $f_* f^* A \cong \underline{\mathrm{Hom}}_{\mathrm{Ho}\,\mathcal{C}(S)}(M_S(X), A)$.*
(2) *The natural map*
$$f_!(C \otimes f^* A) \to f_! C \otimes A$$
in $\mathrm{Ho}\,\mathcal{C}(S)$ is an isomorphism.
(3) *The natural map*
$$f^* \underline{\mathrm{Hom}}(A, B) \to \underline{\mathrm{Hom}}(f^* A, f^* B)$$
in $\mathrm{Ho}\,\mathcal{C}(X)$ is an isomorphism.

Proof. We prove the third point, the first two are similar but easier. Since f^* is also a right Quillen functor it commutes with homotopy limits and fiber sequences, hence we can assume that A is of the form $M_S(U)$, $U \in C(S)$. Let $B \in \mathcal{C}(S)$ be fibrant and cofibrant. The category $\mathcal{C}(S)$ consists of sheaves (maybe with transfers) on $C(S)$ with values in some category $\mathcal{V}$, and there is a functor $v : \mathcal{V} \to \mathbf{Set}$ such that for $F \in \mathcal{C}(S)$ and $V \in C(S)$ we have $\mathrm{Hom}(M_S(V), F) = v(F(V))$. Let $V \in C(X)$. We have
$$f^* \underline{\mathrm{Hom}}_{\mathcal{C}(S)}(M_S(U), B)(V) = \underline{\mathrm{Hom}}_{\mathcal{C}(S)}(M_S(U), B)(V)$$
$$= v(B(U \times_Y V)) = v(B(U_X \times_X V)) = \underline{\mathrm{Hom}}_{\mathcal{C}(X)}(M_X(U_X), f^* B)(V)\ ,$$
which shows the claim. □

15. Applications

In this section we will give applications of the general theory of E_∞-algebras in the model categories $\mathcal{C}(S)$ as in section 14.8. In particular we use Theorem 13.4 to construct what we call limit motives. A special case thereof is a motivic definition of tangential basepoints.

We need some preparations.

Let $\mathcal{C}$ be a cofibrantly generated model category with generating (trivial) cofibrations I (J) which we assume to be almost finitely generated (see [Hov3, Section 4], where we use a slightly stronger condition for an object $F \in \mathcal{C}$ to be finitely presented, namely we require that $\mathrm{Hom}(F, \_)$ commutes with λ-sequences for all ordinals λ, not only for $\lambda = \omega$. Let J' be the second set of trivial cofibrations appearing in the definition of almost finitely generated.

We formulate the following further assumptions on $\mathcal{C}$:

- D1 The cofibrations in $\mathcal{C}$ are monomorphisms. This has the consequence that any subcomplex of a relative I-cell complex (see [Hir, Definition 12.5.7]) is uniquely determined by its set of cells.
- D2 There is a class $\mathcal{F}$ of finitely presented objects in $\mathcal{C}$ containing the domains and codomains of maps of I and J and closed under finite coproducts such that the following assertion is valid: For any $F \in \mathcal{F}$, triangle $B \leftarrow A \rightarrow C$ in $\mathcal{C}$ and map $\varphi : F \rightarrow B \sqcup_A C$ there is an $F' \in \mathcal{F}$ and a map $\psi : F' \rightarrow B$ such that for any cofibration $B' \rightarrow B$ such that $A \rightarrow B$ factors through B' φ factors through $B' \sqcup_A C$ if and only if ψ factors through B'.
- D3 There is a functorial cylinder object $F \mapsto F \otimes I$ such that $F \otimes I$ is finitely presented if $F \in \mathcal{F}$ and which preserves cofibrant objects.

Lemma 15.1. *Let $X : \lambda \rightarrow \mathcal{C}_f$ be a diagram. Then* $\mathrm{colim} X_i$ *also belongs to* $\mathcal{C}_f$*.*

Proof. Immediate from the definition of almost finitely generated. □

Lemma 15.2. *Let $A \in \mathcal{C}$ be finitely presented and cofibrant with a cylinder object $A \otimes I$ which is also finitely presented and cofibrant. Let $X : \lambda \rightarrow \mathcal{C}$ be a diagram. Then the natural map*

$$\mathrm{colim}_i \mathrm{Hom}_{\mathrm{Ho}\,\mathcal{C}}(A, X_i) \rightarrow \mathrm{Hom}_{\mathrm{Ho}\,\mathcal{C}}(A, \mathrm{hocolim}_i Q X_i)$$

is an isomorphism.

Proof. We can always achieve that the diagram X is a λ-sequence in $\mathcal{C}_{cf}$ and that the transition maps are cofibrations. Then $\mathrm{colim} X_i$ is fibrant by the Lemma above and computes the homotopy colimit. Because of the assumptions on A the homotopy classes of maps from A to $\mathrm{colim} X_i$ coincides with the colimit of the homotopy classes of maps from A to the X_i, which is the statement we want to prove. □

Lemma 15.3. *Assume that $\mathcal{C}$ fulfills D1 and D2. Let $X : \lambda \rightarrow \mathcal{C}$ be a relative I-*cell *complex, $F \in \mathcal{F}$ and $f : F \rightarrow \mathrm{colim} X$ a map. Then there is a smallest finite subcomplex of X through which f factors.*

Proof. By transfinite induction on λ: If λ is a limit ordinal or the successor of a limit ordinal then because F is finitely presented f factors through some X_i, so the assertion follows by induction hypothesis. Let $\lambda = \alpha + 2$ and suppose that f does not factor through X_α. Let $X_{\alpha+1}$ be a pushout of X_α by $\varphi : A \rightarrow B$ in I. Choose an F' as in D2 for the triangle $X_\alpha \leftarrow A \rightarrow B$. Then there is a smallest finite subcomplex of $X|_{\alpha+1}$ through which the map $A \sqcup F' \rightarrow X_\alpha$ factors. Then the pushout of this subcomplex by φ is the desired finite subcomplex. □

Corollary 15.4. *Let D1 and D2 be fulfilled. The intersection of subcomplexes of a relative I-*cell *complex (which is defined by the intersection of the set of cells) is again a subcomplex. The union of subcomplexes is a subcomplex.*

Lemma 15.5. *Let D1 and D2 be fulfilled. Let $L: \lambda \to \mathcal{C}$ be an I-*cell *complex and $X \in \mathcal{C}$. Then the map*

$$\mathrm{Hom}(\mathrm{colim} L, X) \to \lim_{K \subset L} \mathrm{Hom}_{\mathrm{Ho}\,\mathcal{C}}(\mathrm{colim} K, X) \ ,$$

where $K \subset L$ runs over the filtered system of all finite subcomplexes, is an isomorphism.

Proof. Let X be fibrant and $X_\bullet$ a simplicial frame on X. For a subcomplex K of L let $S_K := \mathrm{Hom}(\mathrm{colim} K, X_\bullet) \in \mathbf{SSet}$. For inclusions of subcomplexes $K \subset K'$ the map $S_{K'} \to S_K$ is a fibration in $\mathbf{SSet}$. We have to prove bijectivity of the natural map $\varphi: \ \pi_0(\lim_K S_K) \to \lim_K \pi_0(S_K)$, where K runs through all finite subcomplexes. Therefore we choose a well-ordering on the set of finite subcomplexes $\mathcal{S}$. Then surjectivity of φ follows like this: Let $(c_K)_{K \in \mathcal{S}}$ be an element in the image of φ. We can choose preimages of the c_K in the order of the well-ordering in the way that they are compatible among themselve, which means that the preimages should coincide on the intersection of the subcomplexes where they are already defined. Injectivity follows in the same way by lifting homotopies. □

Proposition 15.6. *Let $\mathcal{C}$ be a left proper cellular symmetric monoidal and let $K \in \mathcal{C}$ be cofibrant. Suppose given a full subcategory $\mathcal{A} \subset \mathrm{Ho}\,\mathcal{C}$ which contains the images of all domains and codomains of the generating cofibrations of $\mathcal{C}$ and is stable under $\_\otimes K$. Suppose further that $\_\otimes K|_{\mathcal{A}}$ induces isomorphisms on homotopy function complexes in* $\mathrm{Ho}\,\mathbf{SSet}$. *Then we have:*

(1) *The composition*

$$\mathcal{A} \subset \mathrm{Ho}\,\mathcal{C} \to \mathrm{Ho}\,Sp^{\Sigma}(\mathcal{C}, K)$$

is a full embedding.

(2) *Suppose further that $\mathcal{C}$ is stable and fulfills D1-D3. Then the functor*

$$\mathrm{Ho}\,\mathcal{C} \to \mathrm{Ho}\,Sp^{\Sigma}(\mathcal{C}, K)$$

is a full embedding.

Proof. First observe that by [Hov1, Theorem 5.6.5] the homotopy function complexes $\mathrm{map}(A \otimes K, B \otimes K)$ and $\mathrm{map}(A, \underline{\mathrm{Hom}}(K, B \otimes K))$ are naturally isomorphic in $\mathrm{Ho}\,\mathbf{SSet}$ for $A, B \in \mathrm{Ho}\,\mathcal{C}$. From this, our hypothesis on $\_\otimes K|_{\mathcal{A}}$ and $\mathcal{A}$ and from [Hov4, Proposition 5.2] it follows that the natural map $A \to \underline{\mathrm{Hom}}(K, A \otimes K)$ is an isomorphism for $A \in \mathcal{A}$ (*).

The next Lemma shows that the Proposition will follow from

Claim: The unit map for the adjunction between $\mathrm{Ho}\,\mathcal{C}$ and $\mathrm{Ho}\,Sp^{\Sigma}(\mathcal{C}, K)$ evaluated on objects from $\mathcal{A}$ is an isomorphism.

Let $A \in \mathcal{C}$ be a cofibrant object which maps to the essential image of $\mathcal{A}$ in $\mathrm{Ho}\,\mathcal{C}$. Let $R'F_0^K A$ be a fibrant replacement for $F_0^K A$ for the projective model structure on $Sp^{\Sigma}(\mathcal{C}, K)$. From (*) it follows that $R'F_0^K A$ is an Ω-spectrum, i.e. is already fibrant for the stable model structure. So the right adjoint to LF_0^K sends $R'F_0^K A$ to $\mathrm{Ev}_0 R'F_0^K A$, and the unit morphism $A \to \mathrm{Ev}_0 R'F_0^K A$ is clearly an isomorphism, which proves the first claim.

For the second claim let X and Y be I-cell complexes in $\mathcal{C}$. For a finite subcomplex $K \subset X$ clearly K and $K \otimes I$ are finitely presented, so by Lemma 15.2

we have $\mathrm{Hom}_{\mathrm{Ho}\,\mathcal{C}}(K,Y) = \mathrm{colim}_i \mathrm{Hom}_{\mathrm{Ho}\,\mathcal{C}}(K,Y_i)$. The same statement is valid in $Sp^{\Sigma}(\mathcal{C},K)$. Now since $\mathcal{C}$ is stable it therefore follows by transfinite induction and the first part of the Proposition that the map $\mathrm{Hom}_{\mathrm{Ho}\,\mathcal{C}}(K,Y) \to \mathrm{Hom}_{\mathrm{Ho}\,Sp^{\Sigma}(\mathcal{C},K)}(K,Y)$ is an isomorphism (we omitted applying the appropriate functor). Finally Lemma 15.5 implies that also the map $\mathrm{Hom}_{\mathrm{Ho}\,\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathrm{Ho}\,Sp^{\Sigma}(\mathcal{C},K)}(X,Y)$ is an isomorphism. □

Lemma 15.7. *Let $L: \mathcal{C} \leftrightarrow \mathcal{D} : R$ be an adjunction and suppose that the unit map is an isomorphism on a full subcategory $i: \mathcal{A} \to \mathcal{C}$. Then $L \circ i$ is a full embedding.*

Proof. This follows immediately from the fact that the counit is also an isomorphism on the image of $\mathcal{A}$ under L because the composition

$$LA \xrightarrow{L(\mathrm{unit}_A)} LRLA \xrightarrow{\mathrm{counit}_{LA}} LA$$

is the identity for all $A \in \mathcal{C}$. □

15.1. **Motives over smooth schemes.** For a separated Noetherian scheme set $SmCor(S) = Cor_{equi}(Sm/S)$. We consider the categories of the last section for this case. Let k be a field and set $SmCor(k) = SmCor(Spec(k))$.

We first want to compare the categories $DM^{\mathit{eff}}(k)$ and $DM(k)$ and the categories defined in [Vo3] (see below).

Recall that a presheaf with transfers F on Sm/k is called *homotopy invariant* if for all $X \in Sm/k$ the map $F(X) \to F(X \times \mathbb{A}^1)$ is an isomorphism. An $F \in Shv_{Nis}(SmCor(k))$ is homotopy invariant if it is homotopy invariant as a presheaf with transfers.

Proposition 15.8. *For a complex $Z \in \mathcal{D}(Shv_{Nis}(SmCor(k)))$ the following conditions are equivalent:*

(1) *Z is $\mathbb{A}^1$-local.*
(2) *The map on homomorphism groups in $\mathcal{D}(Shv_{Nis}(SmCor(k)))$*
$$\mathrm{Hom}(S^n\mathbb{Z}_{tr}(X), Z) \to \mathrm{Hom}(S^n\mathbb{Z}_{tr}(X \times \mathbb{A}^1), Z)$$
is an isomorphism for all $X \in Sm/k$ and $n \in \mathbb{Z}$.
(3) *The map on homomorphism groups in $\mathcal{D}(Shv_{Nis}(SmCor(k)))$*
$$\mathrm{Hom}(F, Z) \to \mathrm{Hom}(F \otimes S^0\mathbb{Z}_{tr}(\mathbb{A}^1), Z)$$
is an isomorphism for all $F \in \mathcal{D}(Shv_{Nis}(SmCor(k)))$.
(4) *The map*
$$Z \to \underline{\mathrm{Hom}}(S^0\mathbb{Z}_{tr}(\mathbb{A}^1), Z)$$
is an isomorphism in $\mathcal{D}(Shv_{Nis}(SmCor(k)))$.

The equivalent conditions imply:

(5) *The cohomology sheaves of Z are homotopy invariant.*

If in addition the field k is perfect and Z is bounded from below then conditions 1-4 are equivalent *to condition 5.*

Proof. The implications $1 \Rightarrow 2$ and $3 \Rightarrow 2$ are obvious and the equivalence between 3 and 4 follows from the adjunction between $\otimes$ and $\underline{\mathrm{Hom}}$ and the Yoneda Lemma.

$2 \Rightarrow 1$: clear.

$4 \Rightarrow 5$: Choose a fibrant representative $Z' \in \mathit{Comp}(\mathit{Shv}_{\mathit{Nis}}(\mathit{SmCor}(k)))$ of Z. Since $S^0\mathbb{Z}_{tr}(\mathbb{A}^1)$ is cofibrant the internal hom complex $Z^A := \underline{\mathrm{Hom}}(S^0\mathbb{Z}_{tr}(\mathbb{A}^1), Z')$ in $\mathit{Comp}(\mathit{Shv}_{\mathit{Nis}}(\mathit{SmCor}(k)))$ represents $\underline{\mathrm{Hom}}(S^0\mathbb{Z}_{tr}(\mathbb{A}^1), Z)$ and is itself fibrant. Observe that Z^A sends $X \in \mathit{Sm}/k$ to the complex of abelian groups $Z'(X \times \mathbb{A}^1)$.

If we now suppose that $Z' \to Z^A$ is a quasi-isomorphism it follows that it is already a quasi-isomorphism for presheaves with transfers since evaluation on objects is a right Quillen functor. Hence the cohomology presheaves of Z' are homotopy invariant, and by [Vo3, Theorem 3.1.12] the associated sheaves are as well.

The implication $5 \Rightarrow 2$ under the additional assumptions follows from statement 2 in the proof of [Vo3, Proposition 3.2.3].

For the implication $1 \Rightarrow 4$ on uses [Hov4, Proposition 5.2] and adjointness. □

Recall from [Vo3] that $\mathit{DM}_{-}^{\mathit{eff}}(k)$ is the full subcategory of $\mathcal{D}(\mathit{Shv}_{\mathit{Nis}}(\mathit{SmCor}(k)))$ consisting of complexes bounded from below (note that our indexing of complexes is opposite to that in [Vo3]) and having homotopy invariant cohomology sheaves. If k is perfect this is a triangulated subcategory. Statement 5 of Lemma 15.8 immediately implies

Proposition 15.9. *If k is perfect the composition*

$$\mathit{DM}_{-}^{\mathit{eff}}(k) \to \mathcal{D}(\mathit{Shv}_{\mathit{Nis}}(\mathit{SmCor}(k))) \to \mathit{DM}^{\mathit{eff}}(k)$$

is a full embedding.

Recall also that $\mathit{DM}_{gm}^{\mathit{eff}} \subset \mathit{DM}_{-}^{\mathit{eff}}(k)$ is the full subcategory generated by bounded complexes in $\mathit{Comp}(\mathit{Shv}_{\mathit{Nis}}(\mathit{SmCor}(k)))$ of representable sheaves. The category $\mathit{DM}_{gm}(k)$ is gotten from $\mathit{DM}_{gm}^{\mathit{eff}}(k)$ by Spanier Whitehead stabilization of $\mathbf{T}$.

Theorem 15.10. *If k is perfect and admits resolution of singularities all maps in*

$$\mathit{DM}_{gm}^{\mathit{eff}}(k) \to \mathit{DM}^{\mathit{eff}}(k) \to \mathit{DM}(k) \leftarrow \mathit{DM}_{gm}(k)$$

are full embeddings.

Proof. The claim will follow from Proposition 15.6 with $\mathcal{A} = \mathit{DM}_{gm}^{\mathit{eff}}(k)$, [Vo3, Theorem 4.3.1] if $M^{\mathit{eff}}(k)$ fulfills D1-D3. Only D2 needs some explanation. We choose $\mathcal{F}$ to be the class of all $S^n\mathbb{Z}_{tr}(X)$ and $D^n\mathbb{Z}_{tr}(X)$ for $X \in \mathit{Sm}/k$, $n \in \mathbb{Z}$. Let $B \leftarrow A \to C$ be a triangle in $M^{\mathit{eff}}(k)$, $F = S^n\mathbb{Z}_{tr}(X)$ or $F = D^n\mathbb{Z}_{tr}(X)$ and $\varphi : F \to (B \oplus C)/A$ a map. Then after a Nisnevich cover $U \to X$ φ lifts to a map $\tilde{\varphi} : F' = D^n\mathbb{Z}_{tr}(U) \to B \oplus C$. Let ψ be the first component of $\tilde{\varphi}$ and $B' \subset B$ a subobject such that A factors through B'. Suppose that ψ factors through B', hence we have a map $F' \to (B' \oplus C)/A$, and since $F' \to F$ is an epimorphism φ factors through $(B' \oplus C)/A$. Suppose that φ factors through $(B' \oplus C)/A$. Because

$$\begin{array}{ccc} B' & \xrightarrow{\subset} & B \\ \downarrow & & \downarrow \\ (B' \oplus C)/A & \xrightarrow{\subset} & (B \oplus C)/A \end{array}$$

is a pullback square ψ factors through B'. □

Let k be a field of characteristic 0 and $X \in Sm/k$. Let $\pi: X \to Spec(k)$ be the structure morphism. We are going to apply Theorem 13.4 in the situation $\mathcal{C} = M(k)$ and $\mathcal{D} = M(X)$. We have to check conditions 1-3 of Theorem 13.4. For $B \in DM(k)$ we have $\pi_* \pi^* B \cong \underline{\mathrm{Hom}}_{DM(k)}(\mathbb{Z}_{tr}(X), B)$. Hence the first of the conditions follows from the rigidity of the tensor category $DM_{gm}(k)$, which means that for $A, B \in DM_{gm}(k)$ the natural map $A^\vee \otimes B \to \underline{\mathrm{Hom}}_{DM_{gm}(k)}(A, B)$ is an isomorphism (here we set $A^\vee = \underline{\mathrm{Hom}}(A, \mathbb{Z})$) (see [Vo3, Theorem 4.3.7]).

For the second condition we have the

Lemma 15.11. *Let $f: X \to S$ be a morphism between separated Noetherian schemes. Then the map $f_*: \mathrm{Ho}\,\mathcal{C}(X) \to \mathrm{Ho}\,\mathcal{C}(S)$ preserves homotopy λ-sequences, where the categories $\mathcal{C}(X)$ and $\mathcal{C}(S)$ are as in section 14.8.*

Proof. We prove the case $\mathcal{C}(X) = M(X)$, $\mathcal{C}(S) = M(S)$, the other cases are similar or easier. Suppose given a λ-sequence $Y: \lambda \to M(X)_{cf}$ with cofibrations as transition maps. Since filtered colimits in $Shv_{Nis}(SmCor(S))$ are created in presheafs with transfers f_* commutes with λ-sequences by definition of f_*. Hence we have to check that $\mathrm{colim}_i f_* Y_i \in M(S) = Sp^{\Sigma}_{\mathbf{T}} Comp(Shv_{Nis}(SmCor(S)))$ computes the homotopy colimit. We can find a λ-sequence $\widetilde{Y}: \lambda \to M(S)_{cf}$ where all transition maps are cofibrations together with an objectwise weak equivalence $\widetilde{Y} \to f_* Y$. Since these maps are weak equivalences between fibrant objects it follows that every map $\widetilde{Y}_i \to f_* Y_i$ is a level quasi isomorphism. Hence using the injective model structure on $M^{eff}(S)$ it follows that the map $\mathrm{colim}_i \widetilde{Y}_i \to \mathrm{colim}_i f_* Y_i$ is a weak equivalence, what we wanted to show. □

The third condition is clear.

Let f be as in the Lemma above. We denote the full subcategory of $\mathrm{Ho}\,\mathcal{C}(X)$ $f^*(\mathrm{Ho}\,\mathcal{C}(S))$-generated by homotopy colimits by $U\mathrm{Ho}\,\mathcal{C}(X/S)$ (see Definition 13.1).

Now let again $\pi: X \to Spec(k)$ be a smooth scheme. Let as above $A(X) = \pi_* \mathbb{1} \in D^{\leq 2}\mathrm{Comm}_{M(k)}$ be the motivic cohomology of X relative to k as a commutative algebra in $M(k)$. Then Theorem 13.4 implies

Corollary 15.12. *There is a natural equivalence of tensor triangulated categories*

$$\tilde{\pi}^*: D(A(X)\text{–}Mod) \to UDM(X/k)$$

such that its composition with $DM(k) \to D(A(X)\text{–}Mod)$ is naturally isomorphic to $\pi^*: DM(k) \to DM(X)$ and such that the composition of the right adjoint $\tilde{\pi}_*$ of $\tilde{\pi}^*$ with $D(A(X)\text{–}Mod) \to DM(k)$ is naturally isomorphic to $\pi_*|_{UDM(X/k)}$.

Remark 15.13. *If we assume Spanier Whitehead duality in $S\mathcal{H}(k)$ (which holds) then a similar statement is valid for the category $US\mathcal{H}(X/k)$.*

More generally we can state the

Corollary 15.14. *Let $f: X \to S$ be a morphism between separated Noetherian schemes and let $\mathcal{C}(S)$ be either $M(S)$ or $Sp^{\Sigma}_{\mathbf{T}}(S)$ (same for X). Assume that for any $M \in \mathrm{Ho}\,\mathcal{C}(S)$ the map $M \otimes f_* \mathbb{1} \to f_* f^*(M)$ is an isomorphism. Then there is a natural equivalence of tensor triangulated categories*

$$\tilde{f}^*: D(f_* \mathbb{1}\text{–}Mod) \to U\mathrm{Ho}\,\mathcal{C}(X/S)$$

with similar compatibilities as in Corollary 15.12. A similar statement is valid if we consider categories A–Mod and $f^* A$–Mod for a cofibrant algebra $A \in \mathrm{Comm}_{\mathcal{C}(S)}$.

There is the following application of the symmetric monoidal functor constructed after Lemma 9.14 and of duality:

Proposition 15.15. *Let $X, Y \in Sm/k$. Then the natural map*

$$A(X) \sqcup A(Y) \to A(X \times_k Y)$$

is a (weak) isomorphism in $D^{(\leq 2)}\mathrm{Comm}_{M(k)}$.

15.2. **Limit Motives.** In this section we change our notation. We fix a separated Noetherian base scheme S and a cofibrant algebra $A \in \mathrm{Comm}_{Sp^{\Sigma}_{\mathbf{T}}(S)}$ (here we work with the site Sm/S and the Nisnevich topology for the definition of $Sp^{\Sigma}_{\mathbf{T}}(S)$). Then for every $X \in Sch/S$ with structure map f we denote by $M(X)$ the symmetric monoidal model category with weak unit f^*A–*Mod*. We set $DM(X) := \mathrm{Ho}\, M(X)$. We set $\mathrm{Comm}_{M(X)} := \mathrm{Comm}(f^*A)$ (see Definition 9.5 for the notation). For example if $S = Spec(\mathbb{Q})$ we can take A to be a cofibrant resolution of the motivic Eilenberg-MacLane spectrum $H\mathbb{Z}$ on S. The reason that we changed notation is that morally we would like to work with our previously defined $DM(X)$, but we will need the following exact triangle, which we do not know to hold in the previous $DM(X)$:

Proposition 15.16. *Let $X \in Sch/S$. Let $i : Z \subset X$ be a closed embedding and $j : U \subset X$ the complementary open embedding and let $F \in DM(X)$. Then there is an exact triangle*

$$j_!j^*F \to F \to i_*i^*F \to j_!j^*F[1]$$

in $DM(X)$ (as defined above).

Proof. We first construct a functorial map φ_F from the (functorial) cofiber of the map $j_!j^*F \to F$ to i_*i^*F. Let $F \in M(X)$ be cofibrant. Then factor the map $j_!j^*F \to F$ functorially as $j_!j^*F \to \widetilde{F} \to F$ into a cofibration followed by a weak equivalence. Let $i^*F \to Ri^*F$ be the functorial fibrant replacement in $M(Z)$. Then we have a canonical map $\widetilde{F}/j_!j^*F \to F/j_!j^*F \to i_*i^*F \to i_*Ri^*F$, which gives the desired map in the homotopy category.

For the rest of the proof we can forget the A-module structure on F. First we suppose that F is a domain or codomain of a generating cofibration of $Sp^{\Sigma}_{\mathbf{T}}(S)$. Then F comes up to a shift with the Tate object from $Spc(S)$ where we have the homotopy pushout square of [MV, Theorem 3.2.21]. By the next Lemma the image of this homotopy pushout square in $S\mathcal{H}(S)$ gives the sequence we are looking at (the Lemma ensures that the right lower corner of the homotopy pushout square is correct).

In general write F as a cell complex in $Sp^{\Sigma}_{\mathbf{T}}(S)$. Now clearly $f_!f^*$ preserves homotopy λ-sequences, and by Lemma 15.11 i_* preserves homotopy λ-sequences. Then one shows by induction on λ that φ_F is an isomorphism applying Lemma 13.2 to the successive pushouts of the given λ-sequence. □

Lemma 15.17. *Let $i: Z \subset X$ be a closed embedding of separated Noetherian schemes. Then the square*

$$\begin{array}{ccc} \mathcal{H}_\bullet(Z) & \xrightarrow{\Sigma^\infty_{\mathbf{T}}} & S\mathcal{H}(Z) \\ \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} \\ \mathcal{H}_\bullet(X) & \xrightarrow{\Sigma^\infty_{\mathbf{T}}} & S\mathcal{H}(X) \end{array}$$

commutes (up to a canonical natural isomorphism).

Proof. We can suppose that we work with non-symmetric spectra. Let $F \in Spc_\bullet(Z)$ be fibrant and cofibrant. Let $r: \Sigma^\infty_{\mathbf{T}} F \to R^p \Sigma^\infty_{\mathbf{T}} F$ be a fibrant replacement for the projective model structure on spectra, i.e. r is a level weak equivalence and $R^p \Sigma^\infty_{\mathbf{T}} F$ is level fibrant. Clearly the functor R^∞ of [Hov3, Proposition 4.4] commutes with i_*, hence this Proposition implies that $i_*(R^p \Sigma^\infty_{\mathbf{T}} F)$ computes the derived direct image of $\Sigma^\infty_{\mathbf{T}} F$ in $S\mathcal{H}(X)$. Let $Qi_*F \to i_*F$ be a cofibrant replacement. Then the second part of the next Lemma shows that the canonical map $\Sigma^\infty_{\mathbf{T}} Qi_*F \to i_*(R^p\Sigma^\infty_{\mathbf{T}} F)$ is a level weak equivalence, which finishes the proof. □

Let i be as in the Lemma and let $j: U \to X$ be the complementary open embedding. We denote by $\mathcal{H}_\bullet(X)_Z$ the symmetric monoidal subcategory of $\mathcal{H}_\bullet(X)$ consisting of objects $F \in \mathcal{H}(X)$ such that $j^*F = *$. We remark that we have in $\mathcal{H}_\bullet(X)$ a homotopy pushout square like in [MV, Theorem 3.2.21].

Lemma 15.18. *Let i be as above. Then $i^*|_{\mathcal{H}_\bullet(X)_Z}$ is a symmetric monoidal equivalence. In particular for $F \in \mathcal{H}_\bullet(Z)$ and $G \in \mathcal{H}_\bullet(X)$ we have $i_*F \wedge G = i_*(F \wedge i^*G)$.*

Proof. Let $F \in \mathcal{H}_\bullet(X)_Z$. By [MV, Theorem 3.2.21] the map $F \to i_*i^*F$ is an isomorphism. Hence i^* is an equivalence onto its image and we have to show that the essential image is everything. First note that for any $Y \in Sm/Z$ we can find a Zariski cover $V \to X$ such that there is a $\widetilde{V} \in Sm/X$ with $\widetilde{V}_Z \cong V$. Since Y is gotten from the covering pieces by successive pushouts it follows that Y_+ is in the image of i^*. Then given a homotopy λ-cell complex C in $\mathcal{H}_\bullet(Z)$ one shows inductively on $\alpha \in \lambda$ that the map $i^*i_*C \to C$ is an isomorphism on the subcomplex given by cells $< \alpha$. The second claim follows like this (we prove it with F replaced by some i^*F): Let $F, G \in \mathcal{H}_\bullet(X)$ with $F|_U = *$. Then $(F \wedge G)|_U = *$, hence $i_*(i^*F \wedge i^*G) = i_*i^*(F \wedge G) = F \wedge G$. □

We are now going to construct limit motives. We begin with some preparations.

Proposition 15.19. *Let $i: Z \subset X$ be a closed embedding in Sm/S and let $j: U \subset X$ be the complementary open embedding. Let $p: N \to Z$ be the normal bundle of Z in X and $p^\circ: N^\circ \to Z$ the complement of the zero section. Then there is a natural isomorphism $i^*j_*\mathbb{1} \cong p^\circ_*\mathbb{1}$ in $D\mathrm{Comm}_{M(Z)}$.*

Proof. This combines the next two Lemmas. □

Lemma 15.20. *Let Z be a separated Noetherian scheme, $p: N \to Z$ a (geometric) vector bundle and $i: Z \subset N$ the zero section. Let $j: N \setminus i(Z) \subset N$ be the open inclusion and $p^\circ: N \setminus i(Z) \to Z$ the projection. Then there is a natural isomorphism $i^*j_*\mathbb{1} \cong p^\circ_*\mathbb{1}$ in $D\mathrm{Comm}_{M(Z)}$.*

Proof. The base change morphism for the diagram

$$\begin{array}{ccc} Z & \xrightarrow{i} & N \\ \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle p} \\ Z & \xrightarrow{\mathrm{Id}} & Z \end{array}$$

applied to the algebra $j_*\mathbb{1}$ gives a map $p^\circ_*\mathbb{1} \cong p_*(j_*\mathbb{1}) \to i^*j_*\mathbb{1}$. We show that this map is an isomorphism. From now on we can forget that we deal with algebras since all functors involved commute with forgetting the algebra structure. Proposition 15.16 applied to $j_*\mathbb{1}$ yields an exact triangle

$$j_!j^*j_*\mathbb{1} \to j_*\mathbb{1} \to i_*i^*j_*\mathbb{1} \to j_!j^*j_*\mathbb{1}[1] ,$$

and there is an isomorphism $j_!j^*j_*\mathbb{1} \cong j_!\mathbb{1}$. So the base change morphism applied to this triangle yields a map of triangles

$$\begin{array}{ccccccc} 0 & \longrightarrow & i^*j_*\mathbb{1} & \xrightarrow{\mathrm{Id}} & i^*j_*\mathbb{1} & \longrightarrow & 0 \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ p_*j_!\mathbb{1} & \longrightarrow & p^\circ_*\mathbb{1} & \longrightarrow & i^*j_*\mathbb{1} & \longrightarrow & p_*j_!\mathbb{1}[1] \end{array} ,$$

hence we are ready if we show that $p_*j_!\mathbb{1} = 0$. We apply p_* to the triangle

$$j_!\mathbb{1} \to \mathbb{1} \to i_*\mathbb{1} \to j_!\mathbb{1}[1] .$$

The second map is mapped to an isomorphism since p is an $\mathbb{A}^1$-weak equivalence, hence $p_*j_!\mathbb{1} = 0$. □

Lemma 15.21. *Let the situation be as in Proposition 15.19 and let $i' : Z \subset N$ and $j' : N^\circ \subset N$ be the zero section and its complement. Then there is a natural isomorphism $i^*j_*\mathbb{1} \cong i'^*j'_*\mathbb{1}$ in $D\mathrm{Comm}_{M(Z)}$.*

Proof. We use a similar construction as in the proof of [MV, Theorem 3.2.23]. Let $\pi : B \to X \times \mathbb{A}^1$ be the blow-up of $X \times \mathbb{A}^1$ in $Z \times \{0\}$, $f : Z \times \mathbb{A}^1 \to B$ the canonical closed embedding which splits $i(Z) \times \mathbb{A}^1$ and $g : X \to B$ the closed embedding which splits $X \times \{1\}$. We have $P := \pi^{-1}(Z \times \{0\}) \cong \mathbb{P}(N \oplus \mathcal{O}) \supset \mathbb{P}(N)$. Set $\widetilde{B} := B \setminus \mathbb{P}(N)$, so we have a closed embedding $h : N \to \widetilde{B}$. The maps f and g factor through $\widetilde{B}$, and we denote the factor maps also by f and g.

Let $\widetilde{B}^\circ := \widetilde{B} \setminus f(Z \times \mathbb{A}^1)$ and $j'' : \widetilde{B}^\circ \subset \widetilde{B}$ the open inclusion. We have pullback squares

$$\begin{array}{ccc} U & \longrightarrow & \widetilde{B}^\circ \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j''} \\ X & \xrightarrow{g} & \widetilde{B} \end{array} \quad \text{and} \quad \begin{array}{ccc} N^\circ & \xrightarrow{h^\circ} & \widetilde{B}^\circ \\ \downarrow & & \downarrow{\scriptstyle j''} \\ N & \xrightarrow{h} & \widetilde{B} \end{array} .$$

Claim 1: The two base change morphisms for these diagrams applied to $\mathbb{1}$ are isomorphisms, i.e. we have isomorphisms $i^*j_*\mathbb{1} \cong i_1^*f^*j''_*\mathbb{1}$ and $i'^*j'_*\mathbb{1} \cong i_0^*f^*j''_*\mathbb{1}$, where $i_k : Z \times \{k\} \to Z \times \mathbb{A}^1$, $k = 0, 1$, are the two inclusions.

Let $q: \ Z \times \mathbb{A}^1 \to Z$ be the projection. The base change morphisms applied to the diagrams

$$\begin{array}{ccc} Z \times \{k\} & \xrightarrow{i_k} & Z \times \mathbb{A}^1 \\ \downarrow & & \downarrow \\ Z & \xrightarrow{\mathrm{Id}} & Z \end{array}$$

yields maps $h_k: \ i_k^* f^* j''_* \mathbb{1} \to q_* f^* j''_* \mathbb{1}$, $k = 0, 1$.

Claim 2: The maps h_k, $k = 0, 1$, are isomorphisms.

Claim 1 and 2 obviously give the desired isomorphism $i^* j_* \mathbb{1} \cong i'^* j'_* \mathbb{1}$, so we are left with proving the claims.

After Zariski localization on X we can assume that there is an etale map $e: \ X \to \mathbb{A}^n_S$ such that $Z = e^{-1}(\mathbb{A}^k_S)$. So by Proposition 14.14 (1) and (3) we are reduced to the case $X = \mathbb{A}^n_S$, $Z = \mathbb{A}^k_S$. By changing S to $\mathbb{A}^k_S$ we can also assume that $k = 0$. Let $x_1, \ldots, x_{n+1}$ be coordinates on $\mathbb{A}^{n+1}_S$ and $y_1, \ldots, y_{n+1}$ homogeneous coordinates on $\mathbb{P}^n_S$. Then $B = \{(x, y) \in (\mathbb{A}^{n+1} \times \mathbb{P}^n)_S \mid x_i y_j = x_j y_i, \, i, j = 1, \ldots, n+1\}$. Let $W \subset B$ be defined by the equations $x_i = 0$, $i = 1, \ldots, n+1$ and $y_1 = 0$. Then $\widetilde{B} = B \setminus W$. We consider $\widetilde{B}$ and $\mathbb{A}^{n+1}_S$ as schemes over $\mathbb{A}^1_S$ via the first projection $\mathbb{A}^{n+1}_S \to \mathbb{A}^1_S$. The assignment $x_1 \mapsto x_1$, $x_i \mapsto \frac{y_i}{y_1}$, $i = 2, \ldots, n+1$, yields an isomorphism $\varphi: \ \widetilde{B} \to \mathbb{A}^{n+1}_S$ over $\mathbb{A}^1_S$ such that $(\varphi \circ f)(\mathbb{A}^1_S)$ is the closed subscheme defined by $x_2 = \cdots = x_{n+1} = 0$. Hence by Lemma 15.20 and Proposition 14.14 (1) we have $f^* j''_* \mathbb{1} = M((\mathbb{A}^n \setminus \{0\})_{\mathbb{A}^1_S})^\vee \cong \mathbb{1} \oplus \mathbb{1}(-n)[-2n+1]$. We have analogous descriptions of $i^* j_* \mathbb{1}$ and $i^* j_* \mathbb{1}$, so claim 1 follows. Now also claim 2 follows since $f^* j''_* \mathbb{1}$ is a pullback from S and $\mathbb{A}^1_S \to S$ is an $\mathbb{A}^1$-weak equivalence. $\square$

Let now $i: \ D \subset X$ be a closed embedding in Sm/S such that D is a divisor, let $J: \ X^\circ \to X$ be the complementary open embedding and let $p^\circ: N^\circ \to D$ be the pointed normal bundle of D in X. The morphism p° obviously satisfies the conditions of Corollary 15.14, hence we have an equivalence $UDM(N^\circ/D) \sim D(p^\circ_* \mathbb{1}\text{–}Mod)$. The functor $i^* j_*: \ DM(X^\circ) \to DM(D)$ factors through $D(i^* j_* \mathbb{1}\text{–}Mod)$. Proposition 15.19 suggests that we have a natural equivalence $D(i^* j_* \mathbb{1}\text{–}Mod) \sim D(p^\circ_* \mathbb{1}\text{–}Mod)$, but we get such an equivalence only if we have a morphism from $i^* j_* \mathbb{1}$ to $p^\circ_* \mathbb{1}$ in the 2-category $D^{\leq 2}\mathrm{Comm}_{M(D)}$. Reexamining the proofs of Lemmas 15.20 and 15.21 we find that we have a chain of weak isomorphisms $i^* j_* \mathbb{1} \to B \leftarrow B' \to p^\circ_* \mathbb{1}$ in $D^{\leq 2}\mathrm{Comm}_{M(D)}$ where all maps are unique up to unique 2-isomorphism. Since there is no way in a 2-category to find an inverse unique up to unique 2-isomorphism of a weak isomorphism this chain of weak isomorphisms is the only thing we get. Nevertheless it follows that there is a natural equivalence $D(i^* j_* \mathbb{1}\text{–}Mod) \sim D(p^\circ_* \mathbb{1}\text{–}Mod)$ unique up to unique natural isomorphism by composing the functors induced by the maps in this chain or their adjoints. Now we can define the functor

$$\mathcal{L}_{X,D}: \ DM(X^\circ) \to DM(N^\circ)$$

to be the composition

$$DM(X^\circ) \to D(i^* j_* \mathbb{1}\text{–}Mod) \sim D(p^\circ_* \mathbb{1}\text{–}Mod) \sim UDM(N^\circ/D) \to DM(N^\circ) \, .$$

Intuitively the functor does the following: We first restrict a given motivic sheaf on X° to a tubular neighborhood of D (which of course does not exist). Then we identify this tubular neighborhood with a tubular neighborhood of the zero section in N (the normal bundle of D in X) and carry over the restricted sheaf. This we finally extend to the whole of N°. As long as we believe that some sort of monodromy action around D is unipotent our above definition makes perfect sense to simulate this intuitive description.

We now would like to generalize this construction to the following situation:

Let $X \in Sm/S$ and $D \subset X$ a divisor with normal crossings relative to S, i.e. $D = \bigcup_{i\in I} D_i$ with $D_i \in Sm/S$ and locally in the etale topology the intersections of the D_i look like intersections of coordinate hyperplanes in some $\mathbb{A}^n_S$. Let $X^\circ := X \setminus D$. For $J \subset I$ let $D_J := \bigcap_{i\in J} D_i$ and $D_J^\circ := D_J \setminus \bigcup_{i\in I\setminus J} D_i$. Let N_i be the normal bundle of D_i in X and N_i° the complement of the zero section. Let N_J be the fiber product of the N_i over D_J for $i \in J$ and N_J° the corresponding product of the N_i°. Finally let $N_J^{\circ\circ}$ be the restriction of N_J° to D_J°. Our goal is to construct a functor

$$\mathcal{L}_{X,J}:\ DM(X^\circ) \to DM(N_J^{\circ\circ})$$

for any $J \subset I$. Of course the situation for a general $J \subset I$ is the same as the situation in which we consider all divisors for $X \setminus \bigcup_{i\in I\setminus J} D_i$.

We would like that various $\mathcal{L}_J$ are compatible in the following sense: Consider disjoint subsets $J, J' \subset I$. On the one hand side we can consider the functor $\mathcal{L}_{X,J\cup J'}$. For $i \in I \setminus J$ denote by $\widetilde{D}_i$ the restriction of D_i to D_J and by D'_i the preimage of $\widetilde{D}_i$ in N_J° with respect to the natural projection. The D'_i, $i \in I \setminus J$, are again divisors with normal crossings in N_J° relative to D_J with complement $N_J^{\circ\circ}$. Now we apply the definitions above to this situation to get corresponding objects $D'_{J'}$, ${D'}^\circ_{J'}$, ${N'}^\circ_{J'}$ and ${N'}^{\circ\circ}_{J'}$. Clearly we have canonical isomorphisms ${D'}^\circ_{J'} \cong D^\circ_{J\cup J'}$ and ${N'}^{\circ\circ}_{J'} \cong N^{\circ\circ}_{J\cup J'}$.

So on the other hand we have a composition of functors

$$DM(X^\circ) \xrightarrow{\mathcal{L}_{X,J}} DM(N_J^{\circ\circ}) \xrightarrow{\mathcal{L}_{N_J^\circ,J'}} DM({N'}^{\circ\circ}_{J'}) \sim DM(N^{\circ\circ}_{J\cup J'}) \ .$$

We want to have a natural isomorphism $\varphi_{J',J}$ between this composition and $\mathcal{L}_{X,J\cup J'}$. Moreover if we have three disjoint subsets $J, J'J'' \subset I$ we want to have

$$\varphi_{J''\cup J',J} \circ (\varphi_{J'',J'} \underset{0}{\circ} \mathrm{Id}_{\mathcal{L}_J})\circ = \varphi_{J'',J'\cup J} \circ (\mathrm{Id}_{\mathcal{L}_{J''}} \underset{0}{\circ} \varphi_{J',J}) \ .$$

This compatibility implies all other possible compatibilities.

Below we will only sketch the construction of the $\varphi_{J,J'}$ and only indicate the proof of the compatibility.

We introduce some further notation: Let $j:\ X^\circ \to X$, $X^J := X \setminus \bigcup_{i\in J} D_i$ and $j^J:\ X^J \to X$ be the open inclusion. Let $i_J:\ D_J \to X$ and $i_J^\circ:\ D_J^\circ \to X$ be the closed respectively locally closed embedding. Let furthermore $p_J^\circ:\ N_J^\circ \to D_J$ and $p_J^{\circ\circ}:\ N_J^{\circ\circ} \to D_J^\circ$ be the projections.

Proposition 15.22. *Let $A \in D\mathrm{Comm}_{M(X)}$ such that Zariski-locally on X $A^\sharp$ is a pullback of an object from $DM(S)$. Then there is a canonical isomorphism $i_J^* j_*^J j^{J*} A \cong p^\circ_{J*} p_J^{\circ *} i_J^* A$ in $D\mathrm{Comm}_{M(D_J)}$. Furthermore if A is given as an object in $D^{\leq 2}\mathrm{Comm}_{M(X)}$ there is a natural chain of weak isomorphisms in $D^{\leq 2}\mathrm{Comm}_{M(X)}$ connecting $i_J^* j_*^J j^{J*} A$ and $p^\circ_{J*} p_J^{\circ *} i_J^* A$.*

Proof. We have to give analogues of Lemmas 15.20 and 15.21. The analogue of Lemma 15.20 has the same proof and states in our case that there is a natural isomorphism $i'^* j'_* p_J^{\circ *} i_J^* A \cong p^\circ_{J*} p_J^{\circ *} i_J^* A$, where $j' :\ N_J^\circ \to N_J$ is the open inclusion and $i' :\ D_J \to N_J$ the zero section. We prove the analogue of Lemma 15.21: For every $i \in J$ let π_i, B_i, f_i, g_i, $\widetilde{B}_i$ and $\widetilde{B}_i^\circ$ be as in the proof of Lemma 15.21 for $Z = D_i$. Let $\pi_J :\ \widetilde{B}_J \to X \times \mathbb{A}^1$ be the fiber product of the $\widetilde{B}_i$ over $X \times \mathbb{A}^1$ for $i \in J$ and $\widetilde{B}_J^\circ$ the corresponding fiber product of the $\widetilde{B}_i^\circ$. We have a closed embedding $h_J :\ N_J \to \widetilde{B}_J$. Let $j''_J :\ \widetilde{B}_J^\circ \subset \widetilde{B}_J$ be the open inclusion and $f_J : D_J \times \mathbb{A}^1 \to \widetilde{B}_J$ the intersection of the divisors which build the complement of j''_J. Let $g_J :\ X \to \widetilde{B}_J$ be the product of the g_i. Then we have again pullback squares

$$\begin{array}{ccccccc} X^J & \longrightarrow & \widetilde{B}_J^\circ & \quad\text{and}\quad & N_J^\circ & \xrightarrow{h_J^\circ} & \widetilde{B}_J^\circ \\ \downarrow{\scriptstyle j^J} & & \downarrow{\scriptstyle j''_J} & & \downarrow & & \downarrow{\scriptstyle j''_J} \\ X & \xrightarrow{g} & \widetilde{B} & & N_J & \xrightarrow{h_J} & \widetilde{B}_J \end{array} \quad .$$

Similarly to the proof of Lemma 15.21 one shows that the base change morphisms of these diagrams yield isomorphisms $i_J^* j_*^J j^{J*} A \cong i_1^* f_J^* j''_{J*} \pi_J^* A$ and $i'^* j'_* p_J^{\circ *} i_J^* A \cong i_0^* f_J^* j''_{J*} \pi_J^* A$, $i_k :\ D_J \times \{k\} \to D_J \times \mathbb{A}^1$, $k = 0, 1$, the inclusions. Also one shows that the corresponding maps $h_k :\ i_k^* f_J^* j''_{J*} \pi_J^* A \to q_* f_J^* j''_{J*} \pi_J^* A$, $q :\ D_J \times \mathbb{A}^1 \to D_J$ the projection, are isomorphisms. □

Hence we can define the functor

$$\mathcal{L}_{X,J} :\ DM(X^\circ) \to DM(N_J^{\circ\circ})$$

as the composition

$$DM(X^\circ) \to D(i_J^{\circ *} j_* \mathbb{1}\text{–}Mod) \sim D(p^{\circ\circ}_{J*} \mathbb{1}\text{–}Mod) \sim UDM(N_J^{\circ\circ}/D_J^\circ) \to DM(N_J^{\circ\circ}) \; .$$

We are going to sketch the construction of the natural isomorphism $\varphi_{J',J}$. We can assume that $J \cup J' = I$. First consider the cartesian squares

$$\begin{array}{ccccc} D'_{J'} & \xrightarrow{i'} & N_J^\circ & \xleftarrow{j'} & N_J^{\circ\circ} = (N_J^\circ)^{J'} \\ \downarrow & & \downarrow{\scriptstyle p_J^\circ} & & \downarrow \\ D_I & \xrightarrow{\tilde{i}} & D_J & \xleftarrow{\tilde{j}} & D_J^\circ \end{array} \quad .$$

Let $\tilde{p}^\circ :\ N_{J'}^\circ|_{D_I} \to D_I$ and $p'^\circ :\ N'^\circ_{J'} \to D'_{J'}$ be the projections. We apply Proposition 15.22 to the algebra $A := p^\circ_{J*}\mathbb{1}$ to get an isomorphism $\tilde{p}^\circ_* \tilde{p}^{\circ *} \tilde{i}^* A \cong \tilde{i}^* \tilde{j}_* \tilde{j}^* A$ (*) and also a connecting chain of weak isomorphisms in $D^{\leq 2}\mathrm{Comm}_{M(D_I)}$. The same relation for the algebra $\mathbb{1}$ on N_J° yielded the functor $\mathcal{L}_{N_J^\circ;J'}$. We now get a diagram of functors with natural isomorphisms in the squares

$$\begin{array}{ccccc} DM((N_J^\circ)^{J'}) & \xrightarrow{i'^* j'_*} & D(i'^* j'_* \mathbb{1}\text{–}Mod) & \xrightarrow{\sim} & D(p'^\circ_* \mathbb{1}\text{–}Mod) \\ \uparrow & & \uparrow & & \uparrow \\ D(\tilde{j}^* A\text{–}Mod) & \xrightarrow{\tilde{i}^* \tilde{j}_*} & D(\tilde{i}^* \tilde{j}_* \tilde{j}^* A\text{–}Mod) & \xrightarrow{\sim} & D(\tilde{p}^\circ_* \tilde{p}^{\circ *} \tilde{i}^* A\text{–}Mod) \end{array} \quad ,$$

where we used appropriate naturality of the equivalence in Corollary 15.14 and of the constructions in the proof of Proposition 15.22. We also have a cartesian square

$$\begin{array}{ccc} {N'}^{\circ}_{J'} = N^{\circ}_I & \xrightarrow{p'^{\circ}} & D'_{J'} = N^{\circ}_J|_{D_I} \\ \downarrow{\scriptstyle q'} & & \downarrow{\scriptstyle q} \\ N^{\circ}_{J'}|_{D_I} & \xrightarrow{\tilde{p}^{\circ}} & D_I \end{array} .$$

Hence base change morphisms induce isomorphisms $p^{\circ}_{I*}\mathbb{1} \cong \tilde{p}^{\circ}_* \circ q'_*\mathbb{1} \cong \tilde{p}^{\circ}_*\tilde{p}^{\circ *}q_*\mathbb{1} \cong \tilde{p}^{\circ}_*\tilde{p}^{\circ *}\tilde{i}^*A$ (**), so we get an equivalence

$$D(\tilde{p}^{\circ}_*\tilde{p}^{\circ *}\tilde{i}^*A\text{–}Mod) \sim UDM(N^{\circ}_I/D_I) .$$

Collecting together we get a naturally commutative square

$$\begin{array}{ccc} DM((N^{\circ}_J)^{J'}) & \xrightarrow{\mathcal{L}_{N^{\circ}_J, J'}} & UDM({N'}^{\circ}_{J'}/D'_{J'}) \\ \uparrow & & \uparrow \\ D(\tilde{j}^*A\text{–}Mod) & \longrightarrow & UDM(N^{\circ}_I/D_I) \end{array} .$$

Now we use the fact that $A \cong i^*_J j^J_* \mathbb{1}$. Another application of the base change morphism to the cartesian square in the diagram

$$\begin{array}{ccccc} & & D^{\circ}_J & \longrightarrow & X^{J'} & \longleftarrow & X^{\circ} \\ & & \downarrow & & \downarrow & & \\ D_I & \longrightarrow & D_J & \longrightarrow & X & & \end{array}$$

yields an isomorphism $\tilde{i}^*\tilde{j}_*\tilde{j}^*A \cong i^*_I j^I_*\mathbb{1}$. Combining this with (*) and (**) we see that we obtain in $D^{\leq 2}\mathrm{Comm}_{M(D_I)}$ a natural (quite long) chain of weak isomorphisms connecting $p^{\circ}_{I*}\mathbb{1}$ and $i^*_I j^I_*\mathbb{1}$. But also Proposition 15.22 provides us with such a chain. Our construction of $\varphi_{J',J}$ will be finished if we construct a natural isomorphism between the two functors induced by these two chains.

Let the notation be as in the proof of Proposition 15.22. Let $\widetilde{B}'_{J'}$ etc. be the analogous objects defined for the situation on $D_J \times \mathbb{A}^1$ with divisors $(D_i \cap D_J) \times \mathbb{A}^1$, $i \in J'$. The divisors $\widetilde{D}_{J,i}$, $i \in I$, on $\widetilde{B}_J$ look as follows: For $i \in J$ we have $\widetilde{D}_{J,i} = f_i(D_i \times \mathbb{A}^1) \times_{X \times \mathbb{A}^1} \widetilde{B}_{J\setminus\{i\}}$ and otherwise $\widetilde{D}_{J,i} = \pi_J^{-1}(D_i \times \mathbb{A}^1)$. Let $\mathcal{B} := \widetilde{B}_J \times_X \widetilde{B}_{J'}$ and $\mathcal{B}^{\circ} := \widetilde{B}^{\circ}_J \times_X \widetilde{B}^{\circ}_{J'}$ We have morphisms $\rho :\ \mathcal{B} \to X \times \mathbb{A}^2$ and $\iota :\ \widetilde{B}_I \to \mathcal{B}$. Furthermore we have pullback squares

$$\begin{array}{ccc} \widetilde{B}'_{J'} & \longrightarrow & \mathcal{B} \\ \downarrow{\scriptstyle {\pi'}_{J'}} & & \downarrow \\ (D_J \times \mathbb{A}^1) \times \mathbb{A}^1 & \longrightarrow & \widetilde{B}_J \times \mathbb{A}^1 \end{array} \quad \text{and} \quad \begin{array}{ccc} \widetilde{B}_I & \xrightarrow{\iota} & \mathcal{B} \\ \downarrow{\scriptstyle \pi_I} & & \downarrow{\scriptstyle \rho} \\ X \times \mathbb{A}^1 & \xrightarrow{\mathrm{Id}\times\triangle} & X \times \mathbb{A}^2 \end{array} .$$

On $\mathcal{B}$ we have the divisors $\mathcal{D}_i := D_{J,i} \times_X \widetilde{B}_{J'}$ for $i \in J$ and $\mathcal{D}_i := \widetilde{B}_J \times_X D_{J',i}$ for $i \in J'$. The $D_{I,i}$ are the pullbacks of the $\mathcal{D}_i$ with respect to ι. Let $\tilde{q} :\ (D_J \times \mathbb{A}^1) \times \mathbb{A}^1 \to D_J \times \mathbb{A}^1$ be the projection. Let $A := f^*_J j''_{J*}\mathbb{1}$, $\widetilde{A} := {\pi'}^*_{J'}\tilde{q}^*A$ and $B := {f'}^*_{J'} j'''_{J'*} j'''^*_{J'}\widetilde{A}$, so B is an algebra on $D_I \times \mathbb{A}^2$.

We have canonical isomorphisms $i^*_{(0,0)}B \cong \tilde{p}^\circ_* \tilde{p}^{\circ *} \tilde{i}^* p^\circ_{J*} \mathbb{1}$, $i^*_{(0,1)}B \cong \tilde{i}^* \tilde{j}_* \tilde{j}^* p^\circ_{J*} \mathbb{1}$, $i^*_{(1,0)}B \cong \tilde{p}^\circ_* \tilde{p}^{\circ *} \tilde{i}^* i^*_J j^J_* \mathbb{1}$ and $i^*_{(1,1)}B \cong \tilde{i}^* \tilde{j}_* \tilde{j}^* i^*_J j^J_* \mathbb{1}$ (***), where $i_{(k,l)} : D_I \times \{(k,l)\} \to D_I \times \mathbb{A}^2$, $k,l = 0,1$, are the inclusions. We have isomorphisms between the $i^*_{(k,l)}B$ by comparing them to $(D_I \times \mathbb{A}^2 \to D_I)_* B$ via base change morphisms, and the isomorphisms between the right hand sides of (***) used above are compatible with these isomorphisms. Again via a base change morphism we have a canonical isomorphism $B \cong (D_I \times \mathbb{A}^2 \subset \mathcal{B})^* (\mathcal{B}^\circ \subset \mathcal{B})_* \mathbb{1}$, and the left square above shows then that we also have canonical isomorphisms $i^*_{(0,0)}B \cong p^\circ_{I*}\mathbb{1}$ and $i^*_{(1,1)}B \cong i^*_I j^I_* \mathbb{1}$. Compatibility of base change morphisms shows now that the two possible identifications of $p^\circ_{I*}\mathbb{1}$ and $i^*_I j^I_* \mathbb{1}$ we constructed above actually coincide. Our arguments have been in homotopy categories and not in homotopy 2-categories, and we leave it to the reader to really extract from the above arguments the required 2-morphisms (actually a huge diagram where 2-morphisms connect many different ways connecting $p^\circ_{I*}\mathbb{1}$ and $i^*_I j^I_* \mathbb{1}$).

For the compatibility of the $\varphi_{J,J'}$ one should consider $\widetilde{B}_J \times_X \widetilde{B}_{J'} \times_X \widetilde{B}_{J''}$ and in there compare the constructed 2-morphisms. This we also leave to the reader. We have to admit that we did not honestly have checked this, but certainly it is correct.

Remark 15.23. *Instead of divisors $D_i \subset X$ we also can take closed $D_i \subset X$, $D_i \in Sm/S$, such that the intersections of the D_i look etale locally like intersections of orthogonal standard affine subspaces of some $\mathbb{A}^n_S$. For these situations all constructions above work in exactly the same way, in particular we also get the functors $\mathcal{L}_{X,J}$.*

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