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MATHEMATICS

Differentials of the Adams Spectral Sequence and the Kervaire Invariant

V. A. Smirnov

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This paper studies the differentials of the Adams spectral sequence for stable homotopy groups of spheres and solves the Kervaire invariant one problem for n-manifolds with $n = 2^i - 2$, where $i > 6$.

The Kervaire invariant was defined by M. Kervaire for almost parallelizable smooth $(4n + 2)$-manifolds [1], where it was used to construct an example of a *PL* 10-manifold not admitting a smooth structure.

Browder [2] proved that the Kervaire invariant vanishes if $n + 2$ is not a power of 2. Barratt et al. [3] showed that the Kervaire invariant equals 1 for some n-manifolds in the case of $n = 2^i - 2$, where $i = 2, 3, 4, 5, 6$. Since then, the question of whether the Kervaire invariant equals 1 for n-manifolds with $n = 2^i - 2$ and $i > 6$ has remained open.

In the language of spectral sequences [4], this question can be reformulated as follows: Do the elements h_n^2 of the term E^2 of the Adams spectral sequence for stable homotopy groups of spheres survive up to the term E^∞?

In this paper, we suggest an algebraic approach to answering this question, which is based on the homotopy theory of coalgebras over operads in the category of chain complexes; the main results of this theory were reported at M.M. Postnikov's algebraic topology seminar.

Recall that the notion of an operad was introduced by May [5] in order to describe the structure on iterated loop spaces. In [6], this notion was used to describe the algebraic structure on the cochain complex of a topological space. In particular, it was proved that the singular cochain complex $C^*(X)$ of a topological space X carries the structure of an algebra over an E_∞-operad E. In [7], the homotopy theory of coalgebras over operads was constructed. It was proved that the structure of a coalgebra over an E_∞-operad on the singular chain complex $C_*(X)$ of a simply connected topological space X completely determines the weak homotopy type of this space. In [8, 9], operad methods were used to describe the Adams spectral sequence for homotopy groups of topological spaces and find its differentials.

To describe the Adams spectral sequence, we use the differential graded algebra Λ [10]. As a graded algebra, it is generated by elements λ_i of dimension $i \geq 0$ and the relations

$$\lambda_{2i+n+1}\lambda_i = \sum_j \binom{n-j-1}{j} \lambda_{2i+j+1}\lambda_{i+n-j}.$$

Thus, as a graded module, the algebra Λ is generated by elements of the form $\lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_k}$, in which $i_1 \leq 2i_2$, $i_2 \leq 2i_3$, …, $i_{k-1} \leq 2i_k$. Such sequences $i_1, i_2, \ldots, i_k$ are said to be admissible.

On the algebra Λ, a differential d is defined; it acts on the generators λ_i as

$$d(\lambda_i) = \sum_k \binom{i-k}{k} \lambda_{k-1}\lambda_{i-k}.$$

The algebra Λ is isomorphic to the term $E^1 = \{E^1_{p,q}\}$ of the Adams spectral sequence for stable homotopy groups of spheres, and its homology with respect to the differential d is isomorphic to the term $E_2 = \{E^2_{p,q}\}$ of this spectral sequence.

Direct calculations show that the unique cycles in the term $E^1_{1,*}$ are the elements λ_{2^n-1}. The corresponding homology classes are denoted by h_n. They generate the term $E^2_{1,*}$.

The term $E^2_{2,*}$ is generated by the products h_nh_m, where $n \leq m$, and the relations $h_nh_{n+1} = 0$.

Moscow State Pedagogical University,
ul. Malaya Pirogovskaya 1, Moscow, 119882 Russia;
e-mail: v.smirnov@ru.net

The algebra Λ makes it possible to calculate not only stable homotopy groups of spheres but also unstable homotopy groups of "nice" topological spaces in the Massey–Peterson sense [11].

Recall that the homology of a nice topological space X is the exterior algebra generated by a graded module M over the Steenrod algebra. Examples of nice spaces are the sphere, classical topological groups, etc.

For a graded module M, let $\Lambda \times M$ denote the submodule of $\Lambda \otimes M$ generated by the elements $\lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_k} \otimes x_n$, where $i_1, i_2, \ldots, i_k$ form an admissible sequence and $i_k < n$.

We say that a module M is a Λ-comodule if an operation $\varphi: M \to \Lambda \times M$ satisfying the relation

$$d(\varphi) + \varphi \cup \varphi = 0$$

is defined, where $\varphi \cup \varphi = (\gamma \times 1)(\varphi \times 1)\varphi: M \to \Lambda \times M$ and $\gamma: \Lambda \otimes \Lambda \to \Lambda$ is multiplication in the algebra Λ.

For a Λ-comodule M, we can define a differential d_φ on the module $\Lambda \times M$ by setting

$$d_\varphi(y \otimes x) = d(y) \otimes x + \varphi \cap y \otimes x,$$

where $\varphi\cap = (\gamma \times 1)(1 \times \varphi): \Lambda \times M \to \Lambda \times M$. We denote the corresponding differential graded module by $\Lambda \times_\varphi M$.

If X is a nice topological space and $H_*(X)$ is the exterior algebra generated by the graded module M, then, on M, the Λ-comodule structure

$$\varphi(x_n) = \sum_i \lambda_{i-1} \otimes Sq_i(x_n)$$

is defined. The term $E^1(X)$ of the Adams spectral sequence for homotopy groups of the topological space X is isomorphic to the module $\Lambda \times M$, and the homology of the differential graded module $\Lambda \times_\varphi M$ is isomorphic to the term $E^2(X)$ of this spectral sequence. In particular, the homology of the complex $\Lambda \times H_*(S^n)$, where S^n is the n-sphere, is isomorphic to the term E^2 of the Adams spectral sequence for homotopy groups of the sphere S^n.

Consider the special orthogonal group SO. Its homology is the exterior algebra over a module isomorphic to the homology module of the real projective space RP^∞. The Λ-comodule structure is defined on the generators $x_n \in H_*(RP^\infty)$ by

$$\varphi(x_n) = \sum_k \binom{n-k}{k} \lambda_{k-1} \otimes x_{n-k}.$$

The homology of the complex $\Lambda \times_\varphi H_*(RP^\infty)$ is isomorphic to the term E^2 of the Adams spectral sequence for homotopy groups of the space SO.

Theorem 1. *On the algebra Λ, a new differential $\tilde{d}$ and a new multiplication $\tilde{\gamma}$ can be defined so that the homology of the corresponding differential graded algebra $\tilde{\Lambda}$ is isomorphic to the term E^∞ of the Adams spectral sequence for stable homotopy groups of spheres. If X is a nice topological space (that is, $H_*(X)$ is the exterior algebra over the graded module M over the Steenrod algebra), then, on the module M, a $\tilde{\Lambda}$-comodule structure $\tilde{\varphi}$ can be defined so that the homology of the complex $\tilde{\Lambda} \times_{\tilde{\varphi}} M$ is isomorphic to the term E^∞ of the Adams spectral sequence of homotopy groups of X.*

A proof of this theorem is contained in [12]. Here, we only describe its main stages. Let $\bar{E}$ denote the comonad in the category of chain complexes corresponding to the E_∞-operad E. The singular chain complex $C_*(X)$ of the topological space X is a coalgebra over the comonad $\bar{E}$. The homotopy groups of X can be calculated by using the co-B-construction

$$F(\bar{E}, C_*(X)): C_*(X) \to \bar{E}(C_*(X)) \to \bar{E}^2(C_*(X)) \to \ldots.$$

Let us pass from this co-B-construction to its homology. The homology $\bar{E}_*$ of the comonad $\bar{E}$ carries an A_∞-comonad structure consisting of operations

$$\delta^n: \bar{E}_* \to \bar{E}_*^{n+2}, \quad n \ge 0,$$

increasing dimension by n. The operations δ^n commute with the suspension $S: \overline{S\bar{E}_*} \to \bar{E}_*$.

The homology $H_*(X)$ of the topological space X carries an A_∞-coalgebra structure over the A_∞-comonad $\bar{E}_*$, which consists of operations

$$\tau^n: H_*(X) \to \bar{E}_*^{n+1}(H_*(X)), \quad n \ge 0,$$

increasing dimension by n and commuting with the suspension. These structures determine a new differential on the co-B-construction

$$F(\bar{E}_*, H_*(X)): H_*(X) \to \bar{E}_*(H_*(X)) \\ \to \bar{E}_*^2(H_*(X)) \to \ldots,$$

the homology with respect to this differential is isomorphic to the term E^∞ of the Adams spectral sequence for homotopy groups of X.

Recall that the homology $H_*(\bar{E}(C_*(X))) = \bar{E}_*(H_*(X))$ is isomorphic to the exterior algebra over the graded module $\mathcal{E}(H_*(X))$, which is generated by elements of the form $e_{j_1}e_{j_2}\ldots e_{j_k} \otimes x_n$ of dimension $2^k n - 2^{k-1}j_k - \ldots - j_1$, where $x_n \in H_n(X)$ and $0 \le j_1 \le j_2 \le \ldots \le j_k \le n$.

The suspension $S: \overline{S\mathcal{E}} \to \mathcal{E}$ is defined by

$$S(e_{j_1}e_{j_2}\ldots e_{j_k}) = e_{j_1+1}e_{j_2+1}\ldots e_{j_k+1}.$$

The A_∞-comonad structure on $\bar{E}_*$ induces an A_∞-comonad structure on $\mathscr{E}$, which consists of operations $\delta^n\colon \mathscr{E} \to \mathscr{E}^{n+2}$, where $n \geq 0$, increasing dimension by n and commuting with the suspension.

If X is a nice topological space, then the structure of an A_∞-coalgebra over the A_∞-comonad $\bar{E}_*$ on the homology $H_*(X)$ induces the structure of an A_∞-coalgebra over the A_∞-comonad $\mathscr{E}$ on the module M, which consists of operations $\tau^n\colon M \to \mathscr{E}^{n+1}(M)$, where $n \geq 0$, increasing dimension by n and commuting with the suspension.

In this case, the co-B-construction $F(\bar{E}_*, H_*(X))$ can be replaced by the co-B-construction

$$F(\mathscr{E}, M)\colon M \to \mathscr{E}(M) \to \mathscr{E}^2(M) \to \dots,$$

whose homology is isomorphic to the term E^∞ of the Adams spectral sequence for homotopy groups of X as well.

Note that the operation $\delta^0\colon \mathscr{E} \to \mathscr{E}^2$ determines a comonad structure on $\mathscr{E}$. If M is treated as the trivial $\mathscr{E}$-coalgebra, then the homology of the corresponding co-B-construction $F(\mathscr{E}, M)$ is isomorphic to the graded module L generated by elements of the form $e_{j_1} e_{j_2} \dots e_{j_k} \otimes x_n$ of dimension $2^k n - i_1 - 2i_2 - \dots - 2^{k-1} i_k - k$, where $x_n \in M$, $j_1 > j_2 > \dots > j_k$, and $j_1 + j_2 + \dots + 2^{k-2} j_k < 2^{k-1} n$.

Let us rewrite these elements in the form $\lambda_{i_1} \lambda_{i_2} \dots \lambda_{i_k} \otimes x_n$, where $i_k = n - j_k - 1$, $i_{k-1} = 2n - j_k - j_{k-1} - 1$, …, $i_1 = 2^{k-1} n - 2^{k-2} j_k - \dots - j_2 - 1$. For these elements, we have $i_k < n$, $i_{k-1} \leq 2i_k$, …, $i_1 \leq 2i_2$. Thus, the graded module L is isomorphic to the graded module $\Lambda \times M$.

It follows from the perturbation theory of differentials [13] that the remaining operations $\delta^n\colon \mathscr{E} \to \mathscr{E}^{n+2}$ with $n \geq 1$ induce a differential $\tilde{d}$ on Λ, and the structure of an A_∞-coalgebra over the A_∞-comonad $\mathscr{E}$ on the module M induces a $\tilde{\Lambda}$-comodule structure $\tilde{\varphi}\colon M \to \tilde{\Lambda} \times M$ on M.

The homology of the corresponding complex $\tilde{\Lambda} \times_{\tilde{\varphi}} M$ is isomorphic to the term E^∞ of the Adams spectral sequence for homotopy groups of X. This completes the proof of the theorem.

In particular, the homology of the complex $\tilde{\Lambda}$ is isomorphic to the term E^∞ of the Adams spectral sequence for stable homotopy groups of spheres, the homology of the complex $\tilde{\Lambda} \times H_*(S^n)$ is isomorphic to the term E^∞ of the Adams spectral sequence for homotopy groups of the sphere S^n, and the homology of the complex $\tilde{\Lambda} \times_{\tilde{\varphi}} H_*(RP^\infty)$ is isomorphic to the term E^∞ of the Adams spectral sequence for homotopy groups of the space SO.

Note that multiplication γ in the algebra Λ is subject to a perturbation too. The value of the perturbed multiplication $\tilde{\gamma}\colon \Lambda \otimes \Lambda \to \Lambda$ at an element $\lambda_j \otimes \lambda_i$ coincides with $\gamma(\lambda_j \otimes \lambda_i) = \lambda_j \lambda_i$ provided that $j \leq 2i$. In the case of $j > 2i$, the value $\tilde{\gamma}(\lambda_j \otimes \lambda_i)$ is obtained by adding some new elements $\lambda_{i_1} \lambda_{i_2} \dots \lambda_{i_k}$, where $k > 2$, to $\gamma(\lambda_j \otimes \lambda_i)$.

Since the operations δ^n commute with the suspension, it follows that the value of the perturbed multiplication $\tilde{\gamma}$ at elements of the form $e_p \otimes e_q$, where $p \leq q$, consists of elements of the form $e_{j_1} e_{j_2} \dots e_{j_k}$ with $j_1 > j_2 > j_k \geq p$. Passing from elements of the form e_j to elements of the form λ_i, we see that, for $j > 2i$, the value $\tilde{\gamma}(\lambda_j \otimes \lambda_i)$ consists of elements $\lambda_{i_1} \lambda_{i_2} \dots \lambda_{i_k}$ with $i_k < j - i$.

Now, let us formulate the main theorem.

Theorem 2. *The elements h_n^2, where $n \geq 1$, of the Adams spectral sequence for stable homotopy groups of spheres survive up to the term E^∞.*

Proof. There is a J-homomorphism $J\colon \pi_*(SO) \to \sigma_*$ from the homotopy groups $\pi_*(SO)$ to the stable homotopy groups of spheres σ_* [10]. On the terms E^1 of the corresponding spectral sequences, it has the form

$$J\colon \Lambda \times H_*(RP^\infty) \to \Lambda$$

and is defined by $J(\lambda_{i_1} \lambda_{i_2} \dots \lambda_{i_k} \otimes x_n) = \lambda_{i_1} \lambda_{i_2} \dots \lambda_{i_k} \lambda_n$.

Let $\tilde{\Lambda}'$ denote the subcomplex of $\tilde{\Lambda}$ generated by the elements $\lambda_{i_1} \lambda_{i_2} \dots \lambda_{i_k}$ for which $i_1 i_2 \dots i_k$ is an admissible sequence and $i_{k-1} < i_k$. It is easy to see that the homomorphism J induces the isomorphism of chain complexes

$$\tilde{\Lambda} \times_\varphi H_*(RP^\infty) \cong \tilde{\Lambda}'.$$

Recall that the homotopy groups $\pi_*(SO)$ vanish in the dimensions $2^{n+1} - 3$ for $n > 1$. Therefore, the homology of the complex $\tilde{\Lambda}'$ vanishes as well in these dimensions.

Let us prove that the value of the differential $\tilde{d}$ at $\lambda_{2^n-1} \lambda_{2^n-1}$ belongs to $\tilde{\Lambda}'$. We have

$$\tilde{d}(\lambda_{2^n-1} \lambda_{2^n-1}) = \tilde{\gamma}(\tilde{d}(\lambda_{2^n-1}) \otimes \lambda_{2^n-1}) + \tilde{\gamma}(\lambda_{2^n-1} \otimes \tilde{d}(\lambda_{2^n-1})).$$

It is easy to see that the first term of the sum on the right-hand side belongs to $\tilde{\Lambda}'$. Let us show that the second term belongs to $\tilde{\Lambda}'$ also. Indeed, $\tilde{d}(\lambda_{2^n-1})$ belongs to $\tilde{\Lambda}'$ and is the sum of elements of the form

$\lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_k}$, where $k > 2$ and $i_1 + i_2 + \ldots + i_k = 2^n - 2$. If $2^n - 1 \le 2i_1$, then the elements $\lambda_{2^n-1}\lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_k}$ belong to $\tilde{\Lambda}'$. If $2^n - 1 > 2i_1$, then, multiplying λ_{2^n-1} by λ_{i_1}, we obtain products $\lambda_{j_1}\lambda_{j_2}\ldots\lambda_{j_l}$ in which the dimension of the last factor j_l is at most $2^n - i_1 - 2$. If $j_l > 2i_2$, then, multiplying this factor λ_{j_l} by λ_{i_2}, we obtain products in which the dimension of the last element is at most $2^n - i_1 - i_2 - 3$. Continuing, we see that the dimension of the element preceding λ_{i_k} is at most $2^n - i_1 - \ldots - i_{k-1} - (k-1) = i_k - k + 2$, which is less than i_k; therefore, the product $\tilde{\gamma}(\lambda_{2^n-1} \otimes \lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_k})$ belongs to $\tilde{\Lambda}'$.

Thus, the element $\tilde{d}(\lambda_{2^n-1}\lambda_{2^n-1})$ is a cycle in $\tilde{\Lambda}'$ of dimension $2^{n+1} - 3$. The vanishing of the homology of the complex $\tilde{\Lambda}'$ in these dimensions implies the existence of an element $b_{2^{n+1}-2}$ belonging to the complex $\tilde{\Lambda}'$ for which $\tilde{d}(b_{2^{n+1}-2}) = \tilde{d}(\lambda_{2^n-1}\lambda_{2^n-1})$. Therefore, the element $\lambda_{2^n-1}\lambda_{2^n-1} + b_{2^{n+1}-2}$ is a cycle in the complex $\tilde{\Lambda}$, i.e., the elements h_n^2 with $n \ge 1$ survive up to the term E^∞.

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