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Secondary Steenrod Operations in Cohomology of Infinite-Dimensional Projective Spaces

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The important role of the Steenrod operations Sq^i in the description of the cohomology of topological spaces, the calculation of various spectral sequences, and the study of homotopy properties of topological spaces and their continuous maps is well known [1, 2].

Projective spaces, in particular $\mathbb{RP}^\infty$ and $\mathbb{CP}^\infty$, play a special role in the construction of Steenrod operations and their computation.

In the present paper, we consider secondary Steenrod operations and give formulas for computing these operations in the cohomology of the spaces $\mathbb{RP}^\infty$ and $\mathbb{CP}^\infty$.

Recall that the Steenrod operations $\mathrm{Sq}^i\colon H^n(X) \to H^{n+i}(X)$ arise in the cohomology $H^*(X)$ of a topological space X (with coefficients in $\mathbb{Z}/2$) from the action of an E_∞-operad on the singular cochain complex $C^*(X)$ (see [1, 3]). An *operad* in the category of chain complexes is a family $E = \{E(j)\}$ of chain complexes $E(j)$, $j \ge 1$, on which the permutation groups Σ_j act, together with maps

$$\gamma\colon E(k) \otimes E(j_1) \otimes \dots \otimes E(j_k) \to E(j_1 + \dots + j_k),$$

compatible with the actions of the permutation groups, and satisfy a certain associativity property [3, 4].

An operad E is an E_∞-*operad* if the complexes $E(j)$ are Σ_j-free and acyclic. In particular, the complex $E(2)$ of an E_∞-operad can be thought of as a Σ_2-free complex spanned by elements e_n of dimension n, on which the differential is defined as

$$d(e_n) = e_{n-1} + (-1)^n e_{n-1} T, \qquad T \in \Sigma_2.$$

An action of an E_∞-operad E on the cochain complex $C^*(X)$ of a topological space X is a set of operations

$$\mu(j)\colon E(j) \otimes_{\Sigma_j} C^*(X)^{\otimes j} \to C^*(X)$$

satisfying a certain associativity property.

In particular, the restriction of the operation $\mu(2)$ to the elements e_i yields the products

$$\bigcup_i\colon C^*(X) \otimes C^*(X) \to C^*(X).$$

In what follows, for the sake of brevity we denote the complex $E(j) \otimes_{\Sigma_j} C^*(X)^{\otimes j}$ simply by $E(j\,; C^*(X))$.

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Passing to homology, we obtain the maps

$$\mu_*(j)\colon E_*(j\,;H^*(X)) \to H^*(X).$$

The elements $e_i \otimes x_n \otimes x_n$, where x_n is a cocycle in $C^n(X)$, are cycles in $E(2\,;C^*(X))$. We denote the corresponding homology class in $E_*(2\,;H^*(X))$ by $e_i \times x_n$, for brevity. The values of the operation $\mu_*(2)$ on these elements are usually denoted by $\mathrm{Sq}^{n-i}(x_n)$.

The operations $\mathrm{Sq}^{n-i}\colon H^n(X) \to H^{2n-i}(X)$ are the desired Steenrod operations. It is easy to show that $\mathrm{Sq}^0(x_n) = x_n$, $\mathrm{Sq}^n(x_n) = (x_n)^2$ and $\mathrm{Sq}^i(x_n) = 0$ if $i > n$. Moreover, the Cartan formula relating the Steenrod operations to the multiplication in the cohomology of X is true, namely,

$$\mathrm{Sq}^i(xy) = \sum_k \mathrm{Sq}^k(x)\,\mathrm{Sq}^{i-k}(y).$$

Using this formula, one can easily compute the Steenrod operations in the cohomology of $\mathbb{RP}^\infty$. Indeed, this cohomology is the algebra of polynomials in a single one-dimensional generator u_1; we denote the nth iterated product of this generator by itself by u_n. Now,

$$\mathrm{Sq}^i(u_n) = \mathrm{Sq}^i(u_1 \cdots u_1) = \sum_{i_1+\dots+i_n=i} \mathrm{Sq}^{i_1}(u_1)\cdots\mathrm{Sq}^{i_n}(u_1).$$

Since the indices $i_1, \dots, i_n$ in the last sum can take only values 0 or 1, and the number of ones is i, we finally obtain the formula

$$\mathrm{Sq}^i(u_n) = \binom{n}{i} u_{n+i}.$$

For the space $\mathbb{CP}^\infty$, in a similar way, we can establish the formula

$$\mathrm{Sq}^{2i}(u_{2n}) = \binom{n}{i} u_{2(n+i)}.$$

Now, let us discuss secondary operations. The secondary Steenrod operations are usually defined in a geometric fashion, by means of bundles [2]. We are going to define the secondary Steenrod opertations using the E_∞-algebra structure on the cochain complex $C^*(X)$ of the topological space X.

Recall that any pair of maps $f_1\colon M_1 \to M_2$, $f_2\colon M_2 \to M_3$ of cochain complexes determines a secondary operation in cohomology

$$\langle f_2, f_1\rangle\colon H^*(M_1) \to H^*(M_3)$$

according to the formula $\langle f_2, f_1\rangle = \eta_3 f_2 h_2 f_1 \xi_1$, where $\xi_i\colon H^*(M_i) \to M_i$ is the map choosing representatives of cohomology classes, $\eta_i\colon M_i \to H^*(M_i)$ is the projection, $\eta_i\xi_i = \mathrm{Id}$, $h_i\colon M_i \to M_i$ is a homotopy, $d(h_i) = \xi_i\eta_i - \mathrm{Id}$ (see [5, 6]).

Any commutative diagram of cochain complexes

$$\begin{array}{ccc} M_1 & \xrightarrow{f_1} & M_2 \\ {\scriptstyle g_1}\big\downarrow & & \big\downarrow{\scriptstyle g_2} \\ N_1 & \xrightarrow{f_2} & N_2 \end{array}$$

determines a secondary operation in cohomology, $H^*(M_1) \to H^*(N_2)$, which is the difference of the operations $\langle g_2, f_1\rangle$ and $\langle f_2, g_1\rangle$.

If this diagram is commutative up to a homotopy $h\colon M_1 \to N_2$, then the operation $\eta_2 h \xi_1$ must be added to the above difference of the operations.

Because of the discussion above, the commutative diagram

$$\begin{array}{ccc} E(2\,;\, E(2\,;\, C^*(X))) & \xrightarrow{\ \gamma\ } & E(4\,;\, C^*(X)) \\ {\scriptstyle E(2\,;\,\mu(2))}\big\downarrow & & \big\downarrow{\scriptstyle \mu(4)} \\ E(2\,;\, C^*(X)) & \xrightarrow{\ \mu(2)\ } & C^*(X) \end{array}$$

induces the secondary operation $\mu^1_*\colon E_*(2\,;\, E_*(2\,;\, H^*(X))) \to H^*(X)$ given by the formula

$$\mu^1_* = \eta\mu(4)h(4)\gamma\xi + \eta\mu(2)h(2)E(2\,;\,\mu(2))\xi,$$

where $\xi\colon E_*(2\,;\, E_*(2\,;\, H^*(X))) \to E(2\,;\, E(2\,;\, C^*(X)))$ is a choice of representative map; the mapping $\eta\colon C^*(X) \to H^*(X)$ is a projection;

$$h(2)\colon E(2\,;\, C^*(X)) \to E(2\,;\, C^*(X)) \qquad \text{and} \qquad h(4)\colon E(4\,;\, C^*(X)) \to E(4\,;\, C^*(X))$$

are the corresponding homotopies. The operation μ^1_* is the desired secondary Steenrod operation. The elements of the form $e_i \otimes e_j^{\otimes 2} \otimes (x_n)^{\otimes 4} \in E(2\,;\, E(2\,;\, C^*(X)))$, where x_n is a cocycle in $C^n(X)$, determine homology classes in $E_*(2\,;\, E_*(2\,;\, H^*(X)))$ that we denote, for brevity, by $e_i \times e_j \times x_n$. Using the Steenrod operations Sq, we denote these operations by $\mathrm{Sq}^{2n-j-i} \otimes \mathrm{Sq}^{n-j} \otimes x_n$, and the value of μ^1_* on these elements is denoted by $\langle \mathrm{Sq}^{2n-j-i}, \mathrm{Sq}^{n-j}, x_n \rangle \in H^{4n-2j-i-1}(X)$.

In order to compute the secondary Steenrod operations, let us make use of the multiplication in the cochain complex $C^*(X)$ and the fact that for an E_∞-operad E a Hopf operad can be chosen, which is equipped, in addition to the operad structure γ, with a coalgebra structure given by maps $\nabla(j)\colon E(j) \to E(j) \otimes E(j)$ that are compatible with the operations γ.

In this case, for the cochain complex $C^*(X)$, we have the homotopy commutative diagrams

$$\begin{array}{ccc} E(j\,;\, C^*(X)) & \xrightarrow{\ \mu(j)\ } & C^*(X) \\ \big\uparrow{\scriptstyle E(j\,;\,\pi)} & & \big\uparrow{\scriptstyle \pi} \\ E(j\,;\, C^*(X)^{\otimes 2}) & & \big| \\ {\scriptstyle \nabla(j)}\big\downarrow & & \big| \\ E(j\,;\, C^*(X))^{\otimes 2} & \xrightarrow{\ \mu(j)^{\otimes 2}\ } & C^*(X)^{\otimes 2} \end{array}.$$

Consider the diagram

$$\begin{array}{ccccc} E(2\,;\, E(2\,;\, C^*(X)) & \xrightarrow{\ E(2\,;\,\mu)\ } & E(2\,;\, C^*(X)) & \xrightarrow{\ \mu\ } & C^*(X) \\ {\scriptstyle E(2\,;\,E(2\,;\,\pi))}\big\uparrow & & \big\uparrow{\scriptstyle E(2\,;\,\pi)} & & \big\uparrow{\scriptstyle \pi} \\ E(2\,;\, E(2\,;\, C^*(X)^{\otimes 2})) & \xrightarrow{\ E(2\,;\,\mu)\ } & E(2\,;\, C^*(X)^{\otimes 2}) & & \big| \\ {\scriptstyle \nabla}\big\downarrow & & {\scriptstyle \nabla}\big\downarrow & & \big| \\ E(2\,;\, E(2\,;\, C^*(X)))^{\otimes 2} & \xrightarrow{\ E(2\,;\,\mu)^{\otimes 2}\ } & E(2\,;\, C^*(X))^{\otimes 2} & \xrightarrow{\ \mu^{\otimes 2}\ } & C^*(X)^{\otimes 2} \end{array}.$$

Its lower left square is commutative, and the right and upper left squares are homotopy commutative.

Denote the secondary operation induced by the right square by

$$\nu_*^1(2)\colon E_*(2\,;\,H^*(X)^{\otimes 2}) \to H^*(X).$$

Then the secondary operation induced by the upper left square has the form

$$E_*(2\,;\,\nu_*^1(2))\colon E_*(2\,;\,E_*(2\,;\,H^*(X)^{\otimes 2})) \to E_*(2\,;\,H^*(X)).$$

The operation induced by the lower left square is denoted by

$$\psi_*^1(2)\colon E_*(2\,;\,E_*(2\,;\,H^*(X)^{\otimes 2})) \to E_*(2\,;\,H^*(X))^{\otimes 2}.$$

Similarly, in the diagram

$$\begin{array}{ccccc}
E(2\,;\,E(2\,;\,C^*(X)) & \xrightarrow{\gamma} & E(4\,;\,C^*(X)) & \xrightarrow{\mu} & C^*(X) \\
{\scriptstyle E(2\,;\,E(2\,;\,\pi))}\uparrow & & \uparrow{\scriptstyle E(4\,;\,\pi)} & & \uparrow{\scriptstyle \pi} \\
E(2\,;\,E(2\,;\,C^*(X)^{\otimes 2})) & \xrightarrow{\gamma} & E(4\,;\,C^*(X)^{\otimes 2}) & & \Big| \\
{\scriptstyle \nabla}\downarrow & & {\scriptstyle \nabla}\downarrow & & \Big| \\
E(2\,;\,E(2\,;\,C^*(X)))^{\otimes 2} & \xrightarrow{\gamma^{\otimes 2}} & E(4\,;\,C^*(X))^{\otimes 2} & \xrightarrow{\mu^{\otimes 2}} & C^*(X)^{\otimes 2}
\end{array},$$

the left squares are commutative, while the right one is homotopy commutative.

Denote the secondary operation induced by the right square by

$$\nu_*^1(4)\colon E_*(4\,;\,H^*(X)^{\otimes 2}) \to H^*(X),$$

and the one induced by the lower square by

$$\psi_*^1(4)\colon E_*(2\,;\,E_*(2\,;\,H^*(X)^{\otimes 2})) \to E_*(4\,;\,H^*(X))^{\otimes 2}.$$

It is easy to show that the secondary operation induced by the upper left square is zero.

The comparison of the two above diagrams implies that for the secondary operations in the cohomology of X, we have the following formula (with coefficients in $\mathbb{Z}/2$):

$$\begin{aligned}
\mu_*^1 E_*(2\,;\,E_*(2\,;\,\pi_*)) &= \pi_*(\mu_*^1 \otimes \mu_* E_*(2\,;\,\mu_*) + \mu_* E_*(2\,;\,\mu_*) \otimes \mu_*^1)\nabla_* \\
&\quad + \nu_*^1(2) E_*(2\,;\,\mu_*) + \mu_* E_*(2\,;\,\nu_*^1(2)) + \nu_*^1(4)\gamma_* + \pi_*\mu_*^{\otimes 2}\psi_*^1(2) + \pi_*\mu_*^{\otimes 2}\psi_*^1(4).
\end{aligned}$$

A similar formula is valid if we replace the multiplication π by the n-fold multiplication

$$\pi(n)\colon C^*(X)^{\otimes n} \to C^*(X).$$

We make use of it for computing the secondary Steenrod operations in the cohomology of $\mathbb{RP}^\infty$.

Since the chain complex of $\mathbb{RP}^\infty$ can be chosen isomorphic to its homology, the secondary operations $\nu_*^1(2)$, $\nu_*^1(4)$, $\psi_*^1(4)$ are zero.

Represent the element $u_n \in \mathbb{RP}^\infty$ as the n-fold product of u_1 by itself. The secondary Steenrod operations μ_*^1 on the element u_1 are zero and, therefore, for the secondary Steenrod operations on the element u_n, we have

$$\mu_*^1(e_i \times e_j \times u_n) = \pi_*(n)\mu_*^{\otimes n}\psi_*^1(2)(e_i \times e_j \times (u_1)^{\otimes n}).$$

By definition, the right-hand side of the above formula is equal to the value on the element $e_i \times e_j \times (e_1)^{\otimes n}$ of the map

$$\eta\pi(n)\mu^{\otimes n}\nabla(n)h(n)E(2\,;\,\mu^{\otimes n})\xi + \eta\pi(n)\mu^{\otimes n}E(2\,;\,\mu^{\otimes n})h^{\otimes n}\nabla(n)\xi,$$

where

$$h(n)\colon E(2\,;\,C^*(\mathbb{RP}^\infty)^{\otimes n}) \to E(2\,;\,C^*(\mathbb{RP}^\infty)^{\otimes n}),$$
$$h\colon E(2\,;\,E(2\,;\,C^*(\mathbb{RP}^\infty))) \to E(2\,;\,E(2\,;\,C^*(\mathbb{RP}^\infty)))$$

are homotopies between $\xi\eta$ and Id.

For an arbitrary graded module M, the homotopy $h\colon E(2\,;\,M) \to E(2\,;\,M)$ is defined by the formula

$$h(e_i \otimes x \otimes y) = \begin{cases} e_{i+1} \otimes y \otimes x, & x > y, \\ 0, & x \le y. \end{cases}$$

This formula immediately implies that the second summand in the expression for

$$\pi_*(n)\mu_*^{\otimes n}\psi_*^1(2)$$

is zero. Substituting the formula for the homotopy h into the first summand, we obtain

$$h(n)E(2\,;\,\mu^{\otimes n})\xi(e_i \times e_j \times (u_1)^{\otimes n}) = \sum_{j'<j''} e_{i+1} \otimes (u_{j'_1} \otimes \dots \otimes u_{j'_n}) \otimes (u_{j''_1} \otimes \dots \otimes u_{j''_n}),$$

where $j' = (j'_1, \dots, j'_n)$, $j'' = (j''_1, \dots, j''_n)$, $j'_k, j''_k = 1, 2$, $\sum_k j'_k = \sum_k j''_k = 2n - j$.

Applying the map $\pi(n)\mu^{\otimes n}\nabla(n)$ to this expression, we can write

$$\begin{aligned}\mu_*^1(e_i \times e_j \times u_n) &= \langle \mathrm{Sq}^{2n-j-i}, \mathrm{Sq}^{n-j}, u_n \rangle \\ &= \sum_{k,\, 0 \le m < j} \frac{1}{2}\binom{n}{m}\binom{n-m}{j-m}\binom{n-j}{j-m}\binom{n-2j+m}{k}\binom{m}{i+1-2k} u_{4n-2j-i-1}.\end{aligned}$$

Use the equations

$$\binom{n}{m}\binom{n-m}{j-m}\binom{n-j}{j-m} = \binom{n}{2j-m}\binom{2j-m}{2j-2m}\binom{2j-2m}{j-m},$$
$$\frac{1}{2}\binom{2j-2m}{j-m} = \begin{cases} 1, & j-m = 2^r, \\ 2l, & j-m \ne 2^r. \end{cases}$$

Substituting them in the formula for μ_*^1, we finally obtain the following statement.

Theorem. *For the secondary Steenrod operations in the cohomology of* $\mathbb{RP}^\infty$, *we have*

$$\begin{aligned}\mu_*^1(e_i \times e_j \times u_n) &= \langle \mathrm{Sq}^{2n-j-i}, \mathrm{Sq}^{n-j}, u_n \rangle \\ &= \sum_{k,r} \binom{n}{j+2^r}\binom{j+2^r}{2^{r+1}}\binom{n-j-2^r}{k}\binom{j-2^r}{i+1-2k} u_{4n-2j-i-1}.\end{aligned}$$

One can show in a similar way that the secondary Steenrod operations in the cohomology of $\mathbb{CP}^\infty$ can be expressed by the formula

$$\begin{aligned}\mu_*^1(e_{2i-1} \times e_{2j} \times u_{2n}) &= \langle \mathrm{Sq}^{4n-2j-2i+1}, \mathrm{Sq}^{2n-2j}, u_{2n} \rangle \\ &= \sum_{k,r} \binom{n}{j+2^r}\binom{j+2^r}{2^{r+1}}\binom{n-j-2^r}{k}\binom{j-2^r}{i-2k} u_{8n-4j-2i}.\end{aligned}$$

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