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# COMPUTATIONS OF COMPLEX EQUIVARIANT BORDISM RINGS

By Dev P. Sinha

*Abstract*. We give explicit computations of the coefficients of homotopical complex equivariant cobordism theory $MU^G$, when $G$ is abelian. We present a set of generators which is complete for any abelian group. We present a set of relations which is complete when $G$ is cyclic and which we conjecture to be complete in general. We proceed by first computing the localization of $MU^G$ obtained by inverting Euler classes of representations. We then define a family of operations which essentially divide by Euler classes and use these operations to define our generating sets. We give geometric applications of these computations to the study of equivariant genera, circle actions on four-manifolds, and cobordism relations between Lens spaces.

**1. Introduction.** Bordism theory is fundamental in algebraic topology and its applications. In the early sixties Conner and Floyd introduced equivariant bordism as a powerful tool in the study of transformation groups. In the late sixties, tom Dieck introduced homotopical bordism in order to refine understanding of the localization techniques employed by Atiyah, Segal and Singer in index theory. Despite the many successful computations and applications of bordism theories, equivariant bordism has been mysterious from a computational point of view, even for cyclic groups of prime order $p$ (see [14] and [15]).

In this paper we present the first computations of the coefficients of equivariant bordism, for abelian groups. The key constructions are operations on equivariant bordism. Analogs of these operations should play an important role in equivariant stable homotopy more generally. Our main techniques involve localization and give some insight into the structure of $MU^G_*$ for nilpotent groups. There has also been recent progress in defining an equivariant version of formal group laws [5] and trying to prove an equivariant version of Quillen's theorem which relates the theory of formal group laws to bordism theory [10]. Understanding the relationship between these approaches and ours should prove fruitful.

We now give a summary of our results. We denote by $MU^G_*$ the homotopical equivariant bordism ring, where $G$ is a compact Lie group. It is defined analogously to $MU_*$ as $\varinjlim_V [S^{\underline{n}\oplus V}, T(\xi^G_{|V|})]^G$, where $V$ ranges over isomorphism classes of complex representations of $G$, $S^{\underline{n}\oplus V}$ is the one-point compactification of the Whitney sum of $\mathbb{C}^n$ with trivial $G$ action and $V$, and $T(\xi^G_{|V|})$ is the Thom space of the universal complex $G$-bundle. In fact, we may use these Thom spaces to define an equivariant spectrum as first done by tom Dieck [8] and hence define

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associated equivariant homology and cohomology theories $MU_*^G(-)$ and $MU_G^*(-)$. We will carefully make these constructions in Section 3.

Euler classes play fundamental roles in our work. The Euler classes which are most important for us are those associated to a complex representation of $G$, considered as a $G$-bundle over a point. More explicitly, the Euler class associated to $V$ is a class $e_V \in MU_G^m(pt.)$, where $m$ is the dimension of $V$ over the reals, represented by the composite $S^0 \hookrightarrow S^V \to T(\xi_{|V|}^G)$, where the second map is "inclusion of a fiber." Euler classes multiply by the rule $e_V \cdot e_W = e_{V \oplus W}$. In homological grading $e_V \in MU_{-m}^G$, so it cannot be in the image of a geometric bordism class under the Pontrijagin-Thom map if it is nontrivial. If $V^G = \{0\}$ then $e_V$ is nonzero, reflecting the fact that $V$ has no nonzero equivariant sections. Therefore, the homotopy groups of $MU^G$ are not bounded below, a feature which already distinguishes it from its nonequivariant counterpart.

More familiar classes in $MU_*^G$ are those in the image of classes in geometric bordism under the Pontrijagin-Thom map. Given a stably complex $G$-manifold $M$, let $[M]$ denote the corresponding class in $MU_*^G$. Complex projective spaces give a rich collection of examples of $G$-manifolds. Given a complex representation $W$ of $G$ let $\mathbb{P}(W)$ denote the space of complex one-dimensional subspaces of $W$ with inherited $G$-action.

The starting point in our work is that after inverting Euler classes, $MU_*^G$ becomes computable by nonequivariant means. That we rely heavily on localization is not surprising because localization techniques have pervaded equivariant topology. Let $R_0$ denote the sub-algebra of $MU_*^G$ generated by the $e_V$ and $[\mathbb{P}(\underline{n} \oplus V)] = Z_{n,v}$ as $V$ ranges over nontrivial irreducible representations. Let $S$ be the multiplicative set in $R_0$ of nontrivial Euler classes. By abuse, denote the same multiplicative set in $MU_*^G$ by $S$. Then the key first result is the following.

THEOREM 1.1. *Let $G$ be nilpotent. The inclusion of $R_0$ into $MU_*^G$ becomes an isomorphism after inverting $S$.*

In other words, we may multiply any class in $MU_*^G$ by some Euler class to get a class in $R_0$ modulo the kernel of the localization map $S$. We are led to study divisibility by Euler classes as well as the kernel of this localization map. We can do so successfully in the case when the group in question is a torus.

Let $T$ be a torus, and let $V$ be a nontrivial irreducible representation of $T$. Let $K(V)$ denote the subgroup of $T$ which acts trivially on $V$. There is a restriction homorphism (of algebras) $res_H^T\colon MU_*^T \to MU_*^H$ for any subgroup $H$. The restriction of $e_V$ to $MU_*^{K(V)}$ is zero, as can be seen using an explicit homotopy. Remarkably, we have the following.

THEOREM 1.2. *The sequence*

$$0 \to MU_*^T \xrightarrow{\cdot e_V} MU_*^T \xrightarrow{res^T_{K(V)}} MU_*^{K(V)} \to 0$$

*is exact.*

The injectivity of multiplication by Euler classes and explicit computation of $S^{-1}MU_*^T$ give rise to the following.

THEOREM 1.?. *There are inclusions of $MU_*$-algebras*

$$R_0 = MU_*[e_v, Z_{n,v}] \subset MU_*^T \subset S^{-1}MU_*^T = MU_*[e_v^{\pm 1}, Z_{n,v}],$$

*where $V$ ranges over nontrivial irreducible representations of $T$ and $n$ ranges over positive integers.*

Using the exact sequence of Theorem 1.2, we define operations which are essentially division by Euler classes. To define these operations we need to split the restriction maps. The restriction map to the trivial group is called the augmentation map $\alpha: MU_*^G \to MU_*$. There is a canonical splitting of this map as rings which defines an $MU_*$-algebra structure on $MU_*^G$. All of the maps we have defined so far are in fact maps of $MU_*$-modules. The restriction maps to other sub-groups are not canonically split, but we do know the following from [17].

THEOREM 1.3. (Comezaña) *Let $G$ be abelian. Then $MU_*^G$ is a free $MU_*$-module concentrated in even degrees.*

Hence we may fix a splitting $s_V$ as $MU_*$-modules of the restriction map $res^T_{K(V)}$. Unless $K(V)$ is the trivial group, this splitting is noncanonical and is not a ring homomorphism.

*Definition* 1.4. Let $T$ and $V$ be as above. Define the $MU_*$-linear operation $\Gamma_V$ as follows. Let $x \in MU_*^T$. Then $\Gamma_V(x)$ is the unique class in $MU_*^T$ which satisfies

$$e_V \cdot \Gamma_V(x) = x - s_V(res^T_{K(V)} x).$$

For convenience, let $\beta_V$ denote $s_V \circ res^T_{K(V)}$. If $I = V_1, \ldots, V_k$ is a finite sequence of nontrivial irreducible representations let $\Gamma_I(x) = \Gamma_{V_k}\Gamma_{V_{k-1}} \cdots \Gamma_{V_1} x$. Fix an ordering on the nontrivial irreducible representations of $T$. Call a finite sequence of representations admissible if it respects this ordering.

We are now ready to state our main theorem.

THEOREM 1.5. *With choices of splittings $s_V$ as in Theorem 5.8 below, $MU_*^T$ is generated as an $MU_*$-algebra by the classes $\Gamma_I(e_V)$ and $\Gamma_I([\mathbb{P}(\underline{n} \oplus V)])$, where $V$ ranges over nontrivial irreducible representations, $I$ ranges over all admissisible sequences of nontrivial irreducible representations, and $n$ ranges over natural numbers.*

*Relations include the following:*

(1) $e_V\Gamma_V(x) = x - \beta_V(x)$,

(2) $\Gamma_V(\beta_V(x)) = 0$,

(3) $\Gamma_V(e_V) = 1$,

(4) $\Gamma_V(x)y = (x - \beta_V(x))\Gamma_V(y) - \Gamma_V(x)\beta_V(y)$,

(5) $\Gamma_V\Gamma_W x = \Gamma_W\Gamma_V x - \Gamma_W\Gamma_V\beta_W(x) - \Gamma_W\Gamma_V(e_W\beta_V(\Gamma_W x))$,

*where $V$ and $W$ range over nontrivial irreducible representations of $T$ and $x$ and $y$ are any classes in $MU^T_*$. For $T = S^1$, these relations are complete.*

The set of generators given is redundant, as it must be. We may take $I$ above to range in any set $A$ such that any admissible sequence of representations is the initial sequence for some sequence in $A$.

We may recover the structure of $MU^G_*$ for any abelian group $G$ by realizing $G$ as the kernel of an irreducible representation of some torus and using the exact sequence of Theorem 1.2.

We give both algebraic and geometric applications of our main computation. For $G = S^1$ and $\rho$ its standard representation, we present a geometric model of $\Gamma_\rho([M])$. This geometric model allows us to compute the completion map $MU^G_* \to (MU^G_*)_{\hat{I}}$, where $I$ is the kernel of the augmentation map from $MU^G_*$ to $MU_*$. The completion theorem of Löffler, as proved by Comezaña and May, states that for $G$ abelian, $(MU^G_*)_{\hat{I}} \cong MU^*(B_G)$, where $B_G$ is the classifying space of $G$. Hence this completion map gives a connection between equivariant bordism and any complex-oriented equivariant theory which is defined using a Borel construction $E_G \times_G -$. We also give more classically-styled applications to the understanding of group actions on manifolds. For example, a current topic of great interest in equivariant cohomology is the investigation of $G$-manifolds with isolated fixed points, essentially extending Smith theory. We prove the following.

THEOREM 1.6. *Let $M$ be a stably-complex four dimensional $S^1$-manifold with three isolated fixed points. Then $M$ is equivariantly cobordant to $\mathbb{P}(\underline{1} \oplus V \oplus W)$ for some distinct nontrivial irreducible representations $V$ and $W$ of $S^1$.*

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**2. Preliminaries.** The group $G$ will always be a compact Lie group. All $G$ actions are assumed to be continuous, and $G$-actions on manifolds are assumed to be smooth. For any $G$-space X, we let $X^G$ denote the subspace of $X$ fixed under the action of $G$. The space of maps between two $G$-spaces, which we denote $\mathrm{Maps}(X, Y)$ has a $G$-action by conjugation. We denote its subspace of $G$-fixed maps by $\mathrm{Maps}^G(X, Y)$. We will often work with based spaces, in which case we assume that the basepoints are fixed by $G$. Throughout, $E_G$ will be a contractible space on which $G$ is acting freely, and $B_G$, the classifying space of $G$, is the quotient of $E_G$ by the action of $G$.

We will always let $V$ and $W$ be finite-dimensional complex representations of $G$. Our $G$-vector bundles will always have paracompact base spaces, so we may define a $G$-invariant inner product on the fibers. The constructions we make using such an inner product will be independent of the choice of inner product up to homotopy. We will use the same notation for a $G$-bundle over a point as for the corresponding representation. We let $|V|$ denote the dimension of $V$ as a complex vector space. The sphere $S^V$ is the one-point compactification of $V$, based at 0 if a base point is needed, and the sphere $S(V)$ is the unit sphere in $V$ with inherited $G$-action. For a $G$-vector bundle $E$, let $T(E)$ denote its Thom space, which is the cofiber of the unit sphere bundle of $E$ included in the unit disk bundle of $E$. Thus for $V$ a representation $T(V) = S^V$.

Let $R^+(G)$ denote the monoid (under direct sum) of isomorphism classes of complex representations of $G$, and let $R(G)$ denote the associated Grothendieck ring (where multiplication is given by tensor product). We let $Irr(G)$ denote the set of isomorphism classes of irreducible complex representations of $G$, and let $Irr^*(G)$ be the subset of nontrivial irreducible representations. If $W = \sum a_i V_i \in R(G)$ where $V_i$ are distinct irreducible representations, let $\nu_V(W)$ for an irreducible $V$ be $a_j$ if $V$ is isomorphic to $V_j$ or zero if $V$ is not isomorphic to any of the $V_i$. We let $I_{R(G)}$ be the augmentation ideal of $R(G)$, that is the subgroup of elements of virtual dimension zero. Recall from the introduction that $\rho$ is the standard representation of $S^1$. We will by abuse use $\rho$ to denote the standard representation restricted to any subgroup of $S^1$. We use $\underline{n}$ or $\mathbb{C}^n$ to denote the trivial $n$-dimensional complex representation of a group. We will sometimes think of representations as group homomorphisms, and talk of their kernels, images, and so forth. In particular, if $W$ and $V$ are distinct irreducible representations, we say that $W$ divides $V$ if the kernel of $W$ is contained in the kernel of $V$. And we say that $V$ is primitive if there are no irreducible representations which divide it.

We rely on techniques from equivariant stable homotopy theory, as described in [17]. Let $\Omega^W(X)$ denote the space of based maps from $S^W$ to $X$. By fixing a representation $\mathcal{U}$ with inner product, of which a countably infinite direct sum of any representation of $G$ appears as a summand, we define a $G$-spectrum $X$ to be a family of spaces $X_V$ indexed on subspaces of $\mathcal{U}$ equipped with bonding maps, which are $G$-homeomorphisms $X_V \to \Omega^{V^\perp} X_W$ for all $V \subseteq W$, where $V^\perp$ is the complementary subspace of $V$ in $W$. A $G$-pre-spectrum is a similarly indexed family of spaces in which the bonding maps are not required to be homeomorphisms. A basic passage from equivariant to ordinary stable homotopy theory is by taking the fixed-points spectrum. Consider only subspaces $V \subset \mathcal{U}^G$. Then we may define the fixed-points spectrum $X^G$ using the family of spaces $(X_V)^G$, where the bonding maps are restrictions to fixed sets of the given bonding maps.

**3. Basic properties of $MU^G$.** There are two basic definitions of bordism, geometric and homotopy theoretic. Equivariantly these two theories are not equivalent, and we will comment on this difference later in this section.

Our main concern is the homotopy theoretic version of complex equivariant bordism, as first defined by tom Dieck [8]. A good reference for a modern treatment of the foundations of complex equivariant bordism is [17], in particular the later chapters. Fix $\mathcal{U}$, a complex representation of which a countably infinite direct sum of any representation of $G$ appears as a summand. If there is ambiguity possible we specify the group by writing $\mathcal{U}(G)$. Let $BU^G(n)$ be the Grassmanian of complex $n$-dimensional linear subspaces of $\mathcal{U}$. Let $\xi_n^G$ denote the tautological complex $n$-plane bundle over $BU^G(n)$. As in the nonequivariant setting, the bundle $\xi_n^G$ over $BU^G(n)$ serves as a model for the universal complex $n$-plane bundle. If $V$ is a complex representation, set $\xi_V^G = \xi_{|V|}^G$.

*Definition* 3.1. We let $TU^G$ be the pre-spectrum, indexed on all complex subrepresentations of $\mathcal{U}$, defined by taking the $V$th entry to be $T(\xi_V^G)$ (it suffices to define entries of a prespectrum only for complex representations). Define the bonding maps by noting that for $V \subseteq W$ in $\mathcal{U}$, letting $V^\perp$ denote the complement of $V$ in $W$, we have

$$S^{V^\perp} \wedge T(\xi_V^G) \cong T(V^\perp \times \xi_V^G).$$

Then use the classifying map

$$V^\perp \times \xi_V^G \to \xi_W^G$$

to define the corresponding map of Thom spaces. Pass to a spectrum in the usual way, so that the $V$th de-looping is given by

$$(MU^G)_V = \lim_{W \supseteq V} \Omega^{V^\perp}(T(\xi_W^G)),$$

to obtain the homotopical equivariant bordism spectrum $MU^G$.

From this spectrum indexed by subspaces of $\mathcal{U}$ we may pass to an $RO(G)$-graded homology theory $MU_\bullet^G(-)$. We will mostly be concerned with the coefficient ring in integer gradings, which we denote $MU_*^G$. For some arguments, we will need groups graded by complex representations of $G$, giving rise to the need for the following proposition, which is proved in the real setting in chapter 15 of [17].

Proposition 3.2. *Let $V$ be a complex representation of $G$. The group $MU_V^G(X)$ is naturally isomorphic to $MU_{2|V|}^G(X)$.*

We prove this proposition after defining the needed multiplicative structure on $MU^G$. The classifying map of the Whitney sum

$$\xi_V^G \times \xi_W^G \to \xi_{V \oplus W}^G$$

gives rise to a map

$$T(\xi_V^G) \wedge T(\xi_W^G) \to T(\xi_{V \oplus W}^G),$$

which defines a multiplication on $MU^G$. The unit element is represented by the maps $S^V \to T(\xi_V^G)$ induced by passing to Thom spaces the classifying map of $V$

viewed as a $G$-bundle over a point. Thus in the usual way the coefficients $MU^G_\bullet$ form a ring and $MU^G_\bullet(X)$ is a module over $MU^G_\bullet$.

*Definition* 3.3. Let $V \subset \mathcal{U}$ be of dimension $n$. Then the classifying map $V \to \xi^G_n$ induces a map of Thom spaces $S^V \to T(\xi^G_n)$, which represents an element $t_V \in MU^G_{V-2n}$ known as a *Thom* class.

*Proof of Proposition* 3.2. We show that the Thom class $t_V$ is invertible. The isomorphism between the groups $MU^G_V(X)$ and $MU^G_{2|V|}(X)$ is then given by multiplication by this Thom class.

The class in $MU^G_{2n-V}$ represented by the map $S^{2n} \to T(\xi^G_V)$ induced by the classifying map $\mathbb{C}^n \to \xi^G_V$ is the multiplicative inverse of $t_V$. The product of this class with $t_V$ is homotopic to the unit map $S^{V\oplus\mathbb{C}^n} \to T(\xi^G_{V\oplus\mathbb{C}^n})$. $\square$

The most pleasant way to produce classes in $MU^G_*$ is from stably complex $G$-manifolds. Recall that there is an real analog of $BU^G(n)$, which we call $BO^G(n)$, and which is the classifying space for all $G$-vector bundles.

*Definition* 3.4. A stably complex $G$-manifold is a pair $(M, \tau)$ where $M$ is a smooth $G$-manifold and $\tau$ is a lift to $BU^G(n)$ of the map to $BO^G(2n)$ which classifies $TM \times V$ for some real representation $V$.

We can define bordism equivalence in the usual way to get a geometric version of equivariant bordism.

*Definition* 3.5. Let $\Omega^{U,G}_*$ denote the ring of stably complex $G$-manifolds up to bordism equivalence.

Classes in geometric bordism give rise to classes in homotopical bordism through the Pontrijagin-Thom construction.

*Definition* 3.6. Define a map $PT\colon \Omega^{U,G}_* \to MU^G_*$ as follows. Choose a representative $M$ of a bordism class. Embed $M$ in some sphere $S^V$, avoiding the basepoint and so that the normal bundle $\nu$ has a complex structure. Identify the normal bundle with a tubular neighborhood of $M$ in $S^V$. Define $PT([M])$ as the composite

$$S^V \xrightarrow{c} T(\nu) \xrightarrow{T(f)} T(\xi_{|\nu|}),$$

where $c$ is the collapse map which is the identity on $\nu$ and sends everything outside $\nu$ to the basepoint in $T(\nu)$, and $T(f)$ is the map on Thom spaces given rise to by the classifying map $\nu \to \xi_{|\nu|}$.

The proof of the following theorem translates almost word-for-word from Thom's original proof.

THEOREM 3.7. *The map PT is a well-defined graded ring homomorphism.*

The Pontrijagin-Thom homomorphism is not an isomorphism equivariantly as it is in the ordinary setting. But, a theorem of Comezaña, namely Theorem 5.4 in Chapter 28 of [17] states that $PT$ is split injective for abelian groups. The following classes illustrate the failure of the Pontrijagin-Thom map to be an isomorphism.

*Definition* 3.8. Compose the map $S^V \to T(\xi_n^G)$, in Definition 3.3 of the Thom class with the evident inclusion $S^0 \to S^V$ to get an element $e_V \in MU^G_{-2n}$ which is called the *Euler class* associated to $V$.

We will see that Euler classes $e_V$ associated to representations $V$ such that $V^G = \{0\}$ are nontrivial. Thus $MU^G_*$ is not connective, a feature which already distinguishes it from $\Omega^{U,G}_*$ as well as $MU_*$. The key difference between the equivariant and ordinary settings is the lack of transversality equivariantly. For example, if $V^G = \{0\}$ the inclusion of $S^0$ into $S^V$ cannot be deformed equivariantly to be transverse regular to $0 \in S^V$.

Finally, we introduce maps relating bordism rings for different groups. Recall that ordinary homotopical bordism $MU$ can be defined using Thom spaces as in our definition of $MU^G$ but without any group action present.

*Definition* 3.9. Define the augmentation map $\alpha: MU^G \to MU$ by forgetting the $G$-action on $MU^G$. When $G$ is abelian and $H$ is a subgroup of $G$ define $res^G_H$ to be the map from $MU^G_* \to MU^H_*$ by restricting the $G$-action to an $H$-action.

We need to have $G$ abelian for the map $res^G_H$ to be so defined. In the abelian setting, any complex representation of $H$ extends to a complex representation of $G$, so that when its $G$-action is restricted to an $H$-action the Thom space $T(\xi_n^G)$ coincides with $T(\xi_n^H)$.

*Definition* 3.10. Define the inclusion map $\iota: MU \to MU^G$ by composing a map $S^n \to T(\xi_n)$ with the inclusion $T(\xi_n) \to T(\xi_n^G)$.

On coefficients, $\iota$ defines an $MU_*$-algebra structure on $MU^G_*$. The kernel of $\alpha$ on coefficients is called the *augmentation ideal*. For example, the Euler class $e_V$ is in the augmentation ideal as the map $S^0 \to S^V$ in its definition is null-homotopic when forgetting the $G$-action. On the other hand, $\iota$ is injective, which follows from the following proposition which is proved for example at the end of Chapter 26 of [17].

PROPOSITION 3.11. *The composite $\alpha \circ \iota: MU \to MU$ is homotopic to the identity map.*

**4. The connection between taking fixed sets and localization.** The connection between localization, in the commutative algebraic sense, and "taking fixed sets" has been a fruitful theme in equivariant topology. We develop this connection in the setting of bordism in this section.

The main goal of this section is to prove Theorem 1.1, which we restate here for convenience. Let $R_0$ denote the $MU_*$ sub-algebra of $MU_*^G$ generated by the classes $e_V$ and $[\mathbb{P}(\underline{n} \oplus V)]$ as $V$ ranges over nontrivial irreducible representations. Let $S$ be the multiplicative set in $R_0$ of nontrivial Euler classes.

THEOREM. (Restatement of Theorem 1.1) *Let $G$ be nilpotent. The inclusion of $R_0$ into $MU_*^G$ becomes an isomorphism after inverting $S$.*

We prove this theorem by first explicitly computing $S^{-1}MU_*^G$ and then computing the images of generators of $R_0$ in $S^{-1}MU_*^G$. We start with the following lemma, which provides translation between localization and topology. For any commutative ring $R$ and element $e \in R$ let $R[\frac{1}{e}]$ denote the localization of $R$ obtained by inverting $e$.

LEMMA 4.1. *For any $G$, $\widetilde{MU_*^G}(S^{\oplus\infty V}) \cong MU_*^G[\frac{1}{e_V}]$ as rings.*

*Proof*. The left-hand side $\widetilde{MU_*^G}(S^{\oplus\infty V})$ is a ring because $S^{\oplus\infty V}$ is an $H$-space via the equivalence

$$S^{\oplus\infty V} \wedge S^{\oplus\infty V} \cong S^{\oplus\infty V}.$$

To compute the left-hand side, apply $\widetilde{MU_*^G}$ to the identification $S^{\oplus\infty V} = \varinjlim S^{\oplus nV}$. After applying the suspension isomorphisms $\widetilde{MU_*^G}(S^{\oplus kV}) \cong \widetilde{MU^G_{*+|V|}}(S^{\oplus k+1 V})$, the maps in the resulting directed system are multiplication by the $e_V$. ◻

We will see that after inverting Euler classes, equivariant bordism is computable for nilpotent groups. If $G$ is nilpotent, any subgroup is a proper subgroup of its normalizer (see [13], page 101). Hence all maximal subgroups are normal, so that any proper subgroup $H$ is contained in a proper normal subgroup $N$. Thus for any subgroup $H$ there will be a nontrivial representation of $G$ which is trivial when restricted to $H$, namely a nontrivial representation of $G/N$ pulled back to $G$. Therefore, if $G$ is nilpotent and $\{W_i\}$ are the nontrivial irreducible representations of $G$ then $Z = S^{\oplus_i(\oplus_\infty W_i)}$ has fixed sets $Z^G = S^0$ while $Z^H$ is contractible for any $H \subset G$. Robert Stong has pointed out to us that for $G$ finite, the only groups with this property are nilpotent groups.

Our next lemma, taken with Lemma 4.1, establishes the strong link between localization and taking fixed sets.

LEMMA 4.2. *Let $X$ be a finite $G$-complex and let $Z$ be a $G$-complex such that $Z^G \simeq S^0$ and $Z^H$ is contractible for any proper subgroup of $G$. Then the restriction map*

$$\mathrm{Maps}^G(X, Y \wedge Z) \to \mathrm{Maps}(X^G, (Y \wedge Z)^G) = \mathrm{Maps}(X^G, Y^G)$$

*is a homotopy equivalence.*

*Proof*. As shown in Chapter 1 of [17], homotopy extension and lifting properties hold for $G$-complexes. Hence, the restriction map is a fibration whose fiber at a given point is the space of $G$-maps which are specified on $X^G$. Using the skeletal filtration of $X$, we can then filter this mapping space by spaces

$$\mathrm{Maps}^G(D^k \times G/H, Y \wedge Z),$$

such that the maps are specified on the boundary of $D^k \times G/H$, and where $H$ is a proper subgroup of $G$. A standard change-of-groups argument yields that this mapping space is homeomorphic to $\mathrm{Maps}(D^k, (Y \wedge Z)^H)$, again with the map specified on the boundary. But $(Y \wedge Z)^H$ is contractible, and thus so are these mapping spaces. Thus, the fiber of the restriction map is contractible. □

We now translate this lemma to the stable realm. For simplicity, let us suppose that our $G$-spectra are indexed over the real representation ring. We can do so by choosing specific representatives of isomorphism classes of representations. Let $K_n \subset K_{n+1}$ denote a sequence of representations which eventually contain all irreducible representations infinitely often and such that $K_n{}^\perp \subset K_{n+1}$ contains precisely one copy of the trivial representation. If $G$ is finite, we can let $K_n$ be the direct sum of $n$ copies of the regular representation.

*Definition* 4.3. Let $X$ be a $G$-prespectrum. We define the geometric fixed sets spectrum $\Phi^G X$ by passing from a prespectrum $\phi^G X$ defined as follows. We let the entry $\{\phi^G X\}_n$ be $(X_{K_n})^G$, the $G$-fixed set of the $K_n$-entry of $X$. The bonding maps are composites

$$(X_{K_n})^G \longrightarrow (\Omega^{K_n{}^\perp} X_{K_{n+1}})^G \longrightarrow \Omega^{(K_n{}^\perp)^G} (X_{K_{n+1}})^G = \Omega (X_{K_{n+1}})^G,$$

where the first map is a restriction of a bonding map of $X$, and the second map is restriction to fixed sets of the loop space.

While the prespectrum $\phi^G(X)$ depends on the choice of filtration $K_*$, the spectrum $\Phi^G X$ is independent of this choice.

LEMMA 4.4. *Let $Z$ be as in Lemma 4.2. Then for any $G$-prespectrum $X$, the prespectra $(X \wedge Z)^G$ and $\Phi^G X$ are homotopy equivalent.*

*Proof*. From the definition of $(X \wedge Z)^G$, consider

$$(\Omega^W (X_{W \oplus V} \wedge Z))^G.$$

Applying Lemma 4.2, the restriction from this mapping space to $\Omega^{W^G}(X_{W\oplus V})^G$ is a homotopy equivalence. Choosing $V = K_n$, we see that $\Omega^{W^G}(X_{W \oplus K_n})^G$ is an entry of $\phi^G X$. The bonding maps clearly commute with these restriction to fixed sets maps, so we have an equivalence of spectra. □

Note that any $Z$ as in Lemma 4.2 is an (equivariant) H-space as $Z \wedge Z \simeq Z$. Hence if $X$ is a ring spectrum so is $(X \wedge Z)^G$. Taking Lemma 4.1 and Lemma 4.4 together, we have the following.

PROPOSITION 4.5. *Let $G$ be a nilpotent group and let $S$ be the multiplicative set of nontrivial Euler classes in $MU^G_*$. As rings $S^{-1}MU^G_* \cong (\Phi^G MU^G)_*$.*

To compute $(\Phi^G MU^G)_*$, we can use the geometry of Thom spaces. Because smashing a weak equivalence of prespectra with a complex yields another weak equivalence, we have $Z \wedge MU^G \simeq Z \wedge TU^G$ as prespectra, where $TU^G$ denotes the equivariant Thom prespectrum and $Z$ is as in Lemma 4.2. Hence,

$$\Phi^G MU^G \simeq (Z \wedge MU^G)^G \simeq (Z \wedge TU^G)^G \simeq \Phi^G TU^G.$$

As required by the definition of $\Phi^G TU^G$, we proceed with analysis of fixed-sets of Thom spaces.

We need the following basic fact about equivariant vector bundles.

PROPOSITION 4.6. *Let $E$ be a $G$-vector bundle over a base space with trivial $G$-action $X$. Then $E$ decomposes as a direct sum*

$$E \cong \bigoplus_{V \in Irr(G)} E_V,$$

*where $E_V \cong \widetilde{E} \otimes V$ for some vector bundle $\widetilde{E}$.*

The following result is due to tom Dieck [8].

LEMMA 4.7. *For any compact Lie group $G$, the $G$-fixed set of the Thom space of $\xi^G_n$ is homotopy equivalent to*

$$\bigvee_{W \in R^+(G)_n} T(\xi_{|W^G|}) \wedge \left( \prod_{V \in Irr^*(G)} BU(\nu_V(W)) \right)_+ ,$$

*where we define $R^+(G)_n$ as the subset of dimension $n$ representations in $R^+(G)$, and we recall that $\nu_V(W)$ is the greatest number $m$ such that $\oplus_m V$ appears as a summand of $W$.*

*Proof*. The universality of $\xi^G_n$ implies that $(BU^G(n))^G$ is a classifying space for $n$-dimensional complex $G$-vector bundles over trivial $G$-spaces. Using Proposition 4.6 we see that this classifying space is weakly equivalent to

$$\coprod_{W \in R^+(G)_n} \left( \prod_{V \in Irr(G)} BU(\nu_V(W)) \right)$$

Over each component of this union, the universal bundle decomposes as $\xi_1 \times \xi_2$, where $\xi_1$ is the universal vector bundle over the factor of $\prod BU(n)$ corresponding to the trivial representation. The fixed set ${\xi_1}^G$ is all of $\xi_1$ while the fixed set ${\xi_2}^G$ is the zero section. The result now follows by passing to Thom spaces. $\square$

For convenience, we define the following spectrum.

*Definition* 4.8. Recall that $I_{R(G)}$ is the augmentation ideal of $R(G)$. Let

$$\underline{\mathbf{I}}_{\mathbf{R(G)}} = \bigvee_{W \in I_{R(G)}} S^{2(\nu_{\underline{1}} W)}.$$

Define a ring spectrum structure on $\underline{\mathbf{I}}_{\mathbf{R(G)}}$ by sending the $V$ summand smashed with the $W$ summand to the $V+W$ summand.

THEOREM 4.9. *For any compact Lie group $G$,*

$$\Phi^G MU^G \simeq \underline{\mathbf{I}}_{\mathbf{R(G)}} \wedge MU \wedge \left( \prod_{V \in Irr^*(G)} BU \right)_+ .$$

*Proof*. After Lemma 4.7 the proof of this theorem is simply a passage from prespectra to spectra.

By Lemma 4.7,

$$(\phi^G TU^G)_n = \bigvee_{W \in R^+(G)_{d(n)}} T(\xi_{|W^G|}) \wedge \left( \prod_{V \in Irr^*(G)} BU(\nu_V(W)) \right)_+ ,$$

where $d(n)$ is the dimension of $K_n$ in the definition of $\phi^G$. The bonding maps respect the wedge summand decomposition, sending the $W$th wedge summand to the $W'$th wedge summand, where $W - K_n = W' - K_{n+1}$ in $I_{R(G)} \subset R(G)$. Hence, $\Phi^G TU^G$ splits as a wedge sum of factors indexed by elements of $I_{R(G)}$ defined by these differences. Moreover, because any representation appears as a summand of some $K_n$, each element of $I_{R(G)}$ will appear as an index.

Restricted to each wedge summand, a bonding map of $\phi^G TU^G$ is a bonding map for the prespectrum $TU$ on the $T(\xi_{|W^G|})$ factor smashed with a standard inclusion of products of classifying spaces on the $(\prod BU(\nu_V(W)))_+$ factor. Therefore, upon passage to spectra each such wedge summand gives rise to a copy of $MU \wedge (\prod_{V \in Irr^*(G)} BU)_+$, suspended by a factor of $\nu_{\underline{1}}(V)$ where $V \in I_{R(G)}$ is the index of the summand in question.

Finally, note that the product on $\phi^G TU^G$ arising from the product on $TU^G$ also respects wedge summands. The smash product $P = (\phi^G TU^G)_n \wedge (\phi^G TU^G)_m$ splits as

$$\bigvee_{\substack{V \in R^+(G)_{d(n)} \\ W \in R^+(G)_{d(m)}}} T(\xi_{|V^G|}) \wedge T(\xi_{|W^G|}) \wedge \left( \prod_{R \in Irr^*(G)} BU(\nu_R(V)) \times BU(\nu_R(W)) \right)_+ .$$

Under the product on $\phi^G TU^G$, the $V, W$th summand of $P$ gets mapped to the $V+W$th summand of $(\phi^G TU^G)_{m+n}$ by the smash product of the multiplication map $T(\xi_k) \wedge T(\xi_l) \to T(\xi_{k+l})$ which defines the product on $TU$ with the multiplication maps $BU(r)_+ \wedge BU(s)_+ \to BU(r+s)_+$ which are defined through classifying the Whitney sum of vector bundles. □

For a nontrivial irreducible representation $V$, let $f_V$ be the map from $\mathbb{CP}^k$ mapping to $\prod_{W \in Irr^*(G)} BU$ by the canonical inclusion to $BU(1) \subset BU$ on the $V$th factor and by the trivial map on the other factors. Define $Y_{i,V}$ to be the class in $\pi_{2i-|V|}((\bigvee_{W-|W| \in I_{R(G)}} S^{-|W|}) \wedge MU \wedge \prod_{W \in Irr^*(G)} BU)_+)$ represented by $\mathbb{CP}^{i-1}$ mapped to $\prod_{W \in Irr^*(G)} BU$ by $f_V$. By abuse of notation, let $Y_{i,V}$ also denote the image of this class in $\pi_{2i}(\Phi^G MU^G)$ under the standard inclusion of a wedge summand.

We may now complete the central computation of this section.

THEOREM 4.10. *The ring $(\Phi^G MU^G)_*$ is a Laurent algebra tensored with a polynomial algebra as follows:*

$$(\Phi^G MU^G)_* \cong MU_* \left[ e_V^{\pm 1}, Y_{i,V} \right].$$

*Here $V$ ranges over irreducible representations of $G$, $i$ ranges over the positive integers, where as indicated by notation $e_V$ is the image of the Euler class $e_V \in MU_*^G$ under the canonical map to the localization and where $Y_{i,V}$ are as above.*

*Proof.* This theorem is straightforward computation after Theorem 4.9. We use the well-known computation $MU_*(BU) \cong MU_*[Y_i]$ as rings, where $Y_i$ is represented by $\mathbb{CP}^i$ mapping to $BU$ via its inclusion into $BU(1)$, which is standard as in [1]. Because $MU_*(BU)$ is a free $MU_*$-module, it follows from the Künneth theorem that $MU_*(\prod_{Irr^*(G)} BU)$ is a polynomial algebra as well.

Next let $\iota_V \in \pi_*(\underline{\mathbf{I}}_{\mathbf{R(G)}})$ be the generator on the $V - \overline{|V|}$th summand of $\underline{\mathbf{I}}_{\mathbf{R(G)}}$. Then $MU_*(\underline{\mathbf{I}}_{\mathbf{R(G)}}) \cong MU_*[\iota_V, \iota_V^{-1}]$, so that by a simple application of the Künneth theorem, $(\Phi^G MU^G)_*$ is the tensor product of a Laurent algebra and a polynomial algebra as stated. To finish the computation, note that under the map from $MU^G$ to $\Phi^G MU^G$ the Euler class $e_V$ maps to the $\iota_V$ smashed with the unit in $MU_*(\prod_{Irr^*(G)} BU)$. □

We have shown the intimate relation between localization and taking fixed sets for homotopical equivariant bordism. We will also need the following geometric point of view, which dates back to Conner and Floyd.

PROPOSITION 4.11. *Let $M$ be a stably complex $G$-manifold. The normal bundle $\nu$ of $M^G$ in $M$ is a stably complex vector bundle.*

*Proof.* Let $\eta$ be a complex $G$-bundle over $M$ whose underlying real bundle is $TM \times V$, as given by the stably complex $G$-structure of $M$. Then by Proposition 4.6,

$\eta|_{M^G}$ decomposes as a complex $G$-bundle

$$\eta|_{M^G} \cong \eta_1 \oplus \bigoplus_{\rho \in Irr^0(G)} \eta_\rho,$$

where $\eta_1$ has trivial $G$-action. But we can identify $\eta_1$ as having underlying real bundle equal to $TM^G \times V^G$. So the direct sum of the normal bundle $\nu$ with some trivial bundles underlies $V/V^G \bigoplus_{\rho \in Irr^*(G)} \eta_\rho$, which gives $\nu$ the desired stable complex structure. □

*Definition* 4.12. Let

$$F_* = \bigoplus_{W \in R^+(G)} MU_{*-|W|} \left( \prod_{V \in Irr^*(G)} BU \right).$$

Define the homomorphism $\varphi\colon \Omega_*^{U,G} \to F_*$ as sending a class $[M] \in \Omega_n^{U,G}$ to the class represented by $M^G$ with reference map which classifies its normal bundle.

This geometric picture of taking fixed sets of $G$-actions on manifolds fits nicely with the homotopy theoretic picture we have been developing so far.

PROPOSITION 4.13. (tom Dieck) *The following diagram commutes*

$$\begin{array}{ccc} \Omega_*^{U,G} & \xrightarrow{\varphi} & F_* \\ \Big\downarrow{\scriptstyle PT} & & \Big\downarrow{\scriptstyle i} \\ MU_*^G & \xrightarrow{\lambda} & (\Phi^G MU^G)_*, \end{array}$$

*where $i$ is the inclusion map which sends the summand indexed by $W \in R^+(G)$ in $F_*$ to the summand indexed by $W - |W| \in I_{R(G)}$ in $(\Phi^G MU^G)_*$.*

We may now compute the images of geometric classes in $MU_*^G$ under localization by geometric means.

PROPOSITION 4.14. *Let $V$ be an irreducible representation of $G$. The image of $[\mathbb{P}(n \oplus V)]$ in $(\Phi^G MU^G)_*$ is $Y_{n,V} + X$, where $X$ is $(e_{V^*})^{-n}$ if $V$ is one-dimensional, and $X$ is zero otherwise.*

*Proof*. We use homogeneous coordinates on $\mathbb{P}(n \oplus V)$ respecting the direct sum decomposition of $n \oplus V$. There are two possible components of the fixed sets. The points whose coordinates "in $V$" are zero, constitute a fixed $\mathbb{CP}^{n-1}$, whose normal bundle is the tautological line bundle over $\mathbb{CP}^{n-1}$ whereon each fiber is isomorphic to $V$ as a representation of $G$. As a class in $(\Phi^G MU^G)_*$, this manifold with reference map to $\prod_{Irr^*(G)} BU$ represents $Y_{n,V}$. Alternately, when all other coordinates are zero the resulting submanifold is the space of lines in $V$, which is an isolated fixed point when $V$ is one-dimensional and is a projective space with no fixed points, as $V$ has no nontrivial invariant subspaces, when $V$ has higher dimension. As a class in $(\Phi^G MU^G)_*$, an isolated fixed point whose normal bundle is $\oplus W_i$ represents the class $\prod e_{W_i}^{-1}$. □

The following corollary now follows from Proposition 4.5 and Theorem 4.10 as well as the previous theorem.

COROLLARY 4.15. *$S^{-1}MU_*^T \cong MU_*[e_V^{\pm 1}, Z_{i,V}]$ where $Z_{i,V}$ is the image under localization of* $[\mathbb{P}(i \oplus V)]$.

Now recall that $R_0$ is the sub-algebra of $MU_*^G$ generated by classes $e_V$ and $[\mathbb{P}(n \oplus V)]$. Its image in $S^{-1}MU_*^T$ along with inverses of Euler classes clearly generate $S^{-1}MU_*^T$, so we have established Theorem 1.1.

**5. Computation of $MU_*^G$.** By Theorem 1.1, for nilpotent groups $G$ any $x \in MU_*^G$ can be multiplied by an Euler class to get a class in $R_0$ modulo the annihilator of some Euler class. Our plan, which we carry out for abelian groups, is to build $MU_*^G$ from $R_0$ by division by Euler classes. We are faced with two questions: "when can one divide by an Euler class?" and "what are annhilators of Euler classes?"

We now answer both of these questions for any abelian group. Recall that for an irreducible representation $V$, $K(V)$ is the subgroup of $T$ which acts trivially on $V$.

THEOREM 5.1. *Let $G$ be an abelian group and $V$ a nontrivial irreducible representation of $G$. The sequence*

$$MU_*^G \xrightarrow{\cdot a_V} MU_*^G \xrightarrow{\cdot e_V} MU_*^G \xrightarrow{res^G_{K(V)}} MU_*^{K(V)} \to 0$$

*is exact, where $a_V$ is zero if $S(V)/G$ is a point and is the class defined as the composite*

$$S^V \to S^V/G \cong S^2 \xrightarrow{1} MU^G$$

*otherwise.*

Note that Theorem 1.2 is just the case of this theorem in which $G$ is a torus, in which case $a_V$ is always zero.

*Proof.* We construct and analyze the appropriate Gysin sequence.

Apply $\widetilde{MU_G}^*$ to the cofiber sequence $S(V)_+ \xrightarrow{i} S^0 \xrightarrow{j} S^V$ to get the long exact sequence

$$\cdots \to \widetilde{MU_G}^n(S^V) \xrightarrow{j^*} MU_G{}^n \xrightarrow{i^*} MU_G^n(S(V)) \xrightarrow{\delta} \widetilde{MU_G}^{n+1}(S^V) \to \cdots.$$

As $MU_G$ has suspension isomorphisms for any representation, $\widetilde{MU_G}^n(S^V) \cong MU_G^{n-V}$. By Proposition 3.2, $MU_G^{n-V} \cong MU_G^{n-2}$. The map $j^*$ is by definition multiplication by $e_V$.

To compute $MU_G^n(S(V))$, we note that for a nontrivial irreducible representation $V$ of an abelian group $S(V)$ is a free $G/K(V)$ space with $G$ action defined through the projection $G \to G/K(V)$. Hence the $G$-maps from $S(V)$ to $MU_G$ are in one-to-one correspondence with maps from $S(V)/G$, which is homeomorphic to either $S^1$ or a point, to the $K(V)$-fixed set of $MU_G$, which is homeomorphic to the $K(V)$-fixed set of $MU_{K(V)}$. We deduce that $MU_G^n(S(V)) \cong MU_{K(V)}^n(S(V)/G)$.

By Comezaña's theorem (Theorem 1.3), both $MU_G^*$ and $MU_{K(V)}^*$ are concentrated in even degrees. Hence $i^*$ is surjective. In even degrees, $MU_{K(V)}^n(S(V)/G)$ coincides with $MU_{K(V))}^n$ and $i^*$ is the restriction map.

In odd degrees, we have that $MU_{K(V)}^{2k+1}(S(V)/G)$ is zero if $S(V)/G$ is a point and is isomorphic to $MU_{K(V)}^{2k}$ if $S(V)/G$ is homeomorphic to $S^1$. To understand the boundary homomorphism we look at the Barratt-Puppe sequence, and hence the map $S^V \to S^V/S^0$. Under the isomorphisms

$$MU_{K(V)}^{2k} \cong \widetilde{MU}_{K(V)}^{2k+1}(S^1_+) \cong \widetilde{MU}_G^{2k+1}(S(V)_+) \cong \widetilde{MU}_G^{2k+2}(S^V/S^0),$$

any class in $MU_{K(V)}^{2k}$ corresponds to a class in $\widetilde{MU}_G^{2k+2}(S^V/S^0)$ which when pulled back to $S^V$ is a composite with the map $S^V \to S^V/G$, and thus is divisible by $a_V$. Because the kernel of multiplication is an ideal, the image of the boundary homomorphism must be the ideal generated by $a_V$. □

COROLLARY 5.2. *Let $T$ be a torus. The canonical map from $MU_*^T$ to $S^{-1}MU_*^T$ is injective.*

Theorem 1.? follows immediately from this and Corollary 4.15. We also have the following:

COROLLARY 5.3. *$MU_*^T$ is a domain.*

*Remark*. For $G$ finite, the sub-ring of $MU_*^G$ generated by all $a_V$ is the image of the unit map $\pi_0^G \to MU_0^G$, isomorphic to the Burnside ring. Geometrically, $a_V$ is represented by the $G$-set (zero-dimensional $G$-manifold) $G/K(V)$.

*Remark*. Let $T = S^1$ and $\rho^{\otimes n}$ be the $n$th tensor power of the standard representation. There is a general construction in unstable equivariant homotopy theory which reflects the fact that, by Theorem 1.2, the kernel of the restriction map from $MU^{S^1}$ to $MU^{\mathbb{Z}/n}$ is principal, generated by $e_{\rho^{\otimes n}}$.

Let $f: X \to Y$ be an $S^1$-equivariant map of based spaces which is null-homotopic upon restricting the action to $\mathbb{Z}/n$. Let $F: X \times I \to Y$ be a null-homotopy. Construct an $S^1$-equivariant map $f_{\Sigma(F)}: X \times I \times S^1 \to Y$, where the $S^1$

action on itself is through the degree $n$ map, by sending

$$(x, t, \zeta) \mapsto \zeta \cdot F(\zeta^{-1} \cdot x, t).$$

This map passes to the quotient

$$X \times I \times S^1 / \left( \{X \times 0 \times S^1\} \cup \{X \times 1 \times S^1\} \cup \{* \times I \times S^1\} \right),$$

which is $S^{\rho^{\otimes n}} \wedge X$. When restricted to $S^0 \wedge X \subset S^{\rho^{\otimes n}} \wedge X$ this map coincides with the original $f$, and thus gives a "quotient" of $f$ by the class $S^0 \hookrightarrow S^{\rho^{\otimes n}}$.

As in the introduction, let $s_V$ be a splitting of $res_{K(V)}$ as a map of $MU_*$-modules. Let $\beta_V = s_V \circ res_{K(V)}$. For any $x \in MU^T_n$ let $\Gamma_V(x)$ be the unique class in $MU^T_{n+2}$ such that $e_V \cdot \Gamma_V(x) = x - \beta_V(x)$. The existence and uniqueness of $\Gamma_V(x)$ follow from Theorem 1.2 and the fact that $x - \beta_V(x)$ is in the kernel of $res_{K(V)}$. Both $\beta_V$ and $\Gamma_V$ depend on our choice of splitting $s_V$, but we will suppress this dependence from notation. If $I = V_1, \cdots, V_k$ is a finite sequence of nontrivial irreducible representations let $\Gamma_I(x) = \Gamma_{V_k}\Gamma_{V_{k-1}} \cdots \Gamma_{V_1} x$. These operations $\Gamma_V$ give rise to a filtration of $MU^T_*$.

Fix an ordering on the nontrivial irreducible representations of $T$. For $T = S^1$, we choose an ordering in which $\rho^{\otimes n}$ is less than $\rho^{\otimes m}$ if $|n| < |m|$. Call a finite sequence of representations admissible if it respects this ordering. Let $\#I$ be the number of representations in the sequence $I$.

*Definition* 5.4. Let $R_i$ be the sub-module of $MU^T_*$ generated by all $\Gamma_I(y)$ where $y \in R_0$ and $I$ is admissible with $\#I \leq i$.

LEMMA 5.5. *For any choice of splittings $s_V$, and hence of operations $\Gamma_V$, the filtration by $R_i$ exhausts $MU^T_*$.*

*Proof*. By Theorem 4.10, any class in $MU^T_*$ can be multiplied by an Euler class to give a class in $R_0$ modulo the kernel of the canonical map from $MU^T_*$ to $S^{-1}MU^T_*$, where $S$ is the multiplicative set of nontrivial Euler classes. By Theorem 1.2, multiplication by an Euler class is injective, so it follows that the map to $S^{-1}MU^T_*$ is injective. Hence, for any $x \in MU^T_*$ there is some Euler class $e_W$ such that $x \cdot e_W = y \in R_0$. It follows that $x = \Gamma_I(y)$, where $I$ is such that $W = \bigoplus_{V_i \in I} V_i$. We may choose the $V_i$ to be in any order, in particular so that $I$ is admissible. □

An alternate characterization of $R_i$ is as the sub-module generated by all $x \in MU^T_*$ such that $(\prod e_{W_j})x \in R_0$ for some product of fewer than or equal to $i$ Euler classes $e_{W_j}$. Let $\mathcal{R}_i \subset S^{-1}MU^T_* \cong MU_*[e_V^{\pm 1}, Z_{i,V}]$ be the sub-module generated by $x$ such that $(\prod e_{W_j})x \in MU_*[e_V, Z_{i,V}]$, that is to say that $(\prod e_{W_j})x$ is in the image of $R_0$ under the localization map. Alternately, $\mathcal{R}_i$ is the sub-module generated by monomials in which, in reduced form, at most $i$ inverted Euler classes appear. The fact that the map from $MU_*^T$ to $S^{-1}MU_*^T$ is injective leads to the following.

PROPOSITION 5.6. *Some $x \in MU_*^T$ is in $R_i$ if and only if its image in $S^{-1}MU_*^T$ is in $\mathcal{R}_i$.*

It is clear that the operations $\Gamma_V$ play a central role in equivariant bordism. In order to understand them better, we make refined choices of the splittings $\beta_V$. We first decompose $MU_*^T$ in a way which will be convenient for choosing these splittings. Recall that if $W$ and $V$ are distinct irreducible representations, we say that $W$ divides $V$ if the kernel of $W$ is contained in the kernel of $V$, and that an irreducible representation is primitive if there are no other representations which divide it.

LEMMA 5.7. *Let $T$ be a torus, $V$ be an irreducible representation of $T$ and $G$ be the kernel of $V$. Then $MU_*^T \cong I \oplus A_*^V$ as $MU_*$-modules, where $I$ is the ideal generated by $e_V$ when $V$ is primitive and is the ideal generated by all $\Gamma_W(e_V)$ for $W$ which divide $V$ otherwise, and $A_*^V$ is a sub-algebra which is isomorphic to $MU_*^T/I$. Moreover, the restriction map gives rise to an isomorphism from $A_*^V$ to its image, which is a split summand of $MU_*^G$.*

*Proof.* Our starting point is that while the restriction map from $MU_*^T$ to $MU_*^G$ cannot be split multiplicatively, the corresponding map from $\widetilde{S}^{-1}MU_*^T$ to $S^{-1}MU_*^G$ can be split multiplicatively when $S$ is the multiplicative set of all (nonzero) Euler classes in $MU_*^G$ and $\widetilde{S}$ is the multiplicative set of all Euler-classes in $MU_*^T$ which restrict to nonzero classes in $MU_*^G$.

Recall that by Corollary 4.15, $S^{-1}MU_*^G$ is isomorphic to $MU_*[e_W^{\pm 1}, Z_{i,W}]$ where $W$ ranges over irreducible representations of $G$. We claim that $\widetilde{S}^{-1}MU_*^T$ has a sub-algebra isomorphic to $MU_*[e_V^{\pm 1}, Z_{i,V}]$ where $V$ ranges over irreducible representations which restrict nontrivially to $G$, and on this sub-algebra the restriction map is simply restriction of the indexing representations to $G$. The class $Z_{i,V}$ is the image of $[\mathbb{P}(i \oplus V)]$ in this localization. The fact that there are no relations involving the classes $e_V$ and $Z_{i,V}$ follows from the same fact about their images in the localization obtained by inverting all nontrivial Euler classes in $MU_*^T$, which we get from Proposition 4.14. Moreover, $Z_{i,V}$ maps to $Z_{i,res_G(V)}$ under restriction. Hence, we may split this restriction map multiplicatively by choosing lifts of nontrivial irreducible representations of $G$. For $T = S^1$, $G = \mathbb{Z}/n$, we make this choice of lifts definite, taking $\rho^{\otimes i}$ for $0 < i < n$ as the set of lifts of the representations.

Now consider the following commutative diagram:

$$\begin{array}{ccccc} & & MU_*^T & \xrightarrow{\phi_T} & \widetilde{S}^{-1}MU_*^T \\ & & \Big\downarrow r & & S^{-1}r \Big\downarrow \\ J = (\{a_W\}) & \longrightarrow & MU_*^G & \xrightarrow{\phi_G} & S^{-1}MU_*^G, \end{array}$$

where $W$ ranges over representations of $G$, $a_W$ is as in Theorem 5.1, and the rows are exact by Theorem 5.1. As we just remarked, $S^{-1}r$ can be split multiplicatively. Let us call a splitting $s$. Let $P = \phi_T^{-1} \circ s \circ \phi_G \circ r$, where by abuse of notation $\phi_T^{-1}$ is the inverse of the isomorphism from $MU_*^T$ to its image in $\widetilde{S}^{-1}MU_*^T$. Note that $P$ is a ring homomorphism. Because $\phi_G \circ r \circ \phi_T^{-1} \circ s$ is the identity on the image of $\phi_G$, $P$ is idempotent. The image of $P$ in $MU_*^T$ is isomorphic to the image of the map $\phi_G$, which by Theorem 5.1 is $MU_*^G/J$, where $J$ is is the ideal generated by the classes $a_W$. Because the restriction map from $MU_*^T$ is surjective, $MU_*^G/J$ is isomorphic to $MU_*^T$ modulo the preimage of $J$, which is $I$. By setting $A_*^V$ to be the image of $P$, the decomposition of $MU_*^T$ by $P$ gives the stated decomposition.

We may decompose $MU_*^G$ in a similar way, through the idempotent map $r \circ \phi_T^{-1} \circ s \circ \phi_G$, as $r(A_*^V) \oplus I$. The inverse of the map from $A_*^V$ to its image under $r$ is simply $\phi_T^{-1} \circ s \circ \phi_G$. □

THEOREM 5.8. *With notation as in Lemma 5.7, the map $\beta_V$ may be chosen to be multiplicative on $A_*^V$ and in fact on the internal direct sum $A_*^V \oplus (MU_*^T \cdot e_V)$. Moreover, it may be chosen to send any $x\Gamma_W(e_V)$ to $\beta_W(x)\Gamma_W(e_V)$, respecting the filtration by $R_i$.*

*Proof.* That $\beta_V$ may be chosen to be multiplicative on $A_*^V$ follows from the fact that, by Lemma 5.7, $A_*^V$ maps isomorphically to its image under $r$, which is a split summand of $MU_*^G$. We may choose $s_V$ to be the inverse to this isomorphism on $r(A_*^V)$. With such an $s_V$, $\beta_V$ is multiplicative on $A_*$ or in fact on $A_* \oplus (MU_*^T \cdot e_V)$, as $e_V$ maps to zero under $r$.

Next we show that $\beta_V$ sends $A_*^V \cap R_i$ to itself. Let $\Gamma_I(y) \in A_*^V$ for some admissible $I$ with $\#I \leq i$ and $y \in R_0$. By Proposition 5.6, $x \in R_i$ if and only if its image in $S^{-1}MU_*^T \cong MU_*[e_V^{\pm 1}, Z_{i,V}]$ has less than or equal to $i$ negative powers of Euler classes occurring in each monomial which appears. For $\beta_V(\Gamma_I(x))$, the number of negative powers of Euler classes which appear are bounded by the number for $\Gamma_I(x)$, which is $i$.

We must now choose $\beta_V$ on $I$. Note that in $MU_*^G$, $\Gamma_W(e_V)$ annihilates $e_W$, so that

$$r(x\Gamma_W(e_V)) = r(\beta_W(x)\Gamma_W(e_V)).$$

Hence, we may choose $\beta_V(x\Gamma_W(e_V))$ to be $\beta_W(x)\Gamma_W(e_V)$. With this choice, $\beta_V$ respects the filtration by $R_i$ when $\beta_W$ does for all $W$ which divide $V$. By induction, it suffices to show that $\beta_W$ preserves the filtration for primitive $W$, which means that the kernel of $W$ is a torus. But in this case $A_*^W \oplus (MU_*^T \cdot e_W)$ is all of $MU_*^T$, so this follows from what we have shown for $A_*^W$. □

COROLLARY 5.9. *With choice of $\beta_V$ as in Theorem 5.8, $\Gamma_V(x \cdot \Gamma_W(e_V)) = \Gamma_W(x)$.*

*Proof.* We have $\beta_V(x \cdot \Gamma_W(e_V)) = \beta_W(x)\Gamma_W(e_V)$ so that

$$\Gamma_V(x \cdot \Gamma_W(e_V)) = \Gamma_V((x - \beta_W(x))\Gamma_W(e_V)),$$

where $(x - \beta_W(x))\Gamma_W(e_V)$ is divisible by $e_V$. But by calculating their images in $S^{-1}MU_*^T$ we can see that $\Gamma_V((x - \beta_W(x))\Gamma_W(e_V))$ and $\Gamma_W(x)$ are equal. $\square$

*Remark*. Note that it is not possible to split the restriction map multiplicatively on all of $MU_*^G$, as in $MU_*^G$ there is the relation $a_W \cdot e_W = 0$, but this relation cannot be lifted to $MU_*^T$, which is a domain by Corollary 5.3.

We are now ready to prove our main theorem.

THEOREM. (Restatement of Theorem 1.5) With a choice of splitting $s_V$ as in Theorem 5.8, $MU_*^T$ is generated as an $MU_*$-algebra by the classes $\Gamma_I(e_V)$ and $\Gamma_I([\mathbb{P}(\underline{n} \oplus V)])$, where $V$ ranges over nontrivial irreducible representations, $I$ ranges over all admissisible sequences of nontrivial irreducible representations, and $n$ ranges over natural numbers. Relations include the following:

(1) $e_V\Gamma_V(x) = x - \beta_V(x)$,

(2) $\Gamma_V(\beta_V(x)) = 0$,

(3) $\Gamma_V(e_V) = 1$,

(4) $\Gamma_V(x)y = (x - \beta_V(x))\Gamma_V(y) - \Gamma_V(x)\beta_V(y)$,

(5) $\Gamma_V\Gamma_W x = \Gamma_W\Gamma_V x - \Gamma_W\Gamma_V\beta_W(x) - \Gamma_W\Gamma_V(e_W\beta_V(\Gamma_W x))$,

where $V$ and $W$ range over nontrivial irreducible representations of $T$ and $x$ and $y$ are any classes in $MU_*^T$. For $T = S^1$, these relations are complete.

*Proof*. Let $C_i$ be the set consisting of $\Gamma_I(e_V)$ and $\Gamma_I([\mathbb{P}(\underline{n} \oplus V)])$ where $I$ ranges over admissible sequences of length less than or equal to $i$. We will show by induction that $R_i$ is contained in the sub-algebra generated by $C_i$. Passing to limits, we will have that $C = \bigcup C_i$ is a generating set for $MU_*^T$ as stated. We will inductively prove two statements:

(1) $R_i$ is contained in the sub-algebra generated by $C_i$.

(2) For any $x, y \in R_{i-1}$ and $V$ irreducible, we may choose the splitting $s_V$ so that $\Gamma_V(\beta_V(x)\beta_V(y)) \in R_i$.

Both of our induction statements are immediately true for $i = 0$. For each $i$ we will prove both of these statements through a separate induction on $V$, for which we need to replace the first statement by the following, which is equivalent for $i > 0$.

(1) For any $x$ in $R_{i-1}$ and any irreducible $V$, $\Gamma_V(x)$ is in the sub-algebra generated by $C_i$.

The logic of our induction on $V$ with $i$ fixed is that for every $V$ for which (2) is true, we will show (1) holds. And for every $V$ such that (1) holds for all $W$ which divide $V$, we will show (2) is true. It will follow, after establishing the induction steps and showing (2) is true for any primitive $V$, that both statements are true for all $V$.

If $V$ is primitive, then the map $\beta_V$ is in fact multiplicative by Theorem 5.8, so that $\Gamma_V(\beta_V(x)\beta_V(y)) = \Gamma_V(\beta_V(xy)) = 0$. Hence, the second statement is immediately true for primitive $V$.

For the inductive proof of statement (1), consider $\Gamma_I(y)$ where $y \in R_0$ is a monomial in the elements of $C_0$ and $I = V_1, \cdots, V_k$. By definition, $\Gamma_I(y) = \Gamma_{V_1}(\Gamma_{I'}(y))$ where $I' = V_2, \cdots, V_k$. By the induction hypothesis, $\Gamma_{I'}(y)$ is in the sub-algebra generated by $C_{i-1}$. Namely $\Gamma_{I'}(y)$ is a sum of terms of the form $\prod_{i=1}^n \Gamma_{J_i}(z_i)$ where $z_i \in C_0$, $\#J \leq k-1$, and all of the representations which appear in some $J_i$ are less than $V_1$ in our ordering (because in fact they must be in $I'$). By linearity of $\Gamma_{V_1}$, it suffices to show the following:

(1) $\Gamma_{V_1}(\prod_{i=1}^n \Gamma_{J_i}(z_i))$ is in the sub-algebra generated by $C_i$ for any $z_i$ and $J_i$ as above.

We prove this claim by induction on $n$, with the case $n = 1$ being immediate. First note that for any $V$, as can be verified by multiplying by $e_V$,

$$\Gamma_V(xy) = \Gamma_V(x)y + \beta_V(x)\Gamma_V(y) + \Gamma_V(\beta_V(x)\beta_V(y)).$$

Hence we compute

$$\begin{aligned}\Gamma_{V_1}\left(\prod_{i=1}^n \Gamma_{J_i}(z_i)\right) &= \Gamma_{V_1}(\Gamma_{J_1}(z_1)\cdot X) \text{ where } \quad X = \prod_{i=2}^n \Gamma_{J_i}(z_i) \\ &= \Gamma_{V_1}(\Gamma_{J_1}(z_1))X - \beta_{V_1}(\Gamma_{J_1}(z_1))\Gamma_{V_1}(X) \\ &\quad - \Gamma_{V_1}\left(\beta_{V_1}(\Gamma_{J_1}(z_1))\beta_{V_1}(X)\right).\end{aligned}$$

Now we check that $\Gamma_{V_1}(\Gamma_{J_1}(z_1))$ is in $C_i$ by definition. $X$ is in the sub-algebra generated by $C_{i-1}$ and hence by $C_i$. As $\Gamma_{J_1}(z_1)$ is in the sub-algebra generated by $C_{i-1}$ so is $\beta_{V_1}(\Gamma_{J_1}(z_1))$, by Theorem 5.8. We have $\Gamma_{V_1}(X)$ is in the sub-algebra generated by $C_i$ by induction on $n$, and $\Gamma_{V_1}\left(\beta_{V_1}(\Gamma_{J_1}(z_1))\beta_{V_1}(X)\right)$ is in the sub-algebra generated by $C_i$ by the second induction hypothesis for $V_1$. Hence, $\Gamma_{V_1}\left(\prod_{i=1}^n \Gamma_{J_i}(z_i)\right)$ is in the sub-algebra generated by $C_i$ as claimed.

Next, we prove statement (2), inductively assuming statement (1) for any $W$ whose kernel is contained in that of $V$. First note that since $\beta_V(\beta_V(x)\beta_V(y)) = \beta_V(xy)$, we have

$$\Gamma_V(\beta_V(x)\beta_V(y)) = \Gamma_V(\beta_V(x)\beta_V(y) - \beta_V(xy)),$$

so the fact that $\Gamma_V(\beta_V(x)\beta_V(y))$ may be nonzero arises from the failure of $\beta_V$ to be multiplicative. Hence we decompose $x$ and $y$ each as $a + be_V + \Sigma c_W\Gamma_W(e_V)$, where $a \in A_*^V \cap R_{i-1}$ and $c_W \in R_{i-1}$. Because $\beta_V$ is multiplicative on $A_* \oplus (MU_*^T \cdot e_V)$ and is linear in general it suffices to consider $x = c\Gamma_W(e_V)$. But in this case we may proceed directly, as

$$\begin{aligned}\Gamma_V(\beta_V(x)\beta_V(y)) &= \Gamma_V(\beta_V(c\Gamma_W(e_V))\beta_V(y)) \\ &= \Gamma_V(\beta_W(c)\Gamma_W(e_V)\beta_V(y)), \text{ by Theorem 5.8} \\ &= \Gamma_W(\beta_W(c)\beta_V(y)), \text{ by Corollary 5.9,}\end{aligned}$$

and $\Gamma_W(\beta_W(c)\beta_V(y))$ is in $R_i$ by inductive assumption. This computation concludes the inductive argument that $C$ is a complete set of generators.

Next we note that the relations are readily verifiable. Relation (1) holds by definition. And we may use the fact that multiplication by nontrivial Euler classes is a monomorphism to verify relations (2), (3) and (4) by multiplying them by $e_V$, and (5) by multiplying it by $e_V e_W$. For $T = S^1$ we claim that these relations are complete. To show this, we first exhibit an additive basis for $MU_*^{S^1}$. Define the isotropy group of an element $x \in MU_*^T$ to be the largest subgroup $H$ of $T$ such that the restriction of $x$ to $MU_*^H$ is in the image of $MU_*$. Choose an ordering on $C_0$ with $x < y$ if the isotropy groups of $x$ and $y$ are $\mathbb{Z}/n$ and $\mathbb{Z}/m$, respectively, and $n < m$. We show that an additive basis for $MU_*^{S^1}$ is given by monomials of the form $m = \Gamma_I(z_1)z_2 \cdots z_k$, where $z_i \in C_0$, $I$ is admissible, $z_1$ is greater than all other $z_i$ in the ordering above, and if some $z_j = e_W$ then $W \notin I$. The fact that these monomials $m$ are linearly independent over $MU_*$ follows by checking the same fact for their images in $S^{-1}MU_*^T$, which is straightforward. Hence, we are left to show that our relations suffice to reduce to this basis.

Let $M$ be the set of all monomials built from our generating set $C$. Let $m = \Gamma_{I_1}(z_1) \cdots \Gamma_{I_k}(z_k)$ be an element of $M$. We will use relation (4) to reduce $m$ to a sum of monomials each of which will have only one nontrivial $\Gamma_I$ appearing, using relation (5) to reduce further so that this $I$ is in appropriate order. The difficulty is capturing the sense in which relation (4) can be used to simplify such a monomial $m$. To do this we define a partial ordering on $M$. Let $m$ be as above and let $n = \Gamma_{J_1}(w_1) \cdots \Gamma_{J_l}(w_l)$. For convenience, if $I$ is a $k$-tuple of nontrivial representations of $S^1$, namely $\rho^{\otimes n_1}, \cdots, \rho^{\otimes n_k}$ we set $v(I) = \Sigma|n_i|$. We say $m \prec n$ if the following hold.

- $k \leq l$.
- After permuting indices, $z_i$ is less than or equal to $w_i$ in our ordering on $C_0$ and moreover the dimension of $z_i$ is less than or equal to that of $w_i$ as a class in $MU_*^T$.
- $\Sigma v(I_i) < \Sigma v(J_i)$, where $i$ ranges over all indices except that of the $z_i$ (respectively $w_i$) which is maximal in our ordering on $C_0$.

Now we establish the fact that we can reduce any monomial in $M$ to a sum of monomials for each of which only one nontrivial $\Gamma_I$ appears, and the element of $C_0$ on which it operates is maximal with respect to our ordering on $C_0$. First note that for any $m \in M$ there are only finitely many monomials which are less than $m$ in our partial ordering $\prec$. Secondly, suppose $m = \Gamma_I(z)\Gamma_J(w)m'$ where $z$ is maximal in our ordering on $C_0$, $J = W_1, \cdots W_l$ is nontrivial, and $m'$ is some monomial in $M$. Let $J' = W_2, \cdots W_l$. Applying relation (4), we have

$$\Gamma_I(z)\Gamma_{W_1}(\Gamma_{J'}(w))m' = \left[(\Gamma_{J'}(w) - \beta_{W_1}(\Gamma_{J'}(w))\Gamma_{W_1}\Gamma_I(z) - \beta_{W_1}(\Gamma_I(z))\Gamma_J(w)\right] m'.$$

In order to have monomials in $M$, we must apply relation (5) to $\Gamma_{W_1}\Gamma_I(z)$, as $W_1, V_1, \cdots V_k$ is not necessarily admissible. These monomials are all less than $m$ in the ordering $\prec$. We may apply this procedure whenever there is any nontrivial $\Gamma_I$ applied to a nonmaximal element of a monomial. Because there are only finitely many monomials less than a given one under $\prec$ this process will terminate.

Finally, note that once we have a monomial in the form $\Gamma_I(z_0)z_1 \cdots z_k$, where $z_0$ is maximal there still may be some $V$ such that $V \in I$ and $z_i = e_V$ for some $i$. We may use relation (5) to rearrange the order of $I$ temporarily so that $V$ is the first representation in $I$. Then we may use relation (5) to reduce to monomials in which the $e_V$ does not occur, and use (5) once again to restore admissibility. □

*Remark*. For the reduction procedure to show that the relations for $T = S^1$ are complete, it is crucial to have an ordering on monomials such that for any $V$, $\beta_V(m)$ is either equal to $m$ or is strictly less than $m$. Hence this reduction procedure does not extend immediately to higher rank tori.

**6. The completion map and a construction of Conner and Floyd.** From our computations, it is clear that the structure of $MU_*^G$ is governed by the operations $\Gamma_V$. We call these operations Conner-Floyd operations because in the special case in which $G = S^1$, $V = \rho$ the standard representation, and $[M]$ is a geometric class, there is a construction of $\Gamma_\rho([M])$, which dates back to work of Conner and Floyd.

*Definition* 6.1. Define $\gamma(M)$ for any stably complex $S^1$-manifold to be the stably complex manifold

$$\gamma(M) = M \times_{S^1} S^3 \sqcup (-\overline{M}) \times \mathbb{P}(1 \oplus \rho)$$

where $S^3$ has the standard Hopf $S^1$-action, $-\overline{M}$ is the $S^1$-manifold obtained from $M$ by imposing a trivial action on $M$ and taking the opposite orientation, and the $S^1$-action on $M \times_{S^1} S^3$ is given by

$$\zeta \cdot [m, z_1, z_2] = [\zeta \cdot m, z_1, \zeta z_2]\,. \tag{1}$$

Inductively define $\gamma^i(M)$ to be $\gamma(\gamma^{i-1}(M))$, where $\gamma^0(M) = M$.

Proposition 6.2. *Let $\rho$ be the standard representation of $S^1$. And let $M$ be a stably complex $S^1$-manifold. Then $\Gamma_\rho[M] = [\gamma(M)]$ in $MU_*^{S^1}$*

*Proof*. As the localization map is injective it suffices to check the equality in $S^{-1}MU_*^{S^1}$. By Proposition 4.13 we can compute the image of $[M]$, $[\overline{M} \times \mathbb{P}(1 \oplus \rho)]$ and $[\Gamma(M)]$ in the localization at a full set of Euler classes by computing fixed sets with normal bundle data.

There are two types of fixed sets of $\Gamma(M)$. To describe these, we refer to equation (1) above. One set of fixed points of $\Gamma(M)$ are points $[m, z_1, z_2]$ such that $m$ is fixed in $M$ and $z_2 = 0$. This fixed set is diffeomorphic to $M^G$, and its normal bundle is the normal bundle of $M^G$ in $M$ crossed with the representation $\rho$. In the localization, crossing with $\rho$ coincides with multiplying by $e_\rho^{-1}$. The second set of fixed points are $[m, z_1, z_2]$ such that $z_1 = 0$. This set of fixed points is diffeomorphic to $M$, and its normal bundle is the trivial bundle $\rho^{-1}$.

Hence, if $x \in (\Phi^{S^1} MU^{S^1})_*$ is the image of $[M]$, then the image of $[\Gamma(M)]$ is $xe_\rho^{-1} + [\overline{M}]e_{\rho^{-1}}^{-1}$. By subtracting the image of $[\overline{M} \times \mathbb{P}(1 \oplus \rho)]$ we are left with $xe_\rho^{-1} - [\overline{M}]e_\rho^{-1}$, which by definition is the image of $\Gamma_\rho([M])$. The result follows as the map from $MU_*^{S^1}$ to $(\Phi^{S^1} MU^{S^1})_*$ is injective. □

This geometric construction of a single Conner-Floyd operation gives us an explicit understanding of the most important representation of $MU_*^T$, namely the map from $MU_*^T$ to its completion at its augmentation ideal. As a special case of Theorem 1.2, we know that the augmentation ideal of $MU_*^{S^1}$ is principal, generated by $e_\rho$. Because the augmentation map is split and multiplication by $e_\rho$ is a monomorphism, the completion of $MU_*^{S^1}$ at its augmentation ideal is a power series ring over $MU_*$ where $e_\rho$ maps to the power series variable under completion. As an immediate consequence of Proposition 6.2 we have the following.

THEOREM 6.3. *Let $[M]$ be class in $MU_*^{S^1}$ which is the image under the Pontrijagin-Thom map of the class in geometric bordism represented by the complex $S^1$-manifold $M$. The image of $[M]$ under the map from $MU_*^{S^1}$ to its completion at its augmentation ideal, which is isomorphic to $MU_*[[x]]$, is the power series*

$$[\alpha(M)] + [\alpha(\gamma(M))]x + [\alpha(\gamma^2(M))]x^2 + \cdots,$$

*where $\alpha(\gamma^i(M))$ is the manifold obtained from $\gamma^i(M)$ simply by forgetting the $G$-action.*

Understanding this completion map for geometric classes is important for some geometric applications. For example, let $\epsilon$ be a genus, that is a ring homomorphism from $MU_*$ to some ring $E_*$. For an extensive introduction to genera, see [18]. For $G = S^1$, we may extend $\epsilon$ to an equivariant genus $\Omega_*^{U,G} \to E_*[[x]]$, by taking the image of a class $[M]$ under completion, namely $f \in (MU_*^{S^1})_{\hat{I}} \cong MU_*[[x]]$, and applying $\epsilon$ term-wise.

A genus $\epsilon$ is strongly multiplicative if for any fiber bundle of stably complex manifolds $F \to E \to B$, $\epsilon(E) = \epsilon(F) \cdot \epsilon(B)$. The following theorem is a starting point in the study of genera, saying essentially that strongly multiplicative genera are rigid when extended to equivariant genera as above.

THEOREM 6.4. *Let $\epsilon$ be a strongly multiplicative genus. Then for any class $[M] \in \Omega_*^{U,S^1}$, the equivariant extension $\epsilon([M])$ is equal to $\epsilon([\alpha(M)]) \in E_* \subset E_*[[x]]$.*

*Proof.* By Theorem 6.3 the image of $[M]$ under completion is

$$[\alpha(M)] + [\alpha(\gamma(M))]x + [\alpha(\gamma^2(M))]x^2 + \cdots.$$

For any $X \in \Omega_*^{U,S^1}$ we have that $\epsilon([\alpha(\gamma(X))] = 0$ because $\epsilon$ is strongly multiplicative and by definition $\gamma(X)$ is the difference between a twisted product and a trivial product of $X$ and $\mathbb{CP}^1$. $\square$

Returning to computation of the completion map on $MU_*^T$, we now focus on Euler classes.

PROPOSITION 6.5. *The image of the Euler class $e_{\rho^{\otimes n}}$ in the completion $(MU_*^{S^1})_{\hat{I}}$ is $[n]_F x$, the n-series in the formal group law over $MU_*$.*

*Proof*. As the map from $MU_*^G$ to its completion is a map of complex-oriented equivariant cohomology theories, the Euler class of the bundle $V$ over a point gets mapped to the Euler class of $V \times_G E_G$ over $B_G$. For $G = S^1$, $V = \rho^{\otimes n}$ the resulting bundle is the $n$th-tensor power of the tautological bundle over $B_{S^1}$, whose Euler class is by definition the $n$-series. $\square$

We are now ready to state our theorem about the image of the completion map for $MU_*^T$. When $T = (S^1)^k$, the completion of $MU_*^T$ at its augmentation ideal is isomorphic to $MU_*[[x_1, \cdots x_k]]$.

*Definition* 6.6. Let $Y_n(x) \in MU_*[[x]]$ be the image of the class $[\mathbb{P}(\underline{n} \oplus \rho)]$ under the completion map.

THEOREM 6.7. *Let E be the set of all series*

$$[m_1]_F x_1 +_F \cdots +_F [m_k]_F x_k \in MU_*[[x_1, \cdots x_k]].$$

*The image of $MU_*^T$ in its completion at the augmentation ideal is contained in the minimal sub-ring A of $MU_*[[x_1, \cdots, x_k]]$ which satisfies the following two properties:*

- *$E \subset A$, and A contains the series $Y_i(f)$ where $f \in E$ and $Y_i(f)$ are defined above.*
- *If $\alpha f \in A$ then $\alpha \in A$, for any $f \in E$.*

We can recover the image of $MU_*^G$ in its completion at the augmentation ideal for general $G$ by reducing $MU_*[[x_i]]$ modulo the ideal $([d_i]_F x_i)$, where $d_i$ are the orders of the cyclic factors of $G$.

*Proof*. The first condition on $A$ says that the image contains all images of classes in $R_0$. Indeed, $E$ is the image of the Euler classes. We check that the image of $[\mathbb{P}(i \oplus \rho^{\otimes n})]$ in $(MU_*^{S^1})_{\hat{I}}$ is $Y_i([n]_F x)$, which follows from the fact that the $S^1$ action on $[\mathbb{P}(i \oplus \rho^{\otimes n})]$ is pulled back from the $S^1$ action on $[\mathbb{P}(i \oplus \rho)]$ by the degree $n$ homomorphism from $S^1$ to itself. By Theorem 4.10 we may build any class in $MU_*^T$ by dividing classes in $R_0$ by Euler classes. The second condition on $A$ accounts for all possible quotients by Euler classes in the image. $\square$

Suppose $f = a_0 + a_1x + a_2x^2 + \cdots$ is the image of $x \in MU_*^{S^1}$ under completion. Then the image of $\Gamma_\rho(x)$ under completion is that $a_1 + a_2x + a_3x^2 + \cdots$ is in the image. More generally, any $a_i + a_{i+1}x + \cdots$ is in the image of the completion map. So the property of a series being in the image of the completion map depends only on the tail of the series. It would be interesting to find an "analytic" way to define this image.

**7. Applications and further remarks.** In this section we give an assortment of applications and indicate directions for further inquiry.

Our first application is in answer to a question posed by Bott. Suppose a group acts on a manifold compatible with a stably complex structure and that the fixed points of the action are isolated. What can one say about the representations which appear as tangent spaces to the fixed points? If there are only two fixed points, the representations must be dual, which one can prove by Atiyah-Bott localization. What happens for three or more fixed points is an active area of inquiry in equivariant cohomology. With our bordism techniques, we can answer some of these questions, as well as go beyond local information.

THEOREM. (Restatement of Theorem 1.6) Let $M$ be a stably-complex four dimensional $S^1$-manifold with three isolated fixed points. Then $M$ is equivariantly cobordant to $\mathbb{P}(\underline{1} \oplus V \oplus W)$ for some distinct nontrivial irreducible representations $V$ and $W$ of $S^1$.

*Proof.* For convenience, we refer to the Euler class $e_{\rho^{\otimes n}} \in MU_*^{S^1}$ by $e_n$. A complex $S^1$ manifold $M$ with three isolated fixed points defines a class in $MU_*^{S^1}$ whose image under $\lambda\colon MU_*^{S^1} \to S^{-1}MU_*^{S^1}$ is

$$\lambda([M]) = e_a^{-1}e_b^{-1} + e_c^{-1}e_d^{-1} + e_f^{-1}e_g^{-1}$$

for some integers $a, \cdots, g$. We let $T$ denote

$$e_a \cdots e_g \cdot \lambda[M] = e_ce_de_fe_g + e_ae_be_fe_g + e_ae_be_ce_d \in MU_*^{S^1} \subset S^{-1}MU_*^{S^1}.$$

Without loss of generality, assume $a$ is the greatest of the integers $a, \cdots, g$ in absolute value. As $T$ is divisible by $e_a$ in $MU_*^{S^1}$, Theorem 1.2 implies that $T$ restricted to $MU_*^{\mathbb{Z}/a}$ must be zero. The Euler class $e_n$ restricts nontrivially to $MU_*^{\mathbb{Z}/a}$ unless $a|n$. Therefore one of $c, d, f, g$—say $c$—must be equal to $\pm a$. We first claim that this number must be $-a$.

Let $S_{\hat{a}}$ denote the multiplicative set generated by all the Euler classes associated to irreducible representations except for $e_a$. By localizing the modules in Theorem 4.10 and Theorem 1.2, we find that $S_{\hat{a}}^{-1}MU_*^T$ is generated over the operation $\Gamma_a$ by $S_{\hat{a}}^{-1}R_0$. Suppose that $|b|, |d|, |f|, |g| < |a|$ and that $c = a$. Then

$$e_a^{-1}e_b^{-1} + e_a^{-1}e_d^{-1} + e_f^{-1}e_g^{-1}$$

is in the image of the canonical map from $S_{\hat{a}}^{-1}MU_*^{S^1}$ to $S_{\hat{a}}^{-1}R_0$, as it is actually in the image of $\lambda$. Then we must have that $e_b^{-1} + e_d^{-1}$ is divisible by $e_a$ and thus is zero in $S^{-1}MU_*^{\mathbb{Z}/a}$ where $S$ here is the multiplicative set of all Euler classes of $\mathbb{Z}/a$. This localization of $MU_*^{\mathbb{Z}/a}$ is the target of the restriction map from $S_{\hat{a}}^{-1}MU_*^{S^1}$. And by abuse we are using the same names for Euler classes for different groups. But because $|b|, |d| < |a|$, $e_b^{-1}$, $e_d^{-1}$ and their sum are nonzero in $S^{-1}MU_*^{\mathbb{Z}/a}$.

It is straightforward to rule out cases where some of $|b|, |d|, |f|, |g|$ are equal to $|a|$.

Next, consider the class

$$\lambda_{\hat{a}}([M]) - \lambda_{\hat{a}}([\mathbb{P}(1 + \rho^{\otimes a})])e_d^{-1} \in S_{\hat{a}}^{-1}MU_*^{S^1},$$

where $\lambda_{\hat{a}}$ is the canonical map to this localization. Its image under the map to the full localization is

$$e_a^{-1}e_b^{-1} - e_a^{-1}e_d^{-1} + e_f^{-1}e_g^{-1},$$

which implies that $e_b^{-1} - e_d^{-1}$ is divisible by $e_a$ in $S_{\hat{a}}^{-1}MU_*^{S^1}$ or that $b \equiv d \pmod{a}$. But because $|b|, |d| < |a|$ we have that $d = a \mp b$ depending on whether $b$ is positive or negative.

Finally, as $c = -a$ and $d = b - a$ consider $\lambda([M] - [\mathbb{P}(1 \oplus \rho^{\otimes a} \oplus \rho^{\otimes b})])$, which will be equal $e_f^{-1}e_g^{-1} - e_{a-b}^{-1}e_{-b}^{-1}$. Case analysis of necessary divisibilities as we have been doing implies that this difference must be zero, so that the fixed-set data of $[M]$ is isomorphic to that of $\mathbb{P}(1 \oplus \rho^{\otimes a} \oplus \rho^{\otimes b})$.

Finally, because the localization map $\lambda$ is injective, this fixed-set data determines $[M]$ as in $S^1$-equivariant homotopical bordism uniquely, so that $[M]$ must equal $[\mathbb{P}(1 \oplus \rho^{\otimes a} \oplus \rho^{\otimes b})]$ in $MU_4^{S^1}$. But a theorem of Comezaña, from chapter 28 of [17], says that the Pontrijagin-Thom map from $\Omega_*^{U,A}$ to $MU_*^A$ is injective for abelian groups $A$. Hence $M$ is cobordant to $\mathbb{P}(1 \oplus \rho^{\otimes a} \oplus \rho^{\otimes b})$. □

Our next application answers a question about bordism of free $\mathbb{Z}/n$-manifolds posed to us by Milgram. It is well-known that the spheres $S(\oplus_k \rho^{\otimes m})$ for any $m$ relatively prime to $n$ generate $MU_*(B_{\mathbb{Z}/n})$ as an $MU_*$-module. How are these bases related?

THEOREM 7.1. *Let $m$ and $n$ be relatively prime. Let $Q(x)$ be a quotient of $x$ by $[m]_F x$ modulo $[n]_F x$ in $MU_*[[x]]$. Define $a_i \in MU_*$ by $(Q(x))^k = a_0 + a_1 x + a_2 x^2 + \cdots$. Then*

$$[S(\oplus_k \rho^{\otimes m})] = a_0[S(\oplus_k \rho)] + a_1[S(\oplus_{k-1} \rho)] + \cdots + a_{k-1}[S(\rho)]$$

*in $MU_*(B_{\mathbb{Z}/n})$.*

*Proof*. We use an analog of the simple fact that if $M$ is a $G$-manifold and $M \setminus M^G$ has a free $G$-action then $[\partial\nu(M^G)] = 0$ in $MU_*(B_G)$, where $\partial\nu(M^G)$ is the boundary of a tubular neighborhood around the fixed set $M^G$. The null-bordism is defined by $M \setminus \nu(M^G)$. If the fixed points of $M$ are isolated, this will give rise to a relation among spheres with free $G$-actions.

Let $\alpha_0 = q^k$ where $q$ is a quotient of $e_\rho$ by $e_{\rho^{\otimes m}}$ in $MU_*^{\mathbb{Z}/n}$. Inductively, let $\alpha_i$ be a quotient of $\alpha_{i-1} - \overline{\alpha_{i-1}}$ by $e_\rho$ (note that this quotient is not unique as we are working in $\mathbb{Z}/n$ equivariant bordism). Then the "fixed sets" of $\alpha_k$ are given by

$$\lambda(\alpha_k) = {e_{\rho^{\otimes m}}}^{-k} - \overline{\alpha_0}{e_\rho}^{-k} - \overline{\alpha_1}{e_\rho}^{-k+1} - \cdots - \overline{\alpha_{k-1}}{e_\rho}^{-1}.$$

As ${e_V}^{-1}$ corresponds to a tubular neighborhood of an isolated fixed point in geometric bordism, we can deduce via transversality arguments for free $G$-actions that

$$[S(\oplus_k \rho^{\otimes m})] - \overline{\alpha_0}[S(\oplus_k \rho)] - \overline{\alpha_1}[S(\oplus_{k-1} \rho)] - \cdots - \overline{\alpha_{k-1}}[S(\rho)] = 0$$

in $MU_*(B_{\mathbb{Z}/n})$. But the image of $\alpha_0$ in $(MU_*^{\mathbb{Z}/n})_{\hat{I}} \cong MU_*[[x]]/[n]_F x$ is $(Q(x))^k$ from which we can read off that $\overline{\alpha_i} = a_i$.

Note that our expressions in $MU_*(B_{\mathbb{Z}/n})$ are independent of the indeterminacy in choosing $q$ and the $\alpha_i$. □

This old idea of using $G$-manifolds to bound and thus give insight into free $G$-manifolds has been codified by Greenlees's introduction of local cohomology to equivariant stable homotopy theory [9]. Moreover, by work of Greenlees and May, the theories we have been studying provide a unified framework in which to study the characteristic classes $E^*(BG)$ for any complex-oriented theory $E$. We hope that our understanding of relevant commutative algebra can lead to new insights into these techniques.

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REFERENCES

[1] J. F. Adams, *Stable Homotopy and Generalized Homology*, University of Chicago Press, Chicago, 1974.

[2] ———, Prerequisites (in equivariant stable homotopy theory) for Carlsson's lecture, *Algebraic Topology, Aarhus 1982, Lecture Notes in Math.*, vol. 1051, Springer-Verlag, New York, 1984, pp. 483–532.

[3] M. F. Atiyah, G. B. Segal and I. Singer, The index of elliptic operators II, III, *Ann. of Math.* **87** (1968), 531–578.

[4] G. Carlsson, A survey of equivariant stable homotopy theory, *Topology* **31** (1992), 1–27.

[5] M. Cole, J. P. C. Greenlees and I. Kriz, Equivariant formal group laws, *Proc. London Math. Soc. (3)* **81** (2000), 355–386.

[6] P. E. Conner and E. E. Floyd, *Differentiable Periodic Maps*, Springer-Verlag, New York, 1964.

[7] S. R. Costenoble and S. Waner, $G$-transversality revisited, *Proc. Amer. Math. Soc*. **116** (1992), 535–546.

[8] T. tom Dieck, Bordism of $G$-manifolds and integrality theorems, *Topology* **9** (1970), 345–358.

[9] J. P. C. Greenlees, $K$-homology of universal spaces and local cohomology of the representation ring, *Topology* **32** (1993), 295–308.

[10] ———, $MU^A$ is $L_A$, preprint, 2000.

[11] J. P. C. Greenlees and J. P. May, Localization and completion theorems for $MU$-module spectra, *Ann. of Math*. **146** (1997) 509–544.

[12] M. Hopkins, N. Kuhn and D. Ravenel, Morava $K$-theories of classifying spaces and generalized characters for finite groups, *Algebraic Topology (San Feliu de Guixols, 1990), Lecture Notes in Math*., vol. 1509, Springer-Verlag, Berlin, 1992, pp. 186–209.

[13] T. Hungerford, *Algebra*, Springer-Verlag, New York, 1974.

[14] C. Kosniowski, Generators of the $\mathbb{Z}/p$ bordism ring, serendipity, *Math. Z.* **149** (1976), 121–130

[15] I. Kriz, On complex equivariant bordism rings for $G = \mathbb{Z}/p$, *Proceedings of Conference in Honor of Michael Boardman* (to appear).

[16] I. Madsen and R. J. Milgram, *Classifying Spaces for Surgery and Cobordism of Manifolds*, *Ann. of Math. Stud*., vol. 92, Princeton University Press, Princeton, 1979.

[17] J. P. May et al., *Equivariant Homotopy and Cohomology Theory, CBMS Regional Conf. Ser. in Math*., Conf. Board Math. Sci., Washington, DC, 1996.

[18] G. Segal, Elliptic cohomology (after Landweber-Stong, Ochanine, Witten, and others), *Séminaire Bourbaki*, 1987/88, *Astérisque* **161–162** (1988), Exp. No. 695, 4, 187–201 (1989).

[19] D. P. Sinha, The geometry of the local cohomology filtration in equivariant bordism, preprint, 2001.

[20] R. E. Stong, *Notes on Cobordism Theory*, Princeton University Press, Princeton, 1968.

[21] G. Wilson, $K$-theory invariants for unitary $G$-Bordism, *Quart. J. Math. Oxford Ser. (2)* **24** (1973), 499–526.