

HOMOTOPY LIMITS AND COLIMITS AND ENRICHED HOMOTOPY THEORY

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ABSTRACT. Homotopy limits and colimits are homotopical replacements for the usual limits and colimits of category theory, which can be approached either using classical explicit constructions or the modern abstract machinery of derived functors. Our first goal in this paper is expository: we explain both approaches and a proof of their equivalence. Our second goal is to generalize this result to enriched categories and homotopy weighted limits, showing that the classical explicit constructions still give the right answer in the abstract sense. This result partially bridges the gap between classical homotopy theory and modern abstract homotopy theory. To do this we introduce a notion of “enriched homotopical categories”, which are more general than enriched model categories, but are still a good place to do enriched homotopy theory. This demonstrates that the presence of enrichment often simplifies rather than complicates matters, and goes some way toward achieving a better understanding of “the role of homotopy in homotopy theory.”

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1. INTRODUCTION

Limits and colimits are fundamental constructions in category theory and are used to some degree nearly everywhere in mathematics. However, when the category in question has a “homotopy theory,” limits and colimits are generally not homotopically well-behaved. For example, they are not invariant under homotopy equivalence. One of the central needs of homotopy theory is thus for a replacement with better properties, usually called a “homotopy limit.” This is part of a more general need for a “homotopy theory of diagrams” which generalizes the standard category theory of diagrams.

There are two natural candidates for a definition of homotopy limit. The first is defined for any category enriched over topological spaces, or over simplicial sets. In this case there are well-known explicit constructions of homotopy limits and colimits dating back to the classical work [BK72]; for a more modern exposition, see [Hir03]. These homotopy limits are objects satisfying a “homotopical” version of the usual universal property: instead of representing *commuting* cones over a diagram, they represent “homotopy coherent” cones. This universal property is *local* in that it characterizes only a single object, although usually the homotopy limit can be extended to a functor.

More recently, treatments of axiomatic homotopy theory have studied categories which are equipped with some notion of “weak equivalence”. One can “invert” the weak equivalences homotopically to get a simplicially enriched localization, but it is usually more convenient to deal with the original category and its weak equivalences directly. In this context, we have the notion of “derived functor,” which is a universal homotopical approximation to some given functor. From this point of view, it is natural to define a homotopy limit to be a derived functor of the usual limit functor. This sort of homotopy limit has a *global* universal property that refers to all possible homotopical replacements for the limit functor.

Both approaches have advantages. For instance, the classical “local” construction, when expressed as a bar construction, has a natural filtration which gives rise to spectral sequences; this makes it very tractable computationally. On the other hand, the universal property of the “global” constructions makes it easier to obtain coherence and preservation results. Ideally, we would like the two definitions to agree, at least up to homotopy, so that we can use whichever is most convenient for a given purpose. Various comparison proofs have been given, dating back to the original work [BK72].

The dichotomy between these two definitions of homotopy limit is, in fact, part of a larger disconnect between the techniques of classical homotopy theory, on the one hand, and those of modern axiomatic homotopy theory, on the other. In order to bridge this gap, we begin by considering the question of what is meant by “homotopy theory” in the first place. We may say loosely that homotopy theory

is the study of categories equipped with “weak equivalences” using a number of technical tools, among which are the following.

- Notions of “homotopy” and “homotopy equivalence”.
- Subcategories of “good” objects on which the weak equivalences are homotopy equivalences.
- Special classes of maps usually called “fibrations” and “cofibrations”.

The roles of the first two of these, at least, can already be seen in two basic problems of homotopy theory: the construction of *derived categories* or *homotopy categories* by inverting the weak equivalences, and the construction of *derived functors* that carry the “homotopical information” from point-set level functors. Of course, a homotopy category can always be constructed formally, but its hom-sets will in general be large; the problem is to ensure that its hom-sets are small. This is most often done by considering a subcategory of “good” objects on which the weak equivalences are homotopy equivalences, so that the derived hom-sets are quotients of the original hom-sets by the homotopy relation.

Similarly, derived functors are generally constructed by applying the original functor to a subcategory of “good” objects on which the original functor preserves weak equivalences; this frequently happens because the functor preserves homotopies and homotopy equivalences, and the weak equivalences between good objects are homotopy equivalences. The role of the cofibrations and fibrations is less basic, but it has much to do with the construction of homotopy limits and colimits and therefore the subject of the present paper; we will encounter them especially in §8–9 and §23.

Probably the most powerful and best-known axiomatization of homotopy theory is Quillen’s theory of *model categories*, first introduced in [Qui67]. This theory combines all of the above structures in a neat package that fits together very precisely. Any two of the classes of fibrations, cofibrations, and weak equivalences determine the third via lifting and factorization properties. The “good” objects are the *cofibrant* and/or *fibrant* ones, which means that the map from the initial object is a cofibration or that the map to the terminal object is a fibration. The factorization properties are used to define *cylinder* and *path* objects, which are then used to define notions of homotopy, and it turns out that weak equivalences between fibrant and cofibrant objects are precisely the homotopy equivalences; in this way the homotopy category is shown to have small hom-sets.

One also has the notion of a *Quillen adjoint pair* of functors, which is defined by preservation of cofibrations and fibrations. It then follows that the left adjoint preserves weak equivalences between cofibrant objects, and the right adjoint preserves weak equivalences between fibrant objects, so that both have canonical derived functors. Moreover, these derived functors form an adjoint pair between the homotopy categories.

The theory of Quillen model categories and Quillen adjoints is very powerful when it applies. Many interesting categories have useful model structures and many interesting functors are part of Quillen adjunctions. However, not infrequently it happens that a category of interest does not admit a useful model structure, or that a functor of interest is not Quillen with respect to any known model structure. This is frequently true for diagram categories, on which limit and colimit functors are defined. Moreover, even when a category has a model structure, sometimes the cofibrations or fibrations it supplies are unnecessarily strong for the desired

applications. For example, in the usual model structure on topological spaces, the cofibrations are the relative cell complexes, but for many purposes the weaker classical notion of Hurewicz cofibration suffices.

For these reasons, we are driven to re-analyze classical homotopy theory and seek new axioms which are not as restrictive as those of model categories, but which nevertheless capture enough structure to enable us to “do homotopy theory.” Several authors have studied such more general situations, in which either the fibrations or cofibrations are missing, or are not characterized as precisely by lifting properties.

More radically, the authors of the recent book [DHKS04] have begun a study of categories with only weak equivalences, which they call *homotopical categories*, and of subcategories of “good objects,” which they call *deformation retracts*. They do not consider notions of homotopy at all. On the other side of the coin, the authors of [CP97] study abstract notions of homotopy, modeled by simplicially enriched categories, but without reference to weak equivalences.

We believe that a synthesis of such approaches is needed, especially to deal with the important subject of *enriched* homotopy theory, which has rarely been considered in recent treatments of abstract homotopy theory, and then usually only for simplicial enrichments. (For example, [BC83] gives a general formulation of homotopy limits for simplicially enriched categories.) We make no claim to have fully achieved such a synthesis. In the present paper we only develop these ideas as far as is necessary to compare various definitions of homotopy limits and colimits, in the enriched as well as the unenriched setting. But even this much, we believe, is quite illuminating, and sheds some light on what one might call “the role of homotopy in homotopy theory.”

The relevance of these ideas to homotopy limits may be described (ahistorically) as follows. Suppose that we are given a category \mathcal{M} equipped with weak equivalences, and also a notion of homotopy and good objects which realize the weak equivalences as homotopy equivalences. We would like categories of diagrams in \mathcal{M} to inherit a similarly good homotopy theory. If we replace a diagram by one which is objectwise good, then a natural transformation which is an objectwise weak equivalence will be replaced by one which is an objectwise homotopy equivalence, but the homotopy inverses of its components may not fit together into a natural transformation. However, they do in general fit together into a transformation which is “natural up to coherent homotopy”. This notion can be made precise using bar constructions, which in fact allow us to realize homotopy-coherent transformations as true natural transformations between “fattened up” diagrams.

Thus, taking these fattened-up diagrams to be the “good” objects gives a homotopy theory on the diagram category, and in particular, a way to construct homotopy limits and colimits in the abstract or “global” sense mentioned earlier. But since coherent transformations are defined using bar constructions, the homotopy limits we obtain in this way are essentially the same as the “local” constructions from classical homotopy theory. Then, since the classical constructions work just as well in an enriched context, they can be used to define a global notion of “weighted homotopy limit” there as well.

The main goal of this paper is to make the above sketch precise, first showing that the local and global notions agree, and then extending the comparison to the enriched situation. If this were all we had to do, this paper would be much shorter,

but due to the existing disconnect between classical and axiomatic homotopy theory mentioned above, we must first develop many of the tools we need, especially in the enriched context. Moreover, we also want to make it clear that our approach is essentially equivalent to both existing approaches.

Thus, there are two threads in this paper interweaving back and forth: the concrete construction of homotopy limits and colimits, on the one hand, and the general philosophy and results about abstract homotopy theory, on the other. We encourage the reader to keep the above sketch in mind, to avoid becoming lost in the numerous technicalities.

This paper can be divided into six main parts. In the first two parts, we restrict ourselves to simplicially enriched categories and unenriched diagrams, both for ease of exposition and because this is the only context in which the existing global approaches are defined. The remaining four parts are concerned with generalizing these results to the enriched context.

The first part, comprising §§2–4, consists of general theory about derived functors. Except for §4, this is largely a review of material from [DHKS04]. The second part, comprising §§5–10, is devoted to an exposition of the existing “global” and “local” approaches to homotopy limits (given in §5 and §§6–7, respectively), and to two proofs of comparison.

One general comparison proof was given in an early online draft of [DHKS04], but no longer appears in the published book. In §8 we describe this proof (with details postponed to appendix A). Then in §9 we present an alternate proof, making use of the fact that the local definition can also be described using a two-sided bar construction. In §10 we explain how this proof is in line with the intuition of “homotopy coherent transformations” described above.

In the last four parts of the paper, we consider the question of *enriched* homotopy theory. The limits which arise naturally in enriched category theory are *weighted* limits, but no general theory of weighted homotopy limits and colimits exists in the literature. For several reasons, weighted homotopy limits are considerably more subtle than ordinary ones.

In particular, the local definition using a two-sided bar construction enriches very naturally, but all global approaches we are aware of founder on various difficulties in the enriched case. Our approach provides a solution to this difficulty: in favorable cases, for categories that are both enriched and homotopical, we can show directly that the enriched version of the classical bar construction defines a derived functor in the global sense. We do this in the third part of the paper, comprising §§11–13. This justifies the classical enriched bar construction from the global point of view of abstract homotopy theory.

The results in the third part are likely to be adequate for many of the applications, but they are not fully satisfactory since they do not seriously address the relationship between the enrichment and the homotopical structure. In particular, while weighted limits and colimits are themselves enriched functors, the derived functors we produce are only unenriched functors. The fourth and fifth parts of the paper remedy this defect.

In the fourth part, comprising §§14–18, we develop a theory of “enriched homotopical categories”, applying ideas developed in [Hov99, ch. 4] for enriched model categories to the more general homotopical categories of [DHKS04]. The main points are an enriched analogue of the theory of “two-variable adjunctions” given

in [Hov99], and a general theory of enrichments of homotopy categories derived from enriched homotopical categories. This abstract framework may be seen as a proposed axiomatization of enriched homotopy theory, which incorporates weak equivalences, enrichment, enriched homotopy and homotopy equivalences, and good objects.

Then in the fifth part, comprising §§19–22, we apply this theory to prove that the total derived homotopy limits and colimits can be enriched over the homotopy category of the enriching category. We also study the homotopy theory of diagrams more generally, including the behavior of enriched homotopy colimits with respect to functors between domain categories.

The sixth and final part of the paper deals with some technical details which arise in consideration of enriched homotopy limits. In §23, we introduce cofibrations and fibrations into our abstract framework in order to describe the necessary cofibrancy conditions on the shape category. Model theoretic cofibrations, when present, generally suffice, but are often too restrictive for the applications. Moreover, there is an added problem that cofibrant approximation functors generally do not preserve the enrichment; in §24, we consider various ways to get around this problem. For clarity, such technical considerations are swept under the rug with an axiomatization in parts three and five.

It is important to note that everything in this paper has dual versions for limits and colimits, right and left Kan extensions, and cotensor and tensor products. We work mostly with the latter, since we believe they are easier to conceptualize for various reasons, leaving dualizations to the reader in many cases. However, we emphasize that our results are often more necessary for limits than colimits, since simplifying model structures on diagram categories are more often present in the context of colimits.

The reader is assumed to have some knowledge of category theory, including the formalisms of ends and Kan extensions, as described in [ML98]. Some familiarity with model categories is also expected, especially in the second part of the paper. This includes an acquaintance with the various model structures that exist on diagram categories. For good introductions to model categories, see [DS95], [Hov99], and [Hir03]. We assume that all model categories are complete and cocomplete and have functorial factorizations.

We also use a number of ideas and results from [DHKS04], but no prior familiarity with that work is required if the reader is willing to take a few of its results on faith.

In various places, we use the terminology and techniques of enriched category theory, as described in [Kel82] and [Dub70]. Their use is concentrated in the second half of the paper, however, and we attempt to explain these concepts as they arise. Enriched category theory, being the sort of category theory which nearly always arises in practical applications, is at least as important as the more elementary unenriched theory.

To reiterate, then, the upshot of this paper is that the classical two-sided bar construction is almost always the “best” way to construct homotopy limits and colimits, for the following reasons.

- It satisfies a local universal property, representing homotopy coherent cones;
- It also satisfies a global universal property, making coherence and preservation results easy to obtain;

- It comes with a natural filtration by simplicial degree, giving rise to spectral sequences which make it computable; and
- It works just as well in enriched situations, which are frequently the cases of most interest.

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2. GLOBAL DEFINITIONS OF DERIVED FUNCTORS

We begin by considering categories with a suitably well-behaved notion of *weak equivalence*. The most common requirement is a 2-out-of-3 property, but in [DHKS04] it was found more technically convenient to assume a slightly stronger property, as follows.

Definition 2.1 ([DHKS04, §33]). A *homotopical category* is a category \mathcal{M} equipped with a class of morphisms called *weak equivalences* that contains all the identities and satisfies the *2-out-of-6 property*: if hg and gf are weak equivalences, then so are f , g , h , and hgf .

It follows easily that the weak equivalences in a homotopical category include all the isomorphisms and satisfy the usual 2-out-of-3 property. The stronger 2-out-of-6 property is satisfied by all examples that arise, including all model categories and categories of diagrams in model categories.

Any category can be made a homotopical category in a trivial way by taking the weak equivalences to be precisely the isomorphisms; in [DHKS04] these are called *minimal* homotopical categories. If \mathcal{M} is a homotopical category and \mathcal{D} is any small category, the diagram category $\mathcal{M}^{\mathcal{D}}$ has a homotopical structure in which a map of diagrams $\alpha: F \rightarrow F'$ is a weak equivalence if $\alpha_d: Fd \rightarrow F'd$ is a weak equivalence in \mathcal{M} for every object $d \in \mathcal{D}$; we call these *objectwise* weak equivalences.

Any homotopical category \mathcal{M} has a *homotopy category* $\text{Ho } \mathcal{M}$, obtained by formally inverting the weak equivalences, with a localization functor $\gamma: \mathcal{M} \rightarrow \text{Ho } \mathcal{M}$ which is universal among functors inverting the weak equivalences. In general, $\text{Ho } \mathcal{M}$ need not have small hom-sets, which is a major impediment in many applications. Usually, however, other methods are available to ensure local smallness, as discussed briefly in §1. For example, as already mentioned, if \mathcal{M} admits a Quillen model structure, then $\text{Ho } \mathcal{M}$ will have small hom-sets. In Corollary 16.15, we will see another way to ensure that homotopy categories have small hom-sets, using an enriched notion of homotopy.

Homotopical categories are a reasonable axiomatization of the first ingredient for homotopy theory: a subcategory of weak equivalences. In this section and the next, we will follow [DHKS04] in developing the theory of derived functors for homotopical categories.

Thus, suppose that \mathcal{M} and \mathcal{N} are homotopical categories. Let $\text{Ho } \mathcal{M}$ and $\text{Ho } \mathcal{N}$ be their homotopy categories, with localization functors $\gamma: \mathcal{M} \rightarrow \text{Ho } \mathcal{M}$ and $\delta: \mathcal{N} \rightarrow \text{Ho } \mathcal{N}$. Following [DHKS04], we say that a functor $F: \mathcal{M} \rightarrow \mathcal{N}$ which preserves weak equivalences is *homotopical*. We will also use this term for

functors $\mathcal{M} \rightarrow \mathrm{Ho} \mathcal{N}$ which take weak equivalences to isomorphisms; this is equivalent to considering $\mathrm{Ho} \mathcal{N}$ as a minimal homotopical category. If F is homotopical, then by the universal property of localization δF induces a unique functor $\tilde{F}: \mathrm{Ho} \mathcal{M} \rightarrow \mathrm{Ho} \mathcal{N}$ making the following diagram commute.

$$(2.2) \quad \begin{array}{ccc} \mathcal{M} & \xrightarrow{F} & \mathcal{N} \\ \gamma \downarrow & & \downarrow \delta \\ \mathrm{Ho} \mathcal{M} & \xrightarrow{\tilde{F}} & \mathrm{Ho} \mathcal{N} \end{array}$$

Frequently, however, we encounter functors $\mathcal{M} \rightarrow \mathcal{N}$ which are not homotopical, but for which we would still like a derived functor $\mathrm{Ho} \mathcal{M} \rightarrow \mathrm{Ho} \mathcal{N}$. One natural approach is to relax the requirement that (2.2) commute and instead ask that it be filled with a *universal* transformation. This amounts to asking for a right or left Kan extension of δF along γ . The reversal of handedness in the following definition is unfortunate, but both terminologies are too well-established to be changed.

Definition 2.3. A right Kan extension of δF along γ is called a *total left derived functor* of F and denoted $\mathbf{L}F$. Dually, a left Kan extension of δF along γ is called a *total right derived functor* of F and denoted $\mathbf{R}F$.

All the types of derived functor we will define in this section come in two types, “left” and “right.” For clarity, from now on we will define only the left versions, leaving the obvious dualizations to the reader.

Often we are interested in derived functors that, like $\tilde{F} \circ \gamma$ in (2.2), are defined on \mathcal{M} and not just on $\mathrm{Ho} \mathcal{M}$. By the universal property of localization, the existence of a total left derived functor is equivalent to the existence of a “left derived functor,” defined as follows. We also write $\mathbf{L}F$ for this notion, despite the difference in the domain.

Definition 2.4. A *left derived functor* of F is a functor $\mathbf{L}F: \mathcal{M} \rightarrow \mathrm{Ho} \mathcal{N}$ equipped with a comparison map $\mathbf{L}F \rightarrow \delta F$ such that $\mathbf{L}F$ is homotopical and terminal among homotopical functors equipped with maps to δF .

Clearly, both left derived functors and total left derived functors are unique up to unique isomorphism when they exist.

We may also want to “lift” the target of the derived functor from $\mathrm{Ho} \mathcal{N}$ to \mathcal{N} . This is not always possible, but frequently it is, and so to clarify the notions we introduce the following oxymoronic definition.

Definition 2.5. A *point-set left derived functor* is a functor $\mathbb{L}F: \mathcal{M} \rightarrow \mathcal{N}$ equipped with a comparison map $\mathbb{L}F \rightarrow F$ such that the induced map $\delta \mathbb{L}F \rightarrow \delta F$ makes $\delta \mathbb{L}F$ into a left derived functor of F .

Such a functor, when it exists, is only unique “up to homotopy”. By no means do all derived functors “lift” to the point-set level. For example, Quillen functors out of model categories whose factorizations cannot be made functorial, such as those of [Isa01], have derived functors, but not point-set derived functors. However, we will need our model categories to have functorial factorization in order to have an “objectwise cofibrant replacement” on diagrams, so all our model-theoretic derived functors will have point-set versions.

Finally, we might replace the universal property on the level of homotopy categories in the preceding three definitions by a “homotopical universal property” on the point-set level.

Definition 2.6 ([DHKS04, 41.1]). A *left approximation* of F is a homotopical functor $F' : \mathcal{M} \rightarrow \mathcal{N}$ equipped with a comparison map $F' \rightarrow F$ that is “homotopically terminal” among such functors, in a sense made precise in [DHKS04, §38].

It is proven in [DHKS04, §38] that objects with such a homotopical universal property are “homotopically unique” in a suitable sense.

Both point-set derived functors and approximations are functors $\mathcal{M} \rightarrow \mathcal{N}$, but in general the existence of one need not imply the existence of the other. If F' is an approximation, it has a universal property referring only to other functors landing in \mathcal{N} , while the universal property of a derived functor refers to all functors landing in $\text{Ho } \mathcal{N}$, even those that do not factor through \mathcal{N} and are thus only “functorial up to homotopy”; thus $\delta F'$ is not necessarily a derived functor. On the other hand, if $\mathbb{L}F$ is a point-set derived functor, then for any homotopical functor F' mapping to F we have a unique natural transformation $\delta F' \rightarrow \delta \mathbb{L}F$, but it need not lift to a natural zig-zag from F' to $\mathbb{L}F$ in \mathcal{N} ; thus $\mathbb{L}F$ is not necessarily a left approximation.

However, it is useful to note that if \mathcal{N} is a minimal homotopical category (for example, \mathcal{N} could be $\text{Ho } \mathcal{P}$ for some other homotopical category \mathcal{P}), then δ is an equivalence and all the above notions coincide. Thus the results of [DHKS04] about approximations can usually be applied to derived functors as well. We will state our results for point-set derived functors, except when it becomes necessary to pass all the way to total derived functors. It should also be noted that the way in which most derived functors are constructed, which we will explain in the next section, produces both derived functors and approximations with equal ease.

3. DERIVED FUNCTORS VIA DEFORMATIONS

Having defined left and right derived functors in the previous section, we of course would like to know how to construct such things. The easiest and most common way to produce a left derived functor of F applies when the categories involved have model category structures for which F is left Quillen. In this case, for any cofibrant replacement functor Q on \mathcal{M} , the composite FQ can be shown to be both a point-set left derived functor and a left approximation. In this section we follow [DHKS04] in axiomatizing the properties of the cofibrant replacement functor Q which make this work, thus disentangling the notion of “good object” from the rest of the model category machinery.

Definition 3.1 ([DHKS04, §40]). Let $F : \mathcal{M} \rightarrow \mathcal{N}$ be a functor between homotopical categories.

- A *left deformation* of \mathcal{M} is a functor $Q : \mathcal{M} \rightarrow \mathcal{M}$ equipped with a natural weak equivalence $q : Q \xrightarrow{\sim} \text{Id}_{\mathcal{M}}$; it follows from the 2-out-of-3 property that Q is homotopical.
- A *left deformation retract* is a full subcategory \mathcal{M}_Q containing the image of some left deformation (Q, q) . We will always use such parallel notation for deformations and deformation retracts.
- A *left F -deformation retract* is a left deformation retract \mathcal{M}_Q such that the restriction of F to \mathcal{M}_Q is homotopical.

- A deformation into an F -deformation retract is called an F -deformation.
- If there exists a left F -deformation retract, we say F is *left deformable*.

Remark 3.2. By the 2-out-of-3 property, a deformation (Q, q) is an F -deformation if and only if FQ is homotopical and $FqQ: FQ \rightarrow FQ$ is a natural weak equivalence.

If F is left Quillen, a cofibrant replacement functor serves as an F -deformation. Our choice of notation for a left deformation is intended to suggest that it is a “generalized cofibrant replacement.”

As an example of a left deformation that is not a cofibrant replacement, it is well-known that when computing derived tensor products in homological algebra (i.e. the modern version of Tor), it suffices to use *flat* resolutions rather than projective ones, although the latter are the cofibrant objects in the usual model structure. Similarly, in topological situations, it often suffices to consider various weaker forms of cofibrancy, such as the “ h -cofibrations” of [MS06, §4.1] or the “tame spectra” of [EKMM97, §I.2]. We will also need to use deformations that do not arise directly from a model structure to compute homotopy limits and colimits.

The following two results justify the above definitions. There are, of course, dual versions for right deformations and right derived functors.

Proposition 3.3. *If \mathcal{M}_Q is a left deformation retract of \mathcal{M} , then the inclusion $I: \mathcal{M}_Q \hookrightarrow \mathcal{M}$ induces an equivalence of categories $\mathrm{Ho}(\mathcal{M}_Q) \simeq \mathrm{Ho} \mathcal{M}$.*

Proof. Let (Q, q) be a left deformation of \mathcal{M} into \mathcal{M}_Q . Since the inclusion $I: \mathcal{M}_Q \hookrightarrow \mathcal{M}$ and $Q: \mathcal{M} \rightarrow \mathcal{M}_Q$ are both homotopical, they induce functors $\mathrm{Ho} \mathcal{M}_Q \rightleftarrows \mathrm{Ho} \mathcal{M}$. The natural weak equivalence $q: \mathrm{Id}_{\mathcal{M}} \rightarrow Q$ then descends to homotopy categories to give natural isomorphisms $\mathrm{Id}_{\mathrm{Ho} \mathcal{M}} \cong IQ$ and $\mathrm{Id}_{\mathrm{Ho} \mathcal{M}_Q} \cong QI$, so Q and I form the desired equivalence. \square

Proposition 3.4 ([DHKS04, §41]). *If (Q, q) is a left F -deformation, then $\mathbb{L}F = FQ$ is both a point-set left derived functor of F and a left approximation of F .*

Proof. This is an obvious generalization of the well-known proof for Quillen functors, e.g. [Hir03, 8.3.6]. \square

Intuition from classical homotopy theory leads us to expect that we could construct the homotopy category $\mathrm{Ho} \mathcal{M}$ using a suitable deformation (Q, R) of the hom-functor $\mathcal{M}(-, -)$. However, unless \mathcal{M} is *enriched*, this has little chance of working. This is because if \mathcal{M} is not enriched, $\mathcal{M}(-, -)$ takes values only in **Set**, in which the only sensible weak equivalences are the isomorphisms. We will see in §16, however, that if \mathcal{M} is enriched over a category with its own suitable notion of weak equivalence, this intuition is often correct.

In the rest of this section, we will mention a few auxiliary notions and results about deformations which will be useful in the second half of the paper. All of this material is from [DHKS04], and can be skipped on a first reading.

We first consider the question of when an adjunction between deformable functors descends to homotopy categories. We frequently write $F \dashv G$ for an adjunction $F: \mathcal{M}_1 \rightleftarrows \mathcal{M}_2 : G$.

Definition 3.5 ([DHKS04, 43.1]). An adjunction $F \dashv G$ between homotopical categories is *deformable* if F is left deformable and G is right deformable.

Proposition 3.6 ([DHKS04, 43.2 and 44.2]). *If $F: \mathcal{M}_1 \rightleftarrows \mathcal{M}_2 : G$ is a deformable adjunction, then there is a unique adjunction*

$$\mathbf{L}F: \mathrm{Ho} \mathcal{M}_1 \rightleftarrows \mathrm{Ho} \mathcal{M}_2 : \mathbf{R}G$$

which is compatible with the given one and the localization functors of \mathcal{M}_1 and \mathcal{M}_2 .

Definition 3.7 ([DHKS04, 41.6]). *If $\alpha: F \rightarrow F'$ is a natural transformation between functors on homotopical categories, a *derived natural transformation* of α is a natural transformation $\mathbf{L}\alpha: \mathbf{L}F \rightarrow \mathbf{L}F'$ such that*

- (i) $\mathbf{L}F \rightarrow \delta F$ and $\mathbf{L}F' \rightarrow \delta F'$ are derived functors of F and F' , respectively;
- (ii) the square

$$\begin{array}{ccc} \mathbf{L}F & \longrightarrow & \delta F \\ \mathbf{L}\alpha \downarrow & & \downarrow \delta(\alpha) \\ \mathbf{L}F' & \longrightarrow & \delta F' \end{array}$$

commutes; and

- (iii) this square is terminal among commutative squares having $\delta(\alpha)$ on the right and such that the two functors on the left are homotopical.

One can similarly define point-set derived natural transformations, approximations of natural transformations, and so on.

Proposition 3.8. *If $\alpha: F \rightarrow G$ is a natural transformation between left deformable functors $\mathcal{M} \rightarrow \mathcal{N}$:*

$$\begin{array}{ccc} & F & \\ \mathcal{M} & \xrightarrow{\quad} & \mathcal{N} \\ & \downarrow \alpha & \\ & G & \end{array}$$

and there exists a deformation (Q, q) of \mathcal{M} which is both an F -deformation and a G -deformation, then α has a derived natural transformation

$$\begin{array}{ccc} & \mathbf{L}F & \\ \mathrm{Ho} \mathcal{M} & \xrightarrow{\quad} & \mathrm{Ho} \mathcal{N} \\ & \downarrow \mathbf{L}\alpha & \\ & \mathbf{L}G & \end{array}$$

given by αQ .

Moreover, if we have another natural transformation

$$\begin{array}{ccc} & G & \\ \mathcal{M} & \xrightarrow{\quad} & \mathcal{N} \\ & \downarrow \beta & \\ & H & \end{array}$$

and (Q, q) is also an H -deformation, then $\mathbf{L}(\beta\alpha) \cong (\mathbf{L}\beta)(\mathbf{L}\alpha)$.

Proof. See [DHKS04, 41.6] for the first statement. The second statement is straightforward. \square

We would like to know when the operation of “left deriving” is “functorial”, i.e. when it preserves composition of functors. This is not the case in general, but there are special circumstances under which it is true.

Definition 3.9 ([DHKS04, §42]). *A composable pair (F_1, F_2) of left deformable functors $\mathcal{M} \xrightarrow{F_1} \mathcal{N} \xrightarrow{F_2} \mathcal{P}$ is called *locally left deformable* if there are deformation retracts \mathcal{M}_Q and \mathcal{N}_Q such that*

- F_1 is homotopical on \mathcal{M}_Q ;
- F_2 is homotopical on \mathcal{N}_Q ; and
- F_2F_1 is homotopical on \mathcal{M}_Q .

If in addition, F_1 maps \mathcal{M}_Q into \mathcal{N}_Q , the pair is called *left deformable*.

Note that a pair of left Quillen functors is always left deformable, so in the well-behaved world of model categories there is no need for this notion.

Proposition 3.10 ([DHKS04, 42.4]). *If (F_1, F_2) is a left deformable pair, then for any left derived functors $\mathbb{L}F_1$ and $\mathbb{L}F_2$ of F_1 and F_2 , respectively, the composite $\mathbb{L}F_2 \circ \mathbb{L}F_1$ is a left derived functor of F_2F_1 .*

When the functors involved have adjoints, part of the work to check deformability of pairs can be shifted across the adjunction, where it often becomes easier. First we need one more definition.

Definition 3.11 ([DHKS04, 33.9]). A homotopical category is *saturated* if any map which becomes an isomorphism in its homotopy category is a weak equivalence.

Any model category is saturated, e.g. by [Hov99, 1.2.10]. Moreover, if \mathcal{M} is saturated and \mathcal{D} is small, then by [DHKS04, 33.9(v)] the functor category $\mathcal{M}^{\mathcal{D}}$ (with objectwise weak equivalences) is also saturated.

The following is the reason for defining “locally left deformable” in addition to “left deformable.”

Proposition 3.12 ([DHKS04, 42.5]). *Let $F_1: \mathcal{M}_1 \rightleftarrows \mathcal{M}_2 : G_1$ and $F_2: \mathcal{M}_2 \rightleftarrows \mathcal{M}_3 : G_2$ be deformable adjunctions in which the pairs (F_1, F_2) and (G_2, G_1) are locally left and right deformable, respectively, while the \mathcal{M}_i are saturated. Then the pair (F_1, F_2) is left deformable if and only if the pair (G_2, G_1) is right deformable.*

The composition of left derived and right derived functors is a much more subtle question, and even under the best of circumstances (such as left and right Quillen functors) composition need not be preserved. See, for example, [MS06, Counterexample 0.0.1]. In [MS06] this question is dealt with using the tools of “middle derived functors”, which we introduce in the next section.

4. MIDDLE DERIVED FUNCTORS

We have defined derived functors by a universal property and then introduced deformations as a means of constructing derived functors, but a more historically faithful introduction would have been to remark that although most functors are not homotopical, they do frequently preserve weak equivalences between “good objects.” Therefore, a natural “homotopical replacement” for F is to first replace the domain object by an equivalent good object and then apply F .

From this point of view, we are left wondering what the real conceptual difference between left derived functors and right derived functors is, and whether if a functor has both left and right derived functors, the two must agree or be related in some way. In search of a unified notion of derived functor, we might be motivated to drop the restriction that q be a *single* weak equivalence in favor of a zigzag of such, leading to the following definitions.

Definition 4.1. Let $F: \mathcal{M} \rightarrow \mathcal{N}$ be a functor between homotopical categories.

- A *middle deformation* of \mathcal{M} is a functor $D: \mathcal{M} \rightarrow \mathcal{M}$ equipped with a natural zigzag of weak equivalences

$$D \xrightarrow{\sim} D_k \xrightarrow{\sim} \cdots \xrightarrow{\sim} D_1 \xrightarrow{\sim} \text{Id}_{\mathcal{M}} .$$

As before, it follows from the 2-out-of-3 property that D is homotopical, as are all the D_i .

- A *middle deformation retract* is a full subcategory \mathcal{M}_D for which there is a middle deformation, as above, such that \mathcal{M}_D contains the image of D , and each D_i maps \mathcal{M}_D into itself.
- A *middle F -deformation retract* is a middle deformation retract on which F is homotopical. A *middle F -deformation* is a middle deformation into a middle F -deformation retract.
- If F has a middle F -deformation D , then the composite FD , which descends to homotopy categories, is a *middle derived functor* of F .

The assumption that each D_i preserves \mathcal{M}_D allows us to prove a version of Proposition 3.3 for middle deformations.

Proposition 4.2. *If \mathcal{M}_D is a middle deformation retract, then the inclusion*

$$I: \mathcal{M}_D \hookrightarrow \mathcal{M}$$

induces an equivalence $\text{Ho}(\mathcal{M}_D) \simeq \text{Ho } \mathcal{M}$.

Proof. The functors $I: \mathcal{M}_D \rightarrow \mathcal{M}$ and $D: \mathcal{M} \rightarrow \mathcal{M}_D$ are both homotopical and so induce functors $\text{Ho } \mathcal{M}_D \rightleftarrows \text{Ho } \mathcal{M}$. The assumed natural zigzag of weak equivalences provides an isomorphism $ID \cong \text{Id}_{\text{Ho } \mathcal{M}}$, and also $DI \cong \text{Id}_{\text{Ho } \mathcal{M}_D}$ since each intermediate functor D_i maps \mathcal{M}_D to itself. \square

Clearly, left and right deformations are both special cases of middle deformations, so if we could prove that middle derived functors are unique, it would follow that if a functor has both left and right derived functors, they agree. Unfortunately, this is not the case, as the following example shows.

Counterexample 4.3. Let \mathcal{M} and \mathcal{N} be homotopical categories, and let \mathcal{I} be the category with two objects 0 and 1 and one nonidentity arrow $0 \rightarrow 1$ which is a weak equivalence. Let $F^0, F^1: \mathcal{M} \rightarrow \mathcal{N}$ be two functors such that F^0 is left deformable and F^1 is right deformable and let $\alpha: F^0 \rightarrow F^1$ be a natural transformation.

Define $F: \mathcal{M} \times \mathcal{I} \rightarrow \mathcal{N}$ using α in the obvious way, and give $\mathcal{M} \times \mathcal{I}$ the product homotopical structure. Then if Q is a left deformation for F^0 , a left deformation for F is given by projecting to $\mathcal{M} \times \{0\}$ and then applying Q ; thus F is left deformable and $\mathbf{L}F \cong \mathbf{L}F^0$. Dually, F is also right deformable and $\mathbf{R}F \cong \mathbf{R}F^1$.

Therefore, F is both left and right deformable, but the only relationship between its left and right derived functors is that they are connected by a natural transformation.

Since left and right derived functors are both middle derived functors, it follows that middle derived functors are not unique, and therefore they cannot be expected to satisfy any universal property similar to that for left and right derived functors.

Nevertheless, there are important cases in which left and right derived functors *do* agree, and which include many situations encountered in applications. One frequently encountered situation is described by the following definition.

Definition 4.4. Let $\mathcal{M}_{D'} \subset \mathcal{M}$ and $\mathcal{M}_D \subset \mathcal{M}$ be middle deformation retracts. We say $\mathcal{M}_{D'}$ is a *middle sub-deformation retract* of \mathcal{M}_D if $\mathcal{M}_{D'} \subset \mathcal{M}_D$ and moreover $\mathcal{M}_{D'}$ is a middle deformation retract of \mathcal{M}_D .

Remark 4.5. This condition is equivalent to the existence, for a suitable choice of middle deformations D and D' , of a zigzag of weak equivalences:

$$(4.6) \quad D' \xrightarrow{\sim} \cdots \xrightarrow{\sim} D \xrightarrow{\sim} \cdots \xrightarrow{\sim} \text{Id}_{\mathcal{M}}$$

in which all the weak equivalences between D' and D lie in \mathcal{M}_D .

Proposition 4.7. *Let $F: \mathcal{M} \rightarrow \mathcal{N}$ and let $\mathcal{M}_{D'}$ be a middle sub-deformation retract of \mathcal{M}_D such that \mathcal{M}_D is a middle F -deformation retract (and hence so is $\mathcal{M}_{D'}$). Then the corresponding middle derived functors of F are equivalent.*

Proof. Apply F to (4.6). Since all the weak equivalences between D' and D lie in \mathcal{M}_D , on which F is homotopical, the image of that part of the zigzag gives a zigzag of weak equivalences between FD' and FD , which are the two middle derived functors in question. \square

This obvious-seeming result has some less obvious-seeming consequences, such as the following.

Proposition 4.8. *Let $F: \mathcal{M} \rightarrow \mathcal{N}$ and let (Q, q) and (R, r) be left and right F -deformations, respectively. If Q maps \mathcal{M}_R into itself, or R maps \mathcal{M}_Q into itself, then $\mathbf{L}F \cong \mathbf{R}F$.*

Proof. Assume Q maps \mathcal{M}_R into itself. Then we have the following square of weak equivalences

$$\begin{array}{ccc} Q & \xrightarrow{\sim} & QR \\ \sim \downarrow & & \downarrow \sim \\ \text{Id}_{\mathcal{M}} & \xrightarrow{\sim} & R \end{array}$$

in which the image of QR lands in $\mathcal{M}_Q \cap \mathcal{M}_R$. This shows that $\mathcal{M}_Q \cap \mathcal{M}_R$ is a middle sub-deformation retract of both \mathcal{M}_Q and \mathcal{M}_R . Therefore, both $\mathbf{L}F = FQ$ and $\mathbf{R}F = FR$ are equivalent to the middle derived functor FQR , hence equivalent to each other. The case when R maps \mathcal{M}_Q into itself is dual. \square

Corollary 4.9. *If \mathcal{M} and \mathcal{N} are model categories and $F: \mathcal{M} \rightarrow \mathcal{N}$ is both left and right Quillen, then $\mathbf{L}F \cong \mathbf{R}F$.*

Proof. If we take Q and R to be the cofibrant and fibrant approximation functors coming from the functorial factorization in \mathcal{M} , then because $q: Q \rightarrow \text{Id}_{\mathcal{M}}$ is a trivial fibration, Q maps \mathcal{M}_R into itself, and dually R maps \mathcal{M}_Q into itself. Therefore we can apply Proposition 4.8. \square

Thus, if our categories are equipped with a “preferred” model structure, any given functor has at most one “canonical” derived functor. Later on, in §15, when we consider enriched homotopical categories and corresponding notions of homotopy, we will see another common situation in which a preferred extra structure on a homotopical category ensures that there is a “canonical” derived functor.

We conclude this section with the following additional example of Proposition 4.7. This is taken from [MS06], which has many examples of functors that have both left and right derived functors that agree.

Example 4.10. It is shown in [MS06, 9.1.2] that the category \mathcal{K}_B of ex-spaces over a space B is equipped with the following collection of middle deformation retracts (although the authors of [MS06] do not use that terminology).

$$\begin{array}{ccc}
 & & (\mathcal{K}_B)_{\text{well-grounded}, \simeq \text{CW}} \\
 & \nearrow & \searrow \\
 \mathcal{W}_B \hookrightarrow \mathcal{V}_B & & \mathcal{K}_B \\
 & \searrow & \nearrow \\
 & & (\mathcal{K}_B)_{qf\text{-fibrant}}
 \end{array}$$

The detailed definitions of all these categories, most of which refer to a certain “ qf -model structure” on \mathcal{K}_B , need not concern us. The important points are that in \mathcal{W}_B , weak equivalences are homotopy equivalences, so that we have an equivalence $h\mathcal{W}_B \simeq \text{Ho } \mathcal{K}_B$, where $h\mathcal{W}_B$ is the quotient of \mathcal{W}_B by homotopy, and that the definition of \mathcal{W}_B is entirely classical, making no reference to the qf -model structure. The result [MS06, 9.2.2], which says that suitably nice functors $\mathcal{K}_A \rightarrow \mathcal{K}_B$ have middle derived functors induced by functors $h\mathcal{W}_A \rightarrow h\mathcal{W}_B$, can then be seen as an instance of Proposition 4.7.

From the perspective of middle derived functors, the “leftness” or “rightness” of any particular derived functor reveals itself as merely an accident—albeit a very common and very useful one! For example, in §21 we will see how useful the universal properties of left and right derived functors are in proving compatibility relations. The question of when a given middle derived functor is left or right is a subtle one. Experience, especially from the case of model categories, leads us to expect that in general, left adjoints will have left derived functors, while right adjoints will have right derived functors, but we have no more general theory explaining why this should be true. However, see also [Shu07].

5. HOMOTOPY COLIMITS AS DERIVED FUNCTORS

We now begin the second part of the paper and descend to the specific case of homotopy limits and colimits. From the “global” point of view, a homotopy limit should be a derived functor of the limit functor. Since the limit is a right adjoint and the colimit is a left adjoint, we expect the one to have a right derived functor and the other a left derived functor.

Now, if for all model categories \mathcal{M} and all small categories \mathcal{D} the diagram category $\mathcal{M}^{\mathcal{D}}$ always had model structures for which the colimit and limit functors were left and right Quillen, respectively, then no further discussion of this approach would be necessary, at least for model categories. There have been steps in the direction of a notion of model category with these properties; see, for example, [Wei01]. But the more common notion of Quillen model category does not have them.

However, there are important special cases in which the diagram category does have a model structure and limit or colimit functors are Quillen.

- (i) If \mathcal{M} is cofibrantly generated, then all categories $\mathcal{M}^{\mathcal{D}}$ have a *projective model structure* in which the weak equivalences and fibrations are object-wise (see [Hir03, ch. 11] for details). When this model structure exists, the *colimit* functor is left Quillen on it.

- (ii) If \mathcal{M} satisfies the stronger hypothesis of being *sheafifiable* (see [Bek00]), then all $\mathcal{M}^{\mathcal{D}}$ have an *injective model structure* in which the weak equivalences and cofibrations are objectwise. When this model structure exists, the *limit* functor is right Quillen on it.
- (iii) If \mathcal{D} is a *Reedy category*, then for any model category \mathcal{M} , the category $\mathcal{M}^{\mathcal{D}}$ has a *Reedy model structure* in which the weak equivalences are objectwise, but the cofibrations and fibrations are generally not (see [Hov99, ch. 5] or [Hir03, ch. 15] for details). If in addition \mathcal{D} has *fibrant constants* as defined in [Hir03, 15.10.1], then the colimit functor is left Quillen for this model structure. Dually, if \mathcal{D} has cofibrant constants, then the limit functor is right Quillen.

When a suitable model structure exists on $\mathcal{M}^{\mathcal{D}}$, the global definition of homotopy limits or colimits is easy. We simply apply a fibrant or cofibrant replacement in the appropriate model structure and take the usual limit or colimit. However, for projective and injective model structures, the cofibrant and fibrant replacements are constructed by small object arguments and so are difficult to get a handle on, making a more explicit construction desirable.

Reedy model structures have the advantage that fibrant and cofibrant replacements are relatively easy to define and compute. Moreover, many common diagram shapes have Reedy structures—for example, the category

$$(\cdot \leftarrow \cdot \rightarrow \cdot)$$

which indexes pushout diagrams—so many global homotopy limits and colimits can be computed in this way by simply replacing a few maps by fibrations or cofibrations.

However, being Reedy is quite a special property of the diagram category, so in the general case it is far too much to expect. Moreover, although most model categories arising in nature are cofibrantly generated, so that projective model structures exist, many (such as those arising from topological spaces) are not sheafifiable, so there is no known model structure for which the limit functor is Quillen. Thus more technical methods are needed to construct global homotopy limit functors at all in this context.

The approach taken in [DHKS04] is to use a suitable “homotopical replacement” for the shape category \mathcal{D} . Recall that for any functor $H: \mathcal{D} \rightarrow \mathcal{E}$ and any object e of \mathcal{E} , the *comma-category* of H over e has for its objects the arrows $Hd \rightarrow e$ in \mathcal{E} , and for its arrows the arrows in \mathcal{D} whose images under H form commutative triangles. We write $(H \downarrow e)$ for the comma-category. When H is the identity functor of \mathcal{D} , we write $(\mathcal{D} \downarrow d)$ for the comma-category; in this case it is also called the “over-category” of d . There is a dual comma-category $(e \downarrow H)$ which specializes to the “under-category” $(d \downarrow \mathcal{D})$.

We define the *category of simplices* of a simplicial set K to be the comma category $\Delta K = (\Delta \downarrow K)$, where $\Delta: \mathbf{\Delta} \rightarrow s\mathcal{S}$ is the standard cosimplicial simplicial set. The objects of ΔK are all the simplices of K , of all dimensions. We write $\Delta^{op} K = (\Delta K)^{op} \cong (\Delta^{op} \downarrow K)$. The following is straightforward.

Lemma 5.1 ([DHKS04, 22.10]). *For any K , the categories ΔK and $\Delta^{op} K$ are Reedy categories with fibrant and cofibrant constants, respectively.*

We have forgetful functors $\Pi: \Delta K \rightarrow \mathbf{\Delta}$ and $\Sigma = \Pi^{op}: \Delta^{op} K \rightarrow \mathbf{\Delta}^{op}$, which send an n -simplex α to the object $[n]$ of $\mathbf{\Delta}$ or $\mathbf{\Delta}^{op}$.

We are primarily interested in ΔK when K is the nerve of a category. Recall that the *nerve* of a small category \mathcal{D} is a simplicial set $N\mathcal{D}$ whose n -simplices are the strings of n composable arrows in \mathcal{D} . We have $N\mathcal{D}_n = \mathbf{Cat}([n], \mathcal{D})$ where $[n]$ is a string of n composable arrows. Note that some authors call $N\mathcal{D}$ the “classifying space” of \mathcal{D} and denote it $B\mathcal{D}$. However, in addition to the potential for confusion with the use of B for the bar construction, we prefer to reserve that notation for “delooping”-type operations.

If \mathcal{D} is a small category, write $\Delta\mathcal{D} = \Delta N\mathcal{D}$ and $\Delta^{op}\mathcal{D} = \Delta^{op} N\mathcal{D}$. Thus, the objects of $\Delta\mathcal{D}$ are the strings of zero or more composable arrows in \mathcal{D} . We have “target” and “source” functors $T: \Delta\mathcal{D} \rightarrow \mathcal{D}$ and $S: \Delta^{op}\mathcal{D} \rightarrow \mathcal{D}$ which map an n -simplex of $N\mathcal{D}$

$$\alpha_0 \longrightarrow \alpha_1 \longrightarrow \dots \longrightarrow \alpha_n$$

to α_n , respectively α_0 .

Remark 5.2. Not all authors define the nerve of a category the same way; some authors’ $N\mathcal{D}$ is other authors’ $N\mathcal{D}^{op}$. Our conventions are those of [Hir03] and [DHKS04], but opposite to those of [BK72]. A discussion of the precise effect of the choice of convention on the local definition of homotopy colimits can be found in [Hir03, 18.1.11].

Up to homotopy, however, the choice of convention doesn’t matter, since the nerve of a category is weakly equivalent to the nerve of its opposite; see, for example, [Hir03, 14.1.6]. Moreover, upon passing to categories of simplices, we actually have an isomorphism $\Delta\mathcal{D} \cong \Delta(\mathcal{D}^{op})$. Thus homotopy limits and colimits can be defined using either convention, and the comparison proofs go through. Note, however, a potential source of confusion: $\Delta^{op}\mathcal{D}$ and $\Delta(\mathcal{D}^{op})$ are quite different categories.

We summarize the functors relating \mathcal{D} and its categories of simplices in the following diagram.

$$(5.3) \quad \begin{array}{ccccc} \Delta & & \mathcal{D} & & \Delta^{op} \\ & \nwarrow \Pi & \nearrow T & \nwarrow S & \nearrow \Sigma \\ & \Delta\mathcal{D} & & \Delta^{op}\mathcal{D} & \end{array}$$

For any category \mathcal{M} , we have a functor $T^*: \mathcal{M}^{\mathcal{D}} \rightarrow \mathcal{M}^{\Delta\mathcal{D}}$ given by precomposing with T , and similarly for all other functors in the diagram. If \mathcal{M} is cocomplete, we also have *left Kan extensions* such as $\text{Lan}_T: \mathcal{M}^{\Delta\mathcal{D}} \rightarrow \mathcal{M}^{\mathcal{D}}$, defined to be a left adjoint to T^* . See [ML98, ch. X] for a development of Kan extensions in general. Left Kan extensions are also sometimes called *prolongation* functors.

Theorem 5.4 ([DHKS04]). *For any model category \mathcal{M} and small category \mathcal{D} , the functor $\text{colim}_{\mathcal{D}}$ preserves all weak equivalences between diagrams of the form $\text{Lan}_T F'$, where F' is a Reedy cofibrant diagram in $\mathcal{M}^{\Delta\mathcal{D}}$. Moreover, if Q is a cofibrant replacement functor in the Reedy model structure on $\mathcal{M}^{\Delta\mathcal{D}}$, the composite*

$$(5.5) \quad \mathcal{M}^{\mathcal{D}} \xrightarrow{T^*} \mathcal{M}^{\Delta\mathcal{D}} \xrightarrow{Q} \mathcal{M}^{\Delta\mathcal{D}} \xrightarrow{\text{Lan}_T} \mathcal{M}^{\mathcal{D}}$$

is a left deformation for the functor $\text{colim}_{\mathcal{D}}: \mathcal{M}^{\mathcal{D}} \rightarrow \mathcal{M}$, and therefore gives rise to a left approximation and left derived functor of it. This is called the homotopy colimit. Homotopy limits are constructed dually.

Sketch of Proof. See [DHKS04, ch. IV] for the full proof. The fact that the composite

$$\operatorname{colim}_{\mathcal{D}} \operatorname{Lan}_T QT^*,$$

is homotopical follows directly from Lemma 5.1. One can also use Lemma 5.1 to show that Lan_T is homotopical on Reedy cofibrant diagrams, and hence the composite (5.5) is also homotopical. The trickier parts are constructing a natural weak equivalence from the composite (5.5) to the identity functor of $\mathcal{M}^{\mathcal{D}}$ and showing that $\operatorname{colim}_{\mathcal{D}}$ in fact preserves *all* weak equivalences between diagrams of the form $\operatorname{Lan}_T F'$, where F' is a Reedy cofibrant $\Delta\mathcal{D}$ -diagram. These proofs can be found in [DHKS04, §23]. \square

A very similar procedure is followed in [CS98], using essentially the Reedy model structure on $\mathcal{M}^{\Delta\mathcal{D}}$, but restricted to the subcategory of “bounded” diagrams, which are easier to analyze explicitly.

These technical results give homotopy limit and colimit functors for any model category (and, moreover, all homotopy Kan extensions, which fit together into a “homotopy limit system” in the terminology of [DHKS04]). So we have a complete solution for the global approach, at least in the unenriched context. But for computations and examples, as well as for dealing with enrichment, the local approach is much more tractable; hence the need for a comparison.

6. LOCAL DEFINITIONS AND TENSOR PRODUCTS OF FUNCTORS

Recall that rather than ask for homotopical properties of the homotopy limit *functor*, the local approach asks the homotopy limit *object* to represent something homotopically coherent. In the case of colimits, instead of gluing the objects together directly as in the usual colimit, we glue them together with homotopies in between.

To have a precise notion of “homotopy,” we focus on *simplicially enriched categories* \mathcal{M} . These are categories with simplicial sets of morphisms $\mathcal{M}(M, M')$ and simplicial composition maps $\mathcal{M}(M', M'') \times \mathcal{M}(M, M') \rightarrow \mathcal{M}(M, M'')$ which are unital and associative; we think of the 0-simplices of $\mathcal{M}(M, M')$ as the maps $M \rightarrow M'$ and the higher simplices as homotopies and higher homotopies between them.

Remark 6.1. Many authors call a simplicially enriched category a “simplicial category.” However, there is an unfortunate collision with the use of the term “simplicial object in \mathcal{X} ” to denote a functor $\Delta^{op} \rightarrow \mathcal{X}$. A small simplicially enriched category can be identified with a simplicial object in **Cat** which is “object-discrete.”

In order to be able to glue these homotopies together in \mathcal{M} , we need to be able to “multiply” objects of \mathcal{M} by simplicial sets, as in the following definition.

Definition 6.2. A simplicially enriched category \mathcal{M} is *tensoried* if we have for every simplicial set K and object M of \mathcal{M} , an object $K \odot M$ of \mathcal{M} , together with an isomorphism of simplicial sets

$$(6.3) \quad \mathcal{M}(K \odot M, M') \cong \operatorname{Map}(K, \mathcal{M}(M, M'))$$

where Map is the usual simplicial mapping space. Tensors are unique up to unique isomorphism, when they exist, and can be made functorial in a unique way such that (6.3) becomes a natural isomorphism.

For example, $s\mathcal{S}$ is simplicially enriched by Map , and tensored by the ordinary cartesian product. The category $s\mathcal{S}_*$ of based simplicial sets is enriched by the pointed mapping space and tensored with $K \odot X = K_+ \wedge X$. The category of (compactly generated) topological spaces is simplicially enriched by taking the singular complex of the mapping space, and tensored with $K \odot X = |K| \times X$. Similarly, most categories of spectra are simplicially enriched and tensored.

If \mathcal{M} is a tensored simplicially enriched category, there is a well-known construction called the *tensor product of functors*, which assigns an object

$$(6.4) \quad G \odot_{\mathcal{D}} F = \text{coeq} \left(\coprod_{d \twoheadrightarrow d'} Gd' \odot Fd \rightrightarrows \coprod_d Gd \odot Fd \right)$$

of \mathcal{M} to a given pair of functors $F: \mathcal{D} \rightarrow \mathcal{M}$ and $G: \mathcal{D}^{op} \rightarrow s\mathcal{S}$. In favorable cases, this is an example of a *weighted colimit*. We will study homotopy weighted colimits in the second part of this paper, but for now we use tensor products of functors as a tool to construct ordinary homotopy colimits. Note that if we take G to be the functor constant at the terminal simplicial set $*$, which has a unique simplex in each dimension, then we obtain

$$(6.5) \quad * \odot_{\mathcal{D}} F \cong \text{colim } F;$$

thus the tensor product $G \odot_{\mathcal{D}} F$ can be thought of as the colimit of F “fattened up” by G .

Many of the the usual constructions of homotopy theory, such as homotopy pushouts, cones, mapping telescopes, and so on, can be expressed using such tensor products. Inspecting these examples, all of which have some claim to be a “homotopy” version of some colimit, we notice that in most cases the shape of the homotopies we glue in can be described by the nerves of the under-categories $(d \downarrow \mathcal{D})$, for objects d of \mathcal{D} . We have a functor $N(- \downarrow \mathcal{D}): \mathcal{D}^{op} \rightarrow s\mathcal{S}$, and so we make the following definition.

Definition 6.6. The *uncorrected homotopy colimit* of a diagram $F: \mathcal{D} \rightarrow \mathcal{M}$ is the tensor product of functors:

$$(6.7) \quad \text{uhocolim } F = N(- \downarrow \mathcal{D}) \odot_{\mathcal{D}} F$$

if it exists.

The “corrected” version will be given in Definition 8.2. If the tensor products (6.7) exist for all $F: \mathcal{D} \rightarrow \mathcal{M}$, they define an *uncorrected homotopy colimit functor* $\text{uhocolim}: \mathcal{M}^{\mathcal{D}} \rightarrow \mathcal{M}$.

Remark 6.8. Just as an ordinary colimit object is a representing object for cones under a diagram, the uncorrected homotopy colimit is a representing object for *homotopy coherent cones*. This “homotopical universal property” will be made precise in §10; see [CP97] for a similar approach.

Before moving on, we briefly mention the dual notions. A simplicially enriched category \mathcal{M} is *cotensored* if we have for every simplicial set K and object M of \mathcal{M} , an object $\{K, M\}$ of \mathcal{M} and an isomorphism of simplicial sets

$$\mathcal{M}(M, \{K, M'\}) \cong \text{Map}(K, \mathcal{M}(M, M')).$$

For example, $s\mathcal{S}$ is cotensored over itself by Map . Many simplicially enriched categories, including based simplicial sets, (compactly generated) topological spaces,

and spectra, are also cotensored. Like tensors, cotensors are unique and functorial when they exist.

If \mathcal{M} is a cotensored simplicially enriched category and we have functors $F: \mathcal{D} \rightarrow \mathcal{M}$ and $G: \mathcal{D} \rightarrow s\mathcal{S}$, we define the *cotensor product of functors* $\{G, F\}^{\mathcal{D}}$ using equalizers and products in a completely dual way. Finally, the *uncorrected homotopy limit* is defined to be the cotensor product of functors

$$(6.9) \quad \text{uholim } F = \{N(\mathcal{D} \downarrow -), F\}^{\mathcal{D}}.$$

When all these cotensor products exist, they define a functor $\text{uholim}: \mathcal{M}^{\mathcal{D}} \rightarrow \mathcal{M}$.

7. THE BAR CONSTRUCTION

The “generalized bar construction” is quite unreasonably useful in mathematics; here we apply it to rephrase the definition of homotopy colimits in a computationally useful way, which we will show in §10 can also be conceptually helpful. There are many ways to approach the bar construction, and different readers will find different motivations more natural. The introduction given here is chosen to clarify the connection with ordinary colimits and explain why the bar construction is a natural thing to consider in the context of homotopy colimits. Another motivation can be found in §10.

Refer to the diagram (5.3). The functor $S^*: \mathcal{M}^{\mathcal{D}} \rightarrow \mathcal{M}^{\Delta^{op}\mathcal{D}}$ is full and faithful, so that the counit $\text{Lan}_{\mathcal{S}} S^* \rightarrow \text{Id}$ of the adjunction $\text{Lan}_{\mathcal{S}} \dashv S^*$ is an isomorphism. Hence by the universal properties of Kan extensions (see [ML98, X.4 exercise 3]), for any $F \in \mathcal{M}^{\mathcal{D}}$ we have:

$$(7.1) \quad \begin{aligned} \text{colim}_{\mathcal{D}} F &\cong \text{colim}_{\mathcal{D}} \text{Lan}_{\mathcal{S}} S^* F \\ &\cong \text{colim}_{\Delta^{op}\mathcal{D}} S^* F \\ &\cong \text{colim}_{\Delta^{op}} \text{Lan}_{\Sigma} S^* F. \end{aligned}$$

Thus the colimit of a diagram F of any shape is isomorphic to the colimit of a diagram $\text{Lan}_{\Sigma} S^* F$ of shape Δ^{op} . This simplicial object has been studied for many years under another name.

Definition 7.2. The *simplicial bar construction* of F is a simplicial object of \mathcal{M} , denoted $B_{\bullet}(*, \mathcal{D}, F)$, whose object of n -simplices is

$$(7.3) \quad B_n(*, \mathcal{D}, F) = \prod_{\alpha: [n] \rightarrow \mathcal{D}} F(\alpha_0)$$

and whose faces and degeneracies are defined using composition in \mathcal{D} , the evaluation maps $\mathcal{D}(d, d') \times F(d) \rightarrow F(d')$, and insertion of identity arrows.

We refer the reader to [May72, §9-10] and [May75, §7 and §12] for May’s original definitions, and to [Mey84] and [Mey86] for a more abstract point of view. These references deal with the two-sided version, which will be introduced in §9. The term “bar construction” comes from the use of bars for tensor products in the pre- \TeX days of homological algebra, when the symbol \otimes was difficult to typeset. The symbol $*$ on the left is just a placeholder for now; we will see what it represents in §9.

Remark 7.4. Note that when $\mathcal{M} = \mathbf{Set}$, the bar construction is just the nerve of the category of elements of F . In particular, we have $B(*, \mathcal{D}, *) \cong N\mathcal{D}$, where here $*$ denotes the functor constant at a one-element set. In other “set-like” cases (such

as topological spaces), $B_{\bullet}(*, \mathcal{D}, F)$ is the nerve of an internal “enriched category of elements,” which is sometimes called the *transport category*. However, for general \mathcal{M} the bar construction need not be the nerve of anything.

Simplicial bar constructions appear ubiquitously in mathematics. One clue as to the reason is the following characterization in terms of Kan extensions: they can be regarded as a way to replace a diagram of any shape by a “homotopically equivalent” simplicial diagram. Another clue is Lemma 9.2, which will tell us why this is often such a useful replacement to make. Yet another will be explained in §10: they provide a construction of “homotopy coherent natural transformations.” Finally, we will see in §20 that the bar construction is the “universal deformation” for diagram categories, which is perhaps the most satisfying explanation of all.

Proposition 7.5. $\text{Lan}_{\Sigma} S^*F \cong B_{\bullet}(*, \mathcal{D}, F)$.

Proof. By [ML98, X.3 theorem 1], we have the following formula for computing the left Kan extension.

$$(7.6) \quad (\text{Lan}_{\Sigma} S^*F)(n) \cong \text{colim}_{(\Sigma \downarrow [n])} S^*F.$$

However, the comma-category $(\Sigma \downarrow [n])$ is a disjoint union of categories with a terminal object. These terminal objects are the simplices $\alpha: [n] \rightarrow \mathcal{D}$ in $\Delta^{op} \mathcal{D}$, and hence the above Kan extension is a coproduct of $S^*F(\alpha) = F(\alpha_0)$ over all such α . This provides exactly the desired natural isomorphism

$$(\text{Lan}_{\Sigma} S^*F)(n) \cong B_n(*, \mathcal{D}, F). \quad \square$$

Recall from (7.1) that the colimit of $\text{Lan}_{\Sigma} S^*F$ is isomorphic to the colimit of F . But since the colimit of a simplicial diagram is just the coequalizer of the two face maps $d_0, d_1: X_1 \rightrightarrows X_0$, we recover the classical formula for computing colimits using coproducts and coequalizers:

$$(7.7) \quad \text{colim}_{\mathcal{D}} F \cong \text{coeq} \left(\coprod_{d \rightarrow d'} Fd \rightrightarrows \coprod_d Fd \right).$$

Dually, of course, the limit of any functor is isomorphic to the limit of the *cosimplicial cobar construction*, which has a corresponding characterization using the left half of (5.3), and this reduces to the analogue of (7.7) for computing limits using products and equalizers.

The above shows that for usual limits and colimits, the “higher information” contained in the bar construction above level one is redundant. However, in the homotopical situation, this information plays an important role and we have to incorporate it.

In the classical context of topological spaces, the “correct homotopy type” associated to a simplicial space (or spectrum) is usually its geometric realization. We can define the *geometric realization* of a simplicial object X_{\bullet} in any tensored simplicially enriched category as the tensor product of functors

$$|X_{\bullet}| = \Delta \odot_{\Delta^{op}} X_{\bullet}$$

where $\Delta: \Delta \rightarrow s\mathcal{S}$ is the canonical cosimplicial simplicial set. When \mathcal{M} is the category of topological spaces, this reduces to the usual geometric realization. When $\mathcal{M} = s\mathcal{S}$, the realization of a bisimplicial set is isomorphic to its diagonal.

Then we can make the following definition.

Definition 7.8. The *bar construction* is the geometric realization of the simplicial bar construction:

$$\begin{aligned} B(*, \mathcal{D}, F) &= |B_{\bullet}(*, \mathcal{D}, F)| \\ &\equiv \Delta \odot_{\Delta^{op}} B_{\bullet}(*, \mathcal{D}, F) \end{aligned}$$

Before stating the next lemma, we need to clear up some notation. For any object d of \mathcal{D} , we have the hom-functor $\mathcal{D}(d, -): \mathcal{D} \rightarrow \mathbf{Set}$, which we can regard as a functor $\mathcal{D} \rightarrow s\mathcal{S}$ landing in discrete simplicial sets. Since $s\mathcal{S}$ is tensored over itself, we can form the bar construction $B_{\bullet}(*, \mathcal{D}, \mathcal{D}(d, -))$, which by Remark 7.4 is the nerve of the category of elements of $\mathcal{D}(d, -)$, viewed as a horizontally discrete bisimplicial set. Hence its diagonal $B(*, \mathcal{D}, \mathcal{D}(d, -))$ is precisely that nerve. Varying d , we obtain a functor $\mathcal{D}^{op} \rightarrow s\mathcal{S}$ which we denote $B(*, \mathcal{D}, \mathcal{D})$.

Lemma 7.9.

$$B(*, \mathcal{D}, F) \cong B(*, \mathcal{D}, \mathcal{D}) \odot_{\mathcal{D}} F.$$

Proof. Because tensor products of functors, including geometric realization, are all colimits, they all commute with each other. Therefore the right-hand side can be written as $B(*, \mathcal{D}, \mathcal{D} \odot_{\mathcal{D}} F)$, where $\mathcal{D} \odot_{\mathcal{D}} F: \mathcal{D} \rightarrow \mathcal{M}$ is defined by $(\mathcal{D} \odot_{\mathcal{D}} F)(d) = \mathcal{D}(-, d) \odot_{\mathcal{D}} F$. But writing out the definitions, it is easy to see that $\mathcal{D} \odot_{\mathcal{D}} F \cong F$. \square

The punchline is that the bar construction is precisely the uncorrected homotopy colimit from §6.

Proposition 7.10. For $F: \mathcal{D} \rightarrow \mathcal{M}$ where \mathcal{M} is simplicially enriched, tensored, and cocomplete,

$$\begin{aligned} \text{uhocolim } F &= N(- \downarrow \mathcal{D}) \odot_{\mathcal{D}} F \\ &\cong B(*, \mathcal{D}, F). \end{aligned}$$

Proof. Recall that $B(*, \mathcal{D}, \mathcal{D})$ is the nerve of the category of elements of $\mathcal{D}(d, -)$. But that category of elements is precisely the comma-category $(d \downarrow \mathcal{D})$; thus we have a natural isomorphism

$$B_{\bullet}(*, \mathcal{D}, \mathcal{D}) \cong N(- \downarrow \mathcal{D}).$$

Now applying Lemma 7.9, we obtain

$$\begin{aligned} B(*, \mathcal{D}, F) &\cong B(*, \mathcal{D}, \mathcal{D}) \odot_{\mathcal{D}} F \\ &\cong N(- \downarrow \mathcal{D}) \odot_{\mathcal{D}} F = \text{uhocolim } F. \quad \square \end{aligned}$$

So far, this is just a rephrasing of Definition 6.6, and was known as far back as [BK72], who called the simplicial bar construction the *simplicial replacement*. However, as we will see in §9, the bar construction provides an alternate approach to the comparison of local and global homotopy colimits.

8. A COMPARISON BASED ON REEDY MODEL STRUCTURES

In order to compare the local and global definitions, we need to put ourselves in a context in which both are defined, and we also require some compatibility conditions. So suppose now that \mathcal{M} is a simplicial model category. A simplicial model category is a model category which is also simplicially enriched, tensored, and cotensored, and such that the lifting properties in the definition of model category are extended to “homotopy lifting properties” with respect to the simplicial mapping spaces. See [Hir03, ch. 9] for a definition.

One of the consequences of the definition is the following result, which will be crucial later on for analyzing bar constructions.

Lemma 8.1. *If \mathcal{M} is a simplicial model category, then the geometric realization functor $|\cdot|: \mathcal{M}^{\Delta^{op}} \rightarrow \mathcal{M}$ is left Quillen when $\mathcal{M}^{\Delta^{op}}$ is given the Reedy model structure. Therefore, it preserves weak equivalences between Reedy cofibrant simplicial objects.*

Proof. This is a special case of [Hir03, 18.4.11] (which we state below as Lemma A.8). The consequence is stated as [Hir03, 18.6.6]. \square

Now, since Definition 6.6 uses no properties of the model structure, it is unsurprising that we have to modify it a bit. Recall that we assume all model categories have functorial factorizations, so that \mathcal{M} has fibrant and cofibrant replacement functors, which we call R and Q respectively.

Definition 8.2. The *corrected homotopy limit* and *colimit* of F are:

$$(8.3) \quad \text{hocolim } F = \text{uhocolim } QF$$

$$(8.4) \quad \text{holim } F = \text{uholim } RF$$

where Q and R are applied objectwise to F .

The approach to analyzing these functors in [Hir03, §18.4 and 18.5] goes as follows. Because $N(- \downarrow \mathcal{D})$ is cofibrant in the projective model structure on $s\mathcal{S}^{\mathcal{D}^{op}}$, the functor uhocolim preserves weak equivalences between objectwise cofibrant diagrams (and dually for uholim). It follows that the functors hocolim and holim are homotopical.

We will not go into this argument in detail, because while this approach shows that hocolim is homotopical, it is unclear from the argument whether it is defined by applying a deformation, and hence whether it is in fact a derived functor in the sense of §2. The results in this section and the next give alternate proofs of the fact that hocolim is homotopical which also show this stronger result.

In this section we outline a proof which appears in an early draft of [DHKS04] (but not in the published book). The full proof will be given in Appendix A. I would like to thank Phil Hirschhorn for providing me with a copy of the early draft and helping me understand the proof.

Recall from Theorem 5.4 that any cofibrant replacement functor in the Reedy model structure on $\mathcal{M}^{\Delta^{\mathcal{D}}}$ gives rise to a derived functor of colim . The goal of this approach is to show that hocolim is in fact the colimit of a specific Reedy cofibrant replacement for F in $\mathcal{M}^{\Delta^{\mathcal{D}}}$. Define a functor $\mathcal{Q}: \mathcal{M}^{\mathcal{D}} \rightarrow \mathcal{M}^{\Delta^{\mathcal{D}}}$ as follows.

$$\begin{aligned} \mathcal{Q}F(\alpha) &= \text{hocolim } \alpha^* F \\ &\cong N(- \downarrow [n]) \odot_{[n]} \alpha^* QF. \end{aligned}$$

Here we consider an n -simplex α as a functor $\alpha: [n] \rightarrow \mathcal{D}$. The stated isomorphism follows from the fact that $\alpha^* QF = Q\alpha^* F$, since Q is applied objectwise.

Thus the values of $\mathcal{Q}F$ are “generalized mapping cylinders” of F over all the simplices of \mathcal{D} . The mapping cylinder of a map f is naturally homotopy equivalent to the codomain of f , and the same is true for these generalized mapping cylinders.

Specifically, the maps $N(- \downarrow [n]) \rightarrow *$ induce maps

$$\begin{array}{ccc} N(- \downarrow [n]) \odot_{[n]} \alpha^* QF & \longrightarrow & * \odot_{[n]} \alpha^* QF \xrightarrow{\cong} QF(\alpha_n) \\ & \searrow \delta & \downarrow q \\ & & F(\alpha_n) \end{array}$$

which fit together into a natural transformation

$$\delta: \mathcal{Q}F \longrightarrow T^*F.$$

We then prove the following theorem.

Theorem 8.5. *The diagram $\mathcal{Q}F$ is Reedy cofibrant for all F , the functors \mathcal{Q} and $\text{Lan}_T \mathcal{Q}$ are homotopical, and δ is a natural weak equivalence. Moreover,*

$$(8.6) \quad \begin{aligned} \text{hocolim } F &\cong \text{colim}_{\Delta \mathcal{Q}} \mathcal{Q}F \\ &\cong \text{colim}_{\mathcal{Q}} \text{Lan}_T \mathcal{Q}F. \end{aligned}$$

Therefore, $(\text{Lan}_T \mathcal{Q}, \text{Lan}_T \delta)$ is a left deformation of colim , and hence hocolim is a derived functor of the usual colimit.

Proof. It is straightforward to show that δ is a weak equivalence. To wit, the diagram $N(- \downarrow [n])$ is always projective-cofibrant, and because $[n]$ has a terminal object n , the constant $s\mathcal{S}$ -diagram at a point is also projective-cofibrant. Thus by [Hir03, 18.4.5], this implies that the functor $- \odot_{\mathcal{Q}} QF$ preserves the weak equivalence $N(- \downarrow [n]) \rightarrow *$, so upon composing with the weak equivalence q , we see that δ is a weak equivalence.

It follows from the 2-out-of-3 property that \mathcal{Q} and $\text{Lan}_T \mathcal{Q}$ are homotopical. Thus the bulk of the proof consists of proving (8.6) and showing that $\mathcal{Q}F$ is Reedy cofibrant. This involves a lengthy manipulation of Kan extensions and tensor products of functors, which we defer to Appendix A. \square

We do, however, note the following fact, which is crucial to the proof that $\mathcal{Q}F$ is Reedy cofibrant, and will be needed in the next section.

Lemma 8.7. *If $F: \mathcal{D} \rightarrow \mathcal{M}$ is objectwise cofibrant, then S^*F is Reedy cofibrant in $\mathcal{M}^{\Delta^{op} \mathcal{D}}$.*

Proof. Recall from [Hov99] or [Hir03] that a diagram X on a Reedy category is Reedy cofibrant when all the “latching maps” $L_\alpha X \rightarrow X_\alpha$ are cofibrations. Here $L_\alpha X$ is the “latching object” for X at α .

In our situation, let α be an n -simplex in \mathcal{D} . We have $S^*F(\alpha) = F(\alpha_0)$. If α contains no identity maps, then the latching object $L_\alpha S^*F$ is the initial object. Otherwise, it is simply $F(\alpha_0)$. Since F is objectwise cofibrant, in either case the map $L_\alpha S^*F \rightarrow S^*F(\alpha)$ is a cofibration, as required. \square

Recall that S^*F is a character who has already figured in our discussion of the bar construction in §7. Thus, this result suggests we may be able to use the bar construction and deal directly with objectwise cofibrant diagrams, without needing to invoke the replacement category $\Delta \mathcal{D}$. In the next section, we will use this idea to give an alternate proof that hocolim is a derived functor of the usual colimit.

9. A COMPARISON USING THE TWO-SIDED BAR CONSTRUCTION

Let \mathcal{M} be a simplicial model category. We shall give a direct proof of the following theorem (although some final technical steps will be postponed to §13).

Theorem 9.1. *If Q is any cofibrant replacement functor on \mathcal{M} , then*

$$\mathrm{hocolim} \cong B(*, \mathcal{D}, Q-)$$

is a left derived functor of colim .

We stress that Q is only assumed to be a cofibrant replacement functor for the model structure on \mathcal{M} , applied objectwise; *no model structure on $\mathcal{M}^{\mathcal{D}}$ is required.*

We begin by showing that $\mathrm{hocolim}$ is homotopical.

Lemma 9.2. *If $F: \mathcal{D} \rightarrow \mathcal{M}$ is objectwise cofibrant, then the simplicial bar construction $B_{\bullet}(*, \mathcal{D}, F)$ is Reedy cofibrant. Hence the functor*

$$B(*, \mathcal{D}, -): \mathcal{M}^{\mathcal{D}} \longrightarrow \mathcal{M}$$

preserves weak equivalences between objectwise cofibrant diagrams.

Proof. Note that when $\mathcal{M}^{\Delta^{op}}$ and $\mathcal{M}^{\Delta^{op}\mathcal{D}}$ are given the Reedy model structures, the adjunction $\mathrm{Lan}_{\Sigma} \dashv \Sigma^*$ is Quillen. In fact, Σ^* preserves matching objects—that is to say, $M_{\alpha}\Sigma^*G \cong M_{\Sigma\alpha}G$ —from which it follows trivially that Σ^* preserves Reedy fibrations and Reedy trivial fibrations. But by Lemma 8.7, S^*F is Reedy cofibrant whenever F is objectwise cofibrant, so $\mathrm{Lan}_{\Sigma} S^*F$ is also Reedy cofibrant. The second statement follows from the first together with Lemma 8.1. \square

A more general version of this result will be proven in §23.

Results of this sort are as old as the bar construction; it has always been thought of as a sort of cofibrant replacement. May ([May72] and [May75]) shows that under suitable conditions, simplicial bar constructions (on topological spaces) are “proper simplicial spaces,” which is equivalent to being “Reedy h -cofibrant.” We will have more to say about this in §23. For $\mathcal{M} = s\mathcal{S}$, this result is well-known in simplicial homotopy theory, especially in its dual formulation: see [GJ99, Lemma VIII.2.1], the classical [BK72], or more recent [CP97].

To understand its relevance here, recall from Proposition 7.10 that

$$\mathrm{uhocolim} F \cong B(*, \mathcal{D}, F).$$

Thus, as promised after Definition 8.2, we have an alternate proof of the fact that the functor that takes F to $\mathrm{hocolim} F \cong B(*, \mathcal{D}, QF)$ is homotopical: Lemma 9.2 tells us that $B(*, \mathcal{D}, -)$ preserves weak equivalences between objectwise cofibrant diagrams, but Q (applied objectwise) is a homotopical functor replacing every diagram by an objectwise cofibrant one.

Now we want to use the bar construction to show that $\mathrm{hocolim}$ is in fact a derived functor, but first we need to introduce a two-sided generalization of the bar construction. Just as we replaced $*$ by a functor G to generalize ordinary colimits to tensor products of functors in (6.4), here we do the same for the bar construction.

Definition 9.3 ([May75, §12]). Given functors $F: \mathcal{D} \rightarrow \mathcal{M}$ and $G: \mathcal{D}^{op} \rightarrow s\mathcal{S}$, the *two-sided simplicial bar construction* $B_{\bullet}(G, \mathcal{D}, F)$ is a simplicial object of \mathcal{M} whose object of n -simplices is

$$B_n(G, \mathcal{D}, F) = \coprod_{\alpha: [n] \rightarrow \mathcal{D}} G(\alpha_n) \odot F(\alpha_0).$$

and whose faces and degeneracies are defined using composition in \mathcal{D} , the evaluation maps $\mathcal{D}(d, d') \times F(d) \rightarrow F(d')$ and $G(d') \times \mathcal{D}(d, d') \rightarrow G(d)$, and insertion of identity arrows.

As with the one-sided version, the two-sided bar construction can be rephrased using the functors of (5.3). If we write $G \overline{\odot} F: \mathcal{D}^{op} \times \mathcal{D} \rightarrow \mathcal{M}$ for the “external tensor product” of functors:

$$(G \overline{\odot} F)(d', d) = Gd' \odot Fd$$

then we have the following analogue of Proposition 7.5:

$$(9.4) \quad B_{\bullet}(G, \mathcal{D}, F) \cong \text{Lan}_{\Sigma}(T, S)^*(G \overline{\odot} F)$$

and the following analogues of (7.1) and (7.7):

$$(9.5) \quad \begin{aligned} G \odot_{\mathcal{D}} F &= \text{coeq} \left(\coprod_{d \xrightarrow{u} d'} Gd' \odot Fd \rightrightarrows \coprod_d Gd \odot Fd \right) \\ &\cong \text{colim}_{\Delta^{op}} \text{Lan}_{\Sigma}(T, S)^* G \overline{\odot} F. \end{aligned}$$

As in the one-sided case, we define the *two-sided bar construction* to be the realization of the two-sided simplicial bar construction:

$$(9.6) \quad B(G, \mathcal{D}, F) = |B_{\bullet}(G, \mathcal{D}, F)|.$$

The comparison hinges on the following standard result about bar constructions.

Lemma 9.7.

$$\begin{aligned} \text{colim } B(\mathcal{D}, \mathcal{D}, F) &\cong * \odot_{\mathcal{D}} B(\mathcal{D}, \mathcal{D}, F) \\ &\cong B(*, \mathcal{D}, F) \end{aligned}$$

Proof. The first isomorphism was stated as (6.5), and follows directly from the definitions. Now, as in the proof of Lemma 7.9, because tensor products of functors, including geometric realization, are all colimits, they all commute with each other. Thus, we have $* \odot_{\mathcal{D}} B(\mathcal{D}, \mathcal{D}, F) \cong B(* \odot_{\mathcal{D}} \mathcal{D}, \mathcal{D}, F)$. But it is easy to see that $* \odot_{\mathcal{D}} \mathcal{D} \cong *$. \square

Lemma 9.7 suggests we can take $B(\mathcal{D}, \mathcal{D}, Q-)$ as a deformation for the colimit, but before showing that this works, we must recall some facts about simplicial homotopy. Given two maps $X_{\bullet} \rightrightarrows Y_{\bullet}$ of simplicial objects in any category \mathcal{M} , there is an old, purely combinatorial, notion of simplicial homotopy between them; see, for example, [May92, §5]. When \mathcal{M} is complete and cocomplete, we can reformulate this in modern terms by saying that $\mathcal{M}^{\Delta^{op}}$ is simplicially enriched, tensored, and cotensored, and thus has an attendant notion of simplicial homotopy using simplicial cylinders $\Delta^1 \odot X_{\bullet}$. The tensor has a simple description as $(K \odot X)_n = K_n \times X_n$, and the combinatorial description follows by inspection.

For general \mathcal{M} this combinatorial notion of homotopy in $\mathcal{M}^{\Delta^{op}}$ is *a priori* unrelated to any homotopy theory that \mathcal{M} might have. However, when \mathcal{M} is tensored and simplicially enriched, we have

$$(9.8) \quad |K \odot X_{\bullet}| \cong K \odot |X_{\bullet}|$$

for any simplicial set K and simplicial object X_{\bullet} in \mathcal{M} . See, for example, [RSS01, 5.4]. It follows that “geometric realization preserves simplicial homotopy” in the

sense that a simplicial homotopy between maps $X_\bullet \rightrightarrows Y_\bullet$ gives rise to a simplicial homotopy between the realizations $|X_\bullet| \rightrightarrows |Y_\bullet|$ of these maps.

In particular, simplicial homotopy *equivalences* in $\mathcal{M}^{\Delta^{op}}$, in the above purely combinatorial sense, realize to simplicial homotopy equivalences in \mathcal{M} . Moreover, if \mathcal{M} is a simplicial *model* category, simplicial homotopy equivalences are necessarily also weak equivalences for the model structure—see, for example [Hir03, 9.5.16].

With this as background, we can now continue on the path to relating the two-sided bar construction to homotopy colimits, by showing that $B(\mathcal{D}, \mathcal{D}, Q-)$ does in fact define a deformation for the colimit. As with Lemma 9.7, this is a standard result about bar constructions which we merely rephrase in abstract language. The idea is found in [May72, 9.8] and is clarified and generalized in [Mey84, §6-7].

Lemma 9.9. *There is a natural weak equivalence $\varepsilon: B(\mathcal{D}, \mathcal{D}, F) \xrightarrow{\sim} F$. Thus the functor $B(\mathcal{D}, \mathcal{D}, Q-): \mathcal{M}^{\mathcal{D}} \rightarrow \mathcal{M}^{\mathcal{D}}$ is a deformation.*

Proof. Since F is the colimit of $B_\bullet(\mathcal{D}, \mathcal{D}, F)$, we have a natural map

$$\tilde{\varepsilon}: B_\bullet(\mathcal{D}, \mathcal{D}, F) \rightarrow \tilde{F},$$

where \tilde{F} is the constant simplicial diagram on F . Since F is the realization of \tilde{F} , the realization of $\tilde{\varepsilon}$ is our desired map $B(\mathcal{D}, \mathcal{D}, F) \rightarrow F$. It remains only to show that ε is a weak equivalence, and by the above discussion, for this it suffices to show that $\tilde{\varepsilon}$ is objectwise a simplicial homotopy equivalence.

We have a map $\eta: \tilde{F} \rightarrow B_\bullet(\mathcal{D}, \mathcal{D}, F)$ given by inserting identities, and clearly $\tilde{\varepsilon}\eta = 1_{\tilde{F}}$. Moreover, $B_\bullet(\mathcal{D}, \mathcal{D}, F)$ has an *extra degeneracy* induced by inserting identities at the beginning of α (inserting identities elsewhere induces the usual degeneracies). An extra degeneracy for a simplicial object X_\bullet is a collection of maps $s_{-1}: X_n \rightarrow X_{n+1}$ such that $d_0 s_{-1} = 1_{X_n}$, $d_{i+1} s_{-1} = s_{-1} d_i$, and $s_{j+1} s_{-1} = s_{-1} s_j$ for all i and j . This extra degeneracy furnishes a simplicial homotopy

$$(9.10) \quad \eta \tilde{\varepsilon} \simeq 1: B_\bullet(\mathcal{D}(-, d), \mathcal{D}, F) \rightarrow B_\bullet(\mathcal{D}(-, d), \mathcal{D}, F)$$

for each fixed $d \in \mathcal{D}$. Thus each map $B(\mathcal{D}(-, d), \mathcal{D}, F) \rightarrow Fd$ is a simplicial homotopy equivalence in \mathcal{M} , hence a weak equivalence, as desired. (Note, though, that the simplicial homotopies (9.10) are not natural in d .) \square

Taken together, Lemmas 9.7 and 9.9 say that $\text{hocolim } F$ is actually the colimit of a weakly equivalent replacement diagram $B(\mathcal{D}, \mathcal{D}, QF)$. To complete the proof of Theorem 9.1, it remains to show that $B(\mathcal{D}, \mathcal{D}, Q-)$ is a left colim-deformation in the sense of Definition 3.1, that is, that colim is homotopical on its full image. This is a special case of Theorem 13.7 below, which applies to the more general enriched case. We do not repeat the proof here, as it does not simplify much in this case and in any case is not particularly enlightening.

This completes the proof of Theorem 9.1. The reader may find this route to the comparison theorem to be simpler than the comparison proof sketched in §8 and detailed in Appendix A (combined with the proof in [DHKS04, ch. IV] of our Theorem 5.4). However, the real merit of the approach in this section is that it applies nearly verbatim to the more general enriched context, in which the bar construction makes perfect sense but nerves and categories of simplices do not.

10. HOMOTOPY COHERENT TRANSFORMATIONS

There is a different approach we might take for the analysis of the homotopy theory of diagrams, including homotopy limits and colimits. Recall that weak equivalences are frequently handled in homotopy theory by replacing the objects with “good” objects such that weak equivalences become *homotopy* equivalences, which already have inverses up to homotopy. Thus, suppose that \mathcal{M} has this (vaguely defined) property, and consider how we might show that $\mathcal{M}^{\mathcal{D}}$ also has it.

One approach is to imitate the theory of “cell complexes” used in classical homotopy theory, constructing “cell diagrams” using diagrams freely generated by cells placed at different positions. This idea leads to the projective model structure, discussed briefly in §5, with all the attendant disadvantages. In this section we consider a different approach.

Let F and G be diagrams in \mathcal{M} of shape \mathcal{D} , and suppose that we have a natural weak equivalence $\alpha: G \rightarrow F$. We can easily replace F and G objectwise by “good” objects, so that each component $\alpha_d: Gd \rightarrow Fd$ has a homotopy inverse, but in general these homotopy inverses will not fit together into a natural transformation. However, we can reasonably expect that they will fit together into a *homotopy coherent transformation*. Intuitively, this means that the squares

$$(10.1) \quad \begin{array}{ccc} Gd_0 & \xrightarrow{Gf} & Gd_1 \\ \alpha_{d_0}^{-1} \uparrow & \simeq & \uparrow \alpha_{d_1}^{-1} \\ Fd_0 & \xrightarrow{Ff} & Fd_1 \end{array}$$

commute only up to homotopy; that for a pair of composable arrows $d_0 \xrightarrow{f} d_1 \xrightarrow{g} d_2$ in \mathcal{D} , the corresponding homotopies are related by a higher homotopy; and so on.

The notion of homotopy coherent transformation has been made precise in many places, such as [CP97], where it is defined using what is essentially the cobar construction. In our terminology, the definition is as follows. We assume that \mathcal{M} is tensored and cocomplete, so that the bar construction can be defined.

Definition 10.2. A *homotopy coherent transformation* $F \rightsquigarrow G$ is a natural transformation $B(\mathcal{D}, \mathcal{D}, F) \rightarrow G$.

To justify this definition, we unravel the first few levels. Note that by definition of geometric resolution, such a natural transformation is given by a collection of n -simplices $B_n(\mathcal{D}, \mathcal{D}, F) \rightarrow G$ for each n , related by face and degeneracy maps in an appropriate way. For small values of n , what this translates to is the following.

- For each arrow $d_0 \xrightarrow{f} d_1$ in \mathcal{D} , we have a morphism $\alpha_f: Fd_0 \rightarrow Gd_1$.
- These morphisms have the property that for any $d_0 \xrightarrow{f} d_1 \xrightarrow{g} d_2$ in \mathcal{D} , we have $\alpha_{gf} = G(g) \circ \alpha_f$. This comes from the naturality of the transformation $B(\mathcal{D}, \mathcal{D}, F) \rightarrow G$. In particular, we have $\alpha_f = G(f) \circ \alpha_{1_{d_0}}$, so the 0-simplices are determined by morphisms $\alpha_d: Fd \rightarrow Gd$.
- For any $d_0 \xrightarrow{f} d_1 \xrightarrow{g} d_2$ in \mathcal{D} , we have a homotopy $\alpha_{gf} \simeq \alpha_g \circ F(f)$. In particular, for any $d_0 \xrightarrow{f} d_1$ we have $G(f) \circ \alpha_{d_0} = \alpha_f \simeq \alpha_{d_1} \circ F(f)$, so the squares (10.1) commute up to homotopy.
- The above homotopies are preserved by composition with the action of G on arrows, in a straightforward way.

- For any $d_0 \xrightarrow{f} d_1 \xrightarrow{g} d_2 \xrightarrow{h} d_3$ in \mathcal{D} , there is a higher homotopy (a 2-simplex) comparing the homotopy $\alpha_{hgf} \simeq \alpha_h \circ F(gf)$ to the composite homotopy $\alpha_{hgf} \simeq \alpha_{hg} \circ F(f) \simeq \alpha_h \circ F(g) \circ F(f) = \alpha_h \circ F(gf)$.
- And so on. . .

The above remarks lead us to expect that we could produce a “good” replacement for a diagram by replacing its values objectwise by “good” objects in \mathcal{M} , and then applying the bar construction $B(\mathcal{D}, \mathcal{D}, -)$. The results of §9 show that, at least when considering the colimit functor, this strategy works. In §20 we will see that it also works to construct the homotopy category of a diagram category, as we would expect: objectwise “homotopy equivalences” really do have “homotopy coherent” inverses.

We can now also make precise the notion of “homotopy coherent cones” referred to in Remark 6.8. A homotopy coherent cone under a diagram F should consist of a vertex C and a coherent transformation $F \rightsquigarrow C$, where by C we mean the diagram constant at the vertex C . By definition, this is the same as a natural transformation $B(\mathcal{D}, \mathcal{D}, F) \rightarrow C$. Therefore, a representing object for homotopy coherent cones under F is precisely a colimit of $B(\mathcal{D}, \mathcal{D}, F)$; but as we saw in Lemma 9.7, this is precisely the homotopy colimit of F . Thus the latter really does have a “homotopical universal property.”

Remark 10.3. We would like to take this opportunity to briefly mention a misconception which can arise. Since ordinary colimits can be constructed using coproducts and coequalizers, it may seem natural to ask whether homotopy colimits can be constructed using homotopy coproducts and homotopy coequalizers. After having read §7, the reader may be able to guess that this is not true: the naive version of such a construction would correspond to taking the geometric realization of the bar construction truncated at level 1, while we have seen that in general, all the higher information must be included to get the correct result. In the language of this section, the naive construction would be a representing object for cones which commute up to homotopy, but not up to *coherent* homotopy.

There are special cases, however, in which a naive construction does give an equivalent result. For instance, if the category \mathcal{D} is freely generated by some graph, then a cone which commutes with the *generating* arrows up to specified homotopies can be extended, essentially uniquely, to a cone which commutes up to coherent homotopy. In such cases we can use the naive construction, as long as we glue in homotopies only for the generating arrows. A specific example of this is a mapping telescope, which is a homotopy colimit over the category freely generated by the graph $(\cdot \rightarrow \cdot \rightarrow \cdot \rightarrow \dots)$.

This completes the second part of this paper, and we now turn to enriched category theory and weighted homotopy colimits.

11. ENRICHED CATEGORIES AND WEIGHTED COLIMITS

In the majority of categories encountered by the working mathematician, the hom-sets are not just sets but have some extra structure. They may be the vertices of simplicial mapping spaces, as for the simplicially enriched categories considered in §6, but they may also be topological spaces, or abelian groups, or chain complexes, or small categories. A general context for dealing with such situations is given by *enriched category theory*.

The importance of enriched category theory for us is twofold. Firstly, as remarked above, most categories which arise in practice are enriched; thus it is essential for applications to be able to deal with enriched situations. Secondly, however, the presence of enrichment often simplifies, rather than complicates, the study of homotopy theory, since enrichment over a suitable category automatically provides well-behaved notions of homotopy and homotopy equivalence. We will see examples of this in §16–18.

In this section, we will review some notions of enriched category theory. The standard references for the subject are [Kel82] and [Dub70].

Let \mathcal{V}_0 be a bicomplete closed symmetric monoidal category with product \otimes and unit object E . Often the unit object is written I , but in homotopy theory there is potential for confusion with an interval, so we prefer a different notation. Here *closed* means that we have internal hom-objects $[B, C]$ satisfying an adjunction

$$\mathcal{V}_0(A \otimes B, C) \cong \mathcal{V}_0(A, [B, C]).$$

All the above examples give possible choices of \mathcal{V}_0 . A category \mathcal{M} *enriched over* \mathcal{V} , or a \mathcal{V} -category, is defined by a collection of objects M, M', \dots together with objects $\mathcal{M}(M, M')$ of \mathcal{V} , unit morphisms $E \rightarrow \mathcal{M}(M, M)$, and composition morphisms

$$\mathcal{M}(M', M'') \otimes \mathcal{M}(M, M') \rightarrow \mathcal{M}(M, M''),$$

satisfying associativity and unit axioms. Note that if \mathcal{V}_0 is the category **Set** with its cartesian monoidal structure, then a \mathcal{V} -category is the same as an ordinary category (with small hom-sets).

Such a \mathcal{V} -category gives rise to an *underlying* ordinary category \mathcal{M}_0 defined to have the same objects and $\mathcal{M}_0(M, M') = \mathcal{V}_0(E, \mathcal{M}(M, M'))$; experience shows that this always gives the correct answer. The internal-hom of \mathcal{V}_0 gives rise to a \mathcal{V} -category, denoted \mathcal{V} , with $\mathcal{V}(A, B) = [A, B]$. It is easy to check that the underlying category of the \mathcal{V} -category \mathcal{V} is, in fact, the original category \mathcal{V}_0 , justifying the notation. We often abuse notation by referring to the original closed symmetric monoidal category by \mathcal{V} as well.

If \mathcal{M} and \mathcal{N} are \mathcal{V} -categories, a \mathcal{V} -functor $F: \mathcal{M} \rightarrow \mathcal{N}$ consists of an assignation of an object $F(M)$ of \mathcal{N} to each object M of \mathcal{M} together with morphisms

$$\mathcal{M}(M, M') \rightarrow \mathcal{N}(F(M), F(M'))$$

in \mathcal{V}_0 that satisfy functoriality diagrams. As with \mathcal{V} -categories, a \mathcal{V} -functor gives rise to an underlying ordinary functor $F_0: \mathcal{M}_0 \rightarrow \mathcal{N}_0$. Similarly, we have notions of a \mathcal{V} -adjoint pair and a \mathcal{V} -natural transformation which give rise to underlying ordinary adjoint pairs and natural transformations, but are in general stronger.

Now let \mathcal{D} be a small \mathcal{V} -category (i.e. it has a small set of objects), and $F: \mathcal{D} \rightarrow \mathcal{M}$ a \mathcal{V} -functor. We view F as a diagram of shape \mathcal{D} in \mathcal{M} , and would like to take its limit. However, for general \mathcal{V} it may not be possible to define a “cone” over such a diagram in the usual manner; thus we are forced to work with weighted limits from the very beginning. Our approach to weighted colimits is via tensor products of functors, generalizing those used in §6.

We say that a \mathcal{V} -category is *tensored* if we have for every object K of \mathcal{V} and M of \mathcal{M} , an object $K \odot M$ of \mathcal{M} and \mathcal{V} -natural isomorphisms:

$$\mathcal{V}(K, \mathcal{M}(M, M')) \cong \mathcal{M}(K \odot M, M')$$

and *cotensored* if in the same situation we have an object $\{K, M\}$ and \mathcal{V} -natural isomorphisms:

$$\mathcal{V}(K, \mathcal{M}(M', M)) \cong \mathcal{M}(M', \{K, M\}).$$

The \mathcal{V} -category \mathcal{V} is always tensored and cotensored; its tensor is the monoidal product \otimes and its cotensor is its internal-hom. When $\mathcal{V} = \mathbf{Set}$, tensors are copowers and cotensors are powers. As in the simplicial context, tensors are unique up to unique isomorphism when they exist, and can be made functorial in a unique way so that the defining isomorphisms become natural.

We can now make the following definition.

Definition 11.1. If \mathcal{M} is a tensored \mathcal{V} -category, \mathcal{D} is a small \mathcal{V} -category, and $G: \mathcal{D}^{op} \rightarrow \mathcal{V}$ and $F: \mathcal{D} \rightarrow \mathcal{M}$ are \mathcal{V} -functors, their *tensor product* is the following coequalizer:

$$(11.2) \quad G \odot_{\mathcal{D}} F = \text{coeq} \left(\coprod_{d, d'} (Gd' \otimes \mathcal{D}(d, d')) \odot Fd \rightrightarrows \coprod_d Gd \odot Fd \right).$$

When \mathcal{M} is cotensored, we define the *cotensor product of functors* $\{G, F\}^{\mathcal{D}}$ of $F: \mathcal{D} \rightarrow \mathcal{M}$ and $G: \mathcal{D} \rightarrow \mathcal{V}$ to be a suitable equalizer.

Remark 11.3. By the universal properties of tensors, we have the following ‘‘associativity’’ isomorphism:

$$(Gd' \otimes \mathcal{D}(d, d')) \odot Fd \cong Gd' \odot (\mathcal{D}(d, d') \odot Fd).$$

This isomorphism is needed to define one of the two arrows in (11.2). Compared to (6.4), the hom-object $\mathcal{D}(d, d')$ has moved out from under the \coprod , since it is no longer simply a set we can take a coproduct over.

Of course, Definition 11.1 uses ordinary colimits: coproducts and coequalizers. The indexing categories for these colimits are not enriched, however, so we can take these to mean ordinary colimits in the underlying category \mathcal{M}_0 . However, generally we need to require that their universal property is also suitably enriched, as follows. Recall that the universal property of a colimit in \mathcal{M}_0 can be expressed by a natural bijection of sets

$$\mathcal{M}_0 \left(\text{colim}_{d \in D} F(d), Y \right) \cong \lim_{d \in D} \mathcal{M}_0(F(d), Y).$$

We say that this colimit is a \mathcal{V} -*colimit* if this bijection is induced by a \mathcal{V} -natural isomorphism in \mathcal{V}

$$\mathcal{M} \left(\text{colim}_{d \in D} F(d), Y \right) \cong \lim_{d \in D} \mathcal{M}(F(d), Y),$$

where now the limit on the right is an ordinary limit in the category \mathcal{V} .

When the coproducts and coequalizer in (11.2) are \mathcal{V} -colimits, it is proven in [Kel82, §3.10] that the tensor product of functors coincides with the *weighted colimit of F weighted by G* , as defined there. The general definition of weighted colimits includes tensors and \mathcal{V} -colimits as special cases, but we will not need this definition, so we refer the reader to [Kel82, §3] for its development. A \mathcal{V} -category \mathcal{M} is said to be *cocomplete* if it is tensored and admits all small \mathcal{V} -colimits; this is equivalent to its admitting all small weighted colimits.

If \mathcal{M} is cotensored as well, the distinction between ordinary colimits in \mathcal{M}_0 and \mathcal{V} -colimits vanishes; see [Kel82, §3.8]. This applies in particular to the \mathcal{V} -category

\mathcal{V} , so that all colimits in \mathcal{V} are \mathcal{V} -colimits. Thus, if \mathcal{M} is both tensored and cotensored, it is cocomplete if and only if \mathcal{M}_0 is cocomplete in the ordinary sense. This will usually be the case for our \mathcal{V} -categories.

Dually, of course, we define a limit to be a \mathcal{V} -limit when its defining bijection is induced from a \mathcal{V} -natural isomorphism in \mathcal{V} . When products and equalizers are \mathcal{V} -limits, the cotensor product $\{W, F\}^{\mathcal{D}}$ of functors $F: \mathcal{D} \rightarrow \mathcal{M}$ and $W: \mathcal{D} \rightarrow \mathcal{V}$ coincides with the *weighted limit of F weighted by G* . We say a \mathcal{V} -category \mathcal{M} is *complete* if it is cotensored and admits all small \mathcal{V} -limits; this is equivalent to admitting all small weighted limits. If \mathcal{M} is both tensored and cotensored, completeness of \mathcal{M} is equivalent to ordinary completeness of \mathcal{M}_0 .

We can perform a construction similar to (11.2) on two functors $F_1: \mathcal{D} \rightarrow \mathcal{M}$ and $F_2: \mathcal{D} \rightarrow \mathcal{M}$, both landing in \mathcal{M} , by replacing the tensor or cotensor with the enriched hom-objects of \mathcal{M} . This gives an object of \mathcal{V}

$$(11.4) \quad \mathcal{M}^{\mathcal{D}}(F_1, F_2) = \text{eq} \left(\prod_d \mathcal{M}(F_1 d, F_2 d) \rightrightarrows \prod_{d, d'} \mathcal{M}(\mathcal{D}(d, d') \odot F_1 d, F_2 d') \right),$$

which we think of as the \mathcal{V} -object of \mathcal{V} -natural transformations from F_1 to F_2 . These are the hom-objects of a \mathcal{V} -category $\mathcal{M}^{\mathcal{D}}$ whose objects are \mathcal{V} -functors $\mathcal{D} \rightarrow \mathcal{M}$; its underlying category has \mathcal{V} -natural transformations as its morphisms. When \mathcal{M} is cocomplete, the G -weighted colimit for a fixed $G: \mathcal{D}^{op} \rightarrow \mathcal{V}$ extends to a \mathcal{V} -functor $\mathcal{M}^{\mathcal{D}} \rightarrow \mathcal{M}$, and similarly for weighted limits.

Now suppose that \mathcal{M}_0 is a homotopical category. Then $(\mathcal{M}^{\mathcal{D}})_0$ is a homotopical category in which the weak equivalences are objectwise. As in the unenriched case, if \mathcal{M}_0 is saturated, then so is $(\mathcal{M}^{\mathcal{D}})_0$. We may then ask, for a given weight G , does the underlying **Set**-functor $(\mathcal{M}^{\mathcal{D}})_0 \rightarrow \mathcal{M}_0$ of the G -weighted limit or colimit have a (global) derived functor? In other words, is there a *weighted homotopy colimit*?

Pure model category theory has little to say about this question. Even when \mathcal{V}_0 and \mathcal{M}_0 are model categories, related in the most favorable ways known, the category $(\mathcal{M}^{\mathcal{D}})_0$ may not have any relevant model structure. When \mathcal{M}_0 is cofibrantly generated, then $(\mathcal{M}^{\mathcal{D}})_0$ often has a projective model structure, although some smallness and cofibrancy conditions are necessary; we will consider this model structure in Theorem 24.4 for a different reason. However, cofibrant replacements in this model structure are as unwieldy as ever. Moreover, injective and Reedy model structures have no known analogues in the enriched context.

The more technical solution of [DHKS04] does not seem to generalize at all to the enriched case. There seems no way to define a category of simplices for a \mathcal{V} -category \mathcal{D} , so we can't even get off the ground. However, the results of §9 suggest a different approach: define an enriched two-sided bar construction and show that it satisfies the universal property to be a derived functor. We carry out this idea in the next two sections.

12. THE ENRICHED TWO-SIDED BAR CONSTRUCTION

We now proceed to construct enriched homotopy tensor and cotensor products of functors, which will give us weighted homotopy limits and colimits as a special case. As alluded to previously, our approach is to give explicit definitions using bar and cobar constructions and then prove that they are, in fact, derived functors.

We begin by introducing the enriched two-sided bar construction. Henceforth, we assume all our (large) \mathcal{V} -categories are complete and cocomplete.

Definition 12.1. Let \mathcal{M} be a \mathcal{V} -category, let \mathcal{D} be a small \mathcal{V} -category, and let $G: \mathcal{D}^{op} \rightarrow \mathcal{V}$ and $F: \mathcal{D} \rightarrow \mathcal{M}$ be \mathcal{V} -functors. The *two-sided simplicial bar construction* is a simplicial object of \mathcal{M} whose object function is

$$B_n(G, \mathcal{D}, F) = \coprod_{\alpha: [n] \rightarrow \mathcal{D}} (G(\alpha_n) \otimes \mathcal{D}(\alpha_{n-1}, \alpha_n) \otimes \dots \otimes \mathcal{D}(\alpha_0, \alpha_1)) \odot F(\alpha_0)$$

and whose faces and degeneracies are defined using composition in \mathcal{D} , the evaluation maps $\mathcal{D}(d, d') \odot F(d) \rightarrow F(d')$ and $\mathcal{D}(d, d') \otimes G(d') \rightarrow G(d)$, and insertion of identities $E \rightarrow \mathcal{D}(d, d)$.

Remark 12.2. When \mathcal{V} is not cartesian monoidal, we may not be able to define a one-sided bar construction $B(*, \mathcal{D}, F)$. We would want $*$ to be the functor constant at the unit object of \mathcal{V} , but when \mathcal{D} is enriched such a thing need not exist. Thus the two-sided version is forced on us by the enriched point of view.

We would now like to geometrically realize the enriched simplicial bar construction, as we did for the unenriched case in §9. So far, we only know how to take the geometric realization in a simplicially enriched category, but now we are already enriched over another category \mathcal{V} . Inspecting the definition of geometric realization, however, we see that all we need is a canonical functor $\Delta^\bullet: \Delta \rightarrow \mathcal{V}$. We can then define the *geometric realization* of any simplicial object X_\bullet of \mathcal{P} to be

$$|X_\bullet| = \Delta^\bullet \odot_{\Delta^{op}} X_\bullet.$$

Since \mathcal{V} is cocomplete, giving such a functor Δ is equivalent to giving an adjunction $s\mathcal{S} \rightleftarrows \mathcal{V}$ (see [Hov99, 3.1.5]). The left adjoint maps a simplicial set K to the tensor product $K_\bullet \times_\Delta \Delta^\bullet$, while the right adjoint maps an object V of \mathcal{V} to the simplicial set $\mathcal{V}(\Delta^\bullet, V)$.

We will often need to assume more about this adjunction, however. If the left adjoint $s\mathcal{S} \rightarrow \mathcal{V}$ is strong monoidal (as defined in [ML98, XI.2]), then the adjunction makes \mathcal{V} into a monoidal $s\mathcal{S}$ -category (an *s \mathcal{S} -algebra* in the sense of [Hov99, 4.1.8]). In this case, every \mathcal{V} -enriched category \mathcal{M} also becomes simplicially enriched, and if \mathcal{M} is tensored or cotensored over \mathcal{V} , it is also so over $s\mathcal{S}$. As explained after Lemma 9.7, this condition ensures that geometric realization preserves simplicial homotopy.

Furthermore, if \mathcal{V}_0 is a monoidal model category and the adjunction is Quillen (in addition to being strong monoidal), then any \mathcal{V} -model category becomes a simplicial model category. This condition is useful for us primarily because of Lemma 8.1, and because then simplicial homotopy equivalences in \mathcal{M} are necessarily weak equivalences. In a more general homotopical context, we will need to assume the latter property explicitly.

Now we can make the following definitions.

Definition 12.3. Assume the situation of Definition 12.1 and that \mathcal{V} has a canonical cosimplicial object $\Delta^\bullet: \Delta \rightarrow \mathcal{V}$. The *two-sided bar construction* is the geometric realization of the two-sided simplicial bar construction:

$$B(G, \mathcal{D}, F) = |B_\bullet(G, \mathcal{D}, F)|.$$

As in the unenriched case, we have the following result.

Lemma 12.4.

$$\begin{aligned} B(G, \mathcal{D}, F) &\cong G \otimes_{\mathcal{D}} B(\mathcal{D}, \mathcal{D}, F) \\ &\cong B(G, \mathcal{D}, \mathcal{D}) \odot_{\mathcal{D}} F. \end{aligned}$$

Proof. Just like the proofs of Lemma 9.7 and Lemma 7.9. \square

This result means that $B(G, \mathcal{D}, F)$, which we think of as an “uncorrected weighted homotopy colimit”, is in fact an ordinary weighted colimit with a “fattened-up” weight $B(G, \mathcal{D}, \mathcal{D})$. This should be familiar from the unenriched case, when we saw that the uncorrected homotopy colimit could be defined as an ordinary weighted colimit.

13. WEIGHTED HOMOTOPY COLIMITS, I

We think of the two-sided bar construction introduced in the last section as an *uncorrected homotopy tensor product of functors*. In this section, we will construct a corrected version and show that it defines a derived tensor product.

Recall that in the unenriched case, the bar construction (uncorrected homotopy colimit) could easily be shown to preserve weak equivalences between objectwise cofibrant diagrams. The purpose of the ‘correction’ was to construct from it a functor preserving *all* weak equivalences, simply by first replacing all diagrams by objectwise cofibrant ones. We want to do the same thing in the enriched case.

Thus, the first thing we need is a notion of weak equivalence for our \mathcal{V} -categories. In §16 we will consider a very well-behaved notion of “enriched homotopical category”, but for now, to convey the basic ideas and results, we simply suppose that the underlying categories \mathcal{V}_0 and \mathcal{M}_0 are homotopical categories. We also assume in this section that \mathcal{V} comes equipped with a strong monoidal adjunction $s\mathcal{S} \rightleftarrows \mathcal{V}$, as in §12, so that we have a notion of geometric realization in \mathcal{V} -categories which preserves simplicial homotopy.

There are two technical issues in the enriched context that did not arise in the unenriched case. The first is that we need to assume some cofibrancy conditions on our shape category \mathcal{D} in order for even the *uncorrected* bar construction to be well-behaved. These conditions can be somewhat technical, so for now we sidestep the issue by defining \mathcal{D} to be “good” if the bar construction over \mathcal{D} turns out to be well-behaved on objectwise cofibrant diagrams.

In §23 we will discuss what sort of “cofibrancy” we must impose on \mathcal{D} to make this true. We will prove, for example, that for a \mathcal{V} -model categories, any suitably cofibrant \mathcal{V} -category is good. However, in many situations weaker conditions suffice. For example, in topological situations, instead of cofibrancy in the model structure, a sort of Hurewicz cofibrancy is usually enough. The basic idea is that the simplicial bar construction is “Reedy cofibrant” in a suitable sense, just as in Lemma 9.2, but by writing this out explicitly we can make it apply to types of “cofibration” that aren’t necessarily part of a model structure.

We now give the formal definition of goodness. If \mathcal{M}_Q is a full subcategory of \mathcal{M} , such as a deformation retract, we agree to write $\mathcal{M}_Q^{\mathcal{D}}$ for the full sub- \mathcal{V} -category of $\mathcal{M}^{\mathcal{D}}$ consisting of the functors which are objectwise in \mathcal{M}_Q . These will be the counterpart of the “objectwise cofibrant” diagrams on which we expect the bar construction to be well-behaved.

Definition 13.1. We say that \mathcal{D} is *good* for the tensor \odot of \mathcal{M} if there exist left deformation retracts \mathcal{V}_Q and \mathcal{M}_Q of \mathcal{V} and \mathcal{M} , respectively, such that the following conditions hold:

- $B(-, \mathcal{D}, -)$ is homotopical on $\mathcal{V}_Q^{\mathcal{D}^{op}} \times \mathcal{M}_Q^{\mathcal{D}}$;
- If $F \in \mathcal{M}_Q^{\mathcal{D}}$ and $G \in \mathcal{V}_Q^{\mathcal{D}^{op}}$, then

- $B(\mathcal{D}, \mathcal{D}, F) \in \mathcal{M}_Q^{\mathcal{D}}$ and
- $B(G, \mathcal{D}, \mathcal{D}) \in \mathcal{V}_Q^{\mathcal{D}^{op}}$.

The second technical issue that arises in the enriched situation is that in general we may not be able to replace a diagram by one which is objectwise good. Even though there is a deformation Q into \mathcal{M}_Q , the functor Q is not in general a \mathcal{V} -functor, so we cannot apply it objectwise to a \mathcal{V} -functor $F \in \mathcal{M}^{\mathcal{D}}$ to produce a \mathcal{V} -functor $QF \in \mathcal{M}_Q^{\mathcal{D}}$. The problem is that most functorial deformations are constructed using the small-object argument, which involves taking a coproduct over a *set* of maps, thereby losing all information about the enrichment.

We do not have a fully satisfactory solution to this problem. However, there are many cases in which an objectwise cofibrant replacement can be constructed. Sometimes the deformation Q can be made enriched and applied objectwise, while in other cases we can replace diagrams by objectwise cofibrant ones without doing it objectwise. Situations in which objectwise cofibrant replacements exist include the following.

- When \mathcal{D} is unenriched.
- When $\mathcal{M}_Q = \mathcal{M}$ (every object is cofibrant). Note that the dual of this situation, $\mathcal{M}_R = \mathcal{M}$, frequently occurs in topology; this makes homotopy limits often easier to deal with than homotopy colimits.
- When \mathcal{M} is a cofibrantly generated \mathcal{V} -model category and every object of \mathcal{V} is cofibrant (for example, when $\mathcal{V} = s\mathcal{S}$).
- When $\mathcal{M}^{\mathcal{D}}$ has a good projective model structure.

In §24 we will consider various partial solutions of this sort. For now, we again sidestep the issue by assuming, when necessary, that such a replacement is possible. Formally, what we assume is that the subcategories $\mathcal{V}_Q^{\mathcal{D}^{op}}$ and $\mathcal{M}_Q^{\mathcal{D}}$ are left deformation retracts of $\mathcal{V}^{\mathcal{D}^{op}}$ and $\mathcal{M}^{\mathcal{D}}$, respectively. We use the notations $Q_{\mathcal{V}}^{\mathcal{D}^{op}}$ and $Q_{\mathcal{M}}^{\mathcal{D}}$ for a corresponding pair of left deformations.

In many of these cases, the deformations $Q_{\mathcal{V}}^{\mathcal{D}^{op}}$ and $Q_{\mathcal{M}}^{\mathcal{D}}$ are produced by small object arguments, and therefore can be difficult to work with. However, if our given functors F and G are already in $\mathcal{M}_Q^{\mathcal{D}}$ and $\mathcal{V}_Q^{\mathcal{D}^{op}}$, then the corrected homotopy tensor product to be defined below is weakly equivalent to the uncorrected version $B(G, \mathcal{D}, F)$. Thus in this case it is not necessary to apply the replacement first.

Finally, with all these technical details firmly shoved under the rug, we can construct global homotopy tensor products.

Definition 13.2. Let \mathcal{D} be a small \mathcal{V} -category which is good for the tensor \odot of \mathcal{M} , and assume that $\mathcal{V}_Q^{\mathcal{D}^{op}}$ and $\mathcal{M}_Q^{\mathcal{D}}$ are left deformation retracts of $\mathcal{V}^{\mathcal{D}^{op}}$ and $\mathcal{M}^{\mathcal{D}}$, respectively, with corresponding left deformations $Q_{\mathcal{V}}^{\mathcal{D}^{op}}$ and $Q_{\mathcal{M}}^{\mathcal{D}}$. Define the *corrected homotopy tensor product* of $F \in \mathcal{M}^{\mathcal{D}}$ and $G \in \mathcal{V}^{\mathcal{D}^{op}}$ to be

$$(13.3) \quad G \overset{\mathbb{L}}{\odot}_{\mathcal{D}} F = B(Q_{\mathcal{V}}^{\mathcal{D}^{op}} G, \mathcal{D}, Q_{\mathcal{M}}^{\mathcal{D}} F).$$

The cocompleteness of \mathcal{M} ensures that the corrected homotopy tensor product defines a functor

$$\overset{\mathbb{L}}{\odot}_{\mathcal{D}}: (\mathcal{V}^{\mathcal{D}^{op}})_0 \times (\mathcal{M}^{\mathcal{D}})_0 \longrightarrow \mathcal{M}_0.$$

This is the functor that we want to show is the left derived functor of $\odot_{\mathcal{D}}$. Note that since $Q_{\mathcal{V}}^{\mathcal{D}^{op}}$ and $Q_{\mathcal{M}}^{\mathcal{D}}$ will not in general be \mathcal{V} -functors, this bifunctor is *not* in

general induced by a \mathcal{V} -bifunctor $\mathcal{V}^{\mathcal{D}^{\text{op}}} \otimes \mathcal{M}^{\mathcal{D}} \rightarrow \mathcal{M}$. We will have more to say about this later.

The main idea of the proof is the same as for the unenriched case in §9: we show that the deformations $Q_{\mathcal{V}}^{\mathcal{D}^{\text{op}}}$ and $Q_{\mathcal{M}}^{\mathcal{D}}$, followed by $B(\mathcal{D}, \mathcal{D}, -)$, give a left deformation for the tensor product. We first introduce a notation for the corresponding deformation retracts.

Notation 13.4. Recall that we write $\mathcal{M}_Q^{\mathcal{D}}$ for the full subcategory of $\mathcal{M}^{\mathcal{D}}$ determined by the diagrams which are objectwise in \mathcal{M}_Q . We write $\mathcal{M}_{BQ}^{\mathcal{D}}$ for the full image of $B(\mathcal{D}, \mathcal{D}, -)$ applied to $\mathcal{M}_Q^{\mathcal{D}}$. Thus, $\mathcal{M}_{BQ}^{\mathcal{D}}$ is the full subcategory of $\mathcal{M}^{\mathcal{D}}$ determined by diagrams isomorphic to a diagram of the form $B(\mathcal{D}, \mathcal{D}, F)$ for some $F \in \mathcal{M}_Q^{\mathcal{D}}$. Similarly, using $B(-, \mathcal{D}, \mathcal{D})$, we have $\mathcal{V}_{BQ}^{\mathcal{D}^{\text{op}}}$.

The assumptions of goodness ensure that $\mathcal{V}_{BQ}^{\mathcal{D}^{\text{op}}} \subset \mathcal{V}_Q^{\mathcal{D}^{\text{op}}}$ and $\mathcal{M}_{BQ}^{\mathcal{D}} \subset \mathcal{M}_Q^{\mathcal{D}}$. When these are inclusions of left deformation retracts, and when $\mathcal{V}_Q^{\mathcal{D}^{\text{op}}}$ and $\mathcal{M}_Q^{\mathcal{D}}$ are also left deformation retracts of $\mathcal{V}^{\mathcal{D}^{\text{op}}}$ and $\mathcal{M}^{\mathcal{D}}$, respectively (as discussed above), we can conclude that $\mathcal{V}_{BQ}^{\mathcal{D}^{\text{op}}}$ and $\mathcal{M}_{BQ}^{\mathcal{D}}$ are also left deformation retracts of $\mathcal{V}^{\mathcal{D}^{\text{op}}}$ and $\mathcal{M}^{\mathcal{D}}$. The following lemma shows when this holds.

Lemma 13.5. *For any \mathcal{V} -functor $F: \mathcal{D} \rightarrow \mathcal{M}$, there is a natural map*

$$\varepsilon: B(\mathcal{D}, \mathcal{D}, F) \longrightarrow F.$$

which is an objectwise simplicial homotopy equivalence. Therefore, if simplicial homotopy equivalences in \mathcal{M}_0 are weak equivalences, it is an objectwise weak equivalence, and so $B(\mathcal{D}, \mathcal{D}, -)$ is a left deformation.

Proof. Exactly the same as for Lemma 9.9. □

Recall, as remarked after Lemma 9.7, that if \mathcal{M} is a simplicial model category, then simplicial homotopy equivalences are necessarily weak equivalences. This is the case if \mathcal{V} is a simplicial monoidal model category and \mathcal{M} is a \mathcal{V} -model category, for example.

Remark 13.6. Note from the proof of Lemma 9.9 that the map ε is induced by tensoring with F from a natural weak equivalence $\varepsilon: B(\mathcal{D}, \mathcal{D}, \mathcal{D}) \rightarrow \mathcal{D}$. Moreover, $B_{\bullet}(\mathcal{D}, \mathcal{D}, \mathcal{D})$ has *two* extra degeneracies—one on each side—and thus this latter map ε is a weak equivalence for two different reasons. However, each of those reasons is only natural on one side, so only one of them can be used to prove that $\varepsilon: B(\mathcal{D}, \mathcal{D}, F) \rightarrow F$ is a weak equivalence. Dually, however, the other can prove that $\varepsilon: B(G, \mathcal{D}, \mathcal{D}) \rightarrow G$ is a weak equivalence. This fact will be used in the proof of Theorem 13.7 which follows.

We can now prove the general theorems. Notice that we must choose, in defining the deformation, whether to fatten up F or G , but it doesn't matter which. This is analogous to the fact that in defining a derived tensor product of modules, we can replace either module by a resolution.

Theorem 13.7. *Let \mathcal{M} be a \mathcal{V} -category and \mathcal{D} a small \mathcal{V} -category, and make the following assumptions:*

- \mathcal{M}_0 and \mathcal{V}_0 are homotopical categories;
- \mathcal{D} is good for the tensor \odot of \mathcal{M} ;
- Simplicial homotopy equivalences in \mathcal{V} and \mathcal{M} are weak equivalences; and

- $\mathcal{V}_Q^{\mathcal{D}^{op}}$ and $\mathcal{M}_Q^{\mathcal{D}}$ are left deformation retracts of $\mathcal{V}^{\mathcal{D}^{op}}$ and $\mathcal{M}^{\mathcal{D}}$, respectively.

Then

$$(13.8) \quad (\mathcal{V}_Q^{\mathcal{D}^{op}})_0 \times (\mathcal{M}_{BQ}^{\mathcal{D}})_0$$

is a left deformation retract for the functor

$$(13.9) \quad \odot_{\mathcal{D}}: (\mathcal{V}^{\mathcal{D}^{op}})_0 \times (\mathcal{M}^{\mathcal{D}})_0 \longrightarrow \mathcal{M}_0.$$

Therefore the corrected homotopy tensor product

$$\mathbb{L} \odot_{\mathcal{D}}: (\mathcal{V}^{\mathcal{D}^{op}})_0 \times (\mathcal{M}^{\mathcal{D}})_0 \longrightarrow \mathcal{M}_0$$

is a left derived functor of (13.9).

Proof. By Lemma 13.5, the subcategory (13.8) is a left deformation retract of

$$(13.10) \quad (\mathcal{V}_Q^{\mathcal{D}^{op}})_0 \times (\mathcal{M}_Q^{\mathcal{D}})_0.$$

But by assumption, this is in turn a left deformation retract of the whole category

$$(13.11) \quad (\mathcal{V}^{\mathcal{D}^{op}})_0 \times (\mathcal{M}^{\mathcal{D}})_0.$$

Since a left deformation retract of a left deformation retract is a left deformation retract, it follows that (13.8) is a left deformation retract of (13.11). Thus, it suffices to show that $\odot_{\mathcal{D}}$ is homotopical on the subcategory (13.8).

For brevity, write $B_{\mathcal{D}}F = B(\mathcal{D}, \mathcal{D}, F)$. Then by Lemma 13.5, the functor $\text{Id} \times B_{\mathcal{D}}$ is a left deformation of (13.10) into (13.8), and since \mathcal{D} is assumed good, the composite

$$\odot_{\mathcal{D}} \circ (\text{Id} \times B_{\mathcal{D}})$$

is homotopical on (13.10). Thus, by Remark 3.2, it suffices to show that $\odot_{\mathcal{D}}$ preserves the weak equivalence

$$(1, \varepsilon B_{\mathcal{D}}): (G, B_{\mathcal{D}}B_{\mathcal{D}}F) \xrightarrow{\sim} (G, B_{\mathcal{D}}F).$$

for $F \in \mathcal{M}_Q^{\mathcal{D}}$ and $G \in \mathcal{V}_Q^{\mathcal{D}^{op}}$. When we apply $\odot_{\mathcal{D}}$ to $(1, \varepsilon B_{\mathcal{D}})$, we obtain the left-hand map in the following diagram.

$$\begin{array}{ccc} G \odot_{\mathcal{D}} B(\mathcal{D}, \mathcal{D}, \mathcal{D}) \odot_{\mathcal{D}} B(\mathcal{D}, \mathcal{D}, F) & \xrightarrow{\cong} & B(G, \mathcal{D}, \mathcal{D}) \odot_{\mathcal{D}} B(\mathcal{D}, \mathcal{D}, F) \\ \downarrow 1 \odot \varepsilon \odot 1 & & \downarrow \varepsilon \odot 1 \\ G \odot_{\mathcal{D}} \mathcal{D}(-, -) \odot_{\mathcal{D}} B(\mathcal{D}, \mathcal{D}, F) & \xrightarrow{\cong} & G \odot_{\mathcal{D}} B(\mathcal{D}, \mathcal{D}, F) \end{array}$$

But because \mathcal{D} is good, the map $\varepsilon: B(G, \mathcal{D}, \mathcal{D}) \rightarrow G$ is a weak equivalence between diagrams in $\mathcal{V}_Q^{\mathcal{D}^{op}}$. (Here we use Remark 13.6 and the ‘‘other’’ extra degeneracy of ε .) Therefore, once again because \mathcal{D} is good, the right-hand map is a weak equivalence, and thus so is the left-hand one, as desired. \square

Corollary 13.12. *Under the hypotheses of Theorem 13.7, if $G \in \mathcal{V}_Q^{\mathcal{D}^{op}}$, then the functor*

$$G \mathbb{L} \odot_{\mathcal{D}} - = B(G, \mathcal{D}, Q-)$$

is a left derived functor of the weighted colimit $G \odot_{\mathcal{D}} -$.

Proof. Straightforward from Theorem 13.7. \square

Applying Corollary 13.12 to the case $\mathcal{V} = s\mathcal{S}$ and $G = *$, we have completed the proof of Theorem 9.1, since in the unenriched case, we can apply cofibrant and fibrant approximation functors objectwise, so that $\mathcal{M}_Q^\mathcal{D}$ and $\mathcal{M}_R^\mathcal{D}$ are always deformation retracts of $\mathcal{M}^\mathcal{D}$.

The same methods applied to the cobar construction will produce homotopy weighted limits. The proofs are all dual, so we merely state the main definition and results.

Definition 13.13. Say that \mathcal{D} is *good* for the cotensor $\{-, -\}$ of \mathcal{M} if there exist a left deformation retract \mathcal{V}_Q of \mathcal{V} and a right deformation retract \mathcal{M}_R of \mathcal{M} such that the following conditions hold:

- $C(-, \mathcal{D}, -)$ is homotopical on $\mathcal{V}_Q^\mathcal{D} \times \mathcal{M}_R^\mathcal{D}$;
- If $F \in \mathcal{M}_R^\mathcal{D}$ and $G \in \mathcal{V}_Q^\mathcal{D}$, then
 - $C(\mathcal{D}, \mathcal{D}, F) \in \mathcal{M}_R^\mathcal{D}$ and
 - $B(G, \mathcal{D}, \mathcal{D}) \in \mathcal{V}_Q^\mathcal{D}$.

We write $\mathcal{M}_{CR}^\mathcal{D}$ for the full image of $C(\mathcal{D}, \mathcal{D}, -)$ applied to $\mathcal{M}_R^\mathcal{D}$.

Theorem 13.14. *Let \mathcal{M} be a \mathcal{V} -category and \mathcal{D} a small \mathcal{V} -category, and make the following assumptions:*

- \mathcal{M}_0 and \mathcal{V}_0 are homotopical categories;
- \mathcal{D} is good for the cotensor $\{-, -\}$ of \mathcal{M} ;
- Simplicial homotopy equivalences in \mathcal{V} and \mathcal{M} are weak equivalences; and
- $\mathcal{V}_Q^\mathcal{D}$ is a left deformation retract of $\mathcal{V}^\mathcal{D}$ and $\mathcal{M}_R^\mathcal{D}$ is a right deformation retract of $\mathcal{M}^\mathcal{D}$.

Then

$$(13.15) \quad (\mathcal{V}_Q^\mathcal{D})_0^{op} \times (\mathcal{M}_{CR}^\mathcal{D})_0$$

is a right deformation retract for the functor

$$(13.16) \quad \{-, -\}^\mathcal{D}: (\mathcal{V}^\mathcal{D})_0^{op} \times (\mathcal{M}^\mathcal{D})_0 \longrightarrow \mathcal{M}_0.$$

Therefore the corrected homotopy cotensor product

$$\mathbb{R}\{-, -\}^\mathcal{D}: (\mathcal{V}^\mathcal{D})_0^{op} \times (\mathcal{M}^\mathcal{D})_0 \longrightarrow \mathcal{M}_0$$

is a right derived functor of (13.16).

Corollary 13.17. *Under the hypotheses of Theorem 20.7, if $G \in \mathcal{V}_Q^\mathcal{D}$, then the functor*

$$\mathbb{R}\{G, -\}^\mathcal{D} = C(G, \mathcal{D}, R-)$$

is a right derived functor of the weighted limit $\{G, -\}^\mathcal{D}$.

14. ENRICHED TWO-VARIABLE ADJUNCTIONS

The results in the last section convey the basic point of this paper: that the bar and cobar constructions, suitably corrected, define homotopy colimits and limits in the “global” derived sense. These results are sufficient for most applications. However, they are still not fully satisfactory, since the original functor $\odot_\mathcal{D}$ was actually a \mathcal{V} -functor, but our derived functor $\mathbb{L}\odot_\mathcal{D}$ is not enriched.

The importance of enrichment on homotopy categories has only been realized relatively recently. One reason for this is that it was not until the introduction of

closed symmetric monoidal categories of spectra in the 1990s that stable homotopy theory could be done in a fully enriched way. See [SS03b, SS02, Dug06] for some examples of the advantages of this perspective.

Thus, we would like to show that the derived functor $\overset{\mathbb{L}}{\odot}_{\mathcal{D}}$ is itself enriched in a suitable sense. To achieve this, in the next few sections we develop a theory of “enriched homotopical categories”, which we then leverage to prove that at least the total derived functor $\overset{\mathbb{L}}{\odot}_{\mathcal{D}}$ can be enriched over the homotopy category $\mathrm{Ho}\mathcal{V}$. We will then use this to prove a number of useful results about homotopy limits and colimits and the homotopy theory of enriched diagrams.

To get a feel for what enriched homotopical categories should look like, we consider first the case of model categories. Even in this case there is no known truly satisfactory notion of an “enriched model category,” only of an enriched category with a model structure on its underlying ordinary category that interacts well with the enrichment. The most convenient way to ensure this is with Quillen conditions on the enriched-hom, tensor, and cotensor. The condition on the enriched-hom was first written down by Quillen himself for the special case $\mathcal{V} = s\mathcal{S}$.

The case of more general \mathcal{V} is considered in [Hov99, ch. 4], in which are defined the notions of *monoidal model category*, *\mathcal{V} -model category*, *\mathcal{V} -Quillen pair*, and so on. Moreover, it is proven that the homotopy category of a \mathcal{V} -model category is enriched over $\mathrm{Ho}\mathcal{V}$, as is the derived adjunction of a \mathcal{V} -Quillen pair, and so on.

Remark 14.1. In [LM04] Lewis and Mandell have described a more general notion of an enriched model category which does not require that it be tensored and cotensored. However, since model categories are usually assumed to have enough limits and colimits, it seems reasonable in the enriched case to assume the existence of weighted limits and colimits, including tensors and cotensors. This is what we will do.

Our notion of “enriched homotopical category” is related to this notion of enriched model category in the same way that deformable functors are related to Quillen functors. Namely, instead of lifting and extension conditions, we assume directly the existence of certain subcategories on which weak equivalences are preserved. In §16 we will give this definition and prove corresponding results about homotopy categories and derived functors, but first we need to build up some technical tools for constructing enrichments. There is no homotopy theory in this section, only (enriched) category theory. We omit standard categorical proofs, but the reader is encouraged to work at least some of them out.

The following notion is the basic tool we will be using.

Definition 14.2 ([Hov99, ch. 4]). Let \mathcal{M}_0 , \mathcal{N}_0 , and \mathcal{P}_0 be unenriched categories. An *adjunction of two variables* $(\otimes, \mathrm{hom}_\ell, \mathrm{hom}_r) : \mathcal{M}_0 \times \mathcal{N}_0 \rightarrow \mathcal{P}_0$ consists of bifunctors

$$\begin{aligned} \otimes &: \mathcal{M}_0 \times \mathcal{N}_0 \rightarrow \mathcal{P}_0 \\ \mathrm{hom}_\ell &: \mathcal{M}_0^{\mathrm{op}} \times \mathcal{P}_0 \rightarrow \mathcal{N}_0 \\ \mathrm{hom}_r &: \mathcal{N}_0^{\mathrm{op}} \times \mathcal{P}_0 \rightarrow \mathcal{M}_0 \end{aligned}$$

together with natural isomorphisms:

$$\mathcal{P}_0(M \otimes N, P) \cong \mathcal{M}_0(N, \mathrm{hom}_\ell(M, P)) \cong \mathcal{N}_0(M, \mathrm{hom}_r(N, P)).$$

If \mathcal{V}_0 is a closed symmetric monoidal category and \mathcal{M} is a tensored and cotensored \mathcal{V} -category, as in §11, then the tensor, cotensor, and hom-objects define a two-variable adjunction $\mathcal{V}_0 \times \mathcal{M}_0 \rightarrow \mathcal{M}_0$. In particular, this applies to $\mathcal{M} = \mathcal{V}$. We encourage the reader to keep these examples in mind. We have chosen the (perhaps peculiar-looking) notation \oplus for a general two-variable adjunction to avoid conflict with the monoidal product \otimes of \mathcal{V} and the tensor \odot of \mathcal{M} , since sometimes all three will occur in the same formula.

We can use two-variable adjunctions to give alternate characterizations of enriched structures. For instance, a closed symmetric monoidal category is essentially a symmetric monoidal category whose monoidal product \otimes is part of a two-variable adjunction. Similarly, for enriched categories, we have the following notion from [Hov99, ch. 4].

Definition 14.3. Let \mathcal{V}_0 be a closed symmetric monoidal category. A *closed \mathcal{V} -module* \mathcal{M} consists a category \mathcal{M}_0 together with the following data.

- A two-variable adjunction $(\odot, \{\}, \text{hom}): \mathcal{V}_0 \times \mathcal{M}_0 \rightarrow \mathcal{M}_0$;
- A natural isomorphism $K \odot (L \odot M) \cong (K \otimes L) \odot M$; and
- A natural isomorphism $E \odot M \cong M$;

such that three obvious coherence diagrams commute.

Given a closed \mathcal{V} -module, there is an essentially unique way to enrich \mathcal{M}_0 over \mathcal{V} (using the given hom to define the hom-objects) such that the given two-variable adjunction becomes the tensor-hom-cotensor adjunction. Thus in the sequel, we frequently blur the distinction between closed \mathcal{V} -modules and tensored-cotensored \mathcal{V} -categories. Note that our closed \mathcal{V} -modules will all be “left” modules, in contrast to the “right” modules of [Hov99].

This correspondence carries over to functors as well, via the following categorical observations.

Proposition 14.4. Let \mathcal{M} and \mathcal{N} be closed \mathcal{V} -modules and $F_0: \mathcal{M}_0 \rightarrow \mathcal{N}_0$ an ordinary functor. Then the following data are equivalent.

- (i) A \mathcal{V} -functor $F: \mathcal{M} \rightarrow \mathcal{N}$ whose underlying ordinary functor is F_0 .
- (ii) A natural transformation $m: K \odot F_0 X \rightarrow F_0(K \odot X)$ satisfying associativity and unit axioms.
- (iii) A natural transformation $n: F_0(\{K, X\}) \rightarrow \{K, F_0 Y\}$ satisfying associativity and unit axioms.

We call any of the above data an *enrichment* of F_0 . If in (ii) m is an isomorphism, we say that F is a *colax \mathcal{V} -module functor*; this is the only one of these notions considered in [Hov99]. Colax \mathcal{V} -module functors correspond to enrichments of F_0 which preserve tensors. Similarly, if in (iii) n is an isomorphism, we call F a *lax \mathcal{V} -module functor*; these correspond to enrichments which preserve cotensors.

The following technical observation will be needed in Remark 22.4.

Proposition 14.5. Let $F_0: \mathcal{M}_0 \rightarrow \mathcal{N}_0$ and let m and n be two transformations as in Proposition 14.4(ii) and (iii) such that the following diagram commutes.

$$\begin{array}{ccccc}
 \mathcal{M}_0(K \odot X, Y) & \xrightarrow{F_0} & \mathcal{N}_0(F_0(K \odot X), F_0 Y) & \xrightarrow{m} & \mathcal{N}_0(K \odot F_0 X, F_0 Y) \\
 \cong \downarrow & & & & \downarrow \cong \\
 \mathcal{M}_0(X, \{K, Y\}) & \xrightarrow{F_0} & \mathcal{N}_0(F_0 X, F_0(\{K, Y\})) & \xrightarrow{n} & \mathcal{N}_0(F_0 X, \{K, F_0 Y\})
 \end{array}$$

Then the two induced enrichments on F_0 are the same.

Most of the functors we are interested in are part of adjunctions, and since in an enriched adjunction the left adjoint automatically preserves tensors and the right adjoint preserves cotensors, these results take on an especially simple form for adjunctions.

Proposition 14.6. *Let \mathcal{M} and \mathcal{N} be closed \mathcal{V} -modules and let $F_0: \mathcal{M}_0 \rightleftarrows \mathcal{N}_0 : G_0$ be an ordinary adjunction. Then the following data are equivalent.*

- (i) *A \mathcal{V} -adjunction $F: \mathcal{M} \rightleftarrows \mathcal{N} : G$ whose underlying ordinary adjunction is the given one;*
- (ii) *A colax \mathcal{V} -module structure on F_0 ; and*
- (iii) *A lax \mathcal{V} -module structure on G_0 .*

This result can be viewed as a special case of the theory of “doctrinal adjunctions” presented in [Kel74]. In view of (i), we call such data an *enrichment* of the adjunction $F_0 \dashv G_0$. An adjunction $F_0 \dashv G_0$ together with (ii) or (iii) is also known as a *\mathcal{V} -module adjunction*. We summarize the above propositions as follows.

Proposition 14.7. *There is an equivalence between the 2-categories of closed \mathcal{V} -modules, \mathcal{V} -module adjunctions, and (suitably defined) \mathcal{V} -module transformations and of tensored-cotensored \mathcal{V} -categories, \mathcal{V} -adjunctions, and \mathcal{V} -natural transformations.*

We can obtain a “fully enriched” notion of two-variable adjunction by replacing everything in Definition 14.2 by its \mathcal{V} -analogue. There is a slight subtlety in that the “correct” definition of an enriched bifunctor does not use the cartesian product of \mathcal{V} -categories, but rather the *tensor product*. The tensor product of two \mathcal{V} -categories \mathcal{M} and \mathcal{N} , as defined in [Kel82, §1.4], is a \mathcal{V} -category $\mathcal{M} \otimes \mathcal{N}$ whose objects are pairs (M, N) where M and N are objects of \mathcal{M} and \mathcal{N} respectively, and whose hom-objects are

$$(\mathcal{M} \otimes \mathcal{N})((M, N), (M', N')) = \mathcal{M}(M, M') \otimes \mathcal{N}(N, N').$$

Note that unless \mathcal{V} is *cartesian* monoidal, we do not in general have $(\mathcal{M} \otimes \mathcal{N})_0 \simeq \mathcal{M}_0 \times \mathcal{N}_0$; all we have is a functor $\mathcal{M}_0 \times \mathcal{N}_0 \rightarrow (\mathcal{M} \otimes \mathcal{N})_0$ which is bijective on objects.

Definition 14.8. A *\mathcal{V} -adjunction of two variables* $(\otimes, \text{hom}_\ell, \text{hom}_r): \mathcal{M} \otimes \mathcal{N} \rightarrow \mathcal{P}$ between \mathcal{V} -categories consists of \mathcal{V} -bifunctors

$$\begin{aligned} \otimes &: \mathcal{M} \otimes \mathcal{N} \rightarrow \mathcal{P} \\ \text{hom}_\ell &: \mathcal{M}^{\text{op}} \otimes \mathcal{P} \rightarrow \mathcal{N} \\ \text{hom}_r &: \mathcal{N}^{\text{op}} \otimes \mathcal{P} \rightarrow \mathcal{M} \end{aligned}$$

together with \mathcal{V} -natural isomorphisms between hom-objects in \mathcal{V} :

$$\mathcal{P}(M \otimes N, P) \cong \mathcal{M}(N, \text{hom}_\ell(M, P)) \cong \mathcal{N}(M, \text{hom}_r(N, P)).$$

The two-variable adjunctions which arise in \mathcal{V} -category theory, such as the hom-tensor-cotensor adjunction, are generally two-variable \mathcal{V} -adjunctions. Moreover, just as in the one-variable case, a two-variable \mathcal{V} -adjunction $\mathcal{M} \otimes \mathcal{N} \rightarrow \mathcal{P}$ gives rise to an underlying ordinary two-variable adjunction $\mathcal{M}_0 \times \mathcal{N}_0 \rightarrow \mathcal{P}_0$, and we can recover the enriched structure from the underlying two-variable adjunction together with some “module” data, as follows.

Definition 14.9. Let \mathcal{V} be a closed symmetric monoidal category and \mathcal{M} , \mathcal{N} , and \mathcal{P} be closed \mathcal{V} -modules. A \mathcal{V} -bilinear two-variable adjunction consists of a two-variable adjunction $(\otimes, \text{hom}_\ell, \text{hom}_r): \mathcal{M}_0 \times \mathcal{N}_0 \rightarrow \mathcal{P}_0$ together with natural isomorphisms

$$\begin{aligned} m_1: K \odot (M \otimes N) &\xrightarrow{\cong} (K \odot M) \otimes N \\ m_2: K \odot (M \otimes N) &\xrightarrow{\cong} M \otimes (K \odot N) \end{aligned}$$

which make the adjunctions

$$\begin{aligned} (- \otimes N) &\dashv \text{hom}_r(N, -) \\ (M \otimes -) &\dashv \text{hom}_\ell(M, -) \end{aligned}$$

into \mathcal{V} -module adjunctions for all M and N and which are such that the following diagram commutes:

(14.10)

$$\begin{array}{ccccc} (K \otimes L) \odot (M \otimes N) & \xrightarrow{a} & K \odot (L \odot (M \otimes N)) & \xrightarrow{m_2} & K \odot (M \otimes (L \odot N)) \\ \downarrow s & & & & \downarrow m_1 \\ & & & & (K \odot M) \otimes (L \odot N) \\ & & & & \uparrow m_2 \\ (L \otimes K) \odot (M \otimes N) & \xrightarrow{a} & L \odot (K \odot (M \otimes N)) & \xrightarrow{m_1} & L \odot ((K \odot M) \otimes N) \end{array}$$

(Here a denotes use of the associativity isomorphisms of the \mathcal{V} -module structures, and s denotes use of the symmetry isomorphism of \mathcal{V}).

Remark 14.11. By Proposition 14.6, either of the morphisms m_1 and m_2 can be replaced by their conjugates n without changing the substance of the definition. For example, we could replace m_1 by an isomorphism

$$n_1: \text{hom}_r(N, \{K, P\}) \xrightarrow{\cong} \{K, \text{hom}_r(N, P)\}.$$

The Axiom (14.10) would then be replaced by a correspondingly dual version.

Proposition 14.12. A two-variable \mathcal{V} -adjunction $(\otimes, \text{hom}_\ell, \text{hom}_r): \mathcal{M} \otimes \mathcal{N} \rightarrow \mathcal{P}$ gives rise to, and can be reconstructed from, a \mathcal{V} -bilinear two-variable adjunction $\mathcal{M}_0 \times \mathcal{N}_0 \rightarrow \mathcal{P}_0$.

Proof. This is a sequence of diagram chases left to the reader. It is convenient to use the single-variable description of maps out of a tensor product of \mathcal{V} -categories given in [Kel82, §1.4]. \square

15. DERIVED TWO-VARIABLE ADJUNCTIONS

We now want to define a notion of enriched homotopical category in such a way that homotopy categories and derived functors can be enriched. The way such results are proven for enriched model categories in [Hov99] is by phrasing enrichment in terms of two-variable adjunctions, as we have done in §14, and then showing that every “Quillen two-variable adjunction” has a derived two-variable adjunction. Thus, the central notion for us will be, instead of a Quillen two-variable adjunction, a “deformable two-variable adjunction”. In this section, we study derived two-variable adjunctions and derived monoidal structures, remaining

in an unenriched context. Then in §16, we will apply these results to construct derived enrichments.

These two sections are somewhat long and technical, and the reader who is not interested in the details may safely skim them. The following two sections §17 and §18 then use these ideas to provide a perspective on the general role of enrichment in homotopy theory, before we return to the homotopy theory of diagrams in §19.

Recall from Definition 3.1 the notions of *deformation* and *deformation retract* which are used to produce derived functors. Proposition 3.6 gives us a general way to produce derived single-variable adjunctions, but we need derived two-variable adjunctions; thus we begin by defining a notion of deformation for these.

Definition 15.1. Let \mathcal{M} , \mathcal{N} , and \mathcal{P} be homotopical categories. A *deformation retract* for a two-variable adjunction $(\otimes, \text{hom}_\ell, \text{hom}_r): \mathcal{M} \times \mathcal{N} \rightarrow \mathcal{P}$ consists of left deformation retracts \mathcal{M}_Q and \mathcal{N}_Q for \mathcal{M} and \mathcal{N} , respectively, and a right deformation retract \mathcal{P}_R of \mathcal{P} , such that

- \otimes is homotopical on $\mathcal{M}_Q \times \mathcal{N}_Q$;
- hom_ℓ is homotopical on $\mathcal{M}_Q^{\text{op}} \times \mathcal{P}_R$; and
- hom_r is homotopical on $\mathcal{N}_Q^{\text{op}} \times \mathcal{P}_R$.

A *deformation* for a two-variable adjunction consists of a deformation retract, as above, together with corresponding deformations

$$\begin{array}{ll} Q_{\mathcal{M}}: \mathcal{M} \longrightarrow \mathcal{M}_Q & q_{\mathcal{M}}: Q_{\mathcal{M}} \xrightarrow{\sim} \text{Id}_{\mathcal{M}} \\ Q_{\mathcal{N}}: \mathcal{N} \longrightarrow \mathcal{N}_Q & q_{\mathcal{N}}: Q_{\mathcal{N}} \xrightarrow{\sim} \text{Id}_{\mathcal{N}} \\ R_{\mathcal{P}}: \mathcal{P} \longrightarrow \mathcal{P}_R & r_{\mathcal{P}}: \text{Id}_{\mathcal{P}} \xrightarrow{\sim} R_{\mathcal{P}} \end{array}$$

of \mathcal{M} , \mathcal{N} , and \mathcal{P} .

Of course, for a Quillen two-variable adjunction, a deformation is given by cofibrant replacements on \mathcal{M} and \mathcal{N} and a fibrant replacement on \mathcal{P} .

It is proven in [DHKS04, 40.4] than for any single deformable functor F , there is a unique maximal F -deformation retract. Unfortunately, a corresponding result is not true for two-variable adjunctions, as we will see in Counterexample 18.5. Thus, in the sequel, we do not consider “deformability” to be a *property* of a two-variable adjunction; instead, we regard a deformation retract to be *structure* it can be equipped with. However, it does follow from [DHKS04, 40.5] that once a deformation retract for a two-variable adjunction is chosen, it does not matter what deformation we choose; thus we often tacitly choose such a deformation without mentioning it.

Proposition 15.2. *Given a deformation for a two-variable adjunction as above, the functors*

$$\begin{aligned} M \overset{\mathbb{L}}{\otimes} N &= Q_{\mathcal{M}}M \otimes Q_{\mathcal{N}}N \\ \mathbb{R} \text{hom}_\ell(M, P) &= \text{hom}_\ell(Q_{\mathcal{M}}M, R_{\mathcal{P}}P) \\ \mathbb{R} \text{hom}_r(N, P) &= \text{hom}_r(Q_{\mathcal{N}}N, R_{\mathcal{P}}P) \end{aligned}$$

are derived functors of the given ones, and they descend to homotopy categories to define a derived two-variable adjunction $(\overset{\mathbb{L}}{\otimes}, \mathbb{R} \text{hom}_\ell, \mathbb{R} \text{hom}_r)$.

Proof. That they are derived functors follows trivially from Definition 15.1. To construct the derived adjunction isomorphisms, we appeal to Proposition 3.6, which tells us that deformable adjunctions descend to derived adjunctions on the homotopy categories. Therefore, for fixed M , N , and P , we have derived adjunctions:

$$\begin{aligned} (M \overset{\mathbf{L}}{\otimes} -) &\dashv \mathbf{R} \operatorname{hom}_\ell(M, -) \\ (- \overset{\mathbf{L}}{\otimes} N) &\dashv \mathbf{R} \operatorname{hom}_r(N, -) \\ \mathbf{R} \operatorname{hom}_\ell(-, P) &\dashv \mathbf{R} \operatorname{hom}_r(-, P). \end{aligned}$$

Moreover, Proposition 3.6 also tells us that the derived adjunction isomorphisms are the *unique* such natural isomorphisms which are compatible with the given adjunctions. It follows that the composition of two of the above adjunction isomorphisms is the third; hence they give rise to a unique derived two-variable adjunction. \square

We can now apply the above result to the defining two-variable adjunctions of a closed monoidal category to conclude that its homotopy category is also closed monoidal. The notation \mathcal{V}_0 for our closed monoidal category is chosen in anticipation of the enriched context of the next section, but everything is still unenriched for the time being.

Definition 15.3. Let \mathcal{V}_0 be a closed (symmetric) monoidal category which is also a homotopical category. A *closed monoidal deformation retract* for \mathcal{V}_0 is a deformation retract $(\mathcal{V}_Q, \mathcal{V}_Q, \mathcal{V}_R)$ for the two-variable adjunction

$$(\otimes, [-, -], [-, -]): \mathcal{V}_0 \times \mathcal{V}_0 \rightarrow \mathcal{V}_0$$

such that the following conditions are satisfied:

- \otimes maps $\mathcal{V}_Q \times \mathcal{V}_Q$ into \mathcal{V}_Q ;
- $[-, -]$ maps $\mathcal{V}_Q^{op} \times \mathcal{V}_R$ into \mathcal{V}_R ;
- For all $X \in \mathcal{V}_Q$, the natural maps

$$\begin{aligned} QE \otimes X &\longrightarrow E \otimes X \cong X && \text{and} \\ X \otimes QE &\longrightarrow X \otimes E \cong X \end{aligned}$$

are weak equivalences; and

- For all $Y \in \mathcal{V}_R$, the natural map

$$Y \cong [E, Y] \longrightarrow [QE, Y]$$

is a weak equivalence.

If \mathcal{V}_0 is equipped with a closed monoidal deformation retract, we say that it is a *closed (symmetric) monoidal homotopical category*.

As for monoidal model categories, the unit conditions are necessary to ensure that the unit isomorphism descends to the homotopy category. Any monoidal model category (which, recall, is defined to be closed as well) is a closed monoidal homotopical category, using cofibrant and fibrant replacements.

Proposition 15.4. *If \mathcal{V}_0 is a closed (symmetric) monoidal homotopical category, then $\operatorname{Ho}(\mathcal{V}_0)$ is a closed (symmetric) monoidal category.*

Proof. The proof is exactly like the proof for monoidal model categories given in [Hov99, 4.3.2]. By Proposition 15.2, the monoidal product and internal-hom

of \mathcal{V}_0 descend to the homotopy category and define a derived two-variable adjunction, so it remains to construct the associativity, unit, and (if \mathcal{V}_0 is symmetric) symmetry isomorphisms.

Consider the associativity constraint, which is a natural isomorphism

$$a: \otimes \circ (\otimes \times \text{Id}) \xrightarrow{\cong} \otimes \circ (\text{Id} \times \otimes).$$

Because \otimes maps $\mathcal{V}_Q \times \mathcal{V}_Q$ into \mathcal{V}_Q , these two functors have the common deformation retract $\mathcal{V}_Q \times \mathcal{V}_Q \times \mathcal{V}_Q$; therefore by Proposition 3.8, a gives rise to a derived natural isomorphism

$$\mathbf{L}a: \mathbf{L}(\otimes \circ (\otimes \times \text{Id})) \xrightarrow{\cong} \mathbf{L}(\otimes \circ (\text{Id} \times \otimes)).$$

Again because \otimes maps $\mathcal{V}_Q \times \mathcal{V}_Q$ into \mathcal{V}_Q , both pairs $((\text{Id} \times \otimes), \otimes)$ and $((\otimes \times \text{Id}), \otimes)$ are deformable. Thus by Proposition 3.10 there are canonical natural isomorphisms

$$\begin{aligned} \mathbf{L}(\otimes \circ (\otimes \times \text{Id})) &\xrightarrow{\cong} \mathbf{L} \otimes \circ (\mathbf{L} \otimes \times \text{Id}) \quad \text{and} \\ \mathbf{L}(\otimes \circ (\text{Id} \times \otimes)) &\xrightarrow{\cong} \mathbf{L} \otimes \circ (\text{Id} \times \mathbf{L} \otimes). \end{aligned}$$

Composing these, we get the desired natural associativity isomorphism on $\text{Ho}(\mathcal{V}_0)$. Similarly, $\mathcal{V}_Q \times \mathcal{V}_Q \times \mathcal{V}_Q \times \mathcal{V}_Q$ is a common deformation retract for all the functors in the pentagon axiom, hence by Proposition 3.8 its derived version still commutes on the homotopy category level.

For the unit isomorphisms, observe that by the 2-out-of-3 property, the unit axiom for a closed monoidal deformation retract ensures that the pair

$$\mathcal{V}_0 \xrightarrow{E \times \text{Id}} \mathcal{V}_0 \times \mathcal{V}_0 \xrightarrow{\otimes} \mathcal{V}_0$$

is deformable. Thus we can apply similar reasoning to construct the unit isomorphisms and verify their axioms. The symmetry isomorphism, in the case when \mathcal{V}_0 is symmetric, follows in the same way. \square

16. DERIVED ENRICHMENT

For all of this section, we assume that \mathcal{V}_0 is a closed symmetric monoidal homotopical category. Since, by Proposition 15.4, $\text{Ho}(\mathcal{V}_0)$ is also a closed symmetric monoidal category, we can consider categories enriched over $\text{Ho}(\mathcal{V}_0)$. In particular, as for any closed symmetric monoidal category, $\text{Ho}(\mathcal{V}_0)$ has an internal hom which produces a $\text{Ho}(\mathcal{V}_0)$ -enriched category, which we denote $\text{Ho } \mathcal{V}$. The underlying category of $\text{Ho } \mathcal{V}$ is precisely $\text{Ho}(\mathcal{V}_0)$; in other words, we have

$$(\text{Ho } \mathcal{V})_0 \cong \text{Ho}(\mathcal{V}_0).$$

Since there is no real ambiguity, from now on we will write $\text{Ho } \mathcal{V}_0$ for this category. In this section, we will show that the homotopy theory of \mathcal{V} -categories gives rise to derived $\text{Ho } \mathcal{V}$ -categories.

In the next few sections, we will need to be rather pedantic about the distinction between enriched categories and their underlying ordinary categories. It is common in enriched category theory, and usually harmless, to abuse notation and ignore the distinction between \mathcal{V} and \mathcal{V}_0 and between \mathcal{M} and \mathcal{M}_0 , but to avoid confusion amid the proliferation of notation, here we emphasize that distinction.

Definition 16.1. Let \mathcal{M} be a tensored and cotensored \mathcal{V} -category such that \mathcal{M}_0 is a homotopical category. A *deformation retract for the enrichment* of \mathcal{M} consists of left and right deformation retracts \mathcal{M}_Q and \mathcal{M}_R of \mathcal{M}_0 such that $(\mathcal{V}_Q, \mathcal{M}_Q, \mathcal{M}_R)$ is

a deformation retract for the two-variable adjunction $(\odot, \{\}, \text{hom}): \mathcal{V}_0 \times \mathcal{M}_0 \rightarrow \mathcal{M}_0$, and also the following conditions are satisfied.

- \odot maps $\mathcal{V}_Q \times \mathcal{M}_Q$ into \mathcal{M}_Q ;
- $\{\}$ maps $\mathcal{V}_Q^{op} \times \mathcal{M}_R$ into \mathcal{M}_R ;
- For all $X \in \mathcal{M}_Q$, the natural map

$$QE \odot X \longrightarrow E \odot X \cong X$$

is a weak equivalence; and

- For all $Y \in \mathcal{M}_R$, the natural map

$$Y \cong \{E, Y\} \longrightarrow \{QE, Y\}$$

is a weak equivalence.

If \mathcal{M} is equipped with a deformation retract for its enrichment, we say it is a \mathcal{V} -homotopical category.

There are several possible choices we could make in this definition, especially with regards to the dual pairs of axioms relating to the tensor and cotensor. We have chosen to include both for simplicity, but most results depend only on one or the other.

Note that \mathcal{V} itself (meaning the \mathcal{V} -enriched category \mathcal{V}) is a \mathcal{V} -homotopical category. Its closed monoidal deformation retract serves as a deformation retract for its self-enrichment.

Proposition 16.2. *If \mathcal{M} is a \mathcal{V} -homotopical category, then $\text{Ho}(\mathcal{M}_0)$ is the underlying category of a tensored and cotensored $\text{Ho } \mathcal{V}$ -category, which we denote $\text{Ho } \mathcal{M}$.*

Proof. By Proposition 14.7, it suffices to construct a $\text{Ho } \mathcal{V}$ -module structure on $\text{Ho } \mathcal{M}_0$. We do this in a way entirely analogous to Proposition 15.4. The tensor-hom-cotensor two-variable adjunction descends to homotopy categories because it is assumed deformable, and the other axioms of Definition 16.1 ensure that the module isomorphisms and coherence descend to derived versions. \square

As for \mathcal{V} , we have $\text{Ho}(\mathcal{M}_0) \cong (\text{Ho } \mathcal{M})_0$ so there is no ambiguity in writing $\text{Ho } \mathcal{M}_0$ for this category.

We now consider \mathcal{V} -enriched adjunctions between \mathcal{V} -homotopical categories.

Definition 16.3. Let $\Phi: \mathcal{M} \rightarrow \mathcal{N}$ be a \mathcal{V} -functor between \mathcal{V} -homotopical categories. We say Φ is *left \mathcal{V} -deformable* if

- \mathcal{M}_Q is a left deformation retract for Φ_0 ; and
- Φ maps \mathcal{M}_Q into \mathcal{N}_Q .

Similarly, we say Φ is *right \mathcal{V} -deformable* if

- \mathcal{M}_R is a right deformation retract for Φ_0 ; and
- Φ maps \mathcal{M}_R into \mathcal{N}_R .

Proposition 16.4. *If $\Phi: \mathcal{M} \rightarrow \mathcal{N}$ is left \mathcal{V} -deformable, then it has a left derived $\text{Ho } \mathcal{V}$ -functor*

$$\mathbf{L}\Phi: \text{Ho } \mathcal{M} \rightarrow \text{Ho } \mathcal{N}.$$

Similarly, if Φ is right \mathcal{V} -deformable, then it has a right derived $\text{Ho } \mathcal{V}$ -functor

$$\mathbf{R}\Phi: \text{Ho } \mathcal{M} \rightarrow \text{Ho } \mathcal{N}.$$

Proof. We assume Φ is left \mathcal{V} -deformable; the other case is dual. Since Φ_0 is left deformable, it has a left derived functor. We now construct a $\text{Ho } \mathcal{V}$ -enrichment on $\mathbf{L}\Phi_0$ using Proposition 14.4. Let

$$m: (\odot \circ (\text{Id} \times \Phi_0)) \longrightarrow (\Phi_0 \circ \odot)$$

be as in Proposition 14.4(ii) for Φ_0 . Definition 16.3 guarantees that the composites $\odot \circ (\text{Id} \times \Phi_0)$ and $\Phi_0 \circ \odot$ have a common deformation retract $(\mathcal{V}_Q, \mathcal{M}_Q)$. Thus, by Proposition 3.8, m descends to the homotopy category to give a natural transformation

$$(16.5) \quad \mathbf{L}(\odot \circ (\text{Id} \times \Phi_0)) \longrightarrow \mathbf{L}(\Phi_0 \circ \odot).$$

Definition 16.3 also guarantees that the pairs $((\text{Id} \times \Phi_0), \odot)$ and (\odot, Φ_0) are left deformable, so using Proposition 3.10, we can identify the source and target of (16.5) with the composite derived functors $\overset{\mathbf{L}}{\odot} \circ (\text{Id} \times \mathbf{L}\Phi_0)$ and $\mathbf{L}\Phi_0 \circ \overset{\mathbf{L}}{\odot}$. It is straightforward to verify the axioms to show that this transformation is an enrichment of Φ_0 , as desired. \square

Remark 16.6. Note that the point-set derived functor $\mathbb{L}\Phi_0$ is *not* in general a \mathcal{V} -functor, because we have not required deformations of \mathcal{V} -homotopical categories to be \mathcal{V} -functors. In fact, cofibrant and fibrant replacements in \mathcal{V} -model categories are not in general \mathcal{V} -functors, so this would be too much to assume. Proposition 16.4 tells us that the resulting derived functors can nevertheless be enriched “up to homotopy.”

Since most of the functors we are interested in are part of \mathcal{V} -adjunctions, from now on we restrict ourselves to such adjunctions for simplicity. The following definition is the analogue of a Quillen adjoint pair for \mathcal{V} -homotopical categories.

Definition 16.7. Let $\Phi: \mathcal{M} \rightleftarrows \mathcal{N} : \Psi$ be a \mathcal{V} -adjunction between \mathcal{V} -homotopical categories. Say the adjunction is *left \mathcal{V} -deformable* if

- \mathcal{M}_Q is a left deformation retract for Φ ;
- \mathcal{N}_R is a right deformation retract for Ψ ; and
- Φ maps \mathcal{M}_Q into \mathcal{N}_Q .

If instead of the last condition, Ψ maps \mathcal{N}_R into \mathcal{M}_R , we say the adjunction is *right \mathcal{V} -deformable*.

Proposition 16.8. *If $\Phi: \mathcal{M} \rightleftarrows \mathcal{N} : \Psi$ is left or right \mathcal{V} -deformable, then it has a $\text{Ho } \mathcal{V}$ -enriched derived adjunction $\mathbf{L}\Phi: \text{Ho } \mathcal{M} \rightleftarrows \text{Ho } \mathcal{N} : \mathbf{R}\Psi$.*

Proof. We assume the adjunction is left \mathcal{V} -deformable; the other case is dual. Since $\Phi \dashv \Psi$ is deformable, by Proposition 3.6 it has a derived adjunction. The left \mathcal{V} -deformability of the adjunction ensures that Φ is left \mathcal{V} -deformable. Moreover, since Φ is a left \mathcal{V} -adjoint, it preserves tensors, and hence is actually a colax \mathcal{V} -module functor. This implies that $\mathbf{L}\Phi_0$ is also a colax $\text{Ho } \mathcal{V}$ -module functor, so that by Proposition 14.6 we have a canonical enriched adjunction $\mathbf{L}\Phi \dashv \mathbf{R}\Psi$. \square

Proposition 16.9. *Let \mathcal{M} and \mathcal{N} be saturated \mathcal{V} -homotopical categories and let $\Phi_1: \mathcal{M} \rightleftarrows \mathcal{N} : \Psi_1$ and $\Phi_2: \mathcal{N} \rightleftarrows \mathcal{P} : \Psi_2$ be either both left \mathcal{V} -deformable or both right \mathcal{V} -deformable. Then we have $\text{Ho } \mathcal{V}$ -natural isomorphisms*

$$\begin{aligned} \mathbf{L}\Phi_2 \circ \mathbf{L}\Phi_1 &\cong \mathbf{L}(\Phi_2 \circ \Phi_1) \\ \mathbf{R}\Psi_1 \circ \mathbf{R}\Psi_2 &\cong \mathbf{R}(\Psi_1 \circ \Psi_2) \end{aligned}$$

Proof. Suppose both are left \mathcal{V} -deformable; the other case is dual. Then the pair (Φ_1, Φ_2) is left deformable in the sense of Definition 3.9, so the first isomorphism follows from Proposition 3.10. By Proposition 3.12, since \mathcal{M} and \mathcal{N} are saturated, the pair (Ψ_2, Ψ_1) is right deformable, so the second isomorphism follows also from Proposition 3.10. The fact that these isomorphisms are $\text{Ho } \mathcal{V}$ -natural can be deduced by tracing through the construction of the enrichments and the proof of Proposition 3.10 in [DHKS04]. \square

We now consider the analogue of Quillen equivalences.

Proposition 16.10. *Let $\Phi: \mathcal{M} \rightleftarrows \mathcal{N} : \Psi$ be a left or right \mathcal{V} -deformable \mathcal{V} -adjunction between saturated \mathcal{V} -homotopical categories. Then the following conditions are equivalent.*

- (i) *For all $X \in \mathcal{M}_Q$ and $Y \in \mathcal{N}_R$, a map $\Phi X \rightarrow Y$ is a weak equivalence if and only if its adjunct $X \rightarrow \Psi Y$ is.*
- (ii) *For all $X \in \mathcal{M}_Q$ and all $Y \in \mathcal{N}_R$, the composites*

$$\begin{array}{ccccc} X & \xrightarrow{\eta} & \Psi\Phi X & \longrightarrow & \Psi R\Phi X \\ \Phi Q\Psi Y & \longrightarrow & \Phi\Psi Y & \xrightarrow{\varepsilon} & Y \end{array}$$

are weak equivalences.

- (iii) *The derived adjunction $\mathbf{L}\Phi_0: \text{Ho } \mathcal{M}_0 \rightleftarrows \text{Ho } \mathcal{N}_0 : \mathbf{R}\Psi_0$ is an adjoint equivalence of categories.*
- (iv) *The derived $\text{Ho } \mathcal{V}$ -adjunction $\mathbf{L}\Phi: \text{Ho } \mathcal{M} \rightleftarrows \text{Ho } \mathcal{N} : \mathbf{R}\Psi$ is an adjoint equivalence of $\text{Ho } \mathcal{V}$ -categories.*

When these conditions hold, we say that the given adjunction is a \mathcal{V} -homotopical equivalence of \mathcal{V} -homotopical categories.

Proof. The equivalence of (i), (ii), and (iii) is the same as the corresponding result for Quillen equivalences (see, for example, [Hov99, 1.3.13]). This result is stated for general deformable adjunctions between homotopical categories in [DHKS04, §45]. Note that saturation of \mathcal{M} and \mathcal{N} is only needed for the implication (iii) \Rightarrow (i) and (ii). Finally, it follows from the definition of an equivalence of enriched categories (see, for example, [Kel82, §1.11]) that a $\text{Ho } \mathcal{V}$ -adjunction is an adjoint equivalence of $\text{Ho } \mathcal{V}$ -categories precisely when its underlying adjunction is an adjoint equivalence; this proves that (iii) is equivalent to (iv). \square

As in §14, the case of enriched two-variable adjunctions is slightly more subtle. In general, the category $(\mathcal{M} \otimes \mathcal{N})_0$ does not even have an obvious notion of weak equivalence. However, Proposition 14.12 tells us that we should really be considering the induced \mathcal{V} -bilinear two-variable adjunction $\mathcal{M}_0 \times \mathcal{N}_0 \rightarrow \mathcal{P}_0$, and $\mathcal{M}_0 \times \mathcal{N}_0$ does have a notion of weak equivalence induced from \mathcal{M}_0 and \mathcal{N}_0 . Thus we make the following definition.

Definition 16.11. Let $(\otimes, \text{hom}_\ell, \text{hom}_r): \mathcal{M} \otimes \mathcal{N} \rightarrow \mathcal{P}$ be a two-variable \mathcal{V} -adjunction between \mathcal{V} -homotopical categories and suppose that $(\mathcal{M}_Q, \mathcal{N}_Q, \mathcal{P}_R)$ is a deformation retract for the underlying two-variable adjunction $\mathcal{M}_0 \times \mathcal{N}_0 \rightarrow \mathcal{P}_0$. Then:

- If \otimes maps $\mathcal{M}_Q \times \mathcal{N}_Q$ into \mathcal{P}_Q , we say the adjunction is \otimes - \mathcal{V} -deformable;
- If hom_r maps $\mathcal{N}_Q^{\text{op}} \times \mathcal{P}_R$ into \mathcal{M}_R , we say it is hom_r - \mathcal{V} -deformable; and
- If hom_ℓ maps $\mathcal{M}_Q^{\text{op}} \times \mathcal{P}_R$ into \mathcal{N}_R , we say it is hom_ℓ - \mathcal{V} -deformable.

If any one of the above conditions obtains we say the adjunction is \mathcal{V} -deformable.

Note that the tensor-hom-cotensor two-variable \mathcal{V} -adjunction of a \mathcal{V} -homotopical category \mathcal{M} is automatically \odot - \mathcal{V} -deformable and $\{\}$ - \mathcal{V} -deformable.

Remark 16.12. All the notions we have defined so far are generalizations to homotopical categories of notions defined for model categories in [Hov99, ch. 4]. However, enriched two-variable adjunctions are not considered there. We define a *Quillen two-variable \mathcal{V} -adjunction* to be a two-variable \mathcal{V} -adjunction between \mathcal{V} -model categories whose underlying ordinary two-variable adjunction is Quillen. It is straightforward to show that such a Quillen two-variable \mathcal{V} -adjunction is all three kinds of \mathcal{V} -deformable.

Proposition 16.13. *If a two-variable \mathcal{V} -adjunction $\mathcal{M} \otimes \mathcal{N} \rightarrow \mathcal{P}$ is \mathcal{V} -deformable, then it gives rise to a derived two-variable $\text{Ho } \mathcal{V}$ -adjunction $\text{Ho } \mathcal{M} \overset{\mathbf{L}}{\otimes} \text{Ho } \mathcal{N} \rightarrow \text{Ho } \mathcal{P}$.*

Proof. Entirely analogous to Proposition 16.8, using Proposition 14.12 in place of Proposition 14.7. The \otimes - \mathcal{V} -deformable case is most straightforward; for the other two we must invoke Remark 14.11. \square

Note that we write $\text{Ho } \mathcal{M} \overset{\mathbf{L}}{\otimes} \text{Ho } \mathcal{N}$ for the tensor product of $\text{Ho } \mathcal{V}$ -categories, since the tensor product of $\text{Ho } \mathcal{V}$ is $\overset{\mathbf{L}}{\otimes}$. Remark 16.6 applies in the two-variable context as well.

Remark 16.14. While we are dealing with enriched diagrams and enriched functors, and producing $\text{Ho } \mathcal{V}$ -enriched derived functors and adjunctions, the basic notion of “derived functor” in use is still an unenriched one; that is, its universal property is still **Set**-based. The correct enriched universal property is not entirely clear, but in [LM04], Lewis and Mandell introduce some candidate universal properties for enriched derived functors and bifunctors. It would be interesting to interpret tensor products of functors in that framework, especially since monoids in a monoidal category (which they are primarily interested in) can be considered as one-object enriched categories, with left and right modules corresponding to functors of appropriate variance.

We finish this section by mentioning the following remarkable consequence of Proposition 16.2.

Corollary 16.15. *If $\text{Ho } \mathcal{V}_0$ has small hom-sets and \mathcal{M} is a \mathcal{V} -homotopical category, then $\text{Ho } \mathcal{M}_0$ also has small hom-sets.*

Proof. For any two objects M and M' of \mathcal{M} , we have

$$\begin{aligned} \text{Ho } \mathcal{M}_0(M, M') &\cong \text{Ho } \mathcal{M}_0 \left(E \overset{\mathbf{L}}{\odot} M, M' \right) \\ &\cong \text{Ho } \mathcal{V}_0(E, \mathbf{R}\mathcal{M}(M, M')). \end{aligned}$$

But the last set is a hom-set in $\text{Ho } \mathcal{V}_0$ which was assumed to be small; thus $\text{Ho } \mathcal{M}_0(M, M')$ is also small. \square

From one point of view, this result is unexpected; we seem to have gotten something for nothing. However, if we take the point of view that homotopy categories are constructed by replacing the objects by “good” ones so that weak equivalences become homotopy equivalences, and then quotienting by homotopy, we can see

that in a certain sense this is precisely what we have done. Here Q and R give the “good” replacement objects, and the notion of homotopy is that induced from the enrichment over \mathcal{V} .

This is a good example of how the presence of enrichment simplifies, rather than complicates, the study of homotopy theory. We remarked in §3 that for an unenriched category, we cannot hope to construct the homotopy category by simply deforming the hom-functor, but Corollary 16.15 shows that in the enriched situation we can often do precisely that. In the next two sections, we will see more examples of this phenomenon.

17. ENRICHED HOMOTOPY EQUIVALENCES

Let \mathcal{V} be a closed symmetric monoidal homotopical category and let \mathcal{M} be a \mathcal{V} -category. We do *not* assume yet that \mathcal{M} is equipped with any notion of weak equivalence. Rather, in this section we want to explain how such an \mathcal{M} is automatically equipped with a notion of “homotopy equivalence” induced from its enrichment over \mathcal{V} . In the next section, we will return to considering categories \mathcal{M} that have their own notion of weak equivalence compatible with that of \mathcal{V} .

Observe that by definition of the derived tensor product in $\mathrm{Ho} \mathcal{V}_0$, the localization functor $\gamma: \mathcal{V}_0 \rightarrow \mathrm{Ho} \mathcal{V}_0$ is lax symmetric monoidal. Therefore, it can be applied to the hom-objects of any \mathcal{V} -enriched category \mathcal{M} to give a $\mathrm{Ho} \mathcal{V}$ -enriched category, which we call denote $h\mathcal{M}$. We denote its underlying category $(h\mathcal{M})_0$ by simply $h\mathcal{M}_0$, since there is no sense to the notation $h(\mathcal{M}_0)$. Note that there is a canonical functor $\bar{\gamma}: \mathcal{M}_0 \rightarrow h\mathcal{M}_0$ which is just γ applied to the hom-sets $\mathcal{M}_0(M, M') = \mathcal{V}_0(E, \mathcal{M}(M, M'))$.

It may perhaps seem strange that $h\mathcal{M}$ is enriched over $\mathrm{Ho} \mathcal{V}$, especially when we take $\mathcal{M} = \mathcal{V}$ and obtain $h\mathcal{V}$ enriched over $\mathrm{Ho} \mathcal{V}$. However, the $\mathrm{Ho} \mathcal{V}$ -category $h\mathcal{M}$ should be viewed as merely a convenient way to produce the ordinary category $h\mathcal{M}_0$, which is really a classical object, as shown by the following examples.

Example 17.1. If \mathcal{V} is **Top** with the cartesian product, so that \mathcal{M} is topologically enriched, then $h\mathcal{M}_0$ is the quotient of \mathcal{M}_0 by enriched homotopy, since we have

$$\begin{aligned} h\mathcal{M}_0(X, Y) &= \mathrm{Ho} \mathcal{V}_0(*, \mathcal{M}(X, Y)) \\ &= \pi_0(\mathcal{M}(X, Y)). \end{aligned}$$

Similarly, if $\mathcal{V} = s\mathcal{S}$, then $h\mathcal{M}_0$ is the quotient of \mathcal{M}_0 by simplicial homotopy, and if \mathcal{V} is chain complexes with tensor product, so that \mathcal{M} is a dg-category, then $h\mathcal{M}_0$ is the quotient of \mathcal{M}_0 by chain homotopy.

These examples lead us to expect more generally that $h\mathcal{M}_0$ should be the localization of \mathcal{M}_0 at a class of weak equivalences induced from the enrichment. A natural definition to make is the following.

Definition 17.2. A morphism $f: X \rightarrow Y$ in \mathcal{M}_0 is a \mathcal{V} -equivalence if $\bar{\gamma}(f)$ is an isomorphism in $h\mathcal{M}_0$.

It is clear that \mathcal{V} -equivalences satisfy the 2-out-of-6 property, because isomorphisms do, so that \mathcal{M}_0 becomes a homotopical category when equipped with the \mathcal{V} -equivalences. We do not yet know that it is \mathcal{V} -homotopical, however.

Example 17.3. In the example $\mathcal{V} = \mathbf{Top}$, we claim that the \mathcal{V} -equivalences in any topologically enriched category \mathcal{M} are precisely the homotopy equivalences.

Both directions follow from the fact that, by Example 17.1, $h\mathcal{M}_0$ is the quotient of \mathcal{M} by enriched homotopy. Suppose first that f is a homotopy equivalence with homotopy inverse g . Then $\overline{\gamma}(fg) = 1_Y$ and $\overline{\gamma}(gf) = 1_X$ in $h\mathcal{M}_0$, so $\overline{\gamma}(f)$ is an isomorphism, as desired.

Now suppose that $f: X \rightarrow Y$ is a \mathcal{V} -equivalence in \mathcal{M}_0 . Then $\overline{\gamma}(f)$ is an isomorphism in $h\mathcal{M}_0$, so it has an inverse $\overline{g} \in h\mathcal{M}_0(Y, X) \cong \pi_0(\mathcal{M}(Y, X))$. If we choose a $g \in \mathcal{M}_0(Y, X)$ such that $\overline{\gamma}(g) = \overline{g}$, then $\overline{\gamma}(fg) = 1_Y$ and $\overline{\gamma}(gf) = 1_X$. It follows by Example 17.1 that fg and gf are homotopic to 1_Y and 1_X , respectively, in \mathcal{M} , and hence that f is a homotopy equivalence, as desired.

Similarly, one can prove that when \mathcal{V} is simplicial sets or chain complexes, the \mathcal{V} -equivalences are the simplicial homotopy equivalences or chain homotopy equivalences, respectively. Although it seems quite strange, we have managed to access the classical notions of homotopy and homotopy equivalence without referring to intervals or cylinders!

One of the reasons homotopy equivalences are nicer than weak homotopy equivalences in classical homotopy theory is that any topological functor will preserve homotopy equivalences, simply by virtue of its enrichment. A similar fact is true for all \mathcal{V} -equivalences.

Proposition 17.4. *If $F: \mathcal{M} \rightarrow \mathcal{N}$ is any \mathcal{V} -functor between \mathcal{V} -categories, then $F_0: \mathcal{M}_0 \rightarrow \mathcal{N}_0$ preserves \mathcal{V} -equivalences.*

Proof. The following square of functors commutes by definition.

$$\begin{array}{ccc} \mathcal{M}_0 & \xrightarrow{F_0} & \mathcal{N}_0 \\ \downarrow & & \downarrow \\ h\mathcal{M}_0 & \xrightarrow{hF_0} & h\mathcal{N}_0 \end{array}$$

If f in \mathcal{M}_0 becomes an isomorphism in $h\mathcal{M}_0$, it stays an isomorphism in $h\mathcal{N}_0$. Therefore, $F_0(f)$ becomes an isomorphism in $h\mathcal{N}_0$, so $F_0(f)$ is a \mathcal{V} -equivalence. \square

We now consider under what conditions \mathcal{M} is a \mathcal{V} -homotopical category when equipped with the \mathcal{V} -equivalences. By the definition of \mathcal{V} -equivalence, the functor

$$h\mathcal{M}(X, Y) = \gamma(\mathcal{M}(X, Y))$$

takes all \mathcal{V} -equivalences to isomorphisms in $\text{Ho } \mathcal{V}_0$. Thus, if \mathcal{V} is saturated, the functor $\mathcal{M}(-, -)$ takes all \mathcal{V} -equivalences to weak equivalences in \mathcal{V}_0 . Moreover, Proposition 17.4 implies that the functors $K \odot -$ and $\{K, -\}$ preserve all \mathcal{V} -equivalences in \mathcal{M} for any $K \in \mathcal{V}$. Thus, to make \mathcal{M} a \mathcal{V} -homotopical category with the \mathcal{V} -equivalences as weak equivalences and $\mathcal{M}_Q = \mathcal{M}_R = \mathcal{M}_0$, it would suffice to ensure that the functors $- \odot X$ and $\{-, Y\}$ preserve weak equivalences in \mathcal{V}_Q . Unfortunately this is not in general true, except in very special cases.

Proposition 17.5. *Suppose that all the hom-objects $\mathcal{M}(X, Y)$ are in \mathcal{V}_R . Then \mathcal{M} is a saturated \mathcal{V} -homotopical category with the \mathcal{V} -equivalences as weak equivalences. Moreover, we then have an isomorphism $h\mathcal{M} \cong \text{Ho } \mathcal{M}$ of $\text{Ho } \mathcal{V}$ -categories, and therefore $h\mathcal{M}_0$ is the localization of \mathcal{M}_0 at the \mathcal{V} -equivalences.*

Proof. By the remarks above, to show that \mathcal{M} is a \mathcal{V} -homotopical category with the \mathcal{V} -equivalences, it remains only to check that $- \odot X$ and $\{-, Y\}$ preserve weak

equivalences in \mathcal{V}_Q . We prove the first; the other is dual. Let $f: K \xrightarrow{\sim} L$ be a weak equivalence in \mathcal{V}_Q . Let $X, Y \in \mathcal{M}$ and consider the following sequence of naturality squares in \mathcal{V} .

$$\begin{array}{ccccccc}
\mathcal{M}(L \odot X, Y) & \xrightarrow{\cong} & \mathcal{V}(L, \mathcal{M}(X, Y)) & \longrightarrow & \mathcal{V}(QL, R\mathcal{M}(X, Y)) & \equiv & \text{Ho } \mathcal{V}(L, \mathcal{M}(X, Y)) \\
\downarrow f \odot X & & \downarrow f & & \downarrow Qf & & \downarrow f \\
\mathcal{M}(K \odot X, Y) & \xrightarrow{\cong} & \mathcal{V}(K, \mathcal{M}(X, Y)) & \longrightarrow & \mathcal{V}(QK, R\mathcal{M}(X, Y)) & \equiv & \text{Ho } \mathcal{V}(K, \mathcal{M}(X, Y))
\end{array}$$

The first pair of natural isomorphisms come from the definition of the tensor of \mathcal{M} , and the third pair of equalities come from the definition of the closed structure on $\text{Ho } \mathcal{V}_0$.

The second pair of natural transformations

$$\begin{array}{ccc}
\mathcal{V}(L, \mathcal{M}(X, Y)) & \longrightarrow & \mathcal{V}(QL, R\mathcal{M}(X, Y)) \\
\mathcal{V}(K, \mathcal{M}(X, Y)) & \longrightarrow & \mathcal{V}(QK, R\mathcal{M}(X, Y))
\end{array}$$

is induced by the natural transformations $q: Q \xrightarrow{\sim} \text{Id}$ and $r: \text{Id} \xrightarrow{\sim} R$ for the closed monoidal deformation of \mathcal{V} . Since $\mathcal{M}(X, Y) \in \mathcal{V}_R$ and $K, L \in \mathcal{V}_Q$, these natural transformations are weak equivalences. Moreover, since the functor

$$\text{Ho } \mathcal{V}(-, -): \mathcal{V}_0^{op} \times \mathcal{V}_0 \rightarrow \mathcal{V}_0$$

is homotopical, the right vertical map is a weak equivalence as well. By the 2-out-of-3 property, it follows that the left vertical map is a weak equivalence. Applying γ to this map, we obtain an isomorphism

$$h\mathcal{M}(K \odot X, Y) \cong h\mathcal{M}(L \odot X, Y)$$

in $\text{Ho } \mathcal{V}_0$. Since the left vertical map is \mathcal{V} -natural and γ is lax symmetric monoidal, this induced map is $\text{Ho } \mathcal{V}$ -natural in X and Y . Therefore, it induces a natural isomorphism

$$h\mathcal{M}_0(K \odot X, Y) \cong h\mathcal{M}_0(L \odot X, Y).$$

Since this is true naturally in Y , by the Yoneda Lemma the map $\bar{\gamma}(f \odot X)$ must be an isomorphism in $h\mathcal{M}_0$, and therefore $f \odot X$ is a \mathcal{V} -equivalence in \mathcal{M}_0 , as desired.

For the second statement, note that since we are taking $Q = R = \text{Id}_{\mathcal{M}_0}$, we have $h\mathcal{M}(X, Y) = \gamma(\mathcal{M}(X, Y)) = \text{Ho } \mathcal{M}(X, Y)$. Thus $h\mathcal{M} \cong \text{Ho } \mathcal{M}$, and hence $h\mathcal{M}_0 \cong \text{Ho } \mathcal{M}_0$. Since $\text{Ho } \mathcal{M}_0$ is the localization of \mathcal{M}_0 at the \mathcal{V} -equivalences, so is $h\mathcal{M}_0$. This implies directly that \mathcal{M} is saturated, by definition of the \mathcal{V} -equivalences. \square

The condition that all hom-objects are in \mathcal{V}_R corresponds to the frequent requirement in the literature of simplicially enriched categories (such as in [CP97]) that they be *locally Kan*. In the topological literature this requirement is rarely found, since all topological spaces are fibrant. In general, we expect to have to deform the objects of \mathcal{M} to make it a \mathcal{V} -homotopical category, and this may be no easier for the \mathcal{V} -equivalences than for some more general type of weak equivalence. Thus, in the next section, we consider the role of the \mathcal{V} -equivalences in a category that already has a \mathcal{V} -homotopical structure.

18. DERIVED FUNCTORS VIA ENRICHED HOMOTOPY

In this section, let \mathcal{V} be a closed symmetric monoidal homotopical category and let \mathcal{M} be a \mathcal{V} -homotopical category. In this case, $h\mathcal{M}$ is generally different from $\text{Ho}\mathcal{M}$, but just as in classical homotopy theory, it is often a useful stepping-stone to it. The following result is simply a restatement in our more general context of a standard result from model category theory (which, itself, is a generalization of a standard result from classical homotopy theory). We agree from now on to write $\mathcal{M}_{QR} = \mathcal{M}_Q \cap \mathcal{M}_R$.

Proposition 18.1. *Let \mathcal{M} be a \mathcal{V} -homotopical category. Then there is a $\text{Ho}\mathcal{V}$ -functor $\varphi: h\mathcal{M} \rightarrow \text{Ho}\mathcal{M}$ such that the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{M}_0 & & \\ \bar{\gamma} \downarrow & \searrow & \\ h\mathcal{M}_0 & \xrightarrow{\varphi_0} & \text{Ho}\mathcal{M}_0. \end{array}$$

Moreover,

- (i) *The restriction of φ to $h\mathcal{M}_{QR}$ is fully faithful. If either Q maps \mathcal{M}_R into itself or R maps \mathcal{M}_Q into itself, then this restriction is an equivalence.*
- (ii) *If $X, Y \in \mathcal{M}_{QR}$ and $f: X \rightarrow Y$ is a weak equivalence, then it is a \mathcal{V} -equivalence. Conversely, if \mathcal{M} is saturated, then all \mathcal{V} -equivalences in \mathcal{M} are weak equivalences.*

Proof. First we construct the functor φ . By definition of the enrichment in Proposition 16.2, we have $\text{Ho}\mathcal{M}(X, Y) = \mathcal{M}(QX, RY)$. Thus the transformations $QX \rightarrow X$ and $Y \rightarrow RY$ induce maps $\mathcal{M}(X, Y) \rightarrow \mathcal{M}(QX, RY)$. Tracing through the definitions of the composition in $h\mathcal{M}$ and $\text{Ho}\mathcal{M}$, it is straightforward to check that these maps are functorial, and that the given diagram commutes.

We now prove (i). Since $X \in \mathcal{M}_Q$ and $Y \in \mathcal{M}_R$, the hom-functor $\mathcal{M}(-, -)$ preserves the weak equivalences $QX \xrightarrow{\sim} X$ and $Y \xrightarrow{\sim} RY$, so φ restricted to $h\mathcal{M}_{QR}$ is clearly fully faithful. It remains to show it is essentially surjective. If Q preserves \mathcal{M}_R , then for any $M \in \mathcal{M}$ we have $QRM \in \mathcal{M}_{QR}$ and a zigzag of weak equivalences $M \xrightarrow{\sim} RM \xleftarrow{\sim} QRM$, so M is isomorphic in $\text{Ho}\mathcal{M}$ to $QRM \in h\mathcal{M}_{QR}$. The other case is similar.

Finally, to prove (ii), let $f: X \rightarrow Y$ be a weak equivalence in \mathcal{M}_{QR} . Then it becomes an isomorphism in $\text{Ho}\mathcal{M}_0$, and hence also an isomorphism in $h\mathcal{M}_0$, since φ_0 is fully faithful. Thus f is a \mathcal{V} -equivalence by definition. Conversely, if $f: X \rightarrow Y$ is any \mathcal{V} -equivalence in \mathcal{M} , then it becomes an isomorphism in $h\mathcal{M}_0$, hence also in $\text{Ho}\mathcal{M}_0$. Thus if \mathcal{M} is saturated, it must have been a weak equivalence. \square

Recall that we have seen condition (i) before, in Proposition 4.8, and that we saw in Corollary 4.9 that this condition is satisfied in all model categories if Q and R are well chosen. Similarly, in practice most homotopical categories are saturated, so this result is actually quite general. It implies the following fact, which is familiar from classical homotopy theory.

Proposition 18.2. *Let \mathcal{M} and \mathcal{N} be \mathcal{V} -homotopical categories such that*

- (i) *\mathcal{N} is saturated, and*
- (ii) *either Q preserves \mathcal{M}_R or R preserves \mathcal{M}_Q .*

Then any \mathcal{V} -functor $F: \mathcal{M} \rightarrow \mathcal{N}$ has a middle derived functor $\mathrm{Ho} \mathcal{M}_0 \rightarrow \mathrm{Ho} \mathcal{N}_0$.

Proof. Suppose that Q preserves \mathcal{M}_R ; the other case is dual. Then the natural zigzag $\mathrm{Id} \xleftarrow{\sim} Q \xrightarrow{\sim} QR$ is a middle deformation and so \mathcal{M}_{QR} is a middle deformation retract. We aim to show it is a middle F -deformation retract.

By Proposition 17.4, F preserves all \mathcal{V} -equivalences. But by Proposition 18.1, all weak equivalences in \mathcal{M}_{QR} are \mathcal{V} -equivalences, so F maps them to \mathcal{V} -equivalences. Since \mathcal{N} is saturated, by Proposition 18.1, all \mathcal{V} -equivalences in \mathcal{N} are weak equivalences; thus F is homotopical on \mathcal{M}_{QR} as desired. \square

Corollary 18.3. *Let \mathcal{M} and \mathcal{N} be \mathcal{V} -homotopical categories such that*

- \mathcal{N} is saturated; and
- $\mathcal{M}_R = \mathcal{M}$ (“every object is fibrant”).

Then every \mathcal{V} -functor $F: \mathcal{M} \rightarrow \mathcal{N}$ has a left derived functor. Dually, if the second condition is replaced by $\mathcal{M}_Q = \mathcal{M}$ (“every object is cofibrant”) then every \mathcal{V} -functor $F: \mathcal{M} \rightarrow \mathcal{N}$ has a right derived functor.

Proof. Straightforward. \square

Remark 18.4. Even though they were produced using the enrichment, the above middle derived functors are *not* in general $\mathrm{Ho} \mathcal{V}$ -enriched. The functors Q and R are generally not \mathcal{V} -functors, so FQR is not a \mathcal{V} -functor. In adjoint cases like Proposition 16.8, we can use the notion of \mathcal{V} -module functor to get around this problem and construct a $\mathrm{Ho} \mathcal{V}$ -enrichment, but no such tricks are available in the general situation.

Proposition 18.2 shows that the deformations Q and R of a \mathcal{V} -homotopical category are, in a sense, “universal:” they work for *every* (enriched) functor. And while formally speaking, middle derived functors are not unique, as we saw in Counterexample 4.3, certainly these middle derived functors produced from the enrichment have a good claim to be the “correct” ones. However, we must beware of too much enthusiasm; we still have the following enriched version of Counterexample 4.3.

Counterexample 18.5. Let \mathcal{V} be a closed *cartesian* monoidal homotopical category; thus we write its product as \times and its unit (which is the terminal object) as $*$. Let \mathcal{M} and \mathcal{N} be \mathcal{V} -homotopical categories satisfying the conditions of Proposition 18.2, and let \mathcal{I} be the \mathcal{V} -category with two objects 0 and 1 and with $\mathcal{I}(0,0) = \mathcal{I}(1,1) = \mathcal{I}(0,1) = *$ and $\mathcal{I}(1,0) = \emptyset$ (the initial object of \mathcal{V}). The underlying category \mathcal{I}_0 is the ordinary category \mathcal{I} considered in Counterexample 4.3. Let $F^0, F^1: \mathcal{M} \rightarrow \mathcal{N}$ be two \mathcal{V} -functors and let $\alpha: F^0 \rightarrow F^1$ be a \mathcal{V} -natural transformation; then α gives rise to a \mathcal{V} -functor $F: \mathcal{M} \times \mathcal{I} \rightarrow \mathcal{N}$ in an obvious way.

Now, because \mathcal{V} is cartesian monoidal, we have $(\mathcal{M} \times \mathcal{I})_0 \cong \mathcal{M}_0 \times \mathcal{I}_0$, which we give the product homotopical structure. The \mathcal{V} -category $\mathcal{M} \times \mathcal{I}$, however, has two \mathcal{V} -homotopical structures, as follows.

In the “left” \mathcal{V} -homotopical structure, we define deformations

$$\begin{aligned} Q^L(M, i) &= (Q_{\mathcal{M}} M, 0) \\ R^L(M, i) &= (R_{\mathcal{M}} M, i) \end{aligned}$$

The \mathcal{V} -category $\mathcal{M} \times \mathcal{I}$ is tensored and cotensored with tensors and cotensors defined in the obvious way:

$$\begin{aligned} K \odot (M, i) &= (K \odot M, i) \\ \{K, (M, i)\} &= (\{K, M\}, i). \end{aligned}$$

It is straightforward to check that this satisfies Definition 16.1 and condition 18.2(ii) because \mathcal{M} does. Similarly, in the “right” \mathcal{V} -homotopical structure, we define the deformations to be

$$\begin{aligned} Q^L(M, i) &= (Q_{\mathcal{M}}M, i) \\ R^L(M, i) &= (R_{\mathcal{M}}M, 1); \end{aligned}$$

this also satisfies Definition 16.1 and condition 18.2(ii). Applying Proposition 18.2 to these two \mathcal{V} -homotopical structures, we produce two different middle derived functors of F ; one is the “canonical” middle derived functor of F^0 and the other is that of F^1 .

We are forced to conclude that even in the enriched situation, the deformations Q and R must be regarded as part of the structure, in the sense that different choices can produce genuinely different behavior.

This completes our study of general enriched homotopy theory in general. We now return to the case of most interest to us and consider diagram categories.

19. GENERALIZED TENSOR PRODUCTS AND BAR CONSTRUCTIONS

In the next section, we will construct homotopy tensor and cotensor products of functors in a “fully enriched” way, using the techniques of §16. But now that we have the perspective of §14, it is clear that there is no reason to restrict our attention to the tensor-hom-cotensor of a \mathcal{V} -category; we can define homotopy tensor products of functors in the more general setting of an arbitrary two-variable \mathcal{V} -adjunction. This also has the advantage of making the symmetry of the construction more evident. In this section we record the obvious definitions of the tensor product of functors and the bar and cobar constructions in this more general context.

Definition 19.1. Let $(\otimes, \text{hom}_\ell, \text{hom}_r): \mathcal{M} \otimes \mathcal{N} \rightarrow \mathcal{P}$ be a two-variable \mathcal{V} -adjunction, where \mathcal{P} is cocomplete, and let \mathcal{D} be a small \mathcal{V} -category. Given \mathcal{V} -functors $G: \mathcal{D}^{op} \rightarrow \mathcal{M}$ and $F: \mathcal{D} \rightarrow \mathcal{N}$, their *tensor product* is an object of \mathcal{P} defined as the following \mathcal{V} -coequalizer:

$$(19.2) \quad G \otimes_{\mathcal{D}} F = \text{coeq} \left(\coprod_{d, d'} \mathcal{D}(d, d') \odot (Gd' \otimes Fd) \rightrightarrows \coprod_d Gd \otimes Fd \right).$$

Proposition 19.3. *In the above situation, the tensor product defines a functor $\mathcal{M}^{\mathcal{D}^{op}} \otimes \mathcal{N}^{\mathcal{D}} \rightarrow \mathcal{P}$. Moreover, this functor is part of a two-variable \mathcal{V} -adjunction.*

The right adjoints to $\otimes_{\mathcal{D}}$ are the functors hom_r and hom_ℓ applied objectwise; that is, $\text{hom}_r(F, Z)(d) = \text{hom}_r(F(d), Z)$ and similarly.

Clearly, in the case when the given two-variable \mathcal{V} -adjunction is the tensor-hom-cotensor of a \mathcal{V} -category \mathcal{M} , the tensor product of functors $G \otimes_{\mathcal{D}} F$ reduces to that of Definition 11.1. Just as in that case, there is an obvious dual notion of *cotensor product of functors*.

Definition 19.4. In the situation of Definition 19.1, the *two-sided simplicial bar construction* is a simplicial object of \mathcal{P} whose object function is

$$B_n(G, \mathcal{D}, F) = \coprod_{\alpha: [n] \rightarrow \mathcal{D}_0} (\mathcal{D}(\alpha_{n-1}, \alpha_n) \otimes \dots \otimes \mathcal{D}(\alpha_0, \alpha_1)) \odot (G(\alpha_n) \otimes F(\alpha_0))$$

and whose faces and degeneracies are defined using composition in \mathcal{D} , the evaluation maps $\mathcal{D}(d, d') \odot F(d) \rightarrow F(d')$ and $\mathcal{D}(d, d') \odot G(d') \rightarrow G(d)$, and insertion of identities $E \rightarrow \mathcal{D}(d, d)$.

Note the three different bifunctors appearing: the monoidal product \otimes in \mathcal{V} , the tensor \odot of \mathcal{P} , and the given bifunctor \otimes . Because \otimes is part of a two-variable \mathcal{V} -adjunction, it “doesn’t matter which way we parenthesize this expression,” so we can think of it as:

$$\text{“ } G(\alpha_n) \otimes \mathcal{D}(\alpha_{n-1}, \alpha_n) \otimes \dots \otimes \mathcal{D}(\alpha_0, \alpha_1) \otimes F(\alpha_0). \text{”}$$

Definition 19.5. Let $(\otimes, \text{hom}_\ell, \text{hom}_r): \mathcal{M} \otimes \mathcal{N} \rightarrow \mathcal{P}$ be a two-variable \mathcal{V} -adjunction, let \mathcal{D} be a small \mathcal{V} -category, and let $G: \mathcal{D} \rightarrow \mathcal{N}$ and $F: \mathcal{D} \rightarrow \mathcal{P}$ be \mathcal{V} -functors. The *two-sided cosimplicial cobar construction* is a cosimplicial object of \mathcal{M} whose object function is

$$C^n(G, \mathcal{D}, F) = \prod_{\alpha: [n] \rightarrow \mathcal{D}_0} \text{hom}_r \left((\mathcal{D}(\alpha_{n-1}, \alpha_n) \otimes \dots \otimes \mathcal{D}(\alpha_0, \alpha_1)) \odot G(\alpha_0), F(\alpha_n) \right)$$

and whose faces and degeneracies are defined in a way similar to the simplicial bar construction.

We refer the reader to [Mey84] for more about the cosimplicial cobar construction; in almost all respects it is dual to the simplicial bar construction.

Definition 19.6. Assume the situation of Definition 19.4 and that \mathcal{V} has a canonical cosimplicial object $\Delta^\bullet: \Delta \rightarrow \mathcal{V}$. The *two-sided bar construction* is the geometric realization of the two-sided simplicial bar construction:

$$B(G, \mathcal{D}, F) = |B_\bullet(G, \mathcal{D}, F)|.$$

Similarly, in the situation of Definition 19.5, the *two-sided cobar construction* is the *totalization* of the cosimplicial cobar construction:

$$\begin{aligned} C(G, \mathcal{D}, F) &= \text{Tot}(C^\bullet(G, \mathcal{D}, F)) \\ &\equiv \{\Delta^\bullet, C^\bullet(G, \mathcal{D}, F)\}^\Delta. \end{aligned}$$

Lemma 19.7. *In the above situations, we have*

$$\begin{aligned} B(G, \mathcal{D}, F) &\cong G \otimes_{\mathcal{D}} B(\mathcal{D}, \mathcal{D}, F) \\ &\cong B(G, \mathcal{D}, \mathcal{D}) \otimes_{\mathcal{D}} F. \end{aligned}$$

and

$$\begin{aligned} C(G, \mathcal{D}, F) &\cong \text{hom}_r^{\mathcal{D}}(G, C(\mathcal{D}, \mathcal{D}, F)) \\ &\cong \text{hom}_r^{\mathcal{D}}(B(\mathcal{D}, \mathcal{D}, G), F) \end{aligned}$$

20. ENRICHED HOMOTOPY TENSOR AND COTENSOR PRODUCTS

As promised, we now put the results of §13 in the context of §16, so that we can apply the results of the latter section to produce $\text{Ho } \mathcal{V}$ -enrichments of the total derived tensor products. Combining the assumptions of those two sections, we assume that \mathcal{V} is a closed symmetric monoidal homotopical category equipped with a strong monoidal adjunction $s\mathcal{S} \rightleftarrows \mathcal{V}$. We work with a general two-variable \mathcal{V} -adjunction, in which the categories \mathcal{M} , \mathcal{N} , and \mathcal{P} will be \mathcal{V} -homotopical categories. Moreover, clearly the deformation retracts in Definition 13.1 should be taken to be those with which \mathcal{V} -homotopical categories are equipped. Thus we modify Definition 13.1 to the following.

Definition 20.1. Let $(\otimes, \text{hom}_\ell, \text{hom}_r): \mathcal{M} \otimes \mathcal{N} \rightarrow \mathcal{P}$ be a \mathcal{V} -deformable two-variable \mathcal{V} -adjunction between \mathcal{V} -homotopical categories, and let \mathcal{D} be a small \mathcal{V} -category. We say that \mathcal{D} is *good* for \otimes if the following conditions hold:

- $B(-, \mathcal{D}, -)$ is homotopical on $\mathcal{M}_Q^{\mathcal{D}^{\text{op}}} \times \mathcal{N}_Q^{\mathcal{D}}$;
- If $F \in \mathcal{N}_Q^{\mathcal{D}}$ and $G \in \mathcal{M}_Q^{\mathcal{D}^{\text{op}}}$, then
 - $B(\mathcal{D}, \mathcal{D}, F) \in \mathcal{N}_Q^{\mathcal{D}}$,
 - $B(G, \mathcal{D}, \mathcal{D}) \in \mathcal{M}_Q^{\mathcal{D}^{\text{op}}}$, and
 - $B(G, \mathcal{D}, F) \in \mathcal{P}_Q$.

Definition 20.2. Let \mathcal{D} be a small \mathcal{V} -category which is good for \otimes and let $F \in \mathcal{N}^{\mathcal{D}}$ and $G \in \mathcal{M}^{\mathcal{D}^{\text{op}}}$. Assume moreover that $\mathcal{M}_Q^{\mathcal{D}^{\text{op}}}$ and $\mathcal{N}_Q^{\mathcal{D}}$ are left deformation retracts of $\mathcal{M}^{\mathcal{D}^{\text{op}}}$ and $\mathcal{N}^{\mathcal{D}}$, respectively, with corresponding left deformations $Q_{\mathcal{M}}^{\mathcal{D}^{\text{op}}}$ and $Q_{\mathcal{N}}^{\mathcal{D}}$. Define the *corrected homotopy tensor product* of G and F to be

$$(20.3) \quad G \overset{\mathbb{L}}{\otimes}_{\mathcal{D}} F = B(Q_{\mathcal{M}}^{\mathcal{D}^{\text{op}}} G, \mathcal{D}, Q_{\mathcal{N}}^{\mathcal{D}} F).$$

The following generalization of Theorem 13.7 is immediate.

Theorem 20.4. *If \mathcal{D} is good for \otimes , simplicial homotopy equivalences in \mathcal{M} and \mathcal{N} are weak equivalences, and $\mathcal{M}_Q^{\mathcal{D}^{\text{op}}}$ and $\mathcal{N}_Q^{\mathcal{D}}$ are left deformation retracts of $\mathcal{M}^{\mathcal{D}^{\text{op}}}$ and $\mathcal{N}^{\mathcal{D}}$, respectively, then the corrected homotopy tensor product $\overset{\mathbb{L}}{\otimes}_{\mathcal{D}}$ is a derived functor of $\otimes_{\mathcal{D}}$.*

We also state here the dual versions. The lemmas and proofs are also dual, so we omit them and give only the definitions and final results.

Definition 20.5. Let $(\otimes, \text{hom}_\ell, \text{hom}_r): \mathcal{M} \otimes \mathcal{N} \rightarrow \mathcal{P}$ be a two-variable \mathcal{V} -adjunction between \mathcal{V} -homotopical categories and let \mathcal{D} be a small \mathcal{V} -category. We say that \mathcal{D} is *good* for hom_r if the following conditions hold:

- $C(-, \mathcal{D}, -)$ is homotopical on $\mathcal{N}_Q^{\mathcal{D}} \times \mathcal{P}_R^{\mathcal{D}}$
- If $G \in \mathcal{N}_Q^{\mathcal{D}}$ and $F \in \mathcal{P}_R^{\mathcal{D}}$, then
 - $B(\mathcal{D}, \mathcal{D}, G) \in \mathcal{N}_Q^{\mathcal{D}}$,
 - $C(\mathcal{D}, \mathcal{D}, F) \in \mathcal{P}_R^{\mathcal{D}}$, and
 - $C(G, \mathcal{D}, F) \in \mathcal{M}_R$.

Note that goodness is still a cofibrancy condition on \mathcal{D} . We will have a little more to say about this in §23.

Definition 20.6. Let \mathcal{D} be a small \mathcal{V} -category which is good for hom_r , and assume that $\mathcal{N}_Q^{\mathcal{D}}$ and $\mathcal{P}_R^{\mathcal{D}}$ are, respectively, a left and a right deformation retract of $\mathcal{N}^{\mathcal{D}}$

and $\mathcal{P}^{\mathcal{D}}$. Then the (corrected) homotopy cotensor product of $G \in \mathcal{N}^{\mathcal{D}}$ and $F \in \mathcal{P}^{\mathcal{D}}$ is

$$\mathbb{R} \operatorname{hom}_r^{\mathcal{D}}(G, F) = C(Q_{\mathcal{N}}^{\mathcal{D}}G, \mathcal{D}, R_{\mathcal{P}}^{\mathcal{D}}F).$$

Theorem 20.7. *If \mathcal{D} is good for hom_r , simplicial homotopy equivalences in \mathcal{N} and \mathcal{P} are weak equivalences, and $\mathcal{N}_Q^{\mathcal{D}}$ and $\mathcal{P}_R^{\mathcal{D}}$ are, respectively, a left and a right deformation retract of $\mathcal{N}^{\mathcal{D}}$ and $\mathcal{P}^{\mathcal{D}}$, then the corrected homotopy cotensor product $\mathbb{R} \operatorname{hom}_r^{\mathcal{D}}$ is a derived functor of $\operatorname{hom}_r^{\mathcal{D}}$.*

Now we want to apply Proposition 16.13 to produce $\operatorname{Ho} \mathcal{V}$ -enrichments for the total derived tensor and cotensor products, but to do this we first need a \mathcal{V} -homotopical structure on diagram categories. The wonderful thing is that we can do both of these things with the very same construction, since the enriched-hom between diagrams is a cotensor product construction. This fulfills the promise made in §10 by showing that we can use “coherent transformations”, as defined there, to invert the objectwise weak equivalences and thereby construct the homotopy category of a diagram category.

Recall that we write $\mathcal{M}_{BQ}^{\mathcal{D}}$ for the full image of $B(\mathcal{D}, \mathcal{D}, -)$ on $\mathcal{M}_Q^{\mathcal{D}}$. Dually, we write $\mathcal{M}_{CR}^{\mathcal{D}}$ for the full image of $C(\mathcal{D}, \mathcal{D}, -)$ on $\mathcal{M}_R^{\mathcal{D}}$.

Theorem 20.8. *Let \mathcal{M} be a \mathcal{V} -homotopical category in which simplicial homotopy equivalences are weak equivalences, let \mathcal{D} be good for the enriched hom-functor $\mathcal{M}(-, -): \mathcal{M}^{\operatorname{op}} \otimes \mathcal{M} \rightarrow \mathcal{V}$, and assume that $\mathcal{M}_Q^{\mathcal{D}}$ and $\mathcal{M}_R^{\mathcal{D}}$ are, respectively, a left and right deformation of $\mathcal{M}^{\mathcal{D}}$. Then $\mathcal{M}^{\mathcal{D}}$ is a \mathcal{V} -homotopical category when equipped with the deformation retracts*

$$\mathcal{M}_{BQ}^{\mathcal{D}} \quad \text{and} \quad \mathcal{M}_R^{\mathcal{D}}$$

and also when equipped with the deformation retracts

$$\mathcal{M}_Q^{\mathcal{D}} \quad \text{and} \quad \mathcal{M}_{CR}^{\mathcal{D}}.$$

We call the first the bar \mathcal{V} -homotopical structure and the second the cobar \mathcal{V} -homotopical structure on $\mathcal{M}^{\mathcal{D}}$.

Proof. By Theorem 20.7, the given deformation retracts, together with \mathcal{V}_Q , define a deformation retract for the tensor-hom-cotensor two-variable adjunction of $\mathcal{M}^{\mathcal{D}}$. Tensors and cotensors in $\mathcal{M}^{\mathcal{D}}$ are objectwise, and the bar (resp. cobar) construction is a colimit (resp. limit) construction, so it is preserved by tensors (resp. cotensors); thus the remaining axioms of Definition 16.1 for $\mathcal{M}^{\mathcal{D}}$ follow from those for \mathcal{M} . \square

Corollary 20.9. *Under the conditions of Theorem 20.8, the homotopy category $\operatorname{Ho}(\mathcal{M}^{\mathcal{D}})$ is enriched, tensored, and cotensored over $\operatorname{Ho} \mathcal{V}$, and if $\operatorname{Ho} \mathcal{V}_0$ has small hom-sets, then so does $\operatorname{Ho}(\mathcal{M}^{\mathcal{D}})_0$.*

Finally, we can complete Theorem 20.4 as follows.

Corollary 20.10. *Let $(\otimes, \operatorname{hom}_\ell, \operatorname{hom}_r): \mathcal{M} \otimes \mathcal{N} \rightarrow \mathcal{P}$ be a \mathcal{V} -deformable two-variable \mathcal{V} -adjunction between \mathcal{V} -homotopical categories, let \mathcal{D} be good for \otimes , and assume that the bar \mathcal{V} -homotopical structure exists on $\mathcal{M}^{\mathcal{D}}$ and $\mathcal{N}^{\mathcal{D}}$. Then the two-variable \mathcal{V} -adjunction*

$$(\otimes_{\mathcal{D}}, \operatorname{hom}_\ell, \operatorname{hom}_r): \mathcal{M}^{\mathcal{D}^{\operatorname{op}}} \otimes \mathcal{N}^{\mathcal{D}} \longrightarrow \mathcal{P}$$

is \mathcal{V} -deformable of the same type as $(\otimes, \text{hom}_\ell, \text{hom}_r)$, and thus has a “total derived” two-variable $\text{Ho } \mathcal{V}$ -adjunction

$$(\overset{\mathbf{L}}{\otimes}_{\mathcal{D}}, \mathbf{R} \text{hom}_\ell, \mathbf{R} \text{hom}_r): \text{Ho}(\mathcal{M}^{\mathcal{D}^{\text{op}}}) \overset{\mathbf{L}}{\otimes} \text{Ho}(\mathcal{N}^{\mathcal{D}}) \longrightarrow \text{Ho } \mathcal{P}.$$

Proof. We showed in Theorem 13.7 that $Q_{\mathcal{M}} \times B(\mathcal{D}, \mathcal{D}, Q_{\mathcal{N}}-)$ is a deformation for $\otimes_{\mathcal{D}}$. Since \mathcal{D} is good, $B(\mathcal{D}^{\text{op}}, \mathcal{D}^{\text{op}}, Q_{\mathcal{M}}-)$ has image objectwise in \mathcal{M}_Q , from which it follows that $\otimes_{\mathcal{D}}$ is homotopical on the image of $B(\mathcal{D}^{\text{op}}, \mathcal{D}^{\text{op}}, Q_{\mathcal{M}}-) \times B(\mathcal{D}, \mathcal{D}, Q_{\mathcal{N}}-)$. Since hom_ℓ and hom_r are applied objectwise, it follows that they are homotopical on the correct deformation retracts as well.

If the given adjunction is hom_ℓ - or hom_r - \mathcal{V} -deformable, then it follows directly that the tensor product one is also. For the \otimes - \mathcal{V} -deformable case, we invoke the final axiom of goodness to see that $\otimes_{\mathcal{D}}$ maps the image of $B(\mathcal{D}^{\text{op}}, \mathcal{D}^{\text{op}}, Q_{\mathcal{M}}-) \times B(\mathcal{D}, \mathcal{D}, Q_{\mathcal{N}}-)$ into \mathcal{P}_Q . \square

Remark 20.11. It is straightforward to show that if one of the diagram categories is given the cobar \mathcal{V} -homotopical structure, the result is still true. If both are given the cobar structure, the result is true for hom_ℓ - and hom_r - \mathcal{V} -deformability but not necessarily for \otimes - \mathcal{V} -deformability.

Of course, there is a dual version:

Corollary 20.12. *Let $(\otimes, \text{hom}_\ell, \text{hom}_r): \mathcal{M} \otimes \mathcal{N} \rightarrow \mathcal{P}$ be a \mathcal{V} -deformable two-variable \mathcal{V} -adjunction between \mathcal{V} -homotopical categories, let \mathcal{D} be good for hom_r , and assume that simplicial homotopy equivalences in \mathcal{M} and \mathcal{N} are weak equivalences. Assume that the bar \mathcal{V} -homotopical structure exists on $\mathcal{N}^{\mathcal{D}}$ and the cobar \mathcal{V} -homotopical structure exists on $\mathcal{P}^{\mathcal{D}}$. Then the two-variable \mathcal{V} -adjunction*

$$(\otimes, \text{hom}_\ell, \text{hom}_r^{\mathcal{D}}): \mathcal{M} \otimes \mathcal{N}^{\mathcal{D}} \longrightarrow \mathcal{P}^{\mathcal{D}}$$

is \mathcal{V} -deformable of the same type as $(\otimes, \text{hom}_\ell, \text{hom}_r)$, and thus has a “total derived” two-variable $\text{Ho } \mathcal{V}$ -adjunction

$$(\overset{\mathbf{L}}{\otimes}, \mathbf{R} \text{hom}_\ell, \mathbf{R} \text{hom}_r^{\mathcal{D}}): \text{Ho } \mathcal{M} \overset{\mathbf{L}}{\otimes} \text{Ho}(\mathcal{N}^{\mathcal{D}}) \longrightarrow \text{Ho}(\mathcal{P}^{\mathcal{D}}).$$

Combined with Proposition 18.2, these results tell us that in favorable cases, the bar and cobar constructions are the “universal” deformations for functors defined on diagram categories. This provides our final clue to the surprising ubiquity of the bar construction. Unfortunately, to actually apply Proposition 18.2 and produce middle derived functors, we would need the bar construction to preserve fibrant objects or the fibrant replacement functor to preserve the bar construction. However, we will see in the next section that many functors of interest have genuine left or right derived functors which can also be produced with the bar construction. We still don’t really understand the meaning of the “leftness” or “rightness” of a derived functor, but it is an undeniably useful and frequently encountered property.

21. WEIGHTED HOMOTOPY COLIMITS, II

We now apply the general notions of homotopy tensor and cotensor products to the specific cases of weighted limits and colimits, which were our original motivation. Note how natural the proofs are in the global context of derived functors; this is another value of having the theorem that local and global notions agree.

In this section, we continue our standing assumption that \mathcal{V} is a closed symmetric monoidal homotopical category with a strong monoidal adjunction $s\mathcal{S} \rightleftarrows \mathcal{V}$,

and we assume that simplicial homotopy equivalences are weak equivalences in all \mathcal{V} -homotopical categories, including \mathcal{V} , and that all homotopical categories are saturated. All these conditions are satisfied if \mathcal{V} is a simplicial monoidal model category and we consider \mathcal{V} -model categories only.

We also assume that for all \mathcal{V} -homotopical categories \mathcal{M} and all small \mathcal{V} -categories \mathcal{D} considered, the subcategories $\mathcal{M}_Q^{\mathcal{D}}$ and $\mathcal{M}_R^{\mathcal{D}}$ are, respectively, a left and right deformation retract of $\mathcal{M}^{\mathcal{D}}$, so that the bar and cobar \mathcal{V} -homotopical structures exist. Unless otherwise specified, we give diagram categories such as $\mathcal{M}^{\mathcal{D}}$ the bar \mathcal{V} -homotopical structure.

Theorem 21.1. *If \mathcal{D} is good for the tensor \odot of \mathcal{M} and $G: \mathcal{D}^{op} \rightarrow \mathcal{V}$ is objectwise in \mathcal{V}_Q , then the \mathcal{V} -adjunction*

$$G \odot_{\mathcal{D}} -: \mathcal{M}^{\mathcal{D}} \rightleftarrows \mathcal{M} : \{G, -\}$$

is both left and right \mathcal{V} -deformable. Therefore the functor

$$G \overset{\mathbf{L}}{\odot}_{\mathcal{D}} - = B(G, \mathcal{D}, Q-)$$

is a left derived functor of the weighted colimit $G \odot_{\mathcal{D}} -$, and we have a derived Ho \mathcal{V} -adjunction

$$G \overset{\mathbf{L}}{\odot}_{\mathcal{D}} -: \text{Ho}(\mathcal{M}^{\mathcal{D}}) \rightleftarrows \text{Ho } \mathcal{M} : \mathbf{R}\{G, -\}$$

Proof. Straightforward from Corollary 20.10. □

We also have a dual version using Theorem 20.7. which constructs homotopy weighted limits, using the cobar \mathcal{V} -homotopical structure, as follows.

Theorem 21.2. *If \mathcal{D} is good for the cotensor $\{\}$ of \mathcal{M} and $G: \mathcal{D} \rightarrow \mathcal{V}$ is objectwise in \mathcal{V}_Q , then the \mathcal{V} -adjunction*

$$G \odot -: \mathcal{M} \rightleftarrows \mathcal{M}^{\mathcal{D}} : \{G, -\}^{\mathcal{D}}$$

is both left and right \mathcal{V} -deformable. Therefore the functor

$$\mathbf{R}\{G, -\}^{\mathcal{D}} = C(G, \mathcal{D}, R-)$$

is a right derived functor of the weighted limit $\{G, -\}^{\mathcal{D}}$, and we have a derived Ho \mathcal{V} -adjunction

$$G \overset{\mathbf{L}}{\odot} -: \text{Ho } \mathcal{M} \rightleftarrows \text{Ho}(\mathcal{M}^{\mathcal{D}}) : \mathbf{R}\{G, -\}^{\mathcal{D}}$$

This dual result is especially important because, as remarked earlier, while projective model structures often exist on the \mathcal{V} -category $\mathcal{M}^{\mathcal{D}}$ of enriched diagrams, so that homotopy colimits of enriched diagrams can often be defined (if not computed) in that way, there is hardly ever an injective model structure on enriched diagram categories. Thus cobar constructions provide the *only* viable way to compute homotopy weighted limits.

An especially interesting sort of weighted colimit is a coend. Recall (from [Kel82, §3.10], for example) that the coend of a functor $H: \mathcal{D}^{op} \otimes \mathcal{D} \rightarrow \mathcal{M}$ is defined to be the weighted colimit

$$\int^{\mathcal{D}} H = \mathcal{D}(-, -) \odot_{\mathcal{D}^{op} \otimes \mathcal{D}} H.$$

Thus, Theorem 21.1 tells us that the *homotopy coend*, defined as

$$(21.3) \quad \begin{aligned} \mathbb{L} \int^{d \in \mathcal{D}} H(d, d) &= \mathcal{D} \overset{\mathbb{L}}{\odot}_{\mathcal{D}^{op} \otimes \mathcal{D}} H \\ &= B(\mathcal{D}, \mathcal{D}^{op} \otimes \mathcal{D}, QH) \end{aligned}$$

is a derived functor of the usual coend functor, at least if \mathcal{D} is good and also all its hom-objects $\mathcal{D}(d, d')$ are in \mathcal{V}_Q . For example, in [EKMM97, §IV.1], the topological Hochschild homology (THH) of an E_∞ ring spectrum R with coefficients in a bimodule M is defined as the derived smash product $R \overset{\mathbb{L}}{\wedge}_{R \wedge R^{op}} M$, which is a homotopy version of the coend $R \wedge_{R \wedge R^{op}} M \cong \int^R M$.

However, there is a more economical construction of homotopy coends which is also sometimes useful. To motivate this, notice that most coends which arise in practice are tensor products of functors. The tensor product which we have been writing $G \otimes_{\mathcal{D}} F$ can be shown to be isomorphic to the coend

$$\int^{\mathcal{D}} G \overline{\otimes} F$$

where the “external tensor product” is defined by $(G \overline{\otimes} F)(d, d') = Gd \otimes Fd'$. Applying (21.3) to this naively, we would obtain the homotopy coend

$$\mathbb{L} \int^{\mathcal{D}} G \overline{\otimes} F = B(\mathcal{D}, \mathcal{D} \otimes \mathcal{D}^{op}, Q(G \overline{\otimes} F)).$$

But we know that the homotopy tensor product of functors may be given instead by the simpler expression $B(QG, \mathcal{D}, QF)$. In fact, there is an analogous simplification that works for all coends, using a generalized notion of bar construction described in [Mey84] and [Mey86], for example. This bar construction associates to a bifunctor $H: \mathcal{D}^{op} \otimes \mathcal{D} \rightarrow \mathcal{M}$ the realization $B(\mathcal{D}, H)$ of the simplicial object

$$B_n(\mathcal{D}, H) = \coprod_{\alpha: [n] \rightarrow \mathcal{D}} (\mathcal{D}(\alpha_{n-1}, \alpha_n) \otimes \dots \otimes \mathcal{D}(\alpha_0, \alpha_1)) \odot H(\alpha_n, \alpha_0).$$

One can then modify the proof of Theorem 13.7 to show that under suitable conditions on \mathcal{D} , this construction also defines a homotopy coend. The “cyclic bar constructions” frequently used to compute Hochschild homology are special cases of this latter type of bar construction.

We end this section by considering how homotopy limits and colimits behave under the action of functors which change the target category \mathcal{M} . First we need to know that \mathcal{V} -deformable functors can be applied objectwise to diagrams and remain deformable. The restriction to adjunctions in the following proposition, rather than more general \mathcal{V} -functors, is merely for simplicity.

Proposition 21.4. *Let $\Phi: \mathcal{M} \rightleftarrows \mathcal{N} : \Psi$ be a left (resp. right) \mathcal{V} -deformable \mathcal{V} -adjunction. Then the induced \mathcal{V} -adjunction*

$$\Phi^{\mathcal{D}}: \mathcal{M}^{\mathcal{D}} \rightleftarrows \mathcal{N}^{\mathcal{D}} : \Psi^{\mathcal{D}}$$

is also left (resp. right) \mathcal{V} -deformable and hence gives rise to a derived $\text{Ho } \mathcal{V}$ -adjunction

$$\mathbf{L}\Phi^{\mathcal{D}}: \text{Ho}(\mathcal{M}^{\mathcal{D}}) \rightleftarrows \text{Ho}(\mathcal{N}^{\mathcal{D}}) : \mathbf{R}\Psi^{\mathcal{D}}.$$

If the original adjunction $\Phi \dashv \Psi$ was a \mathcal{V} -homotopical equivalence of \mathcal{V} -homotopical categories, then so is the induced one $\Phi^{\mathcal{D}} \dashv \Psi^{\mathcal{D}}$.

Proof. Since Φ is homotopical on \mathcal{M}_Q , $\Phi^\mathcal{D}$ is homotopical on $\mathcal{M}_Q^\mathcal{D}$ and therefore on the image of $B(\mathcal{D}, \mathcal{D}, Q-)$. Similarly, Ψ is homotopical on \mathcal{N}_R , so $\Psi^\mathcal{D}$ is homotopical on $\mathcal{N}_R^\mathcal{D}$. Suppose that the given adjunction is left \mathcal{V} -deformable; then Φ maps \mathcal{M}_Q to \mathcal{N}_Q , and since it is a left adjoint it preserves the bar construction, so it maps the image of $B(\mathcal{D}, \mathcal{D}, Q-)$ in $\mathcal{M}^\mathcal{D}$ to the corresponding image in $\mathcal{N}^\mathcal{D}$. Thus the induced adjunction is also left \mathcal{V} -deformable. The case when it is right deformable is even easier.

Finally, since $\Phi^\mathcal{D}$ and $\Psi^\mathcal{D}$ are simply Φ and Ψ applied objectwise, it is clear from condition (i) of Proposition 16.10 that if $\Phi \dashv \Psi$ is a \mathcal{V} -homotopical equivalence of \mathcal{V} -homotopical categories, so is $\Phi^\mathcal{D} \dashv \Psi^\mathcal{D}$. \square

Proposition 21.5. *Let $\Phi_1 \dashv \Psi_1$ and $\Phi_2 \dashv \Psi_2$ be composable \mathcal{V} -adjunctions which are either both left \mathcal{V} -deformable or both right \mathcal{V} -deformable. Then we have $\text{Ho } \mathcal{V}$ -natural isomorphisms*

$$\begin{aligned} \mathbf{L}\Phi_2^\mathcal{D} \circ \mathbf{L}\Phi_1^\mathcal{D} &\cong \mathbf{L}(\Phi_2 \circ \Phi_1)^\mathcal{D} \\ \mathbf{R}\Psi_1^\mathcal{D} \circ \mathbf{R}\Psi_2^\mathcal{D} &\cong \mathbf{R}(\Psi_1 \circ \Psi_2)^\mathcal{D} \end{aligned}$$

Proof. This follows directly from Proposition 16.9. \square

With this result in hand, it is easy to show that left derived functors of left adjoints preserve homotopy colimits, just as left adjoints preserve ordinary colimits, and dually.

Proposition 21.6. *Let $\Phi: \mathcal{M} \rightleftarrows \mathcal{N} : \Psi$ be a left or right \mathcal{V} -deformable \mathcal{V} -adjunction, let \mathcal{D} be good for the tensors of \mathcal{M} and \mathcal{N} , and let $G: \mathcal{D}^{\text{op}} \rightarrow \mathcal{V}$ be objectwise in \mathcal{V}_Q . Then there is a $\text{Ho } \mathcal{V}$ -natural isomorphism*

$$G \overset{\mathbf{L}}{\odot}_{\mathcal{D}} \mathbf{L}\Phi^\mathcal{D}(F) \cong \mathbf{L}\Phi(G \overset{\mathbf{L}}{\odot}_{\mathcal{D}} F).$$

Similarly, if \mathcal{E} is good for the cotensors of \mathcal{M} and \mathcal{N} and $G: \mathcal{E} \rightarrow \mathcal{V}$ is objectwise in \mathcal{V}_Q , then there is a $\text{Ho } \mathcal{V}$ -natural isomorphism

$$\mathbf{R}\{G, \mathbf{R}\Psi^\mathcal{E}(F)\}^\mathcal{E} \cong \mathbf{R}\Psi(\mathbf{R}\{G, F\}^\mathcal{E}).$$

Proof. This follows from Theorems 21.1 and 21.2 and Propositions 21.4 and 16.9. \square

22. HOMOTOPY THEORY OF ENRICHED DIAGRAMS

In this section, we consider the effect of functors that change the shape category \mathcal{D} . These give rise to Kan extension functors on diagram categories. Namely, given $K: \mathcal{D} \rightarrow \mathcal{E}$, we can restrict along K giving $K^*: \mathcal{M}^\mathcal{E} \rightarrow \mathcal{M}^\mathcal{D}$, and this functor has left and right adjoints Lan_K and Ran_K . The homotopical behavior of these functors is important in many different contexts, such as the comparison in [MMSS01] of various types of diagram spectra, including symmetric spectra and orthogonal spectra. The ‘‘prolongation’’ functors in that paper are left Kan extensions for which the enrichment, as considered here, is an essential aspect.

We continue our standing assumptions from the last section. Recall from [Kel82, §4.1] that (enriched) left Kan extensions can be computed as tensor products of functors, $\text{Lan}_K F(d) \cong \mathcal{D}(K-, d) \odot_{\mathcal{D}} F$. Thus, in addition to Definition 20.1 and Definition 20.5, for this section we make the following definition.

Definition 22.1. A small \mathcal{V} -category \mathcal{D} is *very good for \odot* (resp. *very good for $\{\}$*) if it is good for \odot (resp. good for $\{\}$) and moreover all its hom-objects are in \mathcal{V}_Q .

This condition may initially seem unreasonably strong, especially in topological contexts, but the key is that in such contexts \mathcal{V}_Q does not have to be the actual cofibrant objects in a model structure; often something much weaker suffices. We will say more about these situations in §23.

Proposition 22.2. *If $K: \mathcal{D} \rightarrow \mathcal{E}$ is a \mathcal{V} -functor and \mathcal{D} is very good for \odot , then the adjunction*

$$\mathrm{Lan}_K: \mathcal{M}^{\mathcal{D}} \rightleftarrows \mathcal{M}^{\mathcal{E}} : K^*$$

is right \mathcal{V} -deformable. It therefore has a derived Ho \mathcal{V} -adjunction

$$\mathbf{L}\mathrm{Lan}_K: \mathrm{Ho}(\mathcal{M}^{\mathcal{D}}) \rightleftarrows \mathrm{Ho}(\mathcal{M}^{\mathcal{E}}) : \mathbf{R}K^*.$$

Moreover, for any other \mathcal{V} -functor $H: \mathcal{C} \rightarrow \mathcal{D}$, where \mathcal{C} is also very good for \odot , we have a Ho \mathcal{V} -natural isomorphism

$$\mathbf{L}\mathrm{Lan}_K \circ \mathbf{L}\mathrm{Lan}_H \cong \mathbf{L}\mathrm{Lan}_{KH}.$$

Proof. It follows as in Theorem 21.1 that Lan_K is homotopical on the image of $B(\mathcal{D}, \mathcal{D}, Q-)$, and K^* is homotopical everywhere since weak equivalences are objectwise. Moreover, for the same reason K^* maps $\mathcal{M}_R^{\mathcal{E}}$ into $\mathcal{M}_R^{\mathcal{D}}$, so the adjunction is right \mathcal{V} -deformable. The last statement follows from Proposition 16.9. \square

Proposition 22.3. *If $K: \mathcal{D} \rightarrow \mathcal{E}$ is a \mathcal{V} -functor and \mathcal{D} is very good for $\{\}$, then the adjunction*

$$K^*: \mathcal{M}^{\mathcal{E}} \rightleftarrows \mathcal{M}^{\mathcal{D}} : \mathrm{Ran}_K$$

is left \mathcal{V} -deformable when the diagram categories are given the cobar \mathcal{V} -homotopical structure. It therefore has a derived Ho \mathcal{V} -adjunction

$$\mathbf{L}K^*: \mathrm{Ho}(\mathcal{M}^{\mathcal{E}}) \rightleftarrows \mathrm{Ho}(\mathcal{M}^{\mathcal{D}}) : \mathbf{R}\mathrm{Ran}_K.$$

Moreover, for any other \mathcal{V} -functor $H: \mathcal{C} \rightarrow \mathcal{D}$, where \mathcal{C} is also very good for $\{\}$, we have

$$\mathbf{R}\mathrm{Ran}_K \circ \mathbf{R}\mathrm{Ran}_H \cong \mathbf{R}\mathrm{Ran}_{KH}.$$

Remark 22.4. Here we see a relatively common phenomenon. We have a functor K^* which has both left and right adjoints, and both left and right derived functors (when \mathcal{D} is very good for both \odot and $\{\}$). A natural question is whether the two agree. In this case, it is easy to see that they do. Since K^* is already homotopical, it descends to homotopy categories directly, and the result is equivalent to any other derived functor of it; thus $\mathbf{L}K^* \cong \mathbf{R}K^*$. In the terminology of Proposition 4.7, we could say that the identity functor is a middle deformation for K^* .

So far, however, this only gives agreement as unenriched functors, and we would like to know that the Ho \mathcal{V} -enrichments also coincide. It is straightforward, if a bit tedious, to check that the hypothesis of Proposition 14.5 is satisfied for K^* , so that the two enriched functors are also the same; thus we have a derived string of Ho \mathcal{V} -adjunctions:

$$\mathbf{L}\mathrm{Lan}_K \dashv \mathbf{R}K^* \cong \mathbf{L}K^* \dashv \mathbf{R}\mathrm{Ran}_K.$$

Proposition 22.5. *Let $K: \mathcal{D} \rightarrow \mathcal{E}$ be a \mathcal{V} -functor such that*

- \mathcal{D} and \mathcal{E} are both very good for the tensor \odot of \mathcal{M} ;
- Each map $K: \mathcal{D}(d, d') \rightarrow \mathcal{E}(Kd, Kd')$ is a weak equivalence in \mathcal{V} ; and

- K_0 is essentially surjective.

Then the \mathcal{V} -adjunction

$$\mathrm{Lan}_K: \mathcal{M}^{\mathcal{D}} \rightleftarrows \mathcal{M}^{\mathcal{E}} : K^*$$

is a \mathcal{V} -homotopical equivalence of \mathcal{V} -homotopical categories.

Proof. We use condition (ii) of Proposition 16.10. Let $F \in \mathcal{M}_Q^{\mathcal{D}}$, so that $B(\mathcal{D}, \mathcal{D}, F) \in \mathcal{M}_{BQ}^{\mathcal{D}}$; we must show first that for each $d \in D$,

$$B(\mathcal{D}(-, d), \mathcal{D}, F) \longrightarrow B(\mathcal{E}(K-, Kd), \mathcal{D}, F) \longrightarrow RB(\mathcal{E}(K-, Kd), \mathcal{D}, F)$$

is a weak equivalence. But since $F \in \mathcal{M}_Q^{\mathcal{D}}$, \mathcal{D} and \mathcal{E} are both very good, and each $\mathcal{D}(d, -) \rightarrow \mathcal{E}(Kd, K-)$ is a weak equivalence by assumption, the first map is a weak equivalence, and the second is trivially a weak equivalence.

Now let $G \in \mathcal{N}_R^{\mathcal{D}}$; we must show that for each $e \in \mathcal{E}$,

$$(22.6) \quad B(\mathcal{E}(K-, e), \mathcal{D}, GK) \longrightarrow \mathcal{E}(K-, e) \odot_{\mathcal{D}} GK \longrightarrow Ge$$

is a weak equivalence. Since K is essentially surjective, we may assume that $e = Kd$ for some $d \in \mathcal{D}$. Consider the composite

$$B(\mathcal{D}(-, d), \mathcal{D}, GK) \longrightarrow B(\mathcal{E}(K-, Kd), \mathcal{D}, GK) \longrightarrow GKd.$$

Since $K: \mathcal{D}(-, d) \rightarrow \mathcal{E}(K-, Kd)$ is a weak equivalence and \mathcal{D} and \mathcal{E} are very good, the left map is a weak equivalence. The composite is just the map ε from Lemma 13.5, so it is also a weak equivalence. Thus, by the 2-out-of-3 property, the desired map is a weak equivalence. \square

In the case when \mathcal{D} and \mathcal{E} have one object, this result implies that a weak equivalence between “cofibrant” monoids in a monoidal homotopical category induces an equivalence between their categories of modules.

We would like to weaken the condition of essential surjectivity of K in Proposition 22.5 to a “homotopical” one. A natural set of definitions is the following.

Definition 22.7. Let $K: \mathcal{D} \rightarrow \mathcal{E}$ be a \mathcal{V} -functor.

- Say K is *homotopically fully faithful* if each map $K: \mathcal{D}(d, d') \rightarrow \mathcal{E}(Kd, Kd')$ is a weak equivalence in \mathcal{V} .
- Say K is *homotopically essentially surjective* if every object e is connected to some object Kd in the image of K by a zigzag of \mathcal{V} -equivalences (as in Definition 17.2) in \mathcal{E}_0 .
- If K is both homotopically fully faithful and homotopically essentially surjective, say it is a *homotopical equivalence of \mathcal{V} -categories*.

Remark 22.8. If K is homotopically fully faithful, then the induced $\mathrm{Ho} \mathcal{V}$ -functor $hK: h\mathcal{D} \rightarrow h\mathcal{E}$ is fully faithful. The converse is true if \mathcal{V} is saturated. Similarly, if K is homotopically essentially surjective, then hK is essentially surjective. The converse is true if all hom-objects $\mathcal{E}(e, e')$ are in \mathcal{V}_R , since then by Proposition 17.5 $h\mathcal{E}_0$ is the localization of \mathcal{E}_0 at the \mathcal{V} -equivalences, so any isomorphism in $h\mathcal{E}_0$ must be the image of a zigzag of \mathcal{V} -equivalences in \mathcal{E}_0 .

In particular, this shows that in the case $\mathcal{V} = s\mathcal{S}$, a homotopical equivalence of $s\mathcal{S}$ -categories is in particular a *DK-equivalence*, and that the converse is true for simplicially enriched categories which are locally Kan. In our terminology, a DK-equivalence is a $s\mathcal{S}$ -functor K which is homotopically fully faithful and such that $(hK)_0$ is an equivalence. The notion of DK-equivalence was first defined in [DK80], where it was called a “weak equivalence of simplicial categories;” see also [Ber07].

We can now prove the following generalization of Proposition 22.5.

Proposition 22.9. *Let $K: \mathcal{D} \rightarrow \mathcal{E}$ be a homotopical equivalence of \mathcal{V} -categories such that \mathcal{D} and \mathcal{E} are very good for the tensor \odot of \mathcal{M} , and assume that \mathcal{V} and \mathcal{M} are saturated. Then the \mathcal{V} -adjunction*

$$\mathbf{Lan}_K: \mathcal{M}^{\mathcal{D}} \rightleftarrows \mathcal{M}^{\mathcal{E}} : K^*$$

is a \mathcal{V} -homotopical equivalence of \mathcal{V} -homotopical categories.

Proof. The first half of the proof of Proposition 22.5 still works, and the second half shows that (22.6) is a weak equivalence for all $e = Kd$. Thus it remains only to prove that if (22.6) is a weak equivalence for e and $e - e'$ is a \mathcal{V} -equivalence (in either direction), then (22.6) is a weak equivalence for e' . In this case we have the following commuting square:

$$\begin{array}{ccc} B(\mathcal{E}(K-, e), \mathcal{D}, GK) & \xrightarrow{\sim} & Ge \\ \downarrow & & \downarrow \\ B(\mathcal{E}(K-, e'), \mathcal{D}, GK) & \longrightarrow & Ge' \end{array}$$

Since $e - e'$ is a \mathcal{V} -equivalence in \mathcal{E} , the induced map $\mathcal{E}(K-, e) - \mathcal{E}(K-, e')$ is a \mathcal{V} -equivalence in \mathcal{V} , hence a weak equivalence since \mathcal{V} is saturated. Since \mathcal{D} and \mathcal{E} are very good, this makes the left vertical map a weak equivalence. Similarly, since G is a \mathcal{V} -functor, by Proposition 17.4 it preserves all \mathcal{V} -equivalences, so since \mathcal{M} is saturated, the right vertical map is also a weak equivalence. By the 2-out-of-3 property, the bottom horizontal map is also a weak equivalence, as desired. \square

There is, of course, a dual result for right Kan extensions. By Remark 22.4 and the uniqueness of adjoint equivalences, it follows that if K satisfies the hypotheses of Proposition 22.5 or Proposition 22.9, then $\mathbf{L} \mathbf{Lan}_K \cong \mathbf{R} \mathbf{Ran}_K$.

We now consider the associativity of the derived tensor product of functors.

Proposition 22.10. *Let \mathcal{D} and \mathcal{E} be good for the product \otimes of \mathcal{V} and consider \mathcal{V} -functors $H: \mathcal{D}^{op} \rightarrow \mathcal{V}$, $G: \mathcal{E}^{op} \otimes \mathcal{D} \rightarrow \mathcal{V}$, and $F: \mathcal{E} \rightarrow \mathcal{V}$. Then we have a $\mathbf{Ho} \mathcal{V}$ -natural isomorphism*

$$H \otimes_{\mathcal{D}}^{\mathbf{L}} (G \otimes_{\mathcal{E}}^{\mathbf{L}} F) \cong (H \otimes_{\mathcal{D}}^{\mathbf{L}} G) \otimes_{\mathcal{E}}^{\mathbf{L}} F.$$

Proof. We use a two-variable version of Proposition 16.9. Both $\otimes_{\mathcal{D}}$ and $\otimes_{\mathcal{E}}$ are \mathcal{V} -deformable, so it suffices to check that $\otimes_{\mathcal{D}}$ maps $(\mathcal{V}^{\mathcal{D}^{op}})_Q \otimes (\mathcal{V}^{\mathcal{E}^{op} \otimes \mathcal{D}})_Q$ into $(\mathcal{V}^{\mathcal{E}})_Q$, and similarly for $\otimes_{\mathcal{E}}$. By $(-)_Q$ we refer here to the bar \mathcal{V} -homotopical structure. It is straightforward to check that

$$B(\mathcal{D} \otimes \mathcal{E}^{op}, \mathcal{D} \otimes \mathcal{E}^{op}, QG) \cong B(\mathcal{E}^{op}, \mathcal{E}^{op}, B(\mathcal{D}, \mathcal{D}, QG)).$$

Since $\otimes_{\mathcal{D}}$ is applied pointwise with respect to \mathcal{E} , it commutes with $B(\mathcal{E}^{op}, \mathcal{E}^{op}, -)$. Thus $\otimes_{\mathcal{D}}$ maps the image of the deformation

$$B(\mathcal{D}^{op}, \mathcal{D}^{op}, Q-) \times B(\mathcal{D} \otimes \mathcal{E}^{op}, \mathcal{D} \otimes \mathcal{E}^{op}, Q-)$$

into the image of $B(\mathcal{E}^{op}, \mathcal{E}^{op}, Q-)$, as desired. The case of $\otimes_{\mathcal{E}}$ is similar. \square

We note in passing that combining this result with methods similar to those used in Proposition 15.4 produces the following result.

Proposition 22.11. *Let $\mathcal{B}_{\mathcal{V}}$ be the \mathcal{V} -enriched bicategory whose 0-cells are very good small \mathcal{V} -categories, whose 1-cells from \mathcal{D} to \mathcal{E} are distributors (i.e. \mathcal{V} -functors $\mathcal{E}^{\text{op}} \otimes \mathcal{D} \rightarrow \mathcal{V}$), and whose 2-cells are \mathcal{V} -natural transformations. Then $\mathcal{B}_{\mathcal{V}}$ has a homotopy bicategory $\text{Ho}(\mathcal{B}_{\mathcal{V}})$ which is $\text{Ho } \mathcal{V}$ -enriched.*

This “homotopical bicategory” has recently arisen in connection with unpublished work by Kate Ponto involving fixed point theory. The left and right homotopy Kan extensions considered above equip this homotopy bicategory with “base change functors” similar to those considered in [MS06].

23. HOW TO PROVE GOODNESS

In this section, we make the case that being good or very good is basically a cofibrancy condition on the small category \mathcal{D} . The most appropriate notion of “cofibration,” however, can vary widely. In particular, even when a model structure exists, the cofibrations of the model structure are not necessarily the best cofibrations to use for this purpose. In this section, we use the term “cofibration” without prejudice as to meaning, adopting from [EKMM97] and [MS06] the terminology *q-cofibration* for the model structure cofibrations. Another common type of cofibration is an *h-cofibration* (for “Hurewicz”), generally defined by a homotopy lifting property.

Our goal is to justify the following meta-statements.

Meta-definition 23.1. Suppose that \mathcal{V} comes equipped with a notion of “cofibration” and a collection of “good objects”. Say a small \mathcal{V} -category \mathcal{D} is *cofibrant* if each hom-object $\mathcal{D}(d, d')$ is good and each unit inclusion $E \rightarrow \mathcal{D}(d, d)$ is a cofibration.

Meta-theorem 23.2. *In the above situation, if $(\otimes, \text{hom}_\ell, \text{hom}_r): \mathcal{M} \otimes \mathcal{N} \rightarrow \mathcal{P}$ is a two-variable \mathcal{V} -adjunction and \mathcal{M} , \mathcal{N} , and \mathcal{P} also come equipped with notions of “cofibration” and collections of “good objects,” then any cofibrant small \mathcal{V} -category \mathcal{D} is good for \otimes .*

One situation in which these meta-statements apply is when \mathcal{V} , \mathcal{M} , \mathcal{N} , and \mathcal{P} are model categories, and the cofibrations and cofibrant objects are the *q*-cofibrations and *q*-cofibrant objects. In this case, we will prove a version of Meta-theorem 23.2 as Theorem 23.12. However, there are other situations in which the metatheorem applies, but the cofibrations in question do not arise from a model structure.

For example, it applies to the \mathbb{L} -spectra and R -modules of [EKMM97] when the cofibrations are the Hurewicz cofibrations (maps with the homotopy extension property) and every spectrum is “good.” It applies to Lewis-May spectra with the Hurewicz cofibrations, taking the “good” spectra to be the *tame* spectra. It also applies to the “well-grounded topological categories” of [MS06, ch. 5], when the cofibrations are the *cyl*-cofibrations and the “good objects” are the well-grounded objects. These situations are all topological, but we suspect that similar results are true in other contexts, such as for chain complexes.

A proof of goodness generally has three steps.

- (i) Show that if \mathcal{D} is cofibrant and F and G are objectwise cofibrant, then $B_\bullet(G, \mathcal{D}, F)$ is Reedy cofibrant.
- (ii) Show that weak equivalences between functors give rise to weak equivalences between simplicial bar constructions.

- (iii) Show that geometric realization takes weak equivalences between Reedy cofibrant objects to weak equivalences between cofibrant objects.

The notion of *Reedy cofibrant* is an obvious generalization of the model-category definition. When given a notion of cofibration, we say a simplicial object X_\bullet is Reedy cofibrant if each latching map $L_n X \rightarrow X_n$ is a cofibration. Note that in the topological literature, when the cofibrations are the Hurewicz cofibrations, or h -cofibrations, Reedy h -cofibrant simplicial objects have usually been called *proper simplicial objects*. In this case, the latching objects $L_n X$ are often written as sX_n .

To clarify the essential points, we first give a more general definition of bar constructions. This is the most general definition of bar constructions we have seen; it includes the two-sided bar construction we have been using, as well as both the monadic bar constructions used in [May72] and the generalized bar constructions of [Mey84, Mey86].

Definition 23.3. Let $(\mathcal{C}, \otimes, E)$ be a monoidal category and D a monoid in it. A functor $K: \mathcal{C}^{op} \times \mathcal{C} \rightarrow \mathcal{P}$ is called a *D-bimodule* if it is equipped with two natural transformations

$$\begin{aligned} K(D \otimes X) &\longrightarrow K(X) \\ K(X \otimes D) &\longrightarrow K(X) \end{aligned}$$

each satisfying evident associativity and unit conditions and related by the bimodule commutative square:

$$\begin{array}{ccc} K(D \otimes X \otimes D) & \longrightarrow & K(D \otimes X) \\ \downarrow & & \downarrow \\ K(X \otimes D) & \longrightarrow & K(X). \end{array}$$

The reader should keep the following example in mind.

Example 23.4. Let \mathcal{V} be a monoidal category, let \mathcal{O} be a set, and let \mathcal{C} be the category of \mathcal{O} -graphs in \mathcal{V} . This is just the category $\mathcal{V}^{\mathcal{O} \times \mathcal{O}}$. We equip \mathcal{C} with the following monoidal product:

$$(\mathcal{D} \otimes \mathcal{E})(a, b) = \coprod_{c \in \mathcal{O}} \mathcal{E}(c, b) \otimes \mathcal{D}(a, c).$$

Then a monoid in \mathcal{C} is precisely a small \mathcal{V} -category with object set \mathcal{O} .

Let \mathcal{D} be such a small \mathcal{V} -category with object set \mathcal{O} , and let $F: \mathcal{D} \rightarrow \mathcal{N}$, $G: \mathcal{D}^{op} \rightarrow \mathcal{M}$, and $\otimes: \mathcal{M} \otimes \mathcal{N} \rightarrow \mathcal{P}$ be \mathcal{V} -functors. We define a functor $K: \mathcal{C} \rightarrow \mathcal{P}$ as follows.

$$K(\mathcal{E}) = \coprod_{a, b \in \mathcal{O}} \mathcal{E}(a, b) \odot (G(b) \otimes F(a)).$$

Then the action of \mathcal{D} on F and G makes K into a \mathcal{D} -bimodule as above.

Definition 23.5. Given a D -bimodule K as in Definition 23.3, where \mathcal{P} is simplicially enriched, we define the *bar construction* $B(D, K)$ to be the geometric realization of the following simplicial bar construction.

$$B_n(D, K) = K(\overbrace{D \otimes \dots \otimes D}^n)$$

Here the degeneracies are induced by the unit map of D , the inner faces by the multiplication of D , and the outer faces by the action of D on K .

In the situation of Example 23.4, this gives precisely the enriched two-sided bar construction $B(G, \mathcal{D}, F)$ defined in §12.

We now begin our proofs of goodness. We have the following general result about cofibrancy of generalized bar constructions.

Proposition 23.6. *In the situation of Definition 23.3, suppose that \mathcal{C} and \mathcal{P} have subcategories of “cofibrations” satisfying the following properties.*

- *The unit map $E \rightarrow D$ is a cofibration in \mathcal{C} ;*
- *The pushout product of $E \rightarrow D$ with any other cofibration in \mathcal{C} is a cofibration in \mathcal{C} ;*
- *The functor $D \otimes -$ preserves pushouts (for example, when \mathcal{C} is closed monoidal);*
- *K preserves pushouts and cofibrations; and*
- *The map $\emptyset \rightarrow K(E)$ is a cofibration in \mathcal{P} .*

Then the simplicial bar construction $B_\bullet(D, K)$ is Reedy cofibrant in \mathcal{P} .

Proof. We must show that each map

$$L_n B_\bullet(D, K) \longrightarrow B_n(D, K)$$

is a cofibration in \mathcal{P} . When $n = 0$, we have $L_0 B_\bullet(D, K) = \emptyset$, and $B_0(D, K) = K(E)$, so the final condition above gives the desired result in this case. For $n \geq 1$, we observe that since K preserves pushouts, the desired map is K applied to a map

$$(23.7) \quad L_n D \rightarrow D^n$$

in \mathcal{C} . Since K preserves cofibrations, it suffices to show that the maps (23.7) are cofibrations in \mathcal{C} . We do this by induction on n , starting with $n = 1$. We could have started our induction with $n = 0$, and omitted the last condition above, if we were willing to assume that K preserves initial objects and that the map $\emptyset \rightarrow E$ is a cofibration in \mathcal{C} , as are frequently the case.

When $n = 1$, the map (23.7) is simply $E \rightarrow D$, which was assumed to be a cofibration. So suppose that $L_n D \rightarrow D^n$ is a cofibration; then we have the following pushout square.

$$\begin{array}{ccc} L_n D & \longrightarrow & D^n \\ \downarrow & & \downarrow \\ D \otimes L_n D & \longrightarrow & L_{n+1} D \\ & \searrow & \searrow \\ & & D^{n+1} \end{array}$$

The assumption that $D \otimes -$ preserves pushouts ensures that the pushout here is, in fact, $L_{n+1} D$. The induced map from the pushout is what we want to show is a cofibration, but it is evidently a pushout product of the given cofibration $L_n D \rightarrow D^n$ with $E \rightarrow D$, hence a cofibration by assumption. Thus the maps (23.7) are cofibrations for all $n \geq 1$ and hence $B_\bullet(D, K)$ is Reedy cofibrant, as desired. \square

Example 23.8. Consider the situation of Example 23.4. Suppose that \mathcal{V} and \mathcal{P} have subcategories of “cofibrations” and collections of “good objects” satisfying the following conditions.

- The tensor product \otimes of \mathcal{V} preserves colimits;

- \otimes preserves good objects;
- The tensor \odot of \mathcal{P} preserves good objects and cofibrations between good objects;
- The coproduct in \mathcal{P} of good objects is good;
- The pushout product of cofibrations in \mathcal{V} is a cofibration; and
- The coproduct of cofibrations in \mathcal{V} and in \mathcal{P} is a cofibration.

Suppose also that \mathcal{D} is a small \mathcal{V} -category and that $G: \mathcal{D}^{op} \rightarrow \mathcal{M}$, $F: \mathcal{D} \rightarrow \mathcal{N}$, and $\otimes: \mathcal{M} \otimes \mathcal{N} \rightarrow \mathcal{P}$ are \mathcal{V} -functors satisfying the following conditions.

- Each map $\emptyset \rightarrow \mathcal{D}(d, d')$ and each unit inclusion $E \rightarrow \mathcal{D}(d, d)$ is a cofibration in \mathcal{V} ;
- Each object $\mathcal{D}(d, d')$ is good;
- G and F are objectwise good and objectwise cofibrant (i.e. the maps $\emptyset \rightarrow Gd$ and $\emptyset \rightarrow Fd$ are cofibrations); and
- \otimes preserves good objects and cofibrant objects.

Then it is straightforward to check that the conditions of Proposition 23.6 are satisfied, so that the two-sided simplicial bar construction $B_{\bullet}(G, \mathcal{D}, F)$ is Reedy cofibrant in \mathcal{P} . Moreover, in this case it is objectwise good; this will be needed below.

This completes step (i). Step (ii), that objectwise weak equivalences in G and F give rise to weak equivalences between simplicial bar constructions, is easily handled by the following observation.

Proposition 23.9. *In the situation of Example 23.4, suppose that \mathcal{V} , \mathcal{M} , \mathcal{N} , and \mathcal{P} are homotopical and have collections of “good” objects satisfying the following properties.*

- The bifunctor \otimes , and the tensor \odot of \mathcal{P} , preserve good objects and weak equivalences between good objects;
- The coproduct of weak equivalences between good objects in \mathcal{P} is a weak equivalence; and
- Each object $\mathcal{D}(d, d')$ is good.

Then objectwise weak equivalences $G \xrightarrow{\sim} G'$ and $F \xrightarrow{\sim} F'$ between functors which are objectwise good give rise to an objectwise weak equivalence

$$B_{\bullet}(G, \mathcal{D}, F) \xrightarrow{\sim} B_{\bullet}(G', \mathcal{D}, F').$$

Proof. Straightforward from Definition 19.4. □

We now turn to step (iii). The following proposition is simply a categorical reformulation of the classical proofs, such as that in [EKMM97, X.2.4].

Proposition 23.10. *Suppose that \mathcal{P} is a homotopical category, simplicially enriched, and equipped with a subcategory of “cofibrations” and a collection of “good objects” satisfying the following properties.*

- The simplicial tensor \odot of \mathcal{P} preserves good objects, cofibrations between good objects, and weak equivalences between good objects in \mathcal{P} ;
- The pushout product under \odot of a cofibration of simplicial sets (i.e. a monomorphism) and a cofibration in \mathcal{P} is a cofibration in \mathcal{P} ;
- Pushouts of cofibrations in \mathcal{P} are cofibrations;

- (Gluing Lemma) If in the following diagram:

$$\begin{array}{ccc} \cdot & \xleftarrow{\quad} & \cdot & \xrightarrow{\quad} & \cdot \\ \downarrow \sim & & \downarrow \sim & & \downarrow \sim \\ \cdot & \xleftarrow{\quad} & \cdot & \xrightarrow{\quad} & \cdot \end{array}$$

all objects are good, the maps displayed as \succrightarrow are cofibrations, and the vertical maps are weak equivalences, then the pushouts are good and the induced map of pushouts is a weak equivalence. In particular, weak equivalences between good objects are preserved by pushouts along cofibrations.

- (Colimit Lemma) If M and N are the colimits of sequences of cofibrations $M_k \succrightarrow M_{k+1}$ and $N_k \succrightarrow N_{k+1}$ between good objects, then M and N are good. Moreover, a compatible sequence of weak equivalences $f_k: M_k \xrightarrow{\sim} N_k$ induces a weak equivalence $f: M \xrightarrow{\sim} N$.

Then if $X_\bullet \rightarrow Y_\bullet$ is an objectwise weak equivalence between simplicial objects of \mathcal{P} which are Reedy cofibrant and objectwise good, its realization $|X| \rightarrow |Y|$ is a weak equivalence.

Proof. Let $\Delta_{<n}$ be the full subcategory of Δ spanned by the objects m such that $m < n$, and for a simplicial object $X_\bullet: \Delta^{op} \rightarrow \mathcal{M}$ write $X_{<n}$ for its restriction to $\Delta_{<n}$. Write

$$|X|_n = \Delta^{<n} \odot_{\Delta_{<n}^{op}} X_{<n}.$$

for the “partial realization” of X_\bullet . It is straightforward to check that we have

$$|X| = \operatorname{colim}_{n \rightarrow \infty} |X|_n.$$

In other words, the geometric realization is “filtered by simplicial degree.” Therefore, by the colimit lemma, if we can show that each induced map $|X|_n \rightarrow |Y|_n$ is a weak equivalence between good objects, the desired result will follow.

We prove this by induction on n . The case $n = 0$ follows directly because $|X|_0 = X_0$. So assume that $|X|_{n-1} \rightarrow |Y|_{n-1}$ is a weak equivalence between good objects. Consider the following sequence of pushouts, which it is straightforward to check produces $|X|_n$ from $|X|_{n-1}$.

$$\begin{array}{ccccc} \partial\Delta^n \odot L_n X & \longrightarrow & \partial\Delta^n \odot X_n & & \\ \downarrow & & \downarrow & & \\ \Delta^n \odot L_n X & \longrightarrow & P_n X & \longrightarrow & \Delta^n \odot X_n \\ & & \downarrow & & \downarrow \\ & & |X|_{n-1} & \longrightarrow & |X|_n. \end{array}$$

Since \odot preserves good objects and cofibrations between good objects, the first pushout is a pushout along a cofibration between good objects, and since \odot preserves weak equivalences between good objects in \mathcal{P} , the gluing lemma implies that the induced map $P_n X \rightarrow P_n Y$ is a weak equivalence between good objects.

Now, the map $P_n X \rightarrow \Delta^n \odot X_n$ is the pushout product of $\partial\Delta^n \rightarrow \Delta^n$ and $L_n X \rightarrow X_n$, which are both cofibrations, hence is a cofibration. The fact that \odot preserves weak equivalences between good objects, combined with the induction assumption, implies that the conditions of the gluing lemma are again satisfied

for the second pushout. Thus, $|X|_n \rightarrow |Y|_n$ is a weak equivalence between good objects, as desired. \square

It follows that if all the conditions in the above propositions are satisfied, then $B(-, \mathcal{D}, -)$ is homotopical on objectwise good diagrams. The other two conditions in Definition 20.1 also follow, as do those in Definition 22.1, if the “good” objects in the above propositions are the left deformation retracts $\mathcal{V}_Q, \mathcal{M}_Q, \mathcal{N}_Q$, and \mathcal{P}_Q and the left adjoint $s\mathcal{S} \rightarrow \mathcal{V}$ lands in \mathcal{V}_Q . In particular, we have the following special case.

Definition 23.11. Let \mathcal{V} be a monoidal model category. A small \mathcal{V} -category \mathcal{D} is *q-cofibrant* if each hom-object $\mathcal{D}(d, d')$ is *q-cofibrant* in \mathcal{V} and all the unit inclusions $E \rightarrow \mathcal{D}(d, d)$ are *q-cofibrations* in \mathcal{V} .

Theorem 23.12. Let \mathcal{V} be a simplicial monoidal model category, \mathcal{M}, \mathcal{N} , and \mathcal{P} be \mathcal{V} -model categories, and $(\otimes, \text{hom}_\ell, \text{hom}_r): \mathcal{M} \otimes \mathcal{N} \rightarrow \mathcal{P}$ be a Quillen two-variable \mathcal{V} -adjunction. Then a *q-cofibrant* \mathcal{V} -category \mathcal{D} is good for any Quillen two-variable \mathcal{V} -adjunction, and very good for the tensor and cotensor of any \mathcal{V} -model category.

Proof. Taking the cofibrations to be the *q-cofibrations* and the good objects to be the *q-cofibrant* objects, it is well-known that all the conditions of Example 23.8, Proposition 23.9, and Proposition 23.10 are satisfied. \square

This completes our justification of the meta-statements at the beginning of this section.

In the dual situation, to prove goodness for cotensor products, the arguments are analogous. The cofibrations in \mathcal{V} remain cofibrations, but in \mathcal{P} we must consider fibrations instead, with properties dual to those enumerated above. Similarly, realization of simplicial objects becomes totalization of cosimplicial objects, expressed as an inverse limit of partial totalizations.

Note also that when we consider goodness for the hom-functor $\mathcal{M}(-, -): \mathcal{M}^{op} \otimes \mathcal{M} \rightarrow \mathcal{V}$ of a \mathcal{V} -homotopical category, \mathcal{M} itself no longer needs any notion of cofibration or fibration. Instead, \mathcal{V} must have both cofibrations and fibrations which interact in a suitable way. Thus, if \mathcal{V} is nice enough, *all* \mathcal{V} -homotopical categories have a well-behaved homotopy theory of diagrams. This shows the essential role of the enrichment, which puts the real “homotopy” in homotopy theory.

24. OBJECTWISE GOOD REPLACEMENTS

In this section, we investigate under what conditions $\mathcal{M}_Q^\mathcal{D}$ is a left deformation retract of $\mathcal{M}^\mathcal{D}$, so that every diagram can be replaced by one which is “objectwise good.” Our first observation is trivial.

Proposition 24.1. *If \mathcal{D} is not enriched, then $\mathcal{M}_Q^\mathcal{D}$ is a left deformation retract of $\mathcal{M}^\mathcal{D}$.*

Proof. Apply the functor Q objectwise. \square

The problem in the enriched case is that frequently in the case of model categories, the functor Q is produced by a small object argument. While the small object argument excels at producing functorial replacements, it generally fails to produce *enriched* functorial replacements, and when Q is not an enriched functor, we cannot

compose it with an enriched functor $F: \mathcal{D} \rightarrow \mathcal{M}$ to produce a new enriched functor $QF: \mathcal{D} \rightarrow \mathcal{M}$.

There are some cases, however, in which Q can be chosen to be an enriched functor. One trivial case is when $\mathcal{M}_Q = \mathcal{M}$ (every object is cofibrant), so that Q can be chosen to be the identity. Dually, if $\mathcal{M}_R = \mathcal{M}$, then R can be chosen to be the identity. This latter case arises frequently in topology, which makes homotopy limits easier to deal with; this is fortunate because, as we have mentioned frequently, our results are generally more necessary for limits than for colimits.

Another case in which Q can be chosen enriched is when every object of \mathcal{V} is cofibrant. This allows us to actually perform a small-object-argument factorization in an enriched-functorial way. The only commonly occurring \mathcal{V} s for which this is true are $\mathcal{V} = s\mathcal{S}$ and $\mathcal{V} = \mathbf{Cat}$, but in these cases it simplifies things greatly. We briefly sketch a proof below; it is essentially the same as [Hir03, 4.3.8], although the latter is stated specifically for localizations of simplicial model categories.

Proposition 24.2. *Let \mathcal{V} be a monoidal model category, let \mathcal{M} be a \mathcal{V} -model category which is cofibrantly generated, and assume that for all $K \in \mathcal{V}$, the tensor $K \odot -$ preserves cofibrations in \mathcal{M} . For example, this is true if all objects of \mathcal{V} are cofibrant. Then there exists a \mathcal{V} -functorial cofibrant replacement functor Q on \mathcal{M} . Dually, if $K \odot -$ preserves trivial cofibrations for all K , there is a \mathcal{V} -functorial fibrant replacement functor R on \mathcal{M} .*

Proof. The construction is essentially the same as the usual small object argument. Given X , we construct QX as the colimit of a transfinite sequence $\{X_\beta\}$ starting with $X_0 = \emptyset$ and using the following pushouts.

$$(24.3) \quad \begin{array}{ccc} \coprod_{A \rightarrow B} (\mathcal{M}(A, X_\beta) \times_{\mathcal{M}(A, X)} \mathcal{M}(B, X)) \odot A & \longrightarrow & X_\beta \\ \downarrow & & \downarrow \\ \coprod_{A \rightarrow B} (\mathcal{M}(A, X_\beta) \times_{\mathcal{M}(A, X)} \mathcal{M}(B, X)) \odot B & \longrightarrow & X_{\beta+1} \end{array}$$

Here the coproducts are over all maps $A \rightarrow B$ in a set of generating cofibrations. The crucial difference from the usual small object argument is the use of the enriched ‘objects of commuting squares’ $\mathcal{M}(A, X_\beta) \times_{\mathcal{M}(A, X)} \mathcal{M}(B, X)$ instead of a coproduct over all such commuting squares in \mathcal{M}_0 . It is straightforward to check by induction that this change makes $X \mapsto QX$ \mathcal{V} -functorial, and the usual arguments work to show that $QX \rightarrow X$ is a trivial fibration. The assumption that \odot preserves cofibrations is needed to ensure that the left-hand maps are all still cofibrations, so that QX is in fact cofibrant. The construction of R is dual. \square

Unfortunately, having to use a small-object-argument factorization to correct the bar construction makes it much less explicit, thus partially nullifying one of its advantages. However, as noted after Definition 13.2, if the diagrams we begin with are already objectwise cofibrant, there is no need for a correction at all. The mere *existence* of the deformation $Q_{\mathcal{M}}^{\mathcal{D}}$ tells us that the results we get this way agree with the global derived functor.

There are other situations in which Q can be made \mathcal{V} -functorial, notably those in which a weaker notion than model-category-theoretic cofibrancy suffices. In these cases, the replacement functors are often given by explicit constructions, which can

usually be made \mathcal{V} -functorial. Frequently, in fact, these explicit constructions are themselves a sort of bar construction.

There is also a situation in which $\mathcal{M}_Q^{\mathcal{D}}$ is a deformation retract of $\mathcal{M}^{\mathcal{D}}$ even though Q is *not* a \mathcal{V} -functor, and that is when $\mathcal{M}^{\mathcal{D}}$ admits a model structure of its own in which the cofibrant diagrams are, in particular, objectwise cofibrant. In this case, a cofibrant replacement functor for $\mathcal{M}^{\mathcal{D}}$ serves as a deformation into $\mathcal{M}_Q^{\mathcal{D}}$.

As remarked in §11, the only type of model structure we can reasonably expect to have on an enriched diagram category is a projective one, but these exist in reasonable generality. There is, however, something a little strange about the idea of applying a projective-cofibrant replacement functor before the bar construction, since projective cofibrancy is sometimes a good enough deformation for homotopy colimits on its own. However, there are a couple of reasons why this is still a useful thing to consider.

Firstly, projective-cofibrant replacements are generally quite hard to compute with. But as in the situation of Proposition 24.2, if the diagram we started with is already objectwise cofibrant, we don't need to make it projective-cofibrant before applying the bar construction. The value of the existence of the projective model structure for us, then, is that it tells us that the bar construction actually defines a derived functor on the whole diagram category, rather than just the objectwise cofibrant diagrams.

Secondly, projective model structures are *not* directly useful for computing homotopy *limits*, because the limit functor is not right Quillen for them. Moreover, as noted in §11, injective model structures do not generally exist in the enriched situation. However, by combining a projective model structure with the cobar construction, we can compute homotopy limits: since projective-fibrant diagrams are, by definition, objectwise fibrant, a projective-fibrant replacement works perfectly well to correct the cobar construction.

With this as prelude, we now state a theorem about the existence of projective model structures on enriched diagram categories. This is a straightforward application of the principle of transfer of model structures along adjunctions, which was first stated in [Cra95]. Various special cases of this result are scattered throughout the literature, but we have been unable to find a statement of it in full generality.

For example, essentially the same result for the case $\mathcal{V} = \mathcal{M}$ can be found in [SS03a, 6.1]. In the case when \mathcal{D} has one object and $\mathcal{V} = \mathcal{M}$, it reduces to a result like those of [Hov98] and [SS00] for modules over a monoid in a monoidal model category. It can also be viewed as a special case of the model structures considered in [BM07] for algebras over colored operads.

Theorem 24.4. *Let \mathcal{V} be a monoidal model category and let \mathcal{M} be a cofibrantly generated \mathcal{V} -model category with generating cofibrations I and generating trivial cofibrations J , whose domains are assumed small with respect to all of \mathcal{M} . Let \mathcal{D} be a small \mathcal{V} -category such that $\mathcal{D}(d, d') \odot -$ preserves trivial cofibrations (for example, this occurs if each $\mathcal{D}(d, d')$ is cofibrant in \mathcal{V} , or if $\mathcal{M} = \mathcal{V}$ and \mathcal{V} satisfies the “monoid axiom”). Then $\mathcal{M}^{\mathcal{D}}$ has a cofibrantly generated \mathcal{V} -model structure in which the weak equivalences and fibrations are objectwise. If $\mathcal{D}(d, d') \odot -$ also preserves cofibrations (as also occurs if each $\mathcal{D}(d, d')$ is cofibrant) then cofibrant diagrams are in particular objectwise cofibrant.*

Proof. We use [Cra95, Theorem 3.3], or more precisely its restatement in [BM03, 2.5]. The forgetful functor $U: \mathcal{M}^{\mathcal{D}} \rightarrow \mathcal{M}^{\text{ob } \mathcal{D}}$ has both left and right adjoints, given

by left and right Kan extension. Write F for the left adjoint; we want to use F to induce a model structure on $\mathcal{M}^{\mathcal{D}}$.

First we observe that $\mathcal{M}^{\text{ob } \mathcal{D}}$ has a model structure which is just a power of the model structure on \mathcal{M} . This model structure is cofibrantly generated; sets of generating cofibrations and trivial cofibrations are given by

$$\begin{aligned} I' &= \{L_d f \mid f \in I, d \in \text{ob } \mathcal{D}\} \\ J' &= \{L_d f \mid f \in J, d \in \text{ob } \mathcal{D}\} \end{aligned}$$

where $L_d f$ consists of f at spot d and \emptyset elsewhere.

Since U has a right adjoint, it preserves colimits, and therefore its left adjoint F preserves smallness of objects. Therefore it remains to check that relative FJ' -cell complexes are weak equivalences in $\mathcal{M}^{\mathcal{D}}$. However, U preserves colimits and reflects weak equivalences, so it suffices to check that UFJ' -cell complexes are weak equivalences in $\mathcal{M}^{\text{ob } \mathcal{D}}$. Now by definition of F as a left Kan extension, we have

$$UF(L_d f)_{d'} = \mathcal{D}(d, d') \odot f$$

and so our assumption ensures that the maps in UFJ' are trivial cofibrations, hence relative UFJ' -cell complexes are also trivial cofibrations in $\mathcal{M}^{\text{ob } \mathcal{D}}$ and thus weak equivalences.

This shows that the model structure exists. To see that it is a \mathcal{V} -model structure, recall from [Hov99, 4.2.2] that it suffices to check the lifting extension properties for the cotensor. But since the fibrations, trivial fibrations, and cotensor are all objectwise, this follows from the corresponding property for \mathcal{M} . Similarly, the unit condition, when expressed using the cotensor, follows from the unit condition for \mathcal{M} .

For the last statement, we use an adjointness argument to show more generally that projective-cofibrations are objectwise cofibrations. Consider U and its *right* adjoint which we denote G ; we want to show that U preserves cofibrations, which is equivalent to showing that G preserves trivial fibrations. Now any trivial fibration in $\mathcal{M}^{\text{ob } \mathcal{D}}$ is a product of maps of the form $R_d f$ for trivial fibrations f in \mathcal{M} , where $R_d f$ consists of f at spot d and 1 elsewhere, and by definition of G as a right Kan extension, we have

$$G(R_d f)_{d'} = \{\mathcal{D}(d', d), f\}.$$

But since $\mathcal{D}(d', d) \odot -$ preserves cofibrations, by adjointness $\{\mathcal{D}(d', d), -\}$ preserves trivial fibrations, so G must preserve trivial fibrations, as desired. \square

It follows, for instance, that when \mathcal{D} is cofibrant, $\mathcal{M}_Q^{\mathcal{D}}$ and $\mathcal{M}_R^{\mathcal{D}}$ are, respectively, a left and a right deformation retract of $\mathcal{M}^{\mathcal{D}}$, and so (in addition to the projective \mathcal{V} -model structure) the bar and cobar \mathcal{V} -homotopical structures exist. As remarked above, the projective model structure can sometimes be used to compute homotopy colimits, but not homotopy limits; for this we need to use the cobar construction.

APPENDIX A. PROOF OF THEOREM 8.5

Here we will complete the proof of Theorem 8.5. The proof we give is slightly modified from the proof given in an early online draft of [DHKS04]. First, we recall the statement of the theorem.

Theorem 8.5. *The diagram $\mathcal{Q}F$ is Reedy cofibrant for all F , the functors \mathcal{Q} and $\text{Lan}_T \mathcal{Q}$ are homotopical, and δ is a natural weak equivalence. Moreover,*

$$(8.6) \quad \begin{aligned} \text{hocolim } F &\cong \text{colim}_{\Delta \mathcal{Q}} \mathcal{Q}F \\ &\cong \text{colim}_{\mathcal{Q}} \text{Lan}_T \mathcal{Q}F. \end{aligned}$$

Therefore, $(\text{Lan}_T \mathcal{Q}, \text{Lan}_T \delta)$ is a left deformation of colim , and hence hocolim is a derived functor of the usual colimit.

We will need some technical lemmas. Let $H: \mathcal{E} \rightarrow \mathcal{D}$ be a functor between small categories. Applying the nerve functor N , we obtain a map $NH: N\mathcal{E} \rightarrow N\mathcal{D}$ of simplicial sets. Given a simplex α in $N\mathcal{D}$, let $\Lambda_H(\alpha)$ be the fiber of NH over α , so that the following square is a pullback:

$$(A.1) \quad \begin{array}{ccc} \Lambda_H(\alpha) & \longrightarrow & N\mathcal{E} \\ \downarrow & & \downarrow NH \\ \Delta^n & \xrightarrow{\alpha} & N\mathcal{D} \end{array}$$

This defines a functor $\Lambda_H: \Delta \mathcal{D} \rightarrow s\mathcal{S}$.

Lemma A.2. *In the above situation, we have $\text{Lan}_{S^{op}} \Lambda_H \cong N(- \downarrow H)$.*

Proof. As in the proof of Proposition 7.5, left Kan extension can be computed as a colimit over a comma-category. Here, the formula is:

$$(A.3) \quad (\text{Lan}_{S^{op}} \Lambda_H)(d) \cong \text{colim}_{(S^{op} \downarrow d)} \Lambda_H.$$

Note moreover that we have $(S^{op} \downarrow d) \cong \Delta(d \downarrow \mathcal{D})$. Now the colimit (7.6) in $s\mathcal{S}$ can also be viewed as a colimit in $(s\mathcal{S} \downarrow N\mathcal{D})$, and pullback along NH , viewed as a functor $(s\mathcal{S} \downarrow N\mathcal{D}) \rightarrow (s\mathcal{S} \downarrow N\mathcal{E})$, preserves colimits because it has a right adjoint. Thus, applying colimits to the left vertical arrow in (A.1), and noting that $\text{colim}_{\Delta \mathcal{C}} \Delta^n \cong N\mathcal{C}$ for any category \mathcal{C} , we get the following pullback square.

$$\begin{array}{ccc} \text{Lan}_{S^{op}} \Lambda_H & \longrightarrow & N\mathcal{E} \\ \downarrow & & \downarrow NH \\ N(c \downarrow \mathcal{D}) & \longrightarrow & N\mathcal{D} \end{array}$$

It is straightforward to check that there is also a similar pullback square with $\text{Lan}_{S^{op}} \Lambda_H$ replaced by $N(c \downarrow H)$. \square

A particular case of this lemma is when $\mathcal{E} = \mathcal{D}$ and $H = \text{Id}_{\mathcal{D}}$. Then we have $\Lambda_H = \Delta \Pi: \Delta \mathcal{D} \rightarrow s\mathcal{S}$ (where $\Delta: \mathbf{\Delta} \rightarrow s\mathcal{S}$ is the canonical cosimplicial simplicial set, and Π is the forgetful functor from (5.3)), and the result says that

$$(A.4) \quad \text{Lan}_{S^{op}} \Delta \Pi \cong N(- \downarrow \mathcal{D}).$$

Now, given two simplices α and β in \mathcal{D} , let $\Lambda(\alpha, \beta)$ be their pullback over $N\mathcal{D}$, so that the following diagram is a pullback:

$$\begin{array}{ccc} \Lambda(\alpha, \beta) & \longrightarrow & \Delta^m \\ \downarrow & & \downarrow \beta \\ \Delta^n & \xrightarrow{\alpha} & N\mathcal{D}. \end{array}$$

This defines a functor $\Lambda: \Delta \mathcal{D} \times \Delta \mathcal{D} \rightarrow s\mathcal{S}$.

Lemma A.5.

$$\operatorname{colim}_{\Delta\mathcal{D}} \Lambda \cong \Delta\Pi: \Delta\mathcal{D} \rightarrow s\mathcal{S}$$

Proof. Similar to the proof of Lemma A.2. \square

Lemma A.6. *For any functors $K: \mathcal{D} \rightarrow \mathcal{E}$, $G: \mathcal{D}^{op} \rightarrow s\mathcal{S}$, and $F: \mathcal{E} \rightarrow \mathcal{M}$,*

$$(\operatorname{Lan}_{K^{op}} G) \odot_{\mathcal{D}} F \cong G \odot_{\mathcal{E}} (K^*F).$$

Proof. Note that because the tensor \odot has adjoints on both sides (the cotensor and the enriched hom-functor), it preserves colimits in both variables. The result follows from this, the fact that colimits commute with colimits, and the fact that $\mathcal{E}(-, Kd) \odot_{\mathcal{E}} F \cong F(Kd)$, as remarked in the proof of Lemma 7.9. \square

We now attack the proof of Theorem 8.5. Recall that in §8, we reduced the proof to showing (8.6) and that $\mathcal{Q}F$ is Reedy cofibrant. We combine the above lemmas to manipulate hocolim into a form looking more like $\mathcal{Q}F$ as follows.

$$\begin{aligned} \operatorname{hocolim} F &= N(- \downarrow \mathcal{D}) \odot_{\mathcal{D}} \mathcal{Q}F && \text{(by definition)} \\ &\cong \operatorname{Lan}_{S^{op}} \Delta\Pi \odot_{\mathcal{D}} \mathcal{Q}F && \text{(by (A.4))} \\ &\cong \Delta\Pi \odot_{\Delta\mathcal{D}} S^* \mathcal{Q}F && \text{(by Lemma A.6)} \\ &\cong \left(\operatorname{colim}_{\alpha \in \Delta\mathcal{D}} \Lambda(\alpha, -) \right) \odot_{\Delta\mathcal{D}} S^* \mathcal{Q}F && \text{(by Lemma A.5)} \\ &\cong \operatorname{colim}_{\alpha \in \Delta\mathcal{D}} \left(\Lambda(\alpha, -) \odot_{\Delta\mathcal{D}} S^* \mathcal{Q}F \right) \end{aligned}$$

because colimits commute with tensor products of functors. Thus hocolim F is the colimit of the $\Delta\mathcal{D}$ -diagram

$$(A.7) \quad \Lambda(\alpha, -) \odot_{\Delta\mathcal{D}} S^* \mathcal{Q}F$$

and it remains to identify that diagram with $\mathcal{Q}F$. To begin with, we have

$$\begin{aligned} \Lambda(\alpha, -) \odot_{\Delta\mathcal{D}} S^* \mathcal{Q}F &\cong \operatorname{Lan}_{S^{op}} \Lambda(\alpha, -) \odot_{\mathcal{D}} \mathcal{Q}F && \text{(by Lemma A.6)} \\ &\cong N(- \downarrow \alpha) \odot_{\mathcal{D}} \mathcal{Q}F && \text{(by Lemma A.2).} \end{aligned}$$

Now note that for any functor $H: \mathcal{E} \rightarrow \mathcal{D}$, we have

$$\operatorname{Lan}_{H^{op}} N(- \downarrow \mathcal{E}) \cong N(- \downarrow H).$$

This is easily verified using the formula $N(- \downarrow H) \cong B(*, \mathcal{D}, H)$. Therefore for $H = \alpha$, we have:

$$\begin{aligned} N(- \downarrow \alpha) \odot_{\mathcal{D}} \mathcal{Q}F &\cong \left(\operatorname{Lan}_{\alpha^{op}} N(- \downarrow [n]) \right) \odot_{[n]} \mathcal{Q}F \\ &\cong N(- \downarrow [n]) \odot_{[n]} \alpha^* \mathcal{Q}F && \text{(by Lemma A.6)} \\ &\cong \mathcal{Q}F, \end{aligned}$$

as desired.

Finally, we want to know that $\mathcal{Q}F$ is Reedy cofibrant. Here we must use some more technical model category theory. Recall the following result from [Hir03, 18.4.11]:

Lemma A.8. *Let \mathcal{R} be a Reedy category and \mathcal{M} a simplicial model category. If $G_1 \rightarrow G_2$ is a Reedy cofibration of \mathcal{R}^{op} -diagrams of simplicial sets, and $F_1 \rightarrow F_2$ is a Reedy cofibration of \mathcal{R} -diagrams in \mathcal{M} , then the pushout corner map:*

$$(G_1 \odot_{\mathcal{R}} F_2) \bigsqcup_{G_1 \odot_{\mathcal{R}} F_1} (G_2 \odot_{\mathcal{R}} F_1) \longrightarrow G_2 \odot_{\mathcal{R}} F_2$$

is a cofibration in \mathcal{M} that is a weak equivalence if either given map is a Reedy weak equivalence.

Note that since Δ is a Reedy cofibrant Δ -diagram in $s\mathcal{S}$, Lemma A.8 implies Lemma 8.1. The methods used in the proof of Lemma A.8 can be used to show the following result. Recall from [Hir03, 15.5.2] that if \mathcal{R}_1 and \mathcal{R}_2 are Reedy categories, then $\mathcal{R}_1 \times \mathcal{R}_2$ has a canonical Reedy structure, and the Reedy model structure on $\mathcal{M}^{\mathcal{R}_1 \times \mathcal{R}_2}$ is the same whether we regard it as $\mathcal{R}_1 \times \mathcal{R}_2$ -diagrams in \mathcal{M} , \mathcal{R}_1 -diagrams in $\mathcal{M}^{\mathcal{R}_2}$, or \mathcal{R}_2 -diagrams in $\mathcal{M}^{\mathcal{R}_1}$.

Lemma A.9. *Let \mathcal{R}_1 and \mathcal{R}_2 be Reedy categories and let \mathcal{M} be a simplicial model category. If $G_1 \rightarrow G_2$ is a Reedy cofibration of $\mathcal{R}_1^{\text{op}}$ -diagrams of simplicial sets, and $F_1 \rightarrow F_2$ is a Reedy cofibration of $\mathcal{R}_1 \times \mathcal{R}_2$ -diagrams in \mathcal{M} , then the pushout corner map:*

$$(G_1 \circ_{\mathcal{R}} F_2) \bigsqcup_{G_1 \circ_{\mathcal{R}} F_1} (G_2 \circ_{\mathcal{R}} F_1) \longrightarrow G_2 \circ_{\mathcal{R}} F_2$$

is a Reedy cofibration of \mathcal{R}_2 -diagrams in \mathcal{M} that is a Reedy weak equivalence if either given map is a Reedy weak equivalence.

These lemmas are both special cases of a more general result about two-variable adjunctions which the reader is invited to state and prove.

Taking G_1 and F_1 to be diagrams constant at the initial object, we see that the tensor product of two Reedy cofibrant diagrams is Reedy cofibrant. Now recall from (A.7) that we have

$$\mathcal{Q}F(\alpha) = \Lambda(\alpha, -) \circ_{\Delta \mathcal{D}} S^*QF.$$

By Lemma 8.7, we know that S^*QF is a Reedy cofibrant $\Delta^{\text{op}} \mathcal{D}$ -diagram in \mathcal{M} . Thus by Lemma A.9, to conclude that $\mathcal{Q}F$ is Reedy cofibrant in $\mathcal{M}^{\Delta \mathcal{D}}$, it suffices to observe that $\Lambda(-, -)$ is a Reedy cofibrant $(\Delta \mathcal{D} \times \Delta \mathcal{D})$ -diagram in $s\mathcal{S}$.

This completes the proof of Theorem 8.5. We have given the proof only for a simplicial model category, for ease of exposition. However, a careful examination shows that it goes through without a hitch for general model categories, as long as the simplicial tensor is interpreted appropriately, using the by-now standard techniques of framings or resolutions, as described in [Hov99, ch. 5] or [Hir03, ch. 16].

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