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A transchromatic proof of Strickland's theorem

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A B S T R A C T

In [15] Strickland proved that the Morava E-theory of the symmetric group has an algebro-geometric interpretation after taking the quotient by a certain transfer ideal. This result has influenced most of the work on power operations in Morava E-theory and provides an important calculational tool. In this paper we give a new proof of this result as well as a generalization by using transchromatic character theory. The character maps are used to reduce Strickland's result to representation theory.

1. Introduction and outline

The coefficient ring of Morava E-theory carries the universal deformation of a height n formal group over a perfect field of characteristic p. This formal group seems to determine the Morava E-theory of a large class of spaces. An example of this is the important result of Strickland's [15] that describes the E-theory of the symmetric group (modulo a transfer ideal) as the scheme that classifies subgroups in the universal deformation. This result plays a critical role in the study of power operations for Morava E-theory [9–11] and explicit calculations of the E-theory of symmetric groups [8,18] and the spaces $L(k)$ [4].

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We exploit a method that reduces facts such as the existence of Strickland's isomorphism into questions in representation theory by using the transchromatic generalized character maps of [13]. In this paper we illustrate the method by giving a new proof of Strickland's result as well as a generalization to wreath products of abelian groups with symmetric groups. The new feature here is more than the generalization of Strickland's result to certain p-divisible groups, it is a method for reducing a class of hard problems in E-theory to representation theory.

We explain some of the ideas. Let G be a finite group. There is an endofunctor of finite G-CW complexes $\mathcal{L}$ called the (p-adic) inertia groupoid functor that has some very useful properties:

- Given a cohomology theory E_G on finite G-CW complexes, the composite $E_G(\mathcal{L}(-))$ is a cohomology theory on finite G-CW complexes.
- Let $*$ be a point with a G-action. There is an equivalence

$$EG \times_G \mathcal{L}(*) \simeq \mathrm{Map}(B\mathbb{Z}_p, BG).$$

  The right hand side is the (p-adic) free loop space of BG.
- If E is p-complete, characteristic 0, and complex oriented with formal group $\mathbb{G}_E$ then (working Borel equivariantly) the isomorphisms

$$E^0_{\mathbb{Z}/p^k}(\mathcal{L}(*)) \cong E^0(\coprod_{\mathbb{Z}/p^k} B\mathbb{Z}/p^k) \cong \prod_{\mathbb{Z}/p^k} E^0(B\mathbb{Z}/p^k)$$

  imply that, as k varies, the algebro-geometric object associated to $E_{\mathbb{Z}/p^k}(\mathcal{L}(-))$ is the p-divisible group $\mathbb{G}_E \oplus \mathbb{Q}_p/\mathbb{Z}_p$.
- The target of the character maps of [6] and [13] take values in a cohomology theory built using $\mathcal{L}$.

Because of the second property we feel safe abusing notation and writing $EG \times_G \mathcal{L}(*)$ and $\mathcal{L}BG$ interchangeably. The latter is certainly easier on the eyes.

Now let E be Morava E_n. The p-divisible group associated to $\mathbb{G}_E$ is the directed system built out of the p^k-torsion as k varies

$$\mathbb{G}_E[p] \to \mathbb{G}_E[p^2] \to \dots .$$

We will be interested in finite subgroups of $\mathbb{G}_E$ and related p-divisible groups. A subgroup will always mean a finite flat subgroup scheme of constant rank (order). Given such a finite flat subgroup scheme H we will denote its order by $|H|$.

Precomposing with the inertia groupoid h times gives a cohomology theory $E(\mathcal{L}^h(-))$ with associated p-divisible group $\mathbb{G}_E \oplus \mathbb{Q}_p/\mathbb{Z}_p^h$, where $\mathbb{Q}_p/\mathbb{Z}_p^h = (\mathbb{Q}_p/\mathbb{Z}_p)^h$. In [6] and [13] rings called C_t for $0 \leq t < n$ are constructed with three important properties:

- The ring C_t is a faithfully flat $L_{K(t)}E^0$-algebra and further E^0 injects into C_t.
- There is an isomorphism of p-divisible groups

$$C_t \otimes_{E^0} (\mathbb{G}_E \oplus \mathbb{Q}_p/\mathbb{Z}_p^h) \cong (C_t \otimes_{L_{K(t)}E^0} \mathbb{G}_{L_{K(t)}E}) \oplus \mathbb{Q}_p/\mathbb{Z}_p^{n+h-t}.$$

- Furthermore, for X a finite G-CW complex there is an isomorphism

$$C_t \otimes_{E^0} E_G^0(X) \cong C_t \otimes_{L_{K(t)}E^0} (L_{K(t)}E)_G^0(\mathcal{L}^{n-t}X).$$

In this paper we are interested in comparing two schemes. The first is Spec$(-)$ of the cohomology of the symmetric group $E^0(\mathcal{L}^h B\Sigma_{p^k})$ modulo the ideal I_{tr} generated by the image of the transfer maps along the inclusions $\Sigma_i \times \Sigma_j \subset \Sigma_{p^k}$, where $i, j > 0$ and $i + j = p^k$. Thus the scheme is the functor

$$\mathrm{Spec}(E^0(\mathcal{L}^h B\Sigma_{p^k})/I_{tr}) : E^0\text{-algebras} \longrightarrow \mathrm{Set}$$

mapping

$$R \mapsto \mathrm{Hom}_{E^0\text{-alg}}(E^0(\mathcal{L}^h B\Sigma_{p^k})/I_{tr}, R).$$

The second scheme classifies subgroups of the p-divisible group $\mathbb{G}_E \oplus \mathbb{Q}_p/\mathbb{Z}_p^h$ of order p^k. It is the functor

$$\mathrm{Sub}_{p^k}(\mathbb{G}_E \oplus \mathbb{Q}_p/\mathbb{Z}_p^h) : E^0\text{-algebras} \longrightarrow \mathrm{Set}$$

mapping

$$R \mapsto \{H \subseteq R \otimes \mathbb{G}_E \oplus \mathbb{Q}_p/\mathbb{Z}_p^h | H \text{ a subgroup scheme with } |H| = p^k\}.$$

We begin by proving that these schemes are both finite and flat over E^0. We construct a map between the schemes by using properties of the ring C_0 to embed the rings of functions on these schemes as lattices inside the (generalized) class functions on Σ_{p^k}. By embedding both rings in the same large ring we are able to see that one is a sublattice of the other. Because C_0 is faithfully flat over $p^{-1}E = L_{K(0)}E$ the map we construct is an isomorphism after inverting p.

We prove that the map is an isomorphism by using the third property of the ring C_1 to reduce the computation to height 1. It suffices to prove the map is an isomorphism after reduction to height 1 because the schemes are finite and flat and the determinant of the map between them is a power of p times a unit. We wish to show that the power of p is zero. After base change to C_1 we extend coefficients even further to identify with a faithfully flat extension of p-adic K-theory. This uses the main result of [3]. Thus we have produced a character map from E to a form of p-adic K-theory. This allows us to reduce to a problem in representation theory that ends up being equivalent to the canonical isomorphism

$$\mathrm{Spec}(RA) \xrightarrow{\cong} \mathrm{Hom}(A^*, \mathbb{G}_m),$$

where RA is the representation ring of a finite abelian group A, $\mathbb{G}_m$ is the multiplicative group scheme, and A^* is the Pontryagin dual.

The main result of the paper is the following:

Theorem. *Let $\mathbb{G}_{E_n}$ be the universal deformation formal group, let $\mathcal{L}$ be the (p-adic) free loop space functor, and let*

$$\mathrm{Sub}_{p^k}(\mathbb{G}_{E_n} \oplus \mathbb{Q}_p/\mathbb{Z}_p^h) : E_n^0\text{-}algebras \longrightarrow Set$$

be the functor mapping

$$R \mapsto \{H \subseteq R \otimes \mathbb{G}_{E_n} \oplus \mathbb{Q}_p/\mathbb{Z}_p^h \,||H| = p^k\}.$$

There is an isomorphism

$$\mathrm{Spec}(E_n^0(\mathcal{L}^h B\Sigma_{p^k})/I_{tr}) \cong \mathrm{Sub}_{p^k}(\mathbb{G}_{E_n} \oplus \mathbb{Q}_p/\mathbb{Z}_p^h),$$

where I_{tr} is the ideal generated by the image of the transfer along the inclusions $\Sigma_i \times \Sigma_j \subseteq \Sigma_{p^k}$.

This recovers Strickland's Theorem 9.2 from [15] when $h = 0$. Let $\alpha : \mathbb{Z}_p^h \longrightarrow \Sigma_{p^k}$ and let

$$\mathrm{pr} : \mathrm{Sub}_{p^k}(\mathbb{G}_{E_n} \oplus \mathbb{Q}_p/\mathbb{Z}_p^h) \longrightarrow \mathrm{Sub}_{\leq p^k}(\mathbb{Q}_p/\mathbb{Z}_p^h)$$

be the projection onto the constant étale factor. For $A \subset \mathbb{Q}_p/\mathbb{Z}_p^h$ let

$$\mathrm{Sub}_{p^k}^A(\mathbb{G}_{E_n} \oplus \mathbb{Q}_p/\mathbb{Z}_p^h) : E_n^0\text{-algebras} \longrightarrow \mathrm{Set}$$

be the functor mapping

$$R \mapsto \{H \subseteq R \otimes \mathbb{G}_{E_n} \oplus \mathbb{Q}_p/\mathbb{Z}_p^h \,||H| = p^k, \mathrm{pr}(H) = A\}.$$

We say a map $\alpha : \mathbb{Z}_p^h \longrightarrow \Sigma_{p^k}$ is monotypical if the corresponding $\mathbb{Z}_p^h$-set of size p^k is a disjoint union of isomorphic transitive $\mathbb{Z}_p^h$-sets. Assume that α is monotypical and let $A = \mathrm{im}\,\alpha$, then $C(\mathrm{im}\,\alpha) \cong A \wr \Sigma_{p^j}$. Let $I_{tr} \subset E_n^0(BA \wr \Sigma_{p^j})$ be the ideal generated by the image of the transfers along $A \wr (\Sigma_l \times \Sigma_m) \subset A \wr \Sigma_{p^j}$ with $l, m > 0$ and $l + m = p^j$.

A corollary of the theorem above is the E-theory version of the second author's Theorem 3.11 from [12].

Corollary. *Let $\alpha : \mathbb{Z}_p^h \longrightarrow \Sigma_{p^k}$ be monotypical, A the image of α, and $p^j = p^k/|A|$. Then there are isomorphisms*

$$\operatorname{Spec}(E_n^0(BA \wr \Sigma_{p^j})/I_{tr}) \cong \operatorname{Spec}(E_n^0(BC(\operatorname{im}\alpha)/I_{tr}^{[\alpha]})) \cong \operatorname{Sub}_{p^k}^{A^*}(\mathbb{G}_{E_n} \oplus \mathbb{Q}_p/\mathbb{Z}_p^h),$$

where $I_{tr}^{[\alpha]}$ is described in Theorem 2.5 below.

Outline. In Section 2 we review the transchromatic character maps and we recall a specialization of the main result of [3] which shows that, for good groups, the character map from height n to height 1 can be modified to land in a faithfully flat extension of p-adic K-theory.

Then in Section 3 we develop a transchromatic character theory for the E-theory of the unitary group. We provide an algebro-geometric interpretation of the character map in terms of divisors of $\mathbb{G}_{E_n}$ and divisors of $C_t \otimes \mathbb{G}_{L_{K(t)}E_n} \oplus (\mathbb{Z}/p^k)^{n-t}$.

In Section 4 we develop the theory of centralizers in symmetric groups. These arise in the decomposition of the iterated free loop space $\mathcal{L}^h B\Sigma_{p^k}$. The centralizers that we are interested in all have the form $A \wr \Sigma_{p^j}$ with $j < k$. In order to understand certain transfer maps, we also study the free loops of the maps $B\Sigma_i \times B\Sigma_j \to B\Sigma_{p^k}$, where $i, j > 0$ and $i + j = p^k$.

In Section 5 we show that $E_n^0(BA \wr \Sigma_{p^j})/I_{tr}$ is finitely generated and free as an E_n^0-module. This relies on work of Rezk in [10] that reduces the question to Strickland's result (Thm. 8.6, [15]) that $E_n^0(B\Sigma_{p^k})/I_{tr}$ is finite and free.

In Section 6 we show that the ring of functions on $\operatorname{Sub}_{p^k}^{A}(\mathbb{G}_{E_n} \oplus \mathbb{Q}_p/\mathbb{Z}_p^h)$ is free of the same rank as an E_n^0-module. This follows by reduction to Strickland's result (Thm. 10.1, [14]) concerning $\operatorname{Sub}_{p^k}(\mathbb{G}_{E_n})$. Thus in Sections 5 and 6 we rely on two freeness results of Strickland's.

Finally in Section 7 we construct an injective map

$$f_{p^k} : \Gamma \operatorname{Sub}_{p^k}(\mathbb{G}_{E_n} \oplus \mathbb{Q}_p/\mathbb{Z}_p^h) \hookrightarrow E_n^0(\mathcal{L}^h B\Sigma_{p^k})/I_{tr},$$

where $\Gamma(-)$ is the global sections of the structure sheaf, by embedding both of the rings into the ring of class functions on $(n+h)$-tuples of commuting elements in Σ_{p^k} and exhibiting the domain as a subset of the codomain. The map has the property that it becomes an isomorphism after inverting p. Since the domain and codomain are free of the same rank this means that the failure of the determinant to be a unit is only a power of the prime p. Thus, to prove that the map is an isomorphism, it suffices to base change the map to any E_n^0-algebra in which p is not a unit and is not nilpotent and prove that the resulting map is an isomorphism. We then use the transchromatic character maps to reduce to height 1 and further base change in order to identify with a form of p-adic K-theory. This converts proving the map is an isomorphism to a question in representation theory that is easy to solve.

There is an appendix that includes an elementary proof of Strickland's theorem when $k = 1$.

2. From E-theory to K-theory

We recall the basics of the transchromatic character maps. Then we recall the modification of the character map to height 1 that lands in a faithfully flat extension of p-adic K-theory. This is due to [3].

2.1. Character theory recollections

Fix a prime p. We fix inclusions

$$\mathbb{Q}_p/\mathbb{Z}_p \subset \mathbb{Q}/\mathbb{Z} \subset S^1 \subset \mathbb{C}^*,$$

where the middle inclusion is the exponential map $e^{2\pi i x}$. Thus for a finite abelian group A there is a fixed isomorphism between the characters of A and the Pontryagin dual of A

$$\hat{A} \cong A^*.$$

Throughout the rest of the paper we will use the notation A^* for either of these.

Now let G be a finite group and X a finite G-CW complex. We may produce from this the topological groupoid $X//G$. Let $|X//G|$ be the geometric realization of the nerve of $X//G$. Thus we have an equivalence

$$|X//G| \simeq EG \times_G X.$$

For any cohomology theory E, in this paper, we always set

$$E^*(X//G) = E^*(|X//G|) = E^*(EG \times_G X).$$

There is an endofunctor of topological groupoids called the (p-adic) inertia groupoid functor that we will denote by $\mathcal{L}$. It takes finite G-CW complexes to finite G-CW complexes by mapping

$$X//G \mapsto \mathcal{L}(X//G) := \mathrm{Hom}_{top.gpd}(*//\mathbb{Z}_p, X//G),$$

where the right hand side is the internal mapping topological groupoid. It is an action groupoid by the isomorphism

$$\mathrm{Hom}_{top.gpd}(*//\mathbb{Z}_p, X//G) \cong \Big(\coprod_{\alpha \in \mathrm{Hom}(\mathbb{Z}_p, G)} X^{\mathrm{im}\,\alpha}\Big)//G,$$

where $X^{\mathrm{im}\,\alpha}$ is the fixed points of X with respect to the image of α and G acts by mapping $x \in X^{\mathrm{im}\,\alpha}$ to $gx \in X^{\mathrm{im}\,g\alpha g^{-1}}$.

It is notationally convenient to equate $\mathcal{L}(X//G)$ and $\mathcal{L}(EG \times_G X)$ and we will use these interchangeably. Note that $\mathcal{L}(EG \times_G X)$ is not quite (but is closely related to) the free loop space of $EG \times_G X$. In particular, when X is a space with an action by the trivial group then $\mathcal{L}X = X$ and when $X = *$ and G is a p-group $\mathcal{L}(X//G) = \mathrm{Map}(S^1, BG)$.

For a finite group G, the transchromatic generalized character maps approximate the Morava E-theory of BG by a certain height t cohomology theory for any $t < n$. For arbitrary t these are constructed in [13], when $t = 0$ this is in [6], and when $n = 1$ it goes back to [1]. These papers construct a faithfully flat extension of $L_{K(t)}E_n$ called C_t^0 (for $t > 0$ the faithfully flatness is Proposition 5.3 in [3]) and a map of cohomology theories on finite G-CW complexes called "the character map"

$$E_n^0(X//G) \longrightarrow C_t^0 \otimes_{L_{K(t)}E_n^0} L_{K(t)}E_n^0(\mathcal{L}^{n-t}X//G).$$

The main theorem regarding these maps is the following:

Theorem 2.2. *(See [13].) The character map has the property that the map induced by tensoring the domain up to C_t^0*

$$C_t^0 \otimes_{E_n^0} E_n^0(X//G) \xrightarrow{\cong} C_t^0 \otimes_{L_{K(t)}E_n^0} L_{K(t)}E_n^0(\mathcal{L}^{n-t}X//G)$$

is an isomorphism.

There is a decomposition

$$\mathcal{L}^h BG \cong \coprod_{[\alpha] \in \mathrm{Hom}(\mathbb{Z}_p^h, G)/\sim} BC(\mathrm{im}\,\alpha),$$

where $[\alpha]$ is the conjugacy class of a map $\alpha : \mathbb{Z}_p^h \longrightarrow G$ and $C(\mathrm{im}\,\alpha)$ is the centralizer of the image of α.

Corollary 2.3. *When $X = *$ the character map takes the form*

$$E_n^0(BG) \longrightarrow \prod_{[\alpha] \in \mathrm{Hom}(\mathbb{Z}_p^{n-t}, G)/\sim} C_t^0 \otimes_{L_{K(t)}E_n^0} L_{K(t)}E_n^0(BC(\mathrm{im}\,\alpha)).$$

Remark. When $t = 0$ the ring C_0^0 is an $L_{K(0)}E_n^0 = p^{-1}E_n^0$-algebra and so $C_0^0(BG) = C_0^0$ for all finite G. Thus the codomain of the character map is just the product

$$\prod_{[\alpha] \in \mathrm{Hom}(\mathbb{Z}_p^n, G)/\sim} C_0^0,$$

which is the ring of generalized class functions on G with values in C_0^0. Because of this we will often rewrite this ring as $Cl(G_p^n, C_0^0)$, where G_p^n is shorthand for $\mathrm{Hom}(\mathbb{Z}_p^n, G)$.

Furthermore, there is a version of the theorem that includes taking the quotient by a transfer along $i : H \subset G$. Let $I_{tr} \subset E_n^0(BG)$ be the image of the transfer along i. Let

$\alpha : \mathbb{Z}_p^{n-t} \to G$ and let α' be a lift of α to H up to conjugacy so that there exists $g \in G$ so that $gi\alpha' g^{-1} = \alpha$. There is an induced inclusion

$$gC_H(\operatorname{im}\alpha')g^{-1} \longrightarrow C_G(\operatorname{im}\alpha).$$

Definition 2.4. Let $J_{tr}^{[\alpha]} \subseteq C_t^0 \otimes_{L_{K(t)}E_n^0} L_{K(t)}E_n^0(BC_G(\operatorname{im}\alpha))$ be the ideal generated by the image of the transfer along the inclusions induced by the lifts of α to H up to conjugacy.

Theorem 2.5. *(See Theorem 2.18, [12].) The character map induces an isomorphism*

$$C_t^0 \otimes_{E_n^0} E_n^0(BG)/I_{tr} \cong \prod_{[\alpha]\in G_p^{n-t}/\sim} C_t^0 \otimes_{L_{K(t)}E_n^0} L_{K(t)}E_n^0(BC(\operatorname{im}\alpha))/J_{tr}^{[\alpha]}.$$

Remark. There is an analogous result for the cohomology theory $E_n^0(\mathcal{L}^h(-))$:

$$E_n^0(\mathcal{L}^h BG)/I_{tr} \cong \prod_{[\alpha]\in G_p^h/\sim} E_n^0(BC(\operatorname{im}\alpha))/I_{tr}^{[\alpha]},$$

where $I_{tr}^{[\alpha]}$ is defined as above.

2.6. Landing in K-theory

The work of Barthel and the second author in [3] modifies the character maps introduced above to land in a faithfully flat extension of E_t when G is good. In this subsection we recall the specialization of their construction that gives a character map from height n to height 1. For historical accuracy we note that their work was an offshoot of this project.

Recall the following definition:

Definition 2.7. (See Definition 7.1, [6].) A finite group is *good* if $K(n)^*(BG)$ is generated by transfers of Euler classes of complex representations of subgroups of G.

If G is good then $K(n)^*(BG)$ is concentrated in even degrees (by the remark after Definition 7.1 in [6]) and this implies (Proposition 3.5 in [15]) that $E_n^*(BG)$ is free and concentrated in even degrees.

The cohomology theory $C_t^0 \otimes_{L_{K(t)}E_n^0} L_{K(t)}E_n^*(-)$ defined on finite spaces gives rise to a spectrum C_t and thus a cohomology theory $C_t^*(-)$ defined on all spaces. Of course, this can be quite different from $C_t^0 \otimes_{L_{K(t)}E_n^0} L_{K(t)}E_n^*(-)$ on infinite spaces like BG.

Let $\bar{C}_1 = L_{K(1)}(C_1 \wedge K_p)$. We modify the codomain of the character map from height n to height 1 by changing coefficients further to $\bar{C}_1^0$. The intuition for doing this comes from the following result, which is a special case of a result of Hopkins.

Proposition 2.8. *(See [5].) The ring $\pi_0(C_1 \wedge K_p)$ represents the scheme* $\mathrm{Iso}(\mathbb{G}_{C_1}, \mathbb{G}_m)$.

These results follow from the fact that C_1 and K_p are Landweber exact cohomology theories. Further this implies that both of the maps $C_1^0 \longrightarrow \pi_0(C_1 \wedge K_p)$ and $\mathbb{Z}_p = K_p^0 \longrightarrow \pi_0(C_1 \wedge K_p)$ are flat.

Proposition 2.9. *($t = 1$ in Proposition 5.8 of [3].) The spectrum $\bar{C}_1$ is even periodic and $\bar{C}_1^*$ is faithfully flat as a K_p^*-module.*

This is important for our purposes and used to prove the following result:

Proposition 2.10. *($t = 1$ in Theorem 6.9 of [3].) Let G be a good group and let $\bar{C}_1 = L_{K(1)}(C_1 \wedge K_p)$. The maps $K_p \to \bar{C}_1$ and $C_1 \to \bar{C}_1$ induce isomorphisms*

$$\bar{C}_1^0 \otimes_{\mathbb{Z}_p} K_p^0(BG) \cong \bar{C}_1^0 \otimes_{C_1^0} (C_1^0 \otimes_{L_{K(1)}E_n^0} L_{K(1)}E_n^0(BG)) \cong \bar{C}_1^0(BG).$$

Thus if the target of the character map from E-theory to height 1 has codomain

$$\prod_{[\alpha] \in G_p^{n-1}/\sim} C_1^0 \otimes_{L_{K(1)}E_n^0} L_{K(1)}E^0(BC(\mathrm{im}\,\alpha)),$$

where the groups $C(\mathrm{im}\,\alpha)$ are good, then we may base change further to $\bar{C}_1$ and identify the resulting codomain with the faithfully flat extension of K_p

$$\prod_{[\alpha] \in G_p^{n-1}/\sim} \bar{C}_1^0 \otimes_{\mathbb{Z}_p} K_p^0(BC(\mathrm{im}\,\alpha)).$$

Remark. In this paper we will apply this to groups of the form $A \wr \Sigma_n$.

Convention. Now that we have discussed these points and stated the previous proposition we will return to the second author's conventions in [13] and [12] and abuse notation and write

$$C_t^0(X) := C_t^0 \otimes_{L_{K(t)}E_n^0} L_{K(t)}E_n^0(X)$$

for all spaces X (even $X = BU(n)$).

3. Character theory of the unitary group

3.1. The inertia groupoid and the unitary group

Recall from Subsection 2.1 that the (p-adic) inertia groupoid $\mathcal{L}$ is defined to be the functor on topological groupoids corepresented by $*//\mathbb{Z}_p$. Since we have taken G to be finite, we can replace $*//\mathbb{Z}_p$ by $*//\mathbb{Z}/p^k$ for k large enough.

In this subsection we study the inertia groupoid applied to the unitary group $U(m)$. In order to get a proper grip on $\mathcal{L}BU(m)$, we use a "torsion" version of the inertia groupoid. Thus we define

$$\mathcal{L}_k^h(X//G) := \mathrm{Hom}_{top.gpd}(*//(\mathbb{Z}/p^k)^h, X//G),$$

and we may abuse this notation as described in the previous subsection.

Let $V = \coprod_{n\geq 0} *//U(n)$ be the groupoid of finite dimensional unitary vector spaces. For a finite abelian group A there is an equivalence of groupoids between the complex representations of A and the product over the characters of A of V

$$\mathrm{Hom}(*//A, V) \simeq \prod_{A^*} V$$

given by decomposing a representation of A into one-dimensional representations and grouping isomorphic summands. Let $F(A^*, \mathbb{Z}_{\geq 0})_m$ be the set of functions from A^* to the non-negative integers such that the sum of the values on the characters is $m \in \mathbb{Z}$. The equivalence of groupoids decomposes according to the dimension of the representation to give

$$\mathrm{Hom}(*//A, *//U(m)) \simeq \coprod_{f\in F(A^*,\mathbb{Z}_{\geq 0})_m} \Big(\prod_{l\in A^*} *//U(f(l))\Big).$$

Now let

$$\Lambda_k = ((\mathbb{Z}/p^k)^h)^*,$$

the Pontryagin dual of the abelian group that we use to define $\mathcal{L}_k^h(-)$. The following proposition is immediate from the previous observation.

Proposition 3.2. *There is an equivalence of groupoids*

$$\mathcal{L}_k^h(*//U(m)) \simeq \coprod_{f\in F(\Lambda_k,\mathbb{Z}_{\geq 0})_m} \Big(\prod_{l\in \Lambda_k} *//U(f(l))\Big).$$

3.3. Character theory and divisors

Recall from [13] and [12], that the p-divisible group associated to $C_t^0(\mathcal{L}^{n-t}(-))$ is $\mathbb{G}_{C_t} \oplus \mathbb{Q}_p/\mathbb{Z}_p^{n-t}$.

Let E be either of the cohomology theories E_n or $L_{K(t)}E_n$ and recall that $\Lambda_k = ((\mathbb{Z}/p^k)^h)^*$.

Proposition 3.4. *The p-divisible group associated to $E^0(\mathcal{L}^h(-))$ is $\mathbb{G}_E \oplus \mathbb{Q}_p/\mathbb{Z}_p^h$.*

Proof. We calculate the effect on $*//\mathbb{Z}/p^k$. We have

$$E^0(\mathcal{L}^h(*//\mathbb{Z}/p^k)) \cong E^0(\mathcal{L}^h(B\mathbb{Z}/p^k)) \cong \prod_{\Lambda_k} E^0(B\mathbb{Z}/p^k) \cong \Gamma((\mathbb{G}_E \oplus \mathbb{Q}_p/\mathbb{Z}_p^h)[p^k]),$$

where Γ is the global sections of the structure sheaf. It is easy to check that the Hopf algebra structure on these rings is isomorphic as well. □

There are many divisors of degree m in $\mathbb{G}_E \oplus \mathbb{Q}_p/\mathbb{Z}_p^h$. To get some control over the set, one can consider divisors of degree m in $\mathbb{G}_E \oplus \Lambda_k$.

Definition 3.5. An (effective) divisor of degree m in $\mathbb{G}_E \oplus \Lambda_k$ is a closed subscheme that is finite and flat of degree m over E.

We will use the isomorphism

$$\mathrm{Div}_m(\mathbb{G}_E \oplus \Lambda_k) \cong (\mathbb{G}_E \oplus \Lambda_k)^{\times m}/\Sigma_m.$$

The quotient by Σ_m is the scheme-theoretic quotient. Since $(\mathbb{G}_E \oplus \Lambda_k)^{\times m}$ is affine, the quotient is affine as well. When $h = 0$, $\mathrm{Div}_m(\mathbb{G}_E)$ is the usual divisors of degree m in the formal group $\mathbb{G}_E$.

Proposition 3.6. *The functor* $\mathrm{Div}_m(\mathbb{G}_E \oplus \Lambda_k)$ *is corepresented by*

$$E^0(\mathcal{L}_k^h BU(m)).$$

Proof. This is immediate from Proposition 3.2 and the fact that, in all of these cases, $E^0(BU(m))$ corepresents divisors of degree m on $\mathbb{G}_E$ (Section 9 of [15]). It is worth noting explicitly what is occurring on the level of connected components. A conjugacy class of maps

$$(\mathbb{Z}/p^k)^h \longrightarrow U(m)$$

determines an m-dimensional representation of $(\mathbb{Z}/p^k)^h$ up to isomorphism. The representation decomposes as a sum of m 1-dimensional representations. This corresponds to m maps $(\mathbb{Z}/p^k)^h \longrightarrow S^1$ which determines m elements in Λ_k (counted with multiplicity). The divisor is concentrated on the components of $\mathbb{G}_E \oplus \Lambda_k$ corresponding to these elements. □

We may apply the transchromatic character maps to the unitary groups by using $\mathcal{L}_k^{n-t}$ in place of $\mathcal{L}^{n-t}$. The character map takes the form

$$E_n^0(BU(m)) \longrightarrow C_t^0(\mathcal{L}_k^{n-t} BU(m)).$$

These are defined exactly as the character maps for finite groups. There is no difficulty because we are working with the torsion inertia groupoid.

Note that a map $G \xrightarrow{f} U(m)$ induces $\mathcal{L}_k^{n-t}BG \xrightarrow{\mathcal{L}_k^{n-t}f} \mathcal{L}_k^{n-t}BU(m)$.

Proposition 3.7. *Consider a map $G \xrightarrow{f} U(m)$ and let k be large enough so that every map $\mathbb{Z}_p \longrightarrow G$ factors through $\mathbb{Z}/p^k$. The following square commutes*

$$\begin{tikzcd}
E_n^0(\mathcal{L}_k^h BU(m)) \arrow[r] \arrow[d] & E_n^0(\mathcal{L}^h BG) \arrow[d] \\
C_t^0(\mathcal{L}_k^{h+n-t} BU(m)) \arrow[r] & C_t^0(\mathcal{L}^{h+n-t} BG).
\end{tikzcd}$$

Proof. The character map is the composite of two maps, one is topological and the other is algebraic. The commutativity of the diagram follows from the fact that the topological part of the character map is E_n^0 applied to an evaluation map. That is, the following diagram induced by the map f commutes:

$$\begin{tikzcd}
*//G \arrow[d, "f"] & *//\Lambda_k^* \times \mathrm{Hom}(*//\Lambda_k^*, *//G) \arrow[l, "ev"'] \arrow[d] \\
*//U(m) & *//\Lambda_k^* \times \mathrm{Hom}(*//\Lambda_k^*, *//U(m)). \arrow[l, "ev"']
\end{tikzcd}$$ □

We often say that character maps like these approximate height n cohomology by height t cohomology because when we tensor the domain up to C_t they give an isomorphism. This is *not true* for $U(m)$. However, these maps do have interesting algebro-geometric content.

There is a canonical map of formal groups $\mathbb{G}_{L_{K(t)}E_n} \longrightarrow \mathbb{G}_{E_n}$ and over C_t^0 there is a canonical map $\Lambda_k \longrightarrow \mathbb{G}_{E_n}$ for all k. Put together this gives

$$C_t^0 \otimes (\mathbb{G}_{L_{K(t)}E_n} \oplus \Lambda_k)^{\times m}/\Sigma_m \longrightarrow (\mathbb{G}_{E_n})^{\times m}/\Sigma_m,$$

which we write as $C_t^0 \otimes \mathrm{Div}_m(\mathbb{G}_{L_{K(t)}E_n} \oplus \Lambda_k) \longrightarrow \mathrm{Div}_m(\mathbb{G}_{E_n})$. Here we are thinking of $\mathbb{G}_{L_{K(t)}E_n}$ as a formal group so $C_t^0 \otimes \mathbb{G}_{L_{K(t)}E_n}$ is corepresented by $C_t^0(BS^1) = C_t^0 \otimes_{L_{K(t)}E_n^0} L_{K(t)}E_n^0(BS^1)$.

Proposition 3.8. *Let $\Lambda_k = ((\mathbb{Z}/p^k)^{n-t})^*$. The character map*

$$E_n^0(BU(m)) \longrightarrow C_t^0(\mathcal{L}_k^{n-t}BU(m))$$

fits into a commutative square

$$\begin{tikzcd} E_n^0(BU(m)) \arrow[r, "\cong"] \arrow[d] & \Gamma\,\mathrm{Div}_m(\mathbb{G}_{E_n}) \arrow[d] \\ C_t^0(\mathcal{L}_k^{n-t}BU(m)) \arrow[r, "\cong"] & C_t^0 \otimes \Gamma\,\mathrm{Div}_m(\mathbb{G}_{L_{K(t)}E_n} \oplus \Lambda_k). \end{tikzcd}$$

Proof. Consider the character map applied to the inclusion of a maximal torus $\mathbb{T} \subset U(m)$. As $E_n^0(BU(m))$ injects in $E_n^0(B\mathbb{T})$ and $C_t^0(\mathcal{L}_k^{n-t}BU(m))$ injects into $C_t^0(\mathcal{L}_k^{n-t}B\mathbb{T})$, it is enough to check that the character map applied to S^1 produces the global sections of

$$C_t^0 \otimes \mathbb{G}_{L_{K(t)}E_n} \oplus \Lambda_k \longrightarrow \mathbb{G}_{E_n}$$

and this is the case. $\square$

4. Centralizers in symmetric groups

In this section we develop the theory of centralizers of tuples of commuting elements in wreath products of a finite abelian group and a symmetric group. These arise when studying

$$\mathcal{L}^h B\Sigma_m$$

and play an important role in the rest of the paper. We show that these centralizers are all products of wreath products of finite abelian groups and symmetric groups. We also analyze $\mathcal{L}(-)$ applied to the maps $\Sigma_i \times \Sigma_j \longrightarrow \Sigma_m$, where $i, j > 0$ and $i + j = m$.

For the rest of this section let A be a finite abelian group and let $n \geq 1$ be an integer. Let

$$[n] := \{1, 2, \dots, n\}$$

and consider the set

$$A \times [n] := \coprod_{1 \leq i \leq n} A.$$

The group A^n acts on $A \times [n]$ by multiplication coordinate-wise and the symmetric group Σ_n acts on $A \times [n]$ by permuting the coordinates. The two actions fit together to give an action of $A \wr \Sigma_n$ on $A \times [n]$. This action defines a map

$$s : A \wr \Sigma_n \hookrightarrow \Sigma_{|A|n}.$$

The image of the diagonal map

$$z : A \hookrightarrow A \wr \Sigma_n$$

is the center of $A \wr \Sigma_n$.

Now consider the map

$$d := s \circ z : A \to \Sigma_{|A|n}.$$

Since the image of z is $Z(A \wr \Sigma_n)$ we have

$$\operatorname{im} s = A \wr \Sigma_n \subset C_{\Sigma_{|A|n}}(\operatorname{im} d).$$

Lemma 4.1. *There is an equality* $\operatorname{im}(s) = C_{\Sigma_{|A|n}}(\operatorname{im} d)$ *and thus*

$$A \wr \Sigma_n \cong C_{\Sigma_{|A|n}}(\operatorname{im} d).$$

Proof. We can consider the set $A \times [n]$ as an A-set via the diagonal action. Thus $C_{\Sigma_{|A|n}}(\operatorname{im} d)$ is just the group of automorphisms of $A \times [n]$ as an A-set. As an A-set $A \times [n]$ is a disjoint union of n-copies of A since the group of automorphisms of A as an A-set is isomorphic to A. The group of automorphisms of n-copies is $A \wr \Sigma_n$. $\square$

Let $h \geq 0$ be an integer and let

$$\alpha : \mathbb{Z}^h \to A \wr \Sigma_n$$

be a map. We denote by $\tilde{\alpha}$ the map

$$\tilde{\alpha} := (s \circ \alpha) \oplus d : \mathbb{Z}^h \oplus A \to \Sigma_{n|A|}.$$

Now consider $\tilde{\alpha}$ as an action of $\mathbb{Z}^h \oplus A$ on $A \times [n]$. Given an element $x \in A \times [n]$ and a map α we define the type of x to be the unique surjection

$$t_x : \mathbb{Z}^h \oplus A \twoheadrightarrow A_{t_x}$$

with kernel equal to the stabilizer of x. It is easy to see that the map t_x induces an inclusion

$$A \hookrightarrow A_{t_x}.$$

Note that if x has type t then so does any other element in the $\tilde{\alpha}$-orbit x. Thus given an $\tilde{\alpha}$ orbit O it makes sense to speak of the type of O. Given a type $t : \mathbb{Z}^h \oplus A \twoheadrightarrow A_t$ we denote by N_t the number of $\tilde{\alpha}$-orbits with t as a type.

Definition 4.2. We denote by $T(\alpha)$ the set of all types that occur in $A \times [n]$ (i.e. types t with $N_t > 0$). We call a map $\alpha : \mathbb{Z}^h \longrightarrow A \wr \Sigma_n$ *monotypical* if $|T(\alpha)| = 1$.

Remark. When $A = e$ this is equivalent to saying that the $\mathbb{Z}^h$-set of size n classified by α is isomorphic to a union of isomorphic transitive $\mathbb{Z}^h$-sets.

Lemma 4.3. *There is an isomorphism*

$$C_{A\wr\Sigma_n}(\operatorname{im}\alpha) \cong \prod_{t\in T(\alpha)} A_t \wr \Sigma_{N_t}.$$

Proof. Choose some $h' \geq 0$ and a surjection $\mathbb{Z}^{h'} \to A$. Denote by

$$\alpha' : \mathbb{Z}^h \oplus \mathbb{Z}^{h'} \to \Sigma_{|A|n}$$

the resulting map. By Lemma 4.1 we have isomorphisms

$$C_{A\wr\Sigma_n}(\operatorname{im}\alpha) \cong C_{\Sigma_{n|A|}}(\operatorname{im}\tilde{\alpha}) \cong C_{\Sigma_{n|A|}}(\operatorname{im}\alpha').$$

Also we have a natural bijection

$$T(\alpha) \to T(\alpha')$$

mapping

$$t \mapsto t'$$

with $A_t \cong A_{t'}$ and $N_t = N_{t'}$. Thus it is enough to consider the case of $A = 0$. In this case $A \times [n]$ should be considered as a $\mathbb{Z}^h$-set and $C_{\Sigma_n}(\operatorname{im}\alpha)$ is just the group of automorphisms of $0 \times [n] \cong [n]$ as a $\mathbb{Z}^h$-set. But now $[n]$ decomposes as a disjoint union of $\mathbb{Z}^h$-orbits such that there are exactly N_t of type t. $\square$

Lemma 4.4. *Let $\alpha : \mathbb{Z}^h \to A \wr \Sigma_n$ be a map, then α in monotypical if and only if the action of $C_{A\wr\Sigma_n}(\operatorname{im}\alpha)$ on $A \times [n]$ is transitive.*

Proof. Since the action of $C_{A\wr\Sigma_n}(\operatorname{im}\alpha)$ on $A \times [n]$ preserves types it cannot be transitive if α is not monotypical. However if α is monotypical of type t it is easy to see that the action of $C_{A\wr\Sigma_n}(\operatorname{im}\alpha) \cong A_t \wr \Sigma_{N_t}$ is transitive. Note that in this case

$$N_t|A_t| = n|A|. \qquad \square$$

Let $X \coprod Y = \{1, 2, \ldots, n\}$ be a non-trivial partition and denote by

$$\Sigma_{X,Y} \subset \Sigma_n$$

the subgroup of permutations preserving the partition $\{X, Y\}$.

Let $\alpha : \mathbb{Z}^h \to A \wr \Sigma_n$ be a map. Consider the action $\tilde{\alpha}$ of $\mathbb{Z}^h \oplus A$ on $A \times [n]$. Since for every $i \in [n]$, A acts transitively on $A \times \{i\}$ we have that any $\tilde{\alpha}$-orbit o is of the form $A \times S_o$ for some subset $S_o \subset [n]$. The sets S_o for all the orbits of $\tilde{\alpha}$ form a partition

$$\coprod_o S_o = [n].$$

We shall denote this partition by $P_\alpha := \{S_1, \ldots, S_{k_\alpha}\}$, where k_α is the number of $\tilde{\alpha}$ orbits.

Definition 4.5. We say that a map $\alpha : \mathbb{Z}^h \to A \wr \Sigma_n$ is *well-formed* if $P_\alpha = \{S_1, \ldots, S_{k_\alpha}\}$, is such that every S_i is a set of consecutive numbers.

Lemma 4.6. *Every map $\alpha : \mathbb{Z}^h \to A \wr \Sigma_n$ is conjugate to a well-formed map.*

Proof. Clear. $\square$

Lemma 4.7. *Let $\alpha : \mathbb{Z}^h \to A \wr \Sigma_n$ be a map. If α is not monotypical then there exists some partition $X \coprod Y = \{1, 2, \ldots, n\}$ such that the map*

$$\alpha : \mathbb{Z}^h \to A \wr \Sigma_n$$

factors through the inclusion

$$A \wr \Sigma_{X,Y} \subset A \wr \Sigma_n$$

and

$$C_{A\wr\Sigma_{X,Y}}(\operatorname{im}\alpha) = C_{A\wr\Sigma_n}(\operatorname{im}\alpha).$$

Proof. Now let $\alpha : \mathbb{Z}^h \to A \wr \Sigma_n$ be a map. Let $t \in T(\alpha)$ be a type. Now let $t \in T(\alpha)$. Let $B_t \subset A \times [n]$ be the set of elements of type t. B_t is a union of A_t-orbits. However since we have an inclusion $A \hookrightarrow A_t$ each A_t-orbit is a disjoint union of A-orbits. So there exists some non-empty proper subset $X \subset [n]$ such that

$$B_t = A \times X.$$

Now the action $C_{A\wr\Sigma_n}(\operatorname{im}\alpha)$ on $A \times [n]$ preserves types. Thus for $Y := [n] \setminus X$ we see that

$$\alpha : \mathbb{Z}^h \to A \wr \Sigma_n$$

factors through the inclusion

$$A \wr \Sigma_{X,Y} \subset A \wr \Sigma_n$$

and that

$$C_{A\wr\Sigma_{X,Y}}(\operatorname{im}\alpha) = C_{A\wr\Sigma_n}(\operatorname{im}\alpha). \qquad \square$$

Corollary 4.8. *Let*

$$\alpha : \mathbb{Z}^h \to A \wr \Sigma_n$$

be a well-formed non-monotypical map. Then there exist some $0 < m < n$ such that α factors through the inclusion:

$$A \wr \Sigma_m \times \Sigma_{n-m} \subset A \wr \Sigma_n$$

and we have

$$C_{A\wr\Sigma_m\times\Sigma_{n-m}}(\operatorname{im}\alpha) = C_{A\wr\Sigma_n}(\operatorname{im}\alpha).$$

Lemma 4.9. *Let $\alpha : \mathbb{Z}^h \to A \wr \Sigma_n$ be a monotypical map of type*

$$t : \mathbb{Z}^h \oplus A \to A_t.$$

Let X, Y be a partition of $[n]$. Then α factors through $A\wr\Sigma_{X,Y}$ if and only if the partition $\{X, Y\}$ is coarser then $P_\alpha = \{S_1, \dots, S_{k_\alpha}\}$. Furthermore in this case

$$C_{A\wr\Sigma_{X,Y}}(\operatorname{im}\alpha) = C_{A\wr\Sigma_X}(\operatorname{im}\alpha) \times C_{A\wr\Sigma_Y}(\operatorname{im}\alpha).$$

Proof. It is clear that if X, Y is not coarser then $P_\alpha = \{S_1, \dots, S_{k_\alpha}\}$ the action of α cannot factor through $A \wr \Sigma_{X,Y}$. Otherwise we get that $A \times [n]$ factors as an $\tilde{\alpha}$ set as

$$A \times [n] = A \times X \coprod A \times Y.$$

The result now follows since

$$\begin{aligned} C_{A\wr\Sigma_{X,Y}}(\operatorname{im}\alpha) &= C_{\Sigma_{A\times X, A\times Y}}(\operatorname{im}\tilde{\alpha}) = \\ &= C_{\Sigma_{A\times X}}(\operatorname{im}\tilde{\alpha}) \times C_{\Sigma_{A\times Y}}(\operatorname{im}\tilde{\alpha}) = C_{A\wr\Sigma_X}(\operatorname{im}\alpha) \times C_{A\wr\Sigma_Y}(\operatorname{im}\alpha). \qquad \square \end{aligned}$$

Corollary 4.10. *Let*

$$\alpha : \mathbb{Z}^h \to A \wr \Sigma_n$$

be a well-formed mono-typical map of type t. Let $l = \frac{|A_t|}{|A|}$ note that we have $N_t = \frac{n}{l}$. Let $0 < m < n$, then

(1) *If m is divisible by l then α factors through the inclusion:*

$$A \wr \Sigma_m \times \Sigma_{n-m} \subset A \wr \Sigma_n$$

and we have that the inclusion

$$C_{A\wr\Sigma_m\times\Sigma_{n-m}}(\operatorname{im}\alpha) \subset C_{A\wr\Sigma_n}(\operatorname{im}\alpha)$$

is isomorphic to the natural inclusion

$$A_t \wr \Sigma_{\frac{m}{l}} \times \Sigma_{\frac{n-m}{l}} \subset A_t \wr \Sigma_{\frac{n}{l}}.$$

(2) *If m is not divisible by l then neither α nor any of its conjugates factors through the inclusion*

$$A \wr \Sigma_m \times \Sigma_{n-m} \subset A \wr \Sigma_n.$$

5. The Morava E-theory of centralizers

The goal of this section is to prove a freeness result for the Morava E-theory of centralizers of tuples of commuting elements in symmetric groups modulo a certain transfer ideal.

All of the groups that we study in this paper are good:

Proposition 5.1. *Centralizers of tuples of commuting prime-power order elements in symmetric groups are good.*

Proof. Lemma 4.3 implies that these centralizers are of the form $\prod_i A_i \wr \Sigma_{p^{k_i}}$ where A_i is an abelian p-group. Now the Sylow p-subgroups of this wreath product is $\prod_i A_i \wr \mathbb{Z}/p \wr \ldots \wr \mathbb{Z}/p$, which is good. By Proposition 7.2 of [6] this implies that the group is good. □

Proposition 5.2. *Let $\alpha : \mathbb{Z}_p^h \longrightarrow \Sigma_{p^k}$ be monotypical (the case $A = e$ in Definition 4.2) with centralizer $A \wr \Sigma_{p^i}$. Let $I_{tr} \subset E_n^0(BA \wr \Sigma_{p^i})$ be generated by maps of the form $A \wr (\Sigma_l \times \Sigma_m) \longrightarrow A \wr \Sigma_{p^i}$, where $l + m = p^i$ and $l, m > 0$, then $I_{tr} = I_{tr}^{[\alpha]}$.*

Proof. This follows immediately from both parts of Corollary 4.10. □

Proposition 5.3. *Let $\mathbb{Z}_p^h \xrightarrow{\alpha} \Sigma_{p^k}$. The ring $E_n^0(BC(\operatorname{im}\alpha))/I_{tr}^{[\alpha]}$ is finitely generated and free as an E_n^0-module.*

Proof. Note that Corollary 4.8 implies that this statement is trivial for non-monotypical maps α because $I_{tr}^{[\alpha]} = E_n^0(BC(\operatorname{im}\alpha))$. When α is monotypical Proposition 5.2 implies that we need to study $E_n^0(BA \wr \Sigma_{p^i})/I_{tr}$.

This puts us in a situation that is dual to Section 5 of [10] and thus essentially follows from the discussion there. For the purposes of this proof let $E = E_n$ so that $E_0 = (E_n)_0$. In [11,10] Rezk constructs a functor

$$\mathbb{T}_m : \mathrm{Mod}_{E_0} \longrightarrow \mathrm{Mod}_{E_0}$$

that takes free E_0-modules to free E_0-modules and with the property that

$$\mathbb{T}_m(E_0) \cong E_0(B\Sigma_m).$$

More generally on the free E_0-module $E_0(BA)$ it takes the value

$$\mathbb{T}_m(E_0(BA)) \cong E_0(BA \wr \Sigma_m).$$

Now it follows by dualizing Section 5 of [10] that taking the quotient by the image of the transfer maps along

$$A \wr (\Sigma_i \times \Sigma_j) \longrightarrow A \wr \Sigma_m$$

is the *right* linearization of the functor $\mathbb{T}_m$. Let M be a free E_0-module and let $i_1, i_2 : M \to M \oplus M$ be the canonical inclusions. The right linearization is defined to be the equalizer

$$\mathcal{R}_{\mathbb{T}_m}(M) \longrightarrow \mathbb{T}_m(M) \underset{\mathbb{T}_m(i_1+i_2)}{\overset{\mathbb{T}_m(i_1)+\mathbb{T}_m(i_2)}{\rightrightarrows}} \mathbb{T}_m(M \oplus M).$$

It follows from Theorem 8.6 in [15] that

$$\mathcal{R}_{\mathbb{T}_m}(E_0) \cong (E^0(B\Sigma_m)/I_{tr})^*$$

is free, where $(E^0(B\Sigma_m)/I_{tr})^*$ is the E_0-linear dual. Since $\mathcal{R}_{\mathbb{T}_m}$ is linear it takes free modules to free modules. Thus

$$\mathcal{R}_{\mathbb{T}_m}(E_0(BA)) \cong (E^0(BA \wr \Sigma_m)/I_{tr})^*$$

is free. Taking the E_0-linear dual gives the desired result. $\square$

6. Subgroups of $\mathbb{G}_{E_n} \oplus \mathbb{Q}_p/\mathbb{Z}_p^h$

In this section we prove a freeness result for the ring of functions on the scheme that classifies subgroups of the p-divisible group $\mathbb{G}_{E_n} \oplus \mathbb{Q}_p/\mathbb{Z}_p^h$.

The main algebro-geometric objects of study are the following:

Definition 6.1. For $k \geq 0$ we define

$$\mathrm{Sub}_{p^k}(\mathbb{G}_{E_n} \oplus \mathbb{Q}_p/\mathbb{Z}_p^h) : E_n^0\text{-algebras} \longrightarrow \mathrm{Set}$$

to be the functor mapping

$$R \mapsto \{H \subset R \otimes (\mathbb{G}_{E_n} \oplus \mathbb{Q}_p/\mathbb{Z}_p^h), H \text{ a subgroups scheme with } |H| = p^k\}.$$

Let

$$\mathrm{pr} : \mathbb{G}_{E_n} \oplus \mathbb{Q}_p/\mathbb{Z}_p^h \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p^h$$

be the projection. This induces a surjective map

$$\mathrm{Sub}_{p^k}(\mathbb{G}_{E_n} \oplus \mathbb{Q}_p/\mathbb{Z}_p^h) \longrightarrow \mathrm{Sub}_{\leq p^k}(\mathbb{Q}_p/\mathbb{Z}_p^h)$$

by mapping $H \mapsto \mathrm{pr}(H)$. Since the target of this map is discrete and the map is surjective the fibers disconnect the source. The fibers have the following form:

Definition 6.2. For $A \subset \mathbb{Q}_p/\mathbb{Z}_p^h$ of order less than p^k, define

$$\mathrm{Sub}_{p^k}^A(\mathbb{G}_{E_n} \oplus \mathbb{Q}_p/\mathbb{Z}_p^h) : E_n^0\text{-algebras} \longrightarrow \mathrm{Set}$$

to be the functor mapping

$$R \mapsto \{H \subset R \otimes (\mathbb{G}_{E_n} \oplus \mathbb{Q}_p/\mathbb{Z}_p^h)\}$$

such that H is a subgroup of order p^k and $\mathrm{pr}(H) = A (= R \otimes A)$.

Remark. Note that it is not hard to make sense of projection. For K be the kernel of the composite

$$H \longrightarrow R \otimes (\mathbb{G}_{E_n} \oplus \mathbb{Q}_p/\mathbb{Z}_p^h) \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p^h$$

we see that $\mathrm{pr}(H) = H/K \subset \mathbb{Q}_p/\mathbb{Z}_p^h$.

Example 6.3. When $A = e$ there is an isomorphism $\mathrm{Sub}_{p^k}^A(\mathbb{G}_{E_n} \oplus \mathbb{Q}_p/\mathbb{Z}_p^h) \cong \mathrm{Sub}_{p^k}(\mathbb{G}_{E_n})$.

Remark. Note that $\mathrm{Sub}_{p^k}(\mathbb{G}_{E_n} \oplus \mathbb{Q}_p/\mathbb{Z}_p^h)$ is a closed subscheme of $\mathrm{Div}_{p^k}(\mathbb{G}_{E_n} \oplus \Lambda_k)$.

Next we show that $\mathrm{Sub}_{p^k}^A(\mathbb{G}_{E_n} \oplus \mathbb{Q}_p/\mathbb{Z}_p^h)$ is corepresented by an E_n^0-algebra that is finitely generated and free as an E_n^0-module. We rely on Strickland's result:

Theorem 6.4. *(See Theorem 10.1, [14].) The functor* $\mathrm{Sub}_{p^k}(\mathbb{G}_{E_n})$ *is corepresented by an* E_n^0*-algebra that is free as an* E_n^0*-module of rank equal to the number of subgroups of order* p^k *in* $\mathbb{Q}_p/\mathbb{Z}_p^n$*.*

Proposition 6.5. *The functor* $\mathrm{Sub}^A_{p^k}(\mathbb{G}_{E_n} \oplus \mathbb{Q}_p/\mathbb{Z}_p^h)$ *is corepresentable by an* E_n^0*-algebra that is finitely generated and free as an* E_n^0*-module.*

Proof. Note that there is a surjection (there is a trivial section)

$$\mathrm{Sub}^A_{p^k}(\mathbb{G}_{E_n} \oplus \mathbb{Q}_p/\mathbb{Z}_p^h) \longrightarrow \mathrm{Sub}_{p^k/|A|}(\mathbb{G}_{E_n})$$

given by sending a projection $H \longrightarrow A$ to its kernel. Let $\bar{\mathbb{G}}_{E_n}$ be the pullback of $\mathbb{G}_{E_n}$ to $\mathrm{Sub}_{p^k/|A|}(\mathbb{G}_{E_n})$. The formal group $\bar{\mathbb{G}}_{E_n}$ carries the universal subgroup U of order $p^k/|A|$. For this proof let $S = \Gamma\,\mathrm{Sub}_{p^k/|A|}(\mathbb{G}_{E_n})$. Over $\mathrm{Sub}_{p^k/|A|}(\mathbb{G}_{E_n})$ there is an isomorphism

$$\mathrm{Sub}^A_{p^k}(\mathbb{G}_{E_n} \oplus \mathbb{Q}_p/\mathbb{Z}_p^h) \cong \mathrm{Hom}(A, \bar{\mathbb{G}}_{E_n}/U).$$

We will see this by using the functor of points. Let $\mathrm{Spec}(R)$ be an affine scheme over $\mathrm{Sub}_{p^k/|A|}(\mathbb{G}_{E_n})$. A $\mathrm{Spec}(R)$ point of the left hand side is a subgroup

$$H \subset R \otimes_{E_n^0} (\mathbb{G}_{E_n} \oplus \mathbb{Q}_p/\mathbb{Z}_p^h) = R \otimes_S (\bar{\mathbb{G}}_{E_n} \oplus \mathbb{Q}_p/\mathbb{Z}_p^h)$$

of order p^k that projects onto A. Let K be the kernel of the projection, then $K \cong R \otimes U$. Now we have a map of short exact sequences

$$\begin{tikzcd} K \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & A \arrow[d, dashed] \\ K \arrow[r] & R \otimes (\mathbb{G}_{E_n} \oplus \mathbb{Q}_p/\mathbb{Z}_p^h) \arrow[r] & (R \otimes \mathbb{G}_{E_n})/K \oplus \mathbb{Q}_p/\mathbb{Z}_p^h. \end{tikzcd}$$

Thus we get a $\mathrm{Spec}(R)$ point of the right hand side.

A $\mathrm{Spec}(R)$ point of the right hand side is a map

$$A \longrightarrow R \otimes_S \bar{\mathbb{G}}_{E_n}/U \cong (R \otimes_{E_n^0} \mathbb{G}_{E_n})/K.$$

Combined with the canonical inclusion $A \subset \mathbb{Q}_p/\mathbb{Z}_p^h$ this gives a map

$$A \longrightarrow (R \otimes \mathbb{G}_{E_n})/K \oplus \mathbb{Q}_p/\mathbb{Z}_p^h.$$

Now we pull back to get H

$$\begin{tikzcd} H \arrow[r] \arrow[d] & R \otimes \mathbb{G}_{E_n} \oplus \mathbb{Q}_p/\mathbb{Z}_p^h \arrow[d] \\ A \arrow[r] & (R \otimes \mathbb{G}_{E_n})/K \oplus \mathbb{Q}_p/\mathbb{Z}_p^h \end{tikzcd}$$

a subgroup of order p^k that projects onto A. This produces the map back. Now the scheme $\mathrm{Hom}(A, \bar{\mathbb{G}}_{E_n}/U)$ is finite flat over $\mathrm{Sub}_{p^k/|A|}(\mathbb{G}_{E_n})$ since $\mathbb{G}_{E_n}/U$ is p-divisible. Theorem 6.4 implies that $\mathrm{Sub}_{p^k/|A|}(\mathbb{G}_{E_n})$ is finite flat over E_n^0. Now the composite is finite flat and E_n^0 is complete local, so we are done. □

7. A generalized Strickland's theorem

We show that $E_n^0(\mathcal{L}^h B\Sigma_{p^k})/I_{tr}$ corepresents the scheme $\mathrm{Sub}_{p^k}(\mathbb{G}_{E_n} \oplus \mathbb{Q}_p/\mathbb{Z}_p^h)$. In the first subsection we construct the map between the two objects. The map is clearly an isomorphism after inverting p. In the next subsection we give a direct proof of the main theorem at height 1. In the final subsection we prove that the map is an isomorphism by reduction to the height 1 case.

7.1. Character theoretic construction of the map

The purpose of this subsection is to construct a map

$$f_{p^k} : \Gamma\,\mathrm{Sub}_{p^k}(\mathbb{G}_{E_n} \oplus \mathbb{Q}_p/\mathbb{Z}_p^h) \longrightarrow E_n^0(\mathcal{L}^h B\Sigma_{p^k})/I_{tr}.$$

When $h = 0$ this should be compared to Proposition 9.1 of [15]. We will construct the map by using HKR character theory to embed both sides into class functions. There we will see that the image of the left hand side is contained in the image of the right hand side. Thus there is an injective map between the two rings.

Let

$$\pi : \Sigma_{p^k} \longrightarrow U(p^k)$$

be the permutation representation.

Proposition 7.2. *The following diagram commutes:*

$$\begin{tikzcd}
\Gamma\,\mathrm{Div}_{p^k}(\mathbb{G}_{E_n} \oplus \Lambda_k) \arrow[r, two heads] \arrow[dd, "\cong"] & \Gamma\,\mathrm{Sub}_{p^k}(\mathbb{G}_{E_n} \oplus \mathbb{Q}_p/\mathbb{Z}_p^h) \arrow[r, hook] & \Gamma C_0 \otimes \mathrm{Sub}_{p^k}(\mathbb{G}_{E_n} \oplus \mathbb{Q}_p/\mathbb{Z}_p^h) \arrow[d, "\cong"] \\
& & \Gamma\,\mathrm{Sub}_{p^k}(\mathbb{Q}_p/\mathbb{Z}_p^{n+h}) \arrow[d, "\cong"] \\
E_n^0(\mathcal{L}_k^h BU(p^k)) \arrow[r, "\pi^*"] & E_n^0(\mathcal{L}^h B\Sigma_{p^k})/I_{tr} \arrow[r, hook] & Cl((\Sigma_{p^k})_p^{n+h}, C_0)/I_{tr}.
\end{tikzcd}$$

This implies that there is an injection

$$f_{p^k} : \Gamma\,\mathrm{Sub}_{p^k}(\mathbb{G}_{E_n} \oplus \mathbb{Q}_p/\mathbb{Z}_p^h) \hookrightarrow E_n^0(\mathcal{L}^h B\Sigma_{p^k})/I_{tr}.$$

Proof. We have already seen that the left vertical map is an isomorphism. The right vertical map is given by sending $\alpha : \mathbb{Z}_p^{n+h} \longrightarrow \Sigma_{p^k}$ to the image of the Pontryagin dual

$$\mathbb{Q}_p/\mathbb{Z}_p^{n+h} \hookleftarrow (\mathrm{im}\,\alpha)^*.$$

The right horizontal maps are injections by Propositions 5.3 and 6.5. Now the right vertical isomorphism implies that both of the E_n^0-modules in the middle have the same rank. The left horizontal surjection is due to the fact that we are taking the ring of functions of a closed inclusion.

We can use the results of Section 3 to reduce commutativity of this diagram to a height 0 (and thus combinatorial) problem. Let $\Lambda_k = ((\mathbb{Z}/p^k)^h)^*$ and $\Delta_k = ((\mathbb{Z}/p^k)^{n+h})^*$. Consider the following diagram:

$$\begin{tikzcd}
& \Gamma\,\mathrm{Div}_{p^k}(\mathbb{G}_{E_n} \oplus \Lambda_k) \arrow[rr] \arrow[dl] \arrow[dd] & & \Gamma\,\mathrm{Sub}_{p^k}(\mathbb{G}_{E_n} \oplus \mathbb{Q}_p/\mathbb{Z}_p^h) \arrow[dl, dashed] \arrow[dd] \\
E_n^0(\mathcal{L}_k^h BU(p^k)) \arrow[rr] \arrow[dd] & & E_n^0(\mathcal{L}^h B\Sigma_{p^k})/I_{tr} \arrow[dd] & \\
& C_0^0 \otimes \Gamma\,\mathrm{Div}_{p^k}(\mathbb{G}_{p^{-1}E_n} \oplus \Delta_k) \arrow[rr] \arrow[dl] & & \Gamma\,\mathrm{Sub}_{p^k}(\mathbb{Q}_p/\mathbb{Z}_p^{n+h}) \arrow[dl] \\
C_0^0(\mathcal{L}_k^{n+h} BU(p^k)) \arrow[rr] & & Cl((\Sigma_{p^k})_p^{n+h}, C_0^0)/I_{tr}. &
\end{tikzcd}$$

The top square of this cube is at height n and the bottom square at height 0. The left side commutes by Proposition 3.8. The front side commutes by Proposition 3.7. The back side is completely algebro-geometric. The bottom is at height 0 and thus combinatorial, its commutativity follows from the definition of the permutation representation.

We show that Spec(−) of the bottom square commutes. Let $\mathbb{Z}_p^{n+h} \xrightarrow{\alpha} \Sigma_{p^k}$ be a map classifying a transitive $\mathbb{Z}_p^{n+h}$-set. Let $A = \operatorname{im}\alpha$. Going around the square through $\mathrm{Sub}_{p^k}(\mathbb{Q}_p/\mathbb{Z}_p^{n+h})$ sends this to the image of the Pontryagin dual

$$A^* \longrightarrow \Delta_k \subset \mathbb{Q}_p/\mathbb{Z}_p^{n+h}$$

viewed as a divisor in $C_0^0 \otimes \mathbb{G}_{p^{-1}E_n} \oplus \Delta_k$ that is just 0 in $C_0^0 \otimes \mathbb{G}_{p^{-1}E_n}$.

We show this is the same as going around the bottom square the other way. The map α factors into the following composite

$$\mathbb{Z}_p^{n+h} \longrightarrow (\mathbb{Z}/p^k)^{n+h} \xrightarrow{g} A \longrightarrow \Sigma_{p^k}.$$

Since A is a transitive abelian subgroup of Σ_{p^k} it has order p^k and the composite $\rho : A \to \Sigma_{p^k} \xrightarrow{\pi} U(p^k)$ is the regular representation. Since ρ is the regular representation it is the sum of all characters of A. This defines $|A|$ maps $A \longrightarrow S^1$ are precisely the elements of A^*. Thus we get the divisor

$$A^* \subseteq \Delta_k = ((\mathbb{Z}/p^k)^{n+h})^*.$$

The last thing that needs to be checked is that all of the power series generators (after choosing a coordinate) of

$$C_0^0(\mathcal{L}_k^{n+h} BU(p^k))$$

map to $0 \in Cl((\Sigma_{p^k})_p^{n+h}, C_0^0)/I_{tr}$. But this is clear as they are induced by elements of higher cohomological degree and the codomain is concentrated in degree 0. Thus we get exactly the same divisor of $C_0^0 \otimes \mathbb{G}_{p^{-1}E_n} \oplus \Delta_k$ by going around the square both ways. □

7.3. The height 1 case

Given an abelian group A and an integer $n \geq 0$, there is a map

$$N_A : A \wr \Sigma_n \to A$$

induced by the addition map $A^n \to A$.

Lemma 7.4. *The map N_A induces an isomorphism*

$$R(A) \xrightarrow{\cong} R(A \wr \Sigma_n)/I_{tr}$$

Proof. This isomorphism appears in Section 7 of [17]. Note that given a character ρ of A. The pulled back representation on $A \wr \Sigma_n$ is $\rho^{\otimes n} \wr 1$. □

The conjugacy classes of $A \wr \Sigma_n$ have canonical representatives. It will be useful to know what they are.

Every conjugacy class of Σ_n is determined by a partition of n. We represent this as a non-increasing sequence of integers $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r \geq 1$ such that $\sum \lambda_i = n$. Any such conjugacy class contains a unique element that can be written in cycle notation as

$$\sigma = (1, \ldots, \lambda_1)(\lambda_1 + 1, \ldots, \lambda_1 + \lambda_2) \cdots (n - \lambda_r + 1, \ldots, n).$$

We call such a σ a *canonical representative.*

Now let σ be a canonical representative of Σ_n with r cycles and let $(a_1, \ldots a_r) \in A^r$ be a tuple of elements in the abelian group A. We denote by $[a_1, \ldots, a_r] \wr \sigma \in A \wr \Sigma_n$ the element

$$(a_1, \overbrace{0, \ldots, 0}^{\lambda_1 - 1}, a_2, \overbrace{0, \ldots, 0}^{\lambda_2 - 1}, \ldots, a_r, \overbrace{0, \ldots, 0}^{\lambda_r - 1}) \rtimes \sigma.$$

We call the element of the form $[a_1, \ldots, a_r] \wr \sigma \in A \wr \Sigma_n$ a canonical representative in $A \wr \Sigma_n$. It is well-known and easy to check that the set of canonical representatives in $A \wr \Sigma_n$ is a complete set of representatives for the conjugacy classes of $A \wr \Sigma_n$.

Lemma 7.5. *Let A be an abelian group and $n > 0$ be an integer. The image of all the transfer maps*

$$Cl(A \wr (\Sigma_k \times \Sigma_{n-k}), \mathbb{C}) \to Cl(A \wr \Sigma_n, \mathbb{C})$$

together generate the ideal of functions that vanish on all conjugacy classes in $A \wr \Sigma_n$ which project to the n-cycle in Σ_n (i.e. the conjugacy classes of the form $[a] \wr (1, \ldots, n)$ for some $a \in A$).

Proof. Clear. $\square$

Lemma 7.6. *Assume $|A||r$, then there exists a canonical isomorphism of schemes*

$$c_{r,A} : \mathrm{Hom}(A^*, \mathbb{G}_m) \xrightarrow{\cong} \mathrm{Sub}_r^{A^*}(\mathbb{G}_m \oplus \mathbb{Q}/\mathbb{Z}^h).$$

Proof. Denote $l = \frac{r}{|A|}$ and recall the proof of Proposition 6.5. We get a natural isomorphism

$$\mathrm{Hom}(A^*, \mathbb{G}_m/\mathbb{G}_m[l]) \xrightarrow{\cong} \mathrm{Sub}_r^{A^*}(\mathbb{G}_m \oplus \mathbb{Q}/\mathbb{Z}^h)$$

as schemes over $\mathrm{Sub}_l(\mathbb{G}_m)$. Now $\mathbb{G}_m$ has a unique subgroup $\mathbb{G}_m[l]$ of order l so

$$\mathrm{Sub}_l(\mathbb{G}_m) \cong \mathrm{Spec}\, \mathbb{Z}$$

and we have a canonical isomorphism $\mathbb{G}_m/\mathbb{G}_m[l] \cong \mathbb{G}_m$. $\square$

Lemma 7.7. *There is a canonical isomorphism of schemes*

$$\chi_A : \operatorname{Spec} R(A) \xrightarrow{\cong} \operatorname{Hom}(A^*, \mathbb{G}_m)$$

Proof. Consider the scheme $\operatorname{Spec} R(A)$ as a functor of points, given a test ring T a map $R(A) \to T$ is the same as a map $A^* \to T^\times = \mathbb{G}_m(T)$. □

Let $l = r/|A|$, then there is an injective map

$$g_{r,A} : \Gamma \operatorname{Sub}_r^{A^*}(\mathbb{G}_m \oplus \mathbb{Q}/\mathbb{Z}^h) \hookrightarrow R(A \wr \Sigma_l)/I_{tr}$$

defined by embedding both sides into class functions just as in the proof of Proposition 7.2.

Proposition 7.8. *Let $\alpha : \mathbb{Z}^h \longrightarrow \Sigma_r$ be monotypical and let $A = \operatorname{im} \alpha$ and $l = r/|A|$. There is a commutative diagram*

$$\begin{tikzcd}
\operatorname{Spec} R(A) \arrow[r, "\chi_A", "\cong"'] & \operatorname{Hom}(A^*, \mathbb{G}_m) \arrow[d, "c_{r,A}"', "\cong"] \\
\operatorname{Spec}(R(A \wr \Sigma_l)/I_{tr}) \arrow[u, "N_A", "\cong"'] \arrow[r, "g^*_{r,A}"] & \operatorname{Sub}_r^{A^*}(\mathbb{G}_m \oplus \mathbb{Q}/\mathbb{Z}^h).
\end{tikzcd}$$

Proof. Since all schemes involved are reduced finite and flat over $\operatorname{Spec} \mathbb{Z}$ it is enough to check the commutativity on $\mathbb{C}$-points. By Lemma 7.5 $\operatorname{Spec}(R(A \wr \Sigma_l)/I_{tr})(\mathbb{C})$ can be naturally identified with the set of conjugacy classes of $[a] \wr (1, \dots, l)$ for all $a \in A$. Let $[a] \wr (1, \dots, l)$ be such an element. First we would like to describe $g^*_{r,A}([a] \wr (1, \dots, l))$. The conjugacy class $[a] \wr (1, \dots, l)$ corresponds to a map

$$\alpha_a : \mathbb{Z} \to A \wr \Sigma_l$$

such that $\alpha(a)(1) = [a] \wr (1, \dots, l)$. By Section 4, α_a corresponds to map

$$\tilde{\alpha}_a : \mathbb{Z} \oplus A \to \Sigma_{l|A|}.$$

To understand the kernel of this map note that A embeds into $A \wr \Sigma_l$ via the diagonal map $z : A \to A \wr \Sigma_l$ on the other hand

$$([a] \wr (1, \dots, l))^l = z(a).$$

Thus we get that $B := \operatorname{im} \tilde{\alpha_a}$ is the result of the pushout:

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "1\mapsto a"] \arrow[d, "\times l"] & A \arrow[d, hook] \\
\mathbb{Z} \arrow[r] & B.
\end{tikzcd}$$

Since A comes with the surjection $\alpha : \mathbb{Z}^h \to A$ we can write B also as the pushout of

$$\begin{tikzcd}
\mathbb{Z} \oplus \mathbb{Z}^h \arrow[r, "a\oplus\alpha"] \arrow[d, "\times l\oplus Id"] & A \arrow[d, hook] \\
\mathbb{Z} \oplus \mathbb{Z}^h \arrow[r, two heads] & B.
\end{tikzcd}$$

Dualizing the diagram above we get

$$\begin{tikzcd}
B^* \arrow[r, hook] \arrow[d] & \mathbb{Q}/\mathbb{Z} \oplus \mathbb{Q}/\mathbb{Z}^h \arrow[d, "\times l\oplus Id"] \\
A^* \arrow[r, hook, "a^*\oplus\alpha^*"'] & \mathbb{Q}/\mathbb{Z} \oplus \mathbb{Q}/\mathbb{Z}^h.
\end{tikzcd}$$

Considering B^* as a subgroup in $\mathbb{Q}/\mathbb{Z} \oplus \mathbb{Q}/\mathbb{Z}^h$ we get

$$B^* = g^*_{r,A}([a] \wr (1, \dots, l)) \in \mathrm{Sub}_r^{A^*}(\mathbb{G}_m \oplus \mathbb{Q}/\mathbb{Z}^h)(\mathbb{C}).$$

Now we would like to compare B^* with $c_{r,A} \circ \chi_A \circ N_A([a] \wr (1, \dots, l))$. It is clear that

$$N_A([a] \wr (1, \dots, l)) = a \in A = \mathrm{Spec}(R(A))(\mathbb{C}).$$

The map $\chi_A(a)$ is

$$a^* : A^* \to \mathbb{Q}/\mathbb{Z}$$

mapping $a^*(\phi) = \phi(a)$. To apply $c_{r,A}$ we need to take a pullback as in the proof of Proposition 6.5. We get

$$\begin{tikzcd}
B^* \arrow[r, hook] \arrow[d] & \mathbb{Q}/\mathbb{Z} \oplus \mathbb{Q}/\mathbb{Z}^h \arrow[d, "\times l\oplus Id"] \\
A^* \arrow[r, hook, "a^*\oplus\alpha^*"'] & \mathbb{Q}/\mathbb{Z} \oplus \mathbb{Q}/\mathbb{Z}^h
\end{tikzcd}$$

just as before. □

7.9. The map is an isomorphism

Recall that C_0^0 is a faithfully flat $p^{-1}E_n^0$-algebra. Thus after inverting p the map

$$f_{p^k} : \Gamma\,\mathrm{Sub}_{p^k}(\mathbb{G}_{E_n} \oplus \mathbb{Q}_p/\mathbb{Z}_p^h) \longrightarrow E_n^0(\mathcal{L}^h B\Sigma_{p^k})/I_{tr}$$

constructed in Proposition 7.2 is an isomorphism. Now it suffices to show that the map is an isomorphism after base change to any ring in which p is not nilpotent and not invertible. We use the ring $\bar{C}_1^0$ constructed in Subsection 2.6.

Proposition 7.10. *The isomorphism of Lemma 7.4 induces an isomorphism*

$$K_p^0(BA \wr \Sigma_{p^k})/I_{tr} \xrightarrow{\cong} K_p^0(BA).$$

Proof. Let P be a Sylow p-subgroup of G and J the kernel of the restriction map $RG \longrightarrow RP$. By Proposition 9.7 in [16], $K_p^0(BG) \cong \mathbb{Z}_p \otimes R(G)/J$. Let us write R_pG for $\mathbb{Z}_p \otimes RG$. Assume that A is a p-group, then we have the following diagram

$$\begin{tikzcd}
I_{tr} \arrow[r, two heads] \arrow[d] & J_{tr} \arrow[d] \\
R_pA \wr \Sigma_{p^k} \arrow[r, two heads] \arrow[d] & K_p^0 A \wr \Sigma_{p^k} \arrow[d] \\
R_pA \wr \Sigma_{p^k}/I_{tr} \arrow[r, two heads] \arrow[d, "\cong"] & K_p^0 A \wr \Sigma_{p^k}/J_{tr} \\
R_pA \arrow[r, "\cong"] & K_p^0 A, \arrow[u]
\end{tikzcd}$$

where I_{tr} is the image of the transfer in representation theory and J_{tr} is the image of the transfer in p-adic K-theory. The top two horizontal arrows are surjective by Strickland's result. This implies the third horizontal arrow is surjective. But this is a map between free modules of the same rank so it is an isomorphism. This implies that the bottom right arrow is an isomorphism. □

Thus we have a commutative diagram

$$\begin{tikzcd}
K_p^0(\mathcal{L}^h BU(p^k)) \arrow[d, two heads] & \Gamma\,\mathrm{Div}_{p^k}(\hat{\mathbb{G}}_m \oplus \Lambda_k) \arrow[l, "\cong"'] \arrow[d, two heads] \\
K_p^0(\mathcal{L}^h B\Sigma_{p^k})/I_{tr} & \Gamma\,\mathrm{Sub}_{p^k}(\hat{\mathbb{G}}_m \oplus \mathbb{Q}_p/\mathbb{Z}_p^h), \arrow[l, "\cong", "g_{p^k}"']
\end{tikzcd}$$

where $\hat{\mathbb{G}}_m$ is the formal multiplicative group. The bottom arrow may be defined purely in terms of the other arrows. We choose an element in $\Gamma\,\mathrm{Sub}_{p^k}(\hat{\mathbb{G}}_m \oplus \mathbb{Q}_p/\mathbb{Z}_p^h)$, lift it to the global sections of divisors, pass to $K_p^0(\mathcal{L}_k^h BU(p^k))$ and map down to $K_p^0(\mathcal{L}^h B\Sigma_{p^k})/I_{tr}$. We slightly abuse notation and call the bottom isomorphism g_{p^k}. By Proposition 7.8 it is a product of isomorphisms of the form $g_{p^k,A}$.

Theorem 7.11. *The map*

$$f_{p^k} : \Gamma\,\mathrm{Sub}_{p^k}(\mathbb{G}_{E_n} \oplus \mathbb{Q}_p/\mathbb{Z}_p^h) \longrightarrow E_n^0(\mathcal{L}^h B\Sigma_{p^k})/I_{tr}$$

is an isomorphism.

Proof. Recall from Subsection 2.6 that $\bar{C}_1^0 = \pi_0(L_{K(1)}(C_1 \wedge K_p))$ and that there is an isomorphism of p-divisible groups

$$\bar{C}_1^0 \otimes \mathbb{G}_{C_1} \cong \bar{C}_1^0 \otimes \hat{\mathbb{G}}_m \cong \mathbb{G}_{\bar{C}_1}.$$

Applying the character maps of Theorem 2.2 and Theorem 3.8 to the square

$$\begin{tikzcd} E_n^0(\mathcal{L}_k^h BU(p^k)) \arrow[d] & \Gamma\,\mathrm{Div}_{p^k}(\mathbb{G}_{E_n} \oplus \Lambda_k) \arrow[l, "\cong"'] \arrow[d] \\ E_n^0(\mathcal{L}^h B\Sigma_{p^k})/I_{tr} & \Gamma\,\mathrm{Sub}_{p^k}(\mathbb{G}_{E_n} \oplus \mathbb{Q}_p/\mathbb{Z}_p^h) \arrow[l] \end{tikzcd}$$

and then base changing to $\bar{C}_1^0$ gives the square

$$\begin{tikzcd} \bar{C}_1^0 \otimes_{L_{K(t)}E_n^0} L_{K(t)}E_n^0(\mathcal{L}_k^{n+h-1} BU(p^k)) \arrow[d] & \bar{C}_1^0 \otimes \Gamma\,\mathrm{Div}_{p^k}(\mathbb{G}_{L_{K(t)}E_n} \oplus (\mathbb{Z}/p^k)^{n+h-1}) \arrow[l, "\cong"'] \arrow[d] \\ \bar{C}_1^0(\mathcal{L}^{n+h-1} B\Sigma_{p^k})/I_{tr} & \Gamma\,\mathrm{Sub}_{p^k}(\mathbb{G}_{\bar{C}_1} \oplus \mathbb{Q}_p/\mathbb{Z}_p^{n+h-1}). \arrow[l, "\bar{C}_1^0 \otimes f_{p^k}"'] \end{tikzcd}$$

Proposition 2.10 implies that the bottom arrow of this square is the base change $\bar{C}_1^0 \otimes f_{p^k}$ because all of the groups that appear in $\mathcal{L}^{n+h-1} B\Sigma_{p^k}$ are good. The map from the top arrow to the bottom arrow factors through

$$\begin{tikzcd} \bar{C}_1^0(\mathcal{L}_k^{n+h-1} BU(p^k)) \arrow[d] & \Gamma\,\mathrm{Div}_{p^k}(\mathbb{G}_{\bar{C}_1} \oplus (\mathbb{Z}/p^k)^{n+h-1}) \arrow[l, "\cong"'] \arrow[d] \\ \bar{C}_1^0(\mathcal{L}^{n+h-1} B\Sigma_{p^k})/I_{tr} & \Gamma\,\mathrm{Sub}_{p^k}(\mathbb{G}_{\bar{C}_1} \oplus \mathbb{Q}_p/\mathbb{Z}_p^{n+h-1}), \arrow[l, "\bar{C}_1^0 \otimes f_{p^k}"'] \end{tikzcd}$$

where $\bar{C}_1^0(\mathcal{L}_k^{n+h-1}BU(p^k)) = \pi_0 \bar{C}_1^{\mathcal{L}_k^{n+h-1}BU(p^k)}$. From p-adic K-theory we have the square

$$\begin{tikzcd}
\bar{C}_1^0 \otimes K_p^0(\mathcal{L}_k^{n+h-1}BU(p^k)) \arrow[d] & \bar{C}_1^0 \otimes \Gamma\,\mathrm{Div}_{p^k}(\hat{\mathbb{G}}_m \oplus (\mathbb{Z}/p^k)^{n+h-1}) \arrow[l, "\cong"'] \arrow[d] \\
\bar{C}_1^0 \otimes K_p^0(\mathcal{L}^{n+h-1}B\Sigma_{p^k})/I_{tr} & \bar{C}_1^0 \otimes \Gamma\,\mathrm{Sub}_{p^k}(\hat{\mathbb{G}}_m \oplus \mathbb{Q}_p/\mathbb{Z}_p^{n+h-1}). \arrow[l, "\cong"']
\end{tikzcd}$$

The bottom arrow of this square is $\bar{C}_1^0 \otimes g_{p^k}$. The last square maps to the middle square and the bottom square of the resulting cube is

$$\begin{tikzcd}
\bar{C}_1^0(\mathcal{L}^{n+h-1}B\Sigma_{p^k})/I_{tr} & \Gamma\,\mathrm{Sub}_{p^k}(\bar{C}_1^0 \otimes \mathbb{G}_{C_1} \oplus \mathbb{Q}_p/\mathbb{Z}_p^h) \arrow[l, "\bar{C}_1^0 \otimes f_{p^k}"'] \\
\bar{C}_1^0 \otimes K_p^0(\mathcal{L}^{n+h-1}B\Sigma_{p^k})/I_{tr} \arrow[u, "\cong"] & \bar{C}_1^0 \otimes \Gamma\,\mathrm{Sub}_{p^k}(\hat{\mathbb{G}}_m \oplus \mathbb{Q}_p/\mathbb{Z}_p^{n+h-1}), \arrow[l, "\cong", "\bar{C}_1^0 \otimes g_{p^k}"'] \arrow[u, "\cong"']
\end{tikzcd}$$

where the vertical arrows are isomorphisms by Proposition 2.10. This square commutes because all of the other squares in the cube commute and the maps $\bar{C}_1^0 \otimes g_{p^k}$ and $\bar{C}_1^0 \otimes f_{p^k}$ are determined by the others.

Thus $\bar{C}_1^0 \otimes f_{p^k}$ is an isomorphism and this implies that f_{p^k} is an isomorphism. □

An advantage of the definition of the map f_{p^k} via the character maps is that the following corollary is immediate.

Corollary 7.12. *Let $\alpha : \mathbb{Z}_p^h \longrightarrow \Sigma_{p^k}$ be monotypical, let $A = \mathrm{im}\,\alpha$, and let $p^j = p^k/|A|$. Restricting f gives the isomorphism*

$$E_n^0(BA \wr \Sigma_{p^j})/I_{tr} \cong \Gamma\,\mathrm{Sub}_{p^k}^{A^*}(\mathbb{G}_{E_n} \oplus \mathbb{Q}_p/\mathbb{Z}_p^h).$$

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Appendix A. An elementary proof when $k = 1$

We give a proof of Strickland's theorem when $k = 1$ by direct calculation. The calculation reduces to a congruence regarding Stirling numbers of the first kind. The trick in this case is that the Honda formal group law is easy to describe modulo x^{p^n}.

The mod I_n reduction of the map

$$f_p : \Gamma \operatorname{Sub}_p(\mathbb{G}_{E_n}) \longrightarrow E_n^0(B\Sigma_p)/I_{tr}$$

from Proposition 7.2 is the map

$$\Gamma \operatorname{Sub}_p(\mathbb{G}_{K(n)}) \longrightarrow K(n)^0(B\Sigma_p)/I_{tr}.$$

The domain is still a closed subscheme of divisors. It suffices to show that

$$K(n)^0(BU(p)) \xrightarrow{\pi^*} K(n)^0(B\Sigma_p)/I_{tr}$$

is surjective.

Proposition A.1. *The ring $K(n)^0(B\Sigma_p)/I_{tr}$ is generated by the Chern classes of the permutation representation.*

Proof. The ideal I_{tr} has rank 1 so the rank of $K(n)^0(B\Sigma_p)/I_{tr}$ is $(p^n - 1)/(p - 1)$.

The composite

$$\mathbb{Z}/p \hookrightarrow \Sigma_p \xrightarrow{\pi} U(p)$$

is the regular representation ρ of $\mathbb{Z}/p$. Thus it suffices to show that the Chern classes of ρ generate a subring of rank $(p^n - 1)/(p - 1)$ inside of $K(n)^0(B\mathbb{Z}/p)/I_{tr}$.

Recall that there is an isomorphism

$$K(n)^0(B\mathbb{Z}/p) \cong K(n)^0[x]/(x^{p^n})$$

and the transfer map from e to $\mathbb{Z}/p$ sends 1 to x^{p^n-1}. Thus

$$K(n)^0(B\mathbb{Z}/p)/I_{tr} \cong K(n)^0[x]/(x^{p^n-1}).$$

Let F be the height n Honda formal group law. By Lemma 4.12 in [2], there is a congruence

$$x +_F y = x + y - \sum_{0<j<p} p^{-1} \binom{p}{j} (x^{p^{n-1}})^j (y^{p^{n-1}})^{p-j} \tag{1}$$

modulo x^{p^n}.

Because the regular representation is the sum of the irreducible representations of $\mathbb{Z}/p$, the total Chern class of ρ is

$$c(\rho) = \prod_{0<i<p} (1 - [i]_F(x)t),$$

where $[i]_F(x)$ is the first Chern class of the tensor power of a generating representation $x : \mathbb{Z}/p \hookrightarrow S^1$.

Now Equation (1) implies that $[i]_F(x) = ix \bmod x^{p^n}$. We are left trying to understand

$$c(\rho) = \prod_{0<i<p} (1 - ixt).$$

Thus

$$c_i(\rho) = \Big(\sum_{0<k_1<\ldots<k_i<p} k_1 k_2 \cdots k_i \Big) x^i = s(p,i)x^i,$$

where $s(p,i)$ is the Stirling number of the first kind. It is well-known (see Corollary 4 in [7]) that $s(p,i)$ is divisible by p when $1 < i < p$.

Thus the only Chern class that does not disappear (we are working in characteristic p) is $c_{p-1}(\rho)$. This is congruent to $p-1$.

Now the subring of

$$K(n)^0(B\mathbb{Z}/p)/I_{tr} \cong K(n)^0[x]/x^{p^n-1}$$

generated by x^{p-1} has rank $(p^n - 1)/(p - 1)$. We conclude that $K(n)^0(B\Sigma_p)/I_{tr}$ is generated by the Chern classes of the permutation representation. $\square$

To generalize this, one might want to use the injection of Proposition 9.1 in [15]

$$\Gamma\,\mathrm{Sub}_{p^k}(\mathbb{G}_{E_n}) \hookrightarrow \prod_{A\subset\Sigma_{p^k} \text{ transitive}} \Gamma\,\mathrm{Level}(A, \mathbb{G}_{E_n}),$$

where the product is over abelian transitive subgroups of Σ_{p^k}.

One of the key obstructions to generalizing this proof seems to be the fact that, even for $p = 2$ and $k = 2$, the injection

$$\Gamma\,\mathrm{Sub}_{p^2}(\mathbb{G}_{E_n}) \longrightarrow \Gamma\,\mathrm{Level}(\mathbb{Z}/2 \times \mathbb{Z}/2, \mathbb{G}_{E_n}) \times \Gamma\,\mathrm{Level}(\mathbb{Z}/4, \mathbb{G}_{E_n})$$

does not pass to an injection after taking the quotient by $I_n \subset E_n^0$.

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