CHROMATIC REDSHIFT

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1. Introduction

Consider a commutative ring R, with sum and product operations. The category of representations of R inherits a commutative rig structure, given by direct sum and tensor product. In other words, the category $\mathrm{Mod}(R)$ of R-modules inherits a bipermutative structure. Continuing, one can consider the categorical representations of $\mathrm{Mod}(R)$, and these in turn form a 2-category $\mathrm{Mod}(\mathrm{Mod}(R))$, with a ring-like structure. Iterating, one can consider an n-category of higher representations, for each $n \geq 1$.

All of these constructions can take place within the limiting context of structured ring spectra, or commutative S-algebras. From the category of (finite cell) modules over a commutative S-algebra B we can distill a new commutative S-algebra, the algebraic K-theory spectrum $K(B)$. Continuing, one can form $K(K(B))$, etc. When $B = HR$ is the Eilenberg–Mac Lane spectrum of an ordinary ring, the n-fold algebraic K-theory $K^{(n)}(B)$ is extracted from the n-category of higher representations, as above. In this sense, n-fold iterated algebraic K-theory has something to do with n-categories.

From this point of view it is surprising that n-fold iterated algebraic K-theory also has something to do with formal group laws of height n, i.e., one-dimensional commutative formal group laws F in characteristic p where the series expansion $[p]_F(x)$ for the multiplication-by-p map starts with a unit times x^{p^n}. This is essentially a statement about the formal coproduct on $K^{(n)}(B)^*(\mathbb{C}P^\infty)$ that comes from the product on $\mathbb{C}P^\infty$. Hesselholt–Madsen asked about the chromatic filtration of iterated topological cyclic homology in [HM97, p. 61], but could almost as well have asked about the chromatic filtration of iterated algebraic K-theory.

In a strong form, this connection implies that the algebraic K-theory of a structured ring spectrum related to formal group laws of height n will be related to formal group laws of height $n+1$. In terms of the periodic families of stable homotopy theory, if the homotopy of B is v_n-periodic but not v_{n+1}-periodic, then frequently $K(B)$ is v_{n+1}-periodic but not v_{n+2}-periodic.

Since the (fundamental) period $|v_{n+1}| = 2p^{n+1} - 2$ of v_{n+1}-periodicity is longer than the period $|v_n| = 2p^n - 2$ of v_n-periodicity, we think of this phenomenon as an increase, or lengthening, of wavelengths. This is what we informally call a "redshift". In a related fashion, the v_{n+1}-periodic phenomena are usually hidden or nested behind the v_n-periodic ones, hence more subtle and difficult to detect. Again this corresponds informally to less energetic light, propagating at lower frequencies.

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The height filtration is also related to the sequence of Hopf subalgebras

$$0 \subset \cdots \subset \mathscr{E}(n) = E(Q_0, \dots, Q_n) \subset \dots$$

of the Steenrod algebra $\mathscr{A}$, and their annihilating subalgebras

$$\mathscr{A}_* \supset \cdots \supset (\mathscr{A}//\mathscr{E}(n))^* = P(\bar\xi_k \mid k \ge 1) \otimes E(\bar\tau_k \mid k \ge n+1) \supset \dots .$$

The latter nested sequence of $\mathscr{A}_*$-comodule subalgebras are invariant under the Dyer–Lashof operations that arise from thinking of the dual Steenrod algebra $\mathscr{A}_*$ as $H_*(H)$, where $H = H\mathbb{F}_p$ is a commutative structured ring spectrum.

2. Redshift in the K-theory of rings

We start with examples of chromatic redshift in the algebraic K-theory of discrete rings.

Let k be a finite field of characteristic p, with algebraic closure $\bar k$. Quillen proved [Qui72, §11] that $H_i(BGL(\bar k); \mathbb{F}_p) = 0$ for $i > 0$, so that $K(\bar k)_p \simeq H\mathbb{Z}_p$. Furthermore, he deduced that $\pi_* K(k)_p \cong \pi_* K(\bar k)_p^{hG_k}$ for $* \ge 0$, where the absolute Galois group G_k acts continuously on $K(\bar k)$, so $K(k)_p \simeq H\mathbb{Z}_p$. Multiplication by p acts injectively on $\pi_* K(\bar k)_p$, hence also on $\pi_* K(k)_p$. Think of p as a lift of $p = v_0 \in \pi_* BP$, where BP is the Brown–Peterson ring spectrum with $\pi_* BP = \mathbb{Z}_{(p)}[v_n \mid n \ge 1]$.

For a separably closed field $\bar F$ of characteristic $\ne p$ (including 0), Lichtenbaum conjectured that $\pi_t K(\bar F)_p$ is $\mathbb{Z}_p$ for $t \ge 0$ even and 0 for t odd. This was proved by Suslin [Sus84, Cor. 3.13], and implies that $K(\bar F)_p \simeq ku_p$ and $\hat L_1 K(\bar F) \simeq KU_p$. Here ku is the connective cover of the complex topological K-theory ring spectrum KU, and $\hat L_n = L_{K(n)}$ denotes Bousfield localization [Bou79] with respect to the Morava K-theory ring spectrum $K(n)$. Multiplication by the Bott element $u \in \pi_2 ku_p$ acts bijectively on $\pi_* K(\bar F)_p$, for $* \ge 0$.

Let F be a number field, with a ring of S-integers A.

$$\begin{array}{ccccc} A & \longrightarrow & & \longrightarrow & F \\ \uparrow & & & & \uparrow \\ \mathbb{Z} & \longrightarrow & \mathbb{Z}[1/p] & \longrightarrow & \mathbb{Q} \end{array}$$

Quillen conjectured [Qui75, §9] that there is a spectral sequence

$$E^2_{s,t} = H^{-s}_{\text{ét}}(\operatorname{Spec} A; \mathbb{Z}_p(t/2)) \Longrightarrow \pi_{s+t} K(A)_p$$

converging in total degrees ≥ 1. Here $H^*_{\text{ét}}(-)$ denotes étale cohomology, which is only well-behaved if $1/p \in A$, and $\mathbb{Z}_p(t/2) \cong \pi_t K(\bar F)_p$. For $A = F$ this means that $\pi_* K(F)_p \cong \pi_* K(\bar F)_p^{hG_F}$ for $* \ge 1$, where G_F is the absolute Galois group. The general case requires the more elaborate framework of étale homotopy types. Passing to mod p homotopy, a lift $\beta \in \pi_{2p-2}(S/p)$ of $u^{p-1} \in \pi_{2p-2}(ku; \mathbb{Z}/p)$ would act bijectively on $\pi_*(K(A); \mathbb{Z}/p)$, for $* \ge 1$. Think of $\beta = v_1$ as a lift of $v_1 \in \pi_*(BP; \mathbb{Z}/p)$.

Thomason [Tho85, Thm. 4.1] proved Quillen's conjecture, up to the localization given by inverting β. In particular, $\pi_*(K(F); \mathbb{Z}/p)[1/\beta] \cong \pi_*(K(\bar F)^{hG_F}; \mathbb{Z}/p)$ for $* \ge 2$. It remained to show that $\pi_*(K(A); \mathbb{Z}/p) \to \pi_*(K(A); \mathbb{Z}/p)[1/\beta]$ is an isomorphism for $* \ge 2$. Waldhausen [Wal84, p. 193] noted that this amounts to asking that $K(A) \to L_1 K(A)$ is a p-adic equivalence, in sufficiently high degrees. Here

$L_n = L_{E(n)}$ denotes Bousfield localization with respect to the Johnson–Wilson ring spectrum $E(n)$, or equivalently with respect to $BP[1/v_n]$.

Using topological cyclic homology, Hesselholt–Madsen [HM03, Thm. A] confirmed Quillen's conjecture for valuation rings in local number fields, after special cases were treated by Bökstedt–Madsen [BM94], [BM95] and Rognes [Rog99], [Rog99b].

Finally, Voevodsky's proof [Voe03], [Voe11] of the Milnor and Bloch–Kato conjectures confirmed Quillen's conjecture for rings of integers in global number fields.

3. Redshift in the K-theory of ring spectra

We continue with examples of chromatic redshift in the context of algebraic K-theory of structured ring spectra.

Let $L = E(1)$ be the Adams summand of $KU_{(p)}$, and $\ell = BP\langle 1\rangle$ its connective cover. Using topological cyclic homology, Ausoni–Rognes [AR02, Thm. 0.4] computed $V(1)_*K(\ell_p)$, and Ausoni [Aus10, Thm. 1.1] computed $V(1)_*K(ku_p)$, where $p \geq 5$ and $V(1) = S/(p, v_1)$ is the Smith-Toda spectrum of chromatic type 2. Using a localization sequence of Blumberg–Mandell [BM08, p. 157], this also calculates $V(1)_*K(L_p)$ and $V(1)_*K(KU_p)$. In each case, a lift $v_2 \in \pi_{2p^2-2}V(1)$ of $v_2 \in V(1)_*BP$ acts bijectively on the answer $V(1)_*K(B)$, for $* \geq 2p-2$.

The results are compatible with the existence of a spectral sequence

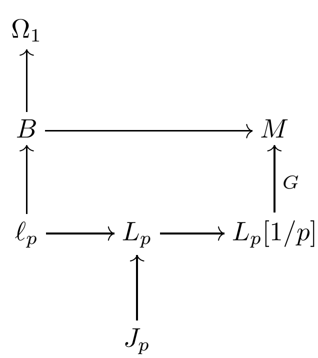
$$E^2_{s,t} = H^{-s}_{\mathrm{mot}}(\operatorname{Spec} B; \mathbb{F}_{p^2}(t/2)) \Longrightarrow V(1)_{s+t}K(B)$$

for suitable "ℓ_p-algebras of S-integers" B, converging in sufficiently high total degrees. Here $H^*_{\mathrm{mot}}(-)$ refers to a hypothetical form of motivic cohomology for commutative structured ring spectra, and $\mathbb{F}_{p^2}(t/2) \cong V(1)_t E_2$ where E_2 is the $K(2)$-local Lubin–Tate ring spectrum with $\pi_* E_2 = \mathbb{W}\mathbb{F}_{p^2}[[u_1]][u]$.

The appearance of the field $\mathbb{F}_{p^2}$ is needed to account for the sign in Ausoni's relation $b^{p-1} = -v_2$ in $V(1)_*K(ku_p)$, since if b is represented by αu^{p+1} and v_2 by u^{p^2-1} then $\alpha^{p-1} = -1$, which cannot be satisfied for $\alpha \in \mathbb{F}_p$.

4. An analogue of the Lichtenbaum–Quillen conjectures

Consider a Galois extension $L_p[1/p] \to M$, like in [Rog08, §4]. By an ℓ_p-algebra of integers in M we mean a connected (only trivial idempotents) commutative ℓ_p-algebra B, with a structure map to M, such that B is semi-finite (retract of a finite cell module), or perhaps dualizable, as an ℓ_p-module:

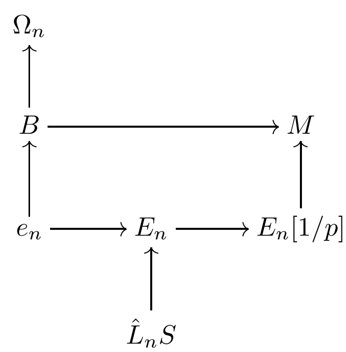


For S-integers we may allow localizations that invert p or v_1. Let Ω_1 be the p-completed homotopy colimit of all such B, i.e., the ℓ_p-algebraic integers.

By analogy with Quillen's conjecture/Voevodsky's theorem we predict that v_2 acts bijectively on $V(1)_*K(B)$, for $* \gg 0$. By analogy with Lichtenbaum's conjecture/Suslin's theorem, we predict that $V(1)_*K(\Omega_1) \cong V(1)_*E_2$, in all sufficiently high degrees, and that $\hat{L}_2K(\Omega_1) \simeq E_2$.

In the case when $B \to \Omega_1$ is an unramified G-Galois extension, the hypothetical motivic cohomology would reduce to group cohomology, and $V(1)_*K(B) \cong V(1)_*K(\Omega_1)^{hG}$ for $* \gg 0$. The general case would require a more elaborate construction than the familiar homotopy fixed points. Even establishing the existence of a ring spectrum map $K(ku) \to E_2$ seems to be an open problem.

Similarly, for $n \geq 1$ let E_n be the $K(n)$-local Lubin–Tate ring spectrum, and let e_n be its connective cover, so that $E_n = e_n[1/u]$. Consider Galois extensions $E_n[1/p] \to M$ and connected commutative e_n-algebras B, with a structure map to M, such that B is semi-finite as an e_n-module:



Let Ω_n be the p-completed homotopy colimit of all such B, i.e., the e_n-algebraic integers.

Let F be a finite p-local spectrum admitting a v_{n+1} self map $v\colon \Sigma^d F \to F$, cf. Hopkins–Smith [HS98, Def. 8]. The finite localization functor L^f_{n+1}, which annihilates all finite $E(n+1)$-acyclic spectra [Mil92, Thm. 4], is a smashing localization such that $F_*L^f_{n+1}X \cong F_*X[1/v]$ for all spectra X.

I stated something like the following at Schloß Ringberg in January 1999 and in Oberwolfach in September 2000:

Conjecture 4.1. *Let $B \to \Omega_n$ and (F, v) be as above.*

(a) Multiplication by v acts bijectively on $F_*K(B)$ for $* \gg 0$, and $K(B) \to L^f_{n+1}K(B)$ is a p-adic equivalence in sufficiently high degrees.

(b) There are isomorphisms $F_*K(\Omega_n) \cong F_*E_{n+1}$ for $* \gg 0$, and $\hat{L}_{n+1}K(\Omega_n) \simeq E_{n+1}$.

The cases $n = -1$ and $n = 0$ correspond to Quillen's results and the proven Lichtenbaum–Quillen conjectures, respectively.

5. The cyclotomic trace map

We can detect chromatic redshift in algebraic K-theory using the cyclotomic trace map to topological cyclic homology, or one of its variants.

The topological Hochschild homology $\mathrm{THH}(B)$ of a commutative S-algebra B is an S^1-equivariant spectrum whose underlying spectrum with S^1-action can be constructed as $B \otimes S^1$, where $\otimes$ refers to the tensored structure of commutative S-algebras over spaces. Let

$$\mathrm{THH}(B)^{hS^1} = F(ES^1_+, \mathrm{THH}(B))^{S^1}$$

be the S^1-homotopy fixed points of $\mathrm{THH}(B)$, and let

$$\mathrm{THH}(B)^{tS^1} = [\widetilde{ES^1} \wedge F(ES^1_+, \mathrm{THH}(B))]^{S^1}$$

be its S^1-Tate construction, also denoted $t_{S^1}\,\mathrm{THH}(B)^{S^1}$ or $\hat{\mathbb{H}}(S^1, \mathrm{THH}(B))$. Here ES^1 is a free contractible S^1-space, and $\widetilde{ES^1}$ is the mapping cone of the collapse map $ES^1_+ \to S^0$. Homotopy fixed point spectra model group cohomology, and the Tate construction models Tate cohomology.

Think of B as a ring spectrum of functions on a brave new scheme X. Then $B \wedge \cdots \wedge B$ is the ring of functions on $X \times \cdots \times X$, so $\mathrm{THH}(B)$ plays the role of the ring of functions on the free loop space $\mathrm{Map}(S^1, X) = \Lambda X$, and $\mathrm{THH}(B)^{hS^1}$ is like the ring of functions on the Borel construction $ES^1_+ \wedge_{S^1} \Lambda X$. The Tate construction is a periodicized version of the Borel construction.

There is a natural trace map

$$K(B) \longrightarrow \mathrm{THH}(B)$$

that factors through the fixed point spectra $\mathrm{THH}(B)^{C_r}$ for all finite subgroups $C_r \subset S^1$. In particular, there is a limiting map

$$K(B) \longrightarrow TF(B;p) = \operatorname*{holim}_n \mathrm{THH}(B)^{C_{p^n}}\,.$$

Continuing with the canonical map from fixed points to homotopy fixed points, the target of

$$\operatorname*{holim}_n \mathrm{THH}(B)^{C_{p^n}} \longrightarrow \operatorname*{holim}_n \mathrm{THH}(B)^{hC_{p^n}}$$

is p-adically equivalent to $\mathrm{THH}(B)^{hS^1}$. The cyclotomic structure of $\mathrm{THH}(B)$ gives a similar map

$$\operatorname*{holim}_n \mathrm{THH}(B)^{C_{p^n}} \longrightarrow \operatorname*{holim}_n \mathrm{THH}(B)^{tC_{p^{n+1}}}$$

whose target is p-adically equivalent to $\mathrm{THH}(B)^{tS^1}$.

The topological Hochschild construction itself does not introduce a redshift, since $\mathrm{THH}(B)$ is a commutative B-algebra. However, in all the computations made so far, any v_{n+1}-periodicity that is seen in the algebraic K-theory $K(B)$ has already been visible in the S^1-Tate construction $\mathrm{THH}(B)^{tS^1}$.

Furthermore, it is possible to see in homological terms where the redshift arises, in terms of these S^1-equivariant constructions.

6. Circle-equivariant redshift

The algebra $H_*(e_n)$ appears to be unwieldy for $n \geq 2$, but there is a map $BP\langle n\rangle \to e_n$ of (not necessarily commutative) S-algebras, covering the usual map $E(n) \to E_n$, and the augmentation $BP\langle n\rangle \to H$ induces an identification

$$H_*(BP\langle n\rangle) \cong P(\bar{\xi}_k \mid k \geq 1) \otimes E(\bar{\tau}_k \mid k \geq n+1)$$

of subalgebras of the dual Steenrod algebra

$$\mathscr{A}_* = P(\bar{\xi}_k \mid k \geq 1) \otimes E(\bar{\tau}_k \mid k \geq 0)\,.$$

Forgetting some structure, we can therefore think of the homology $H_*(B)$ of a commutative e_n-algebra B as a commutative $H_*(BP\langle n\rangle)$-algebra. This makes the Adams spectral sequence

$$E_2^{s,t}(B) = \mathrm{Ext}_{\mathscr{A}_*}^{s,t}(\mathbb{F}_p, H_*(B)) \Longrightarrow \pi_{t-s}(B_p^\wedge)$$

an algebra over the Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_{\mathscr{A}_*}^{s,t}(\mathbb{F}_p, H_*(BP\langle n\rangle)) \Longrightarrow \pi_{t-s}(BP\langle n\rangle_p^\wedge)$$

which collapses at the E_2-term

$$E_2^{*,*} = P(v_0, \dots, v_n)$$

and converges to the homotopy

$$\pi_* BP\langle n\rangle_p^\wedge \cong \mathbb{Z}_p[v_1, \dots, v_n]\,.$$

The Bökstedt spectral sequence

$$E^2_{s,t}(B) = HH_s(H_*(B))_t \Longrightarrow H_{s+t}(\mathrm{THH}(B))$$

is then an algebra spectral sequence over

$$E^2_{*,*} = HH_*(H_*(BP\langle n\rangle)) \cong H_*(BP\langle n\rangle) \otimes E(\sigma\bar\xi_k \mid k \ge 1) \otimes \Gamma(\sigma\bar\tau_k \mid k \ge n+1)$$

converging to $H_*(\mathrm{THH}(BP\langle n\rangle))$. Here σ denotes the suspension operator, coming from the S^1-action on THH, and $\Gamma(x) = \mathbb{F}_p\{\gamma_j x \mid j \ge 0\}$ denotes the divided power algebra on x.

The Dyer–Lashof operations $Q^{p^k}(\bar\tau_k) = \bar\tau_{k+1}$ in $\mathscr{A}_*$ (coming from the commutative S-algebra structure on H), imply multiplicative extensions $(\sigma\bar\tau_k)^p = \sigma\bar\tau_{k+1}$, for $k \ge n+1$, which in turn imply that the Bockstein images $\beta(\sigma\bar\tau_{k+1}) = \sigma\bar\xi_{k+1}$ vanish in the abutment. This argument, see Ausoni [Aus05, Lem. 5.3], implies differentials

$$d^{p-1}(\gamma_j \sigma\bar\tau_k) \doteq \sigma\bar\xi_{k+1} \cdot \gamma_{j-p}\sigma\bar\tau_k$$

for all $j \ge p$, which leave

$$E^p_{*,*} = E^\infty_{*,*} \cong H_*(BP\langle n\rangle) \otimes E(\sigma\bar\xi_1, \dots, \sigma\bar\xi_{n+1}) \otimes P_p(\sigma\bar\tau_k \mid k \ge n+1)$$

converging to

$$H_*(\mathrm{THH}(BP\langle n\rangle)) \cong H_*(BP\langle n\rangle) \otimes E(\sigma\bar\xi_1, \dots, \sigma\bar\xi_{n+1}) \otimes P(\sigma\bar\tau_{n+1})\,.$$

This will still have trivial v_{n+1}-periodic homotopy, but note how building in a circle action gives rise to the class $\sigma\bar\tau_{n+1}$.

The homological Tate spectral sequence

$$E^2_{s,t}(B) = \hat H^{-s}(S^1; H_t(\mathrm{THH}(B))) \Longrightarrow H^c_{s+t}(\mathrm{THH}(B)^{tS^1})$$

converges to a limit that we call the continuous homology of $\mathrm{THH}(B)^{tS^1}$. It is an algebra spectral sequence over

$$E^2_{*,*} = \hat H^{-*}(S^1; H_*(\mathrm{THH}(BP\langle n\rangle))) \cong P(t^{\pm 1}) \otimes H_*(\mathrm{THH}(BP\langle n\rangle))$$

converging to $H^c_*(\mathrm{THH}(BP\langle n\rangle)^{tS^1})$. Here

$$d^2(t^i \cdot x) = t^{i+1} \cdot \sigma x$$

for all x, which leaves

$$\begin{aligned} E^3_{*,*} = P(t^{\pm 1}) &\otimes P(\bar\xi_1^p, \dots, \bar\xi_{n+1}^p, \bar\xi_k \mid k \ge n+2) \\ &\otimes E(\tau'_k \mid k \ge n+2) \otimes E(\bar\xi_1^{p-1}\sigma\bar\xi_1, \dots, \bar\xi_{n+1}^{p-1}\sigma\bar\xi_{n+1}) \end{aligned}$$

where $\tau'_k = \bar\tau_k - \bar\tau_{k-1}(\sigma\bar\tau_{k-1})^{p-1}$ for each $k \ge n+2$. Note that $\bar\tau_{n+1}$ supports a nontrivial d^2-differential to $t \cdot \sigma\bar\tau_{n+1}$, and does not survive to the E^∞-term, while the τ'_k for $k \ge n+2$ are d^2-cycles, due to the known multiplicative extension.

This spectral sequence often collapses at this stage [BR05, Prop. 6.1], and there can be $\mathscr{A}_*$-comodule extensions that combine p^{n+1} shifted copies of

$$P(\bar\xi_1^p, \dots, \bar\xi_{n+1}^p, \bar\xi_k \mid k \ge n+2) \otimes E(\tau'_k \mid k \ge n+2)$$

to a copy of $P(\bar\xi_k \mid k \ge 1) \otimes E(\tau'_k \mid k \ge n+2) \cong H_*(BP\langle n+1\rangle)$. The PhD theses of Sverre Lunøe–Nielsen [LNR12], [LNR11] and Knut Berg (to appear) address these questions. Note the transition from $H_*(BP\langle n\rangle)$ to $H_*(BP\langle n+1\rangle)$, with non-trivial v_{n+1}-periodic homotopy groups. The typical result is that $H^c_*(\mathrm{THH}(B)^{tS^1})$ is an algebra over $H^c_*(\mathrm{THH}(BP\langle n\rangle)^{tS^1})$, which has an associated graded of the form

$$P(t^{\pm p^{n+1}}) \otimes H_*(BP\langle n+1\rangle) \otimes E(\nu_1, \dots, \nu_{n+1})$$

where ν_k is a t-power multiple of $\bar\xi_k^{p-1}\sigma\bar\xi_k$, but that there is room for further $\mathscr{A}_*$-comodule extensions.

This implies that the inverse limit Adams spectral sequence

$$E_2^{s,t}(B) = \mathrm{Ext}^{s,t}_{\mathscr{A}_*}(\mathbb{F}_p, H^c_*(\mathrm{THH}(B)^{tS^1})) \Longrightarrow \pi_{t-s}\,\mathrm{THH}(B)^{tS^1}_p$$

is an algebra over the Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathscr{A}_*}(\mathbb{F}_p, H^c_*(\mathrm{THH}(BP\langle n\rangle)^{tS^1})) \Longrightarrow \pi_{t-s}\,\mathrm{THH}(BP\langle n\rangle)^{tS^1}_p$$

which contains factors like

$$\mathrm{Ext}^{*,*}_{\mathscr{A}_*}(\mathbb{F}_p, H_*(BP\langle n+1\rangle)) \cong P(v_0, \dots, v_n, v_{n+1})\,.$$

Due to the exterior factors $E(\nu_1, \dots, \nu_{n+1})$ there is room for differentials that might truncate the periodic v_{n+1}-action visible above, but empirically this does not happen. A theory that explains the general picture is, however, currently lacking.

7. Beyond elliptic cohomology

Do $K(tmf)$ and $\mathrm{THH}(tmf)^{tS^1}$ detect v_3-periodic families? Work in progress for $p = 2$ with Bruner (2008).

8. Waldhausen's localization tower

The chromatic localization functors (L_n and) $\hat L_n$ and the finite localizations functors L_n^f fit in a diagram of commutative structured ring spectra

$$\begin{array}{ccccccccc}
 & & E_n & & & & KU_p & & \\
 & & \uparrow{\scriptstyle \mathbb{G}_n} & & & & \uparrow{\scriptstyle \mathbb{Z}_p^\times} & & \\
 & & \hat L_n S & & & & J_p & & H\mathbb{Q} \\
 & & \uparrow & & & & \uparrow & & \uparrow{\scriptstyle \simeq} \\
S_{(p)} \longrightarrow & \dots \longrightarrow & L_n^f S & \longrightarrow & L_{n-1}^f S & \longrightarrow \dots \longrightarrow & L_1^f S & \longrightarrow & L_0^f S
\end{array}$$

where $L_n^f S \to L_n S$ is an equivalence for $n \le 1$, but probably not for $n \ge 2$, according to the wisdom concerning Ravenel's telescope conjecture [MRS01]. Applying algebraic K-theory to the lower row one gets a telescopic localization tower

$$K(S_{(p)}) \longrightarrow \dots \longrightarrow K(L_n^f S) \longrightarrow K(L_{n-1}^f S) \longrightarrow \dots \longrightarrow K(L_1 S) \longrightarrow K(\mathbb{Q})$$

similar to that of [Wal84, p. 174], interpolating between the geometrically significant algebraic K-theory of spaces on the left hand side, and the arithmetically significant algebraic K-theory of number fields on the right hand side. Waldhausen worked with L_n, and explicitly assumed that it is a finite localization functor, but we can work with L_n^f instead. This ensures that each finite cell $L_n^f S$-module is L_n^f-equivalent to a finite cell S-module, as can be proved by induction on the number of $L_n^f S$-cells.

Let $\mathscr{C}_0$ be the category of finite p-local spectra, and let $w_n\mathscr{C}_0$ be the subcategory of $E(n)_*$-equivalences, or equivalently of L_n^f-equivalences, for $n \ge 0$. Let $\mathscr{C}_n = \mathscr{C}_0^{w_{n-1}}$ denote the full subcategory of $E(n-1)_*$-acyclic spectra, i.e., the finite spectra of type $\ge n$, for $n \ge 1$. Then $K(\mathscr{C}_0, w_n) \simeq K(L_n^f S)$, and Waldhausen's localization theorem [Wal84, §3] recognizes the homotopy fiber of $K(L_n^f S) \to K(L_{n-1}^f S)$ as $K(\mathscr{C}_n, w_n)$, i.e., the algebraic K-theory of finite spectra of type $\ge n$, with respect to the $E(n)_*$-equivalences. We get a homotopy fiber sequence

$$K(\mathscr{C}_n, w_n) \longrightarrow K(L_n^f S) \longrightarrow K(L_{n-1}^f S)\,.$$

Let $\mathscr{K}_n^{sm}$ be the category of small $K(n)$-local spectra, and let $\mathscr{K}_n'$ be the full subcategory of $K(n)$-localizations of finite spectra of type $\ge n$. Hovey–Strickland [HS99, Thm. 8.5] show that the inclusion $\mathscr{K}_n' \subset \mathscr{K}_n^{sm}$ is an idempotent completion, so the induced map $K(\mathscr{K}_n') \to K(\mathscr{K}_n^{sm})$ induces an isomorphism on π_i for each $i \ge 1$. The localization functors L_n and $\hat{L}_n$ agree on $\mathscr{C}_n$, hence induce an equivalence $K(\mathscr{C}_n, w_n) \simeq K(\mathscr{K}_n')$. Thus we have a map

$$K(\mathscr{C}_n, w_n) \longrightarrow K(\mathscr{K}_n^{sm})\,,$$

which induces a π_i-isomorphism for each $i \ge 1$. We view $\mathscr{K}_n^{sm}$ as a category of suitably small $\hat{L}_n S$-modules.

Let $\mathscr{E}_n^{df}$ be the category of E_n-module spectra that have degreewise finite homotopy groups. Base change along the $K(n)$-local pro-$\mathbb{G}_n$-Galois extension $\hat{L}_n S \to E_n$ takes $\mathscr{K}_n^{sm}$ to $\mathscr{E}_n^{df}$, and conversely [HS99, Cor. 12.16], so it is plausible that a Galois descent comparison map

$$K(\mathscr{K}_n^{sm}) \longrightarrow K(\mathscr{E}_n^{df})^{h\mathbb{G}_n}$$

is close to an equivalence. Finally, $K(\mathscr{E}_n^{df})$ is related to the algebraic K-theory of E_n and its various localizations. For $n = 1$ we have $E_1 = KU_p$, and $K(\mathscr{E}_1^{df})$ is the algebraic K-theory of p-nilpotent finite cell KU_p-modules, which sits [Bar13, Prop. 11.15] in a homotopy fiber sequence

$$K(\mathscr{E}_1^{df}) \longrightarrow K(KU_p) \longrightarrow K(KU_p[1/p])\,.$$

In general, this fiber sequence is replaced by an n-dimensional cubical diagram. Note that the transfer map $K(KU/p) \to K(\mathscr{E}_1^{df})$ associated to $KU_p \to KU/p$ is far from an equivalence, by the calculations of [AR12, Cor. 1.3], so there does not appear to be any easy way to describe the algebraic K-theory of degreewise finite E_n-modules in terms of dévissage, cf. [Wal84, p. 188].

$$
\begin{array}{ccccccc}
 & & K(\mathscr{E}_n^{df}) & \longrightarrow & K(E_n) & & \\
 & & \Big\uparrow{\scriptstyle \mathbb{G}_n} & & & & \\
K(\mathscr{C}_n, w_n) & \longrightarrow & K(\mathscr{K}_n^{sm}) & & & & \\
 & \searrow & & & & & \\
\dots & \longrightarrow & K(L_n^f S) & \longrightarrow & K(L_{n-1}^f S) & \longrightarrow & \dots
\end{array}
$$

Conjecture 4.1 about the structure of the algebraic K-theory of E_n (and various localizations) is therefore also a statement about $K(\mathscr{E}_n^{df})$, and conjecturally about $K(\mathscr{K}_n^{sm})$, which rather precisely measures the difference between $K(L_n^f S)$ and $K(L_{n-1}^f S)$.

9. The spherical case

Calculations of $TC(S;p)$, $K(\mathbb{Z})$ and $TC(\mathbb{Z};p)$ were assembled to a calculation of $K(S)$ at $p = 2$ in [Rog02] and at odd regular primes in [Rog03]. These results describe the cohomology of $K(S)$ as an $\mathscr{A}$-module in all degrees (up to an extension in the odd case), and lead to Adams spectral sequence calculations in a finite range of degrees.

The algebraic K-groups of S are at least as complicated as those of its stable homotopy groups. The complex cobordism spectrum MU has turned out to be a convenient halfway house

$$S \longrightarrow MU \longrightarrow H$$

between homology and homotopy. The Thom equivalence $MU \wedge MU \simeq MU \wedge BU_+$ makes $S \to MU$ a Hopf–Galois extension [Rog08, §12], and the cosimplicial Amitsur resolution

$$[q] \longmapsto MU \wedge MU^{\wedge q}$$

of S is equivalent to the cobar resolution $[q] \longmapsto MU \wedge BU_+^q$ for the $S[BU] = \Sigma^\infty(BU_+)$-comodule algebra MU. Applying algebraic K-theory, an analogue of Quillen's conjecture would predict that $K(S)$ is well approximated by the totalization of the cosimplicial spectrum

$$[q] \longmapsto K(MU \wedge MU^{\wedge q})$$

rewriteable as $[q] \longmapsto K(MU \wedge BU_+^q)$. If, by analogy with the Galois case, there are compatible maps $K(MU \wedge BU_+^q) \to K(MU) \wedge BU_+^q$, then this might in turn be approximated by the totalization of the cobar resolution $[q] \longmapsto K(MU) \wedge BU_+^q$ for an $S[BU]$-comodule algebra structure on $K(MU)$.

Conceivably, this leads to a more conceptual understanding of $\pi_* K(S)$ in terms of $\pi_* K(MU)$ and Hopf–Galois descent, by analogy with the Adams–Novikov spectral sequence description of $\pi_* S$ in terms of $\pi_* MU$ and its $H_*(BU)$-coaction. This has been a motivating factor for the study of $K(MU)$, advertised in [BR05] and [Rog09], and pursued in [LNR11].

10. Higher redshift

For a Lie group G of rank k, consider $(B \otimes G)^{hG}$ or something like $(B \otimes G)^{tG}$. If B is v_n-periodic but not v_{n+1}-periodic, then apparently $(B \otimes G)^{tG}$ is v_{n+k}-periodic. Tested for $B = H$ and $G = T^k$ for small k, as well as for $G = SO(3)$ and $G = S^3$. Work in progress (Rognes, 2008–2011) and in Torleif Veen's PhD thesis (2013).

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