SUPPLEMENTARY NOTES FOR MATH 512 (VERSION 0.17)

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CONTENTS

1. INTRODUCTION

This document is a set of notes made in support of a course I gave at Northwestern in the spring of 2001. The purpose of the course was to say something useful about a cohomology theory called tmf, the "topological modular forms" spectrum.

The approach was to select a characterization of the spectrum tmf, and to compute various things using it. For instance, one might declare tmf to be a ring spectrum with a certain MU-homology. I used a variant of this, with MU replaced by the Thom spectrum

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on $\Omega U(4)$. This is very much like the approach in [HM]; one can regard the present set of notes as a commentary on parts of their paper.

Because of the diverse nature of the audience, I spent much time going over fairly standard material on complex orientable cohomology theories (§§2–7), although from a somewhat more modern viewpoint; see, e.g., [Str99] for a thorough modern treatment. The characterization of tmf used here is based on an observation about the Weierstrass equation which is found in the appendix to [AHS01]; we develop this in §§8–13.

After a characterization of tmf is given in §14, I proceed to set up the calculation of $\pi_*\mathrm{tmf}$ using the Adams-Novikov spectral sequence. Much of the work (§§15–16,18) is the calculation of the E_2-term of this spectral sequence, which are the derived functors of modular forms. I work this out completely except at 2, in which case I hopefully give enough details to recover the method. I refer to [HM] for the calculation of differentials in the spectral sequence (§§17,19).

I also give calculations of $MU_*\mathrm{tmf}$ (in §20) and $H_*(\mathrm{tmf};\mathbb{F}_p)$ (in §21).

I had hoped to cover several more topics: the K-theory of tmf and the homotopy of $L_{K(1)}\mathrm{tmf}$, and the structure of $L_{K(2)}\mathrm{tmf}$, which at primes 2 and 3 is the spectrum EO_2 of Hopkins and Miller. Perhaps I will add this at some point.

This document is a work in progress, although not much work is being done on it at the moment. I appreciate any remarks or corrections.

2. Even periodic ring theories and formal group laws

2.1. **Even periodic ring theories.** Consider a generalized cohomology theory $E^*(-)$ with the following properties:

(1) $E^*(-)$ is a graded commutative ring,
(2) $E^m(\mathrm{pt}) = 0$ when m is odd, and
(3) there exists $u \in E^2(\mathrm{pt})$ and $u^{-1} \in E^{-2}(\mathrm{pt})$ such that $uu^{-1} = 1$.

We'll call such an E an **even periodic ring theory**; it is *even* because the coefficient groups are 0 in odd degrees, and *periodic* by the existence of the element u, which induces isomorphisms $u^n\colon E^q \approx E^{q+2n}$ for all $n \in \mathbb{Z}$. It is important to remember that there can be more than one choice for u.

Proposition 2.2. *Let E be an even periodic ring theory. Then for each $n > 0$ there exist elements $x_n \in \widetilde{E}^0\mathbb{CP}^n$ such that $x_{n+1} \mapsto x_n$ under the inclusion $\mathbb{CP}^n \to \mathbb{CP}^{n+1}$, and such that*

$$E^0[x_n]/(x_n^{n+1}) \xrightarrow{\sim} E^0\mathbb{CP}^n$$

is an isomorphism. Furthermore, $E^q\mathbb{CP}^n = 0$ when q is odd.

Such $\{x_n\}$ give rise to a class $x \in \widetilde{E}^0\mathbb{CP}^\infty$ and an isomorphism

$$E^0[\![x]\!] \xrightarrow{\sim} \lim_n E^0\mathbb{CP}^n \approx E^0\mathbb{CP}^\infty.$$

Again, $E^q\mathbb{CP}^\infty = 0$ for q odd.

A class $x \in E^0\mathbb{CP}^\infty$ as in the proposition is called a **coordinate**. Let $x_i \in E^0(\mathbb{CP}^\infty \times \cdots \times \mathbb{CP}^\infty)$ denote $\pi_i^*(x)$, where π_i is the i-th projection to $\mathbb{CP}^\infty$.

Proposition 2.3. *Let E be an even periodic ring theory. Then the coordinates induce isomorphisms*

$$E^0[x_1,\dots,x_m]/(x_1^{n_1+1},\dots,x_m^{n_m+1}) \xrightarrow{\sim} E^0(\mathbb{CP}^{n_1}\times\cdots\mathbb{CP}^{n_m}),$$

and

$$E^0[\![x_1,\dots,x_m]\!] \xrightarrow{\sim} E^0(\mathbb{CP}^\infty\times\cdots\times\mathbb{CP}^\infty).$$

In particular,

$$E^0(\mathbb{CP}^m\times\mathbb{CP}^n)\approx E^0\mathbb{CP}^n\otimes_{E^0}E^0\mathbb{CP}^m$$

for $m,n<\infty$. The E-cohomology of these spaces vanishes in odd degrees.

The proofs of both of these propositions follow from the Atiyah-Hirzebruch spectral sequence: see [Ada73] for an exposition.

All the isomorphisms described above depend on a choice of coordinate $x\in E^0\mathbb{CP}^\infty$. This choice is not unique. If

$$y=\sum_{i=1}^\infty a_ix^i,\qquad a_i\in E^0,\quad a_1\in(E^0)^\times,$$

then it is straightforward to check that $E^0[y]/(y^{n+1})\xrightarrow{\sim}E^0\mathbb{CP}^n$, and $E^0[\![y]\!]\xrightarrow{\sim}E^0\mathbb{CP}^\infty$. That is, y is another coordinate for E, and all coordinates for E may be represented as a power series $f(x)\in E^0[\![x]\!]$ of a fixed coordinate x with $f(0)=0$ and $f'(0)$ invertible.

2.4. **Formal group laws.** An even periodic ring theory E, together with a coordinate x, give rise to a **formal group law**. A formal group law (1-dimensional, commutative) over a ring A is a power series $F(x_1,x_2)\in A[\![x_1,x_2]\!]$ satisfying

(i) $F(x,0)=x=F(0,x)$,
(ii) $F(x,y)=F(y,x)$,
(iii) $F(x,F(y,z))=F(F(x,y),z)$.

Recall that $\mathbb{CP}^\infty$ is the classifying space for complex line bundles: for each space X and line bundle L over X there is a unique homotopy class of maps $\gamma\colon X\to\mathbb{CP}^\infty$ such that $\gamma^*(L_{\text{univ}})\approx L$, where L_{univ} is the canonical line bundle over $\mathbb{CP}^\infty$. Let

$$\mu\colon\mathbb{CP}^\infty\times\mathbb{CP}^\infty\to\mathbb{CP}^\infty$$

be the map classifying $\pi_1^*(L_{\text{univ}})\otimes\pi_2^*(L_{\text{univ}})$. Now we may define

$$F(x_1,x_2)\stackrel{\text{def}}{=}\mu^*(x)\in E^0(\mathbb{CP}^\infty\times\mathbb{CP}^\infty)\approx E^0[\![x_1,x_2]\!].$$

Proposition 2.5. *This $F(x_1,x_2)$ is a formal group law over E^0.*

If $y=f(x)=\sum_{i=1}^\infty a_ix^i$ is another coordinate for E, we get a different formal group law over E^0 defined by $F'(y_1,y_2)=\mu^*(y)=E^0[\![y_1,y_2]\!]$.

A **homomorphism** $f\colon F\to F'$ of formal group laws over a ring A is a power series $f(x)\in A[\![x]\!]$ with $f(0)=0$ such that

$$f(F(x_1,x_2))=F'(f(x_1),f(x_2)).$$

If $f'(0)$ is invertible in A, then it is easy to see that there exists a unique power series $f^{-1}(x)\in A[\![x]\!]$ with $f(f^{-1}(x))=x=f^{-1}(f(x))$. In such a case, f is an **isomorphism** of formal group laws over A.

In the case of two different coordinates on E as above, we see that the series f relating the coordinates gives rise to a homomorphism $f\colon F \to F'$ of formal group laws. In fact,

$$F'(f(x_1), f(x_2)) = F'(y_1, y_2) = \mu^*(y) = \mu^*(f(x)) = f(\mu^*(x)) = f(F(x_1, x_2)).$$

A simple example of a homomorphism is the n**-series** (or n-th power map) $[n]_F\colon F \to F$, which is defined for all $n \in \mathbb{Z}$. We write $[n]$ for short, when the formal group law F is clear from the context. For positive n,

$$[1](x) = x,$$
$$[n+1](x) = F([n](x), [1](x)).$$

The **inverse** map $[-1]\colon F \to F$ is defined to be the unique power series such that $F(x, [-1](x)) = 0$.

Proposition 2.6. *The inversion map $[-1](x)$ exists and is unique. We have formulas $F([m](x), [n](x)) = [m+n](x)$ and $[n]([m](x)) = [nm](x)$ for all $m, n \in \mathbb{Z}$.*

Proof. Exercise. □

2.7. **Chern classes and examples.** An even periodic ring theory E with chosen coordinate x has a theory of characteristic classes. In particular, there is a "first chern class" for complex line bundles, defined by

$$c_1^{E,x}(L) \in E^0X, \qquad c_1^{E,x}(L) \overset{\text{def}}{=} \gamma^*(x),$$

where $\gamma\colon X \to \mathbb{CP}^\infty$ is the map classifying L. Note that c_1 depends on the choice of coordinate x. If $y = f(x) \in E^0\mathbb{CP}^\infty$ is another coordinate, then

$$c_1^{E,y}(L) = f(c_1^{E,x}(L)).$$

The formal group law is the "addition law" for the chern class of a tensor product. Thus,

$$c_1^{E,x}(L_1 \otimes L_2) = F(c_1^{E,x}(L_1), c_1^{E,x}(L_2)).$$

Here are some examples of even periodic ring theories.

Example 2.8 (Ordinary periodic cohomology). Ordinary cohomology $H^*(X; A)$ is not a periodic ring theory. We can repair this by defining a new theory, $HP^*(-; A)$, which is periodic. Thus, for a finite CW-complex X define

$$HP^*(X; A) = H^*(X; A) \otimes_A A[u, u^{-1}]$$

where $|u| = -2$; that is, $u \in HP^{-2}(\mathrm{pt}; A)$. Thus

$$HP^n(X; A) \approx \bigoplus_{q \in \mathbb{Z}} H^{n+2q}(X; A) \otimes u^q.$$

One computes that $HP^*(\mathbb{CP}^n; A) \approx A[u, u^{-1}, x]/(x^{n+1})$ where

$$-x \in H^2(\mathbb{CP}^n; A) \subset HP^2(\mathbb{CP}^n; A)$$

is the first Chern class of the tautological line bundle. (Note: the tautological line bundle is what geometers call $\mathcal{O}(-1)$, which explains this normalization of the chern class.) If we let $t = xu \in HP^0(\mathbb{CP}^\infty; A)$, we thus obtain

$$HP^0(\mathbb{CP}^n; A) \approx A[t]/(t^{n+1}) \qquad \text{and} \qquad HP^0(\mathbb{CP}^\infty; A) \approx A[\![t]\!].$$

Furthermore, the usual addition formula for chern classes of tensor products of line bundles in $H^*(-;A)$ shows that the associated formal group law is just: $F(t_1,t_2) = t_1 + t_2$. This is called the **additive formal group law**.

Example 2.9 (K-theory). Complex K-theory is certainly a even periodic ring theory, with $K^*(\mathrm{pt}) \approx \mathbb{Z}[u,u^{-1}]$, $|u| = -2$. We have

$$K^0(\mathbb{CP}^n) = \mathbb{Z}[L]/((L-1)^{n+1})$$

where L represents the canonical line bundle. If we set $t = L - 1$ this becomes $K^0\mathbb{CP}^n = \mathbb{Z}[t]/(t^{n+1})$, whence $K^0\mathbb{CP}^\infty \approx \mathbb{Z}[\![t]\!]$.

Over $\mathbb{CP}^\infty \times \mathbb{CP}^\infty$ we have $\pi_1^* L \otimes \pi_2^* L = (1+t_1)(1+t_2) = 1 + t_1 + t_2 + t_1 t_2$, and so we discover that associated to the coordinate t is the formal group law

$$F(t_1,t_2) = \mu^*(t) = t_1 + t_2 + t_1 t_2 \in \mathbb{Z}[\![t]\!].$$

This is called the **multiplicative formal group law**, since it is really multiplication "shifted by -1".

In the examples we have given, the formal group law turned out to be a *polynomial* rather than a power series. These are essentially the only examples for which this happens.

3. Formal groups

Above we described a correspondence

$$\left\{ \begin{array}{c} \text{even periodic ring theories} \\ \text{with chosen coordinate} \end{array} \right\} \Longrightarrow \left\{ \text{ formal group laws } \right\}.$$

We want to develop a "coordinate-free" version of formal group laws, called "formal groups", leading to a correspondence (actually a functor)

$$\{\text{even periodic ring theories}\} \Longrightarrow \{\text{formal groups}\}.$$

3.1. **Some formal geometry.** Suppose that A is a commutative ring. Let $\mathrm{adic}(A)$ denote the category of **adic A-algebras**. An object in this category consists of a commutative ring B together with homomorphisms $i\colon A \to B$ and $r\colon B \to A$ such that $r \circ i = \mathrm{id}_A$, and such that the kernel of r is a nilpotent ideal. A morphism of objects such objects is a map of rings which commutes with the structure, and is the identity on A.

We write $\hat{\mathbb{A}}^1_A(B)$ (or just $\hat{\mathbb{A}}^1(B)$) for the kernel of $r\colon B \to A$. This defines a functor $\hat{\mathbb{A}}^1\colon \mathrm{adic}(A) \to \mathcal{S}\mathrm{et}$. There is a natural isomorphism

$$\hat{\mathbb{A}}^1(B) \approx \operatorname{colim}_n \hom_{\mathrm{adic}(A)}(A[x]/(x^n), B),$$

given by associating each homomorphism $f\colon A[x]/(x^n) \to B$ to the element $f(x) \in \hat{\mathbb{A}}^1(B)$. In other words, the functor $\hat{\mathbb{A}}^1$ is "pro-represented" by $A[\![x]\!]$. Similarly, there is a natural isomorphism

$$\hat{\mathbb{A}}^1(B) \times \hat{\mathbb{A}}^1(B) \approx \operatorname{colim}_{m,n} \hom_{\mathrm{adic}(A)}(A[x,y]/(x^m,y^n), B),$$

whence $\hat{\mathbb{A}}^1 \times \hat{\mathbb{A}}^1$ is pro-represented by $A[\![x,y]\!]$, and so forth.

We write $\hom(\hat{\mathbb{A}}^1, \hat{\mathbb{A}}^1)$ for the set of natural transformations of functors $\hat{\mathbb{A}}^1 \to \hat{\mathbb{A}}^1\colon \mathrm{adic}(A) \to \mathcal{S}\mathrm{et}$.

Lemma 3.2. *There are natural bijections*

$$\hom(\hat{\mathbb{A}}^1, \hat{\mathbb{A}}^1) \approx (x) \subset A[\![x]\!]$$

and

$$\hom(\hat{\mathbb{A}}^1 \times \hat{\mathbb{A}}^1, \hat{\mathbb{A}}^1) \approx (x_1, x_2) \subset A[\![x_1, x_2]\!],$$

defined such that $f(x) \in A[\![x]\!]$ *sends* $b \in \hat{\mathbb{A}}^1(B)$ *to* $f(b) \in \hat{\mathbb{A}}^1(B)$, *resp.* $g(x_1, x_2) \in A[\![x]\!]$ *sends* $(b_1, b_2) \in \hat{\mathbb{A}}^1(B) \times \hat{\mathbb{A}}^1(B)$ *to* $g(b_1, b_2) \in \hat{\mathbb{A}}^1(B)$.

Proof. Exercise. □

3.3. **Formal groups.** A **formal group over** A is a functor

$$G\colon \mathrm{adic}(A) \to \mathcal{A}b$$

with the property that the underlying functor $\mathrm{adic}(A) \to \mathcal{S}et$ is isomorphic to the functor $\hat{\mathbb{A}}^1$. Note that the data of a formal group G does *not* include a specified isomorphism with $\hat{\mathbb{A}}^1$. A choice of such an isomorphism $x\colon G \to \hat{\mathbb{A}}^1$ is called a **coordinate** for G.

Another way to think of G is as a functor $G\colon \mathrm{adic}(A) \to \mathcal{S}et$ which is isomorphic to $\hat{\mathbb{A}}^1$, together with natural transformations $\mu\colon G \times G \to G$, $i\colon G \to G$, and $e\colon * \to G$ satisfying the abelian group axioms; that is, for each $B \in \mathrm{adic}(A)$ there are maps $\mu\colon G(B) \times G(B) \to G(B)$, $i\colon G(B) \to G(B)$, and $e_B \in G(B)$ such that $G(B)$ is an abelian group, and such that for each $f\colon B \to B'$ the induced $G(B) \to G(B')$ is a group homomorphism.

A **homomorphism** of formal groups over A is a natural transformation $\phi\colon G \to G'$ of functors $\mathrm{adic}(A) \to \mathcal{A}b$.

3.4. **Relation between formal group laws and formal groups.** A formal group law $F(x_1, x_2) = x + y + \sum c_{ij} x^i y^j \in A[\![x_1, x_2]\!]$ gives rise to a formal group G, together with a coordinate $x\colon G \to \hat{\mathbb{A}}^1$, as follows. We define $G\colon \mathrm{adic}(A) \to \mathcal{A}b$ to be the functor $\hat{\mathbb{A}}^1\colon \mathrm{adic}(A) \to \mathcal{S}et$ together with an additional abelian group structure. For $B \in \mathrm{adic}(A)$ the abelian group structure $[+]\colon \hat{\mathbb{A}}^1(B) \times \hat{\mathbb{A}}^1(B) \to \hat{\mathbb{A}}^1(B)$ is defined by

$$b_1 [+] b_2 \overset{\text{def}}{=} F(b_1, b_2) = b_1 + b_2 + \sum c_{ij} b_1^i b_2^j, \qquad b_1, b_2 \in \hat{\mathbb{A}}^1(B).$$

The unit of the abelian group structure will be 0. This is well-defined: since b_1 and b_2 lie in the nilpotent augmentation ideal of $r\colon B \to A$, the terms of $F(b_1, b_2)$ become zero in sufficiently large degree. The identification $G \approx \hat{\mathbb{A}}^1$ defines the coordinate x.

A homomorphism $f\colon F \to F'$, $f(x) = \sum c_i x^i \in A[\![x]\!]$, defines a homomorphism of the associated formal groups $f\colon G \to G'$ by

$$G(B) \to G'(B)\colon b \mapsto f(b) = \sum c_i b^i.$$

A formal group G over A, together with a coordinate $x\colon G \to \hat{\mathbb{A}}^1$, gives rise to a formal group law $F(x_1, x_2) \in A[\![x_1, x_2]\!]$ as follows. Let $\mu\colon G \times G \to G$ denote the natural transformation making G a group, and consider the diagram of natural transformations

$$\begin{array}{ccc} G \times G & \xrightarrow{\ \mu\ } & G \\ {\scriptstyle x \times x}\downarrow & & \downarrow{\scriptstyle x} \\ \hat{\mathbb{A}}^1 \times \hat{\mathbb{A}}^1 & \overset{F}{\dashrightarrow} & \hat{\mathbb{A}}^1. \end{array}$$

The vertical arrows are isomorphisms and so the dotted arrow exists and is unique. By the lemma we gave above, $F \in \hom(\hat{\mathbb{A}}^1 \times \hat{\mathbb{A}}^1, \hat{\mathbb{A}}^1) = (x_1, x_2) \subset A[\![x_1, x_2]\!]$ and so we get a formal power series $F(x_1, x_2)$, which is the desired formal group law.

Similarly, given formal groups G and G' with coordinates x and y, a homomorphism $\phi\colon G \to G'$ corresponds via $y \circ \phi = f \circ x$ to a map $f\colon \hat{\mathbb{A}}^1 \to \hat{\mathbb{A}}^1$, and thus to a power series $f(x) \in (x) \subset A[\![x]\!]$.

3.5. **Pullbacks.** We need a way to talk about isogenies of formal groups which are defined over different rings. Given a map $A \to A'$ of rings, there is a functor $\mathrm{adic}(A') \to \mathrm{adic}(A)$ defined by $B' \mapsto B = A \times_{A'} B'$. Note that the augmentation ideal of $B \to A$ is identical to that of $B' \to A'$, i.e. we have a natural isomorphism $\mathbb{A}^1_{A'}(B') \approx \mathbb{A}^1_A(B)$. This functor is right adjoint to a functor $\mathrm{adic}(A) \to \mathrm{adic}(A')$, defined by $B \mapsto A' \otimes_A B$.

Now suppose that G is a formal group over A. Define $\phi^* G$ to be the formal group over A', defined by the formula

$$\phi^* G(B') = G(A \times_{A'} B')$$

for B' any adic A'-algebra. That is, $\phi^* G$ is the composite functor $\mathrm{adic}(A') \to \mathrm{adic}(A) \xrightarrow{G} \mathrm{Ab}$. If we choose a coordinate $x\colon G \to \mathbb{A}^1$, with associated formal group law $F(x_1, x_2)$ in A, then this defines in a natural way a coordinate $x'\colon G' \to \mathbb{A}^1$, with associated formal group law $\phi^* F(x_1, x_2)$ on A', where $\phi^* F$ is the series obtained as the image of $F(x_1, x_2)$ under the natural map $A[\![x_1, x_2]\!] \to A'[\![x_1, x_2]\!]$.

We can define a category of formal groups, the objects of which are pairs (A, G) consisting of a commutative ring A and a formal group G over A, with morphisms $(A, G) \to (A', G')$ being pairs $(\phi\colon A' \to A, \psi\colon G \to \phi^* G')$. We call this simply **the category of formal groups**.

3.6. **Even periodic ring theories give rise to formal groups.** An even periodic ring theory E as above gives rise to a formal group G_E defined for an adic E^0-algebra B by

$$G_E(B) = \mathrm{colim}_n \hom_{\mathrm{adic}(E^0)}(E^0\mathbb{CP}^n, B),$$

with addition law induced by $\mu^*\colon E^0\mathbb{CP}^\infty \to E^0(\mathbb{CP}^\infty \times \mathbb{CP}^\infty)$. A coordinate $x\colon G_E \to \mathbb{A}^1$ corresponds to an coordinate $x \in E^0\mathbb{CP}^\infty$ in the sense used earlier.

Let E and F be even periodic ring theories. A **multiplicative operation** $\phi\colon E \to F$ is a natural transformation $E^0(X) \to F^0(X)$ which for each space X is a ring homomorphism. By substituting $X = \mathbb{CP}^\infty$ or $X = \mathbb{CP}^\infty \times \mathbb{CP}^\infty$, we see that we obtain an isogeny

$$G_F \to \phi^* G_E;$$

by abuse of notation we write $\phi\colon E^0 \to F^0$ for the map induced by the operation ϕ when $X = \mathrm{pt}$. In terms of coordinates $x_E \in E^0\mathbb{CP}^\infty$ and $x_F \in F^0\mathbb{CP}^\infty$, we see that this isogeny corresponds to the map of formal group laws given by the power series $f(T) \in F^0[\![T]\!]$ such that $\phi(x_E) = f(x_F) \in F^0\mathbb{CP}^\infty$.

We have defined a contravariant functor

$$\left\{\begin{matrix}\text{even periodic ring theories and}\\ \text{multiplicative operations}\end{matrix}\right\} \to \left\{\begin{matrix}\text{formal groups}\\ \text{and homomorphisms}\end{matrix}\right\} : E \mapsto (E^0, G_E).$$

3.7. **Examples.**

Example 3.8. Recall the periodic ordinary cohomology $HP^*(X;A)$ defined above. It has $HP^*(\mathrm{pt};A) \approx A[u,u^{-1}]$ with $u \in HP^{-2}$. Given an element $\lambda \in A$ we can define a natural transformation $\psi^\lambda \colon HP^0(-;A) \to HP^0(-;A)$ as follows: $\psi^\lambda(\alpha \otimes u^k) = \lambda^k \alpha \otimes u^k$, where $\alpha \in H^{2k}(X;A)$. This ψ^λ is a multiplicative operation. It is not hard to check that the induced map $(A, \hat{\mathbb{G}}_a) \to (A, \hat{\mathbb{G}}_a)$ of formal groups is that given in terms of the additive coordinate $t \in HP^0(\mathbb{CP}^\infty;A)$ by $f(t) = \lambda t$.

Example 3.9. Now suppose that $A = \mathbb{F}_p$, where p is an odd prime. Then we define an operation $\psi \colon HP^0(-;\mathbb{F}_p) \to HP^0(-;\mathbb{F}_p)$ as follows. For $\alpha \in H^{2k}(X;\mathbb{F}_p)$, let

$$\psi(\alpha \otimes u^k) = \sum_{i=0}^{k} P^i \alpha \otimes u^{k+i(p-1)},$$

where $P^i \colon H^{2k}(X;\mathbb{F}_p) \to H^{2k+i(p-1)}(X;\mathbb{F}_p)$ are the usual mod p Steenrod reduced powers. Then ψ is a multiplicative operation, using the Cartan formula. One computes that, for the generator $t = xu \in HP^0(\mathbb{CP}^\infty;\mathbb{F}_p)$, we have

$$\psi(t) = \psi(x \otimes u) = x \otimes u + x^p \otimes u^p = t + t^p;$$

thus in terms of the additive coordinate the series $f(t) = t + t^p$ defines the induced homomorphism $(\mathbb{F}_p, \hat{\mathbb{G}}_a) \to (\mathbb{F}_p, \hat{\mathbb{G}}_a)$ of formal groups.

Example 3.10. Recall that the formal group associated to complex K-theory is the multiplicative formal group $(\mathbb{Z}, \hat{\mathbb{G}}_m)$. For each $n \in \mathbb{Z}$, the corresponding Adams operation is a multiplicative operation

$$\psi^n \colon K^0(X) \to K^0(X),$$

which is characterized by the property that for line bundles $L \in K^0(X)$, $\psi^n(L) = L^n$. For the multiplicative coordinate $t = L - 1 \in K^0(\mathbb{CP}^\infty)$, the calculation

$$\psi^n(t) = \psi^n(L-1) = L^n - 1 = (1+t)^n - 1 = nT + \dots$$

shows that ψ^n induces the n-th power homomorphism $[n] \colon (\mathbb{Z}, \hat{\mathbb{G}}_m) \to (\mathbb{Z}, \hat{\mathbb{G}}_m)$.

Example 3.11. The Chern character is a multiplicative operation

$$\mathrm{ch} \colon K^0(X) \to HP^0(X;\mathbb{Q}),$$

which is characterized by the property that for line bundles L ,

$$\mathrm{ch}(L) = \exp(u\, c_1(L)).$$

(Note that $c_1(L) \in H^2(X) \subset HP^2(X;\mathbb{Q})$.) Thus it must induce a homomorphism $(\mathbb{Q}, \hat{\mathbb{G}}_a) \to (\mathbb{Z}, \hat{\mathbb{G}}_m)$. We compute

$$\mathrm{ch}(T) = \mathrm{ch}(L-1) = e^{-ux} - 1 = e^{-t} - 1 = -t + \dots,$$

(since $c_1(L) = -x$). This homomorphism is called the **exponential** homomorphism of the multiplicative group.

4. Geometry of formal groups

4.1. **The ring of functions on a formal group.** Given a formal group G over A, we write $\mathcal{O}_G$ for the A-algebra which pro-represents the functor $G\colon \mathrm{adic}(A) \to \mathcal{S}\mathrm{et}$. Similarly, we write $\mathcal{O}_{G\times G}$ for the A-algebra which pro-represents $G \times G\colon \mathrm{adic}(A) \to \mathcal{S}\mathrm{et}$. As was noted above, $\mathcal{O}_G$ is non-canonically isomorphic to $A[\![x]\!]$, and $\mathcal{O}_{G\times G}$ is non-canonically isomorphic to $A[\![x_1, x_2]\!]$. The group law corresponds to a map $\mathcal{O}_G \to \mathcal{O}_{G\times G}$ of A-algebras.

Let $\mathcal{O}_G(-e)$ denote the kernel of the augmentation $\mathcal{O}_G \to A$, and let $\mathcal{O}_G(-ne) = \mathcal{O}_G(-e)^n \subset \mathcal{O}_G$ for $n \geq 0$. Identifying $\mathcal{O}_G \simeq A[\![x]\!]$, we see that $\mathcal{O}_G(-ne) = x^n\mathcal{O}_G$ is the ideal of "functions on G which vanish to order n at e", and we have a chain of ideals

$$\mathcal{O}_G \supset \mathcal{O}_G(-e) \supset \mathcal{O}_G(-2e) \supset \dots.$$

The objects in this chain are natural with respect to the formal group. That is, if we have a map $(A, G) \to (A', G')$ of formal groups, then we get natural maps $\mathcal{O}_{G'}(ne) \to \mathcal{O}_G(ne)$ of $\mathcal{O}_{G'}$-modules for each $n \geq 0$.

4.2. **Invariant differentials.** Recall that if $A \to B$ is a map of commutative rings, the **relative Kähler differentials** is a B-module $\Omega_{B/A}$ defined by

$$\Omega_{B/A} \overset{\mathrm{def}}{=} \bigoplus_{x\in B} B\{dx\} / \left\{ \begin{array}{c} d(xy) = y\,dx + x\,dy, \quad x, y \in B \\ d(a) = 0, \quad a \in A. \end{array} \right\}.$$

The map $d\colon B \to \Omega_{B/A}$ is a derivation. A map $f\colon B \to B'$ of A-algebras induces $\Omega_{B/A} \to \Omega_{B'/A}$ by $dx \mapsto d(f(x))$.

Let Ω_G, the (formal) module of Kähler differentials of $\mathcal{O}_G$ relative to A. By this, we mean the inverse limit

$$\Omega_G \overset{\mathrm{def}}{=} \lim_n \Omega_{(\mathcal{O}_G/\mathcal{O}_G(-ne))/A}.$$

Thus Ω_G is a free $\mathcal{O}_G$-module on one generator dx, where $x \in \mathcal{O}_G$ is any coordinate.

A differential $\eta \in \Omega_G$ is said to be **invariant** if $\mu^*\eta = \pi_1^*\eta + \pi_2^*\eta$, where $\mu^*, \pi_1^*, \pi_2^*\colon \mathcal{O}_G \to \mathcal{O}_{G\times G}$ correspond to the addition and projection maps $\mu, \pi_1, \pi_2\colon G \times G \to G$. In terms of a coordinate x for G, this means that if $\eta = f(x)dx$ is invariant, then

$$f(F(x_1, x_2))d(F(x_1, x_2)) = f(x_1)dx_1 + f(x_2)dx_2.$$

Expanding the left-hand side and evaluating at $x_2 = 0$ gives

$$f(x_1)[dx_1 + F_2(x_1, 0)dx_2] = f(x_1)dx_1 + f(0)dx_2.$$

From the coefficients of the dx_2-terms, we see that $f(x_1)F_2(x_1, 0) = f(0)$.

Proposition 4.3. *Let x be a coordinate on G, with formal group law $F(x_1, x_2)$. A differential $\eta \in \Omega_G$ is invariant if and only if it is of the form*

$$\eta = \frac{a\,dx}{F_2(x, 0)} = \frac{a\,dx}{F_1(0, x)},$$

where $a \in A$.

Proof. We have already shown the only if part. (Note that $F_2(x,0) = F_1(0,x)$ since F is a *commutative* formal group law.) Conversely, suppose $\eta = dx/F_2(x,0)$; we need to show that η is invariant. We must verify the equation $\mu^*\eta = \pi_1^*\eta + \pi_2^*\eta$, which becomes

$$\frac{F_1(x_1,x_2)\,dx_1}{F_2(F(x_1,x_2),0)} + \frac{F_2(x_1,x_2)\,dx_2}{F_2(F(x_1,x_2),0)} = \frac{dx_1}{F_2(x_2,0)} + \frac{dx_2}{F_2(x_2,0)}.$$

To see this, consider the associativity relation $F(F(x_1,x_2),x_3) = F(x_1,F(x_2,x_3))$. Applying the operator $\partial/\partial x_3|_{x_3=0}$ to this gives

$$F_2(F(x_1,x_2),0) = F_2(x_1,x_2)F_2(x_2,0).$$

Similarly, applying $\partial/\partial x_1|_{x_1=0}$, and applying the commutativity relation $F_1(0,x) = F_2(x,0)$ gives

$$F_1(x_2,x_3)F_1(0,x_2) = F_1(0,F(x_2,x_3)).$$

□

We let $\omega_G \subset \Omega_G$ denote the set of invariant differentials of G. It is naturally a module over A, and, as we have proved, it is free on one generator. In fact,

Proposition 4.4. *The function*

$$\omega_G \to \Omega_G \otimes_{\mathcal{O}_G} A$$

induced by the inclusion $\omega_G \subset \Omega_G$ is a natural isomorphism of A-modules.

Proof. The preceding proposition shows that there exists a unique $\eta \in \omega_G$ of the form $\eta = a(1+O(x))dx$ for each $a \in A$, and the function given here is just the one that isolates the constant coefficient. □

A morphism $\psi\colon G \to G'$ of formal groups over A induces a map $\Omega_{G'} \to \Omega_G$ of modules of differentials, and carries $\omega_{G'}$ to ω_G. More generally, a map of formal groups $(\phi,\psi)\colon (A,G) \to (A',G')$ induces natural maps $\Omega_{G'} \to \Omega_G$ of $\mathcal{O}_{G'}$-modules and $\omega_{G'} \to \omega_G$ of A'-modules.

For example, suppose $(A,G) \to (A',G')$ is an isogeny, and let x and x' be coordinates for G and G' with associated formal group laws $F(x_1,x_2)$ and $F'(x_1',x_2')$. Let $\eta = dx/F_2(x,0)$ and $\eta' = dx'/F_2'(x',0)$, and let $f(x') \in A[\![x']\!]$ denote the isogeny. Then $f^*\eta' = f'(0)\eta$.

We say that $(\phi,\psi)\colon (A,G) \to (A',G')$ is **separable** if the induced map $\omega_{G'} \otimes_{A'} A \to \omega_G$ is an isomorphism of A-modules. In terms of coordinates as above, this amounts to saying that $f'(0)$ is invertible in A.

We define a natural map $d\colon \mathcal{O}_G(-e)/\mathcal{O}(-2e) \to \omega_G$ as follows. Start with the derivation $d\colon \mathcal{O}_G(-e) \to \Omega$. If we tensor this map over $\mathcal{O}_G$ with A, we get a map

$$d\colon \mathcal{O}_G(-e)/\mathcal{O}_G(-2e) \approx \mathcal{O}_G(-e) \otimes_{\mathcal{O}_G} A \to \Omega \otimes_{\mathcal{O}_G} A \approx \omega_G.$$

In terms of a coordinate x on G, this map sends

$$d\colon f(x) = ax + O(x^2) \mapsto a\,\eta$$

where η is the unique invariant differential with $\eta = (1+O(x))dx$.

Proposition 4.5. *The map $d\colon \mathcal{O}_G(-e)/\mathcal{O}_G(-2e) \to \omega_G$ defined above is a natural isomorphism of A-modules. Furthermore, d gives rise to a natural identification*

$$\mathcal{O}_G(-ne)/\mathcal{O}_G(-(n+1)e) \xrightarrow{\sim} \omega_G^{\otimes n}$$

for all $n \in \mathbb{Z}$.

Proof. If x is a coordinate on G, then $dx \equiv dx/F_2(x,0) \pmod{x\,dx}$, proving the first statement. The second statement follows from the fact that the natural map $\mathcal{O}_G(-e)^{\otimes n} \to \mathcal{O}_G(-ne)$ induces an identification

$$(\mathcal{O}_G(-e)/\mathcal{O}_G(-2e))^{\otimes n} \approx \mathcal{O}_G(-ne)/\mathcal{O}_G(-(n+1)e)$$

for all $n \in \mathbb{Z}$. □

4.6. **Cohomology of projective spaces and spheres.** Recall that an even periodic ring theory E gives rise to a formal group (E^0, G_E). Essentially by definition, the ring of functions on G_E is $\mathcal{O}_{G_E} = E^0\mathbb{CP}^\infty$. The inclusion $\mathbb{CP}^{n-1} \to \mathbb{CP}^\infty$ induces a cofiber sequence

$$\mathbb{CP}^{n-1} \to \mathbb{CP}^\infty \to \mathbb{CP}^\infty_n$$

where $\mathbb{CP}^\infty_n \overset{\text{def}}{=} \mathbb{CP}^\infty/\mathbb{CP}^{n-1}$, and hence short exact sequence

$$0 \to \widetilde{E}^0\mathbb{CP}^\infty_n \to E^0\mathbb{CP}^\infty \to E^0\mathbb{CP}^{n-1} \to 0,$$

(since these spaces have no odd degree cohomology). Recalling that in terms of a coordinate x we have $\widetilde{E}^0\mathbb{CP}^{n-1} \approx E^0[x]/x^n$, we see that there is a natural identification $E^0\mathbb{CP}^\infty_n = \mathcal{O}_{G_E}(-ne)$.

Similarly, the cofiber sequences

$$\mathbb{CP}^m_n \to \mathbb{CP}^\infty_n \to \mathbb{CP}^\infty_{m+1}$$

(where $\mathbb{CP}^m_n = \mathbb{CP}^m/\mathbb{CP}^{n-1}$ is the stunted projective space) and

$$S^{2n} \to \mathbb{CP}^\infty_n \to \mathbb{CP}^\infty_{n+1}$$

give rise to a short exact sequence and cohomology, and we conclude

Proposition 4.7. *There are isomorphisms*

$$\widetilde{E}^0(\mathbb{CP}^\infty_n) \approx \mathcal{O}_G(-ne), \qquad n \geq 0,$$
$$\widetilde{E}^0(\mathbb{CP}^m_n) \approx \mathcal{O}_G(-ne)/\mathcal{O}_G(-(m+1)e), \qquad m \geq n \geq 0,$$

and

$$\widetilde{E}^0(S^{2n}) \approx \mathcal{O}_G(-ne)/\mathcal{O}_G(-(n+1)e) \approx \omega_G^{\otimes n}, \qquad n \geq 0.$$

Furthermore, these isomorphisms are all natural *with respect to multiplicative operations between cohomology theories.*

By the periodicity of E, the isomorphism $\pi_{2n}E = \widetilde{E}^0(S^{2n}) \approx \omega_G^n$ extends to *all* $n \in \mathbb{Z}$.

Recall that we defined a multiplicative operation to be a natural transformation $\phi\colon E^0(-) \to F^0(-)$ of rings.

Proposition 4.8. *A multiplicative operation $\phi\colon E^0(-) \to F^0(-)$ extends unique to natural transformations $E^{-n}(-) \to F^{-n}(-)$ for $n \geq 0$ which are multiplicative and which commute with suspension.*

Proof. If such an extension exists, we see that the $\phi\colon E^{-n}X \to F^{-n}X$ must factor as

$$E^{-n}X \approx \widetilde{E}^{-n}X_+ \xrightarrow{\sim} \widetilde{E}^0\Sigma^n(X_+) \xrightarrow{\phi} \widetilde{F}^0\Sigma^n(X_+) \xrightarrow{\sim} \widetilde{F}^{-n}X_+ \approx F^{-n}X.$$

We can use this isomorphism as the definition of $\phi\colon E^{-n}X \to F^{-n}X$. □

The above proposition is related to the fact that $E^{-2n}(\mathrm{pt}) \approx \omega_G^n$, and that the homomorphism of the formal group law induces maps between the ω_G^n for $n \geq 0$. A homomorphism does *not* in general induces a map for $n < 0$, and this is an obstruction to extending the operation to positive degrees.

Say that a multiplicative operation $\phi\colon E^0(-) \to F^0(-)$ is **stable** if it extends to a multiplicative natural transformation $E^n(-) \to F^n(-)$ for all $n \in \mathbb{Z}$ and which commutes with suspension. It is not hard to see that if such an extension exists, it must be unique (at least for X a finite complex).

Proposition 4.9. *A multiplicative operation $\phi\colon E^0(-) \to F^0(-)$ is stable if and only if the associated homomorphism $(F^0, G_F) \to (E^0, G_E)$ is separable.*

Proof. (See [Ati89].) If $(F^0, G_F) \to (E^0, G_E)$ is separable, then in terms of coordinates x, y, it is given by a series $y = f(x) \in F^0[\![x]\!]$ with $f(x) = ax + \cdots$ where a is invertible. Thus $\omega_{G_E} \otimes_{E^0} F^0 \to \omega_{G_F}$ is an isomorphism, and hence extends to an isomorphism $\omega_{G_E}^n \otimes_{E^0} F^0 \to \omega_{G_F}^n$ for all $n \in \mathbb{Z}$. Using the argument of the previous proposition, show that you get an operation in all degrees.

Conversely, if the multiplicative operation is stable, then there are maps $\omega_{G_E} \to \omega_{G_F}$ and $\omega_{G_E}^{-1} \to \omega_{G_F}^{-1}$, which by multiplicativity must tensor together to a map

$$E^0 \approx \omega_{G_E} \otimes_{E^0} \omega_{G_E}^{-1} \to \omega_{G_F} \otimes_{F^0} \omega_{G_F}^{-1} \approx F^0$$

which must be the standard map induced by the operation. This implies that $\omega_{G_E} \otimes_{E^0} F^0 \to \omega_{G_F}$ is an isomorphism, and thus the homomorphism is separable. □

5. Heights of formal groups

5.1. **A useful lemma.** We consider formal group laws F, F' over a ring A.

Lemma 5.2. *Let $f \in A[\![x]\!]$ be a homomorphism of formal group laws $f\colon F \to F'$ over a ring A. If $f'(0) = 0$, then $f'(x) = 0$.*

Proof. Apply $(\partial/\partial x_1)_{x_1=0}$ to $f(F(x_1, x_2)) = F'(f(x_1), f(x_2))$; this gives

$$f'(x)F_1(0, x) = F_1'(0, f(x))f'(0).$$

Since $F(x_1, x_2) = x_1 + x_2 + \sum_{i,j\geq 1} c_{ij} x_1^i x_2^j$, we have that $F_1(0, x) \in 1 + (x)$, and in particular has a multiplicative inverse. Therefore, if $f'(0) = 0$ then $f'(x)F_1(0, x) = 0$ and hence $f'(x) = 0$. □

5.3. **Formal group laws over $\mathbb{Q}$-algebras.** Here is a characterization of formal group laws up to isomorphism, when $\mathbb{Q} \subset A$.

Proposition 5.4. *Let $\mathbb{Q} \subset A$, and let F be a formal group law over A. There exists a unique series $\log_F(x) = x + \cdots \in A[\![x]\!]$ inducing an isomorphism.*

$$\log_F \colon F \to \hat{\mathbb{G}}_a.$$

Proof. Let $\log_F(x)$ be the anti-derivative of $1/F_1(0,x)$; this exists because $\mathbb{Q} \subset A$. (The explanation for this choice is that $\log_F^*(dx) = d(\log_F(x)) = \log_F'(x)\,dx = \eta_F$, where $\eta_F = dx/F_1(0,x)$ is an invariant differential for F, and dx is an invariant differential for the additive group.)

To show that $\log_F$ gives a homomorphism, we check that

$$\log_F(F(x_1,x_2)) = \log_F(x_1) + \log_F(x_2).$$

To check this, it suffices to show equality after applying $\partial/\partial x_i$, $i = 1,2$. We carry out the proof for $i = 1$, in which case we must show that

$$F_1(x_1,x_2)/F_1(0,F(x_1,x_2)) = 1/F_1(0,x_1),$$

that is,

$$F_1(x_1,x_2)F_1(0,x_1) = F_1(0,F(x_1,x_2)).$$

But this is exactly what you get if you apply $(\partial/\partial y)_{y=0}$ to the associative law

$$F(F(y,x_1),x_2) = F(y,F(x_1,x_2)).$$

To show uniqueness, suppose that $f\colon F \to \hat{\mathbb{G}}_a$ is another homomorphism with $f(x) = x + \cdots$. Take the difference in $\hom(F,\hat{\mathbb{G}}_a)$, namely $g(x) = \log_F(x) - f(x)$. This is a homomorphism $g\colon F \to \hat{\mathbb{G}}_a$ with $g'(0) = 0$, and by the lemma we see that $g'(x) = 0$, and hence $g(x) = 0$ since $\mathbb{Q} \subset A$. □

The moral is that over a $\mathbb{Q}$-algebra, every formal group law is isomorphic to the additive formal group law. Therefore, over $\mathbb{Q}$-algebras every formal group is isomorphic to the additive formal group.

Exercise 5.5. Let F and F' be formal group laws over a ring $A \supset \mathbb{Q}$. Show that

$$\hom(F,F') \to A \colon f(x) \mapsto f'(0)$$

is an isomorphism of abelian groups.

5.6. **Formal group laws over $\mathbb{F}_p$-algebras.** For formal group laws over an $\mathbb{F}_p$-algebra, we do not have quite as much control. The lemma gives us the following.

Proposition 5.7. *Let $\mathbb{F}_p \subset A$, and let $f\colon F \to F'$ be a homomorphism of formal group laws over A. Then either*

$$f(x) = 0$$

or

$$f(x) = g(x^{p^n})$$

for some $n \geq 0$ and $g(x) \in A[\![x]\!]$ with $g(0) = 0$ and $g'(0) \neq 0$.

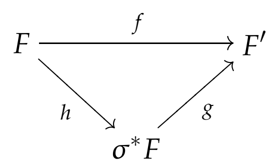
To prove this, we need some ideas. Let $\sigma\colon A \to A$ denote the **Frobenius map** on A, defined by $\sigma(x) = x^p$. Recall that $\sigma^* F$ denotes the formal group law obtain by pulling back along σ; explicitly,

$$(\sigma^* F)(x_1, x_2) = x_1 + x_2 + \sum_{i\geq 1} c_{ij}^p x_1^i x_2^j \quad \text{if} \quad F(x_1, x_2) = x_1 + x_2 + \sum_{i\geq 1} c_{ij} x_1^i x_2^j.$$

Let $h(x) = x^p$; it is immediate to check that $h\colon F \to \sigma^* F$ defines a homomorphism.

Proof. If $f'(0) \neq 0$, then take $g(x) = f(x)$. Similarly if $f(x) = 0$, we are done.

To prove the proposition, we will first prove the following: if $f\colon F \to F'$ is a homomorphism with $f'(0) = 0$, then there exists a factorization



where all maps are homomorphisms. That is, $f(x) = g(x^p)$ for some homomorphism $g\colon \sigma^* F \to F'$, and by induction we conclude that $f(x) = \tilde{g}(x^{p^n})$ for some n.

By the lemma, we know that if $f'(0) = 0$, then $f'(x) = 0$, and since we are over characteristic p, this implies that $f(x) = \sum_{i\geq 1} c_i x^{pi}$. That is, $f(x) = g(x^p)$ for some series $g(x)$, and it suffices to check that $g\colon \sigma^* F \to F'$ is a homomorphism. In fact,

$$g(\sigma^* F(x_1^p, x_2^p)) = g(F(x_1, x_2)^p) = fF(x_1, x_2) = F'(f(x_1), f(x_2)) = F'(g(x_1^p), g(x_2^p)),$$

which on substituting $y_i = x_i^p$ shows that $g(\sigma^* F(y_1, y_2)) = F'(g(y_1), g(y_2))$. □

Suppose that $A = k$ is a field of characteristic p. Then the proposition implies that, since $[p]_F\colon F \to F$ is a homomorphism,

$$[p]_F(x) = v_n x^{p^n} + \cdots$$

for some $v_n \in k^\times$ and some $n = 1, 2, 3, \ldots,$ *or* $[p]_F(x) = 0$. If G is a formal group, then the series $[p](x)$ depends on a choice of coordinate x, but the number n is an invariant of the coordinate. Thus, we make the following definition:

If G is a formal group over a field of characteristic p, we say it has **height** n, $n = 1, 2, 3, \ldots, \infty$ if for any (and hence every) coordinate x we have $[p](x) = v_n x^{p^n} + \cdots$ with $v_n \neq 0$. If G is a formal group over a field of characteristic 0, we say it has **height** 0, since $[p](x) = px + \cdots$ with $p \neq 0$. The height is an isomorphism invariant of the formal group. In fact,

Proposition 5.8. *Over a separably closed field k, formal groups are isomorphic if and only if they have the same height.*

Proof. See, e.g., [Frö68]. □

5.9. **The elements v_n.** For a formal group law F over a general ring A, we have the following.

Proposition 5.10. *Let F be a formal group law over A. We may inductively define elements $v_0 = p \in A$, $v_1 \in A/(p)$, $v_2 \in A/(p, v_1), \ldots, v_n \in A/(p, v_1, \ldots, v_{n-1}), \ldots$, such that*

$$\begin{aligned}
[p]_F(x) &= px + \cdots, \\
[p]_F(x) &\equiv v_1 x^p + \cdots \mod (p), \\
[p]_F(x) &\equiv v_2 x^{p^2} + \cdots \mod (p, v_1), \\
&\vdots \\
[p]_F(x) &\equiv v_n x^{p^n} + \cdots \mod (p, v_1, \ldots, v_{n-1}).
\end{aligned}$$

We want to investigate to what extent these elements are invariants of a formal group G (rather than a formal group law). Thus, given a formal group G over A and a coordinate x, let $v_n^{G,x} \in A/(p, v_1^{G,x}, \ldots, v_{n-1}^{G,x})$ be defined by the above process for the formal group law associated to x.

Similarly, let $\eta_{G,x} \in \omega_G$ denote the invariant 1-form defined by $\eta_{G,x} = dx/F_1(0,x)$, where $F(x_1, x_2)$ is the formal group law associated to x.

Lemma 5.11. *Let $\phi\colon G \to G'$ be an isomorphism of formal groups over A, and let x and y be coordinates on G and G' respectively, so that $\phi^* y = f(x) = cx + \cdots$ for a series f. Then*

$$\begin{aligned}
\phi^*(\eta_{G',y}) &= c\eta_{G,x}, \\
v_n^{G',y} &\equiv c^{1-p^n} v_n^{G,x} \mod (p, v_1^{G,x}, \ldots, v_{n-1}^{G,x}).
\end{aligned}$$

The last line also implies an isomorphism of ideals $(p, v_1^{G,x}, \ldots, v_{n-1}^{G,x}) = (p, v_1^{G',y}, \ldots, v_{n-1}^{G',y})$.

Proof. Let F and F' denote the formal group laws associated to G, x and G', y. For the first part, we have $\phi^*\eta_{G',y} = d(f(x))/F_1'(0, f(x)) = f'(0)dx/F_1'(0,x) = c\eta_{G,x}$, as we have shown before. For the second part, we have by definition $[p]_F(x) = v_n^{G,x} x^{p^n} + \cdots$ and $[p]_{F'}(y) = v_n^{G',y} y^{p^n} + \cdots$. Since ϕ is a homomorphism, we have $f([p]_F(x)) = [p]_{F'}(f(x))$. A little algebra gives

$$c v_n^{G,x} = v_n^{G',y} c^{p^n}$$

and thus the result. □

It is convenient to let

$$A_* \stackrel{\text{def}}{=} \bigoplus_{n \in \mathbb{Z}} \omega_G^{\otimes n},$$

a graded ring, commutative (not up to sign). (Therefore $\pi_{2*}E \approx A_*$ when G is the formal group of a cohomology theory E.)

Corollary 5.12. *For each formal group G over A, prime p, and $n \geq 0$, there is an element $V_n \in \omega_G^{p^n - 1}/(p, V_1, \ldots, V_{n-1})$ with the property that for each isomorphism $\phi\colon (A', G') \to (A, G)$, the element V_n pulls back to V_n.*

Proof. Define $V_n \overset{\text{def}}{=} v_n^{G,x}\eta_{G,x}$, where x is a coordinate. The above remarks show that V_n does not depend on the choice of coordinate. □

Thus in any even periodic ring theory E, there is a canonically defined element $V_n \in \pi_{2(p^n-1)}E/(p, V_1, \dots, V_{n-1})$.

6. Homology of $\mathbb{Z} \times BU$ and related spaces

We want to give an invariant description of $E^0(\mathbb{Z} \times BU)$.

6.1. **Some representable functors.** Let $\mathrm{alg}(A)$ denote the category of A-algebras. Given a formal group G, we define a functor

$$\mathrm{alg}(A) \to \mathcal{S}\mathrm{et}\colon B \mapsto \mathcal{O}_G \otimes_A B.$$

It will be convenient to write $G \otimes B$ for the pullback of G along the map $A \to B$, and thus $\mathcal{O}_G \otimes_A B = \mathcal{O}_{G\otimes B}$. Of course, if we choose a coordinate x for G, we can give an identification $\mathcal{O}_{G\otimes B} \approx B[\![x]\!]$.

Proposition 6.2. *The functor $B \mapsto \mathcal{O}_{G\otimes B}$ is representable. That is, there exists an A-algebra $\mathcal{O}_{\mathrm{Func}_G}$ and an isomorphism*

$$\hom_{\mathrm{alg}(A)}(\mathcal{O}_{\mathrm{Func}_G}, B) \approx \mathcal{O}_{G\otimes B}$$

natural in B.

Proof. Choose a coordinate x for G. Then we have isomorphisms of sets

$$\mathcal{O}_{G\otimes B} \approx B[\![x]\!] \approx \prod_{n\geq 0} B,$$

where a series $f(x) = \sum b_i x^i$ corresponds to the sequence (b_i). Set $\mathcal{O}_{\mathrm{Func}_G} = A[\beta_n,\, n \geq 0] = A[\beta_0, \beta_1, \dots]$, and define

$$\hom_{\mathrm{alg}(A)}(\mathcal{O}_{\mathrm{Func}_G}, B) \to \mathcal{O}_{G\otimes B} \approx B[\![x]\!]$$

by $\phi \mapsto \sum \phi(\beta_n)x^n$. It is clear that this is a bijection. □

The identification $\mathcal{O}_{\mathrm{Func}_G} \approx A[\beta_n,\, n \geq 0]$, and hence the elements β_n, $n \geq 1$, depend on the choice of coordinate. The element $\beta_0 \in \mathcal{O}_{\mathrm{Func}_G}$ does *not* depend on the choice of coordinate. Let $\mathcal{O}_{\mathrm{Units}_G} = \mathcal{O}_{\mathrm{Func}_G}[\beta_0^{-1}] = A[\beta_0, \beta_0^{-1}, \beta_1, \dots]$.

Proposition 6.3. *The functor $B \mapsto \mathcal{O}_{G\otimes B}^\times$, which is a subfunctor of $\mathcal{O}_{G\otimes B}$, is representable by $\mathcal{O}_{\mathrm{Units}_G}$. That is, there is an isomorphism*

$$\hom_{\mathrm{alg}(A)}(\mathcal{O}_{\mathrm{Units}_G}, B) \approx \mathcal{O}_{G\otimes B}^\times$$

natural in B.

There is a further subfunctor of $B \mapsto \mathcal{O}_{G\otimes B}^\times$, namely $1 + \mathcal{O}_{G\otimes B}(-e)$. There is also a quotient functor of $\mathcal{O}_{G\otimes B}^\times$, namely $\mathcal{O}_{G\otimes B}^\times/B^\times$. The natural maps

$$1 + \mathcal{O}_{G\otimes B}(-e) \to \mathcal{O}_{G\otimes B}^\times \to \mathcal{O}_{G\otimes B}^\times/B^\times$$

induce a bijection $1 + \mathcal{O}_{G\otimes B}(-e) \approx \mathcal{O}_{G\otimes B}^\times/B^\times$, so these are really the same functor.

Let $R_1 = \mathcal{O}_{\mathrm{Units}_G}/(\beta_0 - 1) \approx A[\beta_1, \beta_2, \dots]$, and let $R_2 = A[\beta_1/\beta_0, \beta_2/\beta_0, \dots] \subset \mathcal{O}_{\mathrm{Units}_G}$. It is easy to see that the composite $R_2 \to \mathcal{O}_{\mathrm{Units}_G} \to R_1$ is an isomorphism.

Proposition 6.4. *The functors $B \mapsto 1 + \mathcal{O}_{G\otimes B}(-e)$ and $B \mapsto \mathcal{O}^{\times}_{G\otimes B}/B^{\times}$ are represented by R_1 and R_2 respectively.*

Proof. The key observation to make here is that every element in $\mathcal{O}^{\times}_{G\otimes B}$ can be written uniquely as a product $b \cdot f$ with $b \in B^{\times}$ and $f \in 1 + \mathcal{O}_{G\otimes B}(-e)$. □

We are going to identify all these representing rings as the homology of spaces.

6.5. **Homology of $BU(n)$.** Let E be an even periodic ring theory, with formal group $G = G_E$, and choose a coordinate $x \in \widetilde{E}^0\mathbb{CP}^\infty$.

Proposition 6.6. *The "Kronecker pairing" induces a natural isomorphism*

$$E^0\mathbb{CP}^n \xrightarrow{\sim} \hom_{\mathrm{mod}(E_0)}(E_0\mathbb{CP}^n, E_0).$$

Using the coordinate x, we can write

$$E_0\mathbb{CP}^n \approx E_0\{\beta_0, \beta_1, \dots, \beta_{n-1}\},$$

where these elements are defined by $x^j \mapsto (\beta_i \mapsto \delta_{ij})$.

Taking the limit, we get a natural isomorphism

$$E^0\mathbb{CP}^\infty \xrightarrow{\sim} \hom_{\mathrm{mod}(E_0)}(E_0\mathbb{CP}^\infty, E_0),$$

and $E_0\mathbb{CP}^\infty \approx E_0\{\beta_n,\ n \geq 0\}$.

Proof. This follows from the Atiyah-Hirzebruch spectral sequence, together with the fact that it holds for ordinary cohomology. □

Consider the maximal torus $(S^1)^n \to U(n)$. This induces $(BS^1)^n \to BU(n)$, and hence a map

$$E_0(\mathbb{CP}^\infty)^{\otimes n} \to E_0BU(n).$$

Since all maximal tori are conjugate, this map factors through the symmetric coinvariants, and we get

Proposition 6.7.

$$(E_0\mathbb{CP}^\infty)^{\otimes n}_{\Sigma_n} \xrightarrow{\sim} E_0BU(n).$$

6.8. **Homology of $\mathbb{Z} \times BU$.** Let $V \overset{\text{def}}{=} \amalg_{n\geq 0} BU(n)$. This space is a commutative H-space, with map $\mu \colon V \times V \to V$ induced by the maps $BU(m) \times BU(n) \to BU(m+n)$ corresponding to Whitney sum of bundles. Therefore, E_0V is a commutative ring.

Proposition 6.9. *There is an isomorphism of rings*

$$E_0V \approx \bigoplus_{n\geq 0} (E_0\mathbb{CP}^\infty)^{\otimes n}_{\Sigma_n}.$$

Furthermore, there is a isomorphism

$$\hom_{\mathrm{alg}(E_0)}(E_0V, B) \xrightarrow{\sim} \mathcal{O}_{G_E\otimes B},$$

natural in the E_0-algebra B.

Thus this proposition identifies E_0V as the algebra $\mathcal{O}_{\mathrm{Func}_G}$ representing $B \mapsto \mathcal{O}_{G\otimes B}$ which we described above. It means that, after choosing a coordinate, we can identify

$$E_0V \approx E_0[\beta_n,\, n \geq 0].$$

Proof. The first isomorphism is immediate, except perhaps for the fact that it is a ring homomorphism. But this follows by considering the square

$$\begin{array}{ccc} (BS^1)^p \times (BS^1)^q & \longrightarrow & BU(p) \times BU(q) \\ {\scriptstyle\sim}\downarrow & & \downarrow{\scriptstyle\text{Whitney sum}} \\ (BS^1)^{p+q} & \longrightarrow & BU(p+q), \end{array}$$

which commutes up to homotopy; this implies that the product on E_0V is that induced by $(E_0BS^1)^{\otimes p} \otimes (E_0BS^1)^{\otimes q} \approx (E_0BS^1)^{\otimes p+q}$ after passing to symmetric quotients.

The second isomorphism is defined as follows. Given $\phi\colon E_0V \to B$ a map of E_0-algebras, consider the composition

$$E_0\mathbb{CP}^\infty \to E_0V \to B \in \hom_{\mathrm{mod}(E_0)}(E_0\mathbb{CP}^\infty, B)$$

induced by the inclusion $\mathbb{CP}^\infty = BU(1) \to V$. Now use the identification

$$\hom_{\mathrm{mod}(E_0)}(E_0\mathbb{CP}^\infty, B) \approx \hom_{\mathrm{mod}(E_0)}(E_0\mathbb{CP}^\infty, E_0) \otimes_{E_0} B \approx E^0\mathbb{CP}^\infty \otimes_{E_0} B \approx \mathcal{O}_{G_E\otimes B},$$

(using the fact that $E_0\mathbb{CP}^\infty$ is a *free* E_0-module) to define the map $\hom(E_0V, B) \to \mathcal{O}_{G_E\otimes B}$. That this is an isomorphism is just the fact that E_0V is the symmetric algebra on $E_0\mathbb{CP}^\infty$. $\square$

The space $\mathbb{Z} \times BU$ is the classifying space for complex K-theory. It can be identified as the limit of the sequence $V \xrightarrow{\oplus 1} V \xrightarrow{\oplus 1} \cdots$. It inherits the H-space structure corresponding to Whitney sum.

Proposition 6.10. *There is an isomorphism of rings*

$$E_0(\mathbb{Z} \times BU) \approx E_0V[\beta_0^{-1}].$$

Furthermore, there is a isomorphism

$$\hom_{\mathrm{alg}(E_0)}(E_0(\mathbb{Z} \times BU, B) \xrightarrow{\sim} \mathcal{O}^\times_{G_E\otimes B},$$

natural in the E_0-algebra B.

Thus this proposition identifies $E_0(\mathbb{Z} \times BU)$ as the algebra $\mathcal{O}_{\mathrm{Units}_G}$ described above. It means that, after choosing a coordinate, we can identify

$$E_0(\mathbb{Z} \times BU) \approx E_0[\beta_0, \beta_0^{-1}, \beta_n,\, n \geq 1].$$

There are H-space maps

$$\{0\} \times BU \to \mathbb{Z} \times BU \to BU.$$

These give rise to

Proposition 6.11. *There are isomorphisms*

$$\hom_{\mathrm{alg}(E_0)}(E_0BU, B) \xrightarrow{\sim} 1 + \mathcal{O}_{G_E\otimes B}(-e) \approx \mathcal{O}^\times_{G_E\otimes B}/B^\times.$$

This proposition identifies E_0BU with the algebras R_1 and R_2, so that, using a coordinate,

$$E_0BU \approx E_0[(\beta_n/\beta_0),\ n \geq 0] \approx E_0[\beta_0, \beta_0^{-1}, \beta_n,\ n \geq 1]/(\beta_0 - 1).$$

The first isomorphism is "better" (because it corresponds to the inclusion $BU \to \mathbb{Z} \times BU$ which is an infinite loop map, not just an H-space map), but the second one seems more popular.

6.12. **Homology of $\Omega U(n)$.** Let G be a formal group over A, and consider, for each $n \geq 0$, the functors $\mathrm{alg}(A) \to \mathcal{S}\mathrm{et}$ given by

$$B \mapsto (\mathcal{O}_{G\otimes B}/\mathcal{O}_{G\otimes B}(-ne))^{\times}.$$

It is easy to see that, using a coordinate for G, this functor is represented by the ring $R(n) = A[\beta_0^{\pm 1}, \beta_1, \dots, \beta_{n-1}]$. It turns out that this ring is the cohomology of a subspace of $\mathbb{Z} \times BU$.

The Bott periodicity theorem gives a weak equivalence

$$\mathbb{Z} \times BU \xrightarrow{\sim} \Omega U$$

where U is the infinite unitary group. This equivalence is given by an H-space map, with the property that the composite

$$f \colon \mathbb{CP}^\infty \approx \{1\} \times BU(1) \to \mathbb{Z} \times BU \to \Omega U$$

is the colimit of maps $f_n \colon \mathbb{CP}^{n-1} \to \Omega U(n)$ defined by

$$f_n \colon (L \subset \mathbb{C}^n) \mapsto \left(z \in S^1 \subset \mathbb{C}^\times \mapsto \rho_L(z)\right),$$

where $\rho_L(z) \colon \mathbb{C}^n \to \mathbb{C}^n$ is defined by $\rho_L(z)(v) = zv_L + v_L^\perp$, where v_L and $v_L^\perp$ denote the projections of v to each of the factors of $\mathbb{C}^n \approx L \oplus L^\perp$. Note that $f_1 \colon \mathbb{CP}^0 \approx \mathrm{pt} \to \Omega U(1) \approx \mathbb{Z}$ has image $1 \in \mathbb{Z}$. The significance for us is that there is a commutative square

$$\begin{array}{ccc} \mathbb{CP}^{n-1} & \longrightarrow & \Omega U(n) \\ \downarrow & & \downarrow \\ \mathbb{CP}^\infty & \longrightarrow & \Omega U. \end{array}$$

We may define maps

$$\Omega U(n) \to \Omega U \approx \mathbb{Z} \times BU.$$

This is an H-space map.

Proposition 6.13. *There is an isomorphism of rings*

$$E_0\Omega U(n) \approx \left(\bigoplus_{m \geq 0} (E_0\mathbb{CP}^{n-1})^{\otimes m}_{\Sigma_m}\right)[\beta_0^{-1}].$$

Furthermore, there is a isomorphism

$$\hom_{\mathrm{alg}(E_0)}(E_0\Omega U(n), B) \xrightarrow{\sim} (\mathcal{O}_{G_E\otimes B}/\mathcal{O}_{G_E\otimes B}(-ne))^{\times},$$

natural in the E_0-algebra B.

7. The Thom isomorphism

Let X be a space, and V be a (real or complex) vector bundle over X. (We will later specialize to the case of complex vector bundles; for the time being, we should remember that an n-dimensional complex vector bundle has an underlying $2n$-dimensional real vector bundle.) We write $E(V)$ for the total space of the bundle V.

7.1. **Thom spaces.** If X is a finite CW-complex, then the **Thom space** X^V is the 1-point compactification of $E(V)$. If X is an infinite CW-complex, we let $X^V \overset{\text{def}}{=} \operatorname{colim} X_\alpha^{V_\alpha}$, where $X_\alpha \subset X$ are the finite subcomplexes and V is the restriction of V to X_α. We recall the following facts about the Thom space:

(1) If we choose a metric on V, we get a homeomorphism $X^V \approx D(V)/S(V)$, where $S(V) \subset D(V) \subset E(V)$ are the unit sphere and disk bundles, respectively.
(2) If X is a single point, then $X^V \approx S^n$, where n is the real dimension of V.
(3) We have $X^0 \approx X_+ \overset{\text{def}}{=} X \amalg \{\text{pt}\}$.
(4) If $f\colon Y \to X$ is a map, and f^*V is the pullback bundle over Y, then there is an induced map $Y^{f^*V} \to X^V$, which is coherent with respect to composition; i.e., if $g\colon Z \to Y$ is another map, then $Z^{(fg)^*V} \to X^V$ is equal to the composite $Z^{(fg)^*V} \to Y^{f^*V} \to X^V$.
(5) There is a natural isomorphism $(X \times Y)^{V\times W} \approx X^V \wedge Y^W$, where V and W are bundles over X and Y respectively.
(6) For a ring theory $E^*(-)$, the graded group $\widetilde{E}^*(X^V)$ is in a natural way a module over E^*X. The module structure map $E^*X \otimes_{E^*} \widetilde{E}^*X^V \to \widetilde{E}^*X^V$ is induced by the map
$$X^V \to (X\times X)^{V\times 0} \approx X^V \wedge X^0 \approx X^V \wedge X_+,$$
obtained using (4), (5), and (3).

7.2. **Orientations and Thom classes.** We say that a bundle V is E**-orientable** if $\widetilde{E}^*X^V$ is free of rank 1 as a module over E^*X. An E**-orientation** for V is a choice of isomorphism $\widetilde{E}^*X^V \approx E^*X$, or what is the same thing, a choice of generator $x \in \widetilde{E}^*X^V$.

The classical example is when $E^*(-) = H^*(-;\mathbb{Z})$, in which case a bundle is E-orientable if and only if it is orientable in the geometric sense.

It is useful to have a different characterization of orientability. Given a bundle V over X, a **Thom class** is an element $U \in \widetilde{E}^*X^V$ with the property that for each inclusion $x\colon \text{pt} \to X$ the pullback $x^*(U) \in \widetilde{E}^*(\text{pt}^V) \approx \widetilde{E}^*S^n$ is a generator as a module over $E^*\,\text{pt}$.

Note that Thom classes pull back: if $U \in \widetilde{E}^*X^V$ is a Thom class and $f\colon Y \to X$ is a map, then $f^*(U) \in \widetilde{E}^*Y^{f^*V}$ is a Thom class.

Proposition 7.3. *A bundle V over X is E-orientable if it has a Thom class, in which case the Thom classes correspond to the orientations.*

Proof. There is a spectral sequence
$$E_2^{pq}(X^V) = H^p(X, \underline{E^q(D(V_x), S(V_x))}) \Longrightarrow E^{p+q}(D(V), S(V)).$$

The coefficients are a local system over X. This is naturally a spectral sequence of modules over

$$E_2^{pq}(X) = H^p(X, E^q) \Longrightarrow E^{p+q}(X),$$

(the Atiyah-Hirzebruch spectral sequence for X). To show that $E^*(D(V), S(V))$ is free of rank 1 over E^*X, it suffices to show this for the E_2-terms. In fact, the existence of a Thom class $U \in E^*(D(V), S(V))$ gives rise to a generator in each $E^*(D(V_x), S(V_x))$, and the local system is trivial since the generator comes from a globally defined class. This gives the proposition. □

Remark 7.4. The converse of the above proposition is also true: an orientation is always a Thom class.

Corollary 7.5. *If V is an E-orientable bundle over X, and $f\colon X \to Y$ a map, then the natural map $Y^{f^*V} \to X^V$ induces an isomorphism*

$$\widetilde{E}^*X^V \otimes_{E^*X} E^*Y \xrightarrow{\sim} \widetilde{E}^*Y^{f^*V}.$$

Proof. Use the proposition, and the fact that Thom classes pull back to Thom classes. □

Example 7.6. Let $X = \mathbb{CP}^{n-1}$, and let L^* denote the dual of the tautological line bundle over X. (This L^* is the bundle that algebraic geometers like to call $\mathcal{O}(1)$.) A point in L^* conists of a pair

$$(L \subset \mathbb{C}^n, f\colon L \to \mathbb{C}), \quad L \text{ a line in } \mathbb{C}^n,\ f \text{ a } \mathbb{C}\text{-linear map.}$$

Therefore there is a homeomorphism (in fact, a holomorphic map)

$$L^* \to (\mathbb{CP}^n - \{\infty\}), \quad (L, f) \mapsto \Gamma_F = \{\, (v, f(v)) \mid v \in L \,\} \subset \mathbb{C}^n \times \mathbb{C}.$$

Therefore, $(\mathbb{CP}^{n-1})^{L^*} \approx \mathbb{CP}^n$. Hence

$$\widetilde{E}^0(\mathbb{CP}^{n-1})^{L^*} \approx \widetilde{E}^0\mathbb{CP}^n \subset E^0\mathbb{CP}^n.$$

If $x \in \widetilde{E}^0\mathbb{CP}^n$ is a coordinate, we can take x to be an E-orientation for L^* over $\mathbb{CP}^{n-1}$.

Passing to the limit as $n \to \infty$, we see that $(\mathbb{CP}^\infty)^{L^*} \approx \mathbb{CP}^\infty$, and the E-orientations are precisely the coordinates.

The bundles L and L^* are equivalent as *real* bundles. Since the Thom space really only depends on the real bundle, the above remarks continue to hold when L^* is replaced by L. It will turn out to be more convenient to take $[-1](x)$ to be the orientation for L over $\mathbb{CP}^\infty$; this is because L is the pullback of L^* along the map $[-1]\colon \mathbb{CP}^\infty \to \mathbb{CP}^\infty$.

Proposition 7.7. *Let E be an even periodic ring theory, and x a coordinate for its formal group. Then every complex bundle V is E-orientable, and there exist a unique collection of choices $U_V \in \widetilde{E}^0(X^V)$ of Thom classes which are*

1. *natural, so that $U_{f^*V} = f^*(U_V)$,*
2. *multiplicative, so that $U_{V\times W} = U_V \times U_W$, and*
3. *normalized, so that $U_{L^*} = x$ for the bundle L^* over $\mathbb{CP}^\infty$.*

Proof. See [Ada73]. □

In other words, all even periodic ring theories are "complex orientable".

7.8. **Stable bundles and Thom spectra.** Let $\underline{n}$ denote the trivial complex n-plane bundle. Then we have
$$X^{V\oplus\underline{n}} \approx (X\times \mathrm{pt})^{V\times\underline{n}} \approx X^V \wedge (\mathrm{pt})^{\underline{n}} \approx X^V \wedge S^{2n}.$$
This suggests that we should extend the formalism to virtual vector bundles as follows. For a virtual vector bundle $\xi = V - \underline{n}$ over a finite complex define its **Thom spectrum** by
$$X^{\xi} \overset{\text{def}}{=} \Sigma^{\infty}(X^V \wedge S^{-2n}).$$
Thom spectra for for bundles over infinite complexes are defined by passing to a limit as before. Thom spectra have the "same" properties (1)–(6) shared by Thom spaces.

To each map $X \to \mathbb{Z}\times BU$ is associated a virtual complex vector bundle V, and hence a Thom spectrum X^V. Associated to the identity map of $\mathbb{Z}\times BU$ is a spectrum called MUP. Associated to the inclusion $\{0\}\times BU \to \mathbb{Z}\times BU$ is a spectrum called MU. Similarly, associated to $\amalg_{n\geq 0}\{n\}\times BU(n) \to \mathbb{Z}\times BU$ is a spectrum $M \approx \bigvee_{n\geq 0} MU(n)$. The spectrum MU is what is called the **complex cobordism spectrum**. We have $MUP \approx \bigvee_{n\in\mathbb{Z}} \Sigma^{2n} MU$.

Each of these spectra are ring spectra. For instance, consider the map $\mu\colon (\mathbb{Z}\times BU)\times(\mathbb{Z}\times BU) \to \mathbb{Z}\times BU$ classifying Whitney sum of virtual vector bundles. By the multiplicativity property of Thom spectra, the Thom spectrum associated to μ is $MUP\wedge MUP$, and the naturality property gives a map $MUP\wedge MUP \to MUP$. The unit map $S^0 \to MUP$ corresponds to $\mathrm{pt} \to \mathbb{Z}\times BU$ classifying the trivial 0-dimensional bundle.

7.9. **Functorial description of** E_0MUP**.** Let G be a formal group. Let $\mathrm{Coord}_G \subset \mathcal{O}_G(-e)$ denote the set of coordinates for G. This set has a natural action by the multiplicative group of units $\mathcal{O}_G^{\times}$, and in fact is a torsor for this group; that is, if we choose a coordinate x, we get a bijection
$$\mathcal{O}_G^{\times} \xrightarrow{\sim} \mathrm{Coord}_G\colon f \mapsto x\cdot f.$$
Define a functor $\mathrm{Coord}_G\colon \mathrm{alg}(A) \to \mathcal{S}\mathrm{et}$ by $B \mapsto \mathrm{Coord}_{G\otimes B}$. Recall the functor $\mathrm{Units}_G\colon \mathrm{alg}(A) \to \mathcal{A}\mathrm{b}$ sending $B \mapsto \mathcal{O}_{G\otimes B}^{\times}$, which is represented by a ring $\mathcal{O}_{\mathrm{Units}_G}$.

Proposition 7.10. *The functor* Coord_G *is represented by a ring* $\mathcal{O}_{\mathrm{Coord}_G}$*. On choosing a coordinate* x *for* G*, we may give an isomorphism* $\mathcal{O}_{\mathrm{Coord}_G} \approx A[b_0, b_0^{-1}, b_1, b_2, \dots]$*, such that*
$$\hom_{\mathrm{alg}(A)}(\mathcal{O}_{\mathrm{Coord}_G}, B) \xrightarrow{\sim} \mathrm{Coord}_{G\otimes B}$$
is defined by $\phi \mapsto \sum_{n\geq 1}\phi(b_{i-1})x^i$*.*

Proof. Choose a coordinate x for G. This defines bijections $\mathrm{Coord}_{G\otimes B} \approx \mathcal{O}_{G\otimes B}^{\times}$ for all A-algebras B, and hence an isomorphism of functors $\mathrm{Coord}_G \approx \mathrm{Units}_G$. Thus the representing ring for Coord_G exists and is (non-canonically!) isomorphic to the representing ring for Units_G. □

There is a "canonical" map $\iota\colon \mathbb{CP}^{\infty} \to MUP$, defined as follows:
$$\mathbb{CP}^{\infty} \approx (\mathbb{CP}^{\infty})^{L^*} \approx MU(1) \to MUP.$$

Proposition 7.11. *There is a natural isomorphism*
$$\hom_{\mathrm{alg}(E_0)}(E_0MUP, B) \xrightarrow{\sim} \mathrm{Coord}_{G_E\otimes B},$$
and thus a natural isomorphism $E_0MUP \approx \mathcal{O}_{\mathrm{Coord}_{G_E}}$*.*

Proof. Given $\phi\colon E_0MUP \to B$, consider the composite

$$E_0\mathbb{CP}^\infty \xrightarrow{E_0\iota} E_0MUP \to B \in \hom_{\mathrm{mod}(E_0)}(E_0\mathbb{CP}^\infty, B).$$

Now use the identification

$$\hom_{\mathrm{mod}(E_0)}(E_0\mathbb{CP}^\infty, B) \approx \hom_{\mathrm{mod}(E_0)}(E_0\mathbb{CP}^\infty, E_0) \otimes_{E_0} B \approx E^0\mathbb{CP}^\infty \otimes_{E_0} B \approx \mathcal{O}_{G_E \otimes B}$$

to define the natural map. That this is an isomorphism follows from the fact by the Thom isomorphism and the analogous fact for $E_0(\mathbb{Z} \times BU)$, which we already proved. □

7.12. **Quillen's theorem.** There are isomorphisms

$$E^0MUP \xrightarrow{\sim} \hom_{\mathrm{mod}(E_0)}(E_0MUP, E_0)$$

and

$$E^0(MUP \wedge MUP) \xrightarrow{\sim} \hom_{\mathrm{mod}(E_0)}(E_0MUP \otimes_{E_0} E_0MUP, E_0).$$

This implies that for an even periodic ring spectrum E, we have

$$\hom_{\mathrm{RingSpectra}}(MUP, E) \approx \hom_{\mathrm{alg}(E_0)}(E_0MUP, E_0) \approx \mathrm{Coord}_G.$$

That is, coordinates x on the formal group G_E are in natural one-to-one correspondence with maps of ring spectra $MUP \to E$. Similarly, the set $\mathrm{Coord}_{G_E}/(E_0)^\times$ is in natural one-to-one correspondence with maps of ring spectra $MU \to E$. The spectrum MUP admits a canonical coordinate $x\colon \mathbb{CP}^\infty \to MUP$.

Theorem 7.13 (Lazard)**.** *There exists a commutative ring L_0 such that*

$$\hom_{\mathrm{alg}(Z)}(L, A) \approx \{\textit{formal group laws over } A\}.$$

Furthermore, $L_0 \approx \mathbb{Z}[x_n,\ n \geq 1]$.

Let E be an even periodic ring theory. Then for any choice of coordinate x for E (or, what is the same thing, a choice of ring spectrum map $MUP \to E$), there is a map $L_0 \to E_0$ corresponding to the formal group law associated to x.

Theorem 7.14 (Milnor; Quillen)**.**

(1) *The spectrum MUP is an even periodic ring theory, and $\pi_0 MUP \approx \mathbb{Z}[x_n,\ n \geq 1]$.*
(2) *The map $L_0 \to \pi_0 MUP$ associated to the canonical coordinate is an isomorphism.*

7.15. **A word from our sponsor.** Some remarks on what all this says about homotopy theory.

For each ring spectrum E and each spectrum X, there is a Hurewicz map

$$\pi_n X \to E_n X,$$

where π_* denotes stable homotopy groups. This map is induced by the unit map $S^0 \to E$ of the ring spectrum E.

If E and F are even periodic ring spectra, then a stable multiplicative operation ϕ arises from a map $\phi\colon E \to F$ of ring spectra. Therefore, all diagrams

$$\begin{array}{ccc} \pi_n X & \longrightarrow & E_n X \\ & \searrow & \downarrow{\scriptstyle \phi} \\ & & F_n X \end{array}$$

commute.

Example 7.16. If $X = S^0$, then we have maps $\pi_{2n}S^0 \to E_{2n}S^0 \approx \omega^n_{G_E}$. Thus, an even dimensional stable homotopy class gives rise to an element in $\omega^n_{G_E}$ for each E, and this class is invariant under *all* transformations $E \to F$.

Example 7.17. Let $X = M(p) \approx S^0 \cup_p S^1$, the mod-$p$ Moore spectrum. Then, for E_0 torsion free, we have $E_*M(p) \approx E_*/(p)$, and thus Hurewicz maps $\pi_{2n}M(p) \to E_{2n}/(p)$. Recall that earlier we constructed for each formal group G an element $V_1 \in \omega_G^{p-1}/(p)$, and this element was invariant under all isomorphisms between formal groups. In particular, there are such elements $V_1 \in E_{2(p-1)}/(p)$ for all even periodic E. Thus, these elements are candidates for being in the Hurewicz image, and we might guess that there is an element $V_1 \in \pi_{2(p-1)}M(p)$. This turns out to be the case.

Example 7.18. Consider the spectrum $\mathbb{CP}^2$. It sits in a cofiber sequence

$$\cdots \to S^3 \xrightarrow{\eta} S^2 \to \mathbb{CP}^2 \to S^4 \to \cdots .$$

For each even periodic theory it gives rise to a short exact sequence

$$0 \leftarrow \widetilde{E}^0 S^2 \xleftarrow{j} \widetilde{E}^0 \mathbb{CP}^2 \leftarrow \widetilde{E}^0 S^4 \leftarrow 0$$

of E^0-modules, which is canonically isomorphic to

$$0 \leftarrow \mathcal{O}(-e)/\mathcal{O}(-2e) \xleftarrow{j} \mathcal{O}(-e)/\mathcal{O}(-3e) \leftarrow \mathcal{O}(-2e)/\mathcal{O}(-3e) \leftarrow 0.$$

One may ask the question: does the map $S^2 \to \mathbb{CP}^2$ admit a stable retraction $\mathbb{CP}^2 \to S^2$? (Equivalently: is η stably essential?) If this were the case, then the short exact sequence above would admit a splitting, and this splitting would be *natural* with respect to all stable multiplicative operations between such cohomology theories. We can disprove the existence of the retraction by showing that there is no such natural splitting.

In fact, we only need to consider the case when E is complex K-theory (with formal group $(\mathbb{Z}, \hat{\mathbb{G}}_m)$), and the operation ψ^{-1} (corresponding to $[-1]\colon \hat{\mathbb{G}}_m \to \hat{\mathbb{G}}_m$). In terms of the multiplicative coordinate $t \in K^0\mathbb{CP}^\infty$, we have a natural identification of the above sequence with

$$0 \leftarrow (t)/(t^2) \leftarrow (t)/(t^3) \leftarrow (t^2)/(t^3) \leftarrow 0,$$

where these are all ideals in $\mathbb{Z}[\![t]\!]$. The operation $\psi^{-1}(t) = [-1](t) = (1+t)^{-1} - 1 = -t + t^2 - \cdots$. A splitting would give a map $s\colon (t)/(t^2) \to (t)/(t^3)$ with $s(t) = t \bmod (t^2)$, which is determined by $s(t \bmod (t^2)) = t + at^2 \bmod (t^3)$ for some $a \in \mathbb{Z}$. We compute

$$s(\psi^{-1}(t) \bmod (t^2)) = s(-t \bmod (t^2)) = -t - at^2 \bmod (t^3)$$

while

$$\begin{aligned}\psi^{-1}(s(t)) = \psi^{-1}(t + at^2 \bmod (t^3)) &= (-t + t^2 + \cdots) + a(-t + \cdots)^2 \bmod (t^3) \\ &= -t + (1+a)t^2 \bmod (t^3).\end{aligned}$$

In other words, we must have $1 + a = -a$, or $2a = -1$. This equation does not have solutions in $\mathbb{Z}$, so there is no such section. We have just proved that η is stably non-trivial. (There are easier proofs, of course.)

Note that it *is* possible to give a map s with $s(t) \equiv 2t \bmod (t^2)$, by setting $s(t) = 2t - t^2 \bmod (t^3)$. This means we do not get an obstruction to the identity $2\eta = 0$, (which is good, since the identity is true).

8. Elliptic spectra

8.1. **Formal completion of schemes.** Let X be a **pointed** scheme over $\mathrm{Spec}(A)$. That is, X comes with maps

$$\mathrm{Spec}(A) \xrightarrow{e} X \to \mathrm{Spec}(A)$$

whose composite is the identity. Such objects form a category $\mathrm{Spec}(A)\backslash\mathrm{Schemes}/\mathrm{Spec}(A)$, where the morphisms are maps of schemes commuting with the maps to and from $\mathrm{Spec}(A)$. We define a functor

$$\hat{X}_e\colon \mathrm{adic}(A) \to \mathcal{S}\mathrm{et}$$

by

$$B \mapsto \hom_{\mathrm{Spec}(A)\backslash\mathrm{Schemes}/\mathrm{Spec}(A)}(\mathrm{Spec}(B), X).$$

Proposition 8.2. *Suppose that* $e\colon \mathrm{Spec}(A) \to X$ *factors through* $\mathrm{Spec}(A) \to \mathcal{U} \subset X$*, where* $U = \mathrm{Spec}(R)$ *is an open affine subset. Then* $\hat{X}_e$ *is described by*

$$\hat{X}_e(B) \approx \hom_{A\backslash\mathrm{alg}(A)}(R, B) \approx \mathrm{colim}_n \hom_{\mathrm{adic}A}(R/I^n, B),$$

where $I = \mathrm{Ker}(R \to A)$*. In other words,* $\hat{X}^e$ *is pro-represented by the system* $\{R/I^n\}$*.*

We will call $\hat{X}_e$ the **formal completion of** X **along** e.

We will be particularly interested in the case when X is an abelian group scheme over A; i.e., a scheme X over $\mathrm{Spec}(A)$ with identity element $e\colon \mathrm{Spec}(A) \to X$ and group law $\mu\colon X \times_{\mathrm{Spec}(A)} X \to X$. Then $\hat{X}_e$ is actually a functor $\mathrm{adic}(A) \to \mathcal{A}\mathrm{b}$.

Example 8.3. Let $C = \mathbb{A}^1$ over A, and let $e\colon \mathrm{Spec}(A) \to C$ be inclusion of the origin. Then C is an abelian group scheme with group law given by $\mu(x, y) = x + y$, which we call $\mathbb{G}_a$, the **additive group**. The formal completion $\hat{C}_e \approx \hat{\mathbb{G}}_a$, the additive formal group.

Example 8.4. Let $C = \mathbb{A}^1 \setminus \{0\}$, and let $e\colon \mathrm{Spec}(A) \to C$ be inclusion of 1. Then C is an abelian group scheme with group law given by $\mu(x, y) = xy$, which we call $\mathbb{G}_m$, the **multiplicative group**. The formal completion $\hat{C}_e \approx \hat{\mathbb{G}}_m$, the multiplicative formal group.

Example 8.5. Let C be a smooth curve of genus 1, with base point e. This is an **elliptic curve**. Such a C is canonically an abelian group scheme.

Note: when discussing the formal completion of a group scheme C at the identity, we will usually write $\hat{C}$ for $\hat{C}_e$.

8.6. **Elliptic spectra.** An **elliptic spectrum** is a tuple (E, C, ϕ) consisting of

1. an even periodic commutative ring spectrum E,
2. an abelian group scheme C over $\mathrm{Spec}(E_0)$, and
3. an isomorphism $\phi\colon G_E \xrightarrow{\sim} \hat{C}$ of formal groups.

Such objects form a category, in which a morphism

$$(\alpha, \beta) \colon (E, C, \phi) \to (E', C', \phi')$$

consists of

(1) a map $\alpha E \to E'$ of ring spectra (which gives rise to a stable multiplicative operation on the cohomology theories represented by these spectra),
(2) a homomorphism $\beta \colon C' \to \alpha^* C$ of group schemes over $\mathrm{Spec}(E'_0)$, such that
(3) the square

$$\begin{array}{ccc} G_{E'} & \longrightarrow & \alpha^* G_E \\ {\scriptstyle \phi'}\Big\downarrow & & \Big\downarrow{\scriptstyle \alpha^*\phi} \\ \hat{C}' & \longrightarrow & \alpha^* \hat{C} \end{array}$$

commutes.

9. The generalized Weierstrass equation

Fix an affine base scheme $S = \mathrm{Spec} A$. A **generalized Weierstrass equation** over A is a homogeneous equation in X, Y, Z of the form

$$Y^2 Z + a_1 XYZ + a_3 Y^Z = X^3 + a_2 X^2 Z + a_4 X Z^2 + a_6 Z^3$$

with $a_1, a_2, a_3, a_4, a_6 \in A$. We write

$$C = C_{\underline{a}} \subset \mathbb{P}^2_A = \{[X : Y : Z]\}$$

for the curve in the plane associated to this equation, where $\underline{a} = (a_1, \ldots, a_6)$.

We usually prefer to write this in terms of the affine coordinates $x = X/Z$, $y = Y/Z$, in which case the equation becomes

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6.$$

Let $L = \mathbb{P}^2 \smallsetminus \mathbb{A}^2_{x,y} = (Z = 0)$. Then it is easy to check that $C_{\underline{a}} \cap L = \{[0 : 1 : 0]\}$; we will write e for this unique point on the line at infinity.

Example 9.1.

(1) $y^2 = x^3$ defines a curve with a cusp singularity at $(0, 0)$.
(2) $y^2 = x^3 + x^2$ defines a curve with a nodal singularity at $(0, 0)$ (assuming $2 \neq 0$).
(3) $y^2 = x^3 - x$ defines a smooth curve (at least over a field in which $6 \neq 0$).

The curve C is smooth (relative to the base S) exactly when a certain polynomial expression $\Delta(a_1, \ldots, a_6)$ in the a_i's, called the **discriminant**, is invertible in A. For example, if we can write our curve in the form

$$y^2 = (x - e_1)(x - e_2)(x - e_3) = x^3 - (e_1 + e_2 + e_3)x^2 + (e_1 e_2 + e_2 e_3 + e_3 e_1)x - e_1 e_2 e_3$$

for some $e_i \in A$, then it is easy to check that the only possible singularities occur on the x-axis, and that these happen only when $e_i = e_j$ for some $i \neq j$. In this case $\Delta = \pm\left[(e_1 - e_2)(e_2 - e_3)(e_3 - e_1)\right]^2$.

9.2. **Change of coordinates.** Let us consider all isomorphisms $\phi\colon \mathbb{P}^2 \to \mathbb{P}^2$ which carry $C_{\underline{a}'}$ into $C_{\underline{a}}$, and which fix the point e. It is not hard to check that such a map must have the form

$$[X : Y : Z] \mapsto [\lambda^{-2}X + rZ : \lambda^{-3}Y + \lambda^{-2}sX + tZ : Z]$$

for some $r, s, t, \lambda \in A$ with $\lambda^{-1} \in A$. We write $\phi_{r,s,t,\lambda}$ for this map; in terms of affine coordinates, $\phi_{r,s,t,\lambda}(x', y') = (x, y)$, with

$$\begin{aligned} x &= \lambda^{-2}x' + r \\ y &= \lambda^{-3}y' + \lambda^{-2}sx' + t. \end{aligned}$$

Suppose ϕ induces a map $C_{\underline{a}'} \to C_{\underline{a}}$; assuming we know $\underline{a}$, we will derive the $\underline{a}'$. If we write $F_{\underline{a}}(x, y) = y^2 + a_1xy + a_3y - x^3 - a_2x^2 - a_4x - a_6$ we see that

$$\lambda^6 F_{\underline{a}}(x, y) = F_{\underline{a}'}(x', y')$$

for some values $a_1', \dots, a_6'$. If we expand this out, we see that

$$\begin{aligned} \lambda^6 F_{\underline{a}}(x, y) = {}& y'^2 + \lambda(a_1 + 2s)x'y' + \lambda^3(a_3 + a_1r + 2t)y' \\ & - \big[x'^3 + \lambda^2(a_2 - a_1s - s^2 + 3r)x'^2 + \lambda^4(a_4 - a_3s + 2a_2r - a_1t - a_1rs - 2st + 3r^2)x' \\ & \qquad + \lambda^6(a_6 + a_4r - a_3t + a_2r^2 - a_1rt + r^3 - t^2)\big]. \end{aligned}$$

Thus,

$$\begin{aligned} a_1' &= \lambda(a_1 + 2s) \\ a_2' &= \lambda^2(a_2 - a_1s - s^2 + 3r) \\ a_3' &= \lambda^3(a_3 + a_1r + 2t) \\ a_4' &= \lambda^4(a_4 - a_3s + 2a_2r - a_1t - a_1rs - 2st + 3r^2) \\ a_6' &= \lambda^6(a_6 + a_4r - a_3t + a_2r^2 - a_1rt + r^3 - t^2). \end{aligned}$$

If we have a pair of morphisms

$$C_{\underline{a}} \xleftarrow{\phi = \phi_{r,s,t,\lambda}} C_{\underline{a}'} \xleftarrow{\phi' = \phi_{r',s',t',\lambda'}} C_{\underline{a}''}$$

then the composite morphism is given by $\phi \circ \phi'(x'', y'') = (x, y)$ with

$$\begin{aligned} x &= \lambda^{-2}(\lambda'^{-2}x'' + r') + r \\ &= (\lambda\lambda')^{-2}x'' + (r + \lambda^{-2}r') \\ y &= \lambda^{-3}(\lambda'^{-3}y'' + \lambda'^{-2}s'x'' + t') + \lambda^{-2}s(\lambda'^{-2}x'' + r') + t \\ &= (\lambda\lambda')^{-3}y'' + (\lambda\lambda')^{-2}(s + \lambda^{-1}s') + (t + \lambda^{-3}t' + \lambda^{-2}sr'). \end{aligned}$$

In other words, $\phi_{r,s,t,\lambda} \circ \phi_{r',s',t',\lambda'} = \phi_{\nabla(r),\nabla(s),\nabla(t),\nabla(\lambda)}$ where

$$\begin{aligned}\nabla(r) &= r + \lambda^{-2} r' \\ \nabla(s) &= s + \lambda^{-1} s' \\ \nabla(t) &= t + \lambda^{-3} t' + \lambda^{-2} s r' \\ \nabla(\lambda) &= \lambda\lambda'.\end{aligned}$$

9.3. **Invariant 1-form.** Attached to $C_{\underline{a}}$ is a (possibly meromorphic) 1-form

$$\eta_{\underline{a}} = \frac{dx}{2y + a_1 x + a_3} = \frac{dy}{3y^2 + 2a_2 x + a_4 - a_1 y};$$

the equality is obtained from $d(F_{\underline{a}}(x,y)) = 0$.

Proposition 9.4. *Under $\phi = \phi_{r,s,t,\lambda} \colon C_{\underline{a}'} \to C_{\underline{a}}$, we have*

$$\phi^* \eta_{\underline{a}} = \lambda \eta_{\underline{a}'}.$$

Proof. Calculate:

$$\begin{aligned}\phi^* \eta_{\underline{a}} &= \frac{d(\lambda^{-2} x' + r)}{2(\lambda^{-3} y' + \lambda^{-2} s x' + t) + a_1(\lambda^{-2} x' + r) + a_3} \\ &= \frac{\lambda\, dx'}{2y' + \lambda(a_1 + 2s)x' + \lambda^3(a_3 + a_1 r + 2t)} \\ &= \frac{\lambda\, dx'}{2y' + a_1' x' + a_3'} = \lambda \eta_{\underline{a}'}.\end{aligned}$$

□

Let $\omega_C \subset \Omega_C$ denote the free A-submodule generated by $\eta_{\underline{a}}$. It is called the set of **invariant 1-forms**. (If C is smooth, then $\omega_C = \Omega_C$.)

9.5. **Canonical form.** Let us try to find a "canonical form" for the Weierstrass equation for $C_{\underline{a}}$ over A. First note that the left-hand side

$$y^2 + a_1 xy + a_3 y = y^2 + (a_1 x + a_3)y = (y + \tfrac{1}{2}(a_1 x + a_3))^2 - \tfrac{1}{4}(a_1 x + a_3)^2$$

is nearly a perfect square. Thus if we make the substitution

$$y = Y - \tfrac{1}{2}(a_1 x + a_3)$$

then the equation can be rewritten

$$Y^2 = x^3 + (a_2 + \tfrac{1}{4} a_1^2)x^2 + (a_4 + \tfrac{1}{2} a_1 a_3)x + (a_6 + \tfrac{1}{4} a_3^2).$$

Of course, we had to assume that $1/2 \in A$ in order to do this.

It is convenient to set

$$\begin{aligned}b_2 &= 4a_2 + a_1^2 \\ b_4 &= 2a_4 + a_1 a_3 \\ b_6 &= 4a_6 + a_3^2\end{aligned}$$

so that we can write

$$Y^2 = x^3 + \tfrac{1}{4}b_2x^2 + \tfrac{1}{2}b_4x + \tfrac{1}{4}b_6$$

where the $b_i \in A$ (and are at least well-defined even if $1/2 \notin A$).

Now set

$$x = X - \tfrac{1}{12}b_2$$

in order to get rid of the x^2-term (at least if $1/6 \in A$):

$$Y^2 = X^3 + (-\tfrac{1}{48}b_2^2 + \tfrac{1}{2}b_4)X + (\tfrac{1}{864}b_2^3 - \tfrac{1}{24}b_2b_4 + \tfrac{1}{4}b_6).$$

Set

$$\begin{aligned} c_4 &= b_2^2 - 24b_4 \\ c_6 &= -b_2^3 + 36b_2b_4 - 216b_6 \end{aligned}$$

so that $c_4, c_6 \in A$. Thus we have a canonical form

$$Y^2 = X^3 - \tfrac{1}{48}c_4X - \tfrac{1}{864}c_6.$$

Proposition 9.6. *If $1/6 \in A$ then there is an isomorphism*

$$\phi\colon C_{(0,0,0,-c_4/48,-c_6/864)} \to C_{(a_1,\ldots,a_6)}$$

with $\phi(X,Y) = (x,y)$ where

$$\begin{aligned} x &= X - \tfrac{1}{3}a_2 - \tfrac{1}{12}a_1^2 \\ y &= Y - \tfrac{1}{2}a_1X + \tfrac{1}{24}a_1^3 + \tfrac{1}{6}a_1a_2 - \tfrac{1}{2}a_3. \end{aligned}$$

Furthermore, this is the unique *such isomorphism with $\phi^*\eta_{\underline{a}} = dX/2Y$.*

Proof. It is straightforward to check that $\phi^*\eta_{\underline{a}} = dX/2Y$. The only thing left is uniqueness. In fact, it is enough to show that the only map $\phi_{r,s,t,\lambda}$ which takes $Y^2 = X^3 - (c_4/48)X - (c_6/864)$ to itself and which preserves $dX/2Y$ is the identity map $\phi_{0,0,0,1}$, which is a straightforward exercize. □

Remark 9.7. It is nicer to write down the inverse map

$$\phi^{-1}\colon C_{(a_1,\ldots,a_6)} \to C_{0,0,0,-c_4/48,-c_6/864}$$

which is given by $\phi^{-1}(x,y) = (X,Y)$ with

$$\begin{aligned} X &= x + \tfrac{1}{3}a_2 + \tfrac{1}{12}a_1^2 \\ Y &= y + \tfrac{1}{2}a_1x + \tfrac{1}{2}a_3. \end{aligned}$$

9.8. **The discriminant.** The above discussion provides well-defined elements in $c_4, c_6 \in \mathbb{Z}[a_1,\ldots,a_6]$. Let $\Delta = (c_4^3 - c_6^2)/(12)^3$. A straightforward computation shows that $\Delta \in \mathbb{Z}[a_1,\ldots,a_6]$.

Proposition 9.9. *A Weierstrass curve $C_{\underline{a}}$ over A is smooth if and only if Δ is invertible.*

10. Modular forms

10.1. **Generalized elliptic curves.** Let us define a **generalized elliptic curve** to be a morphism of schemes $C \to S$, equipped with a section $e\colon S \to C$, such that there exists a cover $U_i \to S$ by open subsets such that each $C|_{U_i} \to U_i$ is isomorphic to a Weierstrass curve (with base-point e).

Similarly, a morphism of generalized elliptic curves is a pullback square

$$\begin{array}{ccc} C' & \xrightarrow{\phi} & C \\ \downarrow & & \downarrow \\ S' & \xrightarrow{\phi} & S \end{array}$$

such that locally in S and S' the map looks like one of those between Weierstrass curves describe above.

If $C \to S$ is a generalized elliptic curve, then there is a sheaf ω_C of $\mathcal{O}_S$-modules over S, defined locally by $\omega_{C|_{U_i}} \stackrel{\text{def}}{=} \mathcal{O}_{U_i} \cdot \eta_{\underline{a}}$ when $U_i \subset S$ is a sufficiently small open affine over which $C|_{U_i} \approx C_{\underline{a}}$. This sheaf is invertible, i.e., a line bundle over S, and a morphism $\phi\colon C'/S' \to C/S$ induces an isomorphism $\omega_{C'} \xrightarrow{\sim} \phi^*\omega_C$ of sheaves over S.

10.2. **Modular forms.** A **modular form of weight** n is a function f which associates to each generalized elliptic curve C/S an element $f(C/S) \in H^0(S; \omega_C^n)$, and such that for each pullback square

$$\begin{array}{ccc} C' & \xrightarrow{\phi} & C \\ \downarrow & & \downarrow \\ S' & \xrightarrow[\phi]{} & S \end{array}$$

we have $f(C'/S') = \phi^*(f(C/S)) \in H^0(S'; \omega_{C'}^n)$. The set of modular forms of weight n is denoted $\mathcal{M}_n$; these fit together to form a graded, commutative ring $\mathcal{M}_*$.

In particular, modular forms are isomorphism invariants: if C/S and C'/S are two curves which are isomorphic over S, then $f(C/S) = f(C'/S)$.

Let $A = \mathbb{Z}[a_1, a_2, a_3, a_4, a_6]$ and let $A_* = A[\eta, \eta^{-1}]$. The latter is a graded commutative ring, with $\mathrm{wt}(A) = 0$ and $\mathrm{wt}(\eta) = 1$. Let $\Gamma = A[r, s, t, \lambda, \lambda^{-1}]$ and let $\Gamma_* = \Gamma[\eta, \eta^{-1}]$, again with $\mathrm{wt}(\Gamma) = 0$ and $\mathrm{wt}(\eta) = 1$. Define maps

$$A_* \underset{d^1}{\overset{d^0}{\rightrightarrows}} \Gamma_*$$

by letting d^0 be the "obvious" inclusion, and letting $d^1(a_i) = a_i'$ and $d^1(\eta) = \lambda^{-1}\eta$, where the a_i' are given by the formulas from before.

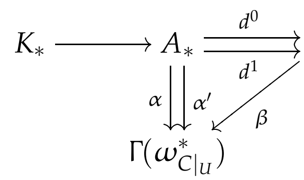
Proposition 10.3. *We have that*

$$\begin{aligned} \mathcal{M}_* &= \mathrm{Ker}[A_* \xrightarrow{d^1 - d^0} \Gamma_*] \\ &= \mathbb{Z}[\bar{c}_4, \bar{c}_6, \overline{\Delta}]/(\bar{c}_4^3 - \bar{c}_6^2 - (12)^3\overline{\Delta}), \end{aligned}$$

where $\overline{c}_i = c_i\eta^i$ *and* $\overline{\Delta} = \Delta\eta^{12}$ *in* A_*.

Proof. For the first equality, note that the "universal" Weierstrass curve $C = C_{(a_1,\dots,a_6)}$ over A determines a map $\iota\colon \mathcal{M}_* \to A_*$, and that since $(d^0)^*C$ and $(d^1)^*C$ are isomorphic as curves over $\operatorname{Spec}\Gamma$ we see that $d^0\iota = d^1\iota$. Thus $\mathcal{M}_*$ maps to the kernel.

We must produce a map $K_* = \operatorname{Ker}(d^0 - d^1) \to \mathcal{M}_*$, that is, we must show that elements in the kernel give rise to modular forms. Consider a curve C/S. For the elements of a cover $\{U\}$ of S we can identify $C|_U \approx C_{\underline{\alpha}}$ for some elements $\alpha_i \in \Gamma(\mathcal{O}_U)$ (hence determining a map $\alpha\colon A \to \Gamma(\mathcal{O}_U)$). If $C|_U \approx C_{\underline{\alpha}'}$ is another such identification (corresponding to $\alpha'\colon A \to \Gamma(\mathcal{O}_U)$), we know that the α_i and the α_i' are related by means of some values $r, s, t, \lambda \in \Gamma(\mathcal{O}_U)$, and hence by a map $\beta\colon \Gamma \to \Gamma(\mathcal{O}_U)$. Thus we have a diagram



with $\beta d^0 = \alpha$ and $\beta d^1 = \alpha'$, and hence a canonical map $K_* \to \Gamma(\omega^*_{C|_U})$, which in particular only depends on $C|_U$ up to isomorphism. These maps thus fit together to give $K_* \to H^0(S, \omega_C^*)$, and it is easy to check that that they are natural, i.e., that $K_* \to \mathcal{M}_*$.

The proof of the second line is a calculation, which will will do later. □

11. ELLIPTIC CURVES AND THE GROUP STRUCTURE

11.1. **Riemann-Roch for curves of genus 1.** Let C be a smooth projective curve over an algebraically closed field k. A **divisor** is a finite formal sum $D = \sum n_P P$ where $P \in C(k)$ and $n_P \in \mathbb{Z}$, and the degree $\deg D$ of a divisor is $\sum n_P$. A meromorphic function f on C, or more generally a meromorphic section of a line bundle over C, has a divisor (f) attached to it, defined by $(f) = \sum n_P P$ where f vanishes to order n_P (or has a pole of order $-n_P$) at P.

If $D = \sum n_P P$ is a divisor on C, then

$$\mathcal{O}(D) \overset{\text{def}}{=} \left\{ \begin{array}{c} \text{meromorphic functions } f \text{ which} \\ \text{have a pole of order at worst } n_p \text{ at } P \end{array} \right\}.$$

We write $\ell(D) = \dim_k \Gamma(C, \mathcal{O}(D))$. Then the Riemann-Roch theorem states that if K is the divisor associated to *any* global meromorphic 1-form on C, then

$$\ell(D) - \ell(K - D) = \deg D - g + 1,$$

where g is the genus of C.

In particular:

(1) We know that $\ell(0) = \dim\Gamma(C, \mathcal{O}) = 1$, by the maximum principle. Thus taking $D = 0$ in the Riemann-Roch formula gives $1 - \ell(K) = 1 - g$, that is $\ell(K) = g$.
(2) Similarly, taking $D = K$ gives $g - 1 = \deg(K) - g + 1$, that is $\deg(K) = 2g - 2$.

Therefore, if $g = 1$, there exists a globally holomorphic 1-form, unique up to scalar: if η is a meromorhic 1-form with divisor K, then Riemann-Roch tells us that there exists a non-zero section of $\mathcal{O}(K)$, i.e. a function with the same zeroes and poles as η; therefore η/f is

a holomorphic form. So we might as well choose η to be this holomorphic form, so $K = 0$, and hence
$$\ell(D) - \ell(-D) = \deg D.$$
We apply this to $D = ne$, where e is a chosen basepoint and $n \geq 0$. Then $\ell(-ne) = 0$ for $n > 1$, and we get

D	$\ell(D)$	element of $\Gamma(\mathcal{O}(D))$
0	1	1
e	1	
$2e$	2	x
$3e$	3	y
$4e$	4	x^2
$5e$	5	xy
$6e$	6	x^3, y^2

Therefore, there must be a relation among $1, x, y, x^2, xy, x^3, y^2$. In fact, by rescaling x and y we may choose them so that $x^3 - y^2 \in \Gamma(\mathcal{O}(5e))$. This shows that every smooth curve of genus 1 actually arises as a Weierstrass equation.

11.2. **The group structure.** Suppose $C \subset \mathbb{P}^2$ is a smooth Weierstrass curve, over a field k. Recall that any line $L \subset \mathbb{P}^2$ intersects C at exactly 3 points, counted with multiplicity. One defines a group law $[+]\colon C(k) \times C(k) \to C(k)$ on the points of C by $P[+]Q = R$, where (P, Q, R') and (e, R, R') are colinear sets of points on C. An inverse $[-1](P) = Q$ is defined when (e, P, Q) are colinear. The identity is e.

This group law can be described in terms of algebraic equations. It thus turns out that for any Weierstrass curve C over any ring A, the open subscheme $C^{\text{smooth}} \subset C$ of smooth points admits the structure of a commutative group scheme, with identity given by the section $e\colon \operatorname{Spec}(A) \to C$.

11.3. **Singular Weierstrass curves.** It is easy to check that a Weierstrass curve $C = C_{\underline{a}}$ is smooth at the point $e = [0 : 1 : 0]$. Suppose $P \in C$ is a singular point. By a coordinate change $(x, y) \mapsto (x + r, y + t)$ we may without loss of generality assume that $P = (0, 0)$. Hence up to isomorphism a singular curve must be given by an equation of the form
$$y^2 + a_1 xy - a_2 x^2 = x^3.$$
By examining the quadratic part, we see there are two cases:

(i) $y^2 + a_1 xy - a_2 x^2 = (y - \alpha_1 x)(y - \alpha_2 x)$ for $\alpha_1 \neq \alpha_2$, in which case $(x, y) \mapsto (y - \alpha_1 x)/(y - \alpha_2 x)$ defines an isomorphism $C^{\text{smooth}} \to \mathbb{A}^1 \setminus \{0\} \approx \mathbb{G}_m$.

(ii) $y^2 + a_1 xy - a_2 x^2 = (y - \alpha x)^2$, in which case $(x, y) \mapsto x/(y - \alpha x)$ defines an isomorphism $C^{\text{smooth}} \to \mathbb{A}^1 \approx \mathbb{G}_a$.

Note that in either of these two cases, the discriminant of the polynomial $y^2 + a_1 xy - a_2 x^2$ is $2(\alpha_1 - \alpha_2)^2 = a_1^2 + 4a_2 = b_2$, and that $c_4 = b_2^2$ and $c_6 = -b_2^3$.

11.4. **The formal group associated to Weierstrass curves.** By completing a Weierstrass curve $C/\mathrm{Spec}(A)$ at the section $e\colon S \to C$, we obtain a formal group $\widehat{C}$ over A. If C is given by an explicit Weierstrass equation $y^2 + \cdots = x^3 + \cdots$, then we can choose an explicit coordinate on $\widehat{C}$, by taking $T = -x/y$. The 1-form $\eta = dx/(2y + a_1x + a_3)$ on C gives an invariant 1-form on the formal group.

Proposition 11.5. *Let C/k be a Weierstrass curve over a field, with* $\mathrm{char} k = p > 0$. *Then $\widehat{C}$ has height* $1, 2$, *or* ∞.

Proof. When C is a singular curve, one checks explicitly that $\widehat{C} \approx \widehat{\mathbb{G}}_a$ or $\widehat{\mathbb{G}}_m$, so that height is either 1 or ∞.

If C is smooth, it suffices to show that the p-th power map $[p]\colon C \to C$ has degree p^2. This means that $[p]^{-1}\{e\}$ contains p^2 points, counted with multiplicities, and in particular e can appear in this preimage at most p^2 times. Thus, in terms of a coordinate T at e we have $[p](T) = aT^n + \dots$ for some $n \leq p^2$, $a \neq 0$, and thus $n = p$ or $n = p^2$ by our previous discussion of the p-series of a formal group law.

To see that the p-th power map has degree p^2, see [Sil86, III.6.2]. □

We say that a generalized elliptic curve C/k is **supersingular** if height $\widehat{C} = 2$. It turns out that at each characteristic p there are (up to isomorphism) only a finite number of supersingular curves; in fact, in each isomorphism class of supersingular curves there is one defined over the field $\mathbb{F}_{p^2}$. (See [Sil86, V.4].) To determine whether a curve is supersingular, compute its p-series mod p. For example:

$$[2](T) = a_1T^2 + (a_3 + a_1a_2)T^4 + \cdots \quad \bmod 2,$$
$$[3](T) = b_2T^3 + (b_2b_6 + 2b_4^2 + 2b_4b_2^2)T^9 + \cdots \quad \bmod 3,$$
$$[5](T) = c_4T^5 + (3c_4^3c_6^2 + 3c_4^6 + 4c_6^4)T^{25} + \cdots \quad \bmod 5,$$
$$[7](T) = 6c_6T^7 + c_4c_6(6c_6^4 + c_4^6)T^{35} + (3c_4^3c_6^6 + 6c_6^8 + 5c_6^4c_4^6 + c_6^2c_4^9 + 6c_4^{12})T^{49} + \cdots \quad \bmod 7,$$

and so forth. When $p > 3$ it is always possible to write v_1 as a modular form of weight $p-1$, but not when $p = 2$ or 3. However, note that

$$v_1^4 \equiv a_1^4 \equiv c_4 \quad \bmod 2,$$
$$v_1^3 \equiv b_2^3 \equiv -c_6 \quad \bmod 3.$$

11.6. **The j-invariant.** It is useful to know about the j-invariant, which classifies isomorphism classes of generalized elliptic curves.

Proposition 11.7 ([Sil86, III.1.4])**.** *Let C/k be a Weierstrass curve over an algebraically closed field. Define $j(C/k) \in k$ by $j = c_4^3/\Delta$. We have the following cases:*

1. *If $c_4 = \Delta = 0$, then C/k is isomorphic to the curve given by $y^2 = x^3$, and $C^{\mathrm{smooth}} \approx \mathbb{G}_a$.*
2. *If $c_4 \neq 0$ and $\Delta = 0$, then C/k is isomorphic to the curve given by $y^2 + xy = x^3$, and $C^{\mathrm{smooth}} \approx \mathbb{G}_m$.*
3. *If $\Delta \neq 0$, then C/k is smooth, and two such curves C and C' are isomorphic if and only if $j(C) = j(C')$.*

11.8. **Examples of elliptic spectra.**

Example 11.9. Let A be a ring, and consider $(HPA, C^{\text{smooth}}, \phi)$ where HPA denotes periodic ordinary cohomology with coefficients in A, C is the Weierstrass curve given by $y^2 = x^3$, and $\phi\colon \widehat{\mathbb{G}}_a \to \widehat{C}$.

Example 11.10. Consider $(K, C^{\text{smooth}}, \phi)$ where K denotes complex K-theory, C is the Weierstrass curve given by $y^2 + xy = x^3$, and $\phi\colon \widehat{\mathbb{G}}_m \to \widehat{C}$.

Another class of examples comes from the Landweber exact functor theorem; several are described in [LRS95]. For instance, let $R = \mathbb{Z}[\frac{1}{6}, c_4, c_6, \Delta^{-1}]$; let C be the Weierstrass curve over this ring. Define a functor from spaces to graded rings by

$$E_*(X) \overset{\text{def}}{=} MUP_*(X) \otimes_{MUP_0} R,$$

where $MUP_0 \to R$ is the map classifying a formal group law associated to a coordinate on $\widehat{C}$. This functor is a generalized cohomology theory.

To see this, we use the Landweber exact functor theorem, which states that a functor such as E_* above is a cohomology theory if for each prime p the sequence $p, v_1, v_2, \dots \in E_*(\text{pt})$ is a regular sequence. In our case, $E_*(\text{pt}) \approx R[\eta, \eta^{-1}]$. It is immediate that the sequence is regular when $p = 2$ or 3. For $p > 3$, we first observe that $E_0/(p) \approx \mathbb{F}_p[c_4, c_6, \Delta^{-1}]$ is an integral domain, and so p, v_1 is a regular sequence. The result then follows from

Lemma 11.11. *For $p > 3$,*

$$v_2 \equiv (-1)^{\frac{p-1}{2}} \Delta^{\frac{p^2-1}{12}} \mod (p, v_1).$$

For a proof, see [Lan88, Thm.2].

12. Weierstrass parameters and Thom spectrum of $\Omega U(4)$

12.1. **Weierstrass parameterizations.** Let C/S be a generalized elliptic curve over a base scheme S. By a **Weierstrass parameterization** of C, we mean an embedding $C \to \mathbb{P}^2 \times S$ over S which identifies C with a curve in $\mathbb{P}^2$ given by a Weierstrass equation. (Recall that a generalized elliptic curve was defined to be something which *locally* in S admits a Weierstrass parameterization.)

A Weierstrass parameterization on C/S effectively amounts to a choice (x, y) of sections $x \in \Gamma(C, \mathcal{O}_C(2e))$ and $y \in \Gamma(C, \mathcal{O}_C(3e))$ such $x^3 - y^2 \in \Gamma(C, \mathcal{O}_C(5e))$.

We write $W(C/S)$ for the set of Weierstrass coordinates on C/S.

If $S = \operatorname{Spec} A$ and $C = C_{\underline{a}}$ is a curve given by a Weierstrass equation over A, then there is a particular choice of $(x, y) \in W(C/S)$ associated to this equation. In this case, there is an isomorphism

$$A^3 \times A^\times \approx \{(r, s, t, \lambda)\} \xrightarrow{\sim} W(C_{\underline{a}}/\operatorname{Spec}(A))$$

given by $(r, s, t, \lambda) \mapsto (\lambda^{-2}x + r, \lambda^{-3}y + \lambda^{-2}sx + t)$.

Recall that if G is a formal group, we write $\operatorname{Coord}_G \subset \mathcal{O}_G$ for the set of coordinates on G, and we write $\operatorname{Coord}^n_G \subset \mathcal{O}_G/\mathcal{O}_G(-(n+1)e)$ for the quotient set. We define a map

$$W(C/S) \to \operatorname{Coord}_{\widehat{C}}\colon (x, y) \mapsto T = -\frac{x}{y}.$$

This passes to a map $W(C/S) \to \mathrm{Coord}^n_{\widehat{C}}$ for all n.

Proposition 12.2. *If $C/\mathrm{Spec}(A)$ is given by a Weierstrass equation, then the map $W(C/S) \to \mathrm{Coord}^4_{\widehat{C}}$ is an isomorphism.*

Proof. Choose Weierstrass coordinates (x, y) for C/S and hence an identification $C \approx C_{\underline{a}}$ for some $a_i \in A$. Given a coordinate $T \in \mathrm{Coord}_{\widehat{C}}$, we want to show that there there exists a unique pair (x', y') of Weierstrass coordinates for C such that $-x'/y' \equiv T \mod T^5$. To demonstrate this, we expand x and y in terms of T:

$$x = T^{-2}(u_0 + u_1T + u_2T^2 + u_3T^3 + \cdots)$$
$$y = -T^{-3}(v_0 + v_1T + v_2T^2 + v_3T^3 + \cdots)$$

with $u_i, v_i \in A$ and $u_0, v_0 \in A^\times$, and $u_0^3 = v_0^2$ (since $x^3 - y^2 \in \Gamma(\mathcal{O}(5e))$). We wish to solve for $(x', y') = (\lambda^{-2}x + r, \lambda^{-3}y + \lambda^{-2}sx + t)$ so that $-x'/y' \equiv T \mod T^5$. A little bit of algebra shows that

$$\lambda = v_0/u_0$$
$$s = v_1/v_0 - u_1/u_0$$
$$r = \frac{(-u_2u_0v_0 + v_2u_0^2 - u_1v_1u_0 + u_1^2v_0)u_0}{v_0^3}$$
$$t = \frac{(-u_3u_0v_0 + v_3u_0^2 - u_2v_1u_0 + u_2u_1v_0)u_0}{v_0^3}$$

is the unique solution which does this. (The trick is to solve for λ, s, r, t in that order.) □

12.3. **The Thom spectrum on $\Omega U(4)$.** Let $\gamma\colon \Omega U(n) \to \Omega U \approx \mathbb{Z} \times BU$, and let $Y = (\Omega U(n))^\gamma$ denote the corresponding Thom spectrum. Recall that if E is an even periodic ring theory, then E_0Y represents Coord^4_G; more precisely,

$$\hom_{\mathrm{alg}(E_0)}(E_0Y, R) \xrightarrow{\sim} \mathrm{Coord}^4_{G_E \otimes R}.$$

Putting together the above remarks gives

Proposition 12.4. *Let (E, C, ϕ) be an elliptic spectrum. Then $(E \wedge Y, C', \phi')$ is an elliptic spectrum, where $C' \overset{\text{def}}{=} C \otimes_{E_0} E_0Y$ and $\phi' \overset{\text{def}}{=} \phi \otimes_{E_0} E_0Y$. The map $E \approx E \wedge S^0 \to E \wedge Y$ extends in a natural way to a map of elliptic spectra.*

Furthermore, there is an isomorphism

$$\hom_{\mathrm{alg}(E_0)}(\pi_0(E \wedge Y), R) \xrightarrow{\sim} W(C \otimes_{E_0} R)$$

which is natural in the elliptic spectrum (E, C, ϕ). If (x, y) is a choice of Weierstrass parameters then we obtain an identification

$$\pi_0(E \wedge Y) \approx \pi_0E[r, s, t, \lambda, \lambda^{-1}]$$

which is determined by associating to a map $\pi_0E[r, s, t, \lambda^\pm] \to R$ the Weierstrass parameters $(x', y') = (\lambda^{-2}x + r, \lambda^{-3}y + \lambda^{-2}sx + t)$ for the curve $C \otimes_{E_0} R$.

Note that the above proposition implies that the curve C' over $\pi_0E \wedge Y$ admits a *canonical* choice of Weierstrass parameters, associated to the identity map $\pi_0E \wedge Y \to \pi_0E \wedge Y$.

13. Hopf algebroids

13.1. **Groupoids.** A **groupoid** is a category in which all maps are isomorphisms. Thus, a (small) groupoid $G = (G_0, G_1)$ consists of a set of objects G_0 and a set of morphism G_1, together with appropriate structure.

More generally, we may consider a **groupoid object** in a category with pullbacks. This consists of a pair G_0, G_1 of objects, together with maps:

$$G_0 \xrightarrow{\epsilon} G_1 \underset{t}{\overset{s}{\rightrightarrows}} G_0 \qquad G_1 \xrightarrow{\chi} G_1$$

$$G_2 \overset{\text{def}}{=} G_1 \overset{s,t}{\underset{G_0}{\times}} G_1 \xrightarrow{\nabla} G_1,$$

which satisfy the axioms appropriate for a groupoid, with s, t, ϵ, ∇, and χ corresponding to source, target, identity, composition, and inverse. (The notation $X \overset{f,g}{\underset{B}{\times}} Y$ means $\lim(X \xrightarrow{f} B \xleftarrow{g} Y)$.)

For $n \geq 0$ define object

$$G_n \overset{\text{def}}{=} \underbrace{G_1 \overset{s,t}{\underset{G_0}{\times}} G_1 \overset{s,t}{\underset{G_0}{\times}} \cdots \overset{s,t}{\underset{G_0}{\times}} G_1}_{n \text{ copies of } G_1}.$$

Thus G_n is the set of functors from the category $[n] = (0 \to 1 \to \ldots \to n)$ to G. This definition is consistent with the previous use of G_0, G_1, G_2. The $G_\bullet$ fit together into a simplicial object, namely the nerve (or bar complex). We will make the convention that $d_0 = t$ and $d_1 = s$.

13.2. **Hopf algebroids.** A **Hopf algebroid** is a groupoid object in the opposite category of commutative rings. Unwinding this, we see that this consists of a pair $\Gamma = (\Gamma^0, \Gamma^1)$ of commutative rings, together with maps:

$$\Gamma^0 \xleftarrow{\epsilon} \Gamma^1 \underset{t}{\overset{s}{\leftleftarrows}} \Gamma^0 \qquad \Gamma^1 \xleftarrow{\chi} \Gamma^1$$

$$\Gamma^2 \overset{\text{def}}{=} \Gamma^1 \overset{s,t}{\underset{\Gamma^0}{\otimes}} \Gamma^1 \xleftarrow{\nabla} \Gamma^1,$$

where $s, t, \epsilon, \nabla, \chi$ satisfy the appropriate axioms. (The notation $A \overset{f,g}{\underset{R}{\otimes}} B$ means $\operatorname{colim}(A \xleftarrow{f} R \xrightarrow{g} B)$.) As in the case of groupoids, we set

$$\Gamma^n \overset{\text{def}}{=} \underbrace{\Gamma^1 \overset{s,t}{\underset{\Gamma^0}{\otimes}} \Gamma^1 \overset{s,t}{\underset{\Gamma^0}{\otimes}} \cdots \overset{s,t}{\underset{\Gamma^0}{\otimes}} \Gamma^1}_{n \text{ copies of } \Gamma^1},$$

and we write $C^*(\Gamma)$ for the corresponding cosimplicial ring, which is called the **cobar complex**. Again, we choose the convention that $d^1 = s$ and $d^0 = t$.

More generally, given a commutative ring R we can consider Hopf algebroids *over* R, which are groupoids in $\operatorname{alg}(R)^{\text{op}}$.

13.3. **The Hopf algebroid of Weierstrass equations.** Consider the functor $G\colon \text{Rings} \to \text{Groupoids}$ defined by $R \mapsto G(R) = (G_0(R), G_1(R))$, with object set

$$G_0(R) = \{\text{Weierstrass equations } F_{\underline{a}} = 0 \text{ over } R\}$$

and with morphism set

$$G_1(R) = \left\{\text{pairs } F_{\underline{a}'}, F_{\underline{a}} \text{ of equations and transformation } \phi \text{ such that } F_{\underline{a}} \circ \phi = \lambda^{-6} F_{\underline{a}'}\right\}.$$

Explicitly as sets, $G_0(R) = \{(a_1, \dots, a_6)\} = R^5$ and $G_1(R) = \{(a_1, \dots, a_6, r, s, t, \lambda)\} = R^5 \times R^3 \times R^\times$. Some of the structure maps are given by

$$\begin{aligned} \text{source}(a_1, \dots, a_6, r, s, t, \lambda) &= (a_1', \dots, a_6') \\ \text{target}(a_1, \dots, a_6, r, s, t, \lambda) &= (a_1, \dots, a_6) \\ \text{identity}(a_1, \dots, a_6) &= (a_1, \dots, a_6, 0, 0, 0, 1). \\ \text{compos}((a_1, \dots, a_6, r, s, t, \lambda), (a_1', \dots, a_6', r', s', t', \lambda')) &= (a_1, \dots, a_6, \nabla(r), \nabla(s), \nabla(t), \nabla(\lambda)). \end{aligned}$$

Here the symbols a_i' and $\nabla(r), \nabla(s), \nabla(t), \nabla(\lambda)$ refer to the expressions given in (9.2).

This groupoid G is clearly represented by a Hopf algebroid (A, Γ), with

$$\begin{aligned} A &= \mathbb{Z}[a_1, a_2, a_3, a_4, a_6] \\ \Gamma &= A[r, s, t, \lambda^{\pm 1}] \end{aligned}$$

and with structure maps $d^0(a_i) = a_i$, $d^1(a_i) = a_i'$, and so forth.

13.4. **Graded Hopf algebroids.** More generally, we may consider a graded Hopf algebroid, which is a groupoid object in the opposite category of commutative graded rings. In our case, we want to consider (A_*, Γ_*) defined by

$$A_{2n} \overset{\text{def}}{=} \Gamma(\operatorname{Spec}(A), \omega_C^n), \qquad \Gamma_{2n} \overset{\text{def}}{=} \Gamma(\operatorname{Spec}(\Gamma), \omega_C^n),$$

with $A_t = \Gamma_t = 0$ if t is odd. Thus we get

$$A_* = A[\eta, \eta^{-1}], \quad \Gamma_* = \Gamma[\eta, \eta^{-1}], \qquad |\eta| = 2, \quad |A| = |\Gamma| = 0,$$

where $\eta = dx/(2y + a_1 x + a_3)$, and we have the additional formulas

$$d^0(\eta) = \eta, \quad d^1(\eta) = \eta' = \lambda^{-1}\eta.$$

13.5. **Comodules.** Let $\Gamma = (\Gamma^0, \Gamma^1)$ be a Hopf algebroid. A **comodule** over Γ is a pair (M, ψ_M) consisting of a left Γ^0-module M and a morphism $\psi_M\colon M \to \Gamma^1 \overset{d^1}{\underset{\Gamma^0}{\otimes}} M$ of left Γ^0-modules making appropriate unit and coassociativity diagrams commute. (Note that $\Gamma^1 \otimes_{\Gamma^0} M$ is regarded as a left Γ^0-module by means of the algebra map $d^0\colon \Gamma^0 \to \Gamma^1$.) We write $\operatorname{Comod}(\Gamma)$ for the category of Γ-comodules.

The category $\operatorname{Comod}(\Gamma)$ is symmetric monoidal: the tensor product of (M, ψ_M) and (N, ψ_N) is defined by $(M \otimes_{\Gamma^0} N, \psi_{M\otimes N})$, with $\psi_{M\otimes N}$ defined by

$$M \otimes_{\Gamma^0} N \xrightarrow{\psi_M \otimes \psi_N} (\Gamma^1 \overset{d^1}{\underset{\Gamma^0}{\otimes}} M) \overset{d^0, d^0}{\underset{\Gamma^0}{\otimes}} (\Gamma^1 \overset{d^1}{\underset{\Gamma^0}{\otimes}} N) \to \Gamma^1 \overset{d^1}{\underset{\Gamma^0}{\otimes}} (M \otimes_{\Gamma^0} N),$$

the second map being defined by the multiplication on Γ^1. The unit of the monoidal structure is (Γ^0, d^1).

13.6. **Homotopy spectral sequence.** Let $X^\bullet$ be a cosimplicial spectrum. Then there is a spectral sequence (the Bousfield-Kan spectral sequence [BK72]), with

$$E_1^{s,t} = \pi_t X^s, \qquad E_2^{s,t} = H^s(\pi_t X^\bullet),$$

and abutting to $\pi_{t-s}\operatorname{holim}(X^\bullet)$.

When $X^\bullet$ is built from a ring spectrum in a nice way, then the E_1-term can often be described in terms of a Hopf algebroid, and hence the E_2-term in terms of the cohomology of this Hopf algebroid. For instance, the Adams-Novikov spectral sequence is a spectral sequence

$$E_2^{s,t} = H^{s,t}(A, \Gamma) \Longrightarrow \pi_{t-s} S^0$$

where (A, Γ) is a graded Hopf algebroid representing the category of formal group laws and isomorphisms between them. It is constructed from the cosimplicial spectrum $X^s = MU^{(s+1)}$ (or from $X^s = MUP^{(s+1)}$ if we prefer).

14. Topological modular forms

14.1. **Spectral sequence for an elliptic spectrum.** Let (E, C, ϕ) be an elliptic spectrum. Let

$$\Delta_*^s = \Delta_*^s(E, C, \phi) \stackrel{\text{def}}{=} \pi_*(E \wedge Y^{\wedge(s+1)})$$

where $Y = (\Omega U(4))^\gamma$ is the Thom spectrum described above. These rings form the E_1-term of the homotopy spectral sequence of the cosimplicial spectrum $X^s = E \wedge Y^{\wedge(s+1)}$, with $d_1\colon \Delta_*^s \to \Delta_*^{s+1}$ a differential.

Proposition 14.2. *Under the above hypotheses,*

(1) $\Delta_* = (\Delta_*^0, \Delta_*^1)$ *is a graded Hopf algebroid over* E_0*,*
(2) (Δ_*^*, d_1) *is precisely the cobar complex* $C^*(\Delta_*)$ *of the Hopf algebroid,*
(3) $H^{s,*}(\Delta_*) = 0$ *for* $s > 0$*, and* $H^{0,t}(\Delta_*) \approx \pi_t E$*.*
(4) *There is a canonical map* $\iota_{(E,C,\phi)}\colon \Gamma_* \to \Delta_*(E, C, \phi)$ *of Hopf algebroids which is natural in the elliptic spectrum* (E, C, ϕ)*, in the sense that for each map* $\rho\colon (E, C, \phi) \to (E', C', \phi')$ *of elliptic spectra we have* $\rho \circ \iota_{(E,C,\phi)} = \iota_{(E',C',\phi')}$*.*

Proof. Let (E, C, ϕ) denote a fixed elliptic spectrum. Define a functor $G\colon \operatorname{alg}(E_0) \to$ Groupoids by

$$G_0(R) = W(C \otimes_{E_0} R)$$
$$G_1(R) = W(C \otimes_{E_0} R)^{\times 2}.$$

In particular, between any two objects in $G(R)$ there is precisely one map. To show (1) and (2), it is effectively enough to show that Δ_0^s naturally represents the functor G_s, which is given by $G_s(R) = W(C \otimes_{E_0} R)^{\times(s+1)}$. (Actually, this ought to be part of the statement of the proposition.) For $s = 0$ this is just (12.4). In general, the $s+1$ maps $Y \to Y^{(s+1)}$ induce $s+1$ maps $\pi_0(E \wedge Y) \to \pi_0(E \wedge Y^{(s+1)})$, and hence a map $\hom(\Delta_*^s, R) \to W(C \otimes R)^{s+1}$; that this map is an isomorphism follows by repeated applications of (12.4).

To show (3), make a choice (x, y) of Weierstrass parameters for C over E_0. This gives a map $\Delta^0_* \to E_*$, and more generally a contracting homotopy for the coaugmented complex $E_* \to \Delta^\bullet_*$.

The map $\Gamma^s \to \Delta^s$ is the one classifying the Weierstrass equations associated to (tautological) choices of Weierstrass parameters for C over $\pi_0(E \wedge Y^{\wedge(s+1)})$. This is clearly natural in the generalized elliptic curve C. □

Remark 14.3. In the above, we could have replaced the role of Y with MUP. Then instead of a Hopf algebroid representing Weierstrass parameterizations of the curve, we would get a Hopf algebroid representing formal coordinates for its formal group.

14.4. **What is** tmf**?** The idea is that there should exist a "universal elliptic spectrum", which maps canonically to all others. Such a spectrum should be associated to a particular family of generalized elliptic curves, namely the "universal family".

Such a spectrum as described above cannot really exist: an elliptic E spectrum is in particular an even periodic ring theory, and thus admits a coordinate $x \in E^0\mathbb{CP}^\infty$. If there we a universal such elliptic spectrum (call it E_{univ}), then a choice of coordinate $x \in E^0_{\mathrm{univ}}\mathbb{CP}^\infty$ would have a *canonical* image in each elliptic spectrum E. But there is no such canonical choice of coordinate on an elliptic curve, since any such choice T maps to another λT, by a map $(x, y) \to (\lambda^{-2}x, \lambda^{-3}y)$ of elliptic curves over the ring $E_0[\lambda, \lambda^{-1}]$.

If we remove the requirement that E_{univ} be even periodic, then we can do something. For instance, any elliptic curve over a ring A containing $1/6$ admits a choice of coordinate T which is canonical *up to scalar* (this follows from the argument made in (9.5)). Thus it turns out that there exists a spectrum $\mathrm{tmf}[1/6]$ which is complex orientable (but *not* even periodic), with the kind of universal property we would like. In particular, $\pi_*\mathrm{tmf}[1/6] = \mathbb{Z}[1/6, c_4, c_6]$ with $c_i \in \pi_{2i}\mathrm{tmf}[1/6]$; i.e., $\mathrm{tmf}[1/6]$ is a connective version of the elliptic cohomology described in [LRS95].

If 6 is not invertible, then it is not necessarily the case that there is a coordinate unique up to scalar. Therefore, we do not expect tmf to be a complex orientable theory.

On the other hand, we do know that if (E, C, ϕ) is an elliptic spectrum, then $E \wedge Y$ has a natural coordinate associated to the canonical Weierstrass parameterization on $C \otimes_{E_0} E_0Y$. Thus, we may suppose that $\mathrm{tmf} \wedge Y$, and more generally $\mathrm{tmf} \wedge Y^{\wedge(s)}$, are even periodic. Furthermore, since tmf should be universal for elliptic spectra, there are are natural maps $\pi_*(\mathrm{tmf} \wedge Y^{\wedge(s+1)}) \to \pi_*(E \wedge Y^{\wedge(s+1)})$ for each elliptic spectrum E. The proposition we proved above thus motivates the following

Theorem 14.5 (Hopkins, Miller, Mahowald, Goerss, et. al)**.** *There exists a connective, commutative ring spectrum* tmf *such that there is an isomorphism of complexes*

$$\pi_*(\mathrm{tmf} \wedge Y^{\wedge(\bullet+1)}) \approx C^*(\Gamma_*).$$

Proof. Deferred. □

Actually, things are better; it can be shown that tmf is an E_∞-ring spectrum.

We will put off the problem of constructing tmf for later. For now, we will use the characterization implicit in the above theorem to compute the homotopy groups of tmf.

14.6. **Derived functors of modular forms.** To compute the spectral sequence

$$E_2^{s,t} = H^{s,t}(A_*, \Gamma_*) \Longrightarrow \pi_{t-s}\mathrm{tmf}$$

we must first compute the E_2-term. We have already noticed (10.3) that

$$H^{0,t}(A_*, \Gamma_*) \approx \begin{cases} \mathcal{M}_{t/2} & \text{if } t \text{ even,} \\ 0 & \text{if } t \text{ odd,} \end{cases}$$

where $\mathcal{M}_*$ is the ring of modular forms. Therefore there will be an edge homomorphism

$$\pi_n\mathrm{tmf} \to \mathcal{M}_{n/2},$$

(where we make the convention that $\mathcal{M}_m = 0$ if m is a half-integer). This homomorphism will not be an isomorphism, although it becomes an isomorphism after inverting 6.

We may say that the higher cohomology groups are the **derived functors of modular forms**. This terminology can be justified; these groups are the cohomology groups of sections of certain line bundles over the "stack of generalized elliptic curves".

15. Calculations in Hopf algebroids, and the calculation with 6 inverted

15.1. **Induced Hopf algebroids.** Let (A, Γ) be a Hopf algebroid, and suppose that $f\colon A \to B$ is a ring homomorphism. Let

$$\Gamma_B \overset{\text{def}}{=} B \underset{A}{\overset{f,d^0}{\otimes}} \Gamma \underset{A}{\overset{d^1,f}{\otimes}} B.$$

Then (B, Γ_B) is again a Hopf algebroid, and there is a map $(A, \Gamma) \to (B, \Gamma_B)$, which induces a functor $f^*\colon \mathrm{Comod}(A, \Gamma) \to \mathrm{Comod}(B, \Gamma_B)$.

In terms of groupoids, this amounts to the following. Let $G(R) = (G_0(R), G_1(R))$ denote the functor from rings to groupoids represented by (A, Γ). Let $H_0(R) = \hom_{\mathrm{Rings}}(B, R)$; there is an evident natural transformation $\phi\colon H_0(R) \to G_0(R)$ induced by f. Then Γ_B represents the functor

$$H_1(R) \overset{\text{def}}{=} H_0(R) \underset{G_0(R)}{\overset{\phi,d_0}{\times}} G_1(R) \underset{G_0(R)}{\overset{d_1,\phi}{\times}} H_0(R).$$

Thus, if $x, y \in H_0(R)$, then $H(R) = (H_0(R), H_1(R))$ is a groupoid with

$$\hom_{H(R)}(x, y) = \hom_{G(R)}(\phi x, \phi y).$$

So $\phi\colon H(R) \to G(R)$ is a full functor.

15.2. **A change of rings theorem.** Recall that a map $f\colon A \to R$ of commutative rings is **faithfully flat** if it is flat, and if $R \otimes_A M = 0$ implies $M = 0$ for any A-module M.

Theorem 15.3. *Let (A, Γ) be a Hopf algebroid, and (B, Γ_B) be the Hopf algebroid induced by a map $f\colon A \to B$. Assume that the morphisms $d^0\colon A \to \Gamma$ and $d^0\colon B \to \Gamma_B$ are flat.* If *there exists a ring R and a morphism g of rings in*

$$A \xrightarrow{1\otimes d^1} B \underset{A}{\overset{f,d^0}{\otimes}} \Gamma \xrightarrow{g} R$$

such that the composite $A \to R$ is faithfully flat, then

$$f^*\colon \mathrm{Comod}(A, \Gamma) \to \mathrm{Comod}(B, \Gamma_B)$$

is an equivalence of categories, and for any $M \in \mathrm{Comod}(A,\Gamma)$ *the induced map*

$$H^*(A,\Gamma;M) \to H^*(B,\Gamma_B;f^*M)$$

is an isomorphism. In particular, $H^*(A,\Gamma) \approx H^*(B,\Gamma_B)$.

Here are two important special cases.

Example 15.4. Suppose that $f\colon A \to B$ is itself faithfully flat. Then we can let $R = B$ and $g = \mathrm{id}\otimes s^0$, so that the composite

$$A \xrightarrow{1\otimes d^1} B \underset{A}{\overset{f,d^0}{\otimes}} \Gamma \xrightarrow{\mathrm{id}\otimes s^0} B \otimes_A A = B$$

is just f. This is an example of "flat descent".

Example 15.5. Suppose that $R = A$, and that the composite

$$A \xrightarrow{1\otimes d^1} B \underset{A}{\overset{f,d^0}{\otimes}} \Gamma \xrightarrow{g} A$$

is the identity on A (which is certainly faithfully flat). Then giving a map g is equivalent to giving a section of

$$H_0(R) \underset{G_0(R)}{\overset{\phi,d_0}{\times}} G_1(R) \xrightarrow{d_1\circ\pi_2} G_0(R)$$

which is natural in R. In particular, for each object $x \in G_0(R)$ we must produce an object $\gamma(x) \in H_0(R)$ and an isomorphism $\alpha_x\colon x \to \phi\gamma(x)$ in $G(R)$. Since ϕ is a full functor, this is effectively the same as giving a functor $\gamma\colon G \to H$ and natural isomorphisms $\phi\circ\gamma \approx \mathrm{id}$ and $\gamma\circ\phi \approx \mathrm{id}$; i.e., an equivalence of categories.

A proof of a variant of this special case (in which the Hopf algebroids are assumed to be complete with respect to the powers of some ideal) is given in [Dev95].

15.6. **Application.** We apply the change of rings theorem to the graded Hopf algebroid (A_*,Γ_*). Define graded subrings $\overline{A}_* \subset A_*$ and $\overline{\Gamma}_* \subset \Gamma_*$ by

$$\overline{A}_* = \mathbb{Z}[\overline{a}_1,\dots,\overline{a}_6], \qquad \overline{\Gamma}_* = \overline{A}_*[\overline{r},\overline{s},\overline{t}],$$

with

$$\overline{a}_i = a_i\eta^i, \quad \overline{r} = r\eta^2, \overline{s} = s\eta, \overline{t} = t\eta^3.$$

(Recall that $\eta \in A_2$.) It is straightforward to check that $(\overline{A}_*,\overline{\Gamma}_*)$ is a sub-Hopf algebroid.

Proposition 15.7. *The conclusion of the change of rings theorem applies to the map* $(\overline{A}_*,\overline{\Gamma}_*) \to (A_*,\Gamma_*)$ *of graded Hopf algebroids. In particular, they have the same cohomology.*

Proof. It is clear that $A_* \approx \overline{A}_*[\eta,\eta^{-1}]$, and that $f\colon \overline{A}_* \to A_*$ is faithfully flat. By the first special case of the change of rings theorem, it suffices to show that (A_*,Γ_*) is the Hopf algebroid induced from $(\overline{A}_*,\overline{\Gamma}_*)$ along f; i.e., that the natural map

$$(\overline{\Gamma}_*)_{A_*} = A_* \underset{\overline{A}_*}{\overset{f,d^0}{\otimes}} \overline{\Gamma}_* \underset{\overline{A}_*}{\overset{d^1,f}{\otimes}} A_* \xrightarrow{d^0\otimes\iota\otimes d^1} \Gamma_*$$

is an isomorphism. This is clear: note that $\eta\otimes 1\otimes 1 \mapsto \eta$ while $1\otimes 1\otimes\eta \mapsto \lambda^{-1}\eta$. □

Remark 15.8. If we forget about the grading on $(\overline{A}_*, \overline{\Gamma}_*)$, then we can identify it with the Hopf algebroid which represents the groupoid $\overline{G}(R)$, whose objects are generalized Weierstrass equations over R, and whose morphisms are variable transformations which preserve the standard invariant 1-form η. In particular, we may regard the groupoid $\overline{G}(R)$ as that having objects the Weierstrass equations $y^2 + \cdots = x^3 + \cdots$, and as having morphisms correpsonding to variable transformations

$$x = x' + r, \qquad y = y' + sx' + t,$$

in other words, the usual formulas with $\lambda = 1$.

Remark 15.9. The Hopf algebroid $(\overline{A}_*, \overline{\Gamma}_*)$ has a topological interpretation: if we let $\overline{Y} = (\Omega SU(4))^\gamma$, then $\pi_*(\mathrm{tmf} \wedge \overline{Y}) \approx \overline{A}_*$ and $\pi_*(\mathrm{tmf} \wedge \overline{Y}^{\wedge(2)}) \approx \overline{\Gamma}_*$.

Henceforward, we will work only with the sub-Hopf algebroid $(\overline{A}_*, \overline{\Gamma}_*)$. In honor of this, we will call it simply (A, Γ), and we will drop the overbar notation, so that henceforth a_i, r, s, t will stand for the elements formerly known as $\overline{a}_i, \overline{r}, \overline{s}, \overline{t}$, which are in gradings $2i, 4, 2, 6$.

The Hopf algebroid (A, Γ) is **connected**, in the sense that $A_t = \Gamma_t = 0$ if $t < 0$ and $A_0 = \Gamma_0 = \mathbb{Z}$. It is also concentrated in even degrees. Therefore,

Proposition 15.10. *We have that*

$$H^{s,t}(A, \Gamma) = 0 \qquad \text{if } 2s > t \text{ (or equivalently } s > t - s\text{)}.$$

Furthermore, each of the groups $H^{s,t}(A, \Gamma)$ is finitely generated.

To compute these cohomology groups, we will proceed by computing the cohomology of $(A \otimes R, \Gamma \otimes R)$, where $R = \mathbb{Z}[1/6], \mathbb{Z}_{(3)}, \mathbb{Z}_{(2)}$.

15.11. **Calculation after inverting 6.** We compute the cohomology of $(A[\frac{1}{6}], \Gamma[\frac{1}{6}])$ by means of the change of rings theorem. Thus, define

$$B \overset{\text{def}}{=} \mathbb{Z}[\tfrac{1}{6}, c_4, c_6], \quad |c_i| = 2i$$

and

$$f \colon A[\tfrac{1}{6}] \to B, \quad a_1, a_2, a_3 \mapsto 0, \quad a_4 \mapsto -c_4/48, \quad a_6 \mapsto -c_6/864.$$

Thus we can form an induced Hopf algebroid (B, Γ_B).

Lemma 15.12. *The map $d^0 \colon B \to \Gamma_B$ is an isomorphism. In particular,*

$$H^{s,t}(B, \Gamma_B) = \begin{cases} B_t & \text{if } s = 0, \\ 0 & \text{if } s > 0. \end{cases}$$

Proof. Let G and H denote the groupoids represented by (A, Γ) and (B, Γ_B) respectively, and $\phi \colon H \to G$ the functor associated to f. It is enough to show that for every R containing $1/6$, the map $d_0 \colon H_1(R) \to H_0(R)$ is an isomorphism, or in other words that in $H(R)$ the only morphisms are identity maps. Let (c_4, c_6) and (c_4', c_6') denote objects in $H(R)$, corresponding to pairs of elements in R. By definition,

$$\hom_{H(R)}\left((c_4', c_6'), (c_4, c_6)\right) = \hom_{G(R)}\left(\phi(c_4', c_6'), \phi(c_4, c_6)\right),$$

and $\phi(c_4, c_6) \in G_0(R)$ corresponds to the Weierstrass equation $y^2 = x^3 - \frac{c_4}{48}x - \frac{c_6}{864}$. It is straightforward to check that the only transformation of the form $(x, y) \mapsto (x+r, y+sx+t)$ which preserves the shape of the equation is that with $r = s = t = 0$; that is to say, the only maps are identity maps. $\square$

Proposition 15.13. *We have*

$$H^{s,*}(A[\tfrac{1}{6}], \Gamma[\tfrac{1}{6}]) \approx \begin{cases} \mathbb{Z}[\frac{1}{6}, c_4, c_6] & \text{if } s = 0, \\ 0 & \text{if } s > 0. \end{cases}$$

Therefore

$$\pi_* \mathrm{tmf}[\tfrac{1}{6}] = \mathbb{Z}[\tfrac{1}{6}, c_4, c_6].$$

Proof. We show that the change of rings theorem applies, and then use the above lemma. We will produce a map g making the composite

$$A \xrightarrow{1\otimes d^1} B \underset{A}{\overset{f,d^0}{\otimes}} \Gamma \xrightarrow{g} A$$

the identity, as in the second special case of the change of rings theorem. Let G and H be as in the proof of the previous lemma. For each $\underline{a} = (a_1, \ldots, a_6) \in G_0(R)$ we want to find $\gamma(\underline{a}) \in H_0(R)$ and $\alpha_{\underline{a}}\colon \underline{a} \to \phi\gamma(\underline{a}) \in G_1(R)$, naturally in R. We define these by

$$\gamma(\underline{a}) = (c_4(\underline{a}), c_6(\underline{a})), \qquad \alpha_{\underline{a}}\colon (x, y) \mapsto (x + \tfrac{1}{3}a_2 + \tfrac{1}{12}a_1^2, y + \tfrac{1}{2}a_1 x + \tfrac{1}{2}a_3),$$

where $c_i(\underline{a}) \in \mathbb{Z}[a_1, \ldots, a_6] \subset A$ as defined in (9.5). We have already carried out the calculations to check this in (9.5), see especially (9.7). $\square$

16. Derived functors of modular forms at 3

16.1. **A reduction.** To compute the cohomology of $(A_{(3)}, \Gamma_{(3)})$, we first reduce to a small Hopf algebroid. Let $B = \mathbb{Z}_{(3)}[b_2, b_4, b_6]$ and define $f\colon A_{(3)} \to B$ by

$$a_1, a_3 \mapsto 0, \quad a_2 \mapsto b_2/4, \quad a_4 \mapsto b_4/2, \quad a_6 \mapsto b_6/4.$$

We obtain an induced Hopf algebroid (B, Γ_B) with

$$\Gamma_B = B \underset{A_{(3)}}{\overset{f,d^0}{\otimes}} \Gamma_{(3)} \underset{A_{(3)}}{\overset{d^1,f}{\otimes}} B \approx \Gamma_{(3)}/(a_1, a_3, a_1', a_3') = B[r, s, t]/(2s, 2t) = B[r].$$

The Hopf algebroid structure is thus captured by

$$d^1(b_i) = b_i', \qquad \nabla(r) = r + r',$$

where

$$\begin{aligned} b_2' &= b_2 + 12r \\ b_4' &= b_4 + b_2 r + 6r^2 \\ b_6' &= b_6 + 2b_4 r + b_2 r^2 + 4r^3. \end{aligned}$$

These are just the transformation formulas for equations of the shape $y^2 = x^3 + \frac{1}{4}b_2 x^2 + \frac{1}{2}b_4 x + \frac{1}{4}b_6$ under $x \mapsto x + r$.

(Note: here and subsequently I write

$$t \mapsto t' \quad \text{instead of} \quad t \mapsto d^n(t) \quad \text{where} \quad d^n \colon \Gamma^{n-1} \to \Gamma^n$$

and

$$t \mapsto t \quad \text{instead of} \quad t \mapsto d^0(t) \quad \text{where} \quad d^0 \colon \Gamma^{n-1} \to \Gamma^n.$$

In other words, use a $'$ to indicate an application of the *last* face operator, and use nothing to indicate an application of the *first* face operator. In particular, for $d^0, d^1 \colon A \to \Gamma$ this reduces to the convention which I have already been using consistently. For $d^1 = \nabla \colon \Gamma \to \Gamma^2 = \Gamma \otimes_A \Gamma$, a formula like $\nabla(t) = t + t' + sr'$ with $r, s, t \in \Gamma$ transates to $\nabla(t) = t \otimes 1 + 1 \otimes t + s \otimes r$.)

Proposition 16.2. $H^*(A_{(3)}, \Gamma_{(3)}) \approx H^*(B, \Gamma_B)$.

Proof. We apply the change of rings theorem as in the second special case. Thus if $(A_{(3)}, \Gamma_{(3)})$ and (B, Γ_B) represent groupoids G and H respectively, then we must find for each $\underline{a} \in G_0$ a $\gamma(\underline{a}) \in H_0$ and $\alpha_{\underline{a}} \colon \underline{a} \to \phi\gamma(\underline{a}) \in G_1$. Take

$$\gamma(\underline{a}) = (b_2(\underline{a}), b_4(\underline{a}), b_6(\underline{a})), \qquad \alpha_{\underline{a}} \colon (x, y) \mapsto (x, y + \tfrac{1}{2}a_1 x + \tfrac{1}{2}a_3)$$

where the $b_i(\underline{a})$ are the polynomials defined in (9.5). Again, this is basically a calculation we did in (9.5). □

16.3. **Some elements in $H^*(B, \Gamma_B)$.** Let us construct some classes in $H^{s,t} = H^{s,t}(B, \Gamma_B)$. We already know about

$$c_4 = b_2^2 - 24b_4 \in H^{0,8}, \quad c_6 = -b_2^3 + 36b_2b_4 - 216b_6 \in H^{0,12}, \quad \Delta = (c_4^3 - c_6^2)/(12)^3 \in H^{0,24}.$$

Write $\delta^s = \sum \pm d^i \colon C^s(\Gamma^\bullet) \to C^{s+1}(\Gamma^\bullet)$ for the differential in the cobar complex. Since $r \in \Gamma$ is primitive, $\delta^1(r) = 0$. Define

$$\alpha = [r] \in H^{1,4}.$$

Clearly $3\alpha = 0$ since $\delta^0(\frac{1}{4}b_2) = 3r$.

There is a cup product in C^*, defined as follows: for $x \in C^p$ and $y \in C^q$, let

$$x \cup y \overset{\text{def}}{=} x y^{[p]} \in C^{p+q}$$

using the shorthand

$$x \in C^{p+q} \text{ for } (d^0)^q(x), \qquad y^{[p]} = y^{\overbrace{'' \cdots ''}^{p}} \in C^{p+q} \text{ for } (d^{\text{last}})^p(y).$$

Since $\delta^{p+q}(x \cup y) = \delta^p x \cup y + (-1)^p x \cup \delta^q y$, this defines a product on cohomology, which turns out to be associative and commutative (see [Rav86, App. A]). Under this product,

$$\alpha \cup \alpha = 0.$$

We have Massey products: if $\alpha \in H^p$, $\beta \in H^q$, $\gamma \in H^r$, then

$$\langle \alpha, \beta, \gamma \rangle \overset{\text{def}}{=} \{[u \cup c + (-1)^p a \cup v]\} \subset H^{p+q+r+1}$$

where the set is taken over all a, b, c, u, v such that $\alpha = [a]$, $\beta = [b]$, $\gamma = [c]$, $\delta(u) = a \cup b$, $\delta(v) = b \cup c$. It is a coset of $\alpha H^{q+r+1} + H^{p+q+1}\gamma$. Since $\delta^1(-\frac{1}{2}r^2) = rr' = r \cup r$, we compute

$$\beta = \langle \alpha, \alpha, \alpha \rangle = [-\tfrac{1}{2}(r^2 r' - r r'^2)] \in H^{2,12},$$

and $3\beta = 0$.

It is relatively easy to check the relations

$$\alpha c_4 = \alpha c_6 = 0, \qquad \beta c_4 = \beta c_6 = 0.$$

For instance, $c_4 \cup \alpha = [c_4 r]$. Now

$$\delta^0 \colon c_4 b_2 = b_2^3 - 24 b_2^2 b_4 \mapsto 12 c_4 r.$$

But also

$$\delta^0 \colon c_6 = -b_2^3 + 36 b_2^2 b_4 - 216 b_6 \mapsto 0,$$

so adding them gives

$$\delta^0 \colon 12(b_2^2 b_4 - 18 b_6) \mapsto 12 c_4 r,$$

so $b_2^2 b_4 - 18 b_6 \to c_4 r$. (In any case, we will have another way to prove these relations later.)

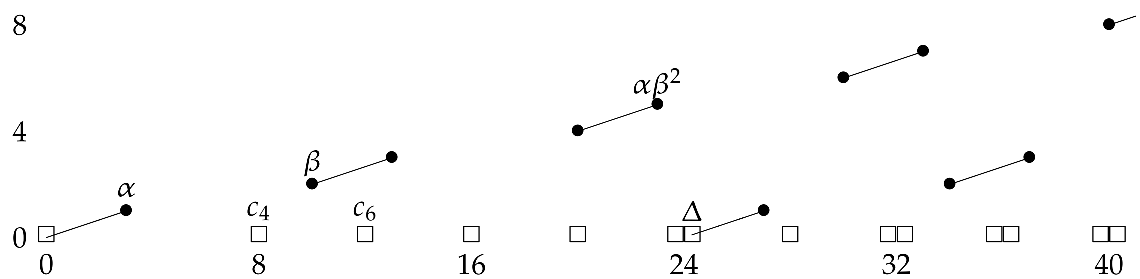
We can now state

Proposition 16.4. *The evident map*

$$\frac{\mathbb{Z}_{(3)}[c_4, c_6, \Delta, \alpha, \beta]}{\begin{pmatrix} c_4^3 - c_6^2 - (12)^3\Delta, \\ \alpha^2 = 0,\ 3\alpha = 3\beta = 0, \\ c_4\alpha = c_4\beta = 0, \\ c_6\alpha = c_6\beta = 0 \end{pmatrix}} \to H^{*,*}(B, \Gamma_B) \approx H^{*,*}(A_{(3)}, \Gamma_{(3)})$$

is an isomorphism.

This will be proved below.



This is a diagram of $H^{s,t}(A_{(3)}, \Gamma_{(3)})$. The horizontal axis is $t - s$ and the vertical axis is s; here $\square = \mathbb{Z}_{(3)}$ and $\bullet = \mathbb{Z}/3$. Lines represent multiplication by α.

16.5. **An algebraic spectral sequence.** If $T = -x/y$ is the usual coordinate at e for a generalized Weierstrass equation, then we have that

$$[3](T) = 3T + \cdots, \qquad [3](T) \equiv b_2 T^3 + \cdots \bmod (3), \qquad [3](T) \equiv 2 b_4^2 T^9 + \cdots \bmod (3, b_2).$$

This suggests that we should consider the ideal $I = (3, b_2, b_4) \subset B$. This is an **invariant ideal**, in the sense that $d^0(I)\Gamma_B = d^1(I)\Gamma_B$, and therefore

$$(\overline{B}, \overline{\Gamma}) \overset{\text{def}}{=} (B/I, \Gamma_B/I\Gamma_B) = (\mathbb{F}_3[b_6], \mathbb{F}_3[b_6, r])$$

is a Hopf algebroid, with

$$b_6' = b_6 + r^3, \qquad \nabla(r) = r + r'.$$

Define a descending filtration $\mathcal{F}^q\Gamma_B^s \subset \Gamma_B^s$ by $\mathcal{F}^q\Gamma_B^s = I^q\Gamma_B^s$. This is a filtration of the complex $C^*(\Gamma^\bullet)$, and the associated graded is $I^q\Gamma_B^s/I^{q+1}\Gamma_B^s$. This is isomorphic to $(I^q/I^{q+1})\otimes_{\overline{B}}\overline{\Gamma}$, since $(d^0)^s\colon B\to\Gamma_s^B$ is flat.

The quotient I/I^2 is naturally a comodule over $(\overline{B},\overline{\Gamma})$, via the restriction of $d^1\colon B\to\Gamma_B$ to $I/I^2 \to I\Gamma_B/I^2\Gamma_B \approx (I/I^2)\otimes_{\overline{B}}\overline{\Gamma}$. Filtering with respect to I gives rise to a spectral sequence (which is a variant of the "algebraic Adams-Novikov spectral sequence".)

Proposition 16.6. *There is a spectral sequence of algebras*

$$E_1^{p,q;t} = H^p(\overline{B},\overline{\Gamma};\mathrm{Sym}^q_{\overline{B}}(I/I^2)) \Longrightarrow H^{p,t}(B,\Gamma_B).$$

Proof. This is the spectral sequence of a filtered chain complex $C^*(\Gamma_B^\bullet)$, using the evident identification $I^q/I^{q+1}\approx \mathrm{Sym}^q_{\overline{B}}(I/I^2)$ as comodules over $(\overline{B},\overline{\Gamma})$. □

Write

$$I/I^2 = \mathbb{F}_3[b_6]\otimes V, \qquad V \overset{\text{def}}{=} \mathbb{F}_3\{B_0,B_2,B_4\},$$

where B_0,B_2,B_4 are representatives of $3,b_2,b_4\in I$. Then the coaction $\psi\colon I/I^2\to I/I^2\otimes_{\overline{B}}\overline{\Gamma}$ is deduced from the formulas $d^1(b_i)=b_i'$ in (B,Γ); we get that $\psi(B_i)=B_i'$ with

$$\begin{aligned} B_0' &= B_0 \\ B_2' &= B_2 + B_0 r \\ B_4' &= B_4 + B_2 r + 2B_0 r^2. \end{aligned}$$

We make one final reduction. Consider the map $\overline{B}=\mathbb{F}_3[b_6]\to\mathbb{F}_3$ sending $b_6\mapsto 0$. This gives rise to an induced Hopf algebroid $(\mathbb{F}_3,C)$ with

$$C = \mathbb{F}_3[b_6,r]/(b_6,b_6') = \mathbb{F}_3[r]/(r^3), \qquad |r|=4, \quad \nabla(r) = r+r'.$$

In particular, this Hopf algebroid is a *Hopf algebra*.

Proposition 16.7. *The change of rings formula applies to the map $(\overline{B},\overline{\Gamma})\to(\mathbb{F}_3,C)$. In particular, there is an isomorphism*

$$H^{p,t}(\overline{B},\overline{\Gamma};\mathrm{Sym}^q_{\overline{B}}(I/I^2)) \approx H^{p,t}(C;\mathrm{Sym}^q_{\mathbb{F}_3}V).$$

Proof. The map $1\otimes d^1\colon \overline{B}=\mathbb{F}_3[b_6]\to\mathbb{F}_3\otimes_{\overline{B}}\overline{\Gamma}\approx\mathbb{F}_3[r]$ sends $b_6\mapsto r^3$, and thus is flat. □

Remark 16.8. The group scheme $\mathrm{Spec}(C)$ is actually the group of automorphisms of the generalized elliptic curve given by $y^2=x^3$ over $\mathbb{F}_3$ which fix the base point and which preserve the 1-form $\eta = dx/2y$.

As is well known,

$$H^{*,*}(C;\mathbb{F}_3)\approx E(\alpha)\otimes P(\beta), \qquad \alpha\in H^{1,4}, \beta\in H^{2,12}.$$

Also, V is a cofree C-comodule, so

$$H^{*,*}(C;V)\approx H^{*,*}(C;\mathbb{F}_3)/(\alpha,\beta)\approx\mathbb{F}_3.$$

We need to understand the structure of the C-comodule $\mathrm{Sym}^*V = \mathbb{F}_3[B_0, B_2, B_4]$. It is sometimes convenient to consider the dual Hopf algebra $C^* = \mathbb{F}_3[P]/P^3$, which acts on a comodule M by operations which decrease degree, i.e., $P\colon M_n \to M_{n-4}$ defined by

$$\psi(x) = x + P(x)r + 2PP(x)r^2, \quad x \in M.$$

There is a Cartan formula for the action on a tensor product $M \otimes N$ of comodules:

$$P(x \otimes y) = P(x) \otimes y + x \otimes P(y), \quad x \in M, y \in N.$$

In our case, $P(B_0) = 0$, $P(B_2) = B_0$, $P(B_4) = B_2$.

Proposition 16.9.

$$H^{0,*}(C; \mathrm{Sym}^*V) \approx \mathbb{F}_3[B_0, C_4, C_6, \delta]/(C_4^3 - C_6^2 - B_0^3\delta),$$

where $C_4 = B_2^2 + B_0B_4 \in H^{0,8}$, $C_6 = -B_2^3 \in H^{0,12}$, *and* $\delta = B_4^3 \in H^{0,24}$. *Furthermore,* B_0, C_4, C_6, δ *are representatives of the modular forms* $3, c_4, c_6, \Delta$ *respectively, and thus are permanent cycles in the spectral sequence.*

Proof. The H^0 is a straightforward computation, using the fact (say) that $H^0(C; M) = \mathrm{Ker}\, P\colon M \to M$. That the generators represent modular forms is immediate by examining the formulae for these in terms of b_2, b_4, b_6: for instance, $c_4 = b_2^2 + (-8)\cdot 3b_4 \in I^2$, and $c_6 = -b_2^3 + 4\cdot 3^2 b_2 b_4 + (-8)\cdot 3^3 b_6 \in I^3$. □

Proposition 16.10. *As a C-comodule,* Sym^*V *is a direct sum of suspensions of* $\mathbb{F}_3$ *and* V. *In particular,*

$$\mathrm{Sym}^*V \approx \mathbb{F}_3[\delta] \otimes M,$$

where

$$M = \mathbb{F}_3 \oplus \bigoplus_{m \in \mathcal{M}} V \cdot m,$$

where

$$\mathcal{M} \overset{\text{def}}{=} \{\, B_0^i C_4^j, B_0^i C_4^j C_6 \mid i, j \geq 0 \,\} \smallsetminus \{1\}$$

is a set of monomials in $H^0(C; M)$.

Putting all this together, we see that

$$H^{*,*}(C; \mathrm{Sym}^*V) \approx \frac{\mathbb{F}_3[B_0, C_4, C_6, \delta, \alpha, \beta]}{\begin{pmatrix} C_4^3 - C_6^2 - B_0^3\delta, \\ \alpha^2 = 0, B_0\alpha = B_0\beta = 0, \\ C_4\alpha = C_4\beta = 0, \\ C_6\alpha = C_6\alpha = 0 \end{pmatrix}}.$$

Since $B_0, C_4, C_6, \delta, \alpha, \beta$ are all permanent cycles, it is immediate that the spectral sequence computing $H^*(B, \Gamma_B)$ collapses at E_1, and we get the answer we want.

17. HOMOTOPY GROUPS OF tmf AT 3

We are now ready to compute the homotopy spectral sequence $E_2^{s,t} \Longrightarrow \pi_{t-s}\mathrm{tmf}_{(3)}$. This is a spectral sequence of rings; the differentials are derivations, and take the form $d_r\colon E_r^{s,t} \to E_r^{s+r,t+r-1}$.

First, it is clear that for degree reasons no differentials are possible until the 24-th stem. In particular, c_4, c_6, α, β are permanent cycles; and thus the first differential to consider is a possible $d_5\colon \Delta \mapsto \pm\alpha\beta^2$.

Both α and β are in the image of the map $\pi_* S^0 \to \pi_*\mathrm{tmf}$, and the "Toda relation" $\alpha\beta^3$ in the sphere implies the same relation in $\pi_*\mathrm{tmf}$. Since the only candidate to hit $\alpha\beta^2$ in the spectral sequence is $\beta\Delta$, we see that we must have

$$d_5\colon \Delta \mapsto \pm\alpha\beta^2,$$

and hence

$$d_5\colon \Delta^2 \mapsto \mp\alpha\beta^2\Delta,$$
$$d_5\colon \Delta^3 \mapsto 0.$$

Proposition 17.1. *Let $[\alpha\Delta] \in \pi_{27}\mathrm{tmf}$ denote the homotopy class represented by $\alpha\Delta \in E_2^{1,28}$. Then $\alpha[\alpha\Delta] = \pm\beta^3$.*

Proof. There is a Toda bracket $\beta = \langle \alpha, \alpha, \alpha \rangle$ in homotopy, so

$$\beta^3 = \beta^2\langle \alpha, \alpha, \alpha \rangle \subset \langle \alpha\beta^2, \alpha, \alpha \rangle.$$

But since $\alpha\beta^2 = 0$ in homotopy, this bracket must be contained in its indeterminacy, which is $\alpha\pi_{27}\mathrm{tmf} = \mathbb{Z}/3\{[\alpha\Delta]\}$. □

As a consequence,

$$d_9(\alpha\Delta^2) = \alpha d_5(\Delta^2) = \alpha(\mp\alpha\beta^2\Delta) = \mp\beta^5.$$

Theorem 17.2. *The spectral sequence for $\pi_*\mathrm{tmf}_{(3)}$ collapses at the E_{10}-term, and $E_{10}^{s,*} = E_\infty^{s,*} = 0$ for $s \geq 10$. The edge homomorphism fits in an exact sequence*

$$0 \to K \to \pi_t\mathrm{tmf}_{(3)} \to \mathcal{M}_{t/2} \otimes \mathbb{Z}_{(3)} \to C \to 0$$

where

$$C = \mathbb{Z}/3[\Delta^3]\{\Delta, \Delta^2\}$$

and

$$K = \mathbb{Z}/3[\Delta^3]\{\alpha, \beta, \alpha\beta, \beta^2, [\alpha\Delta], \beta^3 = \alpha[\alpha\Delta], \beta[\alpha\Delta], \alpha\beta[\alpha\Delta] = \beta^4\}.$$

In particular, the torsion in $\pi_*\mathrm{tmf}_{(3)}$ has order 3, lies in dimensions $3, 10, 13, 20, 27, 30, 37, 40 \bmod 72$, *and is periodic under multiplication by Δ^3. The map*

$$\Delta^3\colon \pi_n\mathrm{tmf}_{(3)} \to \pi_{n+72}\mathrm{tmf}_{(3)}$$

is injective.

18. Derived functors of modular forms at 2

We would like to compute $H^{*,*}(A_{(2)}, \Gamma_{(2)})$. The same strategy as that used at the prime 3 works at the prime 2 as well, but with quite a bit more difficulty: the algebraic spectral sequence does not collapse at E_1 (though it does collapse at E_2), and at any rate the E_1-term is fairly complicated. We will only sketch the argument here.

18.1. **Some elements in $H^{*,*}(A_{(2)}, \Gamma_{(2)})$.** One can construct the following list of elements in $H^{s,t} = H^{s,t}(A_{(2)}, \Gamma_{(2)})$.

$$\begin{aligned} c_4 \in H^{0,8}, \quad c_6 \in H^{0,12}, \quad \Delta \in H^{0,24}, \\ \eta \in H^{1,2}, \quad \nu \in H^{1,4}, \quad \mu \in H^{1,6}, \\ \epsilon \in H^{2,10}, \quad \kappa \in H^{2,16}, \quad \overline{\kappa} \in H^{4,24}. \end{aligned}$$

The first three are modular forms. The second three are easily constructed: they are represented by $[s]$, $[r]$, and $[3a_1 s^2 + 6a_1 s^2 + 4s^2]$. (Alternately, they each represented by a scalar multiple of $[a_1'^k - a_1^k]$, $k = 1, 2, 3$.) ϵ can be constructed as a Massey product $\langle \nu, 2\nu, \eta \rangle$.

The elements $\eta, \nu, \mu, \epsilon, \kappa, 2\overline{\kappa}$ are in the image of the E_2-term of the Adams-Novikov spectral sequence for S^0. The elements $\eta, \nu, \epsilon, \kappa, \overline{\kappa}$ will be permanent cycles in $\pi_*\mathrm{tmf}$, representing the images of the corresponding elements in $\pi_* S^0$ (using Toda's names).

These elements satisfy some relations in $H^{*,*}$. First, the modular relation

$$c_4^3 - c_6^2 - (12)^3 \Delta = 0.$$

The orders of these elements:

$$2\eta = 2\mu = 2\epsilon = 2\kappa = 0, \qquad 4\nu = 0, \qquad 8\overline{\kappa} = 0.$$

Some standard relations involving ν:

$$\eta\nu = 0, \qquad 2\nu^2 = 0, \qquad \nu^4 = 0.$$

Relations involving ϵ and κ:

$$\eta\epsilon = \nu^3, \quad \nu\epsilon = 0, \quad \epsilon^2 = 0, \qquad \eta^2\kappa = 0, \quad \nu^2\kappa = 4\overline{\kappa}, \quad \epsilon\kappa = 0, \quad \kappa^2 = 0.$$

The elements c_4, c_6, and μ kill a lot (but not all) of the torsion:

$$\mu\nu = c_4\nu = c_6\nu = 0, \quad \mu\epsilon = c_4\epsilon = c_6\epsilon = 0, \quad \mu\kappa = c_4\kappa = c_6\kappa = 0.$$

There are relations intertwining η and μ with c_4 and c_6:

$$\mu^2 = \eta^2 c_4, \qquad \mu c_4 = \eta c_6.$$

Finally, there are subtle relations involving $\overline{\kappa}$ with μ, c_4, c_6:

$$c_4\overline{\kappa} = \eta^4\Delta, \qquad c_6\overline{\kappa} = \eta^3\mu\Delta, \qquad \mu^2\overline{\kappa} = \eta^6\Delta.$$

(Note: these are relations in the E_2-term, not in homotopy; several of them fail in homotopy, even when they make sense.)

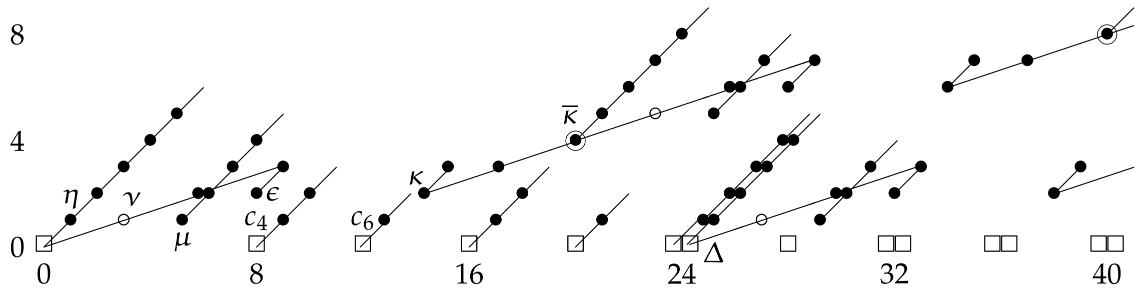
If I'm lucky, I have not left out any relations, and the following is true.

Theorem 18.2. *There is an isomorphism*

$$\mathbb{Z}_{(2)}[c_4, c_6, \Delta, \eta, \nu, \mu, \epsilon, \kappa, \overline{\kappa}]/(\sim) \to H^{*,*}(A_{(2)}, \Gamma_{(2)}),$$

where "$\sim$" denotes the above list of relations.

Note that Δ has no zero-divisors, so that $\Delta\colon H^{s,t} \to H^{s,t+24}$ is injective.



This is a diagram of $H^{s,t}(A_{(2)},\Gamma_{(2)})$. The horizontal axis is $t-s$ and the vertical axis is s; here $\square = \mathbb{Z}_{(2)}$, $\bullet = \mathbb{Z}/2$, $\circ = \mathbb{Z}/4$, and $\odot = \mathbb{Z}/8$. Short lines represent multiplication by η, and long one represent multiplication by ν. Not all η-multiplications from the 0-line are shown.

18.3. **An algebraic spectral sequence.** If $T = -x/y$ is the usual coordinate for a Weierstrass equation, then

$$[2](T) = 2T + \cdots, \qquad [2](T) \equiv a_1 T^2 + \cdots \bmod (2), \qquad [2](T) \equiv a_3 T^4 + \cdots \bmod (2, a_1).$$

Thus we will filter $(A_{(2)},\Gamma_{(2)})$ by powers of the ideal $I = (2, a_1, a_3) \subset A_{(2)}$. We let

$$(\overline{A},\overline{\Gamma}) \overset{\text{def}}{=} (A_{(2)}/I, \Gamma_{(2)}/I)$$

and, as at the prime 3, we obtain

Proposition 18.4. *There exists a spectral sequence of algebras*

$$E_1^{p,q;t} = H^p(\overline{A},\overline{\Gamma}; \mathrm{Sym}^q_{\overline{A}}(I/I^2)) \Longrightarrow H^{p,t}(A_{(2)},\Gamma_{(2)}),$$

with differentials

$$d_r\colon E_r^{p,q;t} \to E_r^{p+1,q+r;t}.$$

We have

$$\overline{A} = \mathbb{F}_2[a_2, a_4, a_6], \qquad \overline{\Gamma} = \overline{A}[r,s,t]$$

with Hopf algebroid structrue given by

$$\begin{aligned} a_2' &= a_2 + s^2 + r, & \nabla(r) &= r + r', \\ a_4' &= a_4 + r^2, & \nabla(s) &= s + s', \\ a_6' &= a_6 + a_4 r + a_2 r^2 + r^3 + t^2, & \nabla(t) &= t + t' + sr'. \end{aligned}$$

We have that $I/I^2 \approx \overline{A} \otimes_{\mathbb{F}_2} V$, where $V = \mathbb{F}_2\{A_0, A_1, A_3\}$ with A_0, A_1, A_3 representing $2, a_1, a_3$. The comodule structure $\psi\colon I/I^2 \to I/I^2 \otimes_{\overline{A}} \overline{\Gamma}$ is given by $\psi(A_i) = A_i'$ with

$$\begin{aligned} A_0' &= A_0, \\ A_1' &= A_1 + A_0 s, \\ A_3' &= A_3 + A_1 r + A_0 t. \end{aligned}$$

As at the prime 3, we consider the Hopf algebroid $(\mathbb{F}_2, C)$ induced by $\overline{A} \to \mathbb{F}_2$ sending $a_2, a_4, a_6 \mapsto 0$. We see that

$$\begin{aligned} C &= \mathbb{F}_2 \otimes_{\overline{A}} \overline{\Gamma} \otimes_{\overline{A}} \mathbb{F}_2 \\ &\approx \mathbb{F}_2[a_2, a_4, a_6, r, s, t]/(a_2, a_4, a_6, a_2', a_4', a_6') \\ &\approx \mathbb{F}_2[r, s, t]/(s^2 + r, r^2, r^3 + t^2) \\ &\approx \mathbb{F}_2[s, t]/(s^4, t^2). \end{aligned}$$

The Hopf algebra structure is given by

$$|s| = 2, \quad |t| = 6, \qquad \nabla(s) = s + s', \quad \nabla(t) = t + t' + ss'^2.$$

Proposition 18.5. *The change of rings theorem applies to the map $(\overline{A}, \overline{\Gamma}) \to (\mathbb{F}_2, C)$. In particular, there is an isomorphism*

$$H^{p,t}(\overline{A}, \overline{\Gamma}; \mathrm{Sym}^q_{\overline{A}}(I/I^2)) \approx H^{p,t}(C; \mathrm{Sym}^q_{\mathbb{F}_2} V).$$

Proof. The map $1 \otimes d^1 \colon \overline{A} = \mathbb{F}_2[a_2, a_4, a_6] \to \mathbb{F}_2 \otimes_{\overline{A}} \overline{\Gamma} = \mathbb{F}_2[r, s, t]$ sends $a_2 \mapsto s^2 + r$, $a_4 \mapsto r^2$, $a_6 \mapsto r^3 + t^2$, and is faithfully flat: the elements $1, s, s^2, s^3, t, st, s^2t, s^3t$ serve as a basis for $\mathbb{F}_2 \otimes_{\overline{A}} \overline{\Gamma}$ over $\overline{A}$. □

Remark 18.6. The group scheme $\mathrm{Spec}(C)$ is actually the group of automorphisms of the generalized elliptic curve given by $y^2 = x^3$ over $\mathbb{F}_2$ which fix the base point and which preserve the 1-form $\eta = dy/y^2$.

The graded Hopf algebra C is isomorphic to the dual of $\mathcal{A}(1)$, the subalgebra of the Steenrod algebra generated by Sq^1 and Sq^2, except that the elements of C are in "double" the gradings of $\mathcal{A}(1)$. It is convenient to calculate with the dual Hopf algebra C^* (which is the double of $\mathcal{A}(1)$), and allow it to act on C-comodules M by operations Sq^1 and Sq^2, sending $\mathrm{Sq}^i \colon M_n \to M_{n-2i}$. With this convention, the comodule structure on V is described by $\mathrm{Sq}^1(A_1) = A_0$, $\mathrm{Sq}^2(A_3) = A_1$, and all other $\mathrm{Sq}^i(A_j) = 0$.

In what follows, we assume the reader is familiar with calculations involving $\mathcal{A}(1)$.

Proposition 18.7.

$$H^{0,*}(C; \mathrm{Sym}^* V) \approx \mathbb{F}_2[A_0, A_1^4, A_3^2].$$

The elements $A_0, A_1^4, A_0^3A_3^2, A_3^4$ are representatives of the modular forms $2, c_4, c_6, \Delta$ respectively, and thus are permanent cycles in the spectral sequence.

Proof. The calculation of H^0 is straightforward. To see how modular forms are detected, we determine which power $I^q \subset A$ they lie in. Thus, for instance,

$$c_4 = \underbrace{a_1^4 + 2^4a_2^2 + (-3)2^4a_4}_{I^4} + \underbrace{2^3a_1^2a_2 + (-3)2^3a_1a_3}_{I^5};$$

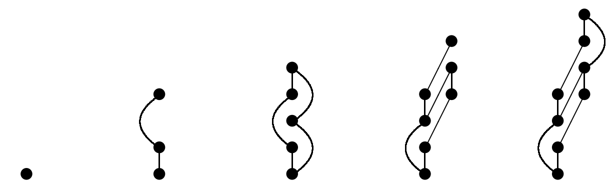
the important term here is a_1^4, since we have reduced a_2, a_4, a_6 to 0. Likewise,

$$c_6 = \underbrace{(-27)2^3a_3^2 + \cdots}_{I^5} + \underbrace{(-1)a_1^6 + 9 \cdot 2^2a_1a_3 + \cdots}_{I^6};$$

the leading term (modulo a_2, a_4, a_6) is $2^3a_3^2$. □

The above calculation means that we should expect $A_3^2, A_0A_3^2, A_0^2A_3^2$ to support differentials in the algebraic spectral sequence, since they do not correspond to modular forms.

Proposition 18.8. *As a C-comodule,* $\mathrm{Sym}^* V$ *is a direct sum of suspensions of comodules* V_k*:*



V_0 V_1 V_2 V_3 $V_n, n \geq 4.$

(The $\bullet$ *represent* $\mathbb{Z}/2$*, and the lines represent* Sq^1 *and* Sq^2*.) In particular,*

$$\mathrm{Sym}^* V \approx \mathbb{F}_2[A_1^4, A_3^2] \otimes M,$$

where

$$M \approx \bigoplus_{k \geq 0} V_k \cdot A_0^k.$$

We have that

$$H^{*,*}(C; \mathbb{F}_2) \approx \mathbb{F}_2[\eta, \nu, e_{11,3}, e_{20,4}]/(\eta\nu, \nu^3, \nu e_{11,3}, e_{11,3} + \eta^2 e_{20,4})$$

with $\eta \in H^{1,2}$, $\nu \in H^{1,4}$, and $e_{t-s,s} \in H^{s,t}$. (In general, $e_{i,j}$ will be used as a name for an element in $H^{j,i+j}(C; M)$, so that (i, j) corresponds to $(t-s, s)$.) The groups $H^{*,*}(C; V_k)$ are naturally modules over the above; we distinguish elements

$$e_{11,3} \in H^{3,14}(C; V_0), \quad e_{20,4} \in H^{4,24}(C; V_0),$$
$$e_{8,2} \in H^{2,10}(C; V_1), \quad e_{17,3} \in H^{3,20}(C; V_1),$$
$$e_{5,1} \in H^{1,6}(C; V_2), \quad e_{14,2} \in H^{2,16}(C; V_2),$$
$$e_{11,1} \in H^{1,12}(C; V_3).$$

$H^{*,*}(C; V_k) \approx \mathbb{F}_2$ if $k \geq 4$.

The multiplicative structure of $H^{*,*}(C; \mathrm{Sym}^* V)$ is fairly complicated; it has $H^{*,*}(C; M) \otimes \mathbb{F}_2[A_1^4, A_3^2]$ as an associated graded. We first note some A_0-multiplications:

$$A_0 e_{20,4} = \nu e_{17,3}, \quad A_0 e_{17,3} = \nu e_{14,2}, \quad A_0 e_{14,2} = \nu e_{11,1}, \quad A_0 e_{11,1} = \nu A_1^4.$$

Note also that $A_0^2 \nu = 0$. There is more structure involving products of the $e_{i,j}$'s. First, the structure as a $H^{*,*}(C; \mathbb{F}_2)$-module gives products with $e_{11,3}$:

$$e_{11,3}^2 = \eta^2 e_{20,4}, \quad e_{8,2} e_{11,3} = \eta^2 e_{17,3}, \quad e_{5,1} e_{11,3} = \eta^2 e_{14,2}.$$

Some more products:

$$e_{5,1}^2 = \eta^2 A_1^4, \quad e_{5,1} e_{8,2} = \eta^2 e_{11,1}, \quad e_{8,2}^2 = \eta^2 e_{14,2} + \eta^4 A_3^2.$$

The last one is the most delicate. It can be proved by considering the short exact sequence $0 \to \Lambda \to V \otimes V \to \mathrm{Sym}^2 V \to 0$, where Λ is an upside-down V: first find the exterior product of $e_{8,2}$ with itself in $H^{4,20}(C; V \otimes V)$ (there is only one non-trivial class in this grading), then compute its image in $H^*(C; \mathrm{Sym}^2 V)$.

To understand some of this structure, it useful to think about what happens when η is formally inverted. In fact,

$$\eta^{-1} H^{*,*}(C; \mathrm{Sym}^* V) \approx \mathbb{F}_2[\eta, \eta^{-1}][e_{5,1}, e_{8,2}, e_{11,3}].$$

Proposition 18.9. *The algebraic spectral sequence admits non-trivial differentials $d^1\colon E_1^{p,q;t} \to E_1^{p+1,q+1;t}$. It collapses at E_2, although there are non-trivial multiplicative extensions.*

To prove this, we first note that $A_0, A_1^4, A_0^3 A_3^2, A_3^4, \eta, \nu, e_{5,1}, e_{8,2}$ are permanent cycles; we have already discussed the first four, and $\eta, \nu, e_{5,1}$ clearly support no differentials. It is not too hard to construct a representative for $e_{8,2}$ (or observe that it comes from the representative for ϵ in the Adams-Novikov E_2-term for the sphere.)

We first establish the differential

$$d_1\colon e_{11,3} \to \eta^2 e_{8,2}$$

(for instance, this is the only way to force $\eta^2 \epsilon = 0$ at the E_2-term). This implies

$$d_1\colon e_{14,2} \to \eta^2 e_{11,1},$$
$$d_1\colon e_{17,3} \to e_{8,2}^2 = \eta^2 e_{14,2} + \eta^4 A_3^2;$$

the first of these is because $\eta^2 e_{14,2} = e_{11,3} e_{5,1} \to \eta^2 e_{8,1} e_{5,1} = \eta^4 e_{11,1}$, and the second because $\eta^2 e_{17,3} = e_{11,3} e_{8,2} \to \eta^2 e_{8,2}^2$. These imply

$$d_1\colon A_3^2 \to e_{11,1},$$

because $e_{8,2}^2 \to 0$ since it is a target, and thus $0 = d_1(e_{8,2}^2) = d_1(\eta^2 e_{14,2} + \eta^4 A_3^2) = \eta^4 e_{11,1} + \eta^4 d_1(A_3^2)$. Thus

$$d_1\colon A_0 A_3^2 \to A_0 e_{11,1} = \nu A_1^4,$$
$$d_1\colon A_0^2 A_3^2 \to A_0^2 e_{11,1} = \nu A_0 A_1^4.$$

All other d_1's are forced by these. We note in particular that the elements

$$\eta^2 A_3^2 + e_{14,2} \in H^{2,16}, \qquad e_{20,4} \in H^{4,24}$$

survive; these represent κ and $\overline{\kappa}$. One can check that $\kappa^2 = 0$:

$$d_1\colon e_{17,3} A_3^2 \to \eta^4 A_3^4 + e_{20,4} A_1^4 = (\eta^2 A_3^2 + e_{14,2})^2 = \kappa^2.$$

This also happens to show that $\overline{\kappa} c_4 = \eta^4 \Delta$.

Several exotic multiplicative extensions can be found at E_2, including

$$\nu^3 = \eta e_{8,2}, \quad \eta(A_0^3 A_3^2) = e_{5,1} A_1^4.$$

Once these are established, one checks that as an algebra E_2 is generated by permanent cycles, and so we are done (more or less).

19. Homotopy groups of tmf at 2

We now have to compute a spectral sequence

$$E_2^{s,t} = H^{s,t}(A_{(2)}, \Gamma_{(2)}) \Longrightarrow \pi_{t-s}\mathrm{tmf}_{(2)}.$$

I will not give a complete exposition of this. The reader should take a look at [HM], especially the full page chart, which is a picture of the spectral sequence starting at the E_4-term, with some of the material on lines $s = 0, 1, 2$ omitted.

Observe first that the classes $\eta, \nu, \epsilon, \kappa, \overline{\kappa}$ come from homotopy classes in $\pi_* S^0$, and so are permanent cycles.

The first possible non-trivial differential is at E_3, given by

$$d_3 \colon \mu \to \eta^4,$$

which forces

$$d_3 \colon c_4 \to 0,$$
$$d_3 \colon c_6 \to \eta^3 c_4,$$
$$d_3 \colon \mu\overline{\kappa} \to \eta^3\overline{\kappa},$$
$$d_3 \colon \Delta \to 0.$$

At E_4 one can start to make sense of multiplicative extensions of the type $4\nu = \eta^3$.

The key differential is

$$d_5 \colon \Delta \to \nu\overline{\kappa}.$$

This can be derived in several ways, for instance using

Lemma 19.1. *In $\pi_{26}S^0$ we have $\nu^2\overline{\kappa} \in \langle \eta_4\sigma, \eta, 2\iota\rangle$.*

Proof. See [HM]. □

Since $\sigma \in \pi_7 S^0$ maps to 0 in tmf we have that $\nu^2\overline{\kappa}$ must lie in the indeterminacy of the bracket, which cannot possibly happen in this dimension unless $\nu^2\overline{\kappa} = 0$ in homotopy, which can only happen if d_5 does as alleged.

This implies

$$d_5 \colon \Delta^2 \to 2\overline{\kappa}\Delta,$$
$$d_7 \colon \Delta^4 \to 4\overline{\kappa}\Delta,$$
$$d_5, d_7 \colon \Delta^8 \to 0.$$

Exercise 19.2 (Mahowald). Compute the remaining differentials in the spectral sequence using elementary Toda bracket considerations.

The last differential turns out to be $d_{23} \colon \eta\Delta^5 \to \overline{\kappa}^6$. Thus

Theorem 19.3. *The spectral sequence for $\pi_*\mathrm{tmf}_{(2)}$ collapses at the E_{24}-term, and $E_{24}^{s,*} = E_\infty^{s,*} = 0$ for $s \geq 24$. The edge homomorphism fits in an exact sequence*

$$0 \to K \to \pi_t\mathrm{tmf}_{(2)} \to \mathcal{M}_{t/2} \otimes \mathbb{Z}_{(2)} \to C \to 0,$$

where

$$C \approx \mathbb{Z}/2[c_4, \Delta]\{c_6\} \oplus \left(\mathbb{Z}/2\{\Delta^4\} \oplus \mathbb{Z}/4\{\Delta^2, \Delta^6\} \oplus \mathbb{Z}/8\{\Delta, \Delta^3, \Delta^5, \Delta^7\}\right) \otimes \mathbb{Z}[\Delta^8].$$

The torsion in $\pi_* \mathrm{tmf}_{(2)}$ *has order at most* 8. *The map*

$$\Delta^8 \colon \pi_n \mathrm{tmf}_{(2)} \to \pi_{n+192} \mathrm{tmf}_{(2)}$$

is injective.

20. Calculation of $MU_* \mathrm{tmf}$

Recall that $Y = (\Omega U(4))^\gamma$ is a Thom spectrum, and thus has a natural map to MUP.

Proposition 20.1. *The spectrum* $MUP \wedge \mathrm{tmf}$ *admits the structure of an elliptic spectrum* $(MUP \wedge \mathrm{tmf}, C, \phi)$ *in such a way that the natural map* $\mathrm{tmf} \wedge Y \approx Y \wedge \mathrm{tmf} \to MUP \wedge \mathrm{tmf}$ *induces a morphism of elliptic spectra. In particular,* $\pi_{odd} MUP \wedge \mathrm{tmf} = 0$, *and*

$$\pi_0 MUP \wedge \mathrm{tmf} = MUP_0 \mathrm{tmf} = \mathbb{Z}[a_1, a_2, a_3, a_4, a_6, e_n, n \geq 4].$$

The generalized elliptic curve $C = C_{\underline{a}}$ *is the Weierstrass equation in* x, y *with coefficients* $a_1, \dots, a_6$. *The natural map* $MUP \to MUP \wedge \mathrm{tmf}$ *corresponds to the coordinate* T_{MUP} *on* $\widehat{C}$ *with the property that*

$$-x/y = T + \sum_{n \geq 5} e_{n-1} T^n.$$

To prove this, we will apply MUP_* to the cosimplicial spectrum $[s] \mapsto \mathrm{tmf} \wedge Y^{(s+1)}$. Recall that if E is an even periodic spectrum, then $E_{\mathrm{odd}} MUP = 0$, and there is a natural isomorphism

$$\hom_{\mathrm{alg}(E_0)}(E_0 MUP, R) \approx \mathrm{Coord}_{G_E \otimes R}.$$

If T_E is a choice of coordinate for E, and T_{MUP} is the canonical coordinate for MUP, we have

$$MUP_0 E = E_0 MUP \approx E_0[b_0^{\pm 1}, b_n, n \geq 1] \qquad \text{where} \qquad T_{MUP} = \sum_{n \geq 1} b_{n-1} T_E^n,$$

and the classes $b_n \in \pi_0 E \wedge MUP$ are in the image of the map $E_0 T_{MUP} \colon E_0 \mathbb{CP}^\infty \to E_0 MUP$. Alternately, we can write

$$MUP_0 E = E_0 MUP \approx E_0[m_0^{\pm 1}, m_n, n \geq 1] \qquad \text{where} \qquad T_E = \sum_{n \geq 1} m_{n-1} T_{MUP}^n,$$

and the classes $m_n \in \pi_0 MUP \wedge E$ are in the image of the map $MUP_0 T_E \colon MUP_0 \mathbb{CP}^\infty \to MUP_0 E$.

Recall that

$$\hom_{\mathrm{Rings}}(\pi_0(\mathrm{tmf} \wedge Y), R) = \{\text{Weierstrass equations } F_{\underline{a}} \text{ over } R\}.$$

$$\hom_{\mathrm{Rings}}(MUP_0 \mathrm{tmf} \wedge Y, R) = \{(F_{\underline{a}}, T_{MUP})\},$$

where $F_{\underline{a}}$ is a Weierstrass equation over R, and T_{MUP} is a coordinate for the curve associated to this equation.

The complex $[s] \mapsto \Delta^s \stackrel{\text{def}}{=} MUP_0(\mathrm{tmf} \wedge Y^{(s+1)})$ is the cobar complex of a Hopf algebroid, representing the groupoid whose objects are as above, and whose morphisms are coordinate transformations relating the two Weierstrass equations.

We can write

$$\Delta^0 = MUP_0(\mathrm{tmf} \wedge Y) \approx \mathbb{Z}[a_1, a_2, a_3, a_4, a_6, m_0^{\pm 1}, m_n, n \geq 1]$$

where $\pi_0(\mathrm{tmf} \wedge Y) \to MUP_0(\mathrm{tmf} \wedge Y)$ is the evident map, and

$$T_C = \sum_{n \geq 1} m_{n-1} T_{MUP}^n$$

where $T_C = -x/y$ is the standard coordinate on the generalized elliptic curve $F_{\underline{a}} = 0$.

Let $R = \mathbb{Z}[a_1, \ldots, a_6, e_n, n \geq 4]$ and define a map $\Delta^0 \to R$ by sending $a_i \mapsto a_i$, $m_0 \mapsto 1$, $m_1, m_2, m_3 \mapsto 0$, and $m_n \mapsto e_n$ for $n \geq 4$.

Proposition 20.2. *The induced Hopf algebroid is (R, R), and the change of rings theorem applies to $(\Delta^0, \Delta^1) \to (R, R)$.*

Proof. The first part is simply the statement that for any coordinate T on a generalized elliptic curve there is a unique pair of Weierstrass parameters (x, y) such that $-x/y = T + O(T^5)$. The second part is also follows from this, since there is an equivalence of groupoids. □

Remark 20.3. There are maps

$$\mathrm{tmf} \wedge Y \xrightarrow{f} MUP \wedge \mathrm{tmf} \wedge Y \xrightarrow{g} \mathrm{tmf} \wedge Y.$$

The map f is induced by $S^0 \to MUP$, and the map g is induced by $Y \to MUP$ and by multiplication in MUP. In homotopy, g corresponds to $\Delta^0 \to R$ as described above. The map f corresponds to a map $R \to \Delta^0$, which is one of the maps involved in the equivalence of groupoids. We will need to be able to compute this map: it sends $a_i \mapsto a_i'$ where we use

$$\begin{aligned}
\lambda &= m_0, \\
s &= m_1/m_0^2, \\
r &= (m_0 m_2 - m_1^2)/m_0^4, \\
t &= -(3m_0 m_1 m_2 + a_1 m_0^3 m_2 - a_1 m_0^2 m_1^2 - m_0^2 m_3 - 2m_1^3)/m_0^6,
\end{aligned}$$

and it sends e_n to m_n/m_0 modulo elements decomposable in the a_i and the m_n for $n \geq 1$. (It would be better if I had calculated these in terms of the b_n's.)

We get a similar description for $MU_*\mathrm{tmf}$, which we will need later.

Proposition 20.4. *There is an isomorphism*

$$MU_*\mathrm{tmf} \approx \mathbb{Z}[a_1, a_2, a_3, a_4, a_6, e_n, n \geq 4], \qquad |a_i| = 2i, |e_n| = 2n.$$

The map $MU_*\mathrm{tmf} \to MUP_*\mathrm{tmf}$ sends $a_i \mapsto \eta^i a_i$ and $e_i \mapsto \eta^i e_i$.

Finally, we calculate the map $MU_*\mathrm{tmf} \to MU_*H\mathbb{Z}$ induced by $\mathrm{tmf} \to H\mathbb{Z}$. Recall that

$$MU_*H\mathbb{Z} \approx \mathbb{Z}[t_n, n \geq 1], \qquad |t_n| = 2n,$$

where

$$T_{MU} = \sum t_{n-1} T_{HZ}^n.$$

Proposition 20.5. *The map $MU_*\mathrm{tmf} \to MU_*H\mathbb{Z}$ sends*

$$a_1 \mapsto -2t_1, \qquad a_2 \mapsto 2t_1^2 - 3t_2, \qquad a_3 \mapsto -2t_1^3 + 4t_1t_2 - 2t_3,$$
$$a_4 \mapsto t_1^4 - 2t_1^2t_2 - 2t_1t_3 + 3t_2^2, \qquad a_6 \mapsto t_1^4t_2 - t_1^2t_2^2 - t_2^3 - t_3^2 - 2t_1^3t_3 + 4t_1t_2t_3,$$

and hence

$$b_2 \mapsto 12t_1^2 - 12t_2, \qquad b_4 \mapsto 6t_1^4 - 12t_1^2t_2 + 6t_2^2, \qquad b_6 \mapsto 4t_1^6 - 12t_1^4t_2 + 12t_1^2t_2^2 - 4t_2^3,$$

and $c_4, c_6 \mapsto 0$. Furthermore,

$$e_n \mapsto t_n \mod (t_1, t_2, t_3).$$

Proof. This is computed by considering the composite $MU \wedge \mathrm{tmf} \to MU \wedge MU \wedge \mathrm{tmf} \to MU \wedge H\mathbb{Z}$. □

21. Computation of $H_*(\mathrm{tmf}; \mathbb{F}_p)$

To carry this out, we use the universal coefficient spectral sequence associated to $H\mathbb{F}_p$ viewed as a module spectrum over MU; see [Ada73]. This has the form

$$E_2^{s,t} = \mathrm{Tor}_s^{MU_*}(MU_tX, \mathbb{F}_p) \Longrightarrow H_{s+t}(X; \mathbb{F}_p).$$

Differentials take the form $d_r\colon E_r^{s,t} \to E_r^{s-r,t+r-1}$. Since $(MU_*X)_{(p)} \approx BP_*X \otimes R_*$ with $R_* \approx \mathbb{Z}[y_n, n \neq p^j - 1]$, we have

$$E_2^{s,t} \approx \mathrm{Tor}_s^{BP_*}(BP_tX, \mathbb{F}_p).$$

It is simpler to use BP rather than MU; however, I need the freedom to be able to do calculations using MU, so I will frame most statements in this form.

We will use the following proposition.

Proposition 21.1. *Let X be a ring spectrum, and let $\phi\colon MU_* \to MU_*X$ denote any complex orientation of MU_*X. Suppose that there is an $n \geq -1$ such that*

1. *$p = \phi(v_0), \phi(v_1), \ldots, \phi(v_n)$ is a regular sequence in MU_*X, and*
2. *$\phi(v_k) \equiv 0 \bmod (p, \phi(v_1), \ldots, \phi(v_n))$.*

Then

$$\begin{aligned} E_2^{*,*}(H_*X) &\approx \big(MU_*X \otimes_{MU_*} \mathbb{F}_p\big) \otimes E(\overline{\tau}_j, j > n) \\ &\approx \big(BP_*X \otimes_{BP_*} \mathbb{F}_p\big) \otimes E(\overline{\tau}_j, j > n) \\ &\approx BP_*X/(p, v_1, \ldots, v_n) \otimes E(\overline{\tau}_j, j > n), \end{aligned}$$

where $\overline{\tau}_j \in \mathrm{Tor}_{1,2(p^j-1)}$. In particular, the universal coefficient spectral sequence collapses at E_2.

Proof. Clearly, if the hypotheses hold for any complex orientation ϕ, they hold for all such, so we may as well assume that ϕ is the map induced by the inclusion $BP \to BP \wedge X$. Let $I = (p, \phi(v_1), \ldots, \phi(v_n))$. Then

$$\begin{aligned} \mathrm{Tor}_*^{BP_*}(BP_*X, \mathbb{F}_p) &\approx \mathrm{Tor}_*^{BP_*/(p,\ldots,v_n)}(BP_*X/I, \mathbb{F}_p) \\ &\approx BP_*X/I \otimes \mathrm{Tor}_*^{BP_*/(p,\ldots,v_n)}(\mathbb{F}_p, \mathbb{F}_p) \\ &\approx (BP_*X/I) \otimes E(\overline{\tau}_j, j > n), \end{aligned}$$

using flat base change twice: first, because $p, \dots, \phi(v_n)$ is a regular sequence in BP_*X, so that BP_*X is flat as a module over $\mathbb{Z}_{(p)}[v_1, \dots, v_n]$; and second, because $BP_*/(p, \dots, v_n) \to BP_*X/I$ factors through $\mathbb{F}_p$. $\square$

Example 21.2. Let's carry this out for $X = H = H\mathbb{F}_p$. We have

$$H_*MU \approx MU_*H \approx MU_*[t_n, n \geq 1] \approx MU_*[m_n, n \geq 1], \qquad |t_n| = |m_n| = 2n.$$

The t_n's are in the image of $T_{MU} \wedge H \colon \Sigma^{-2}\mathbb{CP}^\infty \wedge H \to MU \wedge H$, while the m_n's are in the image of $MU \wedge T_H \colon MU \wedge \Sigma^{-2}\mathbb{CP}^\infty \to MU \wedge H$.

Clearly, using the addendum to the lemma, we have that

$$E_2 \approx (MU_*H \otimes_{MU_*} \mathbb{F}_p) \otimes E(\overline{\tau}_n, n \geq 0).$$

Since

$$MU_*H \otimes_{MU_*} \mathbb{F}_p \approx BP_*H \otimes_{BP_*} \mathbb{F}_p \approx BP_*H \approx \mathbb{F}_p[t_j, j \geq 1], \quad |t_j| = 2(p^j - 1),$$

we see that we get the correct answer (up to associated graded).

One can show, using the definition of the classes $m_n \in MU_{2n}H$, that the natural map $MU_*H \to H_*H$ actually sends $m_{p^j-1} \mapsto \xi_j$ and $m_n \mapsto 0$ if $n \neq p^j - 1$, where ξ_j is the "usual" generator in $H_{2(p^j-1)}H$ (for p odd); or $\xi_j = \zeta_j^2$ for the "usual" $\zeta_j \in H_{2^j-1}H$ (for $p = 2$). Therefore $t_{p^j-1} \mapsto \overline{\xi_j}$ and $t_n \mapsto 0$ if $n \neq p^j - 1$. Similarly, the classes $\overline{\tau_j}$ correspond (up to filtration) to the antipode of the "usual" generator $\tau_j \in H_{2(p^j-1)+1}H$ (for p odd); or $\tau_j = \zeta_j$ (for $p = 2$). (We will need this fact later on.)

We need to verify hypotheses (1) and (2) when $X = \mathrm{tmf}$. First, we need

Proposition 21.3. *Let $C_{\underline{a}}$ be the curve associated to a Weierstrass equation over a ring R, and let T be any coordinate for $\widehat{C}_{\underline{a}}$. Let $v_n \in R$ such that $[p](T) = v_n T^{p^n} + \cdots \mod (p, v_1, \dots, v_{n-1})$ as usual. Then*

$$v_n \equiv 0 \mod (p, v_1, v_2) \quad \textit{for all } n,$$

or what is the same thing, $v_n \equiv 0 \mod (p, v_1, \dots, v_{n-1})$ for $n \geq 3$.

Lemma 21.4. *The proposition holds whenever $R/(p, v_1, v_2)$ is an integral domain.*

Proof. It is enough to embed $R/(p, v_1, v_2) \subset k$ in an algebraically closed field and to prove the lemma there. In this case the lemma follows from the classification of generalized elliptic curves over such fields: such a curve must be isomorphic to $y^2 = x^3$, whose formal group law is $\mathbb{G}_a$. $\square$

Proof of (21.3). Our proof breaks up into cases, depending on the prime. In each case we pass to a universal example of a generalized elliptic curve (at the given prime), and prove the statement of the proposition for it. There really ought to be a better proof than this, but I can't find one.

Case $p = 2$: Every Weierstrass equation arises from the universal one over $R = \mathbb{Z}[a_1, \dots, a_6]$ by base change. Since $v_1 = a_1$ and $v_2 = a_3$, we have $R/(2, v_1, v_2) \approx \mathbb{F}_2[a_2, a_4, a_6]$, which is an integral domain, and so the proposition follows from the lemma.

Case $p = 3$: After localizing at 3 every Weierstrass equation is isomorphic to one induced by base change from $R = \mathbb{Z}_{(3)}[b_2, b_4, b_6]$. Since $v_1 = b_2$ and $v_2 = b_4^2$ we have

$R/(3,v_1,v_2) \approx \mathbb{F}_3[b_4,b_6]/(b_4^2)$. Now v_n is a homogeneous element in this ring: if we grade by $|b_n| = 2n$ then $|v_n| = 2(3^n-1)$. A straightforward calculation shows that $R/(3,v_1,v_2)$ is concentrated in degrees $\equiv 0,8 \bmod 12$, while $|v_n| = 2(3^n-1) \equiv 4 \bmod 12$, and thus $v_n = 0$ in $R/(3,v_1,v_2)$.

Case $p \geq 5$: After localizing at $p \geq 5$ every Weierstrass equation is isomorphic to one induced by base change from $R = \mathbb{Z}_{(p)}[c_4,c_6]$. Again, v_n is a homogeneous element of degree $2(p^n-1)$, assuming $|c_n| = 2n$. One uses the fact that the ideal $(v_1,v_2) \subset \mathbb{F}_p[c_4,c_6]$ is generated by a regular sequence. Then some calculations with poincaré series show that $\mathbb{F}_p[c_4,c_6]/(v_1,v_2)$ vanishes in degrees greater that $2(p^2+p-12)$, which is certainly less that $2(p^n-1)$ for $n \geq 3$. □

Now we verify the hypothesis (2) of the lemma for each prime, using a choice of coordinate for the formal group of the elliptic curve.

Case $p=2$: Since $v_1 = a_1$ and $v_2 = a_3$, it is clear that $2, v_1, v_2$ is a regular sequence in $MU_*\mathrm{tmf}$.

Case $p=3$: We have that $v_1 = b_2$ and $v_2 = 2b_4^2$, so that $(3,v_1,v_2)$ is a regular sequence in $\mathbb{Z}_{(3)}[b_2,b_4,b_6]$. Now $b_i \equiv \pm a_i$ modulo 3 and decomposables for $i = 2,4,6$, and hence $(3,v_1,v_2)$ is a regular sequence in $MU_*\mathrm{tmf}$.

Case $p \geq 5$: We need to use the fact that p, v_1, v_2 is a regular sequence in $\mathbb{Z}_{(p)}[c_4,c_6]$. Now $c_i \equiv a_i$ up to scalar and modulo p and decomposables for $i = 4,6$, and hence (p,v_1,v_2) is a regular sequence in $MU_*\mathrm{tmf}$.

Theorem 21.5. *The mod p ordinary homology of* tmf *is an commutive ring in the category of comodules over the dual steenrod algebra $\mathcal{A}_*$. It is described by*

(1) $H_*(\mathrm{tmf};\mathbb{F}_2) \approx \mathcal{B}_*$, *where*
$$\mathcal{B}_* \approx \mathbb{F}_2[\overline{\zeta}_1^8, \overline{\zeta}_2^4, \overline{\zeta}_3^2, \overline{\zeta}_n, n \geq 4] \subset \mathcal{A}_*,$$
where $\overline{\zeta}_n \in \mathcal{A}_{2^n-1}$ denotes the antipode of the usual generator ζ_n.

(2) $H_*(\mathrm{tmf};\mathbb{F}_3) \approx \mathbb{F}_3[b_4]/(b_4^2) \otimes \mathcal{B}_*$, *where*
$$\mathcal{B}_* \approx \mathbb{F}_3[\overline{\xi}_1^3, \overline{\xi}_n, n \geq 2] \otimes E(\overline{\tau}_n, n \geq 3) \subset \mathcal{A}_*.$$
There is a non-trivial extension
$$0 \to \Sigma^8\mathcal{B}_* \to H_*(\mathrm{tmf};\mathbb{F}_3) \to \mathcal{B}_* \to 0$$
of comodules.

(3) $H_*(\mathrm{tmf};\mathbb{F}_p) \approx \mathbb{F}_p[c_4,c_6]/(v_1,v_2) \otimes \mathcal{B}_*$ *for $p \geq 5$, where*
$$\mathcal{B}_* \approx \mathbb{F}_p[\overline{\xi}_n, n \geq 1] \otimes E(\overline{\tau}_n, n \geq 3) \subset \mathcal{A}_*.$$
As a comodule $H_*(\mathrm{tmf};\mathbb{F}_p)$ has an associated graded which is isomorphic to a sum of suspensions of $\mathcal{B}_*$.

Proof. We apply the lemma in each of the three cases, and we calculate the map induced by $\mathrm{tmf} \to H\mathbb{F}_p$ on the level of E_2-terms. Thus we need to consider the sequence
$$MU_* \xrightarrow{f} MU_*\mathrm{tmf} \xrightarrow{g} MU_*H\mathbb{F}_p,$$
or its BP-analogue. Note that the composite gf is exactly the map described in (21.2).

Case $p = 2$: First we compute g (modulo 2), using the calculations of the previous section:

$$a_1, a_3 \mapsto 0, \qquad a_2 \mapsto t_2, \qquad a_4 \mapsto t_1^4 + t_2^2, \qquad a_6 \mapsto t_3^2 + t_1^4 t_2 + t_1^2 t_2^2 + t_2^3$$

and

$$e_n \mapsto t_n + (\text{decomposables}).$$

In particular, the kernel of g is the ideal $(2, a_1, a_3)$. Therefore, if we pass to the quotient by $(2, v_1, v_2)$:

$$MU_*/(2, v_1, v_2) \xrightarrow{\overline{f}} MU_*\mathrm{tmf}/(2, a_1, a_3) \xrightarrow{\overline{g}} MU_* H\mathbb{F}_2$$

we see that $\overline{g}$ is a monomorphism. Therefore $BP_*\mathrm{tmf}/(p, v_1, v_2) \to BP_* H\mathbb{F}_2$ is also a monomorphism, and hence using the lemma and comparison of spectral sequences we see that $H_*(\mathrm{tmf}; \mathbb{F}_2) \to \mathcal{A}_*$ is a monomorpism.

The image of $MU_*\mathrm{tmf} \to MU_* H\mathbb{F}_2 \to H_* H\mathbb{F}_2$ is $\mathbb{F}_2[\overline{\xi}_1^4, \overline{\xi}_2^2, \overline{\xi}_j, j \geq 3]$, since $t_{2^j-1} \mapsto \overline{\xi}_j$ while $t_n \mapsto 0$ if $n \neq 2^j - 1$, and also $a_4 \mapsto \overline{\xi}_1^4$ and $a_6 \mapsto \overline{\xi}_2^2$.

Case $p = 3$: We compute g modulo 3:

$$a_1 \mapsto t_1, \quad a_2 \mapsto 2t_1^2, \quad a_3 \mapsto t_1^3 + t_1 t_2 + t_3,$$
$$a_4 \mapsto t_1^4 + t_1^2 t_2 + t_1 t_3, \quad a_6 \mapsto t_1^4 t_2 + 2t_1^2 t_2^2 + 2t_2^3 + 2t_3^2 + t_1^3 t_3,$$

while

$$e_n \mapsto t_n + (\text{decomposables}).$$

This implies that

$$b_2, b_4 \mapsto 0, \qquad b_6 \mapsto t_1^6 + 2t_2^3,$$

where $b_{2i} = b_{2i}(\underline{a})$. In particular, the kernel of g is $(3, b_2, b_4)$. Passing to quotients

$$MU_*/(3, v_1, v_2) \xrightarrow{\overline{f}} MU_*\mathrm{tmf}/(3, b_2, b_4^2) \xrightarrow{\overline{g}} MU_* H\mathbb{F}_3.$$

The map $MU_*\mathrm{tmf}/(3, b_2, b_4) \to MU_* H\mathbb{F}_3$ is thus injective, and hence so is

$$BP_*\mathrm{tmf}/(3, b_2, b_4) \to BP_* H\mathbb{F}_3 \approx \mathbb{F}_3[\xi_j, j \geq 1].$$

Thus one computes that that the image is $\mathcal{B}_*$.

Case $p \geq 5$: One checks that g sends

$$a_1 \mapsto -2t_1 + (\text{decomposables}), \quad a_2 \mapsto -3t_2 + (\text{decomposables}),$$
$$a_3 \mapsto -2t_3 + (\text{decomposables}), \quad e_n \mapsto t_n + (\text{decomposables}), n \geq 4.$$

This implies that $MU_*\mathrm{tmf} \to MU_* H\mathbb{F}_p$ is surjective, and thus implies the image of $H_*\mathrm{tmf} \to H_* H\mathbb{F}_p$. We have that $\overline{g}$ sends $c_4, c_6 \mapsto 0$, and so the kernel of $\overline{g}$ is (p, c_4, c_6). □

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