$\pi_*(ko)$ and some of $\pi_*(tmf)$ at the prime 2

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Acknowledgements

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I must also thank Mark Mahowald, though I have never met him, for telling Zhouli Xu that calculating $\pi_*(tmf)$ was a rite of passage for computational homotopy theorists.

I thank Peter May for suggesting the calculation of $\pi_*(ko)$ using the Adams spectral sequence, and his suggestions for me to try a longer calculation. I also thank Peter and Zhouli for emphasizing that my other thesis project on height 3 formal group laws was too complicated to be finished on time, and encouraged me to put together this expository project as my thesis instead.

I thank Eric Peterson and Mike Hill for helping me with an alternative proof of how $H_*(ko) \simeq (A//A(1))_*$. The proof I ended up using is an expanded version of an outline of John Rognes in his unpublished notes.

1 Pep talk

This thesis is meant to fill a gap in the literature, for those starting out on their first larger calculations (ones which are too large for normal paper), and to emphasize some crucial differential finding tricks often left unpublished, or abandoned shivering in a footnote or side comment. This exposition will be repetitive, there will often be multiple proofs of the same things – the aim is to teach methods used by active topologists by showing them in action.

We introduce the May spectral sequence via two examples of its application, toward understanding $\mathrm{Ext}_{A(n)}(\mathbb{F}_2, \mathbb{F}_2)$, a crucial figure in computing the stable homotopy groups of spheres. We do the examples of $A(1)$ and $A(2)$, the latter only in a range.

Further, we hope to illuminate the philosophy behind the May spectral sequence, and show the fun of presenting elements in the May spectral sequence as Massey products.

The May spectral sequence is the systematic destruction of regularity. We find any relations, propagate them, and remove them from the picture. We will be like Michelangelo: starting from the very regular block of marble, and chipping away carefully until our beast emerges. All that can support a differential will support a differential.

It is crucial to emphasize that the calculations we exposite are not difficult, they are only complicated.

This thesis will assume that the reader is familiar with the basics of spectral sequences, and Massey products. For a discussion of the definition of Massey products and their indeterminancy, the author recommends O'Neill [4].

2 Set up of the Exercise

The May spectral sequence converges to the E_2 page of the Adams spectral sequence, so let's talk about the Adams spectral sequence. We work with mod-2 coefficients. The standard Adams spectral sequence says:

$$Ext^{*,*}_{A_*}(H_*(X), H_*(Y)) \implies [X, Y]^{\wedge}_2$$

Where here Ext means comodule Ext. We plug in $X = \mathbb{S}^0$ and $Y = M$, where M is some spectrum, to get:

$$Ext^{*,*}_{A_*}(H_*(\mathbb{S}^0), H_*(M)) \implies \pi_*(M)^{\wedge}_2$$

Note that $H_*(\mathbb{S}^0) \simeq \mathbb{F}_2$. We plug in ko for M and use the isomorphism (which we prove in section 7) $H_*(ko) \simeq A//A(1)$.

$$Ext^{*,*}_{A_*}(\mathbb{F}_2, A//A(1)) \implies \pi_*(ko)^{\wedge}_2$$

Using the change of rings theorem, we slip in:

$$Ext^{*,*}_{A(1)_*}(\mathbb{F}_2, \mathbb{F}_2) \implies \pi_*(ko)^{\wedge}_2$$

3 Why use the May Spectral Sequence?

The calculation of the homotopy ring of connective KO using the Adams or May spectral sequence begins with the homotopy *groups* of ko, and reveals the ring structure and Toda bracket presentations of the generators.

Note that we need that

$$H^*(ko) \simeq A//A(1)$$

where $A(1)$ is the submodule of the Steenrod algebra generated by Sq^1, Sq^2, and $A//A(1)$ represents the Hopf-algebra quotient $A \otimes_{A(1)} \mathbb{F}_2$.

The only way to prove this isomorphism is by using the cofibration $\Sigma ko \xrightarrow{\eta \wedge ko} ko \xrightarrow{c} ku \xrightarrow{\partial} \Sigma^2 ko$, which requires you to know the homotopy groups of ko. There is no way around it! Nevertheless, calculating $\mathrm{Ext}_{A(1)}(\mathbb{F}_2, \mathbb{F}_2)$ is a good exercise, because it immediately collapses to $\pi_*(ko)$.

There are two ways we can go about calculating $\mathrm{Ext}_{A(1)_*}(\mathbb{F}_2, \mathbb{F}_2)$. First, we could try to take an $A(1)_*$ resolution of $\mathbb{F}_2$, and then use this resolution to write down the Adams spectral sequence E_2 page. This is very doable, we find a periodicity in the resolution very quickly, it appears as the cell structure of RP^n with a Joker on the end (this may be seen in Bob Bruner's artwork on resolutions of $A(1)$ [3]. The disadvantage of this hands on approach is the lack of ability to generalize. If we want to compute $\pi_*(tmf)$, for which a key ingredient is $H\mathbb{F}_2^*(tmf) = A//A(2)$, resolution with copies of $A(2)$ by hand is practically impossible and at the very least hellish.

We often like to break up complex computations into bite sized pieces. Instead of calculating an $A(2)$ resolution of $\mathbb{F}_2$ to get directly to the Adams spectral sequence, we can use the May spectral sequence.

The May spectral sequence is an entirely algebraic method of calculation, even the differentials are completely algebraic.

$$E_1^{*,*,*} = \mathbb{F}_2[h_{ij} | i \geq 1, j \geq 0] \Rightarrow Ext_A^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$$

Here we care only about $A(1)$, so our E_1 page will mercifully be a subalgebra of $\mathbb{F}_2[h_{ij} | i \geq 1, j \geq 0]$.

Truly, we use this paper as a "first hit," an exposure to ideas that will allow the reader to calculate $\pi_*(tmf)_2^\wedge$. We will start with the May spectral sequence to stem 13, and then compute the only Adams differential in this range.

4 Calculation of $Ext_{A(1)}(\mathbb{F}_2, \mathbb{F}_2)$ using May Spectral Sequence

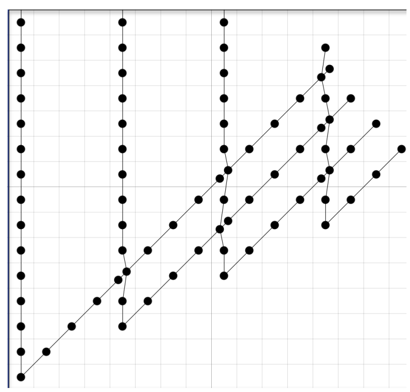
We begin given that the E_1 page of the May spectral sequence is:

$$E_1^{*,*,*} = \mathbb{F}_2[h_{10}, h_{11}, h_{20}]$$

where h_{ij} corresponds to $\xi_i^{2^j}$ in the cobar complex, and the degree in $(t-s, s, u)$ coordinates is $|h_{ij}| = (2^j(2^i - 1) - 1, 1, 1)$. We are further given that the only d_1 differential is:

$$d_1(h_{20}) = h_{10}h_{11}$$

As short hand, we write $h_{in} =: h_n$. We propagate the E_1 page, and draw in the d_1's. We see immediately that our d_1 differential has both h_0 and h_1 ladders. The E_1 page is a bit awful to behold in its full glory without witnessing it being drawn in real time, so we do not picture it here. After propagating the differential, we flip to the E_2 page:

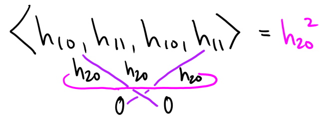


Let's look for d_2's. We needn't look long. The only one which could possibly exist is on the only remaining generator of May degree 2, h_{20}^2.

Theorem 1. We have a Massey product presentation of $h_{20}^2 =: b_{20}$.

$$h_{20}^2 = \langle h_{10}, h_{11}, h_{10}, h_{11} \rangle$$

Proof.



We see 0 on the bottom row because $h_{10}h_{20} + h_{10}h_{20} = 0 \bmod 2$, and $h_{11}h_{20} + h_{11}h_{20} = 0 \bmod 2$ as well. So, 0 kills them both! Yippee! □

Theorem 2. $d_2(h_{20}^2) = h_1^3 + h_0^2 h_2$

Proof. We may take the higher differential of h_{20}^2 by treating it as a 4-fold Massey product. This is May's original proof although he did not include it in his thesis.

One of the main tricks we will use to calculate differentials is using the higher Leibnitz formula. Let's talk about the 3-fold version first.

$$d_?(\langle a_1, a_2, a_3 \rangle) = a_1 c + b a_3$$

where $a_1 a_2 = b$ and $a_2 a_3 = c$.

We are ready for the 4-fold version:

$$d_?(\langle a, b, c, d \rangle) = x \cdot d + y \cdot a$$

where $\langle a, b, c \rangle$ is detected by $x \in$ the E_r page of the May SS, and $\langle b, c, d \rangle$ is detected by $y \in$ the E_r page of the May SS.

In our case, the 3-fold Massey products that appear on the Adams spectral sequence obey the rule $\langle h_i, x, h_i \rangle = h_{i+1} x$:

$$\langle h_0, h_1, h_0 \rangle = h_1^2$$
$$\langle h_1, h_0, h_1 \rangle = h_0 h_2$$

And thus, $d(\langle h_{10}, h_{11}, h_{10}, h_{11} \rangle) = h_1^3 + h_0^2 h_2$, note that they are in the same degree. $\square$

Remark. We use the derivation of this differential as a discussion of the 3 different ways to shed light on differentials: the higher Liebnitz rule, the cobar complex, and Nakamura's lemma. They are parts of a whole. The first way is the most intuitive: we use the Massey product decomposition of an element, this is garunteed to work because there is a Massey product presentation of any element on the May spectral sequence.

May understood the determination of what the differential of a Massey product should be, and his proof of this higher Leibnitz rule uses hypothesis: the vanishing of the elements that could obstruct the proof (the condition is given in Theorem 4.5 pg. 565) as Corollary 4.6 [**?**]. Yet, it works far more generally.

After he had the hueristic for what we know the differential must be, May skirted the technicality in the general theory of differentials of Massey products by embedding in the bar construction. This makes the differential Massey product method perfectly rigorous.

The third way is to use Nakamura's lemma, which is of the form

$$Sq^{i+1}(d_r(a)) = d_{2r}(Sq^i(a))$$

(page 301, Prop 4.3 [5]). These three methods reinforce each other.

There is an indeterminacy subtlety in Nakamura's lemma, but since we have 3 weights in the May spectral sequence, the indeterminacy is negligible.

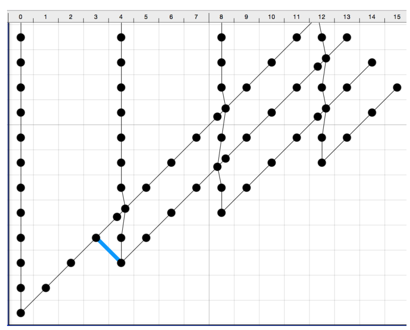
We include an alternative proof of $d_2(h_{20}^2) = h_1^3 + h_0^2 h_2$ using Nakamura's lemma.

Proof.

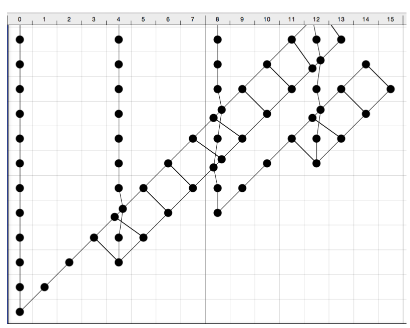
$$\begin{aligned} d_2(Sq^0(h_{20}) &= Sq^1(d_1(h_{20})) \\ &= Sq^1(h_0 h_1) \\ &= Sq^1(h_0)Sq^0(h_1) + Sq^0(h_0)Sq^1(h_1) \\ &= h_1 h_1^2 + h_0^2 h_2 \end{aligned}$$

This uses that $Sq^0(h_{ij}) = h_{ij}^2$, and $Sq^1(h_{ij}) = h_{i,j+1}$ (page 7, [2]). $\square$

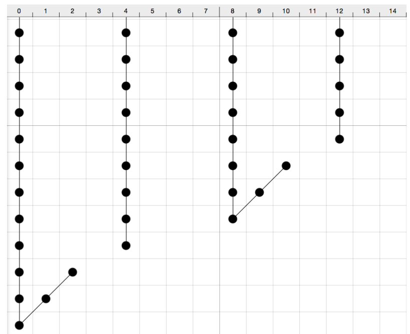
So, we have proven this differential:



and using the h_{11} ladder, we propagate the differential.



Flipping to page 3, we see that there cannot be any more differentials, so the spectral sequence collapses on the 3rd page:

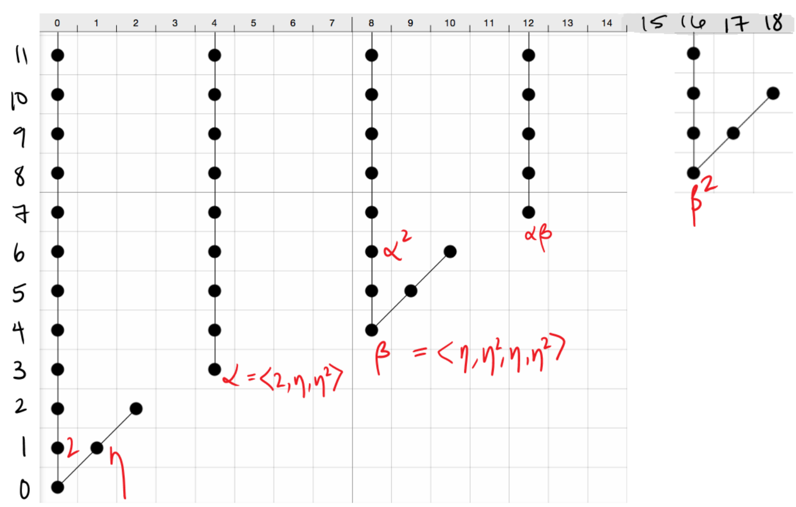


Since the E_∞ page of the May SS is the E_2 page of the Adams SS up to hidden extensions, we check for hidden extensions. This is a highly merciful task because there are no dots in higher filtrations! Therefore, there cannot possibly be hidden extensions.

Now that we have the Adams E_2 page, we look to see if there are any Adams differentials, for grading reasons we see that there are none, so the Adams SS collapses on the 2nd page. So, reading off the resultant homotopy groups:

$$\mathbb{Z}, C_2, C_2, 0, \mathbb{Z}, 0, 0, 0, \mathbb{Z}$$

We are thinking of $2 := h_0$, $\eta := h_1$, $\alpha := h_0 b_{20}$, and $\beta := b_{20}^2$. These are now elements living in the homotopy group of ko, where we are thinking of h_0 as "multiplication by 2 in $\pi_*(ko)$." Thus, just by looking at the below picture, we have $\mathbb{Z}_2^\wedge[\eta, \alpha, \beta]/(\eta^3, 2\eta, \alpha^2 - 4\beta, \alpha\eta)$.



Wow! We really did just get that ring structure with very little work. Now, you'll notice we've also written in the Toda bracket presentations of some of the elements, because they are fun! Let's see how we derived them. We started with the Massey product presentations of the corresponding elements:

Note that when we take the Massey product, we only consider the differentials of highest degree.

Now, how do we get from the Massey product to the Toda bracket? We apply the May convergence theorem (also known as Moss's Theorem), which we state in the 3-fold setting:

Theorem (May's convergence theorem) [Theorem 4.1 , page 561[1]]. Given:

- a, b, c are permanent cycles in May E_r page, and $a \implies \alpha$, $b \implies \beta$, $c \implies \gamma$, where $\alpha, \beta, \gamma \in \pi_*$.
- $\langle a, b, c \rangle$ is defined in May E_r
- $\langle \alpha, \beta, \gamma \rangle$ is defined in Adams E_2 (which is May E_∞ up to extensions)
- No "crossing differentials".

$\implies \exists$ one element in $\langle a, b, c \rangle$ which is a permanent cycle, and it converges to an element in $\langle \alpha, \beta, \gamma \rangle$ Adams E_2.

5 Calculation of $Ext_{A(2)}(\mathbb{F}_2, \mathbb{F}_2)$ to stem 12 using May Spectral Sequence

As adventurers, we must humble ourselves – the case of $A(1)$ was the first little step. When I became confused in the *tmf* case I went back to the *ko* case to get more familiar with the basic mechanics before carrying on with the game.

We next turn to the case of topological modular forms, whose intricacies will from the very outset require of us much greater attention and care. We go out far enough to see something interesting.

Just like the $A(1)$ case, we use the Adam's grading $(t-s, s, u)$, so we grade $|h_{ij}| := (2^j(2^i - 1)) - 1, 1, 1)$. Note that $h_n := h_{1n}$. We take the convention of increasing the May weight by r when we apply the differential r.

We take as given that

$$E_1^{*,*,*} = \mathbb{F}_2[h_0, h_1, h_2, h_{20}, h_{21}, h_{30}]$$

Now, though you may balk at tables of gradings, I emphasize that it is extremely helpful to keep them near your bedside as you compute.

Our first page of generators thus looks like this:

	$t-s$	s	u
h_0	0	1	1
h_1	1	1	1
h_{20}	2	1	1
h_2	3	1	1
h_{21}	5	1	1
h_{30}	6	1	1

We are given as d_1:

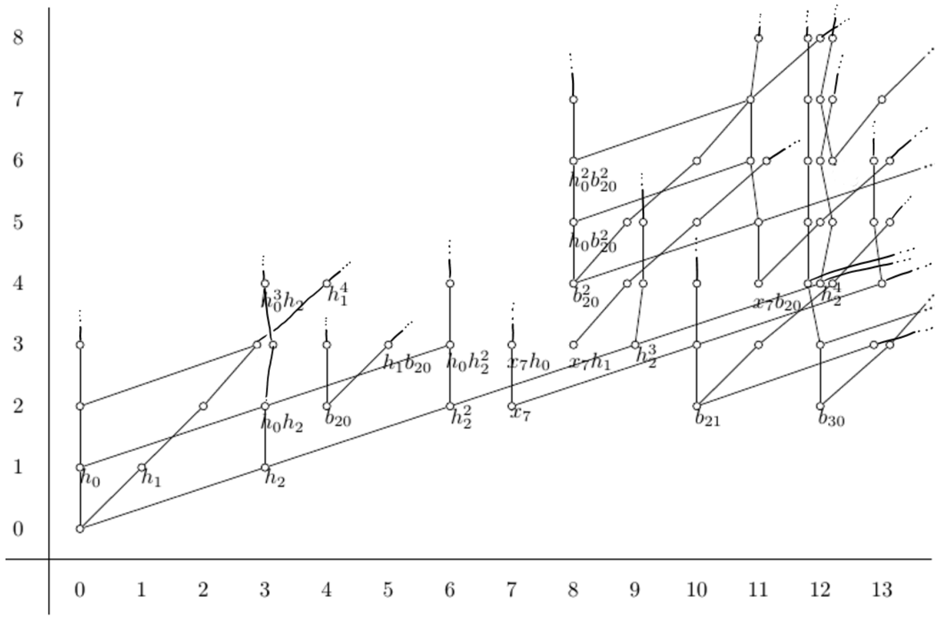
$$\begin{aligned} d_1(h_n) &= 0 \\ d_1(h_{20}) &= h_0 h_1 \\ d_1(h_{21}) &= h_1 h_2 \\ d_1(h_{30}) &= h_0 h_{21} + h_2 h_{20} \end{aligned}$$

We use a computer program, *Ext Chart* by Eric Peterson, to propagate the d_1 differentials via the Leibnitz rule, because it is a task perfect for a computer.

Looking over his program output, we see that $h_1 h_{30} + h_{21} h_{20}$ survives. Indeed, we check and see that the differentials of the individual components ($h_1 h_{30}$ and $h_{21} h_{20}$) both kill $h_{20} h_1 h_{21} + h_1^2 h_{21}$. Since they both kill the same thing, and we are working in the magical mod 2 world, they add and stay as a cycle. We name $x_7 := h_1 h_{30} + h_{21} h_{20}$ because it is in stem 7.

We take the perspective that if c kills $a + b$, then $a = b$. So via d_1 differentials, we get that $x_7 h_0 = h_2 b_{20}$ (thus $h_2 b_{20}^2 = b_{20} x_7 h_0$) and $h_0 h_{21} = h_2 h_{20}$.

When we flip the page to E_2, we see:

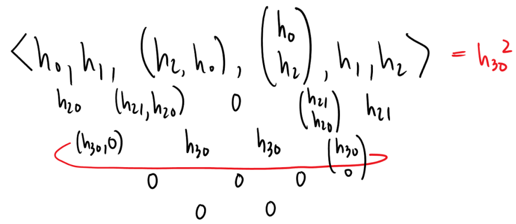


	$t-s$	s	u
h_0	0	1	1
h_1	1	1	1
h_2	3	1	1
b_{20}	4	2	2
b_{21}	10	2	2
b_{30}	12	2	2
x_7	7	2	2

The first thing we do on the E_2 page is uncover the Massey product presentations of our generators, their true form, because this is how we computed the d_2 differentials in the case of $A(1)$.

$$x_7 = \langle h_2, h_1, h_0, h_1 \rangle$$
$$b_{20} = \langle h_0, h_1, h_0, h_1 \rangle$$
$$b_{21} = \langle h_1, h_2, h_1, h_2 \rangle$$
$$b_{30} = \langle h_0, h_1, \begin{pmatrix} h_2 & h_0 \end{pmatrix}, \begin{pmatrix} h_0 \\ h_2 \end{pmatrix}, h_1, h_2 \rangle$$

The Massey product for b_{30} must be a matrix Massey product because h_0 and h_2 pair nontrivially so we cannot do the usual square trick. Instead, we must pair to get $h_0h_2 + h_2h_0$ such that they cancel (we are living in mod 2 land).



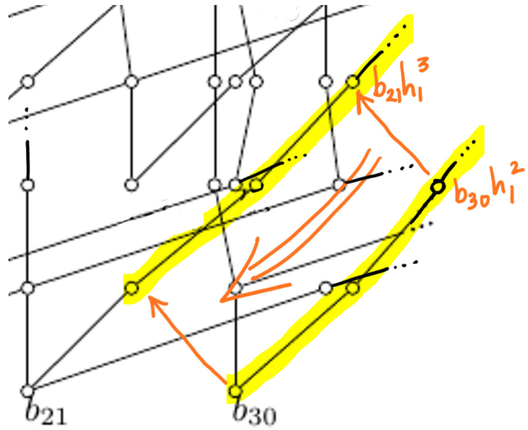
We use these Massey product representations to derive the d_2's on the generators, using the higher Leibnitz rule as we did before:

$$d_2(x_7) = h_0h_2^2$$
$$d_2(b_{20}) = h_1^3 + h_0^2h_2$$
$$d_2(b_{21}) = h_2^3$$

To derive the differential of b_{30}, we use that by the definition of x_7, $x_7^2 = h_1^2b_{30} + b_{20}b_{21}$:

$$\begin{aligned} 0 = d_2(x_7^2) &:= h_1^2 d_2(b_{30}) + b_{20}d_2(b_{21}) + b_{21}d_2(b_{20}) \\ \implies h_1^2 d_2(b_{30}) &= b_{20}(h_2^3) + b_{21}(h_1^3 + h_0^2h_2) \\ &= b_{21}h_1^3 \end{aligned}$$

The other two elements in the sum do not appear in the E_2 page. When we look at the picture below, we may using "h_1^2 division" to get that $d_2(b_{30}) = h_1b_{21}$. That is, $d_2(b_{30}) = 0 \implies d_2(h_1^2b_3) = 0$ which is false.



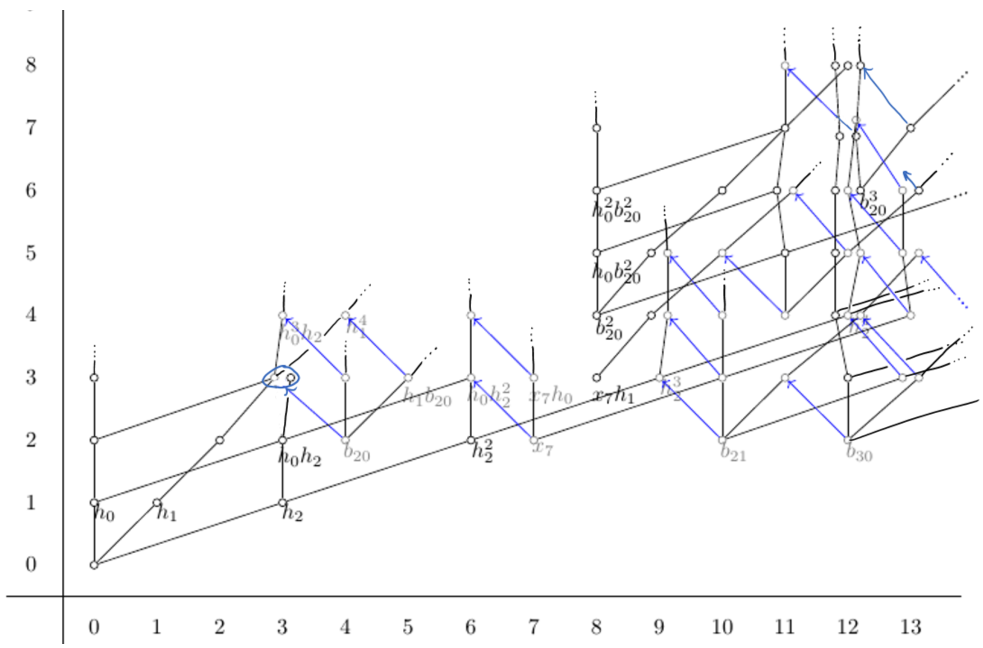
And thus:

$$d_2(b_{30}) = h_1 b_{21}$$

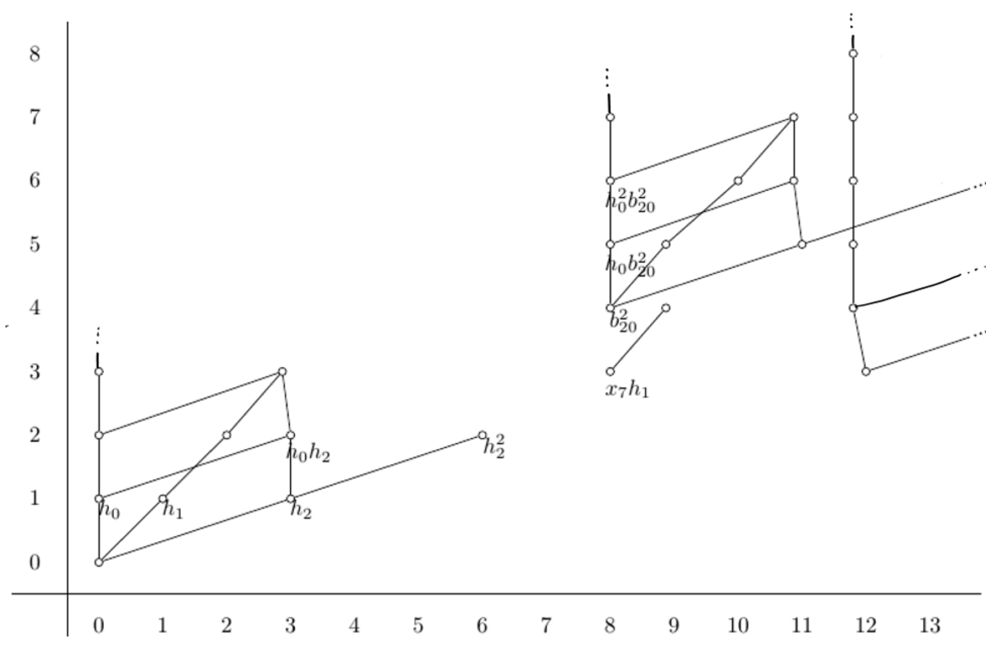
After all of this work, we have our differentials. We propagate these differentials using the Leibniz rule, for example, lets look at $b_{20}x_7$, the lone dot in $(11, 4)$:

$$\begin{aligned} d_2(b_{20}x_7) &= x_7 h_1^3 + h_0^2 h_2 x_7 + b_{20} h_2^2 h_0 \\ &= x_7 h_1^3 \end{aligned}$$

because when we multiply out the other two using that $x_7 := h_1 h_{30} + h_{21} h_{20}$, we see that they died via d_1's.



0 1 2 3 4 5 6 7 8
0 1 2 3 4 5 6 7 8 9 10 11 12 13
h_0
h_1
h_2
h_0h_2
b_{20}
h_1b_{20}
$h_0h_2^2$
x_7h_0
h_2^2
x_7
x_7h_1
b_{20}^2
$h_0b_{20}^2$
$h_0^2b_{20}^2$
b_{20}^3
b_{21}
b_{30}



0 1 2 3 4 5 6 7 8
0 1 2 3 4 5 6 7 8 9 10 11 12 13
h_0
h_1
h_2
h_0h_2
h_2^2
b_{20}^2
$h_0b_{20}^2$
$h_0^2b_{20}^2$
x_7h_1

Taking a head count, we see that all squares of the generators of E_2 are generators on the E_3 page. We also find that we have some special generators in view. We have in sight the families $w_1 := b_{20}^2$, $\alpha := h_1 b_{30}$ (the bottom of the tower in stem 12) and $c_0 = x_7 h_1$. Further, there are no higher differentials because the gradings of possible differential pairs don't have the appropriate/compatible May gradings. Thus, this range of the spectral sequence collapses on E_3.

Why did we stop at 13? Besides the fact that illustrating large spectral sequences is a pain, the range we illustrated is just enough to see an Adams differential! We address this in the next section.

Just for fun, let's uncover some of the surviving elements' true forms! They all have Massey product presentations.

We see that $b_{20}^2 = \langle h_1, h_1^2, h_1, h_1^2 \rangle$ (since b_{20} kills h_1^3), but can we find a representation of b_{20}^2 using the Massey product for b_{20}?

$$\begin{aligned} w_1 = b_{20}^2 &= b_{20}\langle h_0, h_1, h_0, h_1 \rangle \\ &= \langle \begin{pmatrix} h_1 & h_2 \end{pmatrix}, \begin{pmatrix} h_1^2 & h_0^2 \\ h_0^2 & h_1^2 \end{pmatrix}, \begin{pmatrix} h_1 & h_2 \\ h_2 & h_1 \end{pmatrix}, \begin{pmatrix} h_1^2 \\ h_0^2 \end{pmatrix} \rangle \end{aligned}$$

Now, let's do c_0:

$$\begin{aligned} c_0 &= x_7 h_1 \\ &= \langle h_2, h_1, h_0, h_1 \rangle h_1 \\ &= \langle \langle h_1, h_2, h_1 \rangle, h_0, h_1 \rangle \\ &= \langle h_2^2, h_0, h_1 \rangle = \langle h_2, h_1, h_0, h_1^2 \rangle \end{aligned}$$

Remark. Note that the symmetric Massey products are nonzero, this is because we are considering the Massey products as living on the E_∞ page.

Immediately to the right of the range shown (in stem 14) is the element $d_0 := x_7^2 = \langle c_0, h_0, h_, h_2 \rangle$.

Remark. Unrelated to this discussion, it is perhaps useful to note that Sq^0 commutes with Massey products. That is:

$$Sq^0(\langle a, b, c \rangle) = \langle Sq^0(a), Sq^0(b), Sq^0(c) \rangle$$

This is because, if $d_?(e) = ab$, then by Nakamura's lemma, $Sq^0(d_?(e)) = d_?(Sq^0(e)) = Sq^0(ab) = Sq^0(a)Sq^0(b)$. Thus, we may intimately apply Nakamura's lemma inside of Massey products. It is important to note that the Sq^0 case is special, we cannot apply this in the case of Sq^i because $Sq^i(ab) \neq Sq^i(a)Sq^i(b)$.

Remark. If we wish to avoid Massey products, one thing humans do better than computers, we can entirely rely on a computer program and Nakamura's lemma to compute the May spectral sequence converging to $Ext_{A(2)}(\mathbb{F}_2, \mathbb{F}_2)$ *in this range.* Here is how we avoid all of the work (in a way that presently only works for small ranges due to the limitations of existing computer programs and the fact that we need more tricks past the 20 or 30th stem):

Start on the E_1 page, take the d_1's of the generators $h_0, h_1, h_2, h_{20}, h_{21}, h_{30}$. Applying Nakamura's lemma ($Sq^1(d_1(a)) = d_2(Sq^0(a))$ up to some indeterminacy), we get the d_2's of $h_0^2, h_1^2, h_2^2, h_{20}^2, h_{21}^2, h_{30}^2$ on the E_2 page.

After propagating these differentials via the Leibnitz rule, we wish to deduce the differentials on the rest of the generators, in our case and our range, the only generator in E_2 that is not a square of a generator on E_1 is x_7.

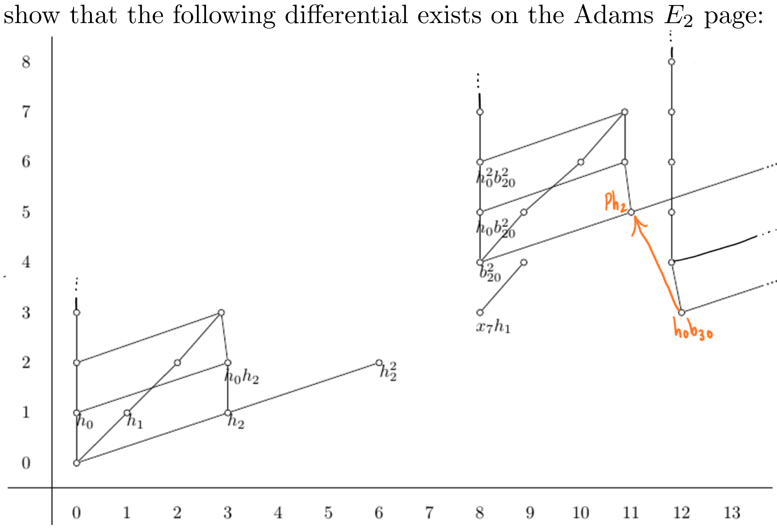
We derived earlier the equality $x_7 h_0 = h_2 b_{20}$, and via the Leibnitz rule, we get: $d_2(h_2 b_{20}) = h_1^3 h_2 + h_0^2 h_2^2$, where the first term is zero since $h_1^3 h_2$ is dead. So, we have: $d_2(h_0 x_7) = h_0^2 h_2^2$.

Then, we apply "h_0 divisibility": that is, $d_2(x_7) = 0 \implies d_2(h_0 x_7) = 0$ which is false. We get this implication because the differential is h_0 linear.

There are no more generators in this range, and no more differentials on future pages because the May weights don't match up, so we are done.

6 On the cell structure of tmf

Now we are out to stem 13, we can see some interesting phenomena. We will show that the following differential exists on the Adams E_2 page:



This is done using the 4-spectral sequence method – an amped up commutative diagram argument. We will show that the differential

$$d_{12} : 8[12] \to Ph_2[0]$$

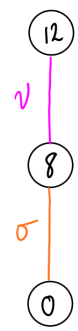
on the Atiyah-Hirzebruch Spectral Sequence (AHSS) implies

$$d_2 : h_0^3[12] \to Ph_2[0]$$

on the Adams Spectral sequence (ASS).

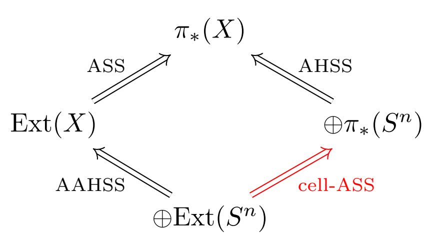
Step 1. This method begins with the creation of a finite cell complex X, a skeleton of tmf. Then, any Adams differential we find in X in this range, we may push forward to tmf.

First of all, we can show what the cell diagram of tmf is up to dimension 12.

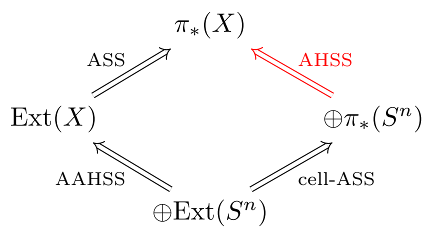


This can be done (assuming $HF_{2*}(tmf) \simeq A//A(2)$) by looking at $A//A(2)$ up to degree 12 and seeing that it is the above, thinking of σ as Sq^8, ν as Sq^4, and so on.

Step 2. Now we begin the argument by going up one side: the cell-ASS collapses on the E_2 page (there are no differentials up to stem 13). The cell-ASS is the cell-wise ASS - different cells have their own ASS, and this is the direct sum of these ASS.

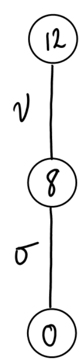


Step 3. We exploit a relationship between 3-fold Toda brackets and differentials in the AHSS of certain 3-cell complexes developed in Section 6 of the 61-sphere paper [6].



In particular, we use Lemma 6.1 to deduce our differential. Our 3-cell complex satisfies the 2 hypotheses of the lemma: $8\nu = 0$ in π_3 because $\pi_3(X) = \mathbb{Z}/8$. The second hypothesis is also satisfied: $8\pi_{11}(X) = 0 \subseteq \sigma\pi_6(X) = 0$. This is because $\pi_{11} = \mathbb{Z}/8$, so $8\pi_{11} = 0$. Further, π_6 is generated by $h_2^2 = \nu^2$, thus $\sigma\pi_6 = \sigma\nu^2 = 0$ (though not pictured, it is indeed the case that $\sigma\nu^2 = 0$).

Applying Lemma 6.1 to the cell complex:



We get:

$$d_{12}(8[12]) = \langle 8, \nu, \sigma \rangle [0]$$

Theorem 2. $\langle 8, \nu, \sigma \rangle = \{Ph_2\}$ up to indeterminacy.

Proof.

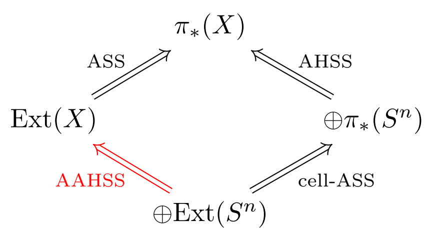
$$\begin{aligned}\langle 8, \nu, \sigma\rangle &= \langle 2, 4\nu, \sigma\rangle \\ &= \langle 2, \eta^3, \sigma\rangle \\ &= \langle 2, \eta, \eta^2\sigma\rangle \\ &= \langle 2, \eta, \nu^3 + \eta\epsilon\rangle \\ &= \langle 2, \eta, \nu^3\rangle + \langle 2, \eta, \eta\epsilon\rangle \\ &= \langle 2, \eta, \nu\rangle\nu^2 + \langle 2, \eta, \eta\epsilon\rangle \\ &= 0 + Ph_2.\end{aligned}$$

The last bracket's well-definedness follows from the May convergence theorem on the following Massey product $\langle h_0, h_1, h_1c_0\rangle = Ph_2$. This Massey product is well defined because x_7b_{20} (in 11, 4) kills $h_1^2c_0$ in the May SS.

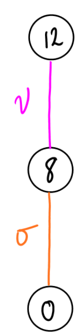
Note that $\nu^3 = \eta^2\sigma + \eta\epsilon$ is a relation in the homotopy groups of spheres.

□

Step 4. On the other side, we do the algebraic AHSS. This will simply give us the name of the element we are attempting to kill. The AAHSS is "coning off pieces" process, i.e., going from shifted copies of Ext(S) to Ext(X).

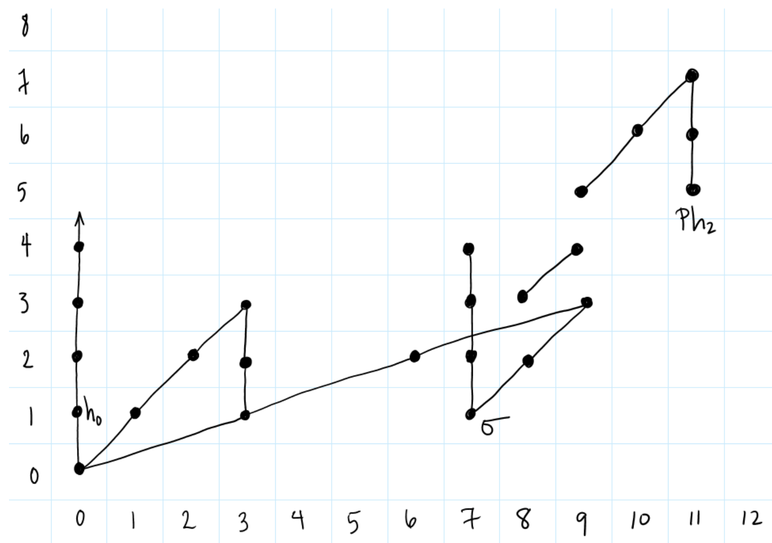


We see that X:

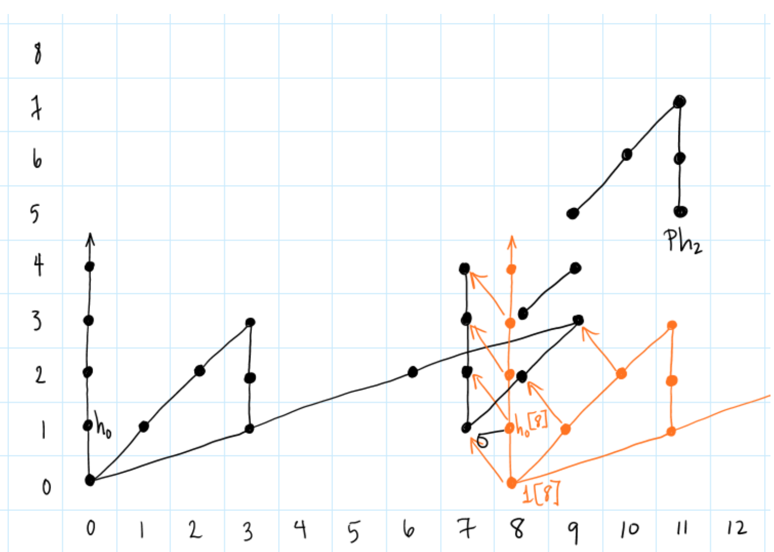


matches the cell structure of tmf quite explicitly by coning off the fellows σ and ν sucessively.

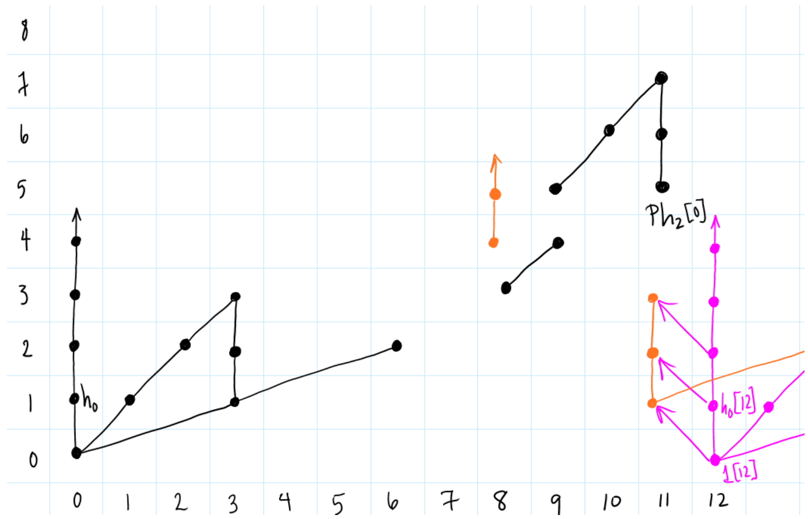
Here's the first 12 stems of the stable homotopy groups of spheres:

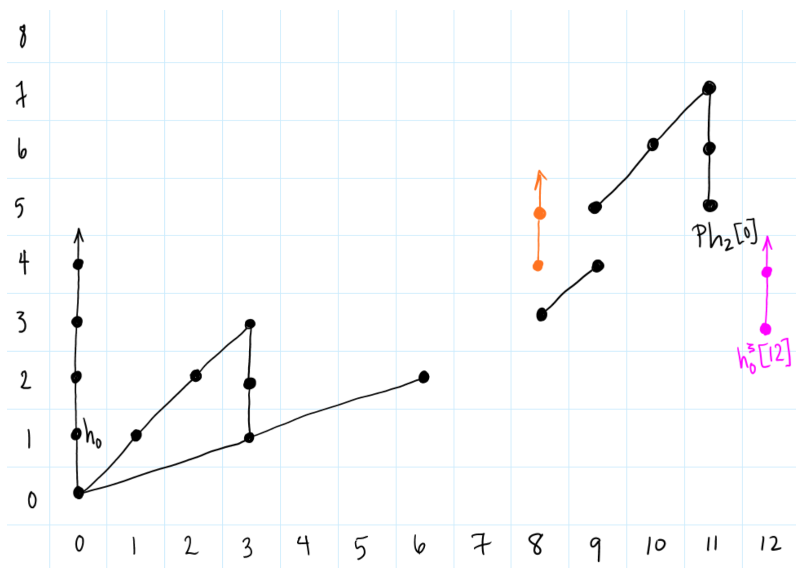


We cone off σ, and by this we mean that we place a shifted copy of the stable stems such that the d_1 on the shifted copy of 1 kills the generator σ. We use 1[8] to denote the copy of 1 coming from the cell in dimension 8.



We cone off $\nu\sigma$, here ν is h_2.

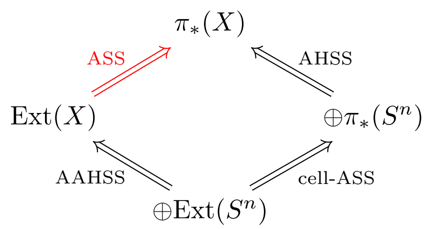




This construction using the cell structure gives us the name of the source and the target of the differential we are attempting to deduce: $h_0^3[12] \to Ph_2[0]$.

Step 5. After we complete the AAHSS, we see that $Ph_2[0]$ is alive and well in Ext(X). However, we know that $Ph_2[0]$ does not survive in $\pi_*(X)$

because of the d_{12} killing it in the AHSS. So, something must kill $Ph_2[0]$ in the Adams spectral sequence.



We use a filtration argument. Note $Ph_2[0]$ is in stem 11, filtration 5, and we see in stem 12, there is only the h_0 tower supported on h_0b_{30} (i.e. $h_0^3[12]$). So, there is no other option for the source element which kills $Ph_2[0]$ – it must be $h_0^3[12]$. Thus,

$$d_2 : h_0^3[12] \to Ph_2[0]$$

exists on the Adams spectral sequence for X, and we push it forward to exist on the Adams spectral sequence for tmf. Tada! Our first Adam's differential!

7 Proof that $A//A(1) \simeq H_*(ko; \mathbb{F}_2)$

This proof is an expansion of a proof in an unpublished paper of John Rognes, which was obtained by Dominic Culver.

Let's orient ourselves before we dive in. We will assume the following two things:

1. The cofiber sequence $\Sigma ko \xrightarrow{\eta\wedge ko} ko \xrightarrow{c} ku \xrightarrow{\partial} \Sigma^2 ko$. We define $d := c\partial$.
2. $H^*(ku; \mathbb{F}_2) \simeq A//E(1)$, where $E(1)$ is the ideal generated by Sq^1 and $[Sq^1, Sq^2] =: Q_1$.

Our proof will consist of the following steps:

1. Lemma 1: Show im $d^* \simeq A(Sq^2)$.
2. Lemma 2: Show $H^*ko \simeq A//E_1/(\text{im } d^*)$

Remark. We can think of this cofiber sequence as coming from the sequence $S^1 \xrightarrow{\eta} S^0 \to C\eta \to S^2$ by smashing with ko. Where η is the Hopf map, and the map from ko to ku is the induced map of complexification of bundles.

Lemma 1. The composite map $c\partial : ku \to \Sigma^2 ku$ induces right multiplication by Sq^2 in cohomology. This composite map is more honestly presented as $\Sigma^2 c \circ \partial$. For it is $ku \xrightarrow{\partial} \Sigma^2 ko$ composed with $\Sigma^2 ko \xrightarrow{\Sigma^2 c} \Sigma^2 ku$.

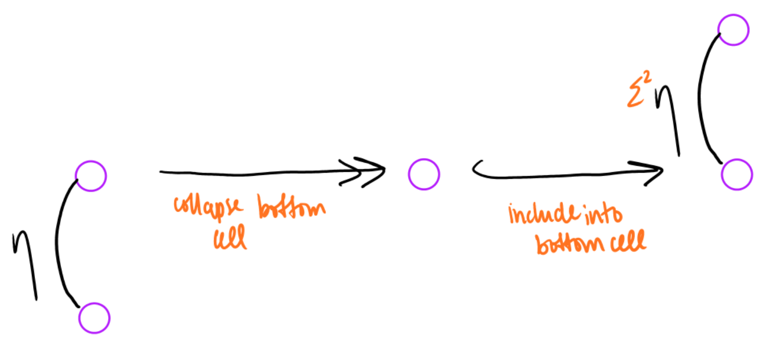
Proof. We look at the map from $C\eta \to ku$, which is the "smash with ko" map. This is nonzero.

$$H^2(\Sigma^2 C\eta) \xleftarrow{(c\partial)^*} H^*(\Sigma^2 ku) \simeq H^2(\Sigma^2 ko) \otimes H^2(\Sigma^2 C\eta)$$

The induced map on cohomology must also be nonzero, as the map behaves like a projection onto one factor $(c\partial)^* : 1 \otimes x \mapsto x$. Thus the generator of $H^2(\Sigma^2 ku)$, which we call $\Sigma^2 1$, pulls back to the generator of $H^2(\Sigma^2 C\eta)$.

Now, we examine the generator $\Sigma^2 1$ of $H^2(\Sigma^2 C\eta)$ as we pull it back to $H^2(C\eta)$. We pull it back over the map

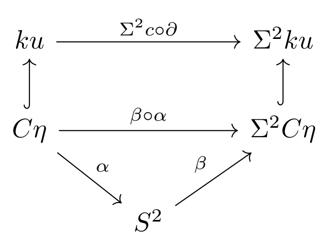
$$C\eta \to S^2 \to \Sigma^2 C\eta$$



We see that the generator $\Sigma^2 1$ pulls back over this map

$$C\eta \xrightarrow{\alpha} S^2 \xrightarrow{\beta} \Sigma^2 C\eta$$

to the nonzero class in degree 2, the only such nonzero class in $H^2(C\eta)$ is Sq^2.



Since $C\eta \hookrightarrow ku$ is an injection, the map $\Sigma^2 c \circ \partial = d$ acts by sending $d^*(\Sigma^2 1) = Sq^2$. So, $im(d^*) \simeq (Sq^2)$.

□

Next, we wish to show that $A//A(1)$ is contained in $H^*(ko; \mathbb{F}_2)$. Finally, we will show that $H^*(ko; \mathbb{F}_2) \simeq A//A(1)$ using an induction argument on degrees.

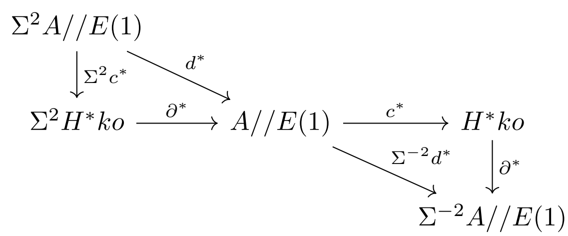
Lemma 2. $A//A(1)$ is isomorphic to $H^*(ko)$ as an A-module.

Step 1. $A//A(1)$ injects into $H^*(ko)$ as an A-module.

Proof. Examine the exact sequence induced on cohomology by η, c, ∂:

$$\Sigma^2 H^*(ko) \xrightarrow{\partial^*} H^*(ku) \xrightarrow{c^*} H^*(ko) \xrightarrow{\eta^*} \Sigma H^*(ko)$$

We are further interested in the map $d = \Sigma^2 c \circ \partial : ku \to \Sigma^2 ku$, so we substitute $H^* ku \simeq A//E(1)$ and add in the map d^*:



Since A is a free $A(1)$-module, the map $d^* : \Sigma^2 A//E(1) \to A//E(1)$ – which by Lemma 1 sends $1 \mapsto Sq^2(1)$ – is induced up from the $A(1)$-module homomorphism $d' : \Sigma^2 A(1)//E(1) \to A(1)//E(1)$ which is also right multiplication by Sq^2.

Let's examine $A(1)//E(1)$ by writing it out in full form, that is,

$$A(Sq^1, Sq^2)//A(Sq^1, Sq^2 Sq^1 - Sq^1 Sq^2) = A(Sq^2)$$

Now, note that this module has only 2 nonzero elements, 1 and Sq^2. Let us convince ourselves of this fact: Since $E(1)$ acts freely on $A(1)$ from the right, $A(1)$ has dim 8, $E(1)$ has dim 4, so $A(1)//E(1)$ has dim 2.

Applying the map $d' : \Sigma^2 A(1)//E(1) \to A(1)//E(1)$, that is, $\Sigma^2\{1, Sq^2\} \to \{1, Sq^2\}$, which sends $\Sigma^2 1 \mapsto Sq^2(1)$, we see that the image of this map is $Sq^2(1)$, and the kernel is $Sq^2(\Sigma^2 1)$ (since $Sq^2(\Sigma^2 1) \mapsto Sq^2 Sq^2 = 0$). That is,

$$\Sigma^2 \mathrm{im}(d') = \ker(d')$$

Therefore,

$$\Sigma^2 \text{im}(d^*) = \ker(d^*)$$

Recall that we defined d^* as $(\Sigma^2 c \circ \partial)^*$. Thus, using the diagram above, and the simple fact that for any maps i, j: $\ker(i) \subseteq \ker(i \circ j)$, and $\text{im}(i \circ j) \subseteq \text{im}(j)$, we see:

$$\ker(\Sigma^2 c^*) \subseteq \ker((\Sigma^2 c \circ \partial)^*) = \Sigma^2 \text{im}((\Sigma^2 c \circ \partial)^*) \subseteq \text{im}(\partial^*) = \ker(c^*)$$

We may rewrite this, subsituting d^* for $(\Sigma^2 c \circ \partial)^*$:

$$\ker(\Sigma^2 c^*) \subseteq \ker(d^*) = \Sigma^2 \text{im}(d^*) \subseteq \text{im}(\partial^*) = \ker(c^*)$$

From this chain of inclusions, we get $\text{im}(d^*) \simeq \ker(c^*)$. By Noether's isomorphism theorem,

$$\text{im}(c^*) \simeq A//E(1)/(\ker(c^*)) \simeq A//E(1)/(\text{im}(d^*))$$

By Lemma 1, im $(d^*) = A(Sq^2)$, and we see that $\text{im}(c^*) \simeq A//E(1)/(Sq^2) \simeq A//A(1)$. Thus,

$$A//A(1) \simeq \text{im}(c^*) \subseteq H^* ko$$

□

Step 2. $A//A(1)$ is isomorphic to $H^*(ko)$ as an A-module.

Proof. We construct the following map of exact sequences. We know by Step 1 that the vertical maps are injective:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Sigma^2 A//A(1) & \xhookrightarrow{e} & A//E(1) & \xrightarrow{a} & A//A(1) & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle \Sigma^2 b} & & \| & & \downarrow{\scriptstyle b} & & \downarrow \\
\Sigma H^* ko & \xrightarrow{\Sigma\eta^*} & \Sigma^2 H^* ko & \xrightarrow{\partial^*} & A//E(1) & \xrightarrow{c^*} & H^* ko & \xrightarrow{\eta^*} & \Sigma H^* ko
\end{array}$$

We see that η^* being zero in degree k forces $A//A(1)$ and H^*ko to be isomorphic in degree k. This is because η^* being zero forces c^* to be surjective, and thus forces $b \circ a$ to be surjective. Thus, b must be surjective in degree k, and b is injective by assumption, so $b : A//A(1) \to H^*ko$ is an isomorphism in degree k.

Our proof will be an inductive proof that η^* is the zero map. We start out our induction very simply: since ko is connective, we see that the map η^* must be zero in degrees ≤ 0. In degree 1, we look at $\eta^* : H^1(ko) \to \Sigma H^1 ko$. We know, given the group structure of $\pi_*(ko)$, that $H^1 ko = \mathbb{Z}_2^\wedge$. Further, just glancing at the first stem of the ASS for ko, we see that $2\eta^* = 0$. There is only one map from $\mathbb{Z}_2^\wedge \to \mathbb{Z}_2^\wedge$ that when doubled is equal to zero, and that is the zero map. Thus η^* must be zero.

We take as the inductive hypothesis that η^* is zero in degrees $\leq n$. We wish to show that this implies that η^* is zero in degree $n + 1$.

We have the following chain of reasoning:

$$
\begin{aligned}
&\eta^* = 0 \text{ in deg } \leq n \\
&\implies c^* \text{ is surjective in deg } \leq n \\
&\implies b \text{ is surjective (and thus is an isomorphism) in deg} \leq n \\
&\implies \Sigma^2 b \text{ is an isomorphism in deg } \leq n+2 \\
&\implies \partial^* \text{ is injective in deg } \leq n+2; \text{ since } e \text{ is injective} \\
&\implies \Sigma\eta^* \text{ is zero in deg } \leq n+2 \\
&\implies \eta^* \text{ is zero in deg } \leq n+1.
\end{aligned}
$$

This concludes the inductive argument.

□

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