Rational homotopy theory*

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CONTENTS

Rational homotopy theory is the study of the rational homotopy category, that is the category obtained from the category of 1-connected pointed spaces by localizing with respect to the family of those maps which are isomorphisms modulo the class in the sense of Serre of torsion abelian groups. As the homotopy groups of spheres modulo torsion are so simple, it is reasonable to expect that there is an algebraic model for rational homotopy theory which is much simpler than either of Kan's models of simplicial sets or simplicial groups. This is what is constructed in the present paper. We prove that rational homotopy theory is equivalent to the homotopy theory of reduced differential graded Lie algebras over $\mathbf{Q}$ and also to the homotopy theory of 2-reduced differential graded cocommutative coalgebras over $\mathbf{Q}$.

In Part I we exhibit a chain of several categories connected by pairs of adjoint functors joining the category $\mathcal{T}_2$ of 1-connected pointed spaces with

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the categories $(\mathrm{DGL})_1$ and $(\mathrm{DGC})_2$ of reduced differential graded Lie algebras and 2-reduced differential graded cocommutative coalgebras over $\mathbf{Q}$ respectively. We prove that these functors induce an equivalence of the rational homotopy category $\mathrm{Ho}_{\mathbf{Q}}\mathcal{T}_2$ with both of the categories $\mathrm{Ho}(\mathrm{DGL})_1$ and $\mathrm{Ho}(\mathrm{DGC})_2$ obtained by localizing with respect to the maps which induce isomorphisms on homology. Moreover these equivalences have the property that the graded Lie algebra $\pi_{*-1}(X) \otimes_{\mathbf{Z}} \mathbf{Q}$ under Whitehead product and the homology coalgebra $H_*(X, \mathbf{Q})$ of a space X are canonically isomorphic to the homology of the corresponding differential graded Lie algebra and coalgebra respectively. An immediate corollary is that every reduced graded Lie algebra (resp. 2-reduced graded coalgebra) over $\mathbf{Q}$ occurs as the rational homotopy Lie algebra (resp. homology coalgebra) of some simply-connected space. This answers a question which is due, we believe, to Hopf.

Part I raises some interesting questions such as how to calculate the maps in the category $\mathrm{Ho}(\mathrm{DGL})_1$ say from one DG Lie algebra to another, and also whether or not there is any relation between fibrations of spaces and exact sequence of DG Lie algebras. In order to answer these questions, we introduced in [21] an axiomatization of homotopy theory based on the notion of a *model category*, which is short for a "category of models for a homotopy theory". A model category is a category endowed with three families of maps called fibrations, cofibrations, and weak equivalences satisfying certain axioms. To a model category $\mathcal{C}$ is associated a homotopy category $\mathrm{Ho}\,\mathcal{C}$, obtained by localizing with respect to the family of weak equivalences, and extra structure on $\mathrm{Ho}\,\mathcal{C}$ such as the suspension and loop functors and the families of fibration and cofibration sequences. The homotopy category together with this structure is called the *homotopy theory* of the model category $\mathcal{C}$. In Part II we show that rational homotopy theory occurs as the homotopy theory of a closed model category, that all of the algebraic categories such as $(\mathrm{DGL})_1$ and $(\mathrm{DGC})_2$ occurring in the proof of Theorem I are closed model categories in a natural way, and that the various adjoint functors induce equivalences of homotopy theories. Combining this result with Theorem I, we obtain a solution to the problem raised by Thom [29] of constructing a commutative cochain functor from the category of simply-connected pointed spaces to the category of (anti-) commutative DG algebras over $\mathbf{Q}$, giving the rational cohomology algebra and having the right properties with respect to fibrations.

Part II contains a number of results of independent interest. In § 2 we show how the Serre mod $\mathcal{C}$ homotopy theory [27], where $\mathcal{C}$ is the class of S-torsion abelian groups and S is a multiplicative system in $\mathbf{Z}$, can be realized as the homotopy theory of a suitable closed model category of simplicial sets. In § 3 we construct another model category for the same homotopy theory out of simplicial groups. In proving the axioms it was necessary to prove the excision property for the homology functor on the category of simplicial groups (II, 3.12).

The category of reduced simplicial sets, with cofibrations defined to be injective maps and with weak equivalences defined to be maps which become homotopy equivalences after the geometric realization functor is applied, turned out to be a closed model category in which it is not true that the base extension of a weak equivalence by a fibration is a weak equivalence. This is reflected in the fact that there exist fibrations with the property that the fiber is not equivalent to the fiber of any weakly equivalent fibration of Kan complexes (II, 2.9). Since the base of such a fibration is never a Kan complex, it does not contribute fibration sequences to the associated homotopy theory. Thus these pathological fibrations are a curiosity forced upon us by the model category axioms. The same phenomenon occurs with DG coalgebras, but not with any of the group-like categories considered here.

In Part II, § 6, we give some applications of the theorems of this paper. In particular we use the DG Lie algebra and DG coalgebra models to derive certain spectral sequences (II, 6.6–6.9) for rational homotopy theory. Of special interest is an unstable rational version (II, 6.9) of the reverse Adams spectral sequence studied in [5]. This raises the question of whether such a spectral sequence holds in general.

In addition to Part I and II, the paper contains two appendices. Appendix A contains the theory of complete Hopf algebras, which is the natural Hopf algebra framework for treating the Malcev completion [18] as well as groups defined by means of the Campbell-Hausdorff formula [17]. Appendix B contains an exposition of some results of DG mathematics in a form particularly suited for our purposes. The main result is that the generalization to DG Lie algebras of the procedure for calculating the homology of a Lie algebra provides a functor $\mathcal{C}$ from DG Lie algebras to DG coalgebras whose adjoint $\mathcal{L}$ is the primitive Lie algebra of the cobar construction, and that the pair $\mathcal{L}$, $\mathcal{C}$ have the same properties of the functors G, $\overline{W}$ of Kan.

Finally we would like to acknowledge the influence on this work of many conversations with Daniel Kan and E.B. Curtis; our debt to their work will be abundantly clear to anyone who reads the proof of Theorem I.

Part I

1. Statement of Theorem I

If $\mathcal{C}$ is a category and S is a family of morphisms of $\mathcal{C}$, then the *localization* [9, Ch. I]; [21, Ch. I, 1.11] of $\mathcal{C}$ with respect to S is a pair consisting of a category $S^{-1}\mathcal{C}$ and a functor $\gamma:\mathcal{C}\to S^{-1}\mathcal{C}$ which carries the maps in S into isomorphisms in $S^{-1}\mathcal{C}$ and which is universal with this property. In general there is a minor set-theoretic difficulty with the existence of $S^{-1}\mathcal{C}$ which may be avoided by use of a suitable set theory with universes. We shall therefore ignore this difficulty and assume the existence of $S^{-1}\mathcal{C}$; for the cases we need this can be verified (see Part II, 1.3a).

Let $\mathcal{T}_r$ be the category of $(r-1)$-connected pointed topological spaces and continuous basepoint preserving maps. (The reason for the notation $\mathcal{T}_r$ is to save space in Part II. The subscript r should be read "begins in dimension r.") We recall the following theorem of Serre [27].

PROPOSITION 1.1. *The following assertions are equivalent for a map $f: X\to Y$ in $\mathcal{T}_2$.*

(i) $\pi_*(f)\otimes_{\mathbf{Z}}\mathbf{Q}: \pi_*(X)\otimes_{\mathbf{Z}}\mathbf{Q}\to\pi_*(Y)\otimes_{\mathbf{Z}}\mathbf{Q}$ *is an isomorphism.*

(ii) $H_*(f,\mathbf{Q}): H_*(X,\mathbf{Q})\to H_*(X,\mathbf{Q})$ *is an isomorphism.*

A map satisfying these conditions will be called a *rational homotopy equivalence*. The localization of $\mathcal{T}_2$ with respect to the family of rational homotopy equivalences will be denoted $\mathrm{Ho}_{\mathbf{Q}}\mathcal{T}_2$ and called the *rational homotopy category*. The study of this category is what Serre calls homotopy theory modulo the class of torsion abelian groups.

The objects of $\mathrm{Ho}_{\mathbf{Q}}\mathcal{T}_2$ are the same as those of $\mathcal{T}_2$ namely 1-connected pointed spaces, however the morphisms are different. If $f: X\to Y$ is a map in $\mathcal{T}_2$, then f determines the map $\gamma(f): X\to Y$ in $\mathrm{Ho}_{\mathbf{Q}}\mathcal{T}_2$. If $f, g: X\to Y$ are homotopic, then $\gamma(f)=\gamma(g)$. In effect consider the maps

$$X \underset{i_1}{\overset{i_0}{\rightrightarrows}} X\wedge I \xrightarrow{\pi} X$$

where I is the unit interval $X\wedge I = X\times I/\{x_0\}\times I$ and $i_j(x)=(x,j)$, $j=0,1$ and $\pi(x,t)=x$. As π is a homotopy equivalence $\gamma(\pi)$ is an isomorphism so $\gamma(i_0)\gamma(\pi)=\mathrm{id}_X=\gamma(i_1)\gamma(\pi)\Rightarrow\gamma(i_0)=\gamma(i_1)$. Therefore if $h: X\wedge I\to Y$ is a homotopy from f to g, we have

$$\gamma(f)=\gamma(h)\gamma(i_0)=\gamma(h)\gamma(i_1)=\gamma(g)\ ,$$

proving the assertion.

As usual in homotopy theory two maps inducing the same map on the functors $\pi_*(\cdot)\otimes_{\mathbf{Z}}\mathbf{Q}$ or $H_*(\cdot,\mathbf{Q})$ do not give the same map in $\mathrm{Ho}_{\mathbf{Q}}\mathcal{T}_2$. It is possible to show that the rational homotopy category is equivalent to the full subcategory of the category of 1-connected pointed CW complexes and homotopy classes of basepoint-preserving maps, consisting of X for which π_*X is a torsion-free divisible abelian group (see Part II, 6.1); however we shall not need this here.

All vector spaces, algebras, tensor products, etc. in this paper are to be understood as being over $\mathbf{Q}$ unless there is indication to the contrary. We shall consider differential graded (DG) vector spaces $V=\bigoplus_q V_q$, $q\in\mathbf{Z}$, where the differential is of degree -1 and where $V_q=0$ for $q<0$. By an element x of V we shall usually mean a homogeneous element whose degree will be denoted $\deg x$. Let (DG) and (G) be the categories of DG and graded vector spaces where the morphisms are homogeneous of degree 0. The tensor product $V\otimes W$ and homology HV of DG vector spaces are defined as usual. There is a canonical isomorphism $T\colon V\otimes W\xrightarrow{\sim} W\otimes V$ called the interchange map given by $T(x\otimes y)=(-1)^{pq}\,y\otimes x$ if $p=\deg x$ and $q=\deg y$. In working with DG objects we shall rigidly adhere to the standard sign rule: whenever something of degree p is moved past something of degree q the sign $(-1)^{pq}$ accrues.

A DG *Lie algebra* is a DG vector space L together with a map $L\otimes L\to L$ denoted $x\otimes y\to[x,y]$ satisfying the antisymmetry and Jacobi identities with signs thrown in according to the sign rule. A DG *coalgebra* is a DG vector space C with a comultiplication map $\Delta\colon C\to C\otimes C$ and an augmentation $\varepsilon\colon C\to\mathbf{Q}[0]$ ($\mathbf{Q}[0]$ is the DG vector space with $\mathbf{Q}[0]_q=\mathbf{Q}$ if $q=0$, and 0 if $q\neq 0$) such that Δ is coassociative, *cocommutative* (i.e., $T\circ\Delta=\Delta$), and ε is a two-sided counit for Δ. Let $\bar{C}=\operatorname{Ker}\varepsilon$. A DG Lie algebra L (resp. DG coalgebra C) will be called *r-reduced* if $L_q=0$ (resp. $\bar{C}_q=0$) for $q<r$. We say reduced instead of 1-reduced. We denote by (DGL)(resp. $(\mathrm{DGL})_r$) and (DGC)(resp. $(\mathrm{DGC})_r$) the categories of DG (resp. r-reduced DG) Lie algebras and DG (resp. r-reduced DG) coalgebras with the obvious morphisms.

By virtue of the Künneth formula $H(V\otimes W)=HV\otimes HW$ homology gives functors $H\colon(\mathrm{DGL})\to(\mathrm{GL})$ and $H\colon(\mathrm{DGC})\to(\mathrm{GC})$. We define a *weak equivalence* of DG objects to be a map f such that H_*f is an isomorphism. The localizations of $(\mathrm{DGL})_1$ and $(\mathrm{DGC})_2$ with respect to their families of weak equivalences will be denoted $\mathrm{Ho}\,(\mathrm{DGL})_1$ and $\mathrm{Ho}\,(\mathrm{DGC})_2$ and called the *homotopy categories* of reduced DG Lie algebras and 2-reduced DG coalgebras respectively.

If X is an object of $\mathcal{T}_2$, then the (singular) homology of X with rational coefficients $H_*(X,\mathbf{Q})$ is a 2-reduced graded coalgebra with comultiplication induced by the diagonal map $X \to X \times X$ and the Künneth isomorphism. The rational homotopy groups $\pi_* X \otimes_{\mathbf{Z}} \mathbf{Q}$ may be made into a graded Lie algebra πX in the following way. Let $\pi_q X = \pi_{q+1} X \otimes_{\mathbf{Z}} \mathbf{Q}$ and let $\tau: \pi_{q+1} X \to \pi_q X$ be given by $\tau x = x \otimes 1$. The *Whitehead product* is a natural bilinear transformation

$$\begin{aligned} \pi_{p+1} X \times \pi_{q+1} X &\longrightarrow \pi_{p+q+1} X \\ \alpha, \beta &\longmapsto [\alpha, \beta] \end{aligned}$$

so there is a unique bilinear operation which we again denoted by [,] on πX such that

$$[\tau\alpha, \tau\beta] = (-1)^{\deg \alpha} \tau[\alpha, \beta] .$$

The anti-symmetry and Jacobi identities for the Whitehead product [10] imply that πX is a graded Lie algebra.

By the definition of $\mathrm{Ho}_{\mathbf{Q}} \mathcal{T}_2$ the functors $X \mapsto H_*(X, \mathbf{Q})$ and $X \to \pi X$ from $\mathcal{T}_2$ to (GC) and (GL) extend uniquely to functors $\mathbf{H}: \mathrm{Ho}_{\mathbf{Q}} \mathcal{T}_2 \longrightarrow$ (GC) and $\pi: \mathrm{Ho}_{\mathbf{Q}} \mathcal{T}_2 \longrightarrow$ (GL) respectively. Let $\pi: \mathrm{Ho}\,(\mathrm{DGL})_1 \to$ (GL) and $\mathbf{H}: \mathrm{Ho}\,(\mathrm{DGL})_2 \to$ (GC) be the unique extensions of the functors $L \mapsto HL$ and $C \mapsto HC$, respectively. We can now state the main result of this paper.

THEOREM I. *There exist equivalences of categories*

$$\mathrm{Ho}_{\mathbf{Q}} \mathcal{T}_2 \xrightarrow{\lambda} \mathrm{Ho}\,(\mathrm{DGL})_1 \xrightarrow{\tilde{\mathcal{C}}} \mathrm{Ho}\,(\mathrm{DGC})_2 .$$

Moreover there are isomorphisms of functors

$$\pi X \xrightarrow{\sim} \pi(\lambda X) \qquad \mathbf{H} X \xrightarrow{\sim} \mathbf{H}(\tilde{\mathcal{C}} \lambda X)$$

from $\mathrm{Ho}_{\mathbf{Q}} \mathcal{T}_2$ *to* (GL) *and* (GC) *respectively.*

COROLLARY. *If L is a reduced graded Lie algebra, then $L \simeq \pi X$ for some 1-connected pointed space X. If C is a 2-reduced graded coalgebra, then $C \simeq H_*(X, \mathbf{Q})$ for some 1-connected pointed space X.*

PROOF. Consider L as a DG Lie algebra with all differentials zero. By the theorem there is a space X in $\mathcal{T}_2$ with $\lambda X \simeq L$ hence $\pi X \simeq \pi \lambda X \simeq \pi L = L$. The second statement is proved similarly.

Remark. By duality one sees that if $A = \bigoplus_{q \geq 0} A_q$ is a graded (anti-) commutative algebra over $\mathbf{Q}$ with A_q finite dimensional for each q and $A_1 = 0$, $A_0 = \mathbf{Q}$, then A is isomorphic to the rational cohomology ring of a space in $\mathcal{T}_2$. This answers affirmatively a conjecture which is originally due, we believe, to Hopf.

$$\mathcal{T}_2 \underset{E_2\,\mathrm{Sing}}{\overset{|\ |}{\rightleftarrows}} \mathcal{S}_2 \underset{\overline{W}}{\overset{G}{\rightleftarrows}} (\mathrm{SGp})_1 \underset{\mathcal{G}}{\overset{\widehat{\mathbf{Q}}}{\rightleftarrows}} (\mathrm{SCHA})_1 \underset{\mathcal{P}}{\overset{\widehat{U}}{\leftrightarrows}} (\mathrm{SLA})_1 \underset{N}{\overset{N^*}{\leftrightarrows}} (\mathrm{DGL})_1 \underset{\mathcal{C}}{\overset{\mathfrak{L}}{\leftrightarrows}} (\mathrm{DGC})_2$$

FIGURE 1

$$\mathrm{Ho}_{\mathbf{Q}}\,\mathcal{T}_2 \underset{(E_2\,\mathrm{Sing})^{\sim}}{\overset{|\ |^{\sim}}{\rightleftarrows}} \mathrm{Ho}_{\mathbf{Q}}\,\mathcal{S}_2 \underset{\overline{W}^{\sim}}{\overset{\tilde{G}}{\rightleftarrows}} \mathrm{Ho}_{\mathbf{Q}}\,(\mathrm{SGp})_1 \underset{\tilde{\mathcal{G}}}{\overset{\mathbf{L}\widehat{\mathbf{Q}}}{\rightleftarrows}} \mathrm{Ho}\,(\mathrm{SCHA})_1 \underset{\tilde{\mathcal{P}}}{\overset{\mathbf{L}\widehat{U}}{\leftrightarrows}} \mathrm{Ho}\,(\mathrm{SLA})_1 \underset{\tilde{N}}{\overset{\mathbf{L}N^*}{\leftrightarrows}} \mathrm{Ho}\,(\mathrm{DGL})_1 \underset{\tilde{\mathcal{C}}}{\overset{\tilde{\mathfrak{L}}}{\leftrightarrows}} \mathrm{Ho}\,(\mathrm{DGC})_2$$

FIGURE 2

2. Outline of the proof of Theorem I

The equivalence λ will be the composition of several equivalences of categories, each equivalence coming by localization from a pair of adjoint functors. The categories and adjoint functors involved are indicated in Figure 1 and listed below. Upon localizing with respect to a suitable family of maps in each category, we obtain Figure 2, where $\sim$ and $\mathbf{L}$ are defined below. The part of Theorem I about the equivalence of categories results from the fact that each functor in Figure 2 is an equivalence of categories. Half of these equivalences are treated in Theorem 2.1. For the others we prove a general categorical result (2.3) whose hypotheses are verified for the remaining cases in § 3 and § 4. Thus the equivalence of categories assertion of Theorem I is proved by § 4. The assertions about the homotopy and homology functors are proved in § 5 and § 6 respectively.

We consider the following categories.

$\mathcal{T}_2$: The category of 1-connected pointed spaces and basepoint preserving continuous maps.

$\mathcal{S}_2$: The category of 2-reduced simplicial sets $=$ full subcategory of the category of simplicial sets consisting of K such that K_q has a single element for $q = 0,1$.

$(\mathrm{SGp})_1$: The category of reduced simplicial groups $=$ full subcategory of the category of simplicial groups consisting of G such that $G_0 = \{e\}$.

$(\mathrm{SCHA})_1$: The category of reduced simplicial complete Hopf algebras over $\mathbf{Q}$. For the definition of complete Hopf algebra see Appendix A. A simplicial CHA R is called reduced if $R_0 \cong \mathbf{Q}$.

$(\mathrm{SLA})_1$: The category of reduced simplicial Lie algebras over $\mathbf{Q}$.

$(\mathrm{DGL})_1$: The category of reduced differential graded Lie algebras over $\mathbf{Q}$.

$(\mathrm{DGC})_2$: The category of 2-reduced differential graded (cocommutative coassociative) coalgebras over $\mathbf{Q}$.

We also consider the following pairs of adjoint functors.

$|\ |$, $E_2\,\mathrm{Sing}$: $|\ |$ is the geometric realization functor [19], [9, Ch. III]. Sing

X is the singular complex of a space X; if K is a pointed simplicial set, then E_2K is the Eilenberg subcomplex consisting of those simplices of K whose 1-skeleton is at the basepoint.

G, $\overline{W}$: If K is a reduced simplicial set, GK is the simplicial group constructed by Kan [12] playing the role of the loop space of K. If G is a simplicial group, $\overline{W}G$ is the simplicial set which acts as its "classifying space" [3], [4], [12].

$\widehat{\mathbf{Q}}$, $\mathcal{G}$: If G is a group then $\widehat{\mathbf{Q}}G$ is the complete Hopf algebra (Appendix A) obtained by completing the group ring $\mathbf{Q}G$ by the powers of its augmentation ideal. If R is a CHA, then $\mathcal{G}R$ is its group of group-like elements. These functors are extended dimension-wise to simplicial groups and simplicial CHA's and denoted by the same letters.

$\hat{U}$, $\mathcal{P}$: If $\mathfrak{g}$ is a Lie algebra over $\mathbf{Q}$, $\hat{U}\mathfrak{g}$ is the CHA obtained by completing the universal enveloping algebra $U\mathfrak{g}$ by powers of its augmentation ideal. If R is a CHA, then $\mathcal{P}R$ is its Lie algebra of primitive elements. These functors are applied dimension-wise to simplicial objects.

N^*, N: If L is a simplicial Lie algebra, its normalized chain complex NL is a DGL with bracket defined by means of the Eilenberg-Zilber map $\underline{\otimes}$ (§ 4). N^* is the left adjoint of N and is constructed in § 4.

$\mathcal{L}$, $\mathcal{C}$: These functors are defined in Appendix B. If C is a DGC, then $\mathcal{L}C$ is the Lie algebra of the primitive elements of the cobar construction of C. $\mathcal{C}L$ is the obvious generalization to DG Lie algebras L of the DG homology coalgebra of a Lie algebra [15].

From each of the above categories we construct the following localizations.

$$\mathrm{Ho}_{\mathbf{Q}}\,\mathcal{T}_2 = S^{-1}\mathcal{T}_2,\ \mathrm{Ho}_{\mathbf{Q}}\,\mathcal{S}_2 = S^{-1}\mathcal{S}_2,\ \mathrm{Ho}_{\mathbf{Q}}\,(\mathrm{sGp})_1 = S^{-1}(\mathrm{sGp})_1$$

where in each case S is the family of *rational homotopy equivalences*, i.e., maps f such that $\pi_* f \otimes_{\mathbf{Z}} \mathbf{Q}$ is an isomorphism.

$\mathrm{Ho}\,(\mathrm{sCHA})_1 = S^{-1}(\mathrm{sCHA})_1$ where S is the set of maps f such that $\pi_*\mathcal{G}f$ (or equivalently $\pi_*\mathcal{P}f$ (3.2)) is an isomorphism.

$\mathrm{Ho}\,(\mathrm{sLA})_1 = S^{-1}(\mathrm{sLA})_1$, $\mathrm{Ho}\,(\mathrm{DGL})_1 = S^{-1}(\mathrm{DGL})_1$, $\mathrm{Ho}\,(\mathrm{DGC})_2 = S^{-1}(\mathrm{DGC})_2$, where in each case S is the set of *weak equivalences*, i.e., maps inducing isomorphisms on homotopy in the case of simplicial Lie algebras and homology in the other cases.

The following notations will be used in this paper. If

$$\mathcal{C}_1 \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{C}_2$$

is a pair of adjoint functors, then the upper arrow F will always denote the left adjoint functor and the canonical adjunction morphisms will be denoted

$$\alpha\colon FG \longrightarrow \mathrm{id}_{\mathcal{C}_2} \qquad \beta\colon \mathrm{id}_{\mathcal{C}_1} \longrightarrow GF .$$

If $\gamma_i\colon \mathcal{C}_i \to S_i^{-1}\mathcal{C}_i$ are localizations $i = 1,2$, and $F\colon \mathcal{C}_1 \to \mathcal{C}_2$ is a functor carrying S_1 into S_2, then F induces a functor $\tilde{F}\colon S_1^{-1}\mathcal{C}_1 \to S_2^{-1}\mathcal{C}_2$ such that $\tilde{F}\gamma_1 = \gamma_2 F$.

We can now take up the easy part of Figure 2.

THEOREM 2.1. *Each of the adjoint functor pairs* $(|\,|, E_2\,\mathrm{Sing})$, $(G, \overline{W})$ *and* $(\mathcal{L}, \mathcal{C})$ *has the property that each functor carries the localizing family of its source into the localizing family of its target, and the property that the adjunction morphisms are in the localizing families. Consequently the* $\sim$ *functors of Figure* 2 *induced by these functors are equivalences of categories.*

PROOF. $(|\,|, E_2\,\mathrm{Sing})$: From the definition of the homotopy groups given by Kan [11], one sees that if K is a 1-connected pointed simplicial set satisfying the extension condition, then the inclusion $E_2K \to K$ is a weak equivalence, i.e., it induces isomorphisms on homotopy groups. Now Milnor [19]; [9, VII, 3] has proved that $K \to \mathrm{Sing}\,|K|$ is always a weak equivalence, hence combining this with Kan's formula $\pi_*(X) = \pi_*(\mathrm{Sing}\,X)$ one has $\pi_*(X) \simeq \pi_*(E_2\,\mathrm{Sing}\,X)$. The assertion of the theorem follows easily.

$(G, \overline{W})$: Kan [12], [4] has proved that $\pi_q K \xrightarrow{\sim} \pi_{q-1}GK$ and that the maps $G\overline{W}(G) \to G$ and $K \to \overline{W}GK$ are weak equivalences, yielding the result.

$(\mathcal{L}, \mathcal{C})$: See Appendix B, 7.5.

The last assertion of the theorem is proved as follows. Suppose that the adjoint functors are F and G. Then for every object X of $\mathcal{C}_1$, α defines an isomorphism

$$\tilde{F}\tilde{G}(\gamma_2 X) = \gamma_2(FGX) \xrightarrow{\gamma_2(\alpha)} \gamma_2 X .$$

As X varies over $\mathrm{Ob}\,\mathcal{C}_2$, this isomorphism gives an isomorphism of functors $\tilde{F}\tilde{G}\gamma_2 \xrightarrow{\sim} \mathrm{id}\,\gamma_2$, and hence by the following proposition an isomorphism $\tilde{F}\tilde{G} \to \mathrm{id}$. Similarly β gives an isomorphism $\mathrm{id} \to \tilde{G}\tilde{F}$ and so $\tilde{F}, \tilde{G}$ are equivalences of categories.

PROPOSITION 2.2. *Let* $\gamma\colon \mathcal{C} \to S^{-1}\mathcal{C}$ *be a categorical localization, and let* $F, G\colon S^{-1}\mathcal{C} \to \mathcal{B}$ *be functors. Then*

$$\mathrm{Hom}\,(F, G) \xrightarrow{\sim} \mathrm{Hom}\,(F\gamma, G\gamma) .$$

PROOF. It follows immediately from the universal property of γ that γ is an isomorphism on objects and that every map in $S^{-1}\mathcal{C}$ is a finite composition of maps of the form $\gamma(g)$ or $\gamma(s)^{-1}$, where g is a map in $\mathcal{C}$ and s is in S.

Let us assume, as is customary, that $\mathcal{C}$ and $S^{-1}\mathcal{C}$ have the same objects and that γ is the identity on objects. Then a natural transformation from $F\gamma$ to $G\gamma$ is a collection of maps $\theta(X)\colon F(X) \to G(X)$ for all objects X such that $\theta F(f) = G(f)\theta$ for all f of the form $\gamma(g)$ where g is a map in $\mathcal{C}$. This formula must also hold for $f = \gamma(s)^{-1}$ and finite compositions, so therefore it is true for all maps f in $S^{-1}\mathcal{C}$, showing that θ is a natural transformation from F to G, and proving the proposition.

For the other pairs of adjoint functors we will not know that the functor F carries S_1 into S_2, so we shall need the following definition and proposition to get the desired equivalence of categories.

Let $\mathcal{C}_1$, $\mathcal{C}_2$ be categories, let S_i be a family of maps in $\mathcal{C}_i$, and let $\gamma_i\colon \mathcal{C}_i \to S_i^{-1}\mathcal{C}_i$ be the corresponding localization functors. If $F\colon \mathcal{C}_1 \to \mathcal{C}_2$ is a functor, then by $\mathbf{L}F$ we shall mean a functor from $S_1^{-1}\mathcal{C}_1$ to $S_2^{-1}\mathcal{C}_2$ together with a natural transformation $\varepsilon\colon (\mathbf{L}F)\gamma_1 \to \gamma_2 F$ having the following universal property: Given a functor $G\colon S^{-1}\mathcal{C}_1 \to S^{-1}\mathcal{C}_2$ and a natural transformation $\eta\colon G\gamma_1 \to \gamma_2 F$ there is a unique natural transformation $\theta\colon G \to \mathbf{L}F$ such that $\eta = \varepsilon(\theta^*\gamma_1)$, where $\theta^*\gamma_1\colon G\gamma_1 \to (\mathbf{L}F)\gamma_1$ is the natural transformation given by $(\theta^*\gamma_1)(X) = \theta(\gamma_1 X)$. The pair $(\mathbf{L}F, \varepsilon)$ if it exists will be called the *left derived functor* of F with respect to S_1 and S_2. It is clear that if F carries the maps of S_1 into S_2, then up to canonical isomorphism $\mathbf{L}F \simeq \tilde{F}$.

PROPOSITION 2.3. *Suppose given localizations and adjoint functors*

$$\begin{tikzcd} \mathcal{C}_1 \arrow[r, shift left, "F"] \arrow[d, "\gamma_1"] & \mathcal{C}_2 \arrow[l, shift left, "G"] \arrow[d, "\gamma_2"] \\ S_1^{-1}\mathcal{C}_1 & S_2^{-1}\mathcal{C}_2 \end{tikzcd}$$

such that

(i) *S_1 contains all isomorphisms of $\mathcal{C}_1$. If f, g are maps of $\mathcal{C}_1$ such that gf is defined, then if any two of the maps f, g, gf are in S_1 so is the third.*

(ii) *A map f in $\mathcal{C}_2$ is in S_2 if and only if $Gf \in S_1$.*

(iii) *There exists a functor $R\colon \mathcal{C}_1 \to \mathcal{C}_1$ and a natural transformation $\xi\colon R \to \mathrm{id}$ such that for all $X \in \mathrm{Ob}\,\mathcal{C}_1$ the maps $\xi\colon RX \to X$ and $\beta\colon RX \to GFRX$ are in S_1.*

Then the left derived functor $\mathbf{L}F$ exists and is quasi-inverse to the functor $\tilde{G}\colon S_2^{-1}\mathcal{C}_2 \to S_1^{-1}\mathcal{C}_1$ induced by G. In particular $\tilde{G}$ and $\mathbf{L}F$ are equivalences of categories.

PROOF. If $f\colon X \to Y$ is in S_1 then by (i), (iii), and the diagram

$$\begin{tikzcd} X & RX \arrow[l, "\xi"'] \arrow[r, "\beta"] & GFRX \\ Y \arrow[u, leftarrow, "f"] & RY \arrow[l, "\xi"'] \arrow[r, "\beta"] \arrow[u, leftarrow, "R"] & GFRY \arrow[u, leftarrow, "GFRf"] \end{tikzcd}$$

one sees that Rf and $GFRf$ are in S_1. Thus the functor $\gamma_2 FR\colon \mathcal{C}_1 \to S^{-1}\mathcal{C}_2$ carries S_1 into isomorphisms and so by the definition of localization there is a unique functor $\mathbf{L}F\colon S_1^{-1}\mathcal{C}_1 \to S_2^{-1}\mathcal{C}_2$ such that $(\mathbf{L}F)\gamma_1 = \gamma_2 FR$. Let $\varepsilon\colon (\mathbf{L}F)\gamma_1 \to \gamma_2 F$ be the natural transformation $\gamma_2 F(\xi)\colon \gamma_2 FR \to \gamma_2 F$. We claim that $(\mathbf{L}F, \varepsilon)$ is a left derived functor of F. Indeed given $H\colon S_1^{-1}\mathcal{C}_1 \to S_2^{-1}\mathcal{C}_2$ and $\eta\colon H\gamma_1 \to \gamma_2 F$ consider the composition

$$H\gamma_1 X \xrightarrow{H\gamma_1(\xi)^{-1}} H\gamma_1 RX \xrightarrow{\eta} \gamma_2 FRX = (\mathbf{L}F)\gamma_1 X$$

for each $X \in \operatorname{Ob}\mathcal{C}_1$. This composition is a natural transformation $H\gamma_1 \to (\mathbf{L}F)\gamma_1$ so by (2.2) it defines a natural transformation $\theta\colon H \to \mathbf{L}F$. It is easily seen that θ satisfies $\varepsilon(\theta * \gamma_1) = \eta$ and is the unique natural transformation with this property.

By (2.2) there are unique natural transformations $\Phi\colon \mathrm{id} \to \widetilde{G}(\mathbf{L}F)$ and $\Psi\colon (\mathbf{L}F)\widetilde{G} \to \mathrm{id}$ given by the compositions

$$\gamma_1 \xrightarrow{\gamma_1(\xi)^{-1}} \gamma_1 R \xrightarrow{\gamma(\beta)} \gamma_1 GF = \widetilde{G}(\mathbf{L}F)\gamma_1$$
$$(\mathbf{L}F)\widetilde{G}\gamma_2 = \gamma_2 FRG \xrightarrow{\gamma_2 F(\xi)} \gamma_2 FG \xrightarrow{\gamma_2(\alpha)} \gamma_2 .$$

By (iii) $\gamma(\beta)$ is an isomorphism, and so Φ is an isomorphism of functors. In order to show Ψ is an isomorphism we show that $FRGY \xrightarrow{F(\xi)} FGY \xrightarrow{\beta} Y$ is in S_2 or by (ii) that G carries this map into S_1. However this follows from (i) using the diagram

$$\begin{tikzcd} RGY \arrow[r, "\xi"] \arrow[d, "\beta"] & GY \arrow[d, "\beta"] \arrow[dr, "\mathrm{id}"] & \\ GFRGY \arrow[r, "\xi"] & GFGY \arrow[r, "G(\alpha)"] & GY \end{tikzcd}$$

and the fact that the left β is in S, by (iii). Hence Φ, Ψ are isomorphisms, $\mathbf{L}F$ and $\widetilde{G}$ are equivalences and the proposition is proved.

3. Application of Curtis' convergence theorems

This section is devoted to proving the hypotheses of (2.3) for the adjoint functor pairs $\mathbf{Q}$, $\mathcal{G}$ and $\hat{U}$, $\mathcal{P}$.

If G is a simplicial group, then its q^{th} homotopy group $\pi_q G$ may be defined either by the formula of Moore

$$\pi_q G = \frac{\operatorname{Ker}\{d_0 \colon N_q G \longrightarrow N_{q-1}G\}}{\operatorname{Im}\{d_0 \colon N_{q+1}G \longrightarrow N_q G\}} \tag{3.1}$$
$$N_q G = \bigcap\nolimits_{0<j\leqq q} \operatorname{Ker}\{d_j \colon G_q \longrightarrow G_{q-1}\} \qquad N_{-1}G = G_{-1} = \{e\}$$

or by the formula of Kan [11] applied to G considered as a pointed simplicial set with basepoint at the identity. The group law on $\pi_q G$ for $q \geqq 1$ is therefore independent of the group law of G and moreover is abelian.

If A is a simplicial complete Hopf algebra (SCHA for short, see Appendix A), then there is a canonical isomorphism of pointed simplicial sets given by the exponential

$$\exp \colon \mathcal{P}A \xrightarrow{\sim} \mathcal{G}A \ ,$$

hence we have

PROPOSITION 3.2. *If A is a* SCHA, *then the exponential induces an isomorphism of homotopy groups $\pi_q(\mathcal{P}A) \xrightarrow{\sim} \pi_q(\mathcal{G}A)$ for $q \geqq 1$.*

$\mathcal{P}A$ is a simplicial vector space over $\mathbf{Q}$, hence so are its homotopy groups. Therefore

COROLLARY 3.3. *$\pi_q(\mathcal{G}A)$ for $q \geqq 1$ is a torsion-free uniquely divisible abelian group and hence is a $\mathbf{Q}$ vector space.*

The following comparison theorem is what started this paper. Free simplicial algebraic objects are defined in [14]; see also the proof of 4.4.

THEOREM 3.4. *If G is a connected free simplicial group, then the adjunction map α induces an isomorphism*

$$\pi(G) \otimes_{\mathbf{Z}} \mathbf{Q} \xrightarrow{\sim} \pi(\mathcal{G}\hat{\mathbf{Q}}G) \ .$$

3.5. *If $\mathfrak{g}$ is a connected free simplical Lie algebra, then α induces an isomorphism*

$$\pi(\mathfrak{g}) \xrightarrow{\sim} \pi(\mathcal{P}\hat{U}\mathfrak{g}) \ .$$

3.6. *If R is a connected free simplicial augmented associative algebra and $\hat{R}$ is the completion of R* (A, 1.2), *then there is an isomorphism*

$$\pi(R) \xrightarrow{\sim} \pi(\hat{R}) \ .$$

The proof requires the following "convergence" or connectivity results based on the work of Curtis.

THEOREM 3.7. *Let G, $\mathfrak{g}$, and R be as in the preceding theorem, let Γ_r be the lower central series filtrations of G and $\mathfrak{g}$ and let $\bar{R}$ be the augmentation ideal of R. Then*

$$\pi_q(\Gamma_r G) \otimes_{\mathbf{Z}} \mathbf{Q} = 0$$
$$\pi_q(\Gamma_r \mathfrak{g}) = 0$$
$$\pi_q(\bar{R}^r) = 0$$

for $r > q$.

PROOF. As pointed out in [7, Remark 4.10], the argument of § 4 of that paper applies in great generality and not just for simplicial groups. By virtue of this argument it suffices to prove (3.7) where G (resp. $\mathfrak{g}$, R) is the free simplicial group (resp. Lie algebra, associative algebra) generated by the simplicial set K which is a finite wedge of 1-spheres $\Delta(1)/\Delta(1)^{\cdot}$, where the basepoint of K is set equal to the identity. Then $R = T(\bar{\mathbf{Q}}K)$, the tensor algebra on the reduced chains on K, so by Künneth $\pi(\bar{R}^r) = \bigoplus_{n \geqq r} T_n(\pi\bar{\mathbf{Q}}K)$, and the connectivity assertion is clear. Also $\mathfrak{g} = L(\bar{\mathbf{Q}}K)$, where L is the free Lie algebra functor, so $U(\mathfrak{g}) = T(\bar{\mathbf{Q}}K) = R$. Now $\mathfrak{g}$ is a retract of $U(\mathfrak{g})$ (B, 3.6) in such a way that $\Gamma_r\mathfrak{g}$ is a retract of $\overline{U(\mathfrak{g})}^r$, so the connectivity assertion for $\Gamma_r\mathfrak{g}$ follows from that of $\bar{R}^r$. One can also use the main result of [6]. Finally for $G = FK$ we have by the main result of [7](for another proof, see [24]) $\pi_q(\Gamma_r G) = 0$ for r sufficiently large. Also $\pi_q(\Gamma_r G/\Gamma_{r+1}G) \otimes \mathbf{Q} \simeq \pi_q\big(L_r(G_{ab} \otimes \mathbf{Q})\big) = 0$ for $r > q$ by what we have just proved for $\mathfrak{g}$. Thus by descending induction on r we have $\pi_q(\Gamma_r G) \otimes \mathbf{Q} = 0$ for $q > r$, and the proof of (3.7) is complete.

The proof of 3.4–3.6 will also require the following. Here $\mathbf{N}$ is the set of integers $\geqq 0$.

PROPOSITION 3.8. *Let* $\{G^r, r \in \mathbf{N}; p^r_s\colon G^r \to G^s\ r \geqq s\}$ *be an inverse system of simplicial groups such that* p^r_s *is surjective. Then there is a canonical exact sequence*

$$0 \longrightarrow R^1 \operatorname{lim-inv}_r \big(\pi_{q+1}(G^r)\big) \longrightarrow \pi_q(\operatorname{lim-inv}_r G^r) \longrightarrow \operatorname{lim-inv}_r \big(\pi_q(G^r)\big) \longrightarrow 0$$

where R^1 lim-inv *is the functor of an inverse system of abelian groups given by* $R^1 \operatorname{lim-inv}_r (A_r) = \operatorname{Coker}\{\theta\colon \prod_0^\infty A_r \to \prod_0^\infty A_r\}$, $\theta((a_r)_{r\in\mathbf{N}}) = (a_r - p^{r+1}_r a_{r+1})_{r\in\mathbf{N}}$.

PROOF. Consider the maps

$$\operatorname{lim-inv} G^r \xrightarrow{\ i\ } \textstyle\prod_0^\infty G^r \xrightarrow{\ \theta\ } \prod_0^\infty G^r$$

where i is the natural inclusion and

$$\theta\big((g_r)_{r\in\mathbf{N}}\big) = (g_r \cdot p^{r+1}_r g^{-1}_{r+1})_{r\in\mathbf{N}}\ .$$

θ is not a simplical group map, but it gives an isomorphism of the left coset simplicial set of $\prod G^r$ by the subgroup $\operatorname{lim-inv}_r G^r$ with $\prod G^r$, since p^r_s is surjective. Thus θ is a principal bundle map and gives rise to a homotopy long exact sequence

$$\xrightarrow{\partial} \pi_q(\text{lim-inv } G^r) \longrightarrow \prod_{r=0}^{\infty} \pi_q(G^r) \xrightarrow{\theta_*} \prod_{r=0}^{\infty} \pi_q(G^r) \xrightarrow{\partial}$$

where $\theta_*((a_r)_{r\in\mathbf{N}}) = (a_r - \pi(p_r^{r+1})a_{r+1})_{r\in\mathbf{N}}$. Taking into account the formula for R^1 lim-inv, the proposition is proved.

PROOF OF (3.4). Write $\hat{G}$ instead of $\mathcal{G}\hat{\mathbf{Q}}G$, and let $F_r\hat{G}$ be the filtration of $\hat{G}$ induced by the canonical filtration of $\hat{\mathbf{Q}}G$. The adjunction map $\alpha: G \rightarrow \hat{G}$ carries $\Gamma_r G$ to $F_r\hat{G}$ and so induces a map of Lie algebras

$$\text{gr}\, G \otimes_{\mathbf{Z}} \mathbf{Q} \longrightarrow \text{gr}\, \hat{G}\ , \tag{3.9}$$

which we will now show is an isomorphism. First note that the logarithm map yields an isomorphism $\text{gr}\,\hat{G} \simeq \text{gr}\,\mathcal{P}\hat{\mathbf{Q}}G$, and that the latter is by (A, 2.14) $\mathcal{P}(\text{gr}\,\hat{\mathbf{Q}}G) \simeq \mathcal{P}(\text{gr}\,\mathbf{Q}G)$. Thus we have to show that $\text{gr}\, G \otimes_{\mathbf{Z}} \mathbf{Q} \simeq \mathcal{P}(\text{gr}\,\mathbf{Q}G)$ which is proved for any group in [23]. Here however things are simpler because G is free, so $\text{gr}\,\mathbf{Q}G$ is the tensor algebra on $\text{gr}_1\,\mathbf{Q}G$ and so $\mathcal{P}(\text{gr}\,\mathbf{Q}G) \simeq L(\text{gr}_1\,\mathbf{Q}G)$. Also $\text{gr}\, G \otimes \mathbf{Q} \simeq L(\text{gr}_1\, G \otimes \mathbf{Q})$, so the isomorphism in question follows from the canonical isomorphism $\text{gr}_1\, G \otimes \mathbf{Q} \simeq \text{gr}_1\,\mathbf{Q}G$.

Consider the diagram

$$\begin{tikzcd}
{} \arrow[r] & \pi_q(\text{gr}_r\, G) \otimes \mathbf{Q} \arrow[r] \arrow[d, "\cong"] & \pi_q(G/\Gamma_{r+1}G) \otimes \mathbf{Q} \arrow[r] \arrow[d] & \pi_q(G/\Gamma_r G) \otimes \mathbf{Q} \arrow[r, "\partial"] \arrow[d] & {} \\
{} \arrow[r] & \pi_q(\text{gr}_r\, \hat{G}) \arrow[r] & \pi_q(\hat{G}/F_{r+1}\hat{G}) \arrow[r] & \pi_q(\hat{G}/F_r\hat{G}) \arrow[r] & {}
\end{tikzcd}$$

where the vertical maps are induced by α, where the tensor product is over $\mathbf{Z}$ and the top row is exact since $\mathbf{Q}$ is flat over $\mathbf{Z}$, and where the first vertical arrow is an isomorphism by (3.9). By induction on r and the five lemma, α induces the isomorphism

$$\pi_q(G/\Gamma_r G) \otimes \mathbf{Q} \xrightarrow{\sim} \pi_q(\hat{G}/F_r\hat{G})\ . \tag{3.10}$$

By (3.7) the inverse system on the left is eventually constant, so $R^1\,\text{lim-inv}_r\, \pi_q(\hat{G}/F_r\hat{G}) = 0$. As $\hat{G} \simeq \text{lim-inv}\, \hat{G}/F_r\hat{G}$, (3.8) shows that $\text{lim-inv}_r \pi_q(\hat{G}/F_r\hat{G}) \simeq \pi_q(\hat{G})$. So taking the inverse limit of the isomorphisms (3.10), we have $\pi_q(G) \otimes \mathbf{Q} \simeq \pi_q(\hat{G})$, which proves (3.4).

The proofs of (3.5) and (3.6) proceed by the same method, filtering so that the associated graded algebras are isomorphic, and passing to the inverse limit by means of (3.7) and (3.8). The details are omitted.

We can now prove the hypotheses of (2.3) for the pair $\hat{\mathbf{Q}}$, $\mathcal{G}$. Recall that we are localizing $(\text{sGp})_1$ (resp. $(\text{sCHA})_1$) with respect to maps f such that

$(\pi f) \otimes_{\mathbf{Z}} \mathbf{Q}$ (resp. $\pi_* \mathcal{G} f$) is an isomorphism. Hypothesis (i) is therefore obvious. For (ii) we must show that if $f: A \to B$ is a map in $(\text{SCHA})_1$, then $\pi \mathcal{G} f$ is an isomorphism if and only if $\pi \mathcal{G} f \otimes_{\mathbf{Z}} \mathbf{Q}$ is an isomorphism. But this is true by (3.3), which implies that $\pi \mathcal{G} A \simeq (\pi \mathcal{G} A) \otimes_{\mathbf{Z}} \mathbf{Q}$ and similarly for B. For (iii) we may take $R = G\overline{W}$ and $\xi =$ the adjunction map α. By Kan's work (see proof of 2.1) if $G \in \operatorname{Ob}(\text{SGp})_1$, $\xi: RG \to G$ is a weak equivalence and hence a rational homotopy equivalence, so it remains to show that $\beta: RG \to \mathcal{G}\mathbf{Q}RG$ is a rational homotopy equivalence. Now RG is free and connected so (3.4) shows that β induces the isomorphism $\theta: \pi(RG) \otimes_{\mathbf{Z}} \mathbf{Q} \xrightarrow{\sim} \pi(\mathcal{G}\hat{\mathbf{Q}}RG)$ given by $\theta(\zeta \otimes q) = q \cdot \pi(\beta)\zeta$, using the $\mathbf{Q}$ module structure of $\pi(\mathcal{G}\hat{\mathbf{Q}}RG)$ afforded by (3.3). However since $\mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{Q} \simeq \mathbf{Q}$, θ is isomorphic to the map $\pi(\beta) \otimes \mathrm{id}: \pi(RG) \otimes_{\mathbf{Z}} \mathbf{Q} \to \pi(\mathcal{G}\hat{\mathbf{Q}}RG) \otimes_{\mathbf{Z}} \mathbf{Q}$, and therefore $\beta: RG \to \mathcal{G}\hat{\mathbf{Q}}RG$ is a rational homotopy equivalence. We have therefore verified the hypotheses of (2.3), so it follows that the functors $\mathbf{L}\hat{\mathbf{Q}}$ and $\tilde{\mathcal{G}}$ in Figure 2 are equivalences of categories.

Remark 3.11. It is perhaps worthwhile to note that, with the exception of the last paragraph, the results of this section generalize immediately to the case where $\mathbf{Q}$ is replaced by a field K of characteristic zero. In fact all the equivalences of Figure 2 to the right of $\operatorname{Ho}(\text{SCHA})_1$ are valid where algebra, Lie algebra, etc., are taken over K. However $\mathbf{L}\hat{K}$ and $\tilde{\mathcal{G}}$ are no longer equivalences, the reason being that $K \otimes_{\mathbf{Z}} K \xrightarrow{\sim} K$ only if $K = \mathbf{Q}$.

We now verify the hypotheses of (2.3) for the pair $\hat{U}, \mathcal{P}$ using some results from the following section. Again (i) is trivial, while from (3.2) we have that a map $f: A \to B$ in $(\text{SCHA})_1$ is such that $\pi \mathcal{G} f$ is an isomorphism if and only if $\pi \mathcal{P} f$ is an isomorphism, proving (ii). For (iii) we shall take $R = N^* \mathcal{L} \mathcal{C} N$ and ξ to be the composite of the adjunction maps $N^* \mathcal{L} \mathcal{C} N \to N^* N \to \mathrm{id}$. If $\mathfrak{g}$ is a reduced simplicial Lie algebra, then $N\mathfrak{g}$ is a reduced DG Lie algebra, so by the properties of $\mathcal{L}$ and $\mathcal{C}$ (B, § 6, Th. 7.5), $\mathcal{L}\mathcal{C}N\mathfrak{g}$ is a free reduced DGL and $\alpha: \mathcal{L}\mathcal{C}N\mathfrak{g} \to N\mathfrak{g}$ is a weak equivalence. By (4.5) $\beta: \mathcal{L}\mathcal{C}N\mathfrak{g} \to NN^* \mathcal{L}\mathcal{C}N\mathfrak{g}$ is a weak equivalence; it is now straightforward to verify that $\xi: R\mathfrak{g} \to \mathfrak{g}$ is a weak equivalence. Moreover by (4.4) $N^* \mathcal{L}\mathcal{C}N\mathfrak{g} = R\mathfrak{g}$ is a free reduced simplicial Lie algebra, so $\beta: R\mathfrak{g} \to \mathcal{P}\hat{U}R\mathfrak{g}$ is a weak equivalence by (3.5). Therefore ξ and R satisfy (iii), and the functors $\mathbf{L}\hat{U}$ and $\mathcal{P}$ of Figure 2 are equivalences of categories.

4. DG and simplical Lie algebras

In this section we shall retain our previous notation. However the results are valid with $\mathbf{Q}$ replaced by any field of characteristic zero.

Let

$$N\colon (sV) \longrightarrow (\mathrm{DG})$$

be the normalization functor from the category of simplicial vector spaces to the category of DG vector spaces. N is given by (3.1) so

$$\pi(V) = H(NV)\ . \tag{4.1}$$

By Dold-Puppe [8], N is an equivalence of categories. We shall denote the inverse functor by N^{-1}. Recall that a simplicial vector space V may be regarded as a chain complex with differential $d = \Sigma(-1)^i d_i$ and that then NV is a subcomplex of V.

Let V, W be simplicial vector spaces, and let $V \otimes W$ be their dimension-wise tensor product. If $x \in V_p$ and $y \in W_q$ let $x \mathbin{\underline{\otimes}} y \in (V \otimes W)_{p+q}$ be the element given by the Eilenberg-Zilber formula

$$x \mathbin{\underline{\otimes}} y = \textstyle\sum_{(\mu,\nu)} \varepsilon(\mu, \nu) s_{\nu_q} \cdots s_{\nu_1} x \otimes s_{\mu_p} \cdots s_{\mu_1} y \tag{4.2}$$

where (μ, ν) runs over all p, q shuffles, i.e., permutations $(\mu_1, \cdots, \mu_p, \nu_1, \cdots, \nu_q)$ of $\{0, \cdots, p+q-1\}$ such that $\mu_1 < \cdots < \mu_p$ and $\nu_1 < \cdots < \nu_q$, and where $\varepsilon(\mu, \nu)$ is the sign of the permutation. The following properties of $\underline{\otimes}$ are well known.

(i) $d(x \mathbin{\underline{\otimes}} y) = dx \mathbin{\underline{\otimes}} y + (-1)^{\deg x} x \mathbin{\underline{\otimes}} dy$

(ii) $x \mathbin{\underline{\otimes}} (y \mathbin{\underline{\otimes}} z) = (x \mathbin{\underline{\otimes}} y) \mathbin{\underline{\otimes}} z$

(iii) If $T\colon V \otimes W \xrightarrow{\sim} W \otimes V$ is given by $T(x \otimes y) = y \otimes x$, then $T(x \mathbin{\underline{\otimes}} y) = (-1)^{pq} y \mathbin{\underline{\otimes}} x$ if $\deg x = p$ and $\deg y = q$.

(iv) If $x \in N_p V$ and $y \in N_q W$, then $x \mathbin{\underline{\otimes}} y \in N_{p+q}(V \otimes W)$ and the map of chain complexes

$$\begin{aligned} (NV) \otimes (NW) &\longrightarrow N(V \otimes W) \\ x, y &\longmapsto x \mathbin{\underline{\otimes}} y \end{aligned}$$

is a chain homotopy equivalence (Eilenberg-Zilber theorem).

Let $\mathfrak{g}$ be a simplicial Lie algebra and if $x \in \mathfrak{g}_p$, $y \in \mathfrak{g}_q$ define $[[x, y]] \in \mathfrak{g}_{p+q}$ to be the image of $x \mathbin{\underline{\otimes}} y$ under the bracket map $\mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$. It follows easily from (i)–(iv) that $\mathfrak{g}$ together with d and $[[\ ,\ ]]$ is a DG Lie algebra and that $N\mathfrak{g}$ is a sub-DG Lie algebra. We thus obtain a functor

$$N\colon (\mathrm{SLA}) \longrightarrow (\mathrm{DGL})\ . \tag{4.3}$$

Similarly $\underline{\otimes}$ defines the structure of a (commutative) DG algebra on NR if R is a simplicial (commutative) algebra.

PROPOSITION 4.4. *The functor* N (4.3) *has a left adjoint* N^*. N^* *carries free* DG *Lie algebras into free simplical Lie algebras.*

PROOF. If $\mathfrak{m}$ is a DG Lie algebra, then considering it as a DG vector space, we may form the simplicial vector space $N^{-1}\mathfrak{m}$ and the simplicial Lie algebra $LN^{-1}\mathfrak{m}$, where L is the free Lie algebra functor applied dimension-wise. If $x \in \mathfrak{m}$ (recall that we only consider homogeneous elements), then we let $N^{-1}x \in N^{-1}\mathfrak{m} \subset LN^{-1}\mathfrak{m}$ be the element corresponding to x under the identification of $\mathfrak{m}$ with $NN^{-1}\mathfrak{m} \subset N^{-1}\mathfrak{m}$. It is clear that if $\mathfrak{g}$ is a simplicial Lie algebra, then there is a one-to-one correspondence between DG vector space maps $\varphi\colon \mathfrak{m} \to N\mathfrak{g}$ and simplicial Lie algebra maps $\theta\colon LN^{-1}\mathfrak{m} \to \mathfrak{g}$ such that $\theta(N^{-1}x) = \varphi(x)$ for all $x \in \mathfrak{m}$. Let

$$N^*\mathfrak{m} = LN^{-1}\mathfrak{m}/I\ ,$$

where I is the smallest simplicial ideal of $LN^{-1}\mathfrak{m}$ containing the elements $[[N^{-1}x, N^{-1}y]] - N^{-1}[x, y]$ for $x, y \in \mathfrak{m}$. Then θ induces a map $N^*\mathfrak{m} \to \mathfrak{g}$ if and only if φ is a Lie homomorphism. Hence there is a one-to-one correspondence between DG Lie algebra maps $\varphi\colon \mathfrak{m} \to N\mathfrak{g}$ and simplicial Lie algebra maps $\theta\colon N^*\mathfrak{m} \to \mathfrak{g}$ and so N^* is a left adjoint functor to N. Note that the adjunction map $\beta\colon \mathfrak{m} \to NN^*\mathfrak{m}$ is given by $x \mapsto N^{-1}x + I$.

We recall that a map $f\colon X \to Y$ of simplicial objects over a category of universal algebras, in particular Lie algebras, is said to be *free* [14] if there are subsets $\Sigma_q \subset Y_q$ for each q such that $\Sigma = \bigcup \Sigma_q$ is stable under the degeneracy operators of Y and such that Y_q is the direct sum of X_q and the free algebra generated by the set Σ_q, $f_q\colon X_q \to Y_q$ being the inclusion of a summand. It may be shown that the class of free maps is closed under direct sums, cobase extension and sequential composition (i.e., if $X_1 \to X_2 \to \cdots$ are all free then $X_1 \to \operatorname{dir\,lim} X_i$ is free). Of course X is free if the map $\varphi \to X$ is free where φ is the initial object.

Now let $\mathfrak{m}$ be a free DG Lie algebra by which we mean that as a graded Lie algebra $\mathfrak{m}$ is isomorphic to $L^g(V)$ where V is a graded vector space and where L^g is the free graded Lie algebra functor (B, § 2). Define $\mathfrak{m}^{(k)}$ to be the subalgebra of $\mathfrak{m}$ generated by V_i, $i \leqq k$. Then $\mathfrak{m}^{(k)}$ is a sub-DG Lie algebra of $\mathfrak{m}$ called the k skeleton. Let $e_j\ j \in J$ be a basis of V_k; we wish to show that the k-skeleton of $\mathfrak{m}$ is obtained from the $k-1$ skeleton by attaching the e_j. Let $S(k-1)$(resp. $D(k)$) be the DG vector space generated by an element y_{k-1} of degree $k-1$ with $dy_{k-1} = 0$ (resp. by an element y_{k-1} of degree $k-1$ and an element x_k of degree k with $dx_k = y_{k-1}$ and $dy_{k-1} = 0$) and let $S(k-1) \hookrightarrow D(k)$ be the obvious inclusion. Then there is a cocartesian diagram

$$\begin{tikzcd} \bigvee_J L^g(S(k-1)) \arrow[r, hook] \arrow[d, "a"] & \bigvee_J L^g(D(k)) \arrow[d, "b"] \\ \mathfrak{m}^{(k-1)} \arrow[r, hook] & \mathfrak{m}^{(k)} \end{tikzcd}$$

in (DGL) where $\bigvee$ denotes direct sum, where a (resp. b) restricted to the j^{th} factor of the direct sum is given by $ay_{k-1} = de_j$ (resp. $bx_k = e_j$, $by_{k-1} = de_j$).

Since N^* is a left adjoint functor it will commute with direct sums, cobase extension, etc., so $N^*\mathfrak{m}$ will be free if we know that $N^*L^gS(k-1) \to N^*L^gD(k)$ is free. Let $A = \Delta(k-1)/\Delta(k-1)^{\cdot}$ (standard $k-1$ simplex with boundary collapsed to a point), let $B = \Delta(k)/V(k, 0)$ (standard k simplex with all faces but the last collapsed to a point) and let $A \to B$ be the map induced by the inclusion of the last face. If X is a pointed simplicial set let $\bar{Q}(X)$ be the simplicial vector space generated by X with basepoint identified with 0. Then it is easy to see that $N^{-1}S(k-1) \to N^{-1}D(k)$ is isomorphic to $\bar{Q}A \to \bar{Q}B$. Since $N^*L^g \simeq LN^{-1}$, the map $N^*L^gS(k-1) \to N^*L^gD(k)$ is isomorphic to $L\bar{Q}A \to L\bar{Q}B$. But the latter is clearly free, the subsets $\Sigma_q \subset L\bar{Q}B_q$ being given by the elements of B_q which are not in A_q. Therefore we have shown that $N^*\mathfrak{m}$ is free and the proof of the proposition is complete.

PROPOSITION 4.5. *Let V be a* DG *vector space and define maps of graded Lie algebras*

$$L^g(HV) \xrightarrow{a} H(L^gV) \xrightarrow{b} \pi(LN^{-1}V)$$

as follows. a is the unique graded Lie algebra map extending the map induced on homology by the inclusion of V in L^gV. As $NLN^{-1}V$ is a DG *Lie algebra, the map $V \to NLN^{-1}V$ given by $x \mapsto N^{-1}x$ extends to a map of* DG *Lie algebras $L^gV \to NLN^{-1}V$, and b is the induced map on homology. Then the maps a and b are isomorphisms.*

PROOF. Consider the diagram

$$\begin{tikzcd} L^g(HV) \arrow[r, "a"] \arrow[d, shift right, "i"'] & H(L^gV) \arrow[r, "b"] \arrow[d, shift right, "H(i)"'] & \pi(LN^{-1}V) \arrow[d, shift right, "\pi(i)"'] \\ T^g(HV) \arrow[r, "a'"] \arrow[u, shift right, "\rho"'] & H(T^gV) \arrow[r, "b'"] \arrow[u, shift right, "H(\rho)"'] & \pi(TN^{-1}V) \arrow[u, shift right, "\pi(\rho)"'] \end{tikzcd}$$

where T^g (resp. T) is the tensor algebra functor from DG (resp. simplicial) vector spaces to DG (resp. simplicial) algebras, where a', b' are defined similarly to a and b, where i is the inclusion of a DG or simplicial Lie algebra into its universal enveloping algebra, and where ρ is the canonical retraction (B, 2.2) of the tensor algebra onto the free Lie algebra given by

$$\rho(x_1 \otimes \cdots \otimes x_n) = \frac{1}{n}[x_1, [\cdots, x_n]] .$$

a' and b' are isomorphisms by Künneth and Eilenberg-Zilber, hence a' and b' being retracts of isomorphisms are also isomorphisms, q.e.d.

THEOREM 4.6. *If* $\mathfrak{m}$ *is a free reduced* DG *Lie algebra, then* $\beta: \mathfrak{m} \to NN^*\mathfrak{m}$ *is a weak equivalence.*

PROOF. Let $\Gamma_r N^*\mathfrak{m}$ be the lower central series filtration of $N^*\mathfrak{m}$. As $[\Gamma_p N^*\mathfrak{m}, \Gamma_q N^*\mathfrak{m}] \subset \Gamma_{p+q} N^*\mathfrak{m}$, it follows that $[[N\Gamma_p N^*\mathfrak{m}, N\Gamma_q N^*\mathfrak{m}]] \subset N\Gamma_{p+q} N^*\mathfrak{m}$ and hence that $\beta\Gamma_r^g\mathfrak{m} \subset N\Gamma_r N^*\mathfrak{m}$, where Γ_r^g is the lower central series filtration in the graded sense for $\mathfrak{m}$. Consequently there is an induced map $\operatorname{gr}\beta: \operatorname{gr}(\mathfrak{m}) \to N \operatorname{gr}(N^*\mathfrak{m})$, where we have used that N is exact. By (4.4) $N^*\mathfrak{m}$ is free so $\operatorname{gr}(N^* \mathfrak{m}) \simeq L(N^*\mathfrak{m})_{ab}$; similarly $\operatorname{gr} \mathfrak{m} = L^g(\mathfrak{m}_{ab})$. But $\operatorname{gr} \beta_1$ induces an isomorphism $\mathfrak{m}_{ab} \xrightarrow{\sim} N(N^*\mathfrak{m})_{ab}$; to see this, note that the canonical maps $\mathfrak{m} \to \mathfrak{m}_{ab}$ and $\mathfrak{m} \to N(N^*\mathfrak{m})_{ab}$ are both universal for DG Lie algebra maps from $\mathfrak{m}$ to abelian DG Lie algebras, and hence are isomorphic. Thus $\operatorname{gr}\beta$ is of the form $L^g V \to NLN^{-1}V$ which by (4.5) is a weak equivalence. By the five lemma and induction, one sees that

$$H_q(\mathfrak{m}/\Gamma_r\mathfrak{m}) \xrightarrow{\sim} \pi_q(N^*\mathfrak{m}/\Gamma_r N^*\mathfrak{m}) .$$

For large enough r, $(\Gamma_r\mathfrak{m})_q = 0$ as $\mathfrak{m}$ is reduced, and $\pi_q(\Gamma_r N^*\mathfrak{m}) = 0$ by (3.5) so $H_q(\mathfrak{m}) \simeq \pi_q(N^*\mathfrak{m})$ and the theorem is proved.

It is now possible to check that the hypotheses of (2.3) hold for the functors N^* ond N. Hypothesis (i) is trivial and (ii) follows from (4.1). For (iii) we take $R = \mathcal{L}\mathcal{C}$ and $\xi =$ the adjunction map α. By (B, 7.5) ξ is always a weak equivalence and the formulas for $\mathcal{L}$ and $\mathcal{C}$ show that $\mathcal{L}\mathcal{C}\mathfrak{m}$ is free and reduced if $\mathfrak{m}$ is reduced. Thus by the above theorem $\beta: R\mathfrak{m} \to NN^*R\mathfrak{m}$ is a weak equivalence and (iii) holds. Therefore by (2.3) we have that the functors $\tilde{N}$ and $\mathbf{L}N^*$ in Figure 2 are equivalences of categories.

Remark. One may show by essentially the same arguments used above that the normalization functor from the category of reduced simplicial commutative algebras to the category of reduced DG commutative algebras induces an equivalence of the corresponding homotopy categories. The filtration Γ_r is replaced by the powers I^r of the argumentation ideal which become higher connected with r by the same argument as (3.6)(see also [25]). Again the really key point is the fact that the symmetric algebra SV is a retract of TV and this uses essentially the fact that $\mathbf{Q}$ has characteristic zero.

5. The Whitehead product

In this section we prove the part of Theorem I relating the rational homotopy Lie algebra of a space with the homology of the associated DG Lie algebra.

Let X be a 1-connected pointed space and let φ be the composition

$$\pi_{q+1}(X) \xrightarrow[\sim]{\partial} \pi_q(\Omega X) \xrightarrow{H} H_q(\Omega X, \mathbf{Z})$$

where ∂ is the boundary operator for the path space fibration $\Omega X \to EX \to X$, and where H is the Hurewicz homomorphism. Samelson [26] has proved the formula

$$\varphi[u, v] = (-1)^p[\varphi u, \varphi v] \qquad \text{if } p = \deg u ,$$

where the bracket on the left is the Whitehead product and on the right is the bracket associated to the Pontrjagin product on $H_*(\Omega X, \mathbf{Z})$. Milnor-Moore [20, appendix] show that H induces an isomorphism of $\pi(\Omega X) \otimes \mathbf{Q}$ with the primitive Lie algebra of the Hopf algebra $H_*(\Omega X, \mathbf{Q})$. Combining these results with the definition of πX given in § 1 we have

PROPOSITION 5.1. *There is a canonical graded Lie algebra isomorphism* $\pi(X) \xrightarrow{\sim} \operatorname{Im}\{\pi(\Omega X) \otimes_{\mathbf{Z}} \mathbf{Q} \to H_*(\Omega X, \mathbf{Q})\} = \mathcal{P}H_*(\Omega X, \mathbf{Q})$.

PROPOSITION 5.2. *If K is a 2-reduced simplicial set, then there is a natural commutative diagram*

$$\begin{tikzcd} \pi(GK) \arrow[r, "h"] \arrow[d, "\cong"] & \pi(\mathbf{Z}GK) \arrow[d, "\zeta"', "\cong"] \\ \pi(\Omega\,|\,K\,|) \arrow[r, "H"] & H(\Omega\,|\,K\,|, \mathbf{Z}) \end{tikzcd}$$

where h is the simplicial Hurewicz homomorphism and ζ is an isomorphism of algebras for the $\underline{\otimes}$ product in $\pi(\mathbf{Z}GK)$ and the Pontrjagin product in $H_*(\Omega\,|\,K\,|, \mathbf{Z})$.

PROOF. We recall that if K is a Kan complex with basepoint $*$, then the Hurewicz homomorphism $h\colon \pi(K) \to \pi(\mathbf{Z}K)$ on the simplicial level is the map on homotopy induced by $x \mapsto x - s_0^n(*)$ if $\deg x = n$. It is easily seen that h is compatible with the topological Hurewicz homomorphism in the sense that the diagram

$$\begin{tikzcd} \pi(X) \arrow[r, "H"] \arrow[d, "\theta", "\wr"'] & H_*(X, \mathbf{Z}) \arrow[d, equal] \\ \pi(\operatorname{Sing} X) \arrow[r, "h"] & \pi(\mathbf{Z} \operatorname{Sing} X) \end{tikzcd}$$

is commutative, where θ is the isomorphism induced by the map which sends a map $u\colon \Delta(q)/\Delta(q)^{\cdot} \to \operatorname{Sing} X$ into the composition $S^q \xrightarrow{\eta} |\Delta(q)/\Delta(q)^{\cdot}| \xrightarrow{|u|} |\operatorname{Sing} X| \xrightarrow{\alpha} X$, where η is any orientation-preserving homotopy equivalence. Using this compatibility, the proposition reduces to showing that there is a natural commutative diagram

$$\begin{tikzcd}
\pi(GK) \arrow[r, "h"] \arrow[d, "\cong"] & \pi(\mathbf{Z}GK) \arrow[d, "\zeta"', "\cong"] \\
\pi(\operatorname{Sing}\Omega\,|K|) \arrow[r, "h"] & \pi(\mathbf{Z}\operatorname{Sing}\Omega\,|K|)
\end{tikzcd}$$

where ζ is an algebra isomorphism for the $\underline{\otimes}$ products coming from the maps $\mu\colon GK \times GK \to GK$ and $\nu\colon \Omega|K| \times \Omega|K| \to \Omega|K|$ furnished by group multiplication and composition of paths. Therefore all we have to do is define in a natural way a homotopy equivalence of $\operatorname{Sing}\Omega|K|$ and GK which up to homotopy is compatible with μ and ν.

At this point we remark that EX is the space of paths in X ending at the basepoint, and that the map $EX \to X$ sends a path into its initial point. Then composition defines a right action of ΩX on EX. Let $GK \to K \times_\tau GK \to K$ be the universal principal GK bundle so that $K \times_\tau GK$ is acyclic. The geometric realization functor carries this fibration into a principal fibration with topological group $|GK|$ (at least in the category of Kelley spaces which is sufficient for our purposes [9, Ch. III]). Hence there is a commutative diagram

$$\begin{tikzcd}
\Omega|K| \arrow[r] \arrow[d, "\rho"] & E|K| \arrow[r] \arrow[d, "\sigma"] & {|K|} \arrow[d, equal] \\
{|GK|} \arrow[r] & {|K \times_\tau GK|} \arrow[r] & {|K|}
\end{tikzcd}$$

where σ exists by the covering homotopy theorem using the contractibility of $E|K|$, and where ρ is induced by σ. As $|K \times_\tau GK|$ is contractible, σ and hence ρ is a homotopy equivalence. Since we have arranged groups to act to the right for principal bundles, it is fairly easy to see that ρ is a map of H-spaces up to homotopy, so taking a homotopy inverse of ρ we obtain a homotopy equivalence $GK \to \operatorname{Sing}\Omega|K|$, which up to homotopy is compatible with μ and ν. This completes the proof of the proposition.

PROPOSITION 5.3. *If A is a simplicial* CHA, *let $i\colon \mathcal{P}A \to A$ be the inclusion map, and let $j\colon \mathcal{G}A \to A$ be the map $j\sigma = \sigma - 1$. Then the diagram*

$$\begin{tikzcd} \pi_q(\mathcal{P}A) \arrow[rr, "\pi(\exp)", "\sim"'] \arrow[dr, "\pi(i)"'] & & \pi_q(\mathcal{G}A) \arrow[dl, "\pi(j)"] \\ & \pi_q A & \end{tikzcd}$$

is commutative for $q > 0$. Moreover $\pi(i)$ is injective.

PROOF. $\pi(i)$ is injective because there is a canonical retraction of a CHA onto its primitive subspace (A, 2.16). To show the diagram commutative, we must show that if an element of $\pi_q(\mathcal{P}A)$ is represented by $x \in (N\mathcal{P}A)_q$, then x and $e^x - 1$ differ by a boundary in NA. Let $S^q = \Delta(q)/\Delta(q)^{\boldsymbol{\cdot}}$, let σ be the canonical q-simplex of S^q, and let $\bar{\mathbf{Q}}S^q$ be the reduced chains on S^q. Then there is a unique map of SCHA's $\hat{T}\bar{\mathbf{Q}}S^q = \hat{U}L\mathbf{Q}S^q \to A$ which sends σ to x. By naturality we may assume $A = \hat{T}\bar{\mathbf{Q}}S^q$ and $x = \sigma$. But by (3.6) $\pi(\hat{T}\bar{\mathbf{Q}}S^q) = \pi(T\bar{\mathbf{Q}}S^q)$, which by Künneth is a tensor algebra on the class of σ. In particular one sees that if I is the augmentation ideal of $T\bar{\mathbf{Q}}S^q$, then $\pi_q I^2 = 0$. Therefore $(e^\sigma - 1) - \sigma \in (I^2)_q$ is a boundary and the proposition is proved.

Combining these propositions we have the following isomorphism of functors from $\mathcal{S}_2$ to (GL):

$$\begin{aligned} \pi(|K|) &\simeq \operatorname{Im}\{\pi(\Omega\,|K|) \otimes \mathbf{Q} \hookrightarrow H_*(\Omega\,|K|, \mathbf{Q})\} && (5.1)\\ &\simeq \operatorname{Im}\{\pi(GK) \otimes \mathbf{Q} \longrightarrow \pi(\mathbf{Q}GK)\} && (5.2)\\ &\simeq \operatorname{Im}\left\{\pi(\mathcal{G}\hat{\mathbf{Q}}GK) \xrightarrow{\pi(i)} \pi(\hat{\mathbf{Q}}GK)\right\} && (3.4), (3.6)\\ &\simeq \pi(\mathcal{P}\hat{\mathbf{Q}}GK) && (5.3)\\ &\simeq H(N\mathcal{P}\hat{\mathbf{Q}}GK)\ . \end{aligned}$$

Therefore if λ is the composition $\tilde{N}\mathcal{P}(\mathbf{L}\hat{\mathbf{Q}})\tilde{G}(E_2\,\mathrm{Sing})\check{}$, we have a canonical isomorphism of functors $\pi X \simeq \pi\lambda X$ from $\mathrm{Ho}_{\mathbf{Q}}\mathcal{T}_2$ to (GL) as asserted in Theorem I.

6. The coproduct on homology

In this section we prove the part of Theorem I relating the rational homology coalgebra of a space X with the homology of the DG coalgebra associated to X by the equivalences in Figure 2. The method is to obtain a formula (6.5) for the rational homology coalgebra of a reduced simplicial set K in terms of $\hat{\mathbf{Q}}GK$.

We begin by reviewing properties of the adjoint functors G and $\bar{W}$ between the categories $\mathcal{S}_1$ of reduced simplicial sets and (SGP) the category of simplicial groups (see [12], [4]). We adhere to the convention adopted in Appendix B that a group acts to the right of a principal bundle; this causes only minor differences in the formulas used here with those of [12] and [4].

If $q\colon E \to K$ is principal fibration of simplicial sets with simplicial group

G and K is reduced, then a lifting function $\rho\colon K \to E$ is by definition a section of q which commutes with all degeneracy operators and with all faces but d_0. Defining $\tau\colon K_q \to G_{q-1}$, $q > 0$ by $d_0\rho x = (\rho d_0 x)\tau x$, one sees that τ satisfies the formulas

$$(6.1) \qquad \begin{aligned} s_i\tau &= \tau s_{i+1} & & i \geqq 0 \\ \tau s_0 x &= \text{the identity in } G_q & & \text{if } x \in K_q \\ d_i\tau &= \tau d_{i+1} & & i > 0 \\ \tau d_1 x &= (\tau d_0 x)(d_0\tau x)\,. \end{aligned}$$

Such a map $\tau\colon K \to G$ will be called a *twisting function* and $\mathcal{T}(K, G)$ will denote the set of twisting functions from K to G. $\mathcal{T}(K, G)$ is a functor contravariant in $K \in \operatorname{Ob} \mathcal{S}_1$ and covariant in $G \in \operatorname{Ob}$ (sGp) and the functors $G\colon \mathcal{S}_1 \to$ (sGp) and $\overline{W}\colon$ (sGp) $\to \mathcal{S}_1$ are defined so that there are natural isomorphisms

$$(6.2) \qquad \operatorname{Hom}_{\mathcal{S}_1}(K, \overline{W}G) \simeq \mathcal{T}(K, G) \simeq \operatorname{Hom}_{(\mathrm{sGp})}(GK, G)\,.$$

If $\tau\colon K \to G$ is a twisting function and K is a Kan complex, then τ induces a homomorphism $\pi_n K \to \pi_{n-1} G$ which will be denoted by $\tilde{\tau}$. If τ arises from a principal G bundle $E \to K$ with a lifting function, then $\tilde{\tau} \simeq \partial\colon \pi_n K \to \pi_{n-1} G$, the boundary operator in the homotopy long exact sequence.

If A, B are simplicial abelian groups, and A is reduced, then by a *twisting homomorphism* $\tau\colon A \to B$ we mean a twisting function such that $\tau\colon A_q \to B_{q-1}$ is a group homomorphism. For example if $\tau\colon K \to G$ is a twisting function, then τ induces a twisting homomorphism $\tau'\colon \overline{\mathbf{Q}}K \to G_{ab} \otimes_{\mathbf{Z}} \mathbf{Q} \simeq I/I^2$, where $\overline{\mathbf{Q}}K$ is the free simplicial $\mathbf{Q}$ module generated by K with basepoint set equal to 0, and where I is the augmentation ideal of $\mathbf{Q}G$.

PROPOSITION 6.3. *Let $\tau\colon K \to GK$ be the canonical twisting function coming from* (6.2). *Then the twisting homomorphism $\tau'\colon \overline{\mathbf{Q}}K \to I/I^2$ induces an isomorphism $\tilde{\tau}'\colon \pi_n(\overline{\mathbf{Q}}K) \to \pi_{n-1}(I/I^2)$ for $n > 0$.*

PROOF. If A is a reduced simplicial abelian group, then there is a canonical exact sequence

$$0 \longrightarrow \Omega A \longrightarrow EA \xrightarrow{\theta} A \longrightarrow 0$$

defined as follows. $(EA)_q = A_{q+1}$ and $d_j\tilde{a} = d_{j+1}a$, $s_j\tilde{a} = s_{j+1}a$ where if $a \in A_{q+1}$, then $\tilde{a}$ is the corresponding element of $(EA)_q$. $\theta\colon EA \to A$ is given by $\theta\tilde{a} = d_0 a$ and $\Omega A = \operatorname{Ker}\theta$. Note that if $\rho\colon A \to EA$ is given by $\rho a = (s_0 a)^\sim$, then ρ is a lifting function and the associated twisting homomorphism is given by $\tau_1 a = (a - s_0 d_0 a)^\sim$. As EA is contractible $\tilde{\tau}_1\colon \pi_n A \xrightarrow{\sim} \pi_{n-1}\Omega A$ for $n > 0$. Taking A to be $\overline{\mathbf{Q}}K$ we have the commutative diagram

$$\begin{tikzcd} & \overline{\mathbf{Q}}K \arrow[dl, "\tau_1"'] \arrow[dr, "\tau'"] & \\ \Omega\overline{\mathbf{Q}}K \arrow[rr, "\varphi"'] & & I/I^2 \end{tikzcd}$$

where $\varphi(\tilde{a}) = \tau' a$ is an isomorphism. In effect $(GK)_q$ is the free group generated by the elements τx and $(\Omega\overline{\mathbf{Q}}K)_q$ is the free $\mathbf{Q}$ module generated by the elements $\tilde{x}$ as x runs over the set $K_{q+1} - s_0 K_q$. Therefore as φ and τ_1 induce isomorphisms on homotopy so does τ', and the proposition follows.

PROPOSITION 6.4. *Let A, A' and B be simplicial abelian groups with A reduced, and let $\tau\colon A \to A'$ be a twisting homomorphism. Then $\tau \otimes d_0\colon A \otimes B \longrightarrow A' \otimes B$ and $d_0 \otimes \tau\colon B \otimes A \to B \otimes A'$ are twisting homomorphisms and we have the formulas*

$$\begin{aligned} (\tau \otimes d_0)(a \underline{\otimes} b) &= \tau a \underline{\otimes} b \\ (d_0 \otimes \tau)(b \underline{\otimes} a) &= (-1)^p b \underline{\otimes} \tau a \qquad \textit{if } \deg b = p\,. \end{aligned}$$

PROOF. The fact that $d_0 \otimes \tau$ is a twisting function is straightforward. On the other hand if $b \in B_p$ and $a \in A_q$ then with the notation of (4.2)

$$(d_0 \otimes \tau)(b \underline{\otimes} a) = \sum\nolimits_{\substack{(\mu,\nu) \\ \nu_1 = 0}} \varepsilon(\mu, \nu) d_0 s_{\nu_q} \cdots s_{\nu_1} b \otimes \tau s_{\mu_p} \cdots s_{\mu_1} a\,.$$

By (6.1) $\tau s_{\mu_p} \cdots s_{\mu_1} a = 0$ if $\mu_1 = 0$ and $s_{\mu_p - 1} \cdots s_{\mu_1 - 1} \tau a$ if $\mu_1 > 0$, so

$$\begin{aligned} (d_0 \otimes \tau)(b \underline{\otimes} a) &= \sum\nolimits_{\substack{(\mu,\nu) \\ \nu_1 = 0}} \varepsilon(\mu, \nu) s_{\nu_q - 1} \cdots s_{\nu_2 - 1} b \otimes s_{\mu_p - 1} \cdots s_{\mu_1 - 1} a \\ &= (-1)^p b \underline{\otimes} \tau a\,. \end{aligned}$$

The proof of the formula for $\tau \otimes d_0$ is similar, q.e.d.

We shall denote the map on homotopy $(d_0 \otimes \tau)^{\sim}\colon \pi_n(A \otimes B) \to \pi_{n-1}(A \otimes B')$ by $1 \otimes \tilde{\tau}$ and similarly denote $(\tau \otimes d_0)^{\sim}$ by $\tilde{\tau} \otimes 1$. Then from the proposition we have

$$\begin{aligned} (\tilde{\tau} \otimes 1)(\alpha \otimes \beta) &= \tilde{\tau}\alpha \otimes \beta \\ (1 \otimes \tilde{\tau})(\beta \otimes \alpha) &= (-1)^q \beta \otimes \tilde{\tau}\alpha \end{aligned}$$

if $\alpha \in \pi_p A$, $\beta \in \pi_q B$, where $\alpha \otimes \beta \in \pi_{p+q}(A \otimes B)$ is the class represented by $a \underline{\otimes} b$ if a represents α and b represents β. Similarly if $\tau_1\colon A \to A'$ and $\tau_2\colon B \to B'$ are twisting homomorphisms $\tilde{\tau}_1 \otimes \tilde{\tau}_2 : \pi_n(A \otimes B) \longrightarrow \pi_{n-2}(A' \otimes B')$ is the homomorphism induced by $a \otimes b \mapsto \tau_1 d_0 a \otimes d_0 \tau_2 b$ so that $(\tilde{\tau}_1 \otimes \tilde{\tau}_2)(\alpha \otimes \beta) = (-1)^p \tilde{\tau}_1 \alpha \otimes \tilde{\tau}_2 \beta$ if $\deg \alpha = p$.

PROPOSITION 6.5. *Let $\tau\colon K \to G$ be a twisting function, let $I = \overline{\mathbf{Q}}G$ be the augmentation ideal of $\mathbf{Q}G$, and let $\tau'\colon \overline{\mathbf{Q}}K \to I/I^2$ be the twisting homomor-*

phism given by $\tau' x = (x-1) + I^2$ if $x \in K$. Then the following diagram is commutative

$$\begin{array}{ccc} \pi_n(\bar{\mathbf{Q}}K) & \xrightarrow{\qquad -\tilde{\tau}' \qquad} & \pi_{n-1}(I/I^2) \\ \downarrow{\scriptstyle \pi_n\Delta} & & \downarrow{\scriptstyle \partial} \\ \pi_n(\bar{\mathbf{Q}}K \otimes \bar{\mathbf{Q}}K) \xrightarrow{\tilde{\tau}' \otimes \tilde{\tau}'} & \pi_{n-2}(I/I^2 \otimes I/I^2) & \xrightarrow{m} \pi_{n-2}(I^2/I^3) \end{array} \tag{6.6}$$

where Δ is the map induced by the diagonal $K \to K \times K$, m is induced by the multiplication in gr $\mathbf{Q}G$, *and where ∂ is the boundary homomorphism for the exact sequence*

$$0 \longrightarrow I^2/I^3 \longrightarrow I/I^3 \longrightarrow I/I^2 \longrightarrow 0 .$$

PROOF. Let $\alpha \in \pi_n(\bar{\mathbf{Q}}K)$ be represented by an element $z = \sum a_x x \in N(\bar{\mathbf{Q}}K)_n$, where x runs over the elements of K_n different from $s_0^n(*)(* =$ basepoint of $K)$ and the a_x are rational numbers. By definition the elements of K_{n-1} different from $s_0^{n-1}(*)$ form a basis for $(\bar{\mathbf{Q}}K)_{n-1}$, so $d_j z = \sum a_x d_j x = 0$ for $j \geqq 0$ implies that for any $y \neq s_0^{n-1}(*)$, the sum of the a_x with $d_j x = y$ is zero. Consequently

$$\sum a_x(\tau d_j x - 1) = 0 \quad \text{in } I , \qquad j > 0 . \tag{6.7}$$

The image of α in $\pi_{n-2}(I^2/I^3)$ obtained by going on the lower path of (6.6) is represented by

$$\sum a_x(\tau' d_0 x)(d_0 \tau' x) . \tag{6.8}$$

To calculate the image by the upper path note that

$$\omega = \sum a_x(\tau x - 1) + I^3 \in (I/I^3)_{n-1}$$

is by (6.7) an element of $N(I/I^3)_{n-1}$ congruent mod I^2 to $\tau' z$. Thus $d_0\omega$ represents $\partial\tilde{\tau}'\alpha$. By (6.1) we have

$$d_0\omega = \sum a_x[(\tau d_0 x)^{-1}(\tau d_1 x) - 1] + I^3 .$$

Using the identity

$$x^{-1}y - 1 \equiv -(x-1) + (y-1) + (x-1)^2 - (x-1)(y-1) \bmod I^3$$

in any group algebra and (6.7) we have

$$\begin{aligned} d_0\omega &= \sum a_x[(\tau' d_0 x)^2 - (\tau' d_0 x)(\tau' d_1 x)] + I^3 \\ &= -\sum a_x(\tau' d_0 x)(d_0 \tau' x) + I^3 \end{aligned}$$

since $\tau' d_1 x = \tau' d_0 x + d_0 \tau' x$. This is the negative of (6.8) so the proposition is proved.

We shall need the following Whitehead-type theorem for simplicial CHA's.

Recall that a CHA is said to be free (A, 2.11) if it is isomorphic to the completed universal enveloping algebra of a free Lie algebra.

PROPOSITION 6.9. *Let $\varphi\colon A \to B$ be a map of reduced simplicial* CHA's *which are both free in every dimension. Then the following assertions are equivalent*:

(i) $\mathrm{gr}_1\,\varphi$ *is a weak equivalence*

(ii) $\mathcal{P}\varphi$ *is a weak equivalence.*

PROOF. Consider the spectral sequence of homotopy groups which arises from the filtration on $\mathcal{P}A$ induced by the standard filtration on A. This is a decreasing filtration so if we index correctly we get a homological type spectral sequence

$$E^2_{pq} = \pi_{p+q}(\mathrm{gr}_q\,\mathcal{P}A) \Longrightarrow \pi_{p+q}(\mathcal{P}A) \qquad d_r\colon E^r_{pr} \longrightarrow E^r_{p-1,q+r-1}\ .$$

Using the fact that A is dimension-wise free, that A is reduced, and (4.5) one sees that this spectral sequence lies in the quadrant $q > 0$, $p \geqq 0$. Hence the inverse system $\pi_n(\mathcal{P}A/F_r\mathcal{P}A)$ is eventually constant and so by (3.8) the spectral sequence is convergent.

Consider the map $E^r_{pq}(\varphi)$ induced by φ from this spectral sequence to the similar one for B. As $E^2(\varphi) = \pi(L\mathrm{gr}_1\,\varphi) = L^g(\pi(\mathrm{gr}_1\,\varphi))$, one sees that if $\mathrm{gr}_1\,\varphi$ is a weak equivalence, then so must $\mathcal{P}\varphi$ by the convergence of the spectral sequence. For the converse, note that if $E^2_{p1}(\varphi)$ is an isomorphism for $p \leqq k$, then so is $E^2_{pq}(\varphi)$ for $p \leqq k$ and all q. Consequently by Zeeman's comparison theorem for spectral sequences, if $\mathcal{P}\varphi$ is a weak equivalence, $E^2_{pq}(\varphi)$ must be an isomorphism for all p and q and therefore $\mathrm{gr}_1\,\varphi$ is a weak equivalence, q.e.d.

We are now in a position to prove the principal result of this section. Let A be a reduced SCHA which is free in each dimension, and let I be the augmentation ideal of A. Define

$$H_q(A) = \begin{cases} \mathbf{Q} & q = 0 \\ \pi_{q-1}(I/I^2) & q > 0 \end{cases} \tag{6.10}$$

and let $\sigma\colon H_q(A) \xrightarrow{\sim} \pi_{q-1}(I/I^2)$ be 0 for $q = 0$ and the identity for $q > 0$. Define a comultiplication Δ on $H_*(A)$ by requiring the diagram

$$\begin{array}{ccccc} H_n(A) & & \xrightarrow{\quad -\sigma \quad} & & \pi_{n-1}(I/I^2) \\ \downarrow{\scriptstyle\Delta} & & & & \downarrow{\scriptstyle\partial} \\ \bigoplus_{i=0}^n H_i(A) \otimes H_{n-i}(A) & \underset{\sim}{\xrightarrow{\sigma\otimes\sigma}} & \bigoplus_{i=1}^{n-1} \pi_{i-1}(I/I^2) \otimes \pi_{n-i-1}(I/I^2) & \underset{\sim}{\xrightarrow{m}} & \pi_{n-2}(I^2/I^3) \end{array} \tag{6.11}$$

to be commutative where (as usual because σ is of degree -1)$(\sigma\otimes\sigma)(u\otimes v) = (-1)^p(\sigma u \otimes \sigma v)$ if $p = \deg u$, where m is induced by the isomorphism

$I/I^2 \otimes I/I^2 \to I^2/I^3$ (since A is free in each dimension) given by multiplication, and where ∂ is the boundary operator for the exact sequence $0 \to I^2/I^3 \to I/I^3 \to I/I^2 \to 0$. It will be shown that Δ is coassociative and cocommutative in the proof of the following.

THEOREM 6.12. *The functor $A \mapsto H_*(A)$ just defined on the full subcategory of* (SCHA) *consisting of dimension-wise free A extends uniquely up to canonical isomorphism to a functor*

$$\mathbf{H}\colon \mathrm{Ho}\,(\mathrm{SCHA})_1 \longrightarrow (\mathrm{GC}) .$$

Moreover if $\mathbf{H}\colon \mathrm{Ho}_{\mathbf{Q}}\,\mathcal{S}_2 \to (\mathrm{GC})$ is the extension of the rational homology coalgebra functor $K \mapsto H_*(K, \mathbf{Q})$ then there is an isomorphism of functors $\mathbf{H} \simeq \mathbf{H}(\mathbf{L}\hat{\mathbf{Q}})\tilde{G}$.

PROOF. We recall that $H_*(K, \mathbf{Q}) = \pi_*(\mathbf{Q}K)$ with the comultiplication defined to be the composition

$$\pi_n(\mathbf{Q}K) \xrightarrow{\pi_n(\Delta)} \pi_n(\mathbf{Q}K \otimes \mathbf{Q}K) \underset{\sim}{\xrightarrow{k^{-1}}} \textstyle\bigoplus_{i=0}^{n} \pi_i(\mathbf{Q}K) \otimes \pi_{n-i}(\mathbf{Q}K) ,$$

where $\Delta x = x \otimes x$ for $x \in K$, and where k is the (Künneth) isomorphism induced by the $\underline{\otimes}$ operation. By (6.3) we have a canonical isomorphism

$$H_n(K, \mathbf{Q}) \simeq \pi_n(\bar{\mathbf{Q}}K) \xrightarrow{\tilde{\tau}'} \pi_{n-1}(I/I^2) \xrightarrow{\sigma^{-1}} H_n(\hat{\mathbf{Q}}GK) \qquad \text{for } n > 0 .$$

By (6.5) and (6.11) we therefore have a canonical isomorphism of functors from $\mathcal{S}_2$ to (GC)

$$H_*(K, \mathbf{Q}) \simeq H_*(\hat{\mathbf{Q}}GK) .$$

This formula shows that $H_*(A)$ is cocommutative and coassociative if A is of the form $\hat{\mathbf{Q}}GK$ for some K. However given a dimension-wise free SCHA B, there is an adjunction map $\varphi\colon A \to B$, where $A = \hat{\mathbf{Q}}G\overline{W}\mathcal{G}B$, which by (3.4) induces an isomorphism for the functor $\mathcal{G}$ and hence also for $\mathcal{P}$ by (3.2). Thus by (6.9), $\mathrm{gr}_1\varphi$ is a weak equivalence so $H_*(A) \xrightarrow{\sim} H_*(B)$ as graded coalgebras. Thus $H_*(B)$ is an object of (GC) for any reduced dimension-wise free SCHA B.

$\mathbf{H}\colon \mathrm{Ho}\,(\mathrm{SCHA})_1 \to (\mathrm{GC})$ is unique up to canonical isomorphism because for any $B \in \mathrm{Ob}\,(\mathrm{SCHA})_1$ we must have

$$\mathbf{H}(\gamma B) \simeq \mathbf{H}(\gamma\hat{\mathbf{Q}}G\overline{W}\mathcal{G}B) \simeq H_*(\hat{\mathbf{Q}}G\overline{W}\mathcal{G}B) \simeq H_*(\overline{W}\mathcal{G}B, \mathbf{Q}) .$$

However by the universal property of γ we can use these isomorphisms to define $\mathbf{H}$. It is then clear that

$$\mathbf{H}(\mathbf{L}\hat{\mathbf{Q}})\tilde{G}(\gamma K) = \mathbf{H}(\gamma\hat{\mathbf{Q}}RGK) \simeq H_*(\overline{W}\mathcal{G}\hat{\mathbf{Q}}RGK, \mathbf{Q}) \simeq H_*(K, \mathbf{Q}) = \mathbf{H}(\gamma K) ,$$

and hence by (2.2) that $\mathbf{H}(\mathbf{L}\hat{\mathbf{Q}})\tilde{G} \simeq \mathbf{H}$, proving the theorem.

PROPOSITION 6.13. *There is a canonical isomorphism of functors from* (DGC)$_2$ *to* (GC)

$$H_*(C) \simeq H_*(\hat{U}N^*\mathfrak{L}C) .$$

PROOF. If $\mathfrak{m}$ is a free reduced DGL we may define a graded coalgebra $\mathbf{H}_*(\mathfrak{m})$ by imitating the definition of $H_*(A)$, namely $\mathbf{H}_q(\mathfrak{m}) \simeq H_{q-1}(\mathrm{gr}_1 U^g(\mathfrak{m}))$ with Δ induced by

$$\partial\colon H_q(\mathrm{gr}_1 U^g(\mathfrak{m})) \longrightarrow H_q(\mathrm{gr}_2 U^g(\mathfrak{m}))$$

via a diagram of the form (6.11). Similarly if $\mathfrak{g}$ is a free reduced SLA we may define $\mathbf{H}_q(\mathfrak{g}) \simeq \pi_{q-1}(\mathrm{gr}_1 U(\mathfrak{g}))$, etc. It is clear that there is a canonical coalgebra isomorphism

$$\mathbf{H}_*(\mathfrak{g}) \simeq H_*(\hat{U}(\mathfrak{g})) . \tag{6.14}$$

Let $\beta'\colon U^g(\mathfrak{m}) \to NU(N^*\mathfrak{m})$ be the obvious extension of $\beta\colon \mathfrak{m} \to NN^*\mathfrak{m}$ and filter $NU(N\mathfrak{m})$ by N applied to the powers of the augmentation ideal. Then β' is a map of filtered DG algebras. Moreover as in the proof of (4.6) one sees that $\mathrm{gr}\,\beta'$ is a weak equivalence. Therefore $\mathrm{gr}_1\,\beta'$ induces a coalgebra isomorphism

$$\mathbf{H}_*(\mathfrak{m}) \xrightarrow{\sim} \mathbf{H}_*(N^*\mathfrak{m}) . \tag{6.15}$$

Finally let $C \in (\mathrm{DGC})_1$, let $\tau\colon C \to \mathfrak{L}C$ be the canonical twisting function, (B, 6.1) so that

$$d\tau + \tau d + m(\tau \otimes \tau)\Delta = 0 , \tag{6.16}$$

where m denotes multiplication in $U(\mathfrak{L}C)$. It is an easy consequence of this formula and the multiplicative isomorphism $U(\mathfrak{L}C) \simeq T(\Omega\bar{C})$ that τ induces as isomorphism

$$H_n(C) \longrightarrow H_{n-1}(\mathrm{gr}_1 U^g(\mathfrak{L}C)) \simeq \mathbf{H}_n(\mathfrak{L}C) .$$

Moreover comparing (6.16) and the diagram (6.11) which has been used to define Δ on $\mathbf{H}_n(\mathfrak{L}C)$, one sees τ induces a canonical coalgebra isomorphism

$$H_*(C) \simeq H_*(\mathfrak{L}C) . \tag{6.17}$$

Combining (6.14), (6.15), and (6.17) the proposition is proved.

The part of Theorem I about homology coalgebras can now be proved. The isomorphism

$$H_*(X, \mathbf{Q}) = H_*(\mathrm{Sing}\, X, \mathbf{Q}) \simeq H_*(E_2\,\mathrm{Sing}\, X, \mathbf{Q})$$

shows that the $\mathbf{H}$ functors on $\mathrm{Ho}_{\mathbf{Q}}\,\mathcal{T}_2$ and $\mathrm{Ho}_{\mathbf{Q}}\,\mathcal{S}_2$ are isomorphic with respect to the equivalences of Figure 2. Theorem 6.12 shows that the $\mathbf{H}$ functors on $\mathrm{Ho}_{\mathbf{Q}}\,\mathcal{S}_2$ and $\mathrm{Ho}\,(\mathrm{SCHA})_1$ are isomorphic, while (6.13) shows that the $\mathbf{H}$ functors on Ho $(\text{SCHA})_1$ and Ho $(\text{DGC})_2$ are isomorphic. Thus the **H** functors on $\mathrm{Ho}_{\mathbf{Q}}\mathcal{T}_2$ and Ho $(\text{DGC})_2$ are isomorphic (in fact canonically isomorphic). The proof of Theorem I is now complete.

PART II.

The purpose of this part is to improve the equivalence of categories of Theorem I to an equivalence of homotopy theories. We use the axiomatization of homotopy theory presented in [21], which will be denoted [HA] in the following. A review of the basic definitions and theorems of [HA] is given in § 1. Theorem II is proved in §§ 2–5. Some applications are presented in § 6.

All diagrams are commutative unless otherwise stated.

1. Closed model categories and statement of Theorem II

We begin by reviewing some of the definitions and theorems of [HA, Ch. I].

Definition. A *closed model category* is a category $\mathcal{C}$ endowed with three distinguished families of maps called cofibrations, fibrations, and weak equivalences satisfying the axioms CM1–CM5 below.

CM1. $\mathcal{C}$ is closed under finite projective and inductive limits.

CM2. If f and g are maps such that gf is defined, then if two of f, g, and gf are weak equivalences, so is the third.

Recall that the maps in $\mathcal{C}$ form the objects of a category $\mathcal{AC}$ having commutative squares for morphisms. We say that a map f in $\mathcal{C}$ is a *retract* of g if there are morphisms $\varphi: f \to g$ and $\psi: g \to f$ in $\mathcal{AC}$ such that $\psi\varphi = \mathrm{id}_f$.

CM3. If f is a retract of g and g is a fibration, cofibration, or weak equivalence, so is f.

A map which is both a fibration (resp. cofibration) and weak equivalence will be called a *trivial fibration* (resp. *trivial cofibration*).

CM4. (Lifting). Given a solid arrow diagram

$$(*) \qquad \begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

the dotted arrow exists in either of the following situations:

(i) i is a cofibration and p is a trivial fibration,

(ii) i is a trivial cofibration and p is a fibration.

CM5. (Factorization). Any map f may be factored in two ways

(i) $f = pi$ where i is a cofibration and p is a trivial fibration,

(ii) $f = pi$ where i is a trivial cofibration and p is a fibration.

We say that a map $i: A \to B$ in a category has the *left lifting property* (LLP) with respect to another map $p: X \to Y$ and p is said to have the *right lifting property* (RLP) with respect to i if the dotted arrow exists in any diagram of the form $(*)$.

Suppose now that $\mathcal{C}$ is a closed model category.

PROPOSITION 1.1. *The cofibrations (resp. trivial cofibrations) are precisely those maps having the* LLP *with respect to all trivial fibrations (resp. fibrations.) The fibrations (resp. trivial fibrations) are precisely those maps having the* RLP *with respect to all trivial cofibrations (resp. cofibrations).*

PROOF. CM4 says that a cofibration has the LLP with respect to any trivial fibration. Conversely if f has the LLP with respect to all trivial fibrations, and if $f = pi$ is as in CM5(i), then f has the LLP with respect to p, so f is a retract of i and therefore f is a cofibration by CM3. The other possibilities are similar, q.e.d.

COROLLARY 1.2. *The class of fibrations (resp. trivial fibrations) is closed under composition and base change and contains all isomorphisms. The class of cofibrations (resp. trivial cofibrations) is closed under composition and cobase change and contains all isomorphisms.*

An object X of $\mathcal{C}$ is called *cofibrant* if the map $\varphi \to X$ ($\varphi =$ initial object of $\mathcal{C}$ which exists by CM1) is a cofibration and *fibrant* if $X \to e$ ($e =$ final object) is a fibration. If $A \vee A$, $in_i: A \to A \vee A$, $i = 1,2$, is the direct sum of two copies of A, we define a *cylinder object* for A to be an object A_I together with maps $\partial_i: A \to A_I$, $i = 0,1$, and $\sigma: A_I \to A$ such that $\partial_0 + \partial_1: A \vee A \to A_I$ is a cofibration, σ is a weak equivalence and $\sigma\partial_i = \mathrm{id}_A$, $i = 0,1$. Here $\partial_0 + \partial_1$ denotes the unique map with $(\partial_0 + \partial_1)in_i = \partial_{i-1}$. If $f, g \in \mathrm{Hom}\,(A, B)$, a *left homotopy* from f to g is defined to be a map $h: A_I \to B$, where A_I is a cyclinder object for A, such that $h\partial_0 = f$ and $h\partial_1 = g$. f is said to be left homotopic to g if such a left homotopy exists. When A is cofibrant, "is left homotopic to" is an equivalence relation (Lemma 4, § 1, *loc. cit.*) on $\mathrm{Hom}\,(A, B)$. The notions of path object and right homotopy are defined in a dual manner. If A is cofibrant and B is fibrant, then the left and right homotopy relations on $\mathrm{Hom}\,(A, B)$ coincide and we denote the set of equivalence classes by $[A, B]$. We let $\pi\mathcal{C}_{cf}$ denote the category whose objects are the objects of $\mathcal{C}$ which are both fibrant, and cofibrant with $\mathrm{Hom}_{\pi\mathcal{C}_{cf}}(A, B) = [A, B]$, and with composition induced from that of $\mathcal{C}$.

The *homotopy category* $\mathrm{Ho}\,\mathcal{C}$ of a closed model category $\mathcal{C}$ is defined to be the localization of $\mathcal{C}$ with respect to the class of weak equivalences. The canonical functor $\gamma: \mathcal{C} \to \mathrm{Ho}\,\mathcal{C}$ induces a functor $\bar{\gamma}: \pi\mathcal{C}_{cf} \to \mathrm{Ho}\,\mathcal{C}$, and we have the following result (Theorem 1, § 1 and Prop. 1, § 5 *loc. cit.*).

THEOREM 1.3. (a) $\mathrm{Ho}\,\mathcal{C}$ *exists.*

(b) $\bar{\gamma}\colon \pi\mathcal{C}_{cf} \to \mathrm{Ho}\,\mathcal{C}$ *is an equivalence of categories.*

(c) *If A is cofibrant and B is fibrant, then*

$$\gamma\colon [A, B] \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Ho}\mathcal{C}}(\gamma A, \gamma B)\ .$$

(d) $\gamma(f)$ *is an isomorphism if and only if f is a weak equivalence.*

If the closed model category $\mathcal{C}$ is pointed, i.e., initial object $\simeq$ final object, then in §§ 2–3 *loc. cit.*, we constructed loop and suspension functors and families of fibration and cofibration sequences in the category $\mathrm{Ho}\,\mathcal{C}$. Such extra structure on the homotopy category is part of the homotopy theory of $\mathcal{C}$. For the purposes of the present paper we shall define the homotopy theory of $\mathcal{C}$ to be the category $\mathrm{Ho}\,\mathcal{C}$ together with the extra structure of loop and suspension functors and the families of fibrations and cofibration sequences. Then we have the following criterion for an equivalence of homotopy theories (§ 4, *loc. cit.*).

THEOREM 1.4. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be closed model categories and let*

$$\mathcal{C}_1 \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{C}_2$$

be a pair of adjoint functors (upper arrow always the left adjoint functor) such that

(i) *F carries cofibrations in $\mathcal{C}_1$ into cofibrations in $\mathcal{C}_2$ and G carries fibrations in $\mathcal{C}_2$ into fibrations in $\mathcal{C}_1$.*

(ii) *If $f\colon A \to B$ is a weak equivalence in $\mathcal{C}_1$ and A and B are cofibrant, then $F(f)$ is a weak equivalence in $\mathcal{C}_2$.*

(iii) *If $g\colon X \to Y$ is a weak equivalence in $\mathcal{C}_2$ and X and Y are fibrant, then $G(g)$ is a weak equivalence in $\mathcal{C}_2$.*

(iv) *If A is a cofibrant object in $\mathcal{C}_1$ and X is a fibrant object in $\mathcal{C}_2$, then a map $f\colon A \to GX$ is a weak equivalence if and only if the corresponding map $f^b\colon FA \to X$ is a weak equivalence.*

Then the derived functors (I, § 2, 2.3),

$$\mathrm{Ho}\,\mathcal{C}_1 \underset{\mathbf{R}G}{\overset{\mathbf{L}F}{\rightleftarrows}} \mathrm{Ho}\,\mathcal{C}_2$$

are equivalences of categories. Moreover if $\mathcal{C}_1$ and $\mathcal{C}_2$ are pointed, then this equivalence preserves the loop and suspension functors and the families of fibration and cofibration sequences.

THEOREM II. *On each of the categories* $\mathcal{S}_2$, $(\mathrm{SGP})_1$, $(\mathrm{SCHA})_1$, $(\mathrm{SLA})_1$, $(\mathrm{DGL})_1$, *and* $(\mathrm{DGC})_2$ *it is possible to define closed model category structures in such a* *way that*

(a) *the families of weak equivalences are precisely as defined in Part* I, § 2,

(b) *the adjoint functors in Figure* 1, *Part* I, § 2, *satisfy the conditions of* 1.4. *Therefore the functors of Figure* 2 *to the right of* $\mathrm{Ho}_{\mathbf{Q}}\,\mathcal{S}_2$ *are equivalences of homotopy theories.*

This will be proved in §§ 2–5. The precise definitions of cofibrations, etc. for each of the categories will be given as they are treated; let us point out here that they are the natural ones.

It is unfortunate that the category $\mathcal{T}_2$ of simply-connected spaces does not satisfy the axioms for the trivial reason that it is not closed under finite limits. However with suitable definitions the remaining axioms hold. This will be discussed in § 6.

2. Serre theory for simplicial sets

Let S be a multiplicative system in $\mathbf{Z}$. An abelian group A will be called S-divisible, S-torsion, S-torsion-free, or S-uniquely-divisible if the canonical map $A \to S^{-1}A$ is surjective, zero, injective, or bijective, respectively. In this section we construct a closed model category consisting of simplicial sets whose associated homotopy theory will be Serre mod $\mathcal{C}$ theory [27] where $\mathcal{C}$ is the class of S-torsion abelian groups.

Let $\mathcal{S}$ be the category of simplicial sets. It is a closed model category [HA, Ch. II, § 3] where the cofibrations are the maps which are injective (in each dimension), where the fibrations are the fiber maps in the sense of Kan, and where the weak equivalences are the maps which are carried into homotopy equivalences by the geometric realization functor. The same is true for the category $\mathcal{S}_0$ of pointed simplicial sets. If X is a pointed simplicial set, we define its q^{th} homotopy group (set if $q = 0$) $\pi_q X$ to be the q^{th} homotopy group of its geometric realization or equivalently (see [13]) the Kan homotopy group $\pi_q Y$ where Y is a Kan complex and there is a weak equivalence $X \to Y$. Using the equivalence of the homotopy theories of spaces and simplicial sets [19], [13], we have the following result of Serre.

PROPOSITION 2.1. *Let* $f\colon X \to Y$ *be a map of* 1-*connected pointed simplicial sets. The following are equivalent*:

(i) $S^{-1}\pi_* f\colon S^{-1}\pi_* X \xrightarrow{\sim} S^{-1}\pi_* Y$

(ii) $S^{-1}H_* f\colon S^{-1}H_* X \xrightarrow{\sim} S^{-1}H_* Y$

(iii) $f^*\colon H^*(Y, A) \xrightarrow{\sim} H^*(X, A)$ *for all* S-*uniquely-divisible abelian groups* A.

A map satisfying these conditions will be called an *S-equivalence*. Let r be an integer $\geqq 1$. A simplical set X will be called *r-reduced* if its $(r-1)$-skeleton is reduced to a point. Let $\mathcal{S}_r$ be the full subcategory of $\mathcal{S}$ consisting of the r-reduced simplicial sets.

Suppose r, S given, and that $S = \{1\}$ if $r = 1$. Let $\mathcal{S}(r, S)$ be the following candidate for a closed model category: $\mathcal{S}(r, S)$ is the category $\mathcal{S}_r$ and its weak equivalences and cofibrations are the S-equivalences and injective maps in $\mathcal{S}_r$. The fibrations in $\mathcal{S}(r, S)$ are (as they must by 1.1) those maps in $\mathcal{S}_r$ with the RLP with respect to the injective S-equivalences in $\mathcal{S}_r$.

THEOREM 2.2. *$\mathcal{S}(r, S)$ is a closed model category.*

PROOF. The axioms CM1, CM2, CM3, and CM4 (ii) are clear. To prove CM5 (i), let $f: X \to Y$ be a map in $\mathcal{S}_r$ and let $X \xrightarrow{i} Z \xrightarrow{p} Y$ be a factorization of f in $\mathcal{S}$ where i is a cofibration and p is a trivial fibration. Give Z the basepoint $i(x_0)$ where x_0 is the basepoint of X, let E_rZ be the Eilenberg subcomplex of Z consisting of those simplices of Z with their $(r-1)$-skeleton at the basepoint, and let $X \xrightarrow{i'} E_rZ \xrightarrow{p'} Y$ be the maps induced by i and p. It is clear that i' is a cofibration in $\mathcal{S}(r, S)$. It is also easily seen that p' has the RLP with respect to $\Delta(q)^{\cdot} \hookrightarrow \Delta(q)$ $q \geqq 0$, hence p' is a map in $\mathcal{S}_r$ which is a trivial fibration in $\mathcal{S}$ and *a fortiori* in $\mathcal{S}(r, S)$, proving CM5 (i). Notice also that if f is a trivial fibration in $\mathcal{S}(r, S)$, then applying CM2 to $f = p'i'$ we find that i' is an S-equivalence, whence i' has the LLP with respect to f, f is a retract of p', and so f is a trivial fibration in $\mathcal{S}$. Hence f has the RLP with respect to cofibrations, which is CM4 (i). We have thus proved CM4 and CM5 (i) as well as the following.

PROPOSITION 2.3. *The trivial fibrations in $\mathcal{S}(r, S)$ are precisely those maps in $\mathcal{S}_r$ which are trivial fibrations in $\mathcal{S}$.*

The proof of CM5 (ii) is in two steps the first of which is the case where $S^{-1}\pi_r f$ is surjective. This uses the theory of minimal fibrations [3].

PROPOSITION 2.4. *The following conditions are equivalent for map f in $\mathcal{S}_r$:*

(i) *f is a fibration in $\mathcal{S}(r, S)$ and $S^{-1}\pi_r f$ is surjective*

(ii) *f is a fibration in $\mathcal{S}$ and π_** Ker f *is S-uniquely-divisible* (Ker f = *fiber of f*).

PROOF. (ii) $\Rightarrow$ (i). First note that $\pi_r f$ is surjective because of the exact homotopy sequence for f and the fact that Ker f is r-reduced. If $S = \{1\}$, the result is clear, so we may assume $r \geqq 2$. By the theory of minimal fibrations, f may be factored, $f = pq$, where q is a trivial fibration and $p: X \to Y$ is a minimal fibration. p in turn may be factored into its Postnikov system

$$\cdots \longrightarrow X_n \xrightarrow{p_n} X_{n-1} \longrightarrow \cdots \longrightarrow X_{r-1} = Y$$
$$X = \lim_{\leftarrow} X_n$$

where p_n is a minimal fibration with fiber $K(A, n)$, $A = \pi_n \operatorname{Ker} p \simeq \pi_n \operatorname{Ker} f$. Let

$$\varphi(A, n)\colon L(A, n) \longrightarrow K(A, n+1)$$

be the fibration which represents the following morphism of functors on $\mathcal{S}$:

$$\begin{array}{ccc} C^n(X, A) = \left\{\begin{matrix}\text{normalized } n\text{-cochains on } X \\ \text{with values in } A\end{matrix}\right\} \\ \Big\downarrow \delta \qquad\qquad\qquad \Big\downarrow \text{coboundary} \\ Z^{n+1}(X, A) = \left\{\begin{matrix}\text{normalized } n\text{-cocycles on } X \\ \text{with values in } A\end{matrix}\right\} \end{array}$$

Then if Z is 1-connected, we have

$$\begin{aligned} H^n(Z, A) &\simeq [Z, K(A, n+1)] \\ &\xrightarrow{\sim} \left\{\begin{matrix}\text{isomorphism classes of minimal} \\ \text{fibrations with base } Z \text{ and fiber } K(A, n)\end{matrix}\right\}, \end{aligned} \tag{1}$$

where the last isomorphism is given by sending a map u to the induced fibration $u^*\varphi(A, n)$. It is clear that each X_n is r-reduced hence 1-connected since $r \geqq 2$, and hence p_n is induced from $\varphi(A, n)$ where A is S-uniquely-divisible. To show that f has the RLP with respect to trivial fibrations in $\mathcal{S}(r, S)$, it suffices to show that $\varphi(A, n)$ does. But if $h\colon U \to V$ is an injective S-equivalence, $h^*\colon C^*(V, A) \to C^*(U, A)$ is a surjective weak equivalence of cochain complexes, hence

$$C^n(V, A) \xrightarrow{(h^*, \delta)} C^n(U, A) \times_{Z^{n+1}(U, A)} Z^{n+1}(V, A)$$

is surjective, and so $\varphi(A, n)$ has the RLP with respect to h by the definition of φ. This proves (i). To finish the proof of 2.4 we need

LEMMA 2.5. *If f is a map in $\mathcal{S}_r$ such that $S^{-1}\pi_r f$ is surjective, then $f = pi$ where i is a trivial cofibration in $\mathcal{S}(r, S)$ and where p satisfies* (ii) *of* 2.4.

PROOF OF LEMMA. It suffices to factor $f = pi$ in $\mathcal{S}_r$ where i is an S-equivalence and p satisfies (ii), for then if we write $i = qj$ using CM5 (i), j is a trivial cofibration in $\mathcal{S}(r, S)$ by CM2 and so $f = (pq)j$ is the factorization required for the lemma.

Factor $f = pi$ in $\mathcal{S}$ where i is a weak equivalence and $p\colon X \to Y$ is a minimal fibration with fiber F. If $S = \{1\}$ then $\pi_r p \simeq \pi_r f$ is surjective so $\pi_q F = 0$ for $q < r$. As p is minimal, F is r-reduced and so X, which is a twisted cartesian product of Y and F, is r-reduced. Then $f = pi$ is a factorization of f in

$\mathfrak{S}_r$ where i is a weak equivalence and p satisfies (ii), and we are done. If $S \neq \{1\}$, then $r \geqq 2$ and we construct by induction a ladder diagram

$$(2)\qquad \begin{tikzcd} \arrow[r] & X_n \arrow[r, "p_n"] \arrow[d, "j_n"] & X_{n-1} \arrow[r] \arrow[d, "j_{n-1}"] & \cdots \arrow[r] & X_{r-1} \arrow[r] \arrow[d, "j_{r-1}"] & Y \arrow[d, equal] \\ \arrow[r] & W_n \arrow[r, "q_n"] & W_{n-1} \arrow[r] & \cdots \arrow[r] & W_{r-1} \arrow[r, equal] & Y \end{tikzcd}$$

where the first row is the Postnikov system of p, where j_n is an S-equivalence, and where q_n is a minimal fibration with fiber $K(S^{-1}\pi_n F, n)$. For $n = r - 1$, we have exact sequences

$$\cdots \longrightarrow \pi_r X \xrightarrow{\pi_r p} \pi_r Y \longrightarrow \pi_{r-1} F \longrightarrow 0$$

$$0 \longrightarrow \pi_{r-1} X_{r-1} \xrightarrow{\pi_r p_{r-1}} \pi_r Y \longrightarrow \pi_{r-1} F \longrightarrow 0$$

$$\pi_q X_{r-1} \xrightarrow[\approx]{\pi_q p_{r-1}} \pi_q Y \qquad\qquad q \geqq r .$$

By hypothesis $S^{-1}\pi_r p \simeq S^{-1}\pi_r f$ is surjective so p_{r-1} is an S-equivalence and we may take $W_{r-1} = Y$, $q_{r-1} = \mathrm{id}_Y$, and $j_{r-1} = p_{r-1}$. Assuming W_{n-1} has been obtained, let $A = \pi_n F$, let $u: X_{n-1} \to K(A, n + 1)$ be a classifying map for p_n (i.e., $p_n \simeq u^*\varphi(A, n)$), and let $\rho: K(A, n + 1) \to K(S^{-1}A, n + 1)$ be induced by the coefficient homomorphism $A \to S^{-1}A$. By the induction hypothesis j_{n-1} is an S-equivalence so by (1) with $Z = W_{n-1}$ and A replaced by $S^{-1}A$, there is a map $v: W_{n-1} \to K(S^{-1}A, n + 1)$ such that vj_{n-1} is homotopic to ρu. Let $q_n: W_n \to W_{n-1}$ be the pull-back $v^*\varphi(S^{-1}A, n)$. Then

$$j_{n-1}^* q_n = (vj_{n-1})^*\varphi(S^{-1}A, n) \simeq (\rho u)^*\varphi(S^{-1}A, n) ,$$

hence there is a map j_n of fibrations

$$\begin{tikzcd} K(A, n) \arrow[r, "\rho"] \arrow[d] & K(S^{-1}A, n) \arrow[d] \\ X_n \arrow[r, "j_n"] \arrow[d, "p_n"] & W_n \arrow[d, "q_n"] \\ X_{n-1} \arrow[r, "j_{n-1}"] & W_{n-1} \end{tikzcd} .$$

The homotopy exact sequence and five lemma show that j_n is an S-equivalence, which completes the inductive construction of (2).

Let $W = \lim_{\leftarrow} W_n$, $j = \lim_{\leftarrow} j_n: X \to W$, and $q = \lim_{\leftarrow} q_{r-1} \cdots q_n$: $W \to Y$. It is clear that q is a map in $\mathfrak{S}_r$ satisfying (ii); j is a map from the fibration p to the fibration q which induces S-equivalences on the base and fibers. Hence j is an S-equivalence. Thus $f = q(ji)$ is the factorization of f required to finish the proof of the lemma.

(i) $\Rightarrow$ (ii). If f satisfies (i), write $f = pi$ as in the lemma. Then i has the LLP with respect to f by the definition of fibration, so f is a retract of p and f satisfies (ii). This completes the proof of Proposition 2.4.

COROLLARY 2.6. *The fibrant objects of $\mathcal{S}(r, S)$ are the r-reduced Kan complexes whose homotopy groups are S-uniquely-divisible.*

LEMMA 2.7. *If f is a map in $\mathcal{S}_r$, then $f = jg$ where j is an injective fibration in $\mathcal{S}(r, S)$ and where $S^{-1}\pi_r g$ is surjective.*

PROOF. We only treat the case $r \geqq 2$; the case $r = 1$ requires only minor modifications. The Hurewicz theorem asserts that $\pi_r X \xrightarrow{\sim} H_r X$ for any r-reduced simplicial set, hence

$$\operatorname{Hom}_{\mathcal{S}_r}\big(X, K(A, r)\big) \simeq \operatorname{Hom}_{ab}(\pi_r X, A) . \tag{3}$$

Given a map $f\colon X \to Y$ in $\mathcal{S}_r$ let $f = j_1(f)f_1$ be the factorization given by the diagram

$$\begin{tikzcd}
X \arrow[dr, "f_1"] \arrow[drr, "\beta"] \arrow[ddr, "f"'] & & \\
& Y' \arrow[r] \arrow[d, "j_1(f)"] & K(\operatorname{Im} S^{-1}\pi_r f, r) \arrow[d, hook, "\rho"] \\
& Y \arrow[r, "\alpha"] & K(S^{-1}\pi_r Y, r)
\end{tikzcd} \tag{4}$$

where the square is cartesian and where α, β, and ρ correspond under the bijection (3) to the obvious maps $\pi_r Y \to S^{-1}\pi_r Y$, $\pi_r X \to \operatorname{Im} S^{-1}\pi_r f$, and $\operatorname{Im} S^{-1}\pi_r f \hookrightarrow S^{-1}\pi_r Y$ respectively. For U in $\mathcal{S}_r$ we have

$$\operatorname{Hom}_{\mathcal{S}_r}\big(U, K(A, r)\big) = Z^r(U, A) = H^r(U, A) ,$$

from which one sees that ρ has the RLP with respect to any weak equivalence in $\mathcal{S}(r, S)$ in addition to being injective. Moreover if $i_1(f)$ is an isomorphism, then we have a diagram

$$\begin{tikzcd}
S^{-1}\pi_r Y' \arrow[r] \arrow[d, "\wr"] & \operatorname{Im} S^{-1}\pi_r f \arrow[d, hook] \\
S^{-1}\pi_r Y \arrow[r, "\mathrm{id}"] & S^{-1}\pi_r Y
\end{tikzcd}$$

which shows that $S^{-1}\pi_r f$ is surjective. Now define by transfinite induction a factorization $f = j_\alpha(f)f_\alpha$ for each ordinal number α by

$$\begin{aligned} j_{\alpha+1}(f) &= j_\alpha(f)j_1(f_\alpha) \\ f_{\alpha+1} &= (f_\alpha)_1 \\ \left.\begin{aligned} j_\beta(f) &= \text{lim-inv}_{\alpha<\beta}\, j_\alpha(f) \\ f_\beta &= \text{lim-inv}_{\alpha<\beta}\, f_\alpha \end{aligned}\right\} & \quad \text{if } \beta \text{ is a limit ordinal .} \end{aligned}$$

Clearly $j_\alpha(f)$ is an injective fibration in $\mathcal{S}(r, S)$ for each α, and since the subobjects of Y form a set, we have $j_1(f_\alpha)$ is an isomorphism for α sufficiently large, so $\operatorname{Im} S^{-1}\pi_r f_\alpha$ is surjective. This proves the lemma.

To prove axiom CM5 (ii), let f be a map in $\mathcal{S}_r$, write $f = jg$ as in Lemma 2.7 and write $g = pi$ as in Lemma 2.5. By Proposition 2.4, p is a fibration and so $f = (jp)i$ is a factorization of f as required by CM5 (ii). This completes the proof of Theorem 2.2.

PROPOSITION 2.8. *The homotopy category* $\operatorname{Ho}\mathcal{S}(r, S)$ *is equivalent to the category whose objects are those r-reduced Kan complexes with S-uniquely-divisible homotopy groups and whose morphisms are simplicial homotopy classes of simplicial set maps.*

PROOF. Every object of $\mathcal{S}(r, S)$ is cofibrant, hence the cofibrant and fibrant objects of $\mathcal{S}(r, S)$ are the r-reduced Kan complexes with S-uniquely-divisible homotopy groups. If Y is an r-reduced Kan complex, let $Y^{\Delta(1)}$ be the "path-space" complex of Y, let $j_e: Y^{\Delta(1)} \to Y$ $e = 0,1$, be the endpoint maps, and let $s: Y \to Y^{\Delta(1)}$ be the map induced by the unique map $\Delta(1) \to \Delta(0)$. Then s is a trivial cofibration and $(j_0, j_1): Y^{\Delta(1)} \to Y \times Y$ is a fibration. As $Y^{\Delta(1)}$ is an $(r-1)$-connected Kan complex, the inclusion $E_r Y^{\Delta(1)} \to Y^{\Delta(1)}$ is a weak equivalence. Hence the map $s: Y \to E_r Y^{\Delta(1)}$ is a weak equivalence. Clearly

$$(j_0, j_1): E_r Y^{\Delta(1)} \longrightarrow Y \times Y$$

is a fibration in $\mathcal{S}_r$, so $E_r Y^{\Delta(1)}$ together with s, j_0, j_1 is a path object Y_I for Y in $\mathcal{S}_r$. It follows that two simplicially homotopic maps from any X to Y in $\mathcal{S}_r$ are right homotopic. But it is a general fact for a closed model category that if f and g are two maps from a cofibrant object X to a fibrant object Y and if Y^I is a path object for Y, then f and g are left (or right, it makes no difference) homotopic if and only if there is an $h: X \to Y^I$ with $j_0 h = f$ and $j_1 h = g$ [HA, I, 1, Lemma 5 (ii)]. Consequently $[X, Y]$ = simplicial homotopy classes of maps from X to Y if Y is fibrant in $\mathcal{S}(r, S)$. The proposition follows from 1.3 (b).

Remark 2.9. The category $\mathcal{S}(1, \{1\})$ is an example of a closed model category in which it is *not* true that the base extension of a weak equivalence by a fibration is a weak equivalence. For example, let K be the reduced simplicial set which is the following quotient of $\Delta(2)$

$$\begin{tikzcd} & \bullet & \\ \bullet \arrow[ur, "a"] \arrow[rr, "a"'] & c & \bullet \arrow[ul, "b"'] \end{tikzcd}$$

all vertices identified

and let $f\colon K \to K(\mathbf{Z}, 1)$ be given by the normalized 1-cocycle $f(a) = 1$ and $f(b) = 0$. If L is the subcomplex isomorphic to $\Delta(1)/\Delta^{\cdot}(1)$ generated by b, then the following square is cartesian

$$\begin{tikzcd} L \arrow[r, "f'"] \arrow[d] & 0 \arrow[d, "p"] \\ K \arrow[r, "f"] & K(\mathbf{Z}, 1) . \end{tikzcd}$$

Moreover f is a weak equivalence, p is a fibration, and f' is not a weak equivalence.

Remark 2.10. Proposition 2.4 shows that the fibrations in $\mathcal{S}(r, S)$ which are surjective look pretty much like fibrations ought to. On the other hand there are fibrations such as the inclusion $* \hookrightarrow K$ in $\mathcal{S}(1, \{1\})$ of Remark 2.9 which do not resemble ordinary fibrations at all. The following proposition shows that fibrations $f\colon X \to Y$ in $\mathcal{S}(r, S)$ when Y is a Kan complex are more reasonable.

PROPOSITION 2.11. *Let $f\colon X \to Y$ be a map in $\mathcal{S}_r$. Then f is a fibration in $\mathcal{S}(r, S)$ and Y is a Kan complex if and only if f has the following properties*:

(i) $\pi_*(\operatorname{Ker} f)$ *is S-uniquely-divisible*

(ii) $\operatorname{Coker} \pi_r f$ *is S-torsion-free*

(iii) $f\colon X \to fX$ *is a fibration in* $\mathcal{S}$

(iv) $Y \to K(\operatorname{Coker} \pi_r f, r)$ *is a fibration in $\mathcal{S}$ with fiber fX.*

PROOF. Suppose f satisfies (i)–(iv). By (iv) Y is a Kan complex. By 2.4, (i) and (iii) imply that $X \to fX$ is a fibration in $\mathcal{S}(r, S)$. Let $A = \operatorname{Coker} \pi_r f$. Then there is a diagram

$$\begin{tikzcd} fX \arrow[r] \arrow[d] & * \arrow[r] \arrow[d] & * \arrow[d] \\ Y \arrow[r] & K(A, r) \arrow[r] & K(S^{-1}A, r) \end{tikzcd}$$

in which the first square is cartesian by (iv) and the second is cartesian by (ii). As the last vertical map is a fibration in $\mathcal{S}(r, S)$, so is $fX \to Y$, hence also f by composition.

Suppose f is a fibration in $\mathcal{S}(r, S)$ where Y is a Kan complex. Let B be the quotient of Coker $\pi_r f$ by its S-torsion subgroup. As Y is a Kan complex the canonical map $Y \to K(\pi_r Y, r)$ is a fibration in $\mathcal{S}$, and hence so is the composite map $Y \to K(\pi_r Y, r) \to K(B, r)$ which we will denote by u. Let $j: Z \to Y$ be the inclusion of the fiber of u, and let $g: X \to Z$ be the unique map with $jg = f$. As f is a fibration in $\mathcal{S}(r, S)$ and j is injective, one sees easily that g is also a fibration in $\mathcal{S}(r, S)$. By definition of B and u, $S^{-1}\pi_r g$ is surjective, so by 2.4, g is a fibration in $\mathcal{S}$. Thus g is surjective, so $fX = Z =$ Ker u and $g =$ the map $f: X \to fX$, proving (iii). Also $\pi_*(\text{Ker} f) = \pi_*(\text{Ker}\, g)$ is S-uniquely-divisible, proving (i). Finally $\pi_r X$ maps into $\pi_r fX$ which in turn maps onto Ker $\{\pi_r u: \pi_r Y \to A\}$, showing that $A =$ Coker $\pi_r f$ and proving (ii) as well as (iv). Thus the proof of the proposition is complete.

When $S = \{1\}$ there is a simpler characterization of fibrations in $\mathcal{S}_r$ with a Kan complex for the base. By $V(n, k)$ denote the subcomplex of the standard n-simplex $\Delta(n)$ which is the union of all the faces but the k^{th}; recall that a map in $\mathcal{S}$ is a fibration if and only if it has the RLP with respect to the inclusion map $V(n, k) \to \Delta(n)$ for $0 \leqq k \leqq n > 0$.

Condition (ii) of the following is therefore a reasonable criterion for a fibration in $\mathcal{S}_r$; however the Example 2.9 can be used to show that the hypothesis that Y is a Kan complex is essential.

PROPOSITION 2.12. *Let $f: X \to Y$ be a map in $\mathcal{S}_r$ where Y is a Kan complex. The following conditions are equivalent.*

(i) *f is a fibration in $\mathcal{S}_r$,*

(ii) *f has the* RLP *with respect to the inclusion $V(n, k) \to \Delta(n)$ for $0 \leqq k \leqq n > r$,*

(iii) *the canonical map $X \to Y \times_{K(\pi_r Y, r)} K(\pi_r X, r)$ is a fibration in $\mathcal{S}$.*

PROOF. (iii) $\Rightarrow$ (i). f is the composition

$$X \longrightarrow Y \times_{K(\pi_r Y, r)} K(\pi_r X, r) \xrightarrow{pr_1} Y .$$

The first map is a fibration in $\mathcal{S}_r$ by (iii) using 2.4, while the second is a base extension of $K(\pi_r X, r) \to K(\pi_r Y, r)$ which is also a fibration in $\mathcal{S}_r$. Thus f is a fibration in $\mathcal{S}_r$.

(i) $\Rightarrow$ (ii). If K is a simplicial set, let $K_{(r)}$ be the r-reduced simplicial set obtained by shrinking the $(r-1)$-skeleton of K to a point. Clearly if $L \in \text{Ob}\, \mathcal{S}_r$

$$\text{Hom}_{\mathcal{S}_r}(K_{(r)}, L) \simeq \text{Hom}_{\mathcal{S}}(K, L) ,$$

so to prove (ii) we must show that f has the RLP with respect to the map $V(n, k)_{(r)} \to \Delta(n)_{(r)}$. But for $r > n$, this map is a weak equivalence since then $V(n,k)$ contains the $(r-1)$-skeleton of $\Delta(n)$. Thus this map is a trivial cofibration in $\mathcal{S}_r$ and f has the RLP with respect to it.

(ii) $\Rightarrow$ (iii). First note that X is a Kan complex. In effect given $\alpha\colon V(n, k) \to X$, if $n \leqq r$, then α is the "zero" map hence α extends to $\Delta(n)$ trivially; if $n > r$, then as Y is a Kan complex we may extend $f\alpha$ to $\beta\colon \Delta(n) \to X$, and then use (ii) to obtain an extension $\Delta(n) \to X$ of α. Let $\varepsilon_X\colon X \to K(\pi_r X, r)$ be the canonical map; to prove (iii) we must construct the dotted arrow γ in any diagram of the form

$$\begin{tikzcd} V(n, k) \arrow[r, "\alpha"] \arrow[d, "i"'] & X \arrow[d, "{(f, \varepsilon_X)}"] \\ \Delta(n) \arrow[r, "{(\beta, z)}"'] \arrow[ur, dashed] & Y \times_{K(\pi_r Y, r)} K(\pi_r X, r) \end{tikzcd} \tag{2.13}$$

for $0 \leqq k \leqq n > 0$. If $n < r$, we may take $\gamma = 0$. If $n > r$, we can by (ii) choose γ so that $\gamma i = \alpha$ and $f\gamma = \beta$. We assert that $\varepsilon_X \gamma = z$. In effect $\varepsilon_X\gamma$ and z are two maps $\Delta(n) \to K(\pi_r X, r)$ hence may be identified with r-cocycles of $\Delta(n)$ with values in $\pi_r X$. These cocycles coincide on $V(n, k)$, hence must be equal; for $n > r + 1$ this is trivial since then $V(n, k)$ contains $\Delta(n)^{(r)}$, but even for $n = r + 1$ it is true, because these cocycles coincide on all but one of the faces of $\Delta(n)$, and therefore on all by the cocycle formula.

If $n = r$, then the map α is necessarily zero, and the map (β, z) is equivalent to an r-simplex y in Y and an element c of $\pi_r X$ such that $(\pi_r f)(c)$ is the homotopy class of y. As X is a Kan complex c is represented an element x of X_r; then fx and y represent the same element of $\pi_r Y$, so there exists an element $z \in Y_{r+1}$ with $d_0 z = y$, $d_1 z = fx$, and $d_j z = *$ for $1 < j \leqq r + 1$. Let $\xi\colon \Delta(r + 1) \to Y$ be the map which sends the canonical $r + 1$ simplex σ_{r+1} to z, and let $\eta\colon V(r + 1, 0) \to X$ be given by $\eta(d_1\sigma) = x$, $\eta(d_j\sigma) = *$ for $1 < j \leqq r + 1$. $f\eta = \xi$ restricted to $V(r + 1, 0)$, so by hypothesis (ii) there is a map $\zeta\colon \Delta(r + 1) \to X$ compatible with ξ and η. Thus letting $x' = d_0\zeta(\sigma_{r+1})$, we have that $fx' = y$ and x' represents c. Therefore we obtain the desired dotted arrow γ in (2.13) by letting γ be the map sending the canonical simplex to x'. The proof of the proposition is therefore complete.

3. Serre theory for simplicial groups

If G is a simplicial group, let N_*G and π_*G be the normalized complex and homotopy groups of G in the sense of Moore. The category $\mathcal{G}$ of simplicial groups is a closed model category, where the weak equivalences are the maps inducing isomorphisms on homotopy groups, where the fibrations are the maps f for which $N_q f$ is surjective for $q > 0$, and where the cofibrations are the maps which are retracts of free simplicial group maps [HA, Ch. II, § 3].

Let

$$\mathcal{S}_1 \underset{\overline{W}}{\overset{G}{\rightleftarrows}} \mathcal{G}$$

be the pair of adjoint functors defined by Kan [12]. Then G preserves cofibrations, $\overline{W}$ preserves fibrations, both G and $\overline{W}$ preserve weak equivalences, and both adjunction morphisms $X \to \overline{W}GX$, $G\overline{W}H \to H$ are weak equivalences. It follows from 1.4 that the homotopy theories of $\mathcal{S}_1$ and $\mathcal{G}$ are equivalent.

A simplicial group is said to be *connected* if $\pi_0 G = 0$. A map $f: G \to H$ of connected simplicial groups will be called an *S-equivalence* if $S^{-1}\pi_* f$ is an isomorphism. Let $\mathcal{G}_r$ be the full subcategory of $\mathcal{G}$ consisting of the r-reduced simplicial groups (i.e., reduced to the identity in dimensions $<r$), and let $\mathcal{G}(r, S)$ $r > 0$ be the following candidate for a closed model category: the category $\mathcal{G}_r$ with cofibrations and weak equivalences defined to be those maps in $\mathcal{G}_r$ which are cofibrations and S-equivalences respectively in $\mathcal{G}$, and with fibrations defined to be those maps in $\mathcal{G}_r$ with the RLP with respect to the maps which are both cofibrations and weak equivalences in $\mathcal{G}(r, S)$.

THEOREM 3.1. *$\mathcal{G}(r, S)$ is a closed model category. The adjoint functors*

$$\mathcal{S}(r+1, S) \underset{\overline{W}}{\overset{G}{\rightleftarrows}} \mathcal{G}(r, S)$$

establish an equivalence of the associated homotopy theories.

The proof of the theorem uses the following proposition whose proof is deferred to the end of this section.

PROPOSITION 3.2. *Let*

$$\begin{tikzcd} H \arrow[r, "i"] \arrow[d, "f"] & G \arrow[d, "f'"] \\ H' \arrow[r, "i'"] & G' \end{tikzcd}$$

be a co-cartesian square in $\mathcal{G}$ where either i or f is a cofibration. If f is a weak equivalence, so is f'. If f is an S-equivalence and G is connected, then f' is an S-equivalence.

PROOF OF THE THEOREM. The axioms CM1, CM2, CM3, and CM4 (ii) are clear. To prove CM4 (i), first note that if H is a simplicial group then the adjunction map $G\overline{W}H \to H$ is surjective. This is because of the diagram

$$\begin{tikzcd} (\overline{W}H)_q \arrow[r, "\tau"] \arrow[d, "\tau'"] & H_{q-1} \\ (G\overline{W}H)_{q-1} \arrow[ur, dashed] & \end{tikzcd}$$

where τ and τ' are the canonical twisting functions, and the fact that τ is the projection, $(\overline{W}H)_q = H_0 \times \cdots \times H_{q-1} \to H_{q-1}$. Any surjective map in $\mathcal{G}$ is a fibration, hence $G\overline{W}H \to H$ is a trivial fibration in $\mathcal{G}$. Now given $f: G \to H$ in $\mathcal{G}_r$ let $\overline{W}G \xrightarrow{u} X \xrightarrow{v} \overline{W}H$ be a cofibration-trivial fibration factorization of $\overline{W}f$ in $\mathcal{S}(r+1, S)$. Then u is injective and v is a surjective weak equivalence, so we obtain a diagram

$$\begin{tikzcd} G\overline{W}G \arrow[r, "\text{cof.}"] \arrow[d, "\text{w. eq.}"] & GX \arrow[r, "\text{surj. w. eq.}"] \arrow[d, "i'"] & G\overline{W}H \arrow[d, "\text{surj. w. eq.}"] \\ G \arrow[r, "\text{cof.}", "i"'] & Z \arrow[r, "p"'] & H \end{tikzcd}$$

where Z is defined so the first square is co-cartesian. By 3.2, i' is a weak equivalence, so p is also. Thus we have shown that $f = pi$ in $\mathcal{G}_r$, where i is a cofibration and p is a trivial fibration in $\mathcal{G}$ and *a fortiori* in $\mathcal{G}(r, S)$, proving CM5 (i). But if f is a trivial fibration in $\mathcal{G}(r, S)$, then writing $f = pi$ as above we have that i is a trivial cofibration in $\mathcal{G}(r, S)$ by CM2, whence i has the LLP with respect to f and f is a retract of p. This proves that f has the RLP with respect to the cofibrations in $\mathcal{G}(r, S)$, which is CM4 (i), as well as the following

PROPOSITION 3.3. *The trivial fibrations in $\mathcal{G}(r, S)$ are those maps in $\mathcal{G}_r$ which are surjective weak equivalences.*

It remains to prove CM5 (ii).

PROPOSITION 3.4. *The following assertions are equivalent for a map f in $\mathcal{G}_r$.*

(i) *f is a fibration in $\mathcal{G}(r, S)$,*

(ii) *$N_q f$ is surjective for $q > r$, $\pi_* \operatorname{Ker} f$ is S-uniquely-divisible and* $\operatorname{Coker} \pi_r f$ *is S-torsion-free.*

LEMMA 3.5. *Any map f in $\mathcal{G}_r$ may be factored $f = pi$ where i is a trivial cofibration in $\mathcal{G}(r, S)$ and where p satisfies condition* (ii) *of* 3.4.

PROOF OF 3.5. Let $f: H \to G$ and suppose first that $S^{-1}\pi_r f$ is surjective. Let $\overline{W}H \xrightarrow{u} K \xrightarrow{v} \overline{W}G$ be a trivial cofibration-fibration factorization of $\overline{W}f$ in $\mathcal{S}(r+1, S)$. Then u is an injective S-equivalence and v is a fibration in $\mathcal{S}$

whose fiber has S-uniquely-divisible homotopy groups (2.4). We claim that Gv is surjective and $\pi_* \operatorname{Ker} Gv$ is S-uniquely-divisible. The surjectivity is clear since v is a fibration, and the rest follows by applying the five lemma to the map of fibrations

$$\begin{tikzcd}
\operatorname{Ker} v \arrow[r] \arrow[d] & K \arrow[r, "v"] \arrow[d] & \overline{W}G \arrow[d] \\
\overline{W} \operatorname{Ker} Gv \arrow[r] & \overline{W}GK \arrow[r, "\overline{W}Gv"] & \overline{W}G\overline{W}G
\end{tikzcd}$$

to show that $\pi_q \operatorname{Ker} Gv = \pi_{q+1}(\overline{W} \operatorname{Ker} Gv) \simeq \pi_{q+1} \operatorname{Ker} v$. Thus if we form the diagram

$$\begin{tikzcd}
G\overline{W}H \arrow[r, "\text{cof.}"] \arrow[d, "\text{w. eq.}"] & GK \arrow[r, "Gv"] \arrow[d, "i'"'] & G\overline{W}G \arrow[d, "\text{surj. w. eq.}"] \\
H \arrow[r, "i"'] & Z \arrow[r, "p"'] & G
\end{tikzcd}$$

so that the first square is co-cartesian, i' is a weak equivalence by 3.2, p is surjective, and $\pi_* \operatorname{Ker} p \simeq \pi_* \operatorname{Ker} Gv$ is S-uniquely-divisible. Also i is an S-equivalence and a cofibration, so $f = pi$ is the factorization required in the lemma.

In case $S^{-1}\pi_r f$ is not surjective, let A be the subgroup of $\pi_r G$ consisting of those elements α for which $s\alpha \in \operatorname{Im} \pi_r f$ for some $s \in S$ and form the diagram

$$(1) \qquad \begin{tikzcd}
H \arrow[dr, "g"] \arrow[drr, "\beta"] \arrow[ddr, "f"'] & & \\
& G' \arrow[r, "\alpha'"] \arrow[d, "j"] & K(A, r) \arrow[d, hook] \\
& G \arrow[r, "\alpha"] & K(\pi_r G, r)
\end{tikzcd}$$

where the square is cartesian and α and β are the obvious maps. As α is surjective and $\pi_r \operatorname{Ker} \alpha = 0$, the same is true for α', so the homotopy exact sequences of α and α' yield that $\pi_r G' \xrightarrow{\sim} A$, and that $S^{-1}\pi_r g$ is surjective. We may then factor $g = pi$ as above. We note that $N_q j$ is an isomorphism for $q > r$ and that $N_q p$ is surjective for all q since p is. Thus $p' = jp$ has $N_q p'$ surjective for $q > r$ and $\pi_* \operatorname{Ker} p' = \pi_* \operatorname{Ker} p$ is S-uniquely-divisible and $\operatorname{Coker} \pi_r p' = \operatorname{Coker} \pi_r j = (\pi_r G)/A$ is S-torsion-free. Thus $f = p'i$ is the factorization of f required in the lemma.

PROOF OF 3.4. (i) $\Rightarrow$ (ii). Let f be a fibration in $\mathcal{G}(r, S)$ and write $f = pi$

as in 3.5. Then i has the LLP with respect to f, so f is a retract of p and so f satisfies (ii).

(ii)$\Rightarrow$(i). Let $f = jg$ be the factorization of f given by diagram (1). Then $A = \operatorname{Im} \pi_r f$, since $\operatorname{Coker} \pi_r f$ is S-torsion-free, so $\pi_r g$ is surjective. Also $N_q g \simeq (N_q j)(N_q g) = N_q f$ is surjective for $q > r$, which together with the surjectivity of $\pi_r g$ implies that $N_* g$ is surjective. Consequently g is surjective with $\pi_* \operatorname{Ker} g \simeq \pi_* \operatorname{Ker} f$ S-uniquely-divisible, so $\overline{W} g$ is a surjective fibration in $\mathcal{S}$ with $\pi_* \operatorname{Ker} \overline{W} g$ S-uniquely-divisible, and so $\overline{W} g$ is a fibration in $\mathcal{S}(r + 1, S)$ by 2.4. $\overline{W}$ carries the cartesian square in (1) into the first square in the diagram

$$\begin{tikzcd}
\overline{W}G' \arrow[r] \arrow[d, "\overline{W}j"] & K(A, r+1) \arrow[r] \arrow[d, hook] & K(S^{-1}A, r+1) \arrow[d, hook, "u"] \\
\overline{W}G \arrow[r] & K(\pi_r G, r+1) \arrow[r] & K(S^{-1}\pi_r G, r+1)
\end{tikzcd}$$

and the second square is also cartesian since $(\pi_r G)/A$ is S-torsion-free. We have seen that the map labelled u is a fibration in $\mathcal{S}(r + 1, S)$, hence $\overline{W} j$ is also. Therefore $\overline{W} f = (\overline{W} j)(\overline{W} g)$ is a fibration in $\mathcal{S}(r + 1, S)$, and so f has the RLP with respect to any map $GK \to GL$ where $K \to L$ is an injective S-equivalence in $\mathcal{S}_{r+1}$. Let $i\colon H \to G$ be a trivial cofibration in $\mathcal{G}(r, S)$ and consider the diagram

$$\begin{tikzcd}
G\overline{W}H \arrow[r, "i'"] \arrow[d] & G\overline{W}G \arrow[d, "q"] \arrow[ddr, "q'"] & \\
H \arrow[r, "i''"] \arrow[drr, "i"'] & G' \arrow[dr, "q''"] & \\
 & & G
\end{tikzcd}$$

where q is a weak equivalence by 3.2, hence q'' is a trivial fibration since q' is; thus i has the LLP with respect to q'' and i is a retract i''. But we have just seen that i' has the LLP with respect to f, hence so does i'' and i. Thus f is a fibration in $\mathcal{G}(r, S)$ and the proof of Proposition 3.4 is complete.

Combining 3.4 and 3.5 we find that $\mathcal{G}(r, S)$ satisfies CM5 (ii) which completes the proof that it is a closed model category. That G and $\overline{W}$ induce an equivalence between the homotopy theories of $\mathcal{G}(r, S)$ and $\mathcal{G}(r + 1, S)$ follows in a straightforward manner from 1.4. Thus Theorem 3.1 is proved.

PROPOSITION 3.6. *The fibrant objects of $\mathcal{G}(r, S)$ are those r-reduced simplicial groups whose homotopy groups are S-uniquely-divisible. The* *cofibrant objects are the r-reduced simplicial groups which are free.* $\mathrm{Ho}\,\mathcal{G}(r, S)$ *is equivalent to the category whose objects are r-reduced free simplicial groups with uniquely S-divisible homotopy groups and whose morphisms are simplicial homotopy classes of maps of simplicial groups.*

PROOF. The first statement results from 3.4, while the second results from the fact that any simplicial subgroup of a free simplicial group is free. The last statement is proved as 2.8 was.

The rest of this section is devoted to the proof of 3.2 and some related results.

If G is a simplicial group and M is a simplicial G module, we let $H_q(G, M)$ be the (group) *homology* of G with values in M [HA, Ch. II, § 6].

There are canonical isomorphisms

$$H_q(G, M) \simeq \pi_q(\mathbf{Z} \otimes_G P) \simeq \pi_q(Q \otimes_G M)$$

where P (resp. Q) is any simplicial G module endowed with a weak equivalence $P \to M$ (resp. $Q \to \mathbf{Z}$) which is flat over $\mathbf{Z}G$ in each dimension. Here $\mathbf{Z}$ denotes the constant simplicial abelian group which is the integers in each dimension with trivial G action. If A is a $\pi_0 G$ module and we also denote by A the simplicial G module which is A in each dimension with G action induced by the $\pi_0 G$ action in the obvious way, then we may take $Q = \mathbf{Z}\,WG$ and we have

$$H_q(G, A) = \pi_q(\mathbf{Z}\,WG \otimes_G A) = H_q(\overline{W}G, A) ,$$

where the last group is the homology of $\overline{W}G$ with values in the local coefficient system coming from A.

PROPOSITION 3.7. *A map $f\colon H \to G$ of simplicial groups is a weak equivalence if and only if $\pi_0 f\colon \pi_0 H \to \pi_0 G$ is an isomorphism and*

$$H_*(f, A)\colon H_*(H, A) \longrightarrow H_*(G, A)$$

is an isomorphism for all $\pi_0 G$ modules A (in fact $A = \mathbf{Z}\pi_0 G$ is all that is required).

PROOF. f is a weak equivalence if and only if $\overline{W}f$ is a weak equivalence which by [HA, Ch. II, § 3, Prop. 4 (vi)] is true if and only if $\pi_1 \overline{W}f \simeq \pi_0 f$ is an isomorphism and $H^*(\overline{W}f, A)$ is an isomorphism for all local coefficient systems A on $\overline{W}G$. But there are universal coefficient spectral sequences

$$E_2^{pq} = \mathrm{Ext}^p_{\mathbf{Z}\pi_0 G}\big(H_q(\overline{W}G, \mathbf{Z}\pi_0 G), A\big) \Longrightarrow H^{p+q}(\overline{W}G, A)$$
$$E^2_{pq} = \mathrm{Tor}_p^{\mathbf{Z}\pi_0 G}\big(H_q(\overline{W}G, \mathbf{Z}\pi_0 G), A\big) \Longrightarrow H_{p+q}(\overline{W}G, A)$$

which permit us to conclude $H^*(\overline{W}f, A)$ is an isomorphism for all A if and only if $H_*(\overline{W}f, A)$ is an isomorphism for all A, in fact only for $A = \mathbf{Z}\pi_0 G$, q.e.d.

LEMMA 3.8. *Let H be a group and let G be the free product of H and the free group with generators σ_i, $i \in I$. Let IH and IG be the augmentation ideals of the group rings $\mathbf{Z}H$ and $\mathbf{Z}G$. Then the map*

$$\begin{aligned} \textstyle\bigoplus_I \mathbf{Z}G \oplus \mathbf{Z}G \otimes_{\mathbf{Z}H} IH &\longrightarrow IG \\ (a_i) + a \otimes x &\longmapsto \textstyle\sum a_i(\sigma_i - 1) + ax \end{aligned}$$

is an isomorphism of left $\mathbf{Z}G$ modules.

PROOF. Recall that if M is a G module, then a derivation $D\colon G \to M$ is a set map such that $D(g_1 g_2) = Dg_1 + g_1 Dg_2$, and that such derivations correspond in one-to-one fashion to left $\mathbf{Z}G$ module homomorphisms $\theta\colon IG \to M$ *via* the formula $Dg = \theta(g - 1)$. Also such derivations D correspond one-to-one to homomorphisms $s\colon G \to M \times_\tau G$ such that $pr_2 \cdot s = \mathrm{id}$ *via* the formula $sg = (Dg, g)$, where $M \times_\tau G$ is the semi-direct product. Using the latter interpretation and the hypothesis on G, we see that derivations D correspond to derivations $D'\colon H \to M$ and elements $z_i \in M$ $i \in I$ *via* the formula $D' = D \mid H$, $z_i = D\sigma_i$. Hence

$$\begin{aligned} \mathrm{Hom}_G(IG, M) &\simeq \mathrm{Der}(G, M) \\ &\simeq M^I \times \mathrm{Hom}_H(IH, M) \\ &\simeq \mathrm{Hom}_G(\textstyle\bigoplus_I \mathbf{Z}G \oplus \mathbf{Z}G \otimes_{\mathbf{Z}H} IH, M) \end{aligned}$$

which proves the lemma.

The lemma implies that $IG/\mathbf{Z}G \cdot IH$ is a free left $\mathbf{Z}G$ module with basis $\sigma_i - 1$. Changing σ_i to σ_i^{-1} and applying the canonical anti-automorphism of $\mathbf{Z}G$ we find

COROLLARY 3.9. *If H and G are as in* 3.8, *then $IG/IH \cdot \mathbf{Z}G$ is a free right $\mathbf{Z}G$ module with basis $\sigma_i - 1$, $i \in I$.*

PROOF OF 3.2. The case where f is both a cofibration and weak equivalence follows from the fact that $\mathcal{G}$ is a closed model category (1.1). The hard part is to show that the cobase extension f' of a weak equivalence f by a cofibration i is again a weak equivalence. We use the criterion 3.7. The functor $\pi_0\colon \mathcal{G} \to$ (groups) is right exact since it is left adjoint to the constant simplicial group functor. Thus $\pi_0 f'$ is the cobase extension of $\pi_0 f$, and so $\pi_0 f'$ is an isomorphism.

To show that f' induces an isomorphism on homology with twisted coefficients, let A be a $\pi_0 G = \pi_0 G'$ module and choose a free simplicial G (resp. G') module P (resp. P') with a trivial fibration $P \to A$ (resp. $P' \to A$). Then $P' \to A$ is trivial fibration of G modules so lifting in

$$\begin{array}{ccc} 0 & \longrightarrow & P' \\ \downarrow & \nearrow & \downarrow \\ P & \longrightarrow & A \end{array}$$

we obtain a map $P \to P'$ over A which is a di-homomorphism for $f'\colon G \to G'$. Tensoring

$$0 \longrightarrow IG/IH\cdot \mathbf{Z}G \longrightarrow \mathbf{Z}G/IH\cdot \mathbf{Z}G \longrightarrow \mathbf{Z} \longrightarrow 0$$

with P and doing similarly with primes, we obtain a map of exact sequences

$$(2)\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & N \otimes_G P & \longrightarrow & (G/IH\cdot \mathbf{Z}G) \otimes_G P & \longrightarrow & \mathbf{Z} \otimes_G P & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle u_1} & & \Big\downarrow{\scriptstyle u_2} & & \Big\downarrow{\scriptstyle u_3} & & \\ 0 & \longrightarrow & N' \otimes_{G'} P' & \longrightarrow & (\mathbf{Z}G'/IH'\cdot \mathbf{Z}G') \otimes_{G'} P' & \longrightarrow & \mathbf{Z} \otimes_{G'} P' & \longrightarrow & 0 \end{array}$$

where $N = IG/IH\cdot \mathbf{Z}G$ and N' is similar. By definition

$$\pi_*(u_3) = H_*(f', A)\colon H_*(G, A) \longrightarrow H_*(G', A)\ .$$

u_2 may be rewritten $\mathbf{Z} \otimes_H P \to \mathbf{Z} \otimes_{H'} P'$ which, since P is also a free H module resolution of A, shows that

$$\pi_*(u_2) = H_*(f, A)\colon H_*(H, A) \longrightarrow H_*(H', A)\ .$$

We now show that u_1 is a weak equivalence. If i happens to be a free map of simplicial groups, then applying 3.8 dimension-wise, we see that $N = IG/IH\cdot \mathbf{Z}G$ is a free simplicial right $\mathbf{Z}G$ module. In general i is a retract of a free map so in any case N is a (dimension-wise) flat right simplicial $\mathbf{Z}G$ module. But tensoring with a flat simplicial module preserves weak equivalences (this follows from [HA, Ch. II, § 6, Th. 6 (a) + (b)]), hence $N \otimes_G P \to N \otimes_G A$ is a weak equivalence. A similar assertion holds with primes so u_1 is weakly equivalent to the map $N \otimes_G A \to N' \otimes_{G'} A$. But this map is an isomorphism. In effect we need only look at a fixed dimension in which case we may assume G (resp. G') is the free product of H (resp. H') and the free group with generators σ_i, whence by 3.9 N (resp. N') is the free right $\mathbf{Z}G$ (resp. $\mathbf{Z}G'$) module with basis $\sigma_i - 1$. Thus u_1 is a weak equivalence. By assumption f is a weak equivalence, so u_2 is a weak equivalence, and so by (2) and the five lemma, u_3 is a weak equivalence. Therefore $\pi_* u_3 = H_*(f', A)$ is an isomorphism, which completes the proof that f' is a weak equivalence.

Before proving the part of 3.2 about S-equivalence we give a consequence of what has been proved so far. If G is a simplicial group, we let $H_*(G) = H_*(G, \mathbf{Z})$.

PROPOSITION 3.10. *If $i\colon H \to G$ is a cofibration with cofiber $G//H$, then there is an exact sequence*

$$\cdots \pi_q\big((G//H)_{ab}\big) \longrightarrow H_q(H) \xrightarrow{i_*} H_q(G) \longrightarrow \pi_{q-1}\big((G//H)_{ab}\big) \cdots$$

which is natural in i.

PROOF. This long exact sequence is the long exact homotopy sequence of the first row of (2) when $A = \mathbf{Z}$, in which case $N \otimes_G P$ is weakly equivalent to $IG/IH \cdot \mathbf{Z}G \otimes_G \mathbf{Z} \simeq (G//H)_{ab}$. The last isomorphism comes from the fact that both sides represent the functors "derivations of G vanishing on H with values in a G module with trivial action". The naturality of the sequence follows from the fact that the choices of P, P', and the map $P \to P'$ are unique up to simplicial homotopy.

COROLLARY 3.11. *If G is a free simplicial group*

$$H_q(G) = \begin{cases} \mathbf{Z} & q = 0 \\ \pi_{q-1}(G_{ab}) & q > 0\,. \end{cases}$$

COROLLARY 3.12. *If $i\colon H \to G$ is a cofibration with cofiber $G//H$, then there is an exact sequence*

$$\cdots H_q(H) \longrightarrow H_q(G) \longrightarrow \tilde{H}_q(G//H) \longrightarrow H_{q-1}(H) \cdots$$

natural in i.

To prove a map f of connected groups is an S-equivalence, it suffices to show that $S^{-1}H_*(f)$ is an isomorphism by virtue of the formulas $\pi_q G = \pi_{q+1}(\overline{W}G)$, $H_*(G) = H_*(\overline{W}G)$, and 2.1. The rest of 3.2 results then from applying the exact sequences of 3.10 to i and i', and using the fact i and i' have the same cofiber.

Remark 3.13. 3.11 is a formula of Kan and immediately implies 3.12 in the case that H is free. If we define $H_*(G, H)$ to be the relative homology $H_*(\overline{W}G, \overline{W}H)$ when H is a sub-simplicial group of G, then 3.12 implies (after analyzing the nature of the maps) that

$$H_q(G, H) \xrightarrow{\sim} H_q\big(G//H, \{e\}\big)$$

when $H \hookrightarrow G$ is a cofibration. Thus we have proved the *excision axiom* for homology of simplicial groups.

4. $(s\mathcal{A})_r$ as a closed model category

Let $\mathcal{A}$ be a category closed under finite limits and having sufficiently many projective objects (Appendix A, Introduction). The natural way of trying to define the structure of a closed model category on $(s\mathcal{A})$ is to define a map $f\colon X \to Y$ to be a fibration (resp. weak equivalence), if for every projective object P of $\mathcal{A}$, the induced map of simplicial sets $\underline{\mathrm{Hom}}(P, X) \to \underline{\mathrm{Hom}}(P, Y)$ is a fibration (resp. weak equivalence) in $\mathcal{S}$. In [HA, Ch. II, § 4, Th. 4] it was shown that in this way $(s\mathcal{A})$ became a closed model category provided at least one of the following conditions was satisfied.

($*$) For every projective object P of $\mathcal{A}$ and $X \in \mathrm{Ob}\,(s\mathcal{A})$ $\underline{\mathrm{Hom}}\,(P, X)$ is a Kan complex. (This holds if every object of $\mathcal{A}$ is an effective quotient of a cogroup object by [HA, II, 4, Prop. 1].

($**$) $\mathcal{A}$ is closed under arbitrary limits and has a set of small projective generators.

Suppose that $\mathcal{A}$ is a pointed category, and let $(s\mathcal{A})_r$ be the category of r-reduced simplicial objects over $\mathcal{A}$, that is, the full subcategory of $(s\mathcal{A})$ consisting of simplicial objects isomorphic to the initial-final object in dimensions $<r$. We are going to prove $(s\mathcal{A})_r$ is a closed model category under hypothesis ($*$). We do not know if this remains true with hypothesis ($**$) and in fact to prove it when $\mathcal{A}$ is the category of pointed sets (§ 2), we used special features of simplicial sets (cohomology and Eilenberg-MacLane complexes).

THEOREM 4.1. *Let $\mathcal{A}$ be a category closed under finite limits and having sufficiently many projective objects. Assume condition ($*$) holds, and call a map $f: X \to Y$ in $(s\mathcal{A})_r$ a fibration (resp. weak equivalence) if and only if for any projective object P of $\mathcal{A}$, the induced map $\underline{\mathrm{Hom}}\,(P, X) \to \underline{\mathrm{Hom}}\,(P, Y)$ is a fibration (resp. weak equivalence) in $\mathcal{S}_r$. Also define f to be a cofibration if and only if it has the* LLP *with respect to all trivial fibrations. Then $(s\mathcal{A})_r$ is a closed model category.*

PROOF. The axioms CM1, 2,3, and 4(i) are trivial.

CM5 (i). By 2.3 a map in $(s\mathcal{A})_r$ is a trivial fibration in $(s\mathcal{A})_r$ if and only if it is a trivial fibration in $(s\mathcal{A})$. Consider the method used in [HA, II, 4, Prop. 3] to factor $f: X \to Y$ into a cofibration $i: X \to Y$ followed by a trivial fibration $p: Z \to Y$ in $(s\mathcal{A})$. i and p were obtained as the inductive limit of maps $X \to Z^n$ and $Z^n \to Y$, where Z^n is obtained by attaching a "projective n-cell" $P_n \otimes \Delta(n)$ to Z^{n-1} *via* a map $P_n \otimes \Delta(n)^{\boldsymbol{\cdot}}$ where P_n is some projective object of $\mathcal{A}$, and where $\underline{\mathrm{Hom}}\,(Q, Z^n) \to \underline{\mathrm{Hom}}\,(Q, Y)$ has the RLP with respect to $\Delta(k)^{\boldsymbol{\cdot}} \to \Delta(k)$ for $k \leqq n$ and any projective object Q. If Y is r-reduced, then we may take $P_n = *$ for $n < r$ in which case Z^n, and hence Z is r-reduced. i is a cofibration in $(s\mathcal{A})$, *a fortiori* a cofibration in $(s\mathcal{A})_r$, and p is a trivial fibration in $(s\mathcal{A})_r$, proving CM5 (i). Note that if f is a cofibration in $(s\mathcal{A})_r$, then f has the LLP with respect to p; hence f is a retract of i and is therefore a cofibration in $(s\mathcal{A})$. Thus we have proved

PROPOSITION 4.2. *A map in $(s\mathcal{A})_r$ is a cofibration, trivial fibration, or weak equivalence if and only if it is so as a map in $(s\mathcal{A})$.*

CM5 (ii). If Z is a simplicial object over $\mathcal{A}$, define its r^{th} "Eilenberg subcomplex" by $(E_r Z)_n = \bigcap_\varphi \operatorname{Ker}\{\varphi^*: Z_n \to Z_{r-1}\}$, where φ runs over all injective monotone maps from $[r-1]$ to $[n]$. We then have the formulas

$$\begin{aligned} \operatorname{Hom}_{(s\mathcal{A})_r}(Y, E_r Z) &\simeq \operatorname{Hom}_{(s\mathcal{A})}(Y, Z) \\ \underline{\operatorname{Hom}}(P, E_r Z) &\simeq E_r \underline{\operatorname{Hom}}(P, Z) \end{aligned} \tag{4.3}$$

for any $Y \in \operatorname{Ob}(s\mathcal{A})_r$ and projective object P of $\mathcal{A}$. Given a map $f: X \to Y$ in $(s\mathcal{A})_r$, factor f into a trivial cofibration $i': X \to Z$ followed by a fibration $p': Z \to Y$ in $(s\mathcal{A})$, and let $i: X \to E_r Z$, $p: E_r Z \to X$ be the induced maps. Using the above formulas it is easy to see that $\underline{\operatorname{Hom}}(P, p)$ has the RLP for trivial cofibrations in $\mathcal{S}_r$; hence p is a fibration in $(s\mathcal{A})_r$. For a pointed $(r-1)$-connected Kan complex K, the map $E_r K \to K$ is a weak equivalence. By hypothesis $(*)$ $\underline{\operatorname{Hom}}(P, Z)$ is a Kan complex, so the map $E_r Z \to Z$ is a weak equivalence. Therefore i is a weak equivalence so if we write $i = qj$ using CM5 (i), in $(s\mathcal{A})_r$, we have $f = (pq)j$ where j is a trivial cofibration and pq is a fibration in $(s\mathcal{A})_r$, proving CM5 (ii).

CM4 (ii). We have to show that a trivial cofibration $i: A \to B$ in $(s\mathcal{A})_r$ has the LLP with respect to a fibration $p: X \to Y$. By 4.2, i is a trivial cofibration in $(s\mathcal{A})$ and by hypothesis $(*)$ every object of $(s\mathcal{A})$ is fibrant. By [HA, II, 2.5] i is a strong deformation retract map, i.e., there exist maps $r: B \to A$ and $h: B \to B^{\Delta(1)}$ such that $ri = \mathrm{id}_A$, $j_1 h = \mathrm{id}_B$, $j_0 h = ir$, and $hi = i^{\Delta(1)} s$. Here $B^{\Delta(1)}$ is the path object of B [HA, II, § 1] and $s: B \to B^{\Delta(1)}$ (resp. $j_e: B^{\Delta(1)} \to B$ $e = 0, 1$) is induced by the unique map $\Delta(1) \to \Delta(0)$ (resp. the e^{th} vertex map $\Delta(0) \to \Delta(1)$). Given $\alpha: A \to X$ and $\beta: B \to Y$ such that $p\alpha = \beta i$ consider the diagram of solid arrows

$$\begin{tikzcd} A \arrow[r, "\alpha^{\Delta(1)} s"] \arrow[d, "i"'] & E_r(X^{\Delta(1)}) \arrow[d, "{(j_0,\, p^{\Delta(1)})}"] \\ B \arrow[r, "{(\alpha r,\, \beta^{\Delta(1)} h)}"'] \arrow[ur, dashed, "k"] & X \times_Y E_r(Y^{\Delta(1)}) \end{tikzcd} \tag{4.4}$$

Assuming for the moment that the right hand vertical arrow is a trivial fibration, it follows that the dotted arrow k exists; setting $\gamma = j_1 k: B \to X$ we have that $p\gamma = \beta$ and $\gamma i = \alpha$, which proves CM4 (ii). To show that $(j_0, p^{\Delta(1)})$ is a trivial fibration, we may apply the functor $\underline{\operatorname{Hom}}(P, \cdot)$ and use formulas 4.3 to reduce to the case of $\mathcal{S}_r$. We must therefore show that if $p: X \to Y$ is a fibration in $\mathcal{S}_r$, then the arrow $(j_0, p^{\Delta(1)})$ in 4.4 has the RLP with respect to $\Delta(n)^\cdot \to \Delta(n)$ for all $n \geqq 0$. For $n < r$ this is trivial, and for $n \geqq r$ $\Delta(n)^\cdot$ contains the $(r-1)$-skeleton of $\Delta(n)$, so it suffices to show that

$$(j_0, p^{\Delta(1)}): X^{\Delta(1)} \longrightarrow X \times_Y Y^{\Delta(1)}$$

has the RLP with respect to $\Delta(n)^{\cdot} \to \Delta(n)$ for $n \geqq r$. This is equivalent to showing that p has the RLP with respect to the injection

$$\Delta(n)^{\cdot} \times \Delta(1) \vee_{\Delta(n)^{\cdot} \times \{0\}} \Delta(n) \times \{0\} \longrightarrow \Delta(n) \times \Delta(1)$$

for $n \geqq r$. Denoting this injection by $L \to K$, as p is a fibration in $\mathcal{S}_r$, it suffices to show that $L/L^{(r-1)} \to K/K^{(r-1)}$ is a trivial cofibration. But this is clear since for $n \geqq r$, L contains the $(r-1)$-skeleton $K^{(r-1)}$ of K. The proof of Theorem 4.1 is now complete.

COROLLARY 4.5. *The homotopy category* $\mathrm{Ho}(s\mathcal{C})_r$ *is equivalent to the category whose objects are the r-reduced cofibrant simplicial objects over* $\mathcal{C}$ *and whose morphisms are simplicial homotopy classes of maps in* $(s\mathcal{C})$.

The proof is the same as that of 2.8.

Remark 4.6. If $\mathcal{C}$ is closed under arbitrary limits and has a small projective generator U, then in the construction of the factorization $f = pi$ for CM5 (i) we could have taken P_n to be a direct sum of copies of U. Thus the map $i: X \to Z$ is *free* [14] in the sense that it is the limit of maps $X \to Z^n$ where $Z^{r-1} = X$ and Z^n is obtained by "attaching n-cells", i.e., copies of $U \otimes \Delta(n)$, to Z^{n-1} *via* maps $U \otimes \Delta(n)^{\cdot} \to Z^{n-1}$. Thus every cofibration in $(s\mathcal{C})_r$ is a retract of a free map with all cells of dimension $\geqq r$ and conversely.

By Appendix A, 2.24, the category (CHA) is closed under limits and has a projective generator which is also a co-Lie algebra object. Therefore we may apply 4.1 to deduce

THEOREM 4.7. *The category* $(\mathrm{SCHA})_r$ *of r-reduced simplicial complete Hopf algebras is a closed model category where a map is a fibration (resp. weak equivalence) if and only if* $\mathcal{P}f$ *is a fibration (resp. weak equivalence) of simplicial vector spaces and where a map is a cofibration if and only if it is a retract of a free map. The homotopy category* $\mathrm{Ho}(\mathrm{SCHA})_r$ *is equivalent to the category whose objects are free simplicial* CHA's *with all cells of dimension* $\geqq r$ *and whose maps are simplicial homotopy classes of maps in* (SCHA).

We leave to the reader to formulate a similar theorem for $(\mathrm{SLA})_r$.

THEOREM 4.8. *The adjoint functors* $\widehat{\mathbf{Q}}$ *and* $\mathcal{G}$ *establish an equivalence of the homotopy theory of* $(\mathrm{SCHA})_r$ *with the rational homotopy theory of* $(\mathrm{SGP})_r$. $\hat{U}$ *and* $\mathcal{P}$ *establish an equivalence of the homotopy theories of* $(\mathrm{SCHA})_r$ *and* $(\mathrm{SLA})_r$.

PROOF. It is only a matter of checking the hypotheses of 1.4. One shows that $\mathcal{G}$ preserves fibrations using 3.4. The other hypotheses follows from Theorems 3.4 and 3.5 of Part I.

5. $(\mathrm{DGL})_r$ and $(\mathrm{DGC})_{r+1}$ as closed model categories

Let r be an integer $\geqq 1$.

THEOREM 5.1. *Define a map in* $(\mathrm{DGL})_r$ *to be a weak equivalence if it induces isomorphisms on homology, a fibration if it is surjective in degrees* $>r$, *and a cofibration if it has the* LLP *with respect to all trivial fibrations. Then* $(\mathrm{DGL})_r$ *is a closed model category.*

THEOREM 5.2. *Define a map in* $(\mathrm{DGC})_{r+1}$ *to be a weak equivalence if it induces isomorphisms on homology, a cofibration if it is injective, and a fibration if it has the* RLP *with respect to all trivial cofibrations. Then* $(\mathrm{DGC})_{r+1}$ *is a closed model category.*

THEOREM 5.3. *The adjoint functors* $\mathfrak{L}$ *and* $\mathcal{C}$ *establish an equivalence of the homotopy theories of* $(\mathrm{DGL})_r$ *and* $(\mathrm{DGC})_{r+1}$.

THEOREM 5.4. *The adjoint functors* N^* *and* N *establish an equivalence of the homotopy theories of* $(\mathrm{SLA})_r$ *and* $(\mathrm{DGL})_r$.

PROOF OF 5.1. The axioms CM1, 2, 3, and 4 (i) are clear. To prove CM5, let $S(q)$ (resp. $D(q)$) be the DG vector space having a basis over $\mathbf{Q}$ consisting of an element σ_q of degree q with $d\sigma_q = 0$ (resp. elements σ_{q-1}, τ_q of degrees $q-1$ and q with $d\tau_q = \sigma_{q-1}$, $d\sigma_{q-1} = 0$). $S(q)$ and $D(q)$ play the role of the q-sphere and q-disk; let $i\colon S(q-1) \to D(q)$ be the obvious inclusion. Let us call a map $f\colon \mathfrak{m} \to \mathfrak{n}$ in (DGL) *free* if as graded Lie algebras $\mathfrak{n}$ is isomorphic to the direct sum of $\mathfrak{m}$ and a free Lie algebra $L(V)$ in such a way that f is isomorphic to the inclusion. Defining the n-skeleton $\mathfrak{n}^{(n)}$ of f to be the graded sub-Lie-algebra of $\mathfrak{n}$ generated by $f(\mathfrak{m})$ and the elements of V of degree $\leqq n$, one sees that $\mathfrak{n}^{(n)}$ is obtained from $\mathfrak{n}^{(n-1)}$ by attaching n-cells, that is copies of $LD(n)$ *via* attaching maps $LS(n-1) \to \mathfrak{n}^{(n-1)}$. As $LS(n-1) \to LD(n)$ $n > r$ and $0 \to LS(r)$ are clearly cofibrations in $(\mathrm{DGL})_r$, it follows that any free map in $(\mathrm{DGL})_r$ is a cofibration. Now by imitating the procedure of attaching cells to kill homotopy groups, one may factor any map f in $(\mathrm{DGL})_r$ into $f = pi$, where i is free and p is a trivial fibration. Therefore we have proved CM5 (i). Moreover if f is already a cofibration, then f has the LLP with respect to p, so f is a retract of i and we have proved the following.

PROPOSITION 5.5. *A map in* $(\mathrm{DGL})_r$ *is a cofibration if and only if it is a retract of a free map.*

Remark. One may show that a sub-graded-Lie algebra of a free reduced graded Lie algebra is again free. It follows that the cofibrations in $(\mathrm{DGL})_r$ are the free maps, but we shall not need this.

To prove CM5 (ii), note that given $f: \mathfrak{m} \to \mathfrak{n}$, we may let V be a huge direct sum of copies of $D(n)$ for various $n > r$ and obtain a map $p: \mathfrak{m} \vee L(V) \to \mathfrak{n}$, which is surjective in degrees $> r$ and hence is a fibration. If $i: \mathfrak{m} \to \mathfrak{m} \vee L(V)$ is the injection of a factor, then i is free and also i is a weak equivalence, since

$$H(\mathfrak{m} \vee L(V)) \simeq H(\mathfrak{m}) \vee H(LV) \simeq H(\mathfrak{m}) \vee L(HV) \simeq H\mathfrak{m} .$$

Thus $f = pi$ is the factorization required for CM5 (ii). Finally note that $0 \to D(n)$ for $n > r$ has the LLP with respect to fibrations and hence so does i. If f is a trivial cofibration, then p a is trivial fibration, so f has the LLP with respect to p; hence f is a retract of i and f has the LLP with respect to fibrations. This proves CM4 (ii) and completes the proof of 5.1.

LEMMA 5.6. *Let* $p: \mathfrak{m} \to \mathfrak{n}$ *be a surjective map in* $(\mathrm{DGL})_r$ *and let*

$$\begin{array}{ccc} Z & \xrightarrow{pr_2} & \mathcal{C}\mathfrak{m} \\ \big\downarrow{\scriptstyle pr_1} & & \big\downarrow{\scriptstyle \mathcal{C}p} \\ Y & \xrightarrow{\xi} & \mathcal{C}\mathfrak{n} \end{array}$$

be a cartesian square in $(\mathrm{DGC})_{r+1}$. *Then if* ξ *is a weak equivalence, so is* pr_2.

PROOF. Let $\mathfrak{a}$ be the kernel of p, and let $q: \mathfrak{m} \to \mathfrak{a}$ be a graded vector space retraction of $\mathfrak{m}$ onto $\mathfrak{a}$. Then recalling (B, § 6) that as coalgebras $\mathcal{C}\mathfrak{g} = S(\Sigma\mathfrak{g})$, we see that p and q induce an isomorphism $\theta: \mathcal{C}(\mathfrak{m}) \xrightarrow{\sim} \mathcal{C}(\mathfrak{n}) \otimes \mathcal{C}(\mathfrak{a})$. This shows that B, 7.1 can be applied to the maps $\mathcal{C}\mathfrak{a} \to \mathcal{C}\mathfrak{m} \to \mathcal{C}\mathfrak{n}$ to give a spectral sequence of coalgebras

$$E^2_{pq} = H_p\mathcal{C}(\mathfrak{n}) \otimes H_q\mathcal{C}(\mathfrak{a}) \Longrightarrow H_{p+q}\mathcal{C}(\mathfrak{m}) .$$

Recalling that $\otimes$ is the direct product in the category of coalgebras, the isomorphism θ induces an isomorphism $Z \xrightarrow{\sim} Y \otimes \mathcal{C}(\mathfrak{a})$; so there is also by B, 7.1, a spectral sequence

$$E^2_{pq} = H_p Y \otimes H_q\mathcal{C}(\mathfrak{a}) \Longrightarrow H_{p+q}(Z)$$

as well as a map of this spectral sequence to the other one induced by ξ and pr_2. As the map is an isomorphism on E^2, it is also an isomorphism on the abutment, so pr_2 is a weak equivalence and the lemma is proved.

PROOF OF 5.2. The axioms CM1, 2, 3, and 4 (ii) are clear.

CM5 (i). Given a map $f: X \to Y$ in $(\mathrm{DGC})_{r+1}$ use CM5 (i) for $(\mathrm{DGL})_r$ to write $\mathcal{L}f = pi$, where $i: \mathcal{L}X \to \mathfrak{m}$ is a cofibration (hence injective by 5.5) and where $p: \mathfrak{m} \to \mathcal{L}Y$ is a trivial fibration hence surjective. Letting $\mathfrak{n} = \mathcal{L}Y$ and $\xi =$ the adjunction map $Y \to \mathcal{C}\mathcal{L}Y$, form the square of Lemma 5.6, and let $j\colon X \to Z$ be be the unique map with $pr_1 j = f$ and $pr_2 j =$ the map $X \to \mathcal{C}\mathfrak{m}$ adjoint to i. As $r \geqq 1$, ξ and $\mathcal{C}p$ are weak equivalences (B, 7.5) and by the lemma so is pr_2. Thus pr_1 is a weak equivalence. As $\mathcal{L}$ carries cofibrations into cofibrations it follows that $\mathcal{C}(p)$ hence also pr_1 has the RLP with respect to all cofibrations; in particular pr_1 is a fibration in $(\text{DGC})_{r+1}$. j is injective because $pr_2 j$ is the composition of the injections $X \to \mathcal{C}\mathcal{L}X \to \mathcal{C}\mathfrak{m}$; thus j is a cofibration in $(\text{DGC})_{r+1}$. Therefore $f = (pr_1)j$ is the factorization required for CM5 (i). Moreover if f is already a trivial fibration, then j is a trivial cofibration by CM2; thus j has LLP with respect to f, so f is a retract of pr_1 and therefore f has the RLP with respect to all cofibrations. This proves CM4 (i).

CM5 (ii). As in the proof of this axiom for $\mathcal{S}_r$ we first consider the case where $f\colon X \to Y$ is such that $H_{r+1}f \simeq H_r\mathcal{L}f$ is surjective. Then $H_r\mathcal{L}f$ is surjective so if we use CM5 (ii) for $(\text{DGL})_r$ to write $\mathcal{L}f = pi$, where i is a trivial cofibration and p is a fibration, then p is surjective. Defining $j\colon X \to Z$, $pr_1\colon Z \to Y$, ξ and pr_2, as above, we have that pr_1 is a fibration in $(\text{DGC})_{r+1}$. By the lemma pr_2 is a weak equivalence; as $i\colon \mathcal{L}X \to \mathfrak{m}$ is a weak equivalence so is $pr_2 j\colon X \to \mathcal{C}\mathfrak{m}$. Therefore j is a trivial cofibration and $f = (pr_1)j$ is the factorization required for CM5 (ii).

In case $H_{r+1}f$ is not surjective, we construct a factorization $f = jg$, where j is an injective fibration and $H_{r+1}g$ is surjective, by following the proof of 2.7. If V is a vector space, let $V[r]$ be the abelian DG Lie algebra having V in dimension r and zero elsewhere, and let $K(V, r+1) = \mathcal{C}V(r)$. Define the factorization $f = j_1(f)f_1$ by the diagram

$$\begin{tikzcd}
X \arrow[dr, "f_1"] \arrow[drr, "\beta"] \arrow[ddr, "f"'] & & \\
& Y^1 \arrow[r] \arrow[d, "j_1(f)"] & K(\operatorname{Im} H_r\mathcal{L}f, r+1) \arrow[d, "\rho"] \\
& Y \arrow[r, "\alpha"] & K(H_r\mathcal{L}Y, r+1)
\end{tikzcd}$$

where the square is cartesian, where α is adjoint to the canonical map $\mathcal{L}Y \to H_r(\mathcal{L}Y)[r]$, where β is adjoint to the map

$$\mathcal{L}X \longrightarrow H_r(\mathcal{L}X)[r] \longrightarrow (\operatorname{Im} H_r\mathcal{L}f)[r] ,$$

and where ρ is the inclusion. It is clear that ρ is a fibration in $(\text{DGC})_{r+1}$, hence so is $j_1(f)$. Repeating this process as in 2.7 one obtains the required factorization $f = jg$; as g may be factored according to the first part of the proof, we have proved CM5 (ii) and the proof of Theorem 5.2 is complete.

Theorems 5.3 and 5.4 are now proved by verifying the hypotheses of 1.4 using results of Appendix B, 4.4 and 4.6. The proof of Theorem II is now complete.

6. Applications

In this section we connect the preceding homotopy theories of algebraic objects with the rational homotopy theory of 1-connected pointed topological spaces, and use the algebraic models to derive results about rational homotopy theory.

It is unfortunate that the category $\mathcal{T}_r$ of $(r-1)$-connected topological spaces is not closed under finite limits, for this prevents us from making $\mathcal{T}_r$ into a closed model category for trivial reasons. However if r, S are as in § 2, let us make the natural definitions and define $\mathcal{T}(r, S)$ to be the category $\mathcal{T}_r$ with the following three distinguished classes of maps.

Cofibrations. These are maps $f\colon X \to Y$ which are cofibrations as maps of topological spaces in the sense of [HA, II, § 3] i.e., f is a retract of a sequential composition of CW maps (see proof of Lemma 3, *loc. cit.*).

Weak equivalences. These are the S-equivalences, i.e., maps inducing isomorphisms for the functor $S^{-1}\pi_*$.

Fibrations. These are Serre fibrations such that the fiber has S-uniquely-divisible homotopy groups.

THEOREM 6.1. (a) *With these definitions $\mathcal{T}(r, S)$ satisfies all of the axioms for a closed model category except* CM1.

(b) *If* $\operatorname{Ho}\mathcal{T}(r, S)$ *is the localization of* $\mathcal{T}(r, S)$ *with respect to the family of weak equivalences, then* $\operatorname{Ho}\mathcal{T}(r, S)$ *is equivalent to the category whose objects are pointed* $(r-1)$*-connected* CW *complexes with S-uniquely-divisible homotopy groups, and whose morphisms are homotopy classes of basepoint-preserving maps.*

(c) *It is possible to define suspension and loop functors and families of fibration and cofibration sequences on the category* $\operatorname{Ho}\mathcal{T}(r, S)$ *by using the fibrations and cofibrations in* $\mathcal{T}(r, S)$ *just as in* [HA, I, § 2–3].

(d) *The functors* $|\cdot|$ *and* $E_r \operatorname{Sing}$ *induce an equivalence of the homotopy theory of* $\mathcal{T}(r, S)$ *with the homotopy theory of* $\mathcal{S}(r, S)$ *as defined in* § 2.

PROOF. The proof of (a) is formally similar to that of 3.1, using the functors $|\cdot|$ and $E_r \operatorname{Sing}$ instead of G and $\overline{W}$. The only point is to show the analogue of 3.2, that the cobase extension of an S-equivalence by a cofibration is again an S-equivalence. But any cofibration is a retract of a sequential composition of CW maps. One therefore is reduced to the case where the cofibration is the map obtained by attaching a single cell, in which case the proof is achieved by using the long exact sequence for homology and the five lemma.

For (b), observe that since the adjunction maps for the functors $|\cdot|$ and $E_r\,\mathrm{Sing}$ are always weak equivalences, they induce an equivalence of the categories $\mathrm{Ho}\,\mathcal{T}(r, S)$ and $\mathrm{Ho}\,\mathcal{S}(r, S)$, and that the latter is by 2.8 and Milnor's theorem [19] equivalent to the category of CW complexes with S-uniquely-divisible homotopy groups and homotopy classes of maps. (c) is a matter of checking that the axiom CM1 enters into the construction of the suspension and loop functors and fibrations and cofibration sequences only in allowing one to form the fiber, pull-back, etc. of various maps. For the suspension functor and cofibration sequences there is no problem because one has only to work with cofibrant objects which have non-degenerate basepoints. For fibrations the problem comes from the fact that the fiber of a map in $\mathcal{T}_r$ is the $(r-1)$-connected covering of the real fiber which need not exist. However one may always replace a fibration of spaces by a weakly equivalent map which is the geometric realization of a fibration f of simplicial sets [22]. If F is the fiber of f, then $|E_rF|$ can be used for the fiber of $|f|$ in $\mathcal{T}_r$, and one may check that the construction of the loop functor and the family of fibration sequences on $\mathrm{Ho}\,\mathcal{T}(r, S)$ still goes through. Finally (d) is proved by the same method as 1.4 (see proof of [HA, I, § 4, Th. 3]), using the fact that $E_r\,\mathrm{Sing}$ preserves fibrations and $|\cdot|$ preserves cofibrations, q.e.d.

Taking $r = 2$ and $S = \mathbf{Z} - \{0\}$, and combining the above with Theorem II we have

COROLLARY 6.2. *The adjoint functors of Figure* 1, *Part* I, § 2, *induce an equivalence of rational homotopy theory, defined to be the homotopy theory of* $\mathcal{T}(2, \mathbf{Z} - \{0\})$ *constructed above, with the homotopy theories of reduced* DG *Lie algebras and* 2*-reduced* DG *coalgebras over* $\mathbf{Q}$.

Remark 6.3. We claim now to have solved the problem raised by Thom in [29]. Suppose that $F \to E \to B$ is a fibration of 1-connected pointed spaces. The problem after translating from cohomology into homology, is to associate DG *cocommutative* coalgebras to F, E, and B in such a way that the Hirsch method for calculating the differentials in the homology spectral sequence can be applied. But this fibration defines a fibration sequence in rational homotopy theory which is equivalent by 6.2 to that of $(\mathrm{DGL})_1$; hence this fibration sequence corresponds to one in $\mathrm{Ho}\,(\mathrm{DGL})_1$ coming from an exact sequence of reduced DG Lie algebras

$$0 \longrightarrow f \longrightarrow e \longrightarrow b \longrightarrow 0\,.$$

However $\mathcal{C}f$, $\mathcal{C}e$, $\mathcal{C}b$ correspond under the equivalence of Theorem I to F, E, and B respectively, and Hirsch's methods apply to the homology spectral sequence of the maps $\mathcal{C}f \to \mathcal{C}e \to \mathcal{C}b$ (in fact this is how the spectral sequence B, 7.1 is derived).

Remark 6.4. We indicate briefly how 6.2 may be used to give an alternative proof of the results of Part I, §§ 5–6, namely that the equivalences λ and $\tilde{\mathcal{C}}$ of Theorem I are compatible with the homotopy Lie algebra and the homology coalgebra functor. The first thing to show is that if V is a vector space and $V[n]$ denotes the DG vector space which is V in dimension n and zero elsewhere, then there are canonical isomorphisms

$$[LV[n], \mathfrak{g}] \simeq \operatorname{Hom}(V, H_n\mathfrak{g}) \qquad \mathfrak{g} \in \operatorname{Ob}(\mathrm{DGL})_1$$
$$[C, \mathcal{C}V[n-1]] \simeq \operatorname{Hom}(H_nC, V) \qquad C \in \operatorname{Ob}(\mathrm{DGC})_2$$

where Hom is maps of vector spaces, and where [,] denotes homotopy classes of maps as defined in § 1, which by 1.3 is the same as maps in the homotopy category. Next one observes that there is an isomorphism of functors

$$\theta_n \colon \pi_n(X) \otimes \mathbf{Q} \simeq H_{n-1}(\lambda X)$$

from $\mathcal{T}_2$ to vector spaces given by the chain of isomorphisms

$$\begin{aligned}\pi_n(X) \otimes \mathbf{Q} &\simeq \pi_n(K) \otimes \mathbf{Q} \simeq \pi_{n-1}(GK) \otimes \mathbf{Q} \\ &\simeq \pi_{n-1}(\mathcal{G}\hat{\mathbf{Q}}GK) \simeq \pi_{n-1}(\mathcal{P}\hat{\mathbf{Q}}GK) \simeq \pi_{n-1}(N\mathcal{P}\hat{\mathbf{Q}}GK)\ ,\end{aligned}$$

where we have put $K = E_2 \operatorname{Sing} X$.

The work of Part I, § 5, went into showing that the collection $\theta = \{\theta_n\}$ gave a Lie algebra isomorphism. However to prove the existence of a functorial Lie algebra isomorphism of $\pi_*(X) \otimes \mathbf{Q}$ with $H_*(\lambda X)$, it is possible to use Hilton's theorem on homotopy operations [10] to show that for some choice of non-zero rational numbers c_n the collection

$$c_n\theta_n \colon \pi_n(X) \otimes \mathbf{Q} \simeq H_{n-1}(\lambda X)$$

is a Lie algebra isomorphism.

When the Lie structure is taken care of, one may take care of the coalgebra structure as follows. First one uses the isomorphism $\pi_n(X) \otimes \mathbf{Q} \simeq H_{n-1}(\lambda X)$ to establish an isomorphism in $\operatorname{Ho}(\mathrm{DGL})_1$ of $\lambda K(V, n)$ with $V[n-1]$, where $K(V, n)$ is the appropriate Eilenberg-MacLane space, and where $V[n-1]$ is regarded as an abelian DG Lie algebra. It follows that there are canonical isomorphisms

$$H^n(X, V) \simeq [X, K(V, n)] \simeq [\lambda X, V[n-1]]$$
$$\simeq [\tilde{\mathcal{C}}\lambda X, \tilde{\mathcal{C}} V[n-1]] \simeq \operatorname{Hom}(H_n(\tilde{\mathcal{C}}\lambda X), V)\ ,$$

and hence a canonical isomorphism $H_n(\lambda X, \mathbf{Q}) \simeq H_n(\tilde{\mathcal{C}}\lambda X)$ of vector spaces. To prove this isomorphism is compatible with comultiplication, it suffices to show that $\tilde{\mathcal{C}}\lambda$ carries the cup product map $\mu\colon K(\mathbf{Q}, p) \times K(\mathbf{Q}, q) \to K(\mathbf{Q}, p+q)$ into the corresponding map $\mu'\colon \mathcal{C}\mathbf{Q}[p-1] \times \mathcal{C}\mathbf{Q}[q-1] \to \mathcal{C}\mathbf{Q}[p+q-1]$ deduced from the comultiplication on coalgebras. The point is that μ can be characterized in terms of the Whitehead product using a Toda bracket. In effect if $\alpha_p\colon S^p \to K(\mathbf{Q}, p)$ is the canonical map giving the orientation of S^p, if β is the composition of the inclusion $S^p \vee S^q \to S^p \times S^q$ with $\alpha_p \times \alpha_q$, and if $[i_p, i_q]\colon S^{p+q-1} \longrightarrow S^p \vee S^q$ is the Whitehead product of the inclusions $i_p\colon S^p \to S^p \vee S^q$ and $i_q\colon S^q \to S^p \vee S^q$, then the Toda bracket $\langle [i_p, i_q], \beta, \mu \rangle = \alpha_{p+q}$ with zero indeterminacy. A similar characterization holds for μ' in terms of the bracket operation for DG Lie algebras. 6.2 implies that $\tilde{\mathcal{C}}\lambda$ will preserve Toda brackets since they are determined by the fibration and cofibration sequences [HA, I, § 3]. Since λ carries Whitehead product into Lie product, it follows that $\tilde{\mathcal{C}}\lambda$ carries μ into μ'.

Remark 6.5. The possibility of viewing rational homotopy theory in terms of both DG Lie algebras and DG coalgebras allows one to give a perfectly Eckmann-Hilton dual treatment for rational homotopy theory. We illustrate this by means of four spectral sequences. Here X and Y are 1-connected pointed spaces, $\pi_*^{\mathbf{Q}}(\Omega X) = \pi_*(\Omega X) \otimes \mathbf{Q}$ with Lie algebra structure given by Samelson product, and $H_* X = H_*(X, \mathbf{Q})$.

6.6 (Serre). *If $p\colon X \to Y$ is a fibration with fiber F and $\pi_2 p$ is surjective, then there is a coalgebra spectral sequence*

$$E^2 = H_* Y \otimes H_* F \Longrightarrow H_* X\ .$$

6.7. *If $i\colon Y \to X$ is a cofibration, then there is a Lie algebra spectral sequence*

$$E^2 = \pi_*^{\mathbf{Q}}(\Omega X) \vee \pi_*^{\mathbf{Q}}(\Omega(X/Y)) \Longrightarrow \pi_*^{\mathbf{Q}}(\Omega X)\ .$$

6.8 (Curtis [7]). *There is a Lie algebra spectral sequence*

$$E^1 = L(\Omega \bar{H}_* X) \Longrightarrow \pi_*^{\mathbf{Q}}(\Omega X)\ .$$

6.9. *There is a coalgebra spectral sequence*

$$E^1 = S(\pi_*^{\mathbf{Q}}(X)) \Longrightarrow H_* X\ .$$

PROOFS. 6.6. Realize the fibration as an exact sequence of DG Lie algebras, apply $\mathcal{C}$, and use B, 7.1 (see 6.3).

6.8. Take the spectral sequence associated to the lower central series filtration of λX.

6.9. Take the spectral sequence associated to the primitive filtration of $\mathcal{C}\lambda X$.

6.7. We may realize i as a cofibration $i: \mathfrak{g}' \to \mathfrak{g}$ of free reduced DG Lie algebras. If $\mathfrak{g}''$ is the cofibre of i, then as graded Lie algebras we may identify $\mathfrak{g}$ with the direct sum $\mathfrak{g}' \vee \mathfrak{g}''$. Define a filtration of $\mathfrak{g}$ by letting $F_p\mathfrak{g}$ be the p-skeleton of $\mathfrak{g}$ with respect to $\mathfrak{g}'$, that is the subalgebra generated by $\mathfrak{g}'$ and $\mathfrak{g}_i$ for $i \leqq p$. Then this filtration gives rise to a spectral sequence of Lie algebras. Note that

$$E^0 = \bigoplus F_p\mathfrak{g}/F_{p-1}\mathfrak{g} \xleftarrow{\sim} \mathfrak{g}'' \vee \mathfrak{g}' .$$

To calculate the differentials, let d, d', d'' be the differentials of $\mathfrak{g}$, $\mathfrak{g}'$, and $\mathfrak{g}''$ respectively and identify $\mathfrak{g}$ with $\mathfrak{g}'' \vee \mathfrak{g}'$. Then d is uniquely determined by its restriction to $\mathfrak{g}'$, which is d', and its restriction to $\mathfrak{g}''$ which is a derivation $d: \mathfrak{g}'' \to \mathfrak{g}'' \vee \mathfrak{g}'$ such that $(d - d'')\mathfrak{g}''_p \subset F_{p-2}\mathfrak{g}$. From this one calculates that d^0 is 0 on $\mathfrak{g}''$ and d' on $\mathfrak{g}'$ so that $E^1 = \mathfrak{g}'' \vee H\mathfrak{g}'$, and that d^1 is d'' on $\mathfrak{g}''$ and 0 on $H\mathfrak{g}'$ so that

$$E^2 = H\mathfrak{g}'' \vee H\mathfrak{g}' \simeq \pi_*^{\mathbf{Q}}\big(\Omega(X/Y)\big) \vee \pi_*^{\mathbf{Q}}(\Omega X) .$$

This proves 6.7.

Remark 6.10. The spectral sequence 6.7 seems to be related to the one derived by Artin and Mazur in [2]. Note also that when $X = Y \vee Z$ then we have

$$\pi_*^{\mathbf{Q}}\big(\Omega(Y \vee C)\big) \simeq \pi_*^{\mathbf{Q}}(\Omega Y) \vee \pi_*^{\mathbf{Q}}(\Omega C)$$

showing that Hilton-Milnor formula holds for rational homotopy groups even if Y and C are not suspensions.

The spectral sequence 6.9 may be rewritten

$$E^1 = H_*\big(K(\pi(X)), \mathbf{Q}\big) \Longrightarrow H_*(X, \mathbf{Q})$$

where $K(\pi(X))$ is the product of the Eilenberg-MacLane spaces $K(\pi_q(X), q)$. We pose the question of whether such a spectral sequence holds with $\mathbf{Q}$ replaced by an arbitrary coefficient group A. In the stable range there does [5].

Final remark 6.11. Combining Theorem I, and Theorems 1.3 and 5.1 of Part II, we obtain the following simple description of the rational homotopy category. It is equivalent to the category with

Objects: free reduced DG Lie algebras over $\mathbf{Q}$

Morphisms: homotopy classes of maps of DG Lie algebras, where two maps $f, g \rightrightarrows \mathfrak{m}$ are said to be homotopic if and only if there exists a diagram of DG Lie algebras

$$\begin{tikzcd} \mathfrak{g} \arrow[d, "\mathrm{id} + \mathrm{id}"'] & \tilde{\mathfrak{g}} \arrow[l, "\sigma"'] \arrow[d, "h"] \\ \mathfrak{g} \vee \mathfrak{g} \arrow[ur, "\partial_1 + \partial_0"] \arrow[r, "f + g"'] & \mathfrak{m} \end{tikzcd}$$

where σ is a weak equivalence.

Appendix A. Complete Hopf Algebras

This appendix is an exposition of the results on complete Hopf algebras used in this paper. Complete Hopf algebras provide the Hopf algebra framework for handling the Malcev completion [18] of a nilpotent group as well as groups defined by the Campbell-Hausdorff formula [17, Ch. II]. In fact it was the proof of the Campbell-Hausdorff formula [28, LA 4.13] that led to the definition of complete Hopf algebras.

Most of the work of this appendix goes into proving that the category (CHA) of complete Hopf algebras is closed under limits and has a projective generator, a result needed in Part II. We review the meanings of these terms. Following Grothendieck, we call a map $f: X \to Y$ in a category an *effective epimorphism* if for any object T, composition with f is a bijection of $\mathrm{Hom}(Y, T)$ with the subset of $\mathrm{Hom}(X, T)$ consisting of all maps φ with the following property. Given two maps $u, v \in \mathrm{Hom}(Z, X)$ such that $fu = fv$, then $\varphi u = \varphi v$. We call an object P *projective* if $\mathrm{Hom}(P, X) \to \mathrm{Hom}(P, Y)$ is surjective for any effective epimorphism $X \to Y$, and we say that P is *small* if the functor $X \mapsto \mathrm{Hom}(P, X)$ commutes with filtered inductive limits. Finally a set $\mathfrak{M}$ of objects is called a set of *generators* if for any object X there is an effective epimorphism $Q \to X$ where Q is a direct sum of members of $\mathfrak{M}$. By a theorem of Lawvere [16] a category closed under limits and having a small projective generator is a category of universal algebras and conversely. Therefore although (CHA) is not a category of universal algebras, it is not far from being one.

The terminology of this appendix is the same as that of Part I with the exception that as we are not in the DG setting, a graded Lie algebra is a Lie algebra in the usual sense without the signs.

A1. Complete augmented algebras

Let K be a field. All modules, tensor products and algebras in this section are to be understood as being over K unless there is mention to the contrary. The augmentation ideal of an augmented algebra R will be denoted $\bar{R}$.

By a *filtration* of an algebra R we mean a decreasing sequence $R = F_0R \supset F_1R \supset \cdots$ of subspaces such that $1 \in F_0R$ and $F_pR \cdot F_qR \subset F_{p+q}R$. Then each F_nR is a two-sided ideal in R and $\operatorname{gr} R = \bigoplus_{n=0}^{\infty} F_nR/F_{n+1}R$ has a natural structure as a graded algebra. By a *complete augmented algebra* we mean an augmented algebra R endowed with a filtration $\{F_nR\}$ such that

(a) $F_1R = \bar{R}$

(b) $\operatorname{gr} R$ is generated as an algebra by $\operatorname{gr}_1 R$

(c) $R = \text{lim-inv}\,(R/F_nR)$.

The class of complete augmented algebras forms a category (CAA), where a map $f\colon R \to R'$ is a map of augmented algebras such that $f(F_nR) \subset F_nR'$.

Condition (b) is easily seen to be equivalent to

(1.1) $$\bar{R}^n + F_rR = F_nR \qquad \text{if } r \geqq n$$

or equivalently that F_nR is the closure of $\bar{R}^n$ for the topology defined by the filtration.

Examples 1.2. If B is an augmented algebra, then $\hat{B} = \text{lim-inv}\, B/\bar{B}^n$ is a complete augmented algebra with $F_nB = \operatorname{Ker}\{\hat{B} \to B/\bar{B}^n\}$. Condition (b) follows from the formula

(1.3) $$\operatorname{gr} \hat{B} = \operatorname{gr} B = \bigoplus \bar{B}^n/\bar{B}^{n+1} .$$

It is clear that the functor $B \mapsto \hat{B}$ is left adjoint to the functor (CAA) $\to$ (AA) which forgets the filtration.

1.4. If J is an ideal in a complete augmented algebra R such that $J \subset \bar{R}$ and J is closed for the topology defined by the filtration, then R/J is a complete augmented algebra with $F_n(R/J) = (F_nR + J)/J$. Condition (b) follows from the formula

(1.5) $$\operatorname{gr} R/J = \operatorname{gr} R/\operatorname{gr} J$$

where J is given the induced filtration. As for (c), first note that R/J is separated since J is closed, i.e., $J = \bigcap (F_nR + J)$. On the other hand, if y_n is a sequence in R/J with $y_n - y_{n-1} \in F_n(R/J)$, then $y_n - y_{n-1} = x_n + J$ with $x_n \in F_nR$, so $x = \sum x_k$ converges in R and

$$(x + J) - y_n = \textstyle\sum_{k<n} x_k + J \in F_{n+1}(R/J) .$$

Thus y_n converges to $x + J$, and R/J is complete.

PROPOSITION 1.6. *The following conditions are equivalent for a map*

$f: R \to R'$ in (CAA).

(i) $\mathrm{gr}_1 f$ *is surjective,*

(ii) f *is surjective,*

(iii) f *induces an isomorphism* $R/\mathrm{Ker}\, f \xrightarrow{\sim} R'$ *where* $R/\mathrm{Ker}\, f$ *is as in* 1.4.

PROOF. If f is surjective, then $\bar{f}: \bar{R} \to \overline{R'}$ is surjective and so $\mathrm{gr}_1 f$ is surjective. If $\mathrm{gr}_1 f$ is surjective, then $\mathrm{gr}\, f$ is surjective by condition (b), and so f is surjective by completeness. It is clear that (iii) implies (ii). For the converse, note that $\mathrm{Ker}\, f = J$ is closed since f is continuous, so R/J is defined by 1.5. Since the map $R/J \to R'$ is an isomorphism of augmented rings, we have only to show that the filtrations are the same. But $F_m R \to F_m R'$ is surjective since the associated graded map is surjective (use (b)). Thus

$$F_m R' \simeq F_m R / F_m R \cap J \simeq F_m R + J/J \simeq F_m(R/J)\ , \qquad \text{q.e.d.}$$

If I is a set, then the algebra $P = K\langle\langle X_i \rangle\rangle_{i \in I}$ of formal power series in the non-commuting indeterminates X_i is a complete augmented algebra because it is the completion of the polynomials in the X_i. If R is a complete augmented algebra and x_i, $i \in I$, is a family of elements of $\bar{R}$ such that $x_i + F_2 R$ is a basis for $\mathrm{gr}_1 R$, then the unique homomorphism $P \to R$ which sends X_i to x_i is surjective by 1.6. Thus

COROLLARY 1.7. *Any complete augmented algebra R is the quotient of a (non-commutative) power series ring P by a closed ideal. Moreover P may be chosen so that* $\mathrm{gr}_1 P \xrightarrow{\sim} \mathrm{gr}_1 R$.

PROPOSITION 1.8. *The effective epimorphisms in* (CAA) *are the maps satisfying the conditions of* 1.6.

PROOF. The map $\pi: R \to R/J$ is an effective epimorphism because a map $f: R \to R'$ factors through π if and only if $f(J) = 0$, and because any element of J is the image of X under a map $K\langle\langle X \rangle\rangle \to R$. Conversely any map f factors $R \xrightarrow{\pi} R/\mathrm{Ker}\, f \xrightarrow{g} R'$, where g is injective and hence a monomorphism. If f is an effective epimorphism, one sees easily that g is an isomorphism, and so f is surjective.

COROLLARY 1.9. *The following conditions are equivalent for a complete augmented algebra R.*

(i) *R is a projective object of* (CAA),

(ii) *R is isomorphic to a non-commutative power series ring,*

(iii) *The natural map $T(\mathrm{gr}_1 R) \to \mathrm{gr}\, R$ is an isomorphism, where T is the tensor algebra functor.*

PROOF. (ii) $\Rightarrow$ (i). There is a one-to-one correspondence between maps

$u: K\langle\langle X_i \rangle\rangle_{i \in I} \to Y$ and families of elements $y_i \in \bar{Y}$ given by $u(X_i) = y_i$. If $v: X \to Y$ is an effective eqimorphism, then $\bar{X} \to \bar{Y}$ is surjective by 1.8, so there are elements $x_i \in \bar{X}$ with $vx_i = y_i$. Thus if $w: K\langle\langle X_i \rangle\rangle \to X$ is given by $wX_i = x_i$, then $vw = u$, proving (i).

(ii) $\Leftrightarrow$ (iii). If P is a power series ring, then $P = TV^{\wedge}$ where V is the vector space having the indeterminates of P for its basis. Hence $\operatorname{gr} P = \operatorname{gr} TV = TV$ is the tensor algebra on $\operatorname{gr}_1 V$. Conversely given R satisfying (iii) choose a surjection $f: P \to R$ as in 1.7 such that $\operatorname{gr}_1 P \xrightarrow{\sim} \operatorname{gr}_1 R$. Then $\operatorname{gr} f$ is an isomorphism, so f is an isomorphism by completeness.

(i) $\Rightarrow$ (iii). Given a projective R, choose a surjection $f: P \to R$ with $\operatorname{gr}_1 f$ an isomorphism. By 1.8, f is an effective epimorphism, so f has a section s. Hence $\operatorname{gr} f$ has a section $\operatorname{gr} s$. But also $(\operatorname{gr} s)(\operatorname{gr} f) = \mathrm{id}$, since this is true in dimension 1 and $\operatorname{gr} P$ is generated by $\operatorname{gr}_1 P$. Hence $\operatorname{gr} R \simeq \operatorname{gr} P$ is a tensor algebra, q.e.d.

PROPOSITION 1.10. *The category* (CAA) *is closed under arbitrary limits and $K\langle\langle X \rangle\rangle$ is a projective generator.*

PROOF. The second assertion is clear from 1.7–1.9. To prove the first, it suffices to show (CAA) closed under sums, cokernels, products, and kernels.

Sums. If R_i is a family write $R_i = P_i/J_i$ where P_i is a power series ring and J_i is a closed ideal. Then the direct sum $P = \bigvee P_i$ exists and is a power series ring whose set of indeterminates is the disjoint union of the set of indeterminates of the P_i. Let J be the closed ideal in P generated by $\sum in_i(J_i)$. Then in_i induces a map $u_i: R_i \to P/J$, and the family $\{u_i\}$ is easily seen to make P/J a direct sum of the X_i.

Cokernels. If $f, g: R \rightrightarrows R'$ are two maps, let J be the closed ideal in R' generated by $(f - g)\bar{R}$. Then $\pi: R' \to R'/J$ is a cokernel for f, g.

Kernels. If $f, g: R \rightrightarrows R'$ are two maps, let P be the power series ring with one indeterminate for each element of $\operatorname{Ker}(\bar{f}, \bar{g})$, and let $u: P \to R$ be the obvious map. We claim that the induced map $u': P/\operatorname{Ker} u \to R$ is a kernel for f, g. Clearly $fu' = gu'$. If $v: X \to R$ satisfies $fv = gv$, then writing $X = P'/J'$, there is a map $w: P' \to P$ such that $\pi w = \pi'$ and w induces a map $w': X \to P/\operatorname{Ker} u$ such that $v = u'w'$. w' is unique because u' is a monomorphism, so $u' = \operatorname{Ker}(f, g)$.

The existence of *products* is similar and left to the reader, q.e.d.

Remark. If $\operatorname{gr}_1 R$ is finite dimensional, then $F_n R = \bar{R}^n$. In effect R is a quotient of a power series ring $P = K\langle\langle X_1, \cdots, X_n \rangle\rangle$ by a closed ideal and $F_n R$ is a quotient of $F_n P$ so one is reduced to the case $R = P$. But then if

$f(X) \in F_nP$, i.e., $f(X) = \sum_{|\alpha| \geqq n} a_\alpha X_\alpha$ (here α runs over finite sequences $(i_1, \cdots, i_r)$ of elements of $\{1, \cdots, n\}$, $|\alpha| = r$ and $X_\alpha = X_{i_1} \cdots X_{i_p}$), we have

$$f(X) = \sum_{i_1, \ldots, i_n} (\sum_\beta a_{\beta i_1 \cdots i_n} X_\beta) X_{i_1} \cdots X_{i_n} ,$$

so $f(X) \in \bar{P}^n$. Thus the category of "finite type" complete augmented algebras is a full subcategory of the category of augmented algebras.

1.11. By a *filtration* (N-sequence in the terminology of Lazard [17]) of a group G (resp. Lie algebra $\mathfrak{g}$), we mean a decreasing sequence of subgroups $G = F_1G \supset F_2G \supset \cdots$ (resp. subspaces $\mathfrak{g} = F_1\mathfrak{g} \supset F_2\mathfrak{g} \supset \cdots$) such that $(F_pG, F_qG) \subset F_{p+q}G$ (resp. $[F_p\mathfrak{g}, F_q\mathfrak{g}] \subset F_{p+q}\mathfrak{g}$). Then $\operatorname{gr} G$ (resp. $\operatorname{gr} \mathfrak{g}$) has a canonical structure of graded Lie ring (resp. Lie algebra over K) where bracket is induced by commutator [28, LA, Ch. II].

If R is a complete augmented algebra let $\mathbf{G}_mR$ be the group $1 + \bar{R}$ under multiplication and let $\mathbf{G}_aR$ be the Lie algebra $\bar{R}$ with $[x, y] = xy - yx$. Then there are adjoint functors

$$(1.12) \qquad (gps) \underset{\mathbf{G}_m}{\overset{\hat{K}}{\rightleftarrows}} (\text{CAA}) \underset{\mathbf{G}_a}{\overset{\hat{U}}{\leftrightarrows}} (\text{LA})$$

where $G \mapsto KG$ (resp. $\mathfrak{g} \to U\mathfrak{g}$) is the group ring (resp. universal enveloping algebra) functor and where $\hat{}$ is completion (1.2).

Letting $F_n(\mathbf{G}_mR) = 1 + F_nR$ and $F_n(\mathbf{G}_aR) = F_nR$, we obtain filtrations of $\mathbf{G}_mR$ and $\mathbf{G}_aR$ such that

$$(1.13) \qquad \operatorname{gr} \mathbf{G}_aR \xrightarrow{\sim} \operatorname{gr} \mathbf{G}_mR$$

as Lie algebras over $\mathbf{Z}$(cf. [28]). In particular $\operatorname{gr} \mathbf{G}_mR$ has a Lie algebra structure over K. This isomorphism is induced by $x \mapsto 1 + x$, but it is also induced by $x \mapsto f(x)$ where $f(X) \in K\langle\langle X \rangle\rangle$ is any power series with $f(X) \equiv 1 + X \bmod X^2$. In particular if K is of characteristic zero, we may take the exponential series

$$e^x = \sum_0^\infty x^n/n!$$

which induces a map of *sets*

$$\exp\colon \mathbf{G}_aR \xrightarrow{\sim} \mathbf{G}_mR$$

which is bijective since the inverse is given by the logarithmic series. We have

$$(1.14) \qquad e^xe^y = e^{x+y} \qquad \text{if } [x, y] = 0 ,$$

but in general only the Campbell-Hausdorff formula.

Complete tensor product. If $V = F_0V \supset F_1V \supset \cdots$ is a filtered vector space, let $\rho_n: F_nV \to \mathrm{gr}_n V$ be the canonical surjection. If W is another filtered vector space, then we filter $V \otimes W$ by

$$F_n(V \otimes W) = \sum_{i+j=n} F_iV \otimes F_jW \subset V \otimes W ,$$

where we identify $F_iV \otimes F_jW$ with its image under the map $F_iV \otimes F_jW \to V \otimes W$, which is injective since K is a field. There is a canonical isomorphism of graded vector spaces

$$\mathrm{gr}\, V \otimes \mathrm{gr}\, W \xrightarrow{\sim} \mathrm{gr}\,(V \otimes W)$$

given by $\rho_px \otimes \rho_qy \mapsto \rho_{p+q}(x \otimes y)$.

If V and W are complete, we let $V \hat{\otimes} W$ be the completion of $V \otimes W$ with respect to this filtration and denote the image of $x \otimes y$ under the map $V \otimes W \to V \hat{\otimes} W$ (which is injective) by $x \hat{\otimes} y$. Then there is an isomorphism

$$\mathrm{gr}\, V \otimes \mathrm{gr}\, W \xrightarrow{\sim} \mathrm{gr}\,(V \hat{\otimes} W) \tag{1.15}$$

given by $\rho_px \otimes \rho_qy \mapsto \rho_{p+q}(x \hat{\otimes} y)$.

As $F_{2n}(V \otimes W) \subset F_nV \otimes W + V \otimes F_nW \subset F_n(V \otimes W)$

$$V \hat{\otimes} W = \text{lim-inv}_n\,(V_n \otimes W_n)$$

where $V_n = V/F_nV$, etc. If $V' \subset V$ and $W' \subset W$ are closed subspaces endowed with induced filtrations then by passing to the inverse limit in the exact sequence of surjective inverse systems

$$\begin{aligned} 0 \longrightarrow V_n' \otimes W_n' \longrightarrow V_n' \otimes W_n \oplus V_n \otimes W_n' \longrightarrow V_n \otimes W_n \\ \longrightarrow (V/V')_n \otimes (W/W')_n \longrightarrow 0 \,, \end{aligned}$$

one sees the validity of the formulas

$$\begin{aligned} (V/V') \hat{\otimes} (W/W') &= (V \hat{\otimes} W)/V \hat{\otimes} W' + V' \hat{\otimes} W \\ (V \hat{\otimes} W') \cap (V' \hat{\otimes} W) &= V' \hat{\otimes} W' \,. \end{aligned} \tag{1.16}$$

If R and R' are complete augmented algebras, then $F_n(R \otimes R')$ is a filtration of $R \otimes R'$ so $R \hat{\otimes} R'$ is an augmented algebra. By (1.15) we have

$$\mathrm{gr}\, R \otimes \mathrm{gr}\, R' \simeq \mathrm{gr}\,(R \hat{\otimes} R') \tag{1.17}$$

which is an isomorphism of graded rings. From this we see $\mathrm{gr}\,(R \hat{\otimes} R')$ is generated by gr_1 and so $R \hat{\otimes} R'$ is a complete augmented algebra. The following properties of the complete tensor product $R \hat{\otimes} R'$ are immediate.

Universal mapping property. Given maps $u: R \to S$ and $v: R' \to S$ in (CAA) such that $[ux, vy] = 0$ for all $x \in R$ and $y \in R'$, there is a unique map $w: R \hat{\otimes} R' \to S$ such that $w(x \hat{\otimes} 1) = ux$ and $w(1 \hat{\otimes} y) = vy$.

(1.18) $\hat{A} \hat{\otimes} \hat{B} = \widehat{A \otimes B}$ if A, B are augmented algebras.

(1.19) $(R/J)\mathbin{\hat{\otimes}}(R'/J') = (R\mathbin{\hat{\otimes}}R')/R\mathbin{\hat{\otimes}}J' + J\mathbin{\hat{\otimes}}R'$ if J and J' are closed ideals of R and R', respectively.

A2. Complete Hopf algebras

A *complete Hopf algebra* is a complete augmented algebra A endowed with a "diagonal" $\Delta\colon A \to A\mathbin{\hat{\otimes}}A$, which is a map of complete augmented algebras and which is coassociative, cocommutative, and has the augmentation map $A \to K$ as a counit. With the evident definition of morphisms the complete Hopf algebras form a category (CHA).

Examples. 2.1. If A is a (coassociative, cocommutative, as always) Hopf algebra, then $\hat{A}$ (1.2) is a CHA with diagonal $\hat{\Delta}\colon \hat{A} \to (A \otimes A)^\wedge \simeq \hat{A}\otimes\hat{A}$ (1.18). In particular if G is a group and $\mathfrak{g}$ is a Lie algebra, then $\widehat{KG}$ and $\widehat{U\mathfrak{g}}$ are CHA's.

2.2. If A is a CHA and J is a closed Hopf ideal of A in the sense that $\Delta J \subset A\mathbin{\hat{\otimes}}J + J\mathbin{\hat{\otimes}}A$, then the complete augmented algebra A/J (1.4) is a CHA with Δ induced by that of A using (1.19).

2.3. If A and A' are CHA's then so is $A\mathbin{\hat{\otimes}}A'$ in the obvious way. Moreover if $pr_1\colon A\mathbin{\hat{\otimes}}A' \to A$ and $pr_2\colon A\mathbin{\hat{\otimes}}A' \to A'$ are the maps induced by the augmentations of A' and A respectively then $A\mathbin{\hat{\otimes}}A'$ with pr_1 and pr_2 is the direct product of A and A' in the category (CHA).

If A is a CHA, we set

$$\begin{aligned} \mathcal{P}A &= \{x \in \bar{A} \mid \Delta x = x\mathbin{\hat{\otimes}}1 + 1\mathbin{\hat{\otimes}}x\} \\ \mathcal{G}A &= \{x \in 1 + \bar{A} \mid \Delta x = x\mathbin{\hat{\otimes}}x\}\,. \end{aligned} \tag{2.4}$$

$\mathcal{P}A$, the set of *primitive* elements of A, is a Lie subalgebra of $\mathbf{G}_a A$, and $\mathcal{G}A$, the set of *group-like* elements, is a subgroup of $\mathbf{G}_m A$ (1.12). Letting $\hat{K}$ and $\hat{U}$ be the completed group and universal enveloping algebra functors with CHA structure as in (2.1), it is straightforward to verify the following.

PROPOSITION 2.5. *There are adjoint functors*

$$(gps) \underset{\mathcal{G}}{\overset{\hat{K}}{\rightleftarrows}} (\text{CHA}) \underset{\mathcal{P}}{\overset{\hat{U}}{\leftrightarrows}} (\text{LA})\,.$$

For the rest of this section we suppose that K has characteristic zero. Then the exponential series e^x is defined.

PROPOSITON 2.6. $x \in \mathcal{P}A \Longleftrightarrow e^x \in \mathcal{G}A$.

PROOF. $x \in \mathcal{P}A \Longleftrightarrow \Delta x = x\mathbin{\hat{\otimes}}1 + 1\mathbin{\hat{\otimes}}x \Longleftrightarrow e^{\Delta x} = e^{x\hat{\otimes}1 + 1\hat{\otimes}x} = e^{x\hat{\otimes}1}\cdot e^{1\hat{\otimes}x}$ $= (e^x\mathbin{\hat{\otimes}}1)(1\mathbin{\hat{\otimes}}e^x) = e^x\mathbin{\hat{\otimes}}e^x$ (using 1.14) $\Longleftrightarrow e^x \in \mathcal{G}A$.

It follows that the exponential and logarithm functions give a canonical isomorphism of sets

$$\text{(2.7)} \qquad \exp\colon \mathcal{P}A \simeq \mathcal{G}A$$

satisfying (1.14). The following is immediate from (1.13).

PROPOSITION 2.8. *Let* $F_r\mathcal{G}A = \{x \in \mathcal{G}A \mid x - 1 \in F_rA\}$ *and* $F_r\mathcal{P}A = \mathcal{P}A \cap F_rA$. *Then* $\{F_r\mathcal{G}A\}$ *and* $\{F_r\mathcal{P}A\}$ *are filtrations* (1.11) *of* $\mathcal{G}A$ *and* $\mathcal{P}A$ *respectively. Moreover the exponential induces an isomorphism*

$$\text{(2.9)} \qquad \operatorname{gr}\mathcal{P}A \xrightarrow{\sim} \operatorname{gr}\mathcal{G}A$$

of Lie rings and defines a K-module structure on $\operatorname{gr}\mathcal{G}A$ *compatible with its bracket. Finally*

$$\text{(2.10)} \qquad A \simeq \text{lim-inv}\,(\mathcal{P}A/F_r\mathcal{P}A),\ \mathcal{G}A \simeq \text{lim-inv}\,(\mathcal{G}A/F_r\mathcal{G}A)\,.$$

Example 2.11. Let S be a set, let FS be the free group generated by S, and let LKS be the free Lie algebra generated by S. Then by the isomorphism of functors (2.7) and by (2.5), there are CHA isomorphisms

$$\hat{K}FS \xrightarrow[\sim]{\varphi} K\langle\langle X_s\rangle\rangle_{r\in S} \xleftarrow[\sim]{\theta} \hat{U}LKS$$

where $K\langle\langle X_s\rangle\rangle$ is the non-commutative power series ring with Δ defined so that the X_s are primitive, and where φ and θ are determined by the formulas $\varphi(s) = e^{X_s}$, $\theta(s) = X_s$ for $s \in S$. Now $ULKS = TKS$ where $T =$ tensor algebra, so $\mathcal{P}(\hat{U}LKS) = \prod_{n=1}^{\infty} L_r(KS)$, from which one deduces that

$$\operatorname{gr}\mathcal{G}(\hat{K}FS) \simeq \operatorname{gr}\mathcal{P}\hat{U}LKS \simeq L(KS)\,.$$

The CHA $K\langle\langle X_s\rangle\rangle_{s\in S}$ will be called the *free* CHA generated by the set S.

The functor gr is compatible with tensor products (1.17), so if A is a CHA, then $\operatorname{gr} A$ is a graded Hopf algebra. $\operatorname{gr} A$ is primitively generated because it is generated by $\operatorname{gr}_1 A$ which consists only of primitive elements for dimensional reasons. By Milnor-Moore there is a Hopf algebra isomorphism.

$$\text{(2.12)} \qquad U(\mathcal{P}\operatorname{gr} A) \xrightarrow{\sim} \operatorname{gr} A\,.$$

PROPOSITION 2.13. *If* A *is a* CHA, *then the Lie algebra* $\mathcal{P}\operatorname{gr} A$ *is generated by* $\operatorname{gr}_1 A$.

PROOF. The canonical map $UL(\operatorname{gr}_1 A) \xrightarrow{\sim} T(\operatorname{gr}_1 A) \to U(\mathcal{P}\operatorname{gr} A)$ is surjective. By the Poincaré-Birkhoff-Witt theorem, any Lie algebra $\mathfrak{g}$ is canonically a retract of $U(\mathfrak{g})$, so if φ is a map of Lie algebras and $U\varphi$ is surjective, then φ is surjective. Thus $L(\mathrm{gr}_1 A) \to \mathcal{P}\,\mathrm{gr}\,A$ is surjective, proving the proposition.

THEOREM 2.14. *The canonical map* $\mathrm{gr}\,\mathcal{P}A \to \mathcal{P}\,\mathrm{gr}\,A$ *is an isomorphism.*

PROOF. The map is injective since $\mathcal{P}A$ has the filtration induced from that of A. By (2.13) it suffices to show that $\mathcal{P}A \to \mathrm{gr}_1 A$ is surjective; in other words the theorem asserts that any CHA has many primitive and hence many group-like elements.

Let $S^2\bar{A} \subset \bar{A} \otimes \bar{A}$ be the symmetric tensors, i.e., the image of the projection operator $x \mathbin{\hat{\otimes}} y \mapsto 1/2(x \mathbin{\hat{\otimes}} y + y \mathbin{\hat{\otimes}} x)$. This projection operator preserves filtration, so if $S^2\bar{A}$ is given the induced filtration, $\mathrm{gr}\,S^2A = S^2(\mathrm{gr}\,A)$. The maps

$$\bar{A} \xrightarrow{\delta_1} S^2\bar{A} \xrightarrow{\delta_2} \bar{A}^{\otimes 3}$$
$$\delta_1 x = \Delta x - x \mathbin{\hat{\otimes}} 1 - 1 \mathbin{\hat{\otimes}} x$$
$$\delta_2(x \mathbin{\hat{\otimes}} y) = \delta_1 x \mathbin{\hat{\otimes}} y - x \mathbin{\hat{\otimes}} \delta_1 y$$

are compatible with filtrations, satisfy $\delta_2\delta_1 = 0$, and are carried by gr into maps

$$\mathrm{gr}\,\bar{A} \xrightarrow{\delta_1'} S^2(\mathrm{gr}\,A) \xrightarrow{\delta_2'} (\mathrm{gr}\,\bar{A})^{\otimes 3} \tag{2.15}$$

given by similar formulas.

LEMMA. *The sequence* (2.15) *is exact.*

PROOF. The maps δ_i' use only the coalgebra structure of $\mathrm{gr}\,A$ which by (2.12) and the Poincaré-Birkhoff-Witt theorem is coalgebra-isomorphic to $S(\mathcal{P}\,\mathrm{gr}\,A)$. Hence we may replace $\mathrm{gr}\,A$ by a commutative polynomial ring $Q = K[X_i]_{i \in I}$ whose coalgebra structure is given by the formula

$$\Delta X_\alpha = \textstyle\sum_{\beta+\gamma=\alpha} X_\beta \otimes X_\gamma ,$$

where we use standard multi-index notation with $X_\alpha = (\alpha!)^{-1}X^\alpha$. Suppose $z = \sum_{\alpha,\beta>0} a_{\alpha\beta} X_\alpha \otimes X_\beta \in S^2\bar{Q}$ and

$$\delta_2' z = \textstyle\sum_{\rho,\sigma,\tau>0} (a_{\rho+\sigma,\tau} - a_{\rho,\sigma+\tau}) X_\rho \otimes X_\sigma \otimes X_\tau = 0 .$$

Then $a_{\rho+\sigma,\tau} = a_{\rho,\sigma+\tau}$ if $\rho, \sigma, \tau > 0$, from which one sees that $a_{\alpha\beta} = a_{\alpha'\beta'}$ if $\alpha + \beta = \alpha' + \beta'$ and $|\alpha| + |\beta| \geqq 3$. Let $u = \sum_{|\gamma| \geqq 3} b_\gamma X_\gamma$ where $b_\gamma = a_{\alpha\beta}$ if $\alpha + \beta = \gamma$ and $\alpha, \beta > 0$. Then

$$z - \delta_1' u = \textstyle\sum_{i,j} a_{ij} X^i \otimes X^j = \delta_1'\Big[\frac{1}{2}\sum_{i,j} a_{ij} X^i X^j\Big] ,$$

since $a_{\alpha\beta} = a_{\beta\alpha}$. Thus $z \in \mathrm{Im}\,\delta_1'$, proving the lemma.

To prove the theorem, we must show that the map $\mathrm{gr}\,\mathcal{P}A \to \mathcal{P}\,\mathrm{gr}\,A$ is surjective. Given $u \in \mathcal{P}\,\mathrm{gr}_r\,A$ we construct by induction on n a sequence x_n in F_rA such that $\delta_1 x_n \in F_n S^2\bar{A}$. Start by choosing $x_r \in F_rA$ with $\rho_r x_r = u$. If x_n has been obtained, then $\delta_2'(\rho_n\delta_1 x_n) = \rho_n\delta_2\delta_1 x = 0$, so there is by the exactness of (2.15) an element $y \in F_nA$ with $\rho_n\delta_1 y = \delta_1'\rho_n y = \rho_n\delta_1 x_n$. We set $x_{n+1} = x_n - y$. As $x_{n+1} - x_n \in F_nA$ the sequence x_n is Cauchy and converges to an element $x \in F_rA$. Then $\delta_1 x = 0$ so $x \in \mathcal{P}A$ and $\rho_r x = u$, which finishes the proof of the theorem.

If W is a complete filtered vector space, let $\hat{S}^n W$ be the image of the symmetrization operator σ on $W \hat{\otimes} \cdots \hat{\otimes} W$, n times. σ preserves filtration and by (1.15) $\mathrm{gr}\,\hat{S}^n W \simeq S^n(\mathrm{gr}\,W)$. Define $\hat{S}W = \prod_0^\infty \hat{S}^n W$. If A is a complete Hopf algebra, define e: $\hat{S}\mathcal{P}A \to A$ by requiring e to be linear, continuous, and such that $e(\sigma(x_1 \hat{\otimes} \cdots \hat{\otimes} x_n)) = (n!)^{-1}\sum x_{\sigma_1} \cdots x_{\sigma_n}$, where σ runs over the symmetric group of degree n and the $x_i \in \mathcal{P}A$. Passing to gr's we get the composition $S(\mathrm{gr}\,\mathcal{P}A) \to S(\mathcal{P}\,\mathrm{gr}\,A) \to \mathrm{gr}\,A$ which is an isomorphism by (1.14), the Poincaré-Birkhoff-Witt theorem for $\mathcal{P}\,\mathrm{gr}\,A$, and (2.12). Thus by completeness we have the following Poincaré-Birkhoff-Witt theorem for CHA's.

COROLLARY 2.16. *e: $\hat{S}\mathcal{P}A \to A$ is an isomorphism of vector spaces. In particular $\mathcal{P}A$ is canonically a vector space retract of A.*

COROLLARY 2.17. *The following conditions are equivalent for a map $f: A \to A'$ of complete Hopf algebras.*

(i) $\mathrm{gr}_1 f$ *is surjective,*

(ii) $F_n\mathcal{P}f: F_n\mathcal{P}A \to F_n\mathcal{P}A'$ *is surjective for all n,*

(iii) $\mathcal{P}f$ *is surjective,*

(ii)′ $F_n\mathcal{G}f$ *is surjective for all n,*

(iii)′ $\mathcal{G}f$ *is surjective,*

(iv) *f induces an isomorphism* $A/\mathrm{Ker}\,f \xrightarrow{\sim} A'$

where $A/\mathrm{Ker}\,f$ is the complete Hopf algebra described in (2.2).

PROOF. (i) $\Leftrightarrow$ (iv) follows easily from (1.6); (ii) $\Leftrightarrow$ (ii)′ is because of the exponential isomorphism (2.7), and similarly for (iii) $\Leftrightarrow$ (iii)′; (ii) $\Rightarrow$ (iii) is trivial; (iii) $\Rightarrow$ (i) because $\mathrm{gr}_1 A = \mathcal{P}A/F_2\mathcal{P}A$ by the theorem; (i) $\Rightarrow$ (ii). By (2.13) and (2.14), $L(\mathrm{gr}_1 A) \to \mathcal{P}\,\mathrm{gr}\,A \simeq \mathrm{gr}\,\mathcal{P}A$ is surjective. Hence $\mathrm{gr}_1 f$ surjective $\Rightarrow$ $\mathrm{gr}\,\mathcal{P}f$ surjective $\Rightarrow$ $F_n\mathcal{P}f$ surjective for all n, q.e.d.

COROLLARY 2.18. *f is an isomorphism if and only if $\mathcal{P}f$ is.*

PROOF. If $\mathcal{P}f$ is an isomorphism, then it is injective and so also is $F_n\mathcal{P}f$. By (2.17) $F_n\mathcal{P}f$ is also surjective so $F_n\mathcal{P}f$ is an isomorphism for all n. Thus $\mathrm{gr}\,\mathcal{P}f = \mathcal{P}(\mathrm{gr}\,f)$ is an isomorphism so $\mathrm{gr}\,f$ is an isomorphism (2.12) and f is an isomorphism.

PROPOSITION 2.19. *The effective epimorphisms in* (CHA) *are the maps satisfying the equivalent conditions of* (2.17).

PROOF. Any map f factors into $A \xrightarrow{\pi} A/\mathrm{Ker}\, f \xrightarrow{g} A'$ where g is injective and hence a monomorphism in (CHA). If f is an effective epimorphism, then g must be an isomorphism so f satisfies (2.17 (iv)).

Conversely given f, let J be the closed ideal generated by $\mathrm{Ker}\, \mathcal{P}f$. J is a closed Hopf ideal and the map f factors into $A \xrightarrow{\pi} A/J \xrightarrow{g} A'$. π is an effective epimorphism, because a map $u: A \to B$ factors through π if and only if $u(\mathrm{Ker}\, \mathcal{P}f) = 0$ and because each element of $\mathcal{P}A$ is the image of X under a map $P \to A$, where $P = K\langle\langle X \rangle\rangle$, $\Delta X = X \hat{\otimes} 1 + 1 \hat{\otimes} X$. By (2.17) $\mathcal{P}\pi$ is surjective and by the definition of J, $\mathrm{Ker}\, \mathcal{P}f \subset \mathrm{Ker}\, \mathcal{P}\pi$. Thus $\mathrm{Ker}\, f = \mathrm{Ker}\, \mathcal{P}\pi$ and $\mathcal{P}g$ is injective. Now if f satisfies the conditions of (2.17) so that $\mathcal{P}f$ and hence $\mathcal{P}g$ is surjective, then $\mathcal{P}g$ is an isomorphism. Thus g is an isomorphism by (2.18) and so $f \simeq \pi$ is an effective epimorphism which finishes the proof of (2.19). We also have proved that $\mathrm{Ker}\, f = J =$ closed ideal generated by $\mathrm{Ker}\, \mathcal{P}f$ when f is surjective.

If we factor a general map f into $A \xrightarrow{\pi} A/\mathrm{Ker}\, f \xrightarrow{g} A'$, then by what we have just showed, $\mathrm{Ker}\, \pi =$ closed ideal generated by $\mathrm{Ker}\, \mathcal{P}\pi$. But g hence also $\mathcal{P}g$ is injective, so $\mathrm{Ker}\, f = \mathrm{Ker}\, \pi$ and $\mathrm{Ker}\, \mathcal{P}f = \mathrm{Ker}\, \mathcal{P}\pi$, and we obtain the following

PROPOSITION 2.20. *If f is any map of complete Hopf algebras, then* $\mathrm{Ker}\, f$ *is the closed ideal generated by* $\mathrm{Ker}\, \mathcal{P}f$.

COROLLARY 2.21. *f is injective if and only if $\mathcal{P}f$ is injective.*

PROPOSITION 2.22. *Any complete Hopf algebra A is isomorphic to the quotient of a free complete Hopf algebra P by a closed Hopf ideal. Moreover we may assume $\mathrm{gr}_1 P \xrightarrow{\sim} \mathrm{gr}_1 A$ in which case P is unique over A up to non-canonical isomorphism.*

PROOF. Choose a basis for $\mathrm{gr}_1 A$ and lift it to elements $x_i \in \mathcal{P}A$ $i \in I$ by (2.14). Then there is a unique map $u: P \to V$ with $u(X_i) = x_i$. $\mathrm{gr}_1 u$ is an isomorphism, so by 2.17, A is isomorphic to $A/\mathrm{Ker}\, u$. If $v: P' \to A$ is a surjective map, then by 2.17, $\mathcal{P}u$ is surjective, so lifting each x_i to an element of P' we obtain a map $w: P \to P'$ such that $vw = u$. If $\mathrm{gr}_1 v$ is an isomorphism and P' is free, then $\mathrm{gr}_1 w$ is an isomorphism so $\mathrm{gr}\, w = T(\mathrm{gr}_1 w)$ is an isomorphism and w is an isomorphism, q.e.d.

From similar arguments one proves

COROLLARY 2.23. *The following conditions are equivalent for a complete Hopf algebra A.*

(i) *A is a projective object of* (CHA),

(ii) $T(\mathrm{gr}_1 A) \xrightarrow{\sim} \mathrm{gr}\, A$,

(iii) $L(\mathrm{gr}_1 A) \xrightarrow{\sim} \mathrm{gr}\, \mathcal{P}A$,

(iv) *A is isomorphic to a complete Hopf algebra P.*

PROPOSITION 2.24. *The category* (CHA) *is closed under arbitrary limits and has $K\langle\langle X\rangle\rangle$, $\Delta X = X \otimes 1 + 1 \otimes X$ as a projective generator.*

The proof is similar to (1.10).

A3. The relation between complete Hopf algebras and Malcev groups

Throughout this section $K = \mathbf{Q}$, although those statements concerning only CHA's and Lie algebras are valid for an arbitrary field of characteristic zero.

Definition 3.1. By a *Malcev group* we mean a group G endowed with a filtration (1.11) $G = F_1G \supset F_2G \supset \cdots$ such that

(i) the associated graded Lie ring $\mathrm{gr}\, G$ is a uniquely divisible abelian group, hence $\mathrm{gr}\, G$ is a Lie algebra over $\mathbf{Q}$.

(ii) $\mathrm{gr}\, G$ is generated as a Lie algebra by $\mathrm{gr}_1 G$.

(iii) $G \simeq \text{lim-inv}_r\, (G/F_rG)$.

Similarly a *Malcev Lie algebra* is a Lie algebra $\mathfrak{g}$ with a filtration such that $\mathrm{gr}\, \mathfrak{g}$ is generated by $\mathrm{gr}_1\, \mathfrak{g}$ and such that $\mathfrak{g}$ is complete for the topology defined by the filtration. With the evident notion of morphisms, we obtain categories (MGp) and (MLA) respectively.

The category of Malcev groups is the full subcategory of Lazard's category of R-groups [17] consisting of those R-groups for which the closures of the terms of the lower central series form a basis for the neighborhoods of the identity (this follows from (3.5)).

Example 3.2. If A is a CHA, endow $\mathcal{G}A$ with the filtration induced by F_rA. Then there are Lie algebra isomorphisms $\mathrm{gr}\, \mathcal{G}A \simeq \mathrm{gr}\, \mathcal{P}A \simeq \mathcal{P}\, \mathrm{gr}\, A$ ((2.8), (2.14)) and $\mathcal{P}\, \mathrm{gr}\, A$ is generated by $\mathrm{gr}_1 A$ (2.13). Thus $\mathcal{G}A$ is a Malcev group. Similarly $\mathcal{P}A$ with the induced filtration is a Malcev Lie algebra.

THEOREM 3.3. *The functors*

$$(\mathrm{MGp}) \xleftarrow{\mathcal{G}} (\mathrm{CHA}) \xrightarrow{\mathcal{P}} (\mathrm{MLA})$$

are equivalences of categories.

The proof will occupy the rest of the section. It will be convenient on several occasions to refer to the following situation. Let H be a group endowed with a filtration, let $j\colon H' \to H$ be a group map, and consider the diagrams

$$
(3.4)\qquad
\begin{array}{ccccccccc}
1 & \longrightarrow & \Gamma_m H'/\Gamma_{m+1}H' & \longrightarrow & H'/\Gamma_{m+1}H' & \longrightarrow & H'/\Gamma_m H' & \longrightarrow & 1 \\
 & & \big\downarrow{\scriptstyle \mathrm{gr}_m j} & & \big\downarrow{\scriptstyle j_{m+1}} & & \big\downarrow{\scriptstyle j_m} & & \\
1 & \longrightarrow & F_m H/F_{m+1}H & \longrightarrow & H/F_{m+1}H & \longrightarrow & H/F_m H & \longrightarrow & 1
\end{array}
$$

induced by j depending on m. Typical diagram chasing arguments (e.g., serpent lemma) will then be applied. For example if $\mathrm{gr}\, j$ is surjective and j_m is an isomorphism for m large, it follows by descending induction on m that j_m is an isomorphism for all m.

PROPOSITION 3.5. *If G is a Malcev group, then*

$$(\Gamma_r G)\cdot F_s G = F_r G \qquad \textit{for } s \geqq r\ .$$

In particular if $F_s G = \{1\}$ for some s, then $\Gamma_r G = F_r G$ for all r.

PROOF. We apply the situation of diagram (3.4) where $H' = H = G/F_s G$, $j=$the identity, and where $F_r H = F_r G\cdot F_s G/F_s G$. Then $\mathrm{gr}\, j$ is surjective by (3.1)(ii) and $j = j_m$ is an isomorphism for $m > s$; thus j_m is an isomorphism for all m and the proposition is proved.

The key technical point in the proof of Theorem 3.3 is the following variant of Ado's theorem.

PROPOSITION 3.6. (a) *If G is a nilpotent group with no non-identity elements of finite order, then the canonical map $G \longrightarrow \mathbf{Q}G/\overline{\mathbf{Q}G}^n$ is injective for $n >$ the class of G, and conversely.*

(b) *If $\mathfrak{g}$ is a nilpotent Lie algebra, then the canonical map $\mathfrak{g} \longrightarrow U\mathfrak{g}/\overline{U\mathfrak{g}}^n$ is injective for $n>$ the class of G, and conversely.*

PROOF. The converse statements are trivial. (b): We may assume $\mathfrak{g}$ finitely generated, in which case the Lie algebra $\mathrm{gr}\,\mathfrak{g}$ associated to the lower central series filtration of $\mathfrak{g}$ is finite dimensional, and so $\mathfrak{g}$ is finite dimensional. By Ado's theorem $\mathfrak{g}$ has a faithful finite dimensional representation V whose composition quotients are trivial $\mathfrak{g}$ modules. Let F be a flag in V stable under $\mathfrak{g}$, and let R be the augmented algebra of endomorphisms of V which preserve F and which induce the same scalar on each of the quotients of F. Then the $\mathfrak{g}$ action on V defines an augmented algebra map $U\mathfrak{g}/\overline{U\mathfrak{g}}^n \longrightarrow R$ where $n = \dim V$; as $\mathfrak{g}$ acts faithfully on V, the map $\mathfrak{g} \longrightarrow U\mathfrak{g}/\overline{U\mathfrak{g}}^n$ is injective. It remains to show this holds for all $n>$ class of $\mathfrak{g}$. Let $\mathrm{gr}'\,\mathfrak{g}$ be the Lie algebra associated to the filtration $F_r\mathfrak{g} = \mathfrak{g} \cap \overline{U\mathfrak{g}}^r$ so that we have maps $\mathrm{gr}\,\mathfrak{g} \longrightarrow \mathrm{gr}'\mathfrak{g} \hookrightarrow \mathcal{P}\,\mathrm{gr}\, U\mathfrak{g}$. Now we have seen (2.13) that $\mathcal{P}\,\mathrm{gr}\, U(\mathfrak{g})$ is generated as a Lie algebra by $\mathrm{gr}_1\,\mathfrak{g}$, hence these maps are surjective. To say $\mathfrak{g}$ has class r means that $\mathrm{gr}_q\,\mathfrak{g} = 0$ for $q > r$; hence $\mathrm{gr}'_q\,\mathfrak{g} = 0$ for $q > r$. As we have shown that $F_n\,\mathfrak{g} = 0$ for n sufficiently large, it follows that $F_n\,\mathfrak{g} = 0$ for $n > r$, proving (b).

(a): By the same arguments used for (b), one reduces to the case where G is finitely generated and to proving that G has a faithful finite dimensional representation V with trivial composition quotients. The only difference is that now $\mathrm{gr}_q G \otimes_{\mathbf{Z}} \mathbf{Q} \to \mathrm{gr}'_q G \otimes_{\mathbf{Z}} \mathbf{Q}$ is surjective, whence for $q >$ class of G, one may show by descending induction on q that $F_q G$ is a torsion group and hence the identity subgroup by the hypotheses on G. To construct V we proceed as in Ado's theorem. Since G is finitely generated and nilpotent, there is a chain $\{G_k\}$ of normal subgroups in G such that the associated quotients are cyclic with trivial G action; we prove the existence of V by induction on the length of this chain. Thus there is an exact sequence

$$1 \longrightarrow G_1 \longrightarrow G \xrightarrow{\pi} C \longrightarrow 1$$

where C is cyclic and where the induction hypothesis applies to G_1, so that the canonical map $\rho\colon G_1 \to \mathbf{Q}G_1/\overline{\mathbf{Q}G}_1^n$ is injective for some n. Write R for the target of ρ, let x be an element of G such that πx generates C, and let θ denote both the automorphism $y \mapsto xyx^{-1}$ of G_1 and its obvious extension to R. The images of the subgroups G_k in $\mathrm{gr}_1 \mathbf{Q}G_1$ generate a chain of subspaces on which G acts trivially; combining this with the $\overline{\mathbf{Q}G}_1$-adic filtration, one sees that R has a flag F stable under the left multiplication representation λ of G_1 and the conjugation action of G such that the associated quotients have trivial action. Consequently both θ and $\lambda(G_1)$ are contained in the group T of endomorphisms of R leaving F stable and inducing the identity on the quotients of F.

If C is infinite cyclic, then we may define an action φ of G on R by the formula $\varphi(yx^i) = \lambda(y)\theta^i$ if $y \in G_1$. It is readily checked that φ is well-defined; as $\varphi(G) \subset T$, R becomes a representation of G faithful on G_1 with trivial composition quotients. Taking the direct sum of R and a faithful representation of C with trivial composition quotients, we obtain the desired V. On the other hand if C is cyclic of order $k > 0$, let u be the unique element of $1 + \bar{R}$ such that $u^k = \rho(x^k)$, and define φ by $\varphi(yx^i) = \rho(y)u^i$ if $y \in G_1$. It is readily verified that $\varphi\colon G \to 1 + \bar{R}$ is a well-defined function; to show that it is a homomorphism one needs the formula $\theta(a) = uau^{-1}$ for $a \in R$. However $a \to \theta a$ and $a \mapsto uau^{-1}$ are elements of T with the same k^{th} power, and as T is uniquely divisible, they coincide. Composing φ with left multiplication we get a faithful action of G on R and R is the desired V. This completes the proof of (a) and the proposition.

COROLLARY 3.7. *Let G be a nilpotent group. The following conditions are equivalent.*

(i) *G is uniquely divisible ($x \mapsto x^n$ is bijective for $n \neq 0$).*

(ii) $\operatorname{gr} G$ *is a Lie algebra over* $\mathbf{Q}$.

(iii) G *is a Malcev group with* $F_r G = \Gamma_r G$.

(iv) $G \xrightarrow{\sim} \mathcal{G}\hat{\mathbf{Q}}G$.

Moreover if these conditions hold,

$$\Gamma_r G = \{x \in G \mid x - 1 \in \overline{\mathbf{Q}G^r}\} \simeq F_r \mathcal{G}\hat{\mathbf{Q}}G$$

and $\operatorname{gr} G \simeq \operatorname{gr} \mathcal{G}\hat{\mathbf{Q}}G \simeq \mathcal{P} \operatorname{gr} \mathbf{Q}G$.

PROOF. (iv) $\Rightarrow$ (i) and (ii) $\Leftrightarrow$ (iii) are trivial.

Assume G uniquely divisible, let $\hat{G} = \mathcal{G}\hat{\mathbf{Q}}G$ with its canonical filtration (3.2), and let $j: G \to \hat{G}$ be the canonical map, and consider the diagram (3.4). $\operatorname{gr} j: \operatorname{gr} G \to \operatorname{gr} \hat{G}$ is surjective because the latter is generated by $\operatorname{gr}_1 \hat{G}$ (2.13) and because G uniquely divisible implies $\operatorname{gr}_1 G \to (\operatorname{gr}_1 G) \otimes \mathbf{Q} \simeq \operatorname{gr}_1 \hat{G}$ is surjective. By induction $j_m: G/\Gamma_m G \to \hat{G}/F_m \hat{G}$ is surjective for all m. For $m >$ the class of G, $\operatorname{gr}_m \hat{G} = 0$, so $F_m \hat{G} = 0$ and $j: G \to \hat{G}$ is surjective. By (3.6)(a) $j^{-1} F_m \hat{G} = 0$, so $j: G \to \hat{G}$ is an isomorphism, proving (iv). Now by descending induction in the diagram $\operatorname{gr}_m j$ and j_m are isomorphisms, proving the "moreover" assertion of the corollary and the implication (iv) $\Rightarrow$ (iii). Finally suppose (ii) holds. Then G has no non-identity elements of finite order so $j: G \to \hat{G}$ is injective by (3.6)(a), and $\operatorname{gr}_1 G$ is divisible so $\operatorname{gr} G \to \operatorname{gr} \hat{G}$ is surjective. We have just seen how these two facts imply (iv), so the corollary is proved.

COROLLARY 3.8. *If G is a nilpotent group, let $\hat{G} = \mathcal{G}\hat{\mathbf{Q}}G$ and let $j: G \to \hat{G}$ be the canonical map. Then*

(1) *j is universal for maps of G into nilpotent uniquely divisible groups.*

(2) *j is characterized up to canonical isomorphism by the following properties*

(a) *$\hat{G}$ is nilpotent and uniquely divisible.*

(b) $\operatorname{Ker} j =$ *the torsion subgroup of* G.

(c) $x \in \hat{G} \Rightarrow x^n \in \operatorname{Im} j$ *for some* $n \neq 0$.

PROOF. (1) is immediate from (3.7). For (2) suppose j has properties (a), (b), (c) and that $k: G \to H$ is another map with these properties. Then by (1) there is a map $\hat{G} \to H$ which one easily sees has properties (b) and (c) and therefore is an isomorphism by the unique divisibility of $\hat{G}$ and H. Hence k is isomorphic to j. It remains to show that j has properties (b) and (c). For (b) let G' be the quotient of G by its subgroup of elements of finite order (it is a subgroup since G is nilpotent). Then by (1) $\hat{G} \xrightarrow{\sim} \hat{G}'$, while by (3.6)(a) $G' \hookrightarrow \hat{G}'$, hence $\operatorname{Ker} j =$ torsion subgroup of G. For (c) we show by induction on m, that $j_m: G/\Gamma_m G \to \hat{G}/F_m \hat{G}$ has property (c) using the diagram (3.4).

Assume j_m has property (c) and let $x \in \hat{G}/F_{m+1}\hat{G}$, so that there is a

$y \in G/\Gamma_{m+1}G$ and $u \in \operatorname{gr}_m G$ with $u \cdot j_{m+1}(y) = x^k$ for some $k \neq 0$. Since $(\operatorname{gr} G) \otimes \mathbf{Q} \longrightarrow \operatorname{gr} \hat{G}$ is surjective, there is a $v \in \operatorname{gr}_m \hat{G}$ with $j_{m+1}(v) = u^p$ for some $p \neq 0$. As u is in the center of $\hat{G}/F_{m+1}\hat{G}$, we have that $x^{kp} = j_{m+1}(vy^p)$, showing that j_{m+1} has property (c). Thus by induction, j has property (c) and the proof of the corollary is complete.

COROLLARY 3.9. *If* $\mathfrak{g}$ *is a nilpotent Lie algebra, then* $\mathfrak{g} \xrightarrow{\sim} \mathcal{P}\hat{U}\mathfrak{g}$. *Moreover* $\Gamma_r\mathfrak{g} = \mathfrak{g} \cap \overline{U\mathfrak{g}}^r \simeq F_r\mathcal{P}\hat{U}\mathfrak{g}$ *and* $\operatorname{gr} \mathfrak{g} \simeq \operatorname{gr} \mathcal{P}\hat{U}\mathfrak{g} \simeq \mathcal{P} \operatorname{gr} U\mathfrak{g}$.

The proof is similar to that of (3.7) but easier.

Remarks. 3.10. $j\colon G \to \hat{G} = \mathcal{G}\hat{\mathbf{Q}}G$ is by (3.8) the *Malcev completion* of G in the sense of [18], [17].

3.11. The second assertion of (3.9) is valid even if $\mathfrak{g}$ is not nilpotent. For a discussion of what happens in the group case and in particular a proof of the isomorphism $(\operatorname{gr} G) \otimes \mathbf{Q} \xrightarrow{\sim} \mathcal{P} \operatorname{gr} \mathbf{Q}G$ in general see [23].

PROOF OF (3.3). Let us call a Malcev group G (resp. Malcev Lie algebra $\mathfrak{g}$, resp. complete Hopf algebra A) *nilpotent* if F_rG (resp. $F_r\mathfrak{g}$, resp. $F_r\mathcal{P}A$) is zero for some r. It follows from (3.7) and (3.9) that the categories of nilpotent Malcev groups (nMGp) and Lie algebras (nMLA) are isomorphic to the categories of nilpotent uniquely divisible groups and nilpotent Lie algebras respectively. Moreover the functors

$$(n\mathrm{MGp}) \xleftarrow{\mathcal{G}} (n\mathrm{CHA}) \xrightarrow{\mathcal{P}} (n\mathrm{MLA}) \tag{3.12}$$

are equivalences of categories, the quasi-inverse functors being $\hat{\mathbf{Q}}$ and $\hat{U}$ respectively. Indeed $G \xrightarrow{\sim} \mathcal{G}\hat{\mathbf{Q}}G$ by (3.7) and to show that $\hat{\mathbf{Q}}\mathcal{G}A \xrightarrow{\sim} A$ it suffices by (2.7) and (2.18) to show that $\mathcal{G}\hat{\mathbf{Q}}\mathcal{G}A \xrightarrow{\sim} \mathcal{G}A$; but this follows from (3.7), since the composition $\mathcal{G}A \to \mathcal{G}\hat{\mathbf{Q}}\mathcal{G}A \to \mathcal{G}A$ is the identity. The case of Lie algebras is similar. Finally the fact that (3.12) are equivalences implies Theorem 3.3, because a Malcev group G (resp. CHA A) may be identified with the inverse system $\{G/F_rG\}$ (resp. $A/A \cdot F_r\mathcal{P}A$) in ($n$MGp) (resp. in ($n$CHA), q.e.d.

APPENDIX B. DG LIE ALGEBRAS AND COALGEBRAS

In this section we give an exposition of the results on DG Lie algebras and coalgebras that are used in the rest of the paper, in particular the functors $\mathcal{L}$ and $\mathcal{C}$. Although the results are presumably well known, we have included proofs (in outline at least) because existing treatments do not directly apply (e.g., in the basic reference [20] only Lie algebras with faithful representation are considered), and because several technical lemmas required for the proofs are needed elsewhere in the paper.

1. Notation

We work over a field K of characteristic zero fixed once and for all. Except in the last section DG objects may be infinite in both directions. The differential is always of degree -1. DG algebras are always associative with unit (a Lie algebra is not an algebra), and DG coalgebras are coassociative with counit and unless otherwise stated *cocommutative*. Here cocommutative means $T\Delta = \Delta$, where $T: V \otimes W \xrightarrow{\sim} W \otimes V$ is the isomorphism $T(v \otimes w) = (-1)^{pq} w \otimes v$, if $p = \deg v$ and $q = \deg w$. When defining maps we shall give formulas involving elements with ambiguous signs which have to be filled in by the standard sign rule. The upper sign is always the one if all elements are of even degree, e.g., $[x, y] = xy \mp yx$.

The *r-fold suspension* $\Sigma^r V (r \in \mathbf{Z})$ of a DG vector space V is defined to be $\Sigma^r \otimes V$, where Σ^r is the DG vector space with $(\Sigma^r)_q = 0$ if $q \neq r$ and $(\Sigma^r)_r =$ the one dimensional vector space over K with basis element e_r. We write $\Sigma^r x$ instead of $e_r \otimes x$ so that $d\Sigma^r x = (-1)^r \Sigma^r dx$. A *map of degree r* from V to W is a map $\Sigma^r V \to W$ and may be identified with a collection $f = \{f_q : V_q \to W_{q+r}\}$ such that $df = (-1)^r f d$.

A *weak equivalence* is a map inducing isomorphisms on homology.

2. The homology of certain functors

If V is a DG vector space, let $T(V)$, $S(V)$, and $L(V)$ be the tensor algebra of, symmetric algebra of, and free Lie algebra generated by V respectively. The functors T, S, and L are left adjoint to the underlying DG vector space functor to (DG) from the categories of DG algebras, DG commutative algebras, DG Lie algebras, respectively, and so there are natural (DG) maps $V \to T(V)$, etc. These give rise to maps $H(V) \to (H(T(V))$, etc., and, as the homology of a DG algebra, etc. is a graded algebra, etc. to natural graded algebra maps $T(H(V)) \to H(T(V))$, etc.

If L is a DG Lie algebra, let $U(L)$ be its universal enveloping algebra. U is the left adjoint of the underlying Lie algebra functor from DG algebras to DG Lie algebras. $U(L)$ is a DG cocommutative Hopf algebra and there is a natural map $U(H(L)) \to H(U(L))$ of graded Hopf algebras.

PROPOSITION 2.1. *If V is a* DG *vector space then the natural maps*
$T(H(V)) \to H(T(V))$ *of graded algebras*
$S(H(V)) \to H(S(V))$ *of graded commutative algebras*
$L(H(V)) \to H(L(V))$ *of graded Lie algebras*
are isomorphisms. If L is a DG *Lie algebra, then the natural map*

$$U(H(L)) \longrightarrow H(U(L))$$

of graded cocommutative Hopf algebras is an isomorphism.

PROOF. The assertion for $T(V)$ follows from the Künneth theorem. We note that $S(V) = \bigoplus_n S_n(V)$ where $S_n(V)$ is the quotient of $V^{\otimes n}$ by the action of the symmetric group $\Sigma(n)$, where $\Sigma(n)$ permutes the factors of $V^{\otimes n}$. As the characteristic of K is zero, the symmetrization operator $(n!)^{-1} \sum \sigma$, $\sigma \in \Sigma(n)$ is defined on $V^{\otimes n}$ and defines a section of the map $V^{\otimes n} \to S_n(V)$, allowing one to identify $S_n(V)$ with the image of the symmetrization operator. As homology is compatible with direct sums, and the Künneth isomorphism is compatible with the interchange map, one sees that both $S_n(H(V))$ and $H(S_n(V))$ are the quotients of $H(V^{\otimes n}) \simeq (H(V))^{\otimes n}$ by $\Sigma(n)$. Hence $S_n(H(V)) \xrightarrow{\sim} H(S_n(V))$ and the assertion for $S(V)$ is proved.

The universal enveloping algebra of $L(V)$ is clearly $T(V)$. Assume for the moment the following

LEMMA 2.2. *The map* $\rho\colon T(V) \to L(V)$ *given by*

$$\rho(x_1 \otimes \cdots \otimes x_n) = \begin{cases} \dfrac{1}{n}[x_1, [x_2, \cdots [x_{n-1}, x_n] \cdots]] & n > 0 \\ 0 & n = 0 \end{cases}$$

is a left inverse for the map $L(V) \to T(V)$. *In particular* $L(V) \to T(V)$ *is injective.*

Regarding $L(V)$ as a sub-DG Lie algebra of $T(V)$ by the lemma, we see that ρ is a projection onto $L(V)$. But the formula for ρ is preserved by the Künneth isomorphism, hence $L(H(V))$ and $H(L(V))$ are both the images of ρ on $T(H(V))$, so the assertion of the proposition for $L(V)$ is proved.

To finish the proof of the proposition we need another fact.

THEOREM 2.3 (Poincaré-Birkhoff-Witt). *Let L be a* DG *Lie algebra and let* $i\colon L \to U(L)$ *be the natural map. Let*

$$e\colon S(L) \longrightarrow U(L)$$

be given by $e(x_1 \cdots x_n) = 1/n! \sum_{\sigma \in \Sigma(n)} \pm i(x_{\sigma 1}) \cdots i(x_{\sigma n})$. *Then e is an isomorphism of* DG *coalgebras.*

It is clear that the following square is commutative

$$\begin{array}{ccc} S(H(L)) & \xrightarrow[\sim]{e} & U(H(L)) \\ \downarrow & & \downarrow \\ H(S(L)) & \xrightarrow[\sim]{H(e)} & H(U(L)) \end{array}$$

and so the assertion for the functor U follows from the assertion for S.

Proposition 1 is therefore proved except for the lemmas.

PROOF OF LEMMA 2.2. $L(V) = \bigoplus_{r=1}^{\infty} L_r(V)$ where $L_r(V)$ is spanned by r^{th} order brackets of elements of V. Hence $L_1(V) = V$ and $[L_r(V), L_s(V)] \subseteq L_{r+s}(V)$. Consequently the endomorphism of $L(V)$ given by $Dx = nx$ if $x \in L_n(V)$ is a derivation, and we can form the semi-direct product $L(V) \oplus KD$ with bracket

$$[x + aD, y + bD] = [x, y] + aDy - bDx \qquad \text{if } x, y \in L(V), a, b \in K .$$

Then $L(V) \oplus KD$ is a $L(V)$ module and hence a $U(L(V)) = T(V)$ module. The map $T(V) \to L(V) \oplus KD$ induced by $u \mapsto uD$ is given by

$$x_1 \otimes \cdots \otimes x_n \longmapsto [x_1, \cdots [x_n, D]] = [x_1, \cdots [x_{n-1}, x_n] \cdots]$$

if $n > 0$ and $x_j \in V$, whereas $z \longmapsto Dz = nz$ if $z \in L_n(V)$. Hence $\rho(z) = z$ if $z \in L(V)$ and the lemma is proved.

The proof of the PBW theorem will be given in the next section.

3. Connected DG coalgebras and the proof of the Poincare-Birkhoff-Witt theorem

Let C be a DG coalgebra with comultiplication $\Delta: C \to C \otimes C$ and counit $\varepsilon: C \to K$. C will be called *connected* if there is an element $1_C \in C$ such that $\Delta 1_C = 1_C \otimes 1_C$, $\varepsilon(1_C) = 1_K$ the unit of K and if $C = \bigcup_{r=0}^{\infty} F_r C$, where $F_r C$ is the filtration of C defined recursively by the formulas

$$F_0 C = K 1_C$$
$$F_r C = \{x \in C \mid \Delta x - x \otimes 1_C - 1_C \otimes x \in F_{r-1} C \otimes F_{r-1} C\} .$$

This definition of connected coalgebras differs from that in [20], however if C is connected in the sense of [20], that is $C_0 = K$ and $C_r = 0$ for all $r < 0$ or all $r > 0$, then C is connected in our sense. Let $\mathcal{P}(C) = \{x \in C \mid \Delta x = 1_C \otimes x + x \otimes 1_C\}$ be the DG subspace of *primitive elements* of C, so that $F_1 C = K \cdot 1_C \oplus \mathcal{P}(C)$.

PROPOSITION 3.1. *If C' is a sub* DG *coalgebra of a connected* DG *coalgebra C, then C' is connected, $F_r C' = F_r C \cap C'$ and $\mathcal{P}(C') = \mathcal{P}(C) \cap C'$. A quotient of a connected* DG *coalgebra is connected and the tensor product of connected* DG *coalgebras is connected.*

PROPOSITION 3.2. *If $\theta: C \to C'$ is a map of* DG *coalgebras, and if C is connected, then θ is injective if and only if θ restricted to $\mathcal{P}(C)$ is injective.*

PROOF. We first show that $1_C \in C'$. Since $x \otimes 1_K = (\mathrm{id} \otimes \varepsilon)\Delta x$ for all $x \in C'$, $\varepsilon: C' \to K$ is surjective and there is an $x \in C$ with $\varepsilon(x) = 1_K$. As C is connected, there is an r such that $x \in F_r C$, and we may assume x chosen so that r is minimal. If $r > 0$,

$$x \otimes 1_K = (\mathrm{id} \otimes \varepsilon)\Delta x = x \otimes 1_K + 1_C \otimes 1_K + \sum x'_i \otimes \varepsilon(x''_i) \qquad \text{in } C' \otimes K\,,$$

where $x'_i, x''_i \in F_{r-1}C'$. Hence $\bar{x} = -\sum \varepsilon(x''_i)x'_i \in C' \cap F_{r-1}C$ has $\varepsilon(\bar{x}) = 1$, and hence by minimality of r, $r = 0$ and so $x = 1_C$ and $1_C \in C'$. A straightforward induction shows that $F_r C' = C' \cap F_r C$ where we take $1_{C'} = 1_C$ hence $\bigcup F_r C' = C'$ and C' is connected. This proves the first assertion of Proposition 3.1 and the other assertions are trivial.

Proposition 3.2 is proved by inductively showing that θ is injective on $F_r C$ hence also on all of C since C is connected.

Remark. It follows from the first assertion of Proposition 3.1 that the element 1_C is uniquely characterized by the formulas $\Delta 1_C = 1_C \otimes 1_C$ and $\varepsilon 1_C = 1_K$. We shall abbreviate 1_C by 1 from now on.

Examples 3.3. Let Δ: $T(V) \to T(V) \otimes T(V)$ be given by

$$\Delta(v_1 \otimes \cdots \otimes v_n) = \bigoplus_{r=0}^{n} (v_1 \otimes \cdots \otimes v_r) \otimes (v_{r+1} \otimes \cdots \otimes v_n) \in \bigoplus_{r=0}^{n} V^{\otimes r} \otimes V^{\otimes(n-r)}$$

where the empty tensor product is to be interpreted as $1 \in V^{\otimes 0} = K$. Let ε: $T(V) \to K$ be the projection onto $V^{\otimes 0}$. Then Δ and ε define a *non-commutative* coalgebra structure on $T(V)$. (Warning: this is *not* the coalgebra structure obtained by regarding $T(V)$ as the universal enveloping algebra of $L(V)$.) It is easily shown that $F_r T(V) = \bigoplus_{n \leqq r} V^{\otimes n}$ whence $T(V)$ is connected.

Let Δ: $S(V) \to S(V) \otimes S(V)$ be the algebra map given by $\Delta v = v \otimes 1 + 1 \otimes v$ and let ε: $S(V) \to K$ be the projection onto $S_0 V = K$. Then Δ and ε define a commutative coalgebra structure on $S(V)$, and a straightforward calculation using shuffle permutations shows that the map N: $S(V) \to T(V)$ given by

$$N(v_1 \cdots v_n) = \sum_{\sigma \in \Sigma(n)} \pm\, v_{\sigma 1} \otimes v_{\sigma 2} \otimes \cdots \otimes v_{\sigma n} \tag{3.4}$$

is an injective map of DG coalgebras. From Proposition 3.1 we conclude that $F_r S(V) = \bigoplus_{n \leqq r} S_n(V)$ whence $S(V)$ is connected. In particular $\mathcal{P}S(V) = V$.

PROOF OF THE PBW THEOREM. e is clearly a DG map hence we may ignore differentials. A calculation with shuffle permutations shows that e is a map of graded coalgebras. Furthermore e is surjective because if we define a filtration on $U(L)$ by $F_r U(L) =$ the subspace of $U(L)$ spanned by products $i(x_1) \cdots i(x_n)$ with $x_i \in L$, $n \leqq r$, then by induction on r, we have $F_r U(L) = e\,(\bigoplus_{n \leqq r} S_n(L))$. It remains to show that e is injective. By Proposition 3.2, it suffices to show that e restricted to $\mathcal{P}S(L) = L$ is injective, or equivalently, to show that the canonical map $i: L \to U(L)$ is injective. By Lemma 2.2 this is true if L is free, that is, of the form $L(V)$ for some graded vector space V, consequently e is an isomorphism if L is free.

Given an arbitrary graded Lie algebra L we construct a diagram of graded Lie algebras

$$\begin{tikzcd} L_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & L_0 \arrow[l, bend right, "s_0"'] \arrow[r, "\rho"] & L \end{tikzcd}$$

where $d_0 s_0 = d_1 s_0 = \mathrm{id}$, $\rho d_0 = \rho d_1$, and L_0, L_1 are free, which is exact in the sense that ρ is a cokernel of the pair d_0, d_1 in the category of graded Lie algebras. This may be done by choosing a surjection $\rho: L_0 \to L$ where L_0 is free, and then factoring $L_0 \xrightarrow{\Delta} L_0 \times_L L_0$ into $L_0 \xrightarrow{s_0} L_1 \xrightarrow{(d_0, d_1)} L_0 \times_L L_0$, where L_1 is obtained by adding more generators to L_0 so that (d_0, d_1) is surjective. Consider the commutative diagram

$$\begin{tikzcd} S(L_1) \arrow[r, shift left, "S(d_0)"] \arrow[r, shift right, "S(d_1)"'] \arrow[d, "e", "\wr"'] & S(L_0) \arrow[r, "S(\rho)"] \arrow[d, "e", "\wr"'] & S(L) \arrow[d, "e"] \\ U(L_1) \arrow[r, shift left, "U(d_0)"] \arrow[r, shift right] & U(L_0) \arrow[r, "U(\rho)"] & U(L) \end{tikzcd}$$

As S is a left adjoint functor, $S(\rho)$ is a cokernel for $S(d_0)$, $S(d_1)$ in the category of commutative graded algebras. Furthermore $S(d_0)$ is surjective, because of $S(s_0)$, hence $S(d_0)\operatorname{Ker} S(d_1)$ is an ideal in $S(L_0)$. It is easily seen that the natural map $S(L_0) \to S(L_0)/S(d_0)\operatorname{Ker} S(d_1)$ is also a cokernel for $S(d_0)$, $S(d_1)$, hence $S(L_0)/S(d_0)\operatorname{Ker} S(d_1) \simeq S(L)$, and so the top row of the above diagram is exact in the category of graded vector spaces. Similar arguments show the same for the bottom row. As e is an isomorphism for L_0 and L_1 since they are free, the five lemma shows that e is an isomorphism for L, q.e.d.

COROLLARY 3.5. *$U(L)$ is connected as a coalgebra and $L \xrightarrow{\sim} \mathcal{P}U(L)$.*

COROLLARY 3.6. *There is a canonical map of* DG *vector spaces $r: U(L) \to L$ which is left inverse to the inclusion $i: L \to U(L)$ and which is functorial as L varies over the category of* DG *Lie algebras.*

Remarks 3.7. This map r is the composition $U(L) \simeq S(L) \xrightarrow{j} L$, where j is the projection onto the tensors of degree 1. If L is free, r is *not* the same as the map ρ of 2.2.

3.8. A curious consequence of the above proof is that $e\colon S(L) \xrightarrow{\sim} U(L)$ for any DG Lie algebra L over a *ring* K containing $\mathbf{Q}$. In effect the reduction to the case where L is free did not use that K is a field, and the free case follows by base extension from $\mathbf{Q}$. Consequently all examples of a Lie algebra $\mathfrak{g}$ over a ring such that $\mathfrak{g} \to U(\mathfrak{g})$ is not injective must occur in characteristic p.

3.9. In the case of ordinary finite dimensional Lie algebras over R, the map $e\colon S(L) \to U(L)$ has the following agreeable interpretation. If G is a Lie group with Lie algebra L, then composition with the exponential map $\exp\colon L \to G$ yields a map from the ring of formal functions on G at the identity to the ring of formal functions on L at 0, that is, a homomorphism $(\exp)^*\colon U(L)^* \to S(L)^*$ where * denotes dual. $(\exp)^*$ is just the transpose of e.

4. A universal coalgebra property of $T(V)$ and $S(V)$ and the theorem of Cartier, Milnor, and Moore

Let $N\colon S(V) \to T(V)$ be the DG coalgebra map 3.4. Let $j\colon T(V) \to V$ and $j\colon S(V) \to V$ denote the projection onto the tensors of degree 1.

PROPOSITION 4.1. *If C is a connected* DG *coalgebra, then the map $\theta \mapsto j\theta$ is a bijection from the set of* DG *coalgebra maps $\theta\colon C \to T(V)$ to the set of* DG *vector space maps $u\colon C \to V$ such that $u(1) = 0$.*

If C is a connected co-commutative DG *coalgebra, then the same is true for* DG *coalgebra maps $C \to S(V)$.*

PROOF. Let $\Delta^{(n)}\colon C \to C^{\otimes n}$ be the composition $C \to C^{\otimes 2} \to \cdots \to C^{\otimes n}$ where the map $C^{\otimes r} \to C^{\otimes(r+1)}$ is any map of the form $(\mathrm{id})^{\otimes p} \otimes \Delta \otimes (\mathrm{id})^{\otimes(r-p-1)}$. Since C is coassociative this composition is independent of any of these choices, and we have the formula

$$(\Delta^{(p)} \otimes \Delta^{(q)})\Delta = \Delta^{(p+q+1)} . \tag{4.2}$$

In particular if $\Delta x = x \otimes 1 + 1 \otimes x + \sum x'_j \otimes x''_j$,

$$\Delta^{(r)}x = (\Delta^{(r-1)} \otimes \mathrm{id})\Delta x = \Delta^{(r-1)}x \otimes 1 + 1^{\otimes r} \otimes x + \textstyle\sum_j \Delta^{(r-1)}x'_j \otimes x''_j ,$$

and so by induction on r we conclude that if $x \in F_rC$, then $\Delta^{(r)}x$ is a linear combination of terms of the form $x_0 \otimes \cdots \otimes x_r$ where $x_j = 1$ for some j. If $u\colon C \to V$ is a DG map with $u(1) = 0$ then $u^{\otimes n}\Delta^{(n-1)}x = 0$ if $x \in F_{n-1}C$, hence since $C = \bigcup F_rC$, the map $\theta\colon C \to T(V)$ given by

$$\theta x = \textstyle\sum_{n=0}^{\infty} u^{\otimes n}\Delta^{(n-1)}x \tag{4.3}$$

is well-defined. It is clear that θ is a DG map and a computation using (4.2) shows that $(\theta \otimes \theta)\Delta = \Delta\theta$. Hence θ is a DG coalgebra map such that $j\theta = u$. It is not hard to show that (4.3) holds for any DG coalgebra map $\theta\colon C \to T(V)$, where $u = j\theta$, and so the first statement of Proposition 4.1 is proved.

If C is co-commutative, then the image of θ is contained in the symmetric tensors in $T(V)$, which is the image of $N: S(V) \to T(V)$. As N is injective θ factors uniquely $\theta = N\bar{\theta}$ where $\bar{\theta}: C \to S(V)$ is a DG coalgebra map, and the second statement of Proposition 4.1 follows from the first, q.e.d.

Let M be a DG comodule under the cocommutative DG coalgebra C, and let $\Delta_M: M \to M \otimes C$, $\Delta_C: C \to C \otimes C$ be the comodule structure and coalgebra structure maps of M and C respectively. By a *coderivation* from M to C, we mean a DG map $\delta: M \to C$ such that $\Delta_C\delta = (1 + T)(\delta \otimes 1)\Delta_M$ where T is the interchange map (see § 1). A *degree r coderivation* from M to C is a degree r map $\delta: M \to C$ of DG vector spaces such that the honest DG map $\Sigma^r M \to C$ associated to δ is a coderivation from $\Sigma^r M$ to C. If we form the semi-direct product coalgebra $M \oplus C$ with comultiplication

$$\begin{aligned}\Delta_{M\oplus C}&(m \oplus c)\\ &= 0 \oplus \Delta_M m \oplus T\Delta_M m \oplus \Delta_C c \in (M \otimes M) \oplus (M \otimes C) \oplus (C \otimes M) \oplus (C \otimes C)\\ &= (M \oplus C) \otimes (M \oplus C)\ ,\end{aligned}$$

then a coderivation δ from M to C may be identified with a DC coalgebra map $M \oplus C \xrightarrow{\theta} C$ such that $\theta i = \mathrm{id}_C$ where $i: C \to M \oplus C$ is given by $i(c) = 0 \oplus c$. As $M \oplus C$ is connected if C is, we obtain from Proposition 4.1 the following

COROLLARY 4.4. *If M is a* DG *comodule under the* DG *coalgebra $S(V)$, then there is a one-to-one correspondence between degree r coderivations $\delta: M \to S(V)$ and degree r maps $v: M \to V$ of* DG *vector spaces given by $v = j\delta$.*

We say that a DG Hopf algebra U is co-commutative or connected if as a coalgebra U is co-commutative or connected. If U is a DG Hopf algebra, then $\mathcal{P}U$ is a sub-DG Lie algebra of the underlying DG Lie algebra of the algebra structure of U.

THEOREM 4.5. *The functor $L \mapsto U(L)$ is an equivalence between the category of* DG *Lie algebras and the category of* DG *co-commutative connected Hopf algebras, the quasi-inverse functor being $U \mapsto \mathcal{P}U$.*

PROOF. By the corollary to the PBW theorem we have that $\mathcal{P}U(L) \simeq L$ so it remains to show that $U\mathcal{P}(U) \xrightarrow{\sim} U$, if U is cocommutative and connected. We may ignore the differentials. By 3.2 and 3.5, the natural map $U\mathcal{P}(U) \to U$ is injective and hence there is a graded vector space map $\alpha: U \to \mathcal{P}(U)$ such that the composition

$$S(\mathcal{P}(U)) \xrightarrow{\sim} U\mathcal{P}(U) \hookrightarrow U \xrightarrow{\alpha} \mathcal{P}(U)$$

is the map $j: S(\mathcal{P}(U)) \to \mathcal{P}(U)$. By Proposition 4.1 there is unique graded coalgebra map $U \xrightarrow{\theta} S(\mathcal{P}(U))$ such that $j\theta = \alpha$. θ is injective by Proposition

3.2 and the composition

$$S(\mathcal{P}(U)) \xrightarrow{\sim} U\mathcal{P}(U) \hookrightarrow U \overset{\theta}{\hookrightarrow} S(\mathcal{P}(U))$$

is the identity so $U\mathcal{P}(U) \xrightarrow{\sim} U$, q.e.d.

5. Principal DG coalgebra bundles and twisting functions

We retain the notation and conventions of the preceding sections with the exception that from now on all DG coalgebras will be assumed to be cocommutative and connected. By virtue of 3.1, the operations that we perform will not lead us out of this category. In particular a DG Hopf algebra will be of the form $U(L)$ by the Cartier-Milnor-Moore theorem.

By a (right) *action* of a DG Lie algebra L on a DG coalgebra E we mean a right $U(L)$ module structure on E such that the module structure map $m: E \otimes U(L) \to E$ is a map of DG coalgebras. By a *principal L bundle with base C* we mean a triple (E, m, π) where m is an action of L on E and $\pi: E \to C$ is a map of DG coalgebras such that $\pi(e\cdot u) = \pi(e)\cdot\varepsilon(u)$ satisfying the following "local triviality" condition: there exists a graded coalgebra map $\rho: C \to E$ with $\pi\rho = \mathrm{id}_C$, which is not necessarily compatible with the differentials of C and E, such that the map $\varphi: C \otimes U(L) \to E$ given by $\varphi(c \otimes u) = \rho(c)\cdot u$ is an isomorphism of graded coalgebras, and right $U(L)$ modules. Such a map ρ will be called a *local cross-section.*

Example. Let m denote the natural L action on the DG coalgebra $C \otimes U(L)$, and let π be given by $\pi(c \otimes u) = c\cdot\varepsilon(u)$. Then $(C \otimes U(L), m, \pi)$ is a principal $U(L)$ bundle with base C and any other isomorphic to this one is said to be *trivial*. It is clear that a principal bundle (E, m, π) is trivial if and only if there exists a local cross section $\rho: C \to E$ such that $d_E\rho = \rho d_C$.

A *twisting function* from a DG coalgebra C to a DG Lie algebra L is a linear map $\tau: C \to L$ of degree -1 such that

$$\tau(1) = 0$$

(5.1)
$$d_L\tau + \tau d_C + \frac{1}{2}[\ ,\ ]\circ(\tau \otimes \tau)\circ\Delta = 0\ .$$

This last equation may be written

(5.2)
$$d_L\tau c + \tau d_C c + \frac{1}{2}\sum_i (-1)^{\deg c_i'}[\tau c_i', \tau c_i''] = 0$$

if $\Delta c = \sum_i c_i' \otimes c_i''$. The following proposition determines the structure of principal bundles in terms of twisting functions.

PROPOSITION 5.3. *Let C be a DG coalgebra and let L be a DG Lie algebra.*

(1) *If* (E, m, π) *is a principal* L *bundle with base* C *and* $\rho: C \to E$ *is a local cross-section, then there is a unique twisting function* $\tau: C \to L$ *such that the differential of* E *is given by*

$$d_E(\rho c \cdot u) = (d_E \rho c) \cdot u + (-1)^{\deg c} \rho c \cdot d_{U(L)} u \tag{5.4}$$

$$d_E \rho c = \rho(d_C c) + \textstyle\sum_i (-1)^{\deg c'_i} \rho c'_i \cdot \tau c''_i \ . \tag{5.5}$$

(2) *The mapping* $(E, m, \pi, \rho) \mapsto \tau$ *defined by* (1) *yields a bijection from the set of isomorphism classes of principal* L *bundles with base* C *and given local cross section to the set of twisting functions from* C *to* L.

PROOF OF (1). Let $\pi': E \to U(L)$ be given by $\pi'(\rho c \cdot u) = \varepsilon(c) u$ and recall that $\pi: E \to C$ is given by $\pi(\rho c \cdot u) = c \cdot \varepsilon(u)$. Then

$$\mathrm{id}_E = m(\pi \otimes \pi') \Delta_E \ . \tag{5.6}$$

If $D: C \to C$ is coderivation of arbitrary degree of the coalgebra E, i.e.,

$$\Delta_E D = (D \otimes 1 + 1 \otimes D) \Delta_E \ , \tag{5.7}$$

then by combining (5.6) and (5.7) we have

$$D = m(\pi D \otimes \pi' + \pi \otimes \pi' D) \Delta_E \ . \tag{5.8}$$

Setting $\tau = \pi' d_E \rho: C \to U(L)$, and taking $D = d_E$ in (5.8) we obtain the formula

$$\begin{aligned} d_E \rho &= m(\pi d_E \otimes \pi' + \pi \otimes \pi' D)(\rho \otimes \rho) \Delta_C \\ &= \rho d_C + m(\rho \otimes \tau) \Delta_C \ , \end{aligned}$$

which is the same as (5.5). If $c \in C$ and $\Delta_C c = \sum c'_i \otimes c''_i$, then we have the formulas

$$\begin{aligned} \Delta_{U(L)} \tau c &= (\pi' \otimes \pi')(d_E \otimes 1 + 1 \otimes d_E)(\rho \otimes \rho) \Delta_C c = \tau c \otimes 1 + 1 \otimes \tau c \\ \pi' d_E^2 \rho c &= \pi' d_E \big(\rho d_C c + \textstyle\sum (-1)^{\deg c'_i} \rho c'_i \cdot \tau c''_i\big) \\ &= \tau d_C c + \textstyle\sum (-1)^{\deg c'_i} \tau c'_i \cdot \tau c''_i + d_{U(L)} \tau c \ . \end{aligned} \tag{5.9}$$

The first formula shows that $\operatorname{Im} \tau \subset L$. By virtue of $d_E^2 = 0$ and the cocommutativity of C, the second shows that τ is a twisting function. We note that (5.4) follows from the fact that E in a DG $U(L)$ module. Finally τ is unique, since (5.5) implies that $\tau = \pi' d_E \rho$; hence the proof of (1) is complete.

PROOF OF (2). The injectivity of the map $(E, m, \pi, \rho) \mapsto \tau$ is clear, since up to isomorphism we may assume that E is the coalgebra $C \otimes U(L)$, m is the natural $U(L)$ module structure on E, π is given by $\pi(c \otimes u) = c \cdot \varepsilon(u)$, and ρ is given by $\rho(c) = c \otimes 1$. Then the only thing needed to determine the isomorphism class of the principal bundle with local cross section is the differential of E, which is determined by τ *via* (5.4) and (5.5). It therefore suffices to show that for any twisting function τ, the endomorphism of $C \otimes U(L)$

given by (5.4) and (5.5) completes $C \otimes U(L)$, m, π, and ρ and to a principal bundle. In other words we must show

(i) d_E is a degree -1 coderivation of E,

(ii) $d_E^2 = 0$,

(iii) d_E is compatible with the $U(L)$ module structure on E, and

(iv) $\pi d_E = d_C \pi$.

(iii) and (iv) are easy; assuming (i) we shall prove (ii). First note that $d_E^2 = \frac{1}{2}[d_E, d_E]$ is a degree -2 coderivation of E, so by (5.8) it is determined by πd_E^2 and $\pi' d_E^2$. However $\pi d_E^2 = d_C^2 \pi = 0$ by (iv). Since the proof (5.9) uses only (i) and (iii), we see that (5.9) holds, and $\pi' d_E^2 \rho = 0$ because τ is a twisting function. But by (iii) $d_E^2(\rho c \cdot u) = d_E^2 \rho c \cdot u \pm \rho c \cdot d_U^2 u = d_E^2 \rho c \cdot u$ and hence $\pi' d_E^2 = 0$. Thus $d_E^2 = 0$ and so (ii) is proved.

It remains to show (i). But the following formulas may be verified rather easily from (5.4).

$$\begin{aligned}(d_E \otimes 1 + 1 \otimes d_E)\Delta_E(\rho c \cdot u) &= (d_E \otimes 1 + 1 \otimes d_E)\Delta_E \rho c \cdot \Delta_U u \\ &\quad + (-1)^{\deg c}\Delta_E \rho c \cdot (d_U \otimes 1 + 1 \otimes d_U)\Delta_U u \\ \Delta_E d_E(\rho c \cdot u) &= \Delta_E d_E \rho c \cdot \Delta_U u + (-1)^{\deg c}\Delta_E \rho c \cdot \Delta_U d_U u\ ,\end{aligned}$$

where $\cdot$ is also used to denote the natural action of $U(L) \otimes U(L)$ on $E \otimes E$. As the last terms of these expressions are equal since d_U is a coderivation of $U(L)$, it suffices in order to show that d_E is a coderivation, to show that $(d_E \otimes 1 + 1 \otimes d_E)\Delta_E \rho c = \Delta_E d_E \rho c$. With patience the following formulas may be deduced from (5.5).

$$\begin{aligned}&(d_E \otimes 1 + 1 \otimes d_E)\Delta_E \rho \\ &\quad = [(\rho \otimes \rho)(d_C \otimes 1 + 1 \otimes d_C)\Delta_C + (m \otimes \rho)(\rho \otimes \tau \otimes 1)(\Delta_C \otimes 1)\Delta_C \\ &\qquad + (\rho \otimes m)(1 \otimes \rho \otimes \tau)(1 \otimes \Delta_C)\Delta_C] \\ &\Delta_E d_E \rho \\ &\quad = [\Delta_E \rho d_C + (m \otimes \rho)(\rho \otimes \tau \otimes 1)(1 \otimes T)(\Delta_C \otimes 1)\Delta_C \\ &\qquad + (\rho \otimes m)(1 \otimes \rho \otimes \tau)(\Delta_C \otimes 1)\Delta_C]\ ,\end{aligned}$$

where $T\colon C \otimes C \to C \otimes C$ is the interchange map. As Δ_C is cocommutative and co-associative, and as d_C is a coderivation of C, we see that these expressions are equal. Consequently d_E is a coderivation, (2) is proved, and the proof of Proposition 5.3 is complete.

6. Universal twisting functions

Let $\mathcal{T}(C, L)$ be the set of twisting functions from the DG coalgebra C to the DG Lie algebra L. $\mathcal{T}(C, L)$ is a bifunctor covariant in L and contravariant in C. If $\tau \in \mathcal{T}(C, L)$, we let $E(C, L, \tau)$ denote the principal L bundle with base C and local cross-section ρ (unique up to isomorphism by 5.3) whose differential is given by (5.4) and (5.5).

A DG coalgebra C will be called *acyclic* if the augmentation $C \to K$ is a homology isomorphism.

PROPOSITION 6.1. *If C is a* DG *coalgebra, then the functor $L \to \mathcal{T}(C, L)$ is represented by a universal twisting function $\tau_C: C \to \mathfrak{L}(C)$. Furthermore $E(C, \mathfrak{L}(C), \tau_C)$ is acyclic.*

PROPOSITION 6.2. *If L is a* DG *Lie algebra, then the functor $C \mapsto \mathcal{T}(C, L)$ is represented by a universal twisting function $\tau_L: \mathcal{C}(L) \to L$. Furthermore $E(\mathcal{C}(L), L, \tau_L)$ is acyclic.*

PROOF. Let $\bar{C} = \operatorname{Ker}\{\varepsilon: C \to K\}$ and let $\Omega = \Sigma^{-1}$ (see § 1), so that $\Omega\bar{C}$ is the (-1)-fold suspension of the DG vector space $\bar{C}$. Let $L(\Omega\bar{C})$ be the free Lie algebra generated by $\Omega\bar{C}$ and let $\tau_C: C \to L(\Omega\bar{C})$ be given by $\tau_C x = \Omega\pi x$, where $\pi: C \to \bar{C}$ is the natural projection. Finally let $\mathfrak{L}(C)$ be the DG Lie algebra which as a graded Lie algebra is $L(\Omega\bar{C})$, but whose differential is given by

$$d_{\mathfrak{L}(C)}\tau_C x = -\tau_C d_C x - \frac{1}{2}\sum_i (-1)^{\deg x_i'}[\tau x_i', \tau x_i''] \,. \tag{6.3}$$

This formula gives $d_{\mathfrak{L}(C)}$ on $\Omega\bar{C}$; it may then be extended uniquely to all of $L(\Omega\bar{C})$ as a degree -1 derivation. Assuming $d^2_{\mathfrak{L}(C)} = 0$ for the moment, we shall show that $\tau_C: C \to \mathfrak{L}(C)$ is a universal twisting function with source C. First of all τ_C is a twisting function by (6.3). If $\tau: C \to L$ is an arbitrary twisting function, then as $\tau(1) = 0$ and $L(\Omega\bar{C})$ is a free Lie algebra, there is a unique homomorphism $\theta: L(\Omega\bar{C}) \to L$ such that $\theta\tau_C = \tau$. Now $\theta d_{\mathcal{C}(L)}$ and $d_L\theta$ are degree -1 derivations of $L(\Omega\bar{C})$ with values in L considered as an $L(\Omega\bar{C})$ module *via* θ; as τ_C and τ are twisting functions $\theta d_{\mathcal{C}(L)} = d_L\theta$ on $\Omega\bar{C}$, hence identically. Thus $\theta: \mathfrak{L}(C) \to L$ is a map of DG Lie algebras such that $\theta\tau_C = \tau$; as θ is determined by τ, we see that τ_C has the desired universal property.

The universal enveloping algebra of $L(\Omega\bar{C})$ is $T(\Omega\bar{C})$, and the extension of $d_{\mathfrak{L}(C)}$ to $T(\Omega\bar{C})$ is the degree -1 derivation given by the formula

$$d_{U(\mathfrak{L}(C))}\tau_C x = -\tau_C d_C x - \sum_i (-1)^{\deg x_i'}\tau x_i' \cdot \tau x_i'' \,. \tag{6.4}$$

by virtue of the cocommutativity of C. Consequently $U(\mathfrak{L}(C))$ is the cobar construction [1] of the DG coalgebra C. Hence $d^2_{U(\mathfrak{L}(C))} = 0$, by coassociativity of C and so $d^2_{\mathfrak{L}(C)} = 0$ as claimed above. Furthermore $E(C, \mathfrak{L}(C), \tau_C)$ is the coalgebra $C \otimes T(\Omega\bar{C})$ with differential given by 5.4, 5.5, and 6.4. Thus $E(C, \mathfrak{L}(C), \tau_C)$ is the "total space" coalgebra for the cobar construction and is acyclic. In fact a contracting homotopy s is given by

$$s(\rho x) = 0$$
$$s(\rho x \cdot \tau_C x_1 \cdots \tau_C x_q) = (-1)^{\deg x} \varepsilon(x) \pi x_1 \cdot \tau_C x_2 \cdots \tau_C x_q \qquad q \geqq 1 .$$

This concludes the proof of Proposition 6.1.

Let $\Sigma L \sharp L$ be the DG Lie algebra constructed from the DG Lie algebra L in the following way. As a graded vector space $\Sigma L \sharp L = \Sigma L \oplus L$, where the elements of ΣL are written Σx, and the elements of L are written θx if x is an element of L. The bracket and differential of L are given by the formulas

$$\begin{aligned} &[\Sigma x, \Sigma y] = 0 && d\Sigma x = \theta x - \Sigma dx \\ &[\Sigma x, \theta y] = \Sigma[x, y] && d\theta x = \theta dx \\ &[\theta x, \theta y] = \theta[x, y] && \end{aligned} \tag{6.5}$$

$\Sigma L \sharp L$ has homology zero, since if h is given by $h\theta x = \Sigma x$, $h\Sigma x = 0$, then $dh + hd = \mathrm{id}$. By Proposition 2.1, $U(\Sigma L \sharp L)$ is acyclic.

Let θ: $U(L) \to U(\Sigma L \sharp L)$ be the Hopf algebra extension of the injection of L into $\Sigma L \sharp L$ given by $x \mapsto \theta x$. Then θ is a DG Hopf algebra map and the right $U(L)$ module structure on $U(\Sigma L \sharp L)$ determined by θ is an action of L on $U(\Sigma L \sharp L)$. Let $\mathcal{C}(L) = U(\Sigma L \sharp L) \otimes_{U(L)} K$ be the "orbit" DG coalgebra of this action, and let π: $U(\Sigma L \sharp L) \to \mathcal{C}(L)$ be the natural surjection. By the PBW theorem we have a coalgebra isomorphism $S(\Sigma L) \otimes U(L) \xrightarrow{\sim} U(\Sigma L \sharp L)$ given by $c \otimes u \mapsto ic \cdot u$, where i: $S(\Sigma L) \to U(\Sigma L \sharp L)$ is the Hopf algebra map which extends the inclusion $\Sigma L \hookrightarrow \Sigma L \sharp L$. (Note that i is not compatible with differentials.) Consequently πi: $S(\Sigma L) \to \mathcal{C}(L)$ is a graded coalgebra isomorphism and the coalgebra map

$$\rho\colon i(\pi i)^{-1}\colon \mathcal{C}(L) \longrightarrow U(\Sigma L \sharp L)$$

is a local cross-section for the action. Therefore $U(\Sigma L \sharp L)$ with this L action, π, and ρ is a principal L bundle with base $\mathcal{C}(L)$. Proposition 5.3 shows that the differential of $U(\Sigma L \sharp L)$ may be calculated by 5.4 and 5.5 in terms of a twisting function $\tau_L\colon \mathcal{C}(L) \to L$ which we shall now determine.

Let π': $U(\Sigma L \sharp L) \to U(L)$ be the Hopf algebra map given by $\Sigma x \mapsto 0$, $\theta x \mapsto x$. Then 5.5 implies $\tau_L = \pi' d_E \rho$, so

$$\begin{aligned} \tau_L(\Sigma x_1 \cdots \Sigma x_q) &= \pi' \textstyle\sum_{j=1}^{q} \pm \Sigma x_1 \cdots (\theta x_j - \Sigma d x_j) \cdots \Sigma x_q \\ &= \begin{cases} 0 & q > 1 \\ x_1 & q = 1 \end{cases} \end{aligned}$$

Consequently τ_L: $\mathcal{C}(L) \to L$ is the composition

$$\mathcal{C}(L) \xrightarrow{(\pi i)^{-1}} S(\Sigma L) \xrightarrow{j} \Sigma L \xrightarrow{\Omega} L .$$

We can now show that τ_L has the desired universal property. In order to simplify the notation a bit we shall identify the coalgebras $S(\Sigma L)$ and $\mathcal{C}(L)$ *via* the map πi in what follows. In particular $\tau_L = \Omega j$ by the preceding calculation. Let $\tau: C \to L$ be an arbitrary twisting function. By 4.1, there is a unique graded coalgebra map $\theta: C \to S(\Sigma L)$ such that $\theta\tau = \tau_L$. Now θd_C and $d_{\mathcal{C}(L)}\theta$ are two degree -1 coderivations to $\mathcal{C}(L)$ from C, which is an $S(\Sigma L)$ comodule *via* θ. But $j\theta d_C = jd_{\mathcal{C}(L)}\theta$, since τ and τ_L are twisting functions, hence $\theta d_C = d_{\mathcal{C}(L)}\theta$, by 4.4, so θ is a DG coalgebra map from $\mathcal{C}$ to $\mathcal{C}(L)$ such that $\theta\tau = \tau_L$. As θ is determined by τ, this proves that τ_L is a universal twisting function with target L. Finally $E(\mathcal{C}(L), L, \tau_L) = U(\Sigma L \# L)$ is acyclic, and so the proof of Proposition 6.2 is achieved.

Remarks 6.6. $\mathcal{L}(C)$ is the DG Lie algebra of primitive elements in the cobar construction of C.

6.7. If $\mathfrak{g}$ is an ordinary Lie algebra over K, and L is the differential graded Lie algebra which is $\mathfrak{g}$ in dimension zero and zero in other dimensions, then $\mathcal{C}(L)_q = \wedge^q \mathfrak{g}$ and the differential on $\mathcal{C}(L)$ is the standard one for computing Lie algebra homology. This may be seen by noting that in the case at hand 6.5 is the well-known formulas $[i(x), i(y)] = [d, \theta(x)] = 0$, $[d, i(x)] = \theta(x)$, $[\theta x, i(y)] = i([x, y])$, etc. Therefore the functor $\mathcal{C}$ is the natural generalization to DG Lie algebras of the standard complex for calculating the homology of a Lie algebra [15].

7. Application of the comparison theorems for spectral sequences

In this section we shall only consider DG objects which are zero in negative dimensions. Recall that a DG coalgebra C (resp. DG Lie algebra L) is r-reduced if $\bar{C}_q = 0$ (resp. $L_q = 0$) for $q < r$, that reduced = 1-reduced, and that $(\text{DGC})_r$ (resp. $(\text{DGL})_h$) are the categories of r-reduced DG coalgebras (resp. Lie algebras).

PROPOSITION 7.1. *Given maps of* DG *coalgebras*

$$C^f \xrightarrow{i} C \xrightarrow{\pi} C^b \tag{7.2}$$

such that

(a) *C is "locally" the product of C^b and C^f in the sense that there exists a coalgebra map $\varphi: C \to C^f$ such that $\varphi i = \mathrm{id}$ and such that $(\pi \otimes \varphi)\Delta: C \to C^b \otimes C^f$ is a coalgebra isomorphism,*

(b) *C, C^f are reduced and C^b is 2-reduced. Then there is a coalgebra spectral sequence*

$$E^2_{pq} = H_p(C^b) \otimes H_q(C^f) \Longrightarrow H_{p+q}(C)$$

independent of the choice of φ and functorial in the diagram (7.2).

PROOF. Let $F_pC^b = \bigoplus_{n \leq p} C_p^b$ and let F_pC be the inverse image F_pC^b under π of F_pC^b in the DG coalgebra sense, i.e., the cotensor product of F_pC^b and C over C^b. We calculate the spectral sequence associated to this filtration of C. i induces an isomorphism $\theta: C^f \to F_0C$ of DG coalgebras. Then $((\mathrm{gr}\,\pi) \otimes \theta^{-1})\Delta$ is a canonical map

$$E^0 = \mathrm{gr}\, C \xrightarrow{\sim} C^b \otimes C^f \tag{7.3}$$

of coalgebras, which is an isomorphism by (a). To calculate the differentials we use the isomorphism $(\pi \otimes \varphi)\Delta$ of (a) to identify C with $C^b \otimes C^f$ as coalgebras, in which case we have

$$\{(\pi \otimes \varphi)\Delta_C\}(x \otimes y) = x \otimes y \qquad \text{if } x \in C^b,\ y \in C^f\ .$$

Denoting the differentials of C^b, C^f, C by d^b, d^f, d respectively, using this formula and the fact that d is a coderivation for Δ_C one calculates the formula

$$d(x \otimes y) = d^b x \otimes y + \sum_i (-1)^{\deg x'_i} x'_i \otimes \varphi d\,(x''_i \otimes y)\ ,$$

where $\Delta x = \sum_i x'_i \otimes x''_i$. If $\deg x'_i < \deg x$, then since $C_1^b = 0$, $\deg x'_i \leqq \deg x - 2$. Consequently if $\deg x = p$

$$d(x \otimes y) - d^b x \otimes y - (-1)^p x \otimes d^f y \in F_{p-2}C\ ,$$

from which one calculates that $E^1_{pq} \simeq C_p^b \otimes H_q(C^f)$ and $E^2_{pq} \simeq H_p(C^b) \otimes H_q(C^f)$. These isomorphisms are induced by (7.3) which was independent of φ, and so the proposition is proved.

COROLLARY 7.4. *Let L be reduced and C 2-reduced, and let (E, m, π) be a principal L bundle with base C. Then there is a coalgebra spectral sequence*

$$E^2_{pq} = H_pC \otimes H_qUL \Longrightarrow H_{p+q}E\ .$$

PROOF. Apply the proposition to

$$U(L) \xrightarrow{i} E \xrightarrow{\pi} C$$

where $i(u) = 1_E \cdot u$. If ρ is a local cross-section, then by means of the coalgebra isomorphism $C \otimes U(L) \to E$ given by $c \otimes u \mapsto \rho c \cdot u$, one may define the map $\varphi: E \to U(L)$ needed for a) by

$$\varphi(\rho c \cdot u) = \varepsilon(c)u\ . \qquad \text{q.e.d.}$$

THEOREM 7.5. *The adjoint functors*

$$(\mathrm{DGL})_1 \underset{\mathcal{C}}{\overset{\mathfrak{L}}{\rightleftarrows}} (\mathrm{DGC})_2$$

carry weak equivalences into weak equivalences. Moreover the adjunction *maps* $\alpha\colon \mathcal{L}\mathcal{C}L \to L$ *and* $\beta\colon C \to \mathcal{C}\mathcal{L}C$ *are always weak equivalences.*

PROOF. Let $f\colon L \to L'$ be a weak equivalence and consider the map of spectral sequences

$$\begin{array}{ccc} H_p\mathcal{C}L \otimes H_q UL & \Longrightarrow & H_{p+q}E(\mathcal{C}L, L, \tau_L) \\ \downarrow & & \downarrow \\ H_p\mathcal{C}L' \otimes H_q UL' & \Longrightarrow & H_{p+q}E(\mathcal{C}L', L', \tau_{L'}) \end{array}$$

By 2.1 and 6.2 the map is an isomorphism on the "fiber" and "total space" so by the comparison theorems [30] for spectral sequences $H_*\mathcal{C}f$ is an isomorphism. Similarly by considering the map of spectral sequences induced by the map of principal bundles $E(\mathcal{C}L, \mathcal{L}\mathcal{C}L, \tau_{\mathcal{C}L}) \to E(\mathcal{C}L, L, \tau_L)$ coming from $\alpha\colon \mathcal{L}\mathcal{C}L \to L$, one sees that α is a weak equivalence. The other assertions of the theorem are proved the same way.

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