# OVERCATEGORIES AND UNDERCATEGORIES OF MODEL CATEGORIES

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If  $\mathcal{M}$  is a model category and Z is an object of  $\mathcal{M}$ , then there are model category structures on the categories  $(\mathcal{M} \downarrow Z)$  (the category of objects of  $\mathcal{M}$  over Z) and  $(Z \downarrow \mathcal{M})$  (the category of objects of  $\mathcal{M}$  under Z) under which a map is a cofibration, fibration, or weak equivalence if and only if its image in  $\mathcal{M}$  under the forgetful functor is, respectively, a cofibration, fibration, or weak equivalence. It is asserted without proof in [1] that if  $\mathcal{M}$  is cofibrantly generated, cellular, or proper, then so is the overcategory  $(\mathcal{M} \downarrow Z)$ . The purpose of this note is to fill in the proofs of those assertions (see Theorem 1.7) and to state and prove the analogous results for undercategories (see Theorem 2.8).

### 1. Overcategories

**Definition 1.1.** If  $\mathcal{M}$  is a category and Z is an object of  $\mathcal{M}$ , then the category  $(\mathcal{M} \downarrow Z)$  of *objects of*  $\mathcal{M}$  *over* Z is the category in which

- an object is a map  $X \to Z$  in  $\mathcal{M}$ ,
- a map from  $X \to Z$  to  $Y \to Z$  is a map  $X \to Y$  in  $\mathcal{M}$  such that the triangle



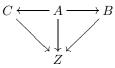
commutes, and

• composition of maps is defined by composition of maps in M.

**Definition 1.2.** If  $\mathcal{M}$  is a category and Z is an object of  $\mathcal{M}$ , then the *forgetful* functor G:  $(\mathcal{M} \downarrow Z) \to \mathcal{M}$  is the functor that takes the object  $A \to Z$  of  $(\mathcal{M} \downarrow Z)$  to the object A of  $\mathcal{M}$  and the map  $A \xrightarrow{A} B Z = B Z$  of  $(\mathcal{M} \downarrow Z)$  to the map  $A \to B$  of  $\mathcal{M}$ .

**Lemma 1.3.** Let  $\mathcal{M}$  be a cocomplete and complete category and let Z be an object of  $\mathcal{M}$ .

(1) The pushout in  $(\mathcal{M} \downarrow Z)$  of the diagram



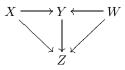
is  $P \to Z$  where P is the pushout in  $\mathfrak{M}$  of the diagram

$$C \longleftarrow A \longrightarrow B$$

and the structure map  $P \to Z$  is the natural map from the pushout in  $\mathcal{M}$ .

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(2) The pullback in  $(\mathcal{M} \downarrow Z)$  of the diagram



is  $P \to Z$  where P is the pullback in  $\mathcal{M}$  of the diagram

$$X \longrightarrow Y \longleftarrow W$$

and the structure map  $P \to Z$  is the composition  $P \to Y \to Z$ .

*Proof.* The described constructions possess the universal mapping properties that characterize the pushout (or pullback) in  $(\mathcal{M} \downarrow Z)$ .

**Lemma 1.4.** Let  $\mathcal{M}$  be a model category and let Z be an object of  $\mathcal{M}$ . If S is a set of maps in  $\mathcal{M}$  and  $S_Z$  is the set of maps in  $(\mathcal{M} \downarrow Z)$  of the form



in which the map  $A \to B$  is an element of S, then a map  $X \xrightarrow{X} Y$  in  $(\mathfrak{M} \downarrow Z)$  is a relative  $S_Z$ -cell complex (see [1, Definition 10.5.8]) if and only if the map  $X \to Y$  in  $\mathfrak{M}$  is a relative S-cell complex.

*Proof.* This follows from Lemma 1.3.

**Theorem 1.5.** Let  $\mathcal{M}$  be a cofibrantly generated model category (see [1, Definition 11.1.2]) with generating cofibrations I and generating trivial cofibration J, and let Z be an object of  $\mathcal{M}$ . If

(1)  $I_Z$  is the set of maps in  $(\mathcal{M} \downarrow Z)$  of the form

in which the map  $A \to B$  is an element of I and

(2)  $J_Z$  is the set of maps in  $(\mathfrak{M} \downarrow Z)$  of the form (1.6) in which the map  $A \to B$  is an element of J,

then the standard model category structure on  $(\mathfrak{M} \downarrow Z)$  (in which a map  $\overset{X \longrightarrow Y}{\underset{Z}{\xrightarrow{}}}$  is

a cofibration, fibration, or weak equivalence in  $(\mathfrak{M} \downarrow Z)$  if and only if the map  $X \to Y$  is, respectively, a cofibration, fibration, or weak equivalence in  $\mathfrak{M}$ ) is cofibrantly generated, with generating cofibrations  $I_Z$  and generating trivial cofibrations  $J_Z$ .

*Proof.* We will show that the set  $I_Z$  permits the small object argument and that a map is a trivial fibration if and only if it has the right lifting property with respect to  $I_Z$ ; the proof of the analogous statement for  $J_Z$  is similar.

Lemma 1.3 implies that the forgetful functor  $G: (\mathcal{M} \downarrow Z) \to \mathcal{M}$  (see Definition 1.2) takes a relative  $I_Z$ -cell complex in  $(\mathcal{M} \downarrow Z)$  to a relative *I*-cell complex in  $\mathcal{M}$ , and so the set  $I_Z$  permits the small object argument.

 $\mathbf{2}$ 

**Theorem 1.7.** Let  $\mathcal{M}$  be a model category and let Z be an object of  $\mathcal{M}$ .

- (1) If  $\mathcal{M}$  is cofibrantly generated, then so is  $(\mathcal{M} \downarrow Z)$ .
- (2) If  $\mathcal{M}$  is cellular, then so is  $(\mathcal{M} \downarrow Z)$ .
- (3) If  $\mathcal{M}$  is left proper, right proper, or proper, then so is  $(\mathcal{M} \downarrow Z)$ .

*Proof.* Part 1 follows from Theorem 1.5, part 2 follows from Theorem 1.5 and Lemma 1.4, and part 3 follows from Lemma 1.3.  $\Box$ 

#### 2. Undercategories

**Definition 2.1.** If  $\mathcal{M}$  is a category and Z is an object of  $\mathcal{M}$ , then the category  $(Z \downarrow \mathcal{M})$  of *objects of*  $\mathcal{M}$  *under* Z is the category in which

- an object is a map  $Z \to X$  in  $\mathcal{M}$ ,
- a map from  $Z \to X$  to  $Z \to Y$  is a map  $X \to Y$  in  $\mathcal{M}$  such that the triangle



commutes, and

• composition of maps is defined by composition of maps in M.

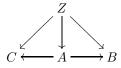
**Proposition 2.2.** If  $\mathfrak{M}$  is a cocomplete category and Z is an object of  $\mathfrak{M}$ , then the forgetful functor U:  $(Z \downarrow \mathfrak{M}) \to \mathfrak{M}$  that takes the object  $Z \to Y$  to Y is right adjoint to the functor F:  $\mathfrak{M} \to (Z \downarrow \mathfrak{M})$  that takes the object X of  $\mathfrak{M}$  to  $Z \to Z \amalg X$ (where that structure map is the natural injection into the coproduct).

*Proof.* If X is an object of  $\mathfrak{M}$  and  $Z \to Y$  is an object of  $(Z \downarrow \mathfrak{M})$ , then the universal mapping property of the coproduct implies that a map  $\overset{Z}{\underset{Z \amalg X}{\longrightarrow} Y}$  in  $(Z \downarrow \mathfrak{M})$  is entirely determined by the choice of a map  $X \to Y$  in  $\mathfrak{M}$ .

**Lemma 2.3.** Let  $\mathcal{M}$  be a cocomplete and complete category and let Z be an object of  $\mathcal{M}$ .

(1) The pushout in  $(Z \downarrow \mathcal{M})$  of the diagram

4

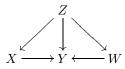


is  $Z \to P$  where P is the pushout in  $\mathcal{M}$  of the diagram

$$C \longleftarrow A \longrightarrow B$$

and the structure map  $Z \to P$  is the composition  $Z \to A \to P$ .

(2) The pullback in  $(Z \downarrow \mathcal{M})$  of the diagram



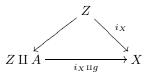
is  $Z \to P$  where P is the pullback in  $\mathcal{M}$  of the diagram

$$X \longrightarrow Y \longleftarrow W$$

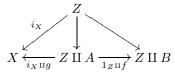
and the structure map  $Z \to P$  is the natural map to the pullback in  $\mathcal{M}$ .

*Proof.* The described constructions possess the universal mapping properties that characterize the pushout (or pullback) in  $(Z \downarrow \mathcal{M})$ .

**Proposition 2.4.** Let  $\mathcal{M}$  be a cocomplete category, let Z be an object of  $\mathcal{M}$ , and let  $F: \mathcal{M} \rightleftharpoons (Z \downarrow \mathcal{M}) : U$  be the adjoint pair of Proposition 2.2. If  $f: A \to B$  is a map in  $\mathcal{M}$  and



is a map in  $(Z \downarrow \mathcal{M})$ , then the pushout in  $(Z \downarrow \mathcal{M})$  of the diagram



is  $Z \to P$  where P is the pushout in  $\mathcal{M}$  of the diagram

$$X \xleftarrow{g} A \xrightarrow{f} B$$

and the structure map  $Z \to P$  is the composition  $Z \xrightarrow{i_X} X \to P$ .

*Proof.* The described construction possesses the universal mapping property required of the pushout in  $(Z \downarrow \mathcal{M})$ .

**Proposition 2.5.** Let  $\mathcal{M}$  be a cocomplete category, let Z be an object of  $\mathcal{M}$ , and let  $F \colon \mathcal{M} \rightleftharpoons (Z \downarrow \mathcal{M}) : U$  be the adjoint pair of Proposition 2.2. If S is a set of maps in  $\mathcal{M}$ , then a relative FS-cell complex (see [1, Definition 10.5.8])  $\underset{X \longrightarrow Y}{\overset{Z}{\longrightarrow}}$  in

 $(Z \downarrow \mathcal{M})$  is a relative S-cell complex  $X \to Y$  in  $\mathcal{M}$  with structure maps defined by composition with the structure map of  $Z \to X$ .

*Proof.* This follows from Proposition 2.4.

**Proposition 2.6.** Let  $\mathcal{M}$  be a cocomplete category, let Z be an object of  $\mathcal{M}$ , and let  $F: \mathcal{M} \rightleftharpoons (Z \downarrow \mathcal{M}) : U$  be the adjoint pair of Proposition 2.2. If S is a set of maps in  $\mathcal{M}$  that permits the small object argument (see [1, Definition 10.5.15]), then FS is a set of maps in  $(Z \downarrow \mathcal{M})$  that permits the small object argument.

*Proof.* This follows from Proposition 2.5 and the adjointness of the functors F and U.  $\hfill \Box$ 

**Theorem 2.7.** Let  $\mathcal{M}$  be a cofibrantly generated model category (see [1, Definition 11.1.2]) with generating cofibrations I and generating trivial cofibrations J. If Z is an object of  $\mathcal{M}$  and  $F \colon \mathcal{M} \rightleftharpoons (Z \downarrow \mathcal{M}) : U$  is the adjoint pair of Proposition 2.2, then the standard model category structure on  $(Z \downarrow \mathcal{M})$  (in which a map

 $\stackrel{Z}{X \to Y}$  is a cofibration, fibration, or weak equivalence in  $(Z \downarrow \mathcal{M})$  if and only if the map  $X \to Y$  is respectively a cofibration fibration or weak equivalence in  $\mathcal{M}$ )

the map  $X \to Y$  is, respectively, a cofibration, fibration, or weak equivalence in  $\mathcal{M}$ ) is cofibrantly generated, with generating cofibrations

$$\mathbf{F}I = \left\{ \begin{array}{c} Z \\ \varkappa \\ Z \amalg X \longrightarrow Z \amalg Y \end{array} \right\} = \mathbf{F}(A \to B) \mid (A \to B) \in I \right\}$$

and generating trivial cofibrations

$$\mathbf{F}J = \left\{ \begin{array}{c} Z \\ \swarrow \\ Z \amalg X \end{array} = \mathbf{F}(A \to B) \mid (A \to B) \in J \right\}$$

*Proof.* We will use [1, Theorem 11.3.2] to show that there is a cofibrantly generated model category structure on  $(Z \downarrow \mathcal{M})$  with generating cofibrations FI and generating trivial cofibration FJ, after which we will show that this coincides with the standard model category structure on  $(Z \downarrow \mathcal{M})$ .

To apply [1, Theorem 11.3.2], we must show that

(1) both of the sets FI and FJ permit the small object argument, and

(2) U takes relative FJ-cell complexes in  $(Z \downarrow \mathcal{M})$  to weak equivalences in  $\mathcal{M}$ . The first condition follows from Proposition 2.6, and the second condition follows

The first condition follows from Proposition 2.6, and the second condition follows from Proposition 2.5, since a relative *J*-cell complex is a trivial cofibration in  $\mathcal{M}$ .

Thus, FI and FJ are the generating cofibrations and generating trivial cofibrations of some model category structure on  $(Z \downarrow \mathcal{M})$ . To see that this is the standard one, we must show that a map in  $(Z \downarrow \mathcal{M})$  is a cofibration, fibration, or weak equivalence if and only if its image under U is, respectively, a cofibration, fibration, or weak equivalence in  $\mathcal{M}$ . For the weak equivalences, this follows from [1, Theorem 11.3.2]. Since the fibrations of  $(Z \downarrow \mathcal{M})$  are the maps with the right lifting property with respect to every element of FJ, the adjointness of F and U implies that these are exactly the maps whose images under U have the right lifting property with respect to J, i.e., exactly the maps whose images under U are fibrations in  $\mathcal{M}$ . Finally, since the fibrations and the weak equivalences of a model category structure determine the cofibrations, the two model category structures on  $(Z \downarrow \mathcal{M})$ must have the same cofibrations as well.

**Theorem 2.8.** Let  $\mathcal{M}$  be a model category and let Z be an object of  $\mathcal{M}$ .

## PHILIP S. HIRSCHHORN

- (1) If  $\mathcal{M}$  is cofibrantly generated, then so is  $(Z \downarrow \mathcal{M})$ .
- (2) If  $\mathcal{M}$  is cellular, then so is  $(Z \downarrow \mathcal{M})$ .
- (3) If  $\mathcal{M}$  is left proper, right proper, or proper, then so is  $(Z \downarrow \mathcal{M})$ .

*Proof.* Part 1 follows from Theorem 2.7, part 2 follows from Theorem 2.7 and Proposition 2.5, and part 3 follows from Lemma 2.3.  $\hfill\square$ 

## References

1. P. S. Hirschhorn, *Model categories and their localizations*, Mathematical Surveys and Monographs, vol. 99, American Mathematical Society, Providence, RI, 2003.

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6