PREREQUISITES (ON EQUIVARIANT STABLE HOMOTOPY) FOR CARLSSONS'S LECTURE.

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§1. Introduction. Three things might be done to help those who wish to understand Carlsson's work on Seqal's Burnside Ring Conjecture [8]. First, one might attempt a general exposition about Seqal's Burnside Ring Conjecture, both in its non-equivariant and in its equivariant forms. Secondly, one might explain the results which Gunawardena, Miller and myself have obtained by calculation for the case $G = (Z_p)^n$. (This is relevant because Carlsson uses these results.) Thirdly, one might attempt a general introduction to equivariant stable homotopy.

In the lecture I gave in Aarhus, I tried to say something on all three topics, but for lack of time I was forced to omit an important part of what I had prepared. In this published text I shall omit the first and second topics, and try to do better justice to the third.

In fact, when I first saw [8] - apart from rejoicing - I thought, "Oh dear; now I shall have to work to understand the fundamentals of this subject". Since I more or less do understand them now (at least, so far as they seem to be needed for Carlsson's work) it may save other topologists trouble if I try to pass on my understanding.

I should stress that I do not claim any originality for what follows; everything I shall explain is or should be known to those who reckon to know about such things. (There is a possible exception in Theorems 5.4 and 8.5, which are recent, and where my statements differ slightly from those in my incoming mail - which I shall acknowledge in due course.)

My survey is arranged as follows. In §2 I shall review unstable equivariant homotopy theory. In §3 I discuss the G-suspension theorem. In §4 I discuss the G-Spanier-Whitehead category. In §5 I discuss certain theorems for reducing problems involving a group G to problems involving a smaller group. In §6 I discuss theories graded over the representation ring $RO(G)$. In §7 I say very little about G-spectra. In §8 I discuss G-Spanier-Whitehead duality.

I shall use the words "ordinary" and "classical" to refer to the non-equivariant case, $G = 1$.

I am very grateful to many correspondents, including G. Carlsson, T. tom Dieck, C. Kosniowski, L.G. Lewis, J.P. May, A. Ranicki and G. Segal. I am particularly grateful to L.G. Lewis for tutorials on G-spectra and to J.P. May for many letters.

§2. Unstable equivariant homotopy. This section must recall how the most elementary part of unstable homotopy theory carries over to the equivariant case.

Let G be a finite group. (Ideally it is desirable to arrange the foundations of equivariant topology so as to cater for compact Lie groups, but for present purposes I will not bother.) A "G-space" is a space on which G acts; for definiteness we agree that groups normally act on the left of spaces. Let X and Y be G-spaces; then a map

$f: X \longrightarrow Y$ is a G-map if

$$f(gx) = g(fx)$$

for all $g \in G$, $x \in X$.

Two G-maps are "G-homotopic" if they are homotopic through G-maps. Alternatively, we can define G-homotopy in terms of G-maps of cylinders; for this purpose, if X is a G-space, we make G act on $I \times X$ by

$$g(t,x) = (t,gx)$$

for all $g \in G$, $t \in I$, $x \in X$.

With these definitions one can carry over a good deal of ordinary homotopy theory. Ordinary homotopy-theory often needs a base-point; at the corresponding places in G-homotopy theory, we suppose given a base-point fixed under G . We then define $[X,Y]^G$ in terms of G-maps and G-homotopies which preserve the base-point.

There are a few simple cases in which problems over G can be reduced to problems over a subgroup H of G . Naturally, we leech onto them to use them in inductive proofs. The technical statement is that if H is a subgroup of G , then the "forgetful functor" from G-spaces to H-spaces has a left adjoint. More precisely, if $i: H \longrightarrow G$ is the inclusion and Y is a G-space, we write i^*Y for the H-space in which H acts on the same space Y via i . Then we have the following natural (1-1) correspondence.

(2.1) $$\text{H-Map}(X, i^*Y) \longleftrightarrow \text{G-Map}(G \times_H X, Y).$$

Here X is supposed to be an H-space, and $G \times_H X$ is the quotient of $G \times X$ in which (g,hx) is identified with (gh,x) . In (2.1) we have no base-points; if we wish to have base-points, then the natural (1-1) correspondence is as follows.

(2.2) $\quad$ Ptd-H-map $(X, i^*Y) \longleftrightarrow$ Ptd-G-Map $((G \sqcup P)\wedge_H X, Y)$.

Here $G \sqcup P$ is the disjoint union of G and a base-point P fixed under G, and $\wedge_H$ is defined as $\times_H$ was before. If we can take X in the form i^*Z where Z is a G-space, then we have the following natural G-homeomorphism.

(2.3)
$$(G \sqcup P)\wedge_H i^*Z \longleftrightarrow (G/H \sqcup P)\wedge Z \ .$$

It is given by

$$(g,z) \longmapsto (g,gz)$$
$$(g,g^{-1}z) \longleftarrow\!\shortmid (g,z).$$

The distinctive features of the equivariant theory begin with the study of fixed-point sets. Let X be a G-space, and let H be a subgroup of G; then the fixed-point set X^H is defined by

$$X^H = \{x \in X \mid hx = x, \ \forall h \in H\} \ .$$

The action of $g \in G$ gives a homeomorphism from X^H to $X^{gHg^{-1}}$; in particular, X^H admits operations from $N(H)/H$, where $N(H)$ is the normaliser of H in G. Any G-map $f: X \longrightarrow Y$ must carry X^H into Y^H and preserve the action of $N(H)/H$; we usually write $f^H: X^H \longrightarrow Y^H$ for the map induced by f on the fixed-point set. If $H \subset K$, then $X^H \supset X^K$.

Many proofs in equivariant homotopy theory are done by induction up the fixed-point sets, beginning with the smallest, X^G, and finishing with the largest, $X^1 = X$. A convenient class of G-spaces in which to do such proofs is the class of G-CW-complexes. We will come to these soon, but first we must mention cells and spheres.

By a "representation of G", we shall mean a finite-dimensional real inner-product space V on which G acts (linearly, and preserving the inner product). Such a representation V has a unit sphere

$S(V)$ and a unit cell $E(V)$ defined by $\|v\|=1$ and $\|v\|\leq 1$ respectively. The usual homeomorphism between $E(V)\times E(W)$ and $E(V\oplus W)$ is equivariant and gives no more nuisance than usual. If we want to use base-points, we normally define S^V to be the one-point compactification of V and put the base-point at infinity. In representation-theory we often write "n" to indicate the representation in which G acts trivially on R^n ; so the new meaning of S^n is S^{R^n} , that is, the old S^n with G acting trivially on it.

The first theorem in ordinary homotopy theory is the theorem that $\pi_r(S^n) = 0$ for $r < n$.

<u>Proposition 2.4</u>. If $\dim V^H < \dim W^H$ for all H , then

$$[S^V, S^W]^G = 0 .$$

The proof will become clear as soon as I have introduced the relevant ideas.

In general, suppose that in the classical case we have some invariant like "dim", which assigns to each suitable space X a value $\dim(X)$ which may be an integer or ∞ . Then the analogue in the equivariant case is to consider "dim X" as a function which assigns to each subgroup $H\subset G$ the value $\dim(X^H)$ (taking equal values on conjugate subgroups H). So the assumption of (2.4) should be thought of as

$$\text{"}\dim(S^V)\leq \dim(S^W)-1\text{"}$$

(with the obvious interpretation of inequality between functions of H). Similarly for the "Hurewicz dimension of X , Hur X" (defined to be the greatest n such that $\pi_r(X) = 0$ for $r < n$). For spheres we have

$$\mathrm{Hur}\,(S^{V^H}) = \dim\,(S^{V^H}) .$$

The following result generalises (2.4).

Proposition 2.5. If $\dim (X^H) \le \mathrm{Hur}\,(Y^H) - 1$ for all H, then

$$[X,Y]^G = 0 .$$

Here it will be prudent to assume that X is a generalised CW-complex of some sort.

On the usual definitions, G-CW-complexes are constructed just like CW-complexes; but instead of using cells of the form E^n, S^{n-1} one uses G-cells of the form

$$(G/H) \times E^n,\ (G/H) \times S^{n-1} .$$

The usual reference for G-CW-complexes is the work of Matumoto [20]. The G-complexes of Bredon [3,5] served the same purpose earlier (for G finite). There is also a thesis by Illman [16], although theses are not usually easily available.

I thank J.P. May for pointing out two possible objections to the usual approach to G-CW-complexes. The first is that the definition of "G-cell" is not wide enough to accommodate the "cells" introduced above. Certainly it would seem worthwhile to make our machinery accept "G-cells" of the form

$$G\times_H E(V),\ G\times_H S(V)$$

where V is a representation of H. This doesn't affect the class of G-spaces considered, because any G-cell of the more general form can be subdivided into G-cells of the special form.

The second objection may be seen from the following example. If H is a subgroup of G, we would like to say that a G-CW-complex "is" an H-CW-complex. Unfortunately, we can't display the G-cell G as a union of H-cells H without choosing coset representatives. This nuisance recurs with products; if X and Y are G-CW-complexes, then $X \times Y$ (with the CW-topology) is likely to come as a complex over the group $G \times G$, and we want it as a complex over the diagonal subgroup

$G \xrightarrow{\Delta} G \times G$.

For present purposes we seem to have a workable way out (though it only serves when G is discrete). We stipulate that the given structure of CW-complex X includes characteristic maps

$$\chi_\alpha : E^{n(\alpha)} \longrightarrow X .$$

If X comes as a G-space, we ask for a commutative diagram of the following form for each $g \in G$ and α .

$$\begin{array}{ccc} E^{n(\alpha)} & \xrightarrow{\chi_\alpha} & X \\ {\scriptstyle \ell(\alpha,g)} \downarrow & & \downarrow {\scriptstyle g} \\ E^{n(\beta)} & \xrightarrow{\chi_\beta} & X \end{array}$$

Clearly β will be unique (so that G will implicitly act on the set of indices α); $\ell(\alpha,g)$ will also be unique, and we ask that it be linear and preserve the inner product. (The last clause is actually redundant.) Then we can choose to organise our characteristic maps into G-orbits

$$G \times_H E(V) \longrightarrow X ,$$

but no such choice is part of the given structure. If we want to insist on G-cells of the form $G/H \times E^n$, we can impose an axiom that $\beta = \alpha$ implies $\ell(\alpha,g) = 1$. If so, we get back to Bredon's G-complexes.

In order to carry over the standard arguments about CW-complexes, one needs to be able to manipulate G-maps of G-cells. Let Y be a G-space; from (2.1) we get the following natural (1-1) correspondence.

(2.6) $$\text{G-Map } ((G/H) \times E^n, Y) \longleftrightarrow \text{Map } (E^n, Y^H) .$$

If on the left we wish to prescribe the values of the G-map on $(G/H) \times S^{n-1}$, that corresponds on the right to prescribing the values of the map on S^{n-1} . So the standard arguments for CW-complexes carry over to G-CW-complexes, using induction over the G-cells plus ordinary homotopy theory in fixed-point subspaces Y^H .

Since our object is to reduce to ordinary homotopy theory, we generally carry out these arguments with G-cells $(G/H)\times E^n$ rather than $G\times_H E(V)$, reducing to that case by subdivision if necessary.

We will forgo a long discussion of those results on CW-complexes which carry over with little change. (For example, the inclusion of a G-subcomplex in a G-CW-complex has the G-homotopy-extension property.)

The first result we do need to mention is the "theorem of J.H.C. Whitehead". Recall that in the ordinary case, a map $f: X \longrightarrow Y$ between path-connected spaces is called an n-equivalence if

$$f_*: \pi_r(X) \longrightarrow \pi_r(Y)$$

is iso for $r < n$ and epi for $r = n$. (If the spaces are not path-connected we modify this definition in an obvious way; see [24 p404].) In the equivariant case, suppose given a function n which assigns to each subgroup $H \subset G$ a value $n(H)$ which may be an integer or ∞ , subject to the condition $n(gHg^{-1}) = n(H)$. Then a G-map $f: X \longrightarrow Y$ is an n-equivalence if $f^H: X^H \longrightarrow Y^H$ is an ordinary $n(H)$-equivalence for each H .

<u>Proposition 2.7</u>. Let W be a G-CW-complex and let $f: X \longrightarrow Y$ be a G-map which is an n-equivalence. Then the induced map

$$f_*: [W,X]^G \longrightarrow [W,Y]^G$$

is onto if

$$\dim W^H \leq n(H) \quad \text{for all } H ;$$

it is a (1-1) correspondence if

$$\dim W^H \leq n(H) - 1 \quad \text{for all } H .$$

As with (2.4), one should think of the assumptions as "dim W ≤ n" and "dim W ≤ n-1", where dim W and n are functions of H .

Results of this sort go back to Bredon [5 Chap. II §5]; see also Matumoto [20 §5], Illman [16 Chapter I §3] and Namboodiri [29 Corollary 2.2].

§3. The G-suspension theorem. This section must recall how the most elementary result of suspension theory carries over to the equivariant case.

When we suspend in equivariant homotopy theory, we have available a variety of actions of G on the suspension coordinates we introduce. For an unreduced suspension, of G-spaces without base-point, it is reasonable to take the join $S(V)*X$. For a reduced suspension, of G-spaces with base-point, it is natural to take the smash product $S^V \wedge X$. Here the action of G on a smash product $X \wedge Y$ is defined by

$$g(x,y) = (gx,gy) ,$$

and similarly for the join.

The relationship between smash and join is much as in the classical case. In fact, classical comparison maps, such as the ordinary quotient map

$$X*Y \longrightarrow X\wedge S^1\wedge Y ,$$

are commonly natural, and therefore equivariant. They can be proved to be G-equivalences by (2.7), provided the following conditions are satisfied.

(a) The restriction of the comparison map to a fixed-point-set is another instance of the same comparison map, for example,

$$X^H * Y^H \longrightarrow X^H \wedge S^1 \wedge Y^H .$$

(b) The comparison map is classically a weak equivalence.

(c) The G-spaces involved are G-CW-complexes.

For this section we will use $S^V \wedge X$. By taking the smash product with the identity map of S^V , we get a function

$$S^V: [X,Y]^G \longrightarrow [S^V \wedge X,\ S^V \wedge Y]^G .$$

We wish to show that S^V is a (1-1) correspondence under suitable conditions. We can follow the classical approach by introducing a function-space. Let $\Omega^V(Z)$ be the space of pointed maps $\omega: S^V \longrightarrow Z$; we make G act on this function space by

$$(g\omega)(s) = g(\omega(g^{-1}s)) .$$

As in the classical case, it is sufficient to study the canonical map

$$Y \longrightarrow \Omega^V(S^V \wedge Y)$$

and prove that it is an n-equivalence for some suitable n . We choose our function $n = n(H)$ so that it has the following properties.

(3.1) For each subgroup $H \subset G$ such that $V^H > 0$ we have

$$n(H) \leq 2\ \mathrm{Hur}(Y^H) - 1 .$$

(3.2) For each pair of subgroups $K \subset H \subset G$ such that $V^K > V^H$ we have

$$n(H) \leq \mathrm{Hur}(Y^K) - 1 .$$

<u>Theorem 3.3</u>. Under these conditions, the map

$$Y \longrightarrow \Omega^V(S^V \wedge Y)$$

is an n-equivalence. It follows that the function

$$S^V: [X,Y]^G \longrightarrow [S^V \wedge X,\ S^V \wedge Y]^G$$

is onto if X is a G-CW-complex and $\dim (X^H) \leq n(H)$ for each H ; it is a (1-1) correspondence if X is a G-CW-complex and $\dim (X^H) \leq n(H) - 1$ for each H .

The second sentence follows from the first by using (2.7), as in the ordinary case.

For $G = Z_2$ the result is due to Bredon [4]. If I may count [4,6] as one paper, then this is the first paper in equivariant stable homotopy theory, and I think it may deserve more credit than it has received. To promote understanding of this subject, I recommend study of the special case $G = Z_2$.

It would have been good if the G-suspension theorem could have come next after [4,6]. The proof does contain ingredients which are additional to those well-known in the ordinary case, including, of course, the use of condition (3.2). The standard reference is Hauschild [14]; see also Namboodiri [29, Theorem 2.3].

One use of the suspension theorem is to show that certain limits are attained. For this purpose we must decide what class of representations to use when we suspend. For the moment we keep our options open; we suppose given some class of "allowable representations" of G , so that our class is closed under passage to sums and summands, and also under passage from any representation to an isomorphic one. We order the allowable representations, writing $W \geq V$ if $W \cong U \oplus V$ for some U . Let X be a finite-dimensional G-CW-complex, and Y a G-space.

Theorem 3.4. There exists an allowable $W_o = W_o(X)$ such that for any allowable $W \geq W_o$ and any allowable V the map

$$S^V \colon [S^W \wedge X, S^W \wedge Y]^G \longrightarrow [S^V \wedge S^W \wedge X, S^V \wedge S^W \wedge Y]^G$$

is a (1-1) correspondence. Indeed, the map

$$S^V \colon [X', S^W \wedge Y]^G \longrightarrow [S^V \wedge X', S^V \wedge S^W \wedge Y]^G$$

is a (1-1) correspondence for any subcomplex X' of $S^W \wedge X$ or of any subdivision of $S^W \wedge X$.

The final sentence about X' is intended to help with [8 p45].

The result will follow from Theorem 3.3, provided we can satisfy the following inequalities on the dimensions.

(i) If for some H there is an allowable V with $V^H > 0$, then

$$\dim W^H + \dim X^H \leq 2 \dim W^H - 2 .$$

It is clear that if there is any allowable V with $V^H > 0$, then by putting sufficiently many copies of it into W we can increase $\dim W^H$ till this inequality is satisfied; it then holds for all larger W.

(ii) If for some $K \subset H$ there is any allowable V with $V^K > V^H$, then

$$\dim W^H + \dim X^H \leq \dim W^K - 2 .$$

It is clear that if there is any allowable V with $V^K > V^H$, then by putting sufficiently many copies of it into W we can increase $\dim W^K - \dim W^H$ till this inequality is satisfied; it then holds for all larger W.

Of course, we have to satisfy inequalities of type (i) for a finite number of subgroups H, and inequalities of type (ii) for a finite number of pairs $K \subset H$, but we can satisfy all these conditions if W is sufficiently large. This proves Theorem 3.4.

§4. The G-analogue of the Spanier-Whitehead category. This section must review how the original approach to stable homotopy theory carries over to the equivariant case.

We wish to pass to a limit and consider stable classes of maps. To take a "limit" of groups we must supply a system of groups and homomorphisms, or equivalently, a functor defined on some suitable category. We take the objects of our category C to be all "allowable" representations V of G (see §3); we take the morphisms $f: V \longrightarrow W$ in C to be the R-linear G-maps which preserve inner products. (Such maps f are necessarily mono.)

Suppose given two G-spaces X, Y with base-points. To each object V of C we associate the set

$$[S^V \wedge X,\ S^V \wedge Y]^G.$$

For any morphism $i: V \longrightarrow W$ in C, we first use i to identify W with $U \oplus V$, where U is the orthogonal complement of the image $i(V)$ under the inner product. We now associate to i the following composite function.

$$\begin{array}{ccc} [S^V \wedge X,\ S^V \wedge Y]^G & \xrightarrow{S^U} & [S^U \wedge S^V \wedge X,\ S^U \wedge S^V \wedge Y]^G \\ & & \updownarrow \\ & & [S^W \wedge X,\ S^W \wedge Y]^G \end{array}$$

(Notice that we can identify $S^U \wedge S^V$ with $S^{U \oplus V}$ and so with S^W.) We get a functor from C to sets.

Next we must check that this functor is such that we can take its limit. First we need to see that if U and V are objects in C, then there is an object in C which receives morphisms from both. This is immediate: it is enough to take $U \oplus V$. Secondly we need to see that if $f,g: U \longrightarrow V$ are two morphisms in C , then there is a further morphism $h: V \longrightarrow W$ such that our functor assigns equal values to hf, hg . It is easy to reduce to the case in which f is an automorphism of V and $g = 1$. One can see by counter-examples that it is not sufficient to take $h = 1$; that is, the composite

$$S^V \wedge X \xrightarrow{f^{-1}\wedge 1} S^V \wedge X \xrightarrow{\phi} S^V \wedge Y \xrightarrow{f\wedge 1} S^V \wedge Y$$

need not be G-homotopic to ϕ . However, we take h to be the injection of V as the second factor in $V \oplus V$. Clearly we have $hf = (1\oplus f)h$. But $1\oplus f$ is homotopic through G-isomorphisms to $f\oplus 1$, for example by

$$\begin{bmatrix} \mathrm{Cos}t & -\mathrm{Sin}t \\ \mathrm{Sin}t & \mathrm{Cos}t \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & f \end{bmatrix} \begin{bmatrix} \mathrm{Cos}t & \mathrm{Sin}t \\ -\mathrm{Sin}t & \mathrm{Cos}t \end{bmatrix} \qquad (0 \le t \le \tfrac{1}{2}\pi).$$

For $f\oplus 1$ we see that $(f\oplus 1)h$ and h induce the same function

$$[S^V \wedge X,\ S^V \wedge Y]^G \longrightarrow [S^{V\oplus V} \wedge X,\ S^{V\oplus V} \wedge Y]^G .$$

This completes the checks.

We may therefore pass to the limit, and define

$$\{X,Y\}^G = \varinjlim_{V\in C} [S^V \wedge X,\ S^V \wedge Y]^G .$$

This definition is due to Segal [23]. Some of my correspondents would prefer to see the category C replaced by an equivalent small category before the limit is taken. In the applications X is finite-dimensional, and the limit is equivalent to taking the common value of $[S^V \wedge X,\ S^V \wedge Y]^G$ for all sufficiently large V - which is perfectly safe, however many V there are.

To continue, composition of G-maps $X \longrightarrow Y \longrightarrow Z$ is compatible with suspension, so it is clear how to define composition of stable maps and make the sets $\{X,Y\}^G$ into the hom-sets of a category.

The analogues of (2.4), (2.5) are as follows.

Proposition 4.1. If $\dim V^H < \dim W^H$ for all H, then

$$\{S^V, S^W\}^G = 0 .$$

Proposition 4.2. If $\dim (X^H) \leq \mathrm{Hur}\,(Y^H) - 1$ for all H,

then $\{X,Y\}^G = 0$.

In fact, in each case one has to take a limit of sets which are all trivial, by (2.4) or (2.5) as the case may be.

The definition of $\{X,Y\}^G$ given above clearly follows that of Spanier and Whitehead for the classical case. Therefore one can only expect it to be useful when X is finite-dimensional. In this case we expect the following result.

Proposition 4.3. If X is a finite-dimensional G-CW-complex, then the limit $\{X,Y\}^G$ is attained by $[S^W \wedge X, S^W \wedge Y]^G$ for all sufficiently large W .

This follows immediately from Theorem 3.4.

For later use, we also need to assure ourselves that our category is really "stable", by verifying that a suitable "external" suspension is a (1-1) correspondence. For this purpose, suppose given two G-spaces X, Y and an allowable representation U . For each object V of C we have a function

$$[S^V \wedge X,\ S^V \wedge Y]^G \xrightarrow{\ \mathrm{Susp}\ V\ } [S^V \wedge X \wedge S^U,\ S^V \wedge Y \wedge S^U]^G$$

carrying f to $f \wedge 1_U$. This function commutes with the maps of our direct system, and defines a function

$$\begin{array}{ccc} \{X,Y\}^G & & \{X\wedge S^U,\ Y\wedge S^U\}^G \\ \| & & \| \\ \varinjlim_{V\in C} [S^V\wedge X,\ S^V\wedge Y]^G & \longrightarrow & \varinjlim_{V\in C} [S^V\wedge X\wedge S^U,\ S^V\wedge Y\wedge S^U]^G . \end{array}$$

Lemma 4.4. If X is a finite-dimensional G-CW-complex then this function

$$\{X,Y\}^G \xrightarrow{\text{Susp}} \{X\wedge S^U,\ Y\wedge S^U\}^G$$

is a (1-1) correspondence.

In fact, the functions Susp_V whose limit is taken are (1-1) correspondences for all sufficiently large V, by Theorem 3.4.

So far we have not said that our sets $\{X,Y\}^G$ are groups. To introduce addition one needs a suspension coordinate on which G acts trivially. From now on we assume that trivial representations are allowable; in this case the sets $\{X,Y\}^G$ become additive groups, and in fact the hom-sets of a preadditive category.

In the applications it is important to have suitable finiteness theorems.

Theorem 4.5. Suppose X is a finite G-CW-complex and Y is a G-space for which each fixed-point-set Y^H is an (ordinary) CW-complex with finitely many cells of each dimension. Then $\{X,Y\}^G$ is a finitely-generated abelian group.

The crucial point is that this limit is attained by

$$[S^W\wedge X,\ S^W\wedge Y]^G$$

for some W, according to (4.3). It is fairly clear how to prove that this group is finitely-generated, by combining the methods indicated in §2 with standard finiteness theorems in the classical case.

§5. Theorems on changing groups. In the unstable case, it is useful to have results such as (2.2) which allow one to reduce suitable problems involving G to problems involving a smaller group. In the stable case, there are more such theorems which are useful; in this section we will consider them.

First suppose given a homomorphism $\theta : G_1 \longrightarrow G_2$. For any G_2-space X, θ^*X will mean the same space considered as a G_1-space, with G_1 acting via θ. In particular, if V is a representation of G_2, then θ^*V is a representation of G_1. We assume that if V is an "allowable" representation of G_2, then θ^*V is an "allowable" representation of G_1. The operation θ^*V commutes with suspension, in the sense that

$$\theta^*(S^V \wedge X) = S^{\theta^*V} \wedge \theta^*X .$$

Therefore θ^* gives a functor from the G_2-Spanier-Whitehead category to the G_1-Spanier-Whitehead category. Here the objects of the "G-Spanier-Whitehead category" are the finite G-CW-complexes; the hom-sets are the sets $\{X,Y\}^G$ introduced in §4.

Until further notice, all representations are allowable.

I will state four results about θ^* before pausing to discuss them. To analyse θ^*, it is reasonable to factor θ through an epimorphism and a monomorphism, and tackle the factors separately. So first, let $i : H \longrightarrow G$ be the inclusion of a subgroup H in the group G; let X run over the H-Spanier-Whitehead category and let Y run over the G-Spanier-Whitehead category.

Theorem 5.1. There is a natural (1-1) correspondence

$$\{X, i^*Y\}^H \longleftrightarrow \{(G_+ P) \wedge_H X, Y\}^G .$$

Theorem 5.2. There is a natural (1-1) correspondence

$$\{i^*Y,\ X\}^H \longleftrightarrow \{Y,\ (G \sqcup P) \wedge_H X\}^G .$$

Secondly, let $j : G \longrightarrow \bar{G}$ be the projection of G on a quotient group $\bar{G}$, and let $N = \operatorname{Ker} j$. (The case most useful for the applications is that in which $N = G$ and $\bar{G} = 1$; but there seems to be no harm in looking for a natural level of generality.) We let X run over finite G-CW-complexes in which the subgroup $N \subseteq G$ acts freely away from the base-point; more precisely, we let X run over the full subcategory of the G-Spanier-Whitehead category determined by these N-free objects. We let Y run over the $\bar{G}$-Spanier-Whitehead category.

<u>Theorem 5.3</u>. There is a natural (1-1) correspondence

$$\{X,\ j^*Y\}^G \longleftrightarrow \{X/N,\ Y\}^{\bar{G}} .$$

<u>Theorem 5.4</u>. There is a natural (1-1) correspondence

$$\{j^*Y,\ X\}^G \longleftrightarrow \{Y,\ X/N\}^{\bar{G}} .$$

Here X/N is of course the usual orbit space.

I will discuss these four results before I proceed to necessary technical details. Theorem 5.1 is a simple analogue of (2.2), and is widely known. Given (5.1), (5.2) says that the forgetful functor from the G-stable world to the H-stable world has a right adjoint which coincides with its known left adjoint. I first heard this principle from L.G. Lewis; of course his "stable worlds" were worlds of spectra, which makes for a better theorem. In that form, the result is to appear in [18]. That work uses the name "Wirthmüller isomorphism", thus giving credit to Wirthmüller for the result from which the authors started; it was only slightly less general than theirs, but I accept that it is more illuminating to state the result in the form I have quoted from Lewis.

I turn to motivation for (5.3). In the Atiyah-Seqal theorem [2] one wishes to know that the ordinary K-theory of the classifying space BG coincides with the equivariant K-theory of the corresponding total space EG . In studying Segal's Burnside Ring Conjecture, one wishes similarly to know that the ordinary cohomotopy of BG coincides with the equivariant cohomotopy of EG . In suitable notation this reads

$$\{EG \sqcup P,\ S^{o}\}^{G} \longleftrightarrow \{\frac{EG \sqcup P}{G},\ S^{o}\}^{1}.$$

This is an instance of (5.3) (with $N = G$, $\bar{G} = 1$) except that (5.3) only gives the result for finite approximations to EG , BG ; the result for EG requires the analogue of (5.3) for spectra, unless you pass to limits from the result for finite approximations. Such results are probably widely known to those who have started work on Segal's conjecture; after (5.1), I regard (5.3) as the second easiest of the four.

Theorem 5.4 is necessary to maintain symmetry, as well as being needed for applications later. I am grateful to L.G. Lewis and J.P. May for letters, to which I owe the case $N = G$, $\bar{G} = 1$ (for the world of spectra).

Theorems 5.3 and 5.4 are not as satisfactory as (5.1) and (5.2); (5.1) and (5.2) each give an honest pair of adjoint functors, but (5.3) and (5.4) do not. (In (5.3) and (5.4) X is restricted to be N-free, but j^*Y cannot be N-free except in trivial cases.) One might like to see the statements of (5.3) and (5.4) improved in some way; I am open to suggestions.

In the rest of this section I will begin by giving some necessary technical details to complete the statements of (5.1) - (5.4), and continue with the proofs. The reader should consider skipping to §6.

The main technical detail concerns the sense in which $(G \sqcup P) \wedge_H X$ is functorial for stable H-maps of X rather than for unstable maps, and similarly for X/N . The careful reader should not take this for granted.

First suppose given a stable H-map $\phi: X_1 \longrightarrow X_2$. Since representations of the form i^*V are cofinal among representations of H , we may suppose given a representative H-map

$$f: S^{i^*V} \wedge X_1 \longrightarrow S^{i^*V} \wedge X_2 .$$

We now define the stable G-map

$$1 \wedge_H \phi: (G \sqcup P) \wedge_H X_1 \longrightarrow (G \sqcup P) \wedge_H X_2$$

to be the class of the following composite.

$$\begin{array}{ccc} S^V \wedge ((G \sqcup P) \wedge_H X_1) & \xrightarrow{\varepsilon^{-1}} & (G \sqcup P) \wedge_H (S^{i^*V} \wedge X_1) \\ & & \big\downarrow 1 \wedge_H f \\ S^V \wedge ((G \sqcup P) \wedge_H X_2) & \xleftarrow{\varepsilon} & (G \sqcup P) \wedge_H (S^{i^*V} \wedge X_2) \end{array}$$

Here the G-homeomorphism

$$(G \sqcup P) \wedge_H (S^{i^*V} \wedge X) \xrightarrow{\varepsilon} S^V \wedge ((G \sqcup P) \wedge_H X)$$

is given by

$$\varepsilon(g, (s, x)) = (gs, (g, x)) .$$

Of course we have to check that the result depends only on ϕ , and that $(G \sqcup P) \wedge_H X$ becomes a functor as stated.

We now wish to copy this procedure for X/N ; the difficulty is that representations of the form j^*W are not cofinal among representations of G . We need the following crucial result.

<u>Proposition 5.5</u>. If X is N-free away from the base-point, then $\{X, Y\}^G$, as defined allowing suspensions of the form S^{j^*W} only, agrees with $\{X, Y\}^G$ as defined using all suspensions S^V .

Proof. Let me begin by fine-tuning some results I explained earlier. In the "theorem of J.H.C. Whitehead", Proposition 2.7, we do not really need the assumption that all the cells of W^H are of dimension $\leq n(H)-1$; it is sufficient if all G-cells of W of the form $(G/H) \times E^m$ have $m \leq n(H)-1$. The same applies to any deduction from (2.7), including the G-suspension theorem, Theorem 3.3. These remarks are due to U. Namboodiri [28,29].

Our object is now to show that the map

$$[S^{j^*W} \wedge X, S^{j^*W} \wedge Y]^G \xrightarrow{S^V} [S^V \wedge S^{j^*W} \wedge X, S^V \wedge S^{j^*W} \wedge Y]^G$$

is iso for all V if the representation W of $\bar{G}$ is sufficiently large (depending on X). Here $S^{j^*W} \wedge X$ is also N-free away from the base-point. Therefore it can have cells $(G/H) \times E^m$ only if $N \cap H = 1$. So it will be sufficient to impose a suitable bound on the dimension of $(S^{j^*W} \wedge X)^H$ just for those subgroups H which satisfy $N \cap H = 1$. We now wish to satisfy the following inequalities on the dimensions.

(i) If H is a subgroup with $N \cap H = 1$ then

$$\dim (j^*W)^H + \dim X^H \leq 2 \dim (j^*W)^H - 2 .$$

We can satisfy this condition by putting sufficiently many copies of the trivial representation into W .

(ii) If $K \subset H$ is a pair of subgroups such that $N \cap H = 1$ and $V^K > V^H$ for some representation V , then

$$\dim (j^*W)^H + \dim X^H \leq \dim (j^*W)^K - 2 .$$

If $V^K > V^H$ for some V then we must have $K < H$. If $N \cap H = 1$ then the images of K, H in $\bar{G}$ satisfy $\bar{K} < \bar{H}$. Therefore there is a representation U of $\bar{G}$ for which $U^{\bar{K}} > U^{\bar{H}}$ (for example, the permutation representation on the cosets of $\bar{K}$). We can satisfy the condition by putting sufficiently many copies of U into W .

Of course, we have to satisfy inequalities of type (i) for a finite number of subgroups H, and inequalities of type (ii) for a finite number of pairs $K \subset H$, but we can satisfy all these conditions if W is sufficiently large. This proves (5.5).

We can now return to the task of making X/N functorial for stable G-maps of X. According to (5.5), any stable G-map $\phi: X_1 \longrightarrow X_2$ has a representative of the form

$$f: S^{j^*W} \wedge X_1 \longrightarrow S^{j^*W} \wedge X_2 .$$

We now define the stable $\overline{G}$-map $\overline{\phi}: X_1/N \longrightarrow X_2/N$ to be the class of the following composite.

$$\begin{array}{ccc} S^W \wedge X_1/N & \longleftrightarrow & \dfrac{S^{j^*W} \wedge X_1}{N} \\ & & \Big\downarrow \overline{f} \\ S^W \wedge X_2/N & \longleftrightarrow & \dfrac{S^{j^*W} \wedge X_2}{N} \end{array}$$

(Here the $\overline{G}$-homeomorphisms of spaces are the obvious ones.) We have to check that the result depends only on ϕ, and that X/N becomes a functor as stated, but these points are trivial.

This completes the technical details needed to explain (5.1)-(5.4).

In what follows we will omit the symbols i^*, j^* which show which groups are supposed to be acting on a given space; it is always easy to work out which groups are supposed to be acting, and these symbols only complicate the notation.

<u>Proof of Theorem 5.1</u>. The (1-1) correspondence is induced by the same two maps that serve in the unstable category. These are the H-map

$$X \xrightarrow{\alpha} (G \sqcup P) \wedge_H X$$

given by $\alpha(x) = (1,x)$, and the G-map

$$(G \sqcup P) \wedge_H Y \xrightarrow{\gamma} Y$$

given by $\gamma(g,y) = gy$. These maps make the following diagrams commute.

$$\begin{array}{ccc} S^V \wedge X & \xrightarrow{S^V \alpha} & S^V \wedge ((G \sqcup P) \wedge_H X) \\ & \searrow^{\alpha} & \uparrow \varepsilon \\ & & (G \sqcup P) \wedge_H (S^V \wedge X) \end{array}$$

$$\begin{array}{ccc} S^V \wedge ((G \sqcup P) \wedge_H Y) & \xrightarrow{S^V \gamma} & S^V \wedge Y \\ \uparrow \varepsilon & \nearrow_{\gamma} & \\ (G \sqcup P) \wedge_H (S^V \wedge Y) & & \end{array}$$

It follows that α is natural, not only for unstable H-maps of X, but also for stable H-maps of X; similarly, γ is natural, not only for unstable G-maps of Y, but also for stable G-maps of Y. Therefore these maps induce natural transformations in the usual way.

These two natural transformations are inverse because the composites

$$Y \xrightarrow{\alpha} (G \sqcup P) \wedge_H Y \xrightarrow{\gamma} Y$$

$$(G \sqcup P) \wedge_H X \xrightarrow{1 \wedge_H \alpha} (G \sqcup P) \wedge_H (G \sqcup P) \wedge_H X \xrightarrow{\gamma} (G \sqcup P) \wedge_H X$$

are already identity maps unstably.

Proof of Theorem 5.2. I have given the proof of (5.1) in the form above so that I can transcribe it by using arrow-reversing duality. First we shall need a stable H-map

$$(G \sqcup P) \wedge_H X \xrightarrow{\alpha} X .$$

This comes as an unstable map; we set

$$\alpha(h, x) = hx$$

$$\alpha(g, x) = x_o \quad \text{if} \quad g \notin H ,$$

where x_o is the base-point. We shall also need a stable G-map

$$Y \xrightarrow{\gamma} (G \sqcup P) \wedge_H Y ,$$

and this we now construct.

First choose an embedding $G/H \longrightarrow W$ of G/H in a representation W of G. For definiteness, we may take W to be a permutation representation, with the elements of G/H as an orthonormal base. Next choose an open equivariant tubular neighbourhood N of G/H in W; for definiteness, we may take the discs of radius 1/2 around the points of G/H. Consider the quotient map of $S^W = W \cup \infty$ in which we identify to a point the complement of N; we obtain a G-map

$$\beta : S^W \longrightarrow S^W \wedge (G/H \sqcup P) .$$

This map β is fixed once for all and does not depend on Y.

Given β, we define γ to be the following composite.

$$\begin{array}{ccc} S^W \wedge Y & \xrightarrow{\beta \wedge 1} & S^W \wedge (G/H \sqcup P) \wedge Y \\ & & \Big\downarrow 1 \wedge \eta^{-1} \\ & & S^W \wedge ((G \sqcup P) \wedge_H Y) . \end{array}$$

Here the G-homeomorphism

$$(G \sqcup P) \wedge_H Y \xrightarrow{\eta} (G/H \sqcup P) \wedge Y$$

is defined by

$$\eta\ (g,\ y) = (g,\ gy)$$

as in (2.3).

The properties of the maps α and γ are as follows.

Lemma 5.6 (i) The following diagram is commutative.

$$\begin{array}{ccc} S^V \wedge ((G \sqcup P) \wedge_H X) & \xrightarrow{S^V\alpha} & S^V \wedge X \\ \uparrow{\scriptstyle \varepsilon} & \nearrow{\scriptstyle \alpha} & \\ (G \sqcup P) \wedge_H (S^V \wedge X) & & \end{array}$$

(ii) The following diagram is commutative.

$$\begin{array}{ccc} S^W \wedge S^V \wedge Y & \xrightarrow{S^V\gamma} & S^W \wedge S^V \wedge (G \sqcup P) \wedge_H Y \\ & {\scriptstyle \gamma}\searrow & \uparrow{\scriptstyle 1 \wedge \varepsilon} \\ & & S^W \wedge ((G \sqcup P) \wedge_H (S^V \wedge Y)) \end{array}$$

(iii) The composite

$$Y \xrightarrow{\gamma} (G \sqcup P) \wedge_H Y \xrightarrow{\alpha} Y$$

is the identity as a stable H-map.

(iv) The composite

$$(G \sqcup P) \wedge_H X \xrightarrow{\gamma} (G \sqcup P) \wedge_H (G \sqcup P) \wedge_H X \xrightarrow{1 \wedge_H \alpha} (G \sqcup P) \wedge_H X$$

is the identity as a stable G-map.

In part (ii), the G-map $S^V\gamma$ is obtained by suspending

$$S^W \wedge Y \xrightarrow{\gamma} S^W \wedge ((G \sqcup P) \wedge_H Y)$$

according to the inclusion $W \longrightarrow W \oplus V$.

Assuming Lemma 5.6, we can complete the proof of Theorem 5.2 as follows. The map α is clearly natural for unstable H-maps of X; using (5.6) (i), we see that α is natural for stable H-maps of X, just as in the proof of (5.1). Similarly, the map γ is natural for unstable G-maps of Y; using (5.6) (ii), and taking a little more care that we are really following the definitions laid down in §4, we see that γ is natural for stable G-maps of Y. Now we follow the standard routine for adjoint functors. The transformation

$$\{Y, X\}^H \longrightarrow \{Y, (G \sqcup P) \wedge_H X\}^G$$

carries a stable H-map

$$Y \xrightarrow{f} X$$

to the composite

$$Y \xrightarrow{\gamma} (G \sqcup P) \wedge_H Y \xrightarrow{1 \wedge_H f} (G \sqcup P) \wedge_H X .$$

The transformation

$$\{Y, (G \sqcup P) \wedge_H X\}^G \longrightarrow \{Y, X\}^H$$

carries a stable G-map

$$Y \xrightarrow{f} (G \sqcup P) \wedge_H X$$

to the composite

$$Y \xrightarrow{f} (G \sqcup P) \wedge_H X \xrightarrow{\alpha} X .$$

These transformations are inverse by (5.6) (iii), (iv).

Proof of Lemma 5.6. It is straightforward to verify parts (i) and (ii) from the definitions.

To prove part (iii), we introduce the map

$$\zeta : (G/H) \sqcup P \longrightarrow S^0$$

which carries g to the base-point if $g \notin H$, to the non-base-point if $g \in H$. We check that the composite

$$S^V \xrightarrow{\beta} S^V \wedge ((G/H) \sqcup P) \xrightarrow{1 \wedge \zeta} S^V$$

is H-homotopic to the identity, and that the diagram

$$\begin{array}{ccc} ((G/H) \sqcup P) \wedge Y & \xrightarrow{\zeta \wedge 1} & Y \\ \uparrow \eta & \nearrow \alpha & \\ (G \sqcup P) \wedge_H Y & & \end{array}$$

is strictly commutative. The result follows by combining these facts.

To prove part (iv), we first note an associative law. Suppose H acts on the right of $A \wedge B$ by acting on the right of B. Then the identity map of $A \wedge B \wedge C$ passes to the quotient to give an identification

$$(A \wedge B) \wedge_H C \longleftrightarrow A \wedge (B \wedge_H C) .$$

Up to this identification, the map

$$\begin{array}{c} S^W \wedge (G/H \sqcup P) \wedge (G \sqcup P) \wedge_H X \\ \downarrow 1 \wedge \eta^{-1} \\ S^W \wedge (G \sqcup P) \wedge_H (G \sqcup P) \wedge_H X \\ \downarrow 1 \wedge 1 \wedge_H \alpha \\ S^W \wedge (G \sqcup P) \wedge_H X \end{array}$$

which occurs in (iv) may be written as $1 \wedge \delta \wedge 1_X$, where

$$(G/H \sqcup P) \wedge (G \sqcup P) \xrightarrow{\delta} (G \sqcup P)$$

carries (g_1, g_2) to g_2 if $g_1 H = g_2 H$, to the base-point otherwise.

It is now sufficient to check that the composite

$$\begin{array}{c} S^W \wedge (G \sqcup P) \\ \Big\downarrow \beta \wedge 1 \\ S^W \wedge (G/H \sqcup P) \wedge (G \sqcup P) \\ \Big\downarrow 1 \wedge \delta \\ S^W \wedge (G \sqcup P) \end{array}$$

is equivariantly homotopic to the identity, where the word "equivariant" means that we preserve both the actions of G on the left of these spaces and the action of H on the right.

In this composite, the map $\beta \wedge 1$ may be regarded as the map of $(S^W \times G)/(\infty \times G)$ which collapses to a point the complement of a tubular neighbourhood of $G/H \times G$. To apply $1 \wedge \delta$ we replace the relevant parts of this map by parts which map to the base-point; we thus obtain the map which collapses to a point the complement of a tubular neighbourhood of G, embedded via $g \longmapsto (gH, g)$. Clearly this embedding is homotopic to the zero cross-section by a linear homotopy $g \longmapsto (tgH, g)$ $(0 \leq t \leq 1)$. In this way we obtain a homotopy with the required equivariance property. This completes the proof of Lemma 5.6, and so finishes the proof of Theorem 5.2.

<u>Proof of Theorem 5.3</u>. In (5.3) and (5.4) we do not have an honest adjunction, and we can only expect to construct a natural transformation in one direction. In (5.3), the transformation is induced by an unstable G-map, namely the quotient map

$$X \xrightarrow{q} X/N .$$

For spaces it is trivial that the induced map

$$[S^W \wedge X/N,\ S^W \wedge Y]^{\overline{G}} \xrightarrow{q^*} [S^W \wedge X,\ S^W \wedge Y]^G$$

is a (1-1) correspondence, because every G-map of $S^W \wedge X$ into a space fixed by N must factor through $S^W \wedge X/N$. (Here W is of course a representation of $\overline{G}$.) Passing to limits, we see that

$$\varinjlim_W [S^W \wedge X/N,\ S^W \wedge Y]^{\overline{G}} \xrightarrow{q^*} \varinjlim_W [S^W \wedge X,\ S^W \wedge Y]^G$$

is a (1-1) correspondence. The left-hand side is $\{X/N,\ Y\}^{\overline{G}}$, and the right-hand side is $\{X,\ Y\}^G$ by Proposition 5.5. This proves Theorem 5.3.

<u>Proof of Theorem 5.4</u>. We shall construct for each finite G-CW-complex X on which N acts freely away from the base-point a stable G-map

$$tr_X \in \{X/N,\ X\}^G$$

with suitable properties. We shall then use this map to induce a natural transformation

$$\{Y,\ X/N\}^{\overline{G}} \longrightarrow \{Y,\ X\}^G .$$

The map tr_X is a "transfer" corresponding to the "covering" $X \longrightarrow X/N$. (I write "covering" because it fails to be an honest covering at the base-point.)

To construct tr_X, we first replace X by an equivalent G-CW-complex if that is thought to ease the next step. We choose a G-embedding

$$\begin{array}{ccc} X & \xrightarrow{(e,q)} & V \times X/N \\ & {\scriptstyle q}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & X/N & \end{array}$$

of the quotient map $q\colon X \longrightarrow X/N$ in the projection π of a trivial vector-bundle, whose fibre V is of course a representation of G. We also choose a G-invariant function $\varepsilon\colon X \longrightarrow R$ which is continuous, zero at the base-point and positive elsewhere, so that as x runs over X the points $(v, qx) \in V \times X/N$ with $\| v - e(x) \| \le \varepsilon(x)$ make up a "tubular neighbourhood" $N(X)$ of X, which is just like an ordinary tubular neighbourhood except that its radius tends to zero as x approaches the base-point. For example, one may choose

$$\varepsilon(x) = \frac{1}{3} \operatorname*{Min}_{n \in N} \| e(nx) - e(x) \| .$$

We now perform the usual "Pontryagin-Thom" construction, and collapse the complement of the open tubular neighbourhood $N(X)$ to the base-point. We obtain a G-map

$$tr_X : S^V \wedge (X/N) \longrightarrow S^V \wedge X .$$

Here we get $S^V \wedge X$ rather than $(S^V \times X)/(\infty \times X)$ precisely because the radius of the tubular neighbourhood goes to zero at the base-point.

<u>Lemma 5.7</u>. The class $tr_X \in \{X/N, X\}^G$ is independent of the choices made in its construction, and natural for unstable G-maps of X.

<u>Proof</u>. It is more or less clear that the choice of ε affects the result only up to a G-homotopy, so it remains to discuss the dependence on V and on the embedding. We handle this together with the proof that tr_X is natural for (unstable) G-maps.

Suppose then that we are given a G-map $f\colon X_1 \longrightarrow X_2$ and embeddings

$$\begin{array}{ccc} X_1 & \xrightarrow{(e_1,q_1)} & V_1 \times X_1/N \\ & {\scriptstyle q_1}\searrow \qquad \swarrow{\scriptstyle \pi} & \\ & X_1/N & \end{array}$$

$$\begin{array}{ccc} X_2 & \xrightarrow{(e_2,q_2)} & V_2 \times X_2/N \\ & {\scriptstyle q_2}\searrow \qquad \swarrow{\scriptstyle \pi} & \\ & X_2/N & \end{array}$$

yielding G-maps

$$tr_1 : S^{V_1} \wedge X_1/N \longrightarrow S^{V_1} \wedge X_1$$

$$tr_2 : S^{V_2} \wedge X_2/N \longrightarrow S^{V_2} \wedge X_2 \ .$$

Then we can embed both embeddings in an embedding

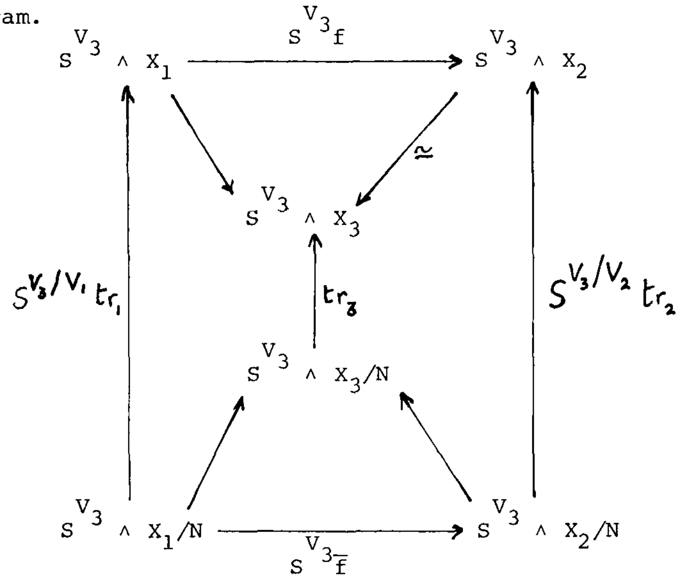
$$\begin{array}{ccc} X_3 & \xrightarrow{(e_3,q_3)} & V_3 \times X_3/N \\ & {\scriptstyle q_3}\searrow \qquad \swarrow{\scriptstyle \pi} & \\ & X_3/N & \end{array}$$

in which X_3 is the mapping-cylinder of f and the injections $X_1 \longrightarrow X_3$, $X_2 \longrightarrow X_3$ are the usual ones. For example, we can take $V_3 = V_1 \times V_2$ and

$$e_3(t, x_1) = ((1-t)\ e_1(x_1)\ ,\ te_2(fx_1))$$

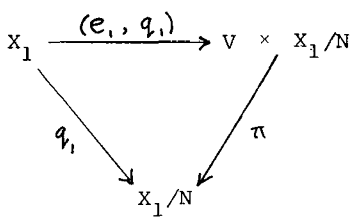
$$e_3(x_2) \quad = \quad (0\ ,\quad e_2(x_2))\ .$$

Performing the same construction on this embedding, we get the following diagram.



This proves Lemma 5.7.

We now wish to show that tr_X has suitable properties for suspension, and of course our discussion is modelled on (5.6) (ii). Suppose given an embedding



leading to the G-map

$$tr_1 : S^V \wedge (X_1/N) \longrightarrow S^V \wedge X_1 .$$

Suppose given also a representation W of $\bar{G} = G/N$. We wish to obtain the map tr_2 for the space $X_2 = S^W \wedge X_1$.

Lemma 5.8. There is a choice of tr_2 which makes the following diagram G-homotopy-commutative.

$$\begin{array}{ccc} S^V \wedge S^W \wedge X_1/N & \xrightarrow{S^W tr_1} & S^V \wedge S^W \wedge X_1 \\ \Big\updownarrow & \nearrow_{tr_2} & \\ S^V \wedge \dfrac{S^W \wedge X_1}{N} & & \end{array}$$

As in (5.6) (ii), the G-map $S^W tr_1$ is obtained by suspending tr_1 according to the inclusion $V \longrightarrow V \oplus W$.

<u>Proof</u>. Let us decompose S^W into the hemisphere $E(W)_o$ given by $|| \, w \, || \leq 1$ and the hemisphere $E(W)_\infty$ given by $|| \, w \, || \geq 1$. Let us choose a real-valued function $\eta(w)$ on S^W which is continuous, G-invariant, 0 at ∞ and positive elsewhere, and 1 on $E(W)_o$; for example, we may take

$$\eta(w) = \frac{1}{|| \, w \, ||} \quad \text{on} \quad E(W)_\infty \ .$$

Then we can construct an embedding

$$\begin{array}{ccc} X_2 & \xrightarrow{(e_2, q_2)} & V \times (X_2/N) \\ & {}_{q_2}\searrow \quad \swarrow_{\pi} & \\ & X_2/N & \end{array}$$

for $X_2 = S^W \wedge X$, by taking

$$e_2 \, (w, \, x) = \eta(w) \, e_1(x) \ .$$

The map tr_2 is given by the corresponding collapsing map, and may be described as follows. For each point $w \in E(W)_o$ we get a copy of

$$tr_1 : S^V \wedge (X_1/N) \longrightarrow S^V \wedge X_1 .$$

Over $E(W)_\infty$ we get some map

$$S^V \wedge E(W)_\infty \wedge (X_1/N) \longrightarrow S^V \wedge E(W)_\infty \wedge X_1 ,$$

but these spaces are G-contractible and so it does not matter what the map is; our map tr_2 is G-homotopic to $S^W tr_1$. This proves Lemma 5.8.

<u>Corollary 5.9</u>. tr_X is natural for G-stable maps of X .

Given (5.5), this follows formally from (5.7) and (5.8). The argument is the same as that for γ in the proof of (5.2).

We now wish to know how tr_X behaves when X is an N-free G-sphere $(G/H \sqcup P) \wedge S^n$. The condition for this G-sphere to be N-free is $N \cap H = 1$; that is, H maps isomorphically to a subgroup $\overline{H}$ of $\overline{G}$. With $X = (G/H \sqcup P) \wedge S^n$ we have $X/N = (\overline{G}/\overline{H} \sqcup P) \wedge S^n$.

We first consider the case $n = 0$. We choose an embedding

$$\begin{array}{ccc} X = G/H \sqcup P & \longrightarrow & V \times (\overline{G}/\overline{H} \sqcup P) \\ & \searrow \quad \swarrow & \\ & \overline{G}/\overline{H} \sqcup P & \end{array}$$

from which to construct tr_X . Since X is discrete, the tubular neighbourhood will consist of a set of discs centred at the points of G/H .

<u>Lemma 5.10</u>. If α_1 and α_2 are as in the proof of (5.2) then the diagram

$$\begin{array}{ccc} S^V \wedge (\overline{G}/\overline{H} \sqcup P) & \xrightarrow{tr_X} & S^V \wedge (G/H \sqcup P) \\ & {}_{S^V\alpha_1}\searrow \quad \swarrow_{S^V\alpha_2} & \\ & S^V & \end{array}$$

is H-homotopy commutative.

Recall that α_1 is only an $\bar{H}$-map and α_2 is only an H-map.

Proof. tr_X is given by the usual collapsing map. To apply $S^V\alpha_2$, we must change our map to the base-point on all discs except that centred on the coset H/H . The result maps $S^V \times \bar{g}\bar{H}$ to the base-point unless $\bar{g}\bar{H}$ is the coset $\bar{H}$; then we get a map of S^V which is H-homotopic to the identity. After this homotopy we reach $S^V\alpha_1$.

Corollary 5.11. If $X = (G/H \sqcup P) \wedge S^n$ then the diagram

$$\begin{array}{ccc} S^V \wedge (\bar{G}/\bar{H} \sqcup P) \wedge S^n & \xrightarrow{tr_X} & S^V \wedge (G/H \sqcup P) \wedge S^n \\ & {\scriptstyle S^V\alpha_1} \searrow \quad \swarrow {\scriptstyle S^V\alpha_2} & \\ & S^V \wedge S^n & \end{array}$$

is H-homotopy-commutative.

Proof. Apply the trivial suspension S^n to (5.10).

Corollary 5.12. The natural transformation

$$\{Y,\ X/N\}^{\bar{G}} \longrightarrow \{Y,\ X\}^G$$

induced by tr_X is iso when $X = (G/H \sqcup P) \wedge S^n$.

Proof. Consider the following diagram.

$$\begin{array}{ccc}
\{Y, (\bar{G}/\bar{H} \sqcup P) \wedge S^n\}^{\bar{G}} & \longrightarrow & \{Y, (G/H \sqcup P) \wedge S^n\}^G \\
\updownarrow \scriptstyle(5.2) & & \updownarrow \scriptstyle(5.2) \\
\{Y, S^n\}^{\bar{H}} & \longleftrightarrow & \{Y, S^n\}^H
\end{array}$$

Here the vertical arrows come from Theorem 5.2, and the lower horizontal isomorphism comes because $\bar{H}$ is isomorphic to H. Corollary 5.11 shows that the diagram is commutative, and the result follows.

Corollary 5.13. The natural transformation

$$\{Y, X/N\}^{\bar{G}} \longrightarrow \{Y, X\}^G$$

induced by tr_X is iso whenever X can be built up by the successive attachment of cones on G-spheres $(G/H \sqcup P) \wedge S^n$ with $N \cap H = 1$.

This follows from (5.12) by an obvious induction over the number of cones, using the five lemma.

Unfortunately, not every finite G-CW-complex can be built up by the successive attachment of cones on G-spheres (think of a finite approximation to $EG \sqcup P$); this is one of the well-known snags of the subject. However, a single trivial suspension S^1 is sufficient to turn any finite G-CW-complex into one which can be constructed in this way. Thus the natural tranformation

$$\{Y \wedge S^1, \overline{\frac{X \wedge S^1}{N}}\}^{G} \longrightarrow \{Y \wedge S^1, X \wedge S^1\}^G$$

is iso. However, the whole result commutes with S^1, so this proves (5.4).

§6. Groups graded over the representation ring RO(G). In this section we will consider the question of theories graded over RO(G).

Lately I've noticed authors writing sentences of the following general form. "Write $\alpha \in RO(G)$ in the form $\alpha = V-W$; then we define

$$\{X, Y\}_{\alpha}^{G} = \{S^V \wedge X, S^W \wedge Y\}^G \text{''}.$$

If you catch anyone writing a sentence like that, make a note that you do not trust his critical faculties. The sentence in quotes is not sufficient. It implies that it is possible to verify that the result obtained depends only on α and does not depend on the choice of V and W; but it is not possible to verify this. In fact, suppose that α is the class of a representation, and that at one point the author wishes to use a representation V, and suppose (as is likely) that at another point he wishes to use a different but isomorphic representation V'. Then he must choose an isomorphism $V \cong V'$ to use in identifying $\{S^V \wedge X, Y\}^G$ with $\{S^{V'} \wedge X, Y\}^G$; and he must say which, for if he chooses a different one it will change his identification by an invertible element of the coefficient ring $\{S^0, S^0\}^G$. If he doesn't say which, then he doesn't know what he is doing and nor do we.

I will list three options suggested for overcoming this difficulty.

(i) Retreat to a notation which displays V and W explicitly.

(ii) Follow the classical precedent. A graded group such as $\pi_n(X)$ is not defined by allowing the use of any old vector space of dimension n; it is defined by using the specific space R^n which is under our control. This suggestion, then, involves an initial choice of preferred representatives. Presumably one begins by choosing one specific irreducible representation in each isomorphism class of irreducible representations.

(iii) "It may appear that $\{X,Y\}_*^G$ is intended to be a function which assigns to each $\alpha \in RO(G)$ a group $\{X,Y\}_\alpha^G$. Indeed, for purposes of planning strategy I like to think of it that way, and I hope you will do the same. But for purposes of rigourous proof, I suggest that $\{X,Y\}_*^G$ is a functor, which assigns a group to each object of some godawful category, and assigns to each morphism in that category a different way of identifying the groups in question".

The merit of (i) is that it is manifestly honest. The drawback is that it does not succeed in justifying notation such as $\{X,Y\}_\alpha^G$, which might be convenient.

The drawback of (ii) is that it may involve unattractive technicalities. Nevertheless, this is probably the best way if anyone seriously needs notation graded over $RO(G)$.

Mathematically, (iii) is indistinguishable from (i). Linguistically, notation with very strong associations, which are totally different from its declared logical meaning, is misleading notation. I suggest we should use misleading notation only when we wish to mislead, for example, on April 1st. Since mathematicians do not normally intend to deceive, misleading notation is especially dangerous to authors capable of self-deception.

Now I will turn to the published record. Bredon's work [4,6] involves homotopy groups which clearly need to be indexed over RO(G) for $G = Z_2$, and it is rigourous by option (ii) because it starts from the two actions of Z_2 on the reals. Notation graded over RO(G) , and the difficulty above, goes back to [23 p60]. Those who read German already possessed the means of implementing option (ii), because the work of tom Dieck [9,10] explicitly says that you choose actual representations, not isomorphism classes. When those who read German started to want to use notation graded over RO(G) , they remembered this [25 p373]; but they forgot it as soon as they could.

Next I point out that questions may arise which need checking from the definitions. I thank J.P. May for drawing my attention to the point which follows.

Authors who write about generalised cohomology theories commonly assume that for each $\alpha \in RO(G)$ and each X there is given a group $\tilde{H}^{\alpha}(X)$. (So, whatever else they are doing, they are not following option (iii).) Such a cohomology theory should come provided with suspension isomorphisms

$$\sigma^V : \tilde{H}^{\alpha}(X) \longrightarrow \tilde{H}^{\alpha+[V]}(S^V \wedge X)$$

where [V] is the class of V in RO(G) . Clearly, $\alpha+[V]+[W]$ is logically the same element of RO(G) as $\alpha+[W]+[V]$, and apart from an axiom saying $\sigma^V\sigma^W = \sigma^{V\oplus W}$, we need an axiom about a diagram of the following form.

$$\begin{array}{ccc}
\tilde{H}^{\alpha}(X) & \xrightarrow{\sigma^V} & \tilde{H}^{\alpha+[V]}(S^V \wedge X) \\
 & & \downarrow \sigma^W \\
\sigma^W \downarrow & & \tilde{H}^{\alpha+[V]+[W]}(S^W \wedge S^V \wedge X) \\
 & & \downarrow (\tau \wedge 1)_* \\
\tilde{H}^{\alpha+[W]}(S^W \wedge X) & \xrightarrow{\sigma^V} & \tilde{H}^{\alpha+[W]+[V]}(S^V \wedge S^W \wedge X)
\end{array}$$

Here $S^W \wedge S^V \xrightarrow{\tau} S^V \wedge S^W$ is of course the switch map. In the ordinary case this diagram only commutes up to a sign $(-1)^{pq}$; in the equivariant case, it has to commute up to an invertible element of the coefficient ring $\{S^0, S^0\}^G$, and we must be told which. For example, if $V = W$, then the composite $\sigma^W\sigma^V$ is logically the same as $\sigma^V\sigma^W$, and the required element is the class of τ . Now τ can be replaced by the map

$$(+1) \oplus (-1): V \oplus V \longrightarrow V \oplus V .$$

Let us write $\varepsilon(V) \in \{S^0, S^0\}^G$ for the element represented by

$$(-1): V \cup (\infty) \longrightarrow V \cup (\infty) ;$$

then the answer in this case must be $\varepsilon(V)$. It can be shown by example that this element may be different both from $+1$ and from -1 (take $G = Z_2$ and take V to be the non-trivial action on the reals).

If any author on this subject had wished to inspire confidence, he should have faced this problem and not tried to skirt it. The answer I would like to see is

$$\prod_{\rho} \varepsilon(\rho)^{<\rho,V> <\rho,W>};$$

here the product runs over irreducibles ρ , and $<\rho,V>$, $<\rho,W>$ are the multiplicities of ρ in V,W respectively.

Of course, the correctness of such an answer, for some well-defined function $\tilde{H}^\alpha$ of $\alpha \in RO(G)$, can only be proved by checking from the definition. However, the inconsistency of certain other answers can be proved without.

I now invite the reader to try to audit works such as [ ] and [ ], and try to determine whether their statements are checked from definitions. In my opinion, uncritical use of $RO(G)$-gradings is likely to lead to treatments which cannot be accepted as satisfactory.

The relevance of all this is as follows. Carlsson's preprint [8] uses groups graded over $RO(G)$. It is suggested by Caruso and May that it might be profitable to rewrite more of Carlsson's proof as an exercise in RO(G)-graded generalised homology and cohomology. Of course, Caruso and May provide a rigourous foundation for the small use of RO(G)-grading they propose. However, we must also consider RO(G)-gradings elsewhere in the subject.

At this point I should perhaps point out one other thing well known to the experts, as follows. This is going to be a splendid subject, but we need to cure it of a certain tendency to minor sloppiness.

Question 6.1. Hey! Wouldn't it be better to deal with that in private?

Answer. I did try, but things seem to have gone too far. Only the other day one of my graduate students brought me his work, and when I checked the main reference, I found it was open to the objections I have explained; and this was from a source I had not previously regarded as suspect. (I wouldn't mind if the only results affected were either (a) so easy that anyone can prove them correctly, or (b) so dull that nobody would ever quote them. But as a defence of mathematical work, "de minimis non curat lex" is less popular than it might be.)

Now, I earnestly desire that if there are going to be theorems in this subject, then this subject should fall in with the rest of topology, and get itself written so that innocent graduate students can tell, without extravagantly much work, what is rigourously proved and what is not. It was so in 1972, why not now?

Question 6.2. But surely anyone can make a mistake?

Answer. Yes, of course, anyone can make a mistake. And anyone can put it right, by publishing some correction or addition to his work. But you want to do it before twenty other people have followed you into the same pitfall. After seven years, the way things move now, your paper is up for the judgement of history.

Question 6.3. But isn't it dangerous to make such sweeping generalizations? You will have all manner of upright citizens pressing you to publish the statement that you intended no slur on their care, rigour or professional standards.

Answer. (a) Everyone knows I don't mean [13] or [18]. The proportion of papers in this subject which are wholly satisfactory is well above the proportion of righteous men for which the Lord would have spared Sodom [30]. (b) I've tried lectures which don't name names and I've tried drafts which do name names, and nothing will please everyone.

I have consulted older and wiser men, and I am moved to preach a sermon to this subject. So, if such of my friends as have favourite pieces of minor sloppiness will please put them down and walk quietly away from them, I will begin.

I earnestly desire that people should not copy out of previous papers without pausing to think whether the passages to be copied make sense. And when we write a sentence which implies that one checks A and B, then we shall take scrap paper and check A and B - from the definitions. And for those of us who have the care of graduate students, I recommend that we give them critical faculties first and their Ph.D.'s afterwards. Here ends my sermon.

§7. G-spectra. In the classical case, the advantage of doing stable homotopy theory in a category of spectra are by now well understood. In this section we will consider very briefly the corresponding equivariant theory.

G-spectra were introduced by tom Dieck in [9,10]; the published account, [12], is less explicit. However, tom Dieck introduced G-spectra merely in order to obtain the associated generalised cohomology theories; he did not treat them as a category in which to do equivariant stable homotopy theory.

A good category of G-spectra exists [18,19]. "Just as the non-equivariant stable category is "Boardman's category", and is still Boardman's category no matter whose construction one actually quotes, so the equivariant stable category is that of Lewis and May". One can have confidence that the work of these authors will be careful, accurate and reliable, and we may hope that it will appear soon.

I need to draw attention to only one snag, and to explain it I must make some preliminary remarks.

For G-spaces X we have fixed-point subspaces X^H and we know what they do under suspension; we have

$$(S^V \wedge X)^H = (S^{V^H}) \wedge X^H .$$

Therefore, passage to fixed-point subspaces defines a functor, say T, from the G-Spanier-Whitehead category of §4 to the $N(H)/H$-Spanier-Whitehead category, where $N(H)$ is the normaliser of H in G.

Any good category of G-spectra must contain the G-Spanier-Whitehead category embedded in it as a full subcategory. In particular, the category of Lewis and May does so. Similarly, the category of N(H)/H-spectra must contain the $N(H)/H$-Spanier-Whitehead category.

We can now consider the following conditions on a hypothetical functor U from G-spectra (whatever they are) to $N(H)/H$-spectra (whatever they are).

(7.1). U extends the functor T defined above.

(7.2). U permits one to carry over to spectra the result (2.6) for spaces, say in the form of a (1-1) correspondence

$$\{(G/H \sqcup P) \wedge S^n,\ Y\}^G \longrightarrow \pi_n(U(Y))$$

(whatever π_n is).

The snag is that these two conditions are inconsistent; you cannot have both and so you must choose.

Lewis and May attach great importance to (7.2). "Since the reduction of equivariant problems to non-equivariant ones by passage to fixed-point spaces is probably the most basic tool in equivariant homotopy theory, it is clearly desirable" to carry over that tool from spaces to spectra. I freely concede the great mathematical interest of the objects $U(Y)$ which Lewis and May construct and call fixed-point spectra. Lewis and May argue further that, to avoid confusion, it would be highly undesirable for anyone to try to attach the name "fixed-points" to a functor U satisfying (7.1).

The relevance of this is as follows. Carlsson, in his preprint [8] p9, says that he will work in a category of G-spectra, and specifically in the category of [18]. If so, then by [8] p44 he wants a functor U with the property (7.1) and he has little or no interest in (7.2). Now, this seems to me a most reasonable request; I see no reason on earth why Carlsson should not have a functor with the property (7.1), and in the first draft of this section I constructed him one.

The reason I have cut this section since the first draft is that it now appears that most of Carlsson's proof can be done without G-spectra.

§8. Equivariant S-duality. In this section we will study the equivariant analogue of ordinary Spanier-Whitehead duality.

In the classical case, there are two standard approaches. In the first, which was historically prior, you suppose given a finite complex X. You choose a good embedding of X in the sphere S^{n+1}, and the complement gives the S-dual of X, up to a shift of n dimensions. In the second, one works not with embeddings, but with structure maps

$X^* \wedge X \longrightarrow S^n$. The standard reference is to Spanier's exercises [24 pp 462-463]. Both approaches carry over to the equivariant case. Of course, in the first, you embed in a sphere with G-action, and in the second, you map to a sphere with G-action. Both approaches have all the good properties a reasonable man would expect.

The standard reference is to Wirthmüller [26], who implements the second approach. [26] was a paper worth writing properly. It is written in RO(G)-graded notation; and at the time it was written, there was no adequate rigourization of RO(G)-graded notation in print so far as I know. Wirthmüller might have written one; alternatively, he might have used different notation. If he had done either, [26] could have been a splendid paper. As it is, I report that it can clearly be rewritten so as to become completely satisfactory.

For the embedding method, I thank J.P. May for recommending a reference to Section 3 of [27].

I will begin by summarising some basic material on G-S-duality.

Question 8.1. We need to begin with the duals of cells and spheres. What is the G-S-dual of $(G/H) \sqcup P$?

Answer. It is $(G/H) \sqcup P$ again. For example, you can embed $(G/H) \sqcup P$ in the sphere corresponding to the permutation representation of G on the elements of G/H ; then the complement is (up to G-equivalence) a wedge, indexed by G/H , of copies of the reduced permutation representation.

If you wish to avoid the embedding method, I suggest that you rely on (5.1) and (5.2) for the following natural (1-1) correspondences.

$$\begin{aligned} &\{(G/H \sqcup P) \wedge X, Y\}^G \\ &\qquad \longleftrightarrow \{X, Y\}^H \\ &\qquad \longleftrightarrow \{X, (G/H \sqcup P) \wedge Y\}^G . \end{aligned}$$

One has to remember that this answer needs modification when G is a compact Lie group. See [26 p 428].

Question 8.2. Should we expect G-S-duality to have good behaviour on cofiberings?

Answer. Yes. Of course, both query and answer beg the question of what we mean by "good behaviour", but we mean, "the same behaviour as for $G = 1$ ".

With the method of embedding in spheres, it is almost clear that the dual of a Mayer-Vietoris diagram

$$\begin{array}{ccccc} & & X & & \\ & \nearrow & & \searrow & \\ X \cap Y & & & & X \cup Y \\ & \searrow & & \nearrow & \\ & & Y & & \end{array}$$

is another Mayer-Vietoris diagram

$$\begin{array}{ccccc} & & X^* & & \\ & \swarrow & & \nwarrow & \\ X^* \cup Y^* & & & & X^* \cap Y^* \\ & \nwarrow & & \swarrow & \\ & & Y^* & & \end{array} .$$

Now take Y (and therefore Y^*) stably G-contractible; we see that the dual of a cofibre diagram

$$A \xrightarrow{f} X \xrightarrow{i} X \cup_f CA = B$$

is another cofibre diagram

$$X^* \cup_{i^*} CB^* \xleftarrow{f^*} X^* \xleftarrow{i^*} B^* .$$

With the Spanier approach, it's one of the lemmas which have to be proved before the method works. See [26 p 429].

I thank J.P. May for pointing out that if you want a cofibering to include all three relevant maps, then on a suitably precise definition, the statement "the S-dual of a cofibering is a cofibering" is actually false if you worry about signs - there is a sign which won't go away. But this is just the same as in the classical case.

Question 8.3. Should we expect G-S-duality to have good behaviour under the forgetful functor i^*, under j^* and under passage to fixed-point sets?

Answer. Yes. Suppose given a homomorphism $\theta: G_1 \longrightarrow G$, and suppose you can embed G-spaces X, X^* in a G-sphere S. Then you can apply θ^* and regard them as G_1-spaces θ^*X, θ^*X^* embedded in the G_1-sphere θ^*S. Similarly, you can pass to fixed-point sets and obtain X^H, $(X^*)^H$ embedded in S^H.

With the Spanier approach, you start from a structure map $X^* \wedge X \longrightarrow S$ and you can again apply θ^* or pass to fixed-point sets. See [26 p 427, p 431].

We turn to more interesting results. First suppose that X is a finite G-CW-complex which is free (away from the base-point) over a normal subgroup $N \subset G$.

Theorem 8.4. Then X admits a G-S-dual $D_G X$ with the following properties.

(i) $D_G X$ is also N-free (away from the base-point).

(ii) The duality is with respect to a "dimension" which is a representation of G/N.

Proof. We first avoid the standard snag mentioned at the end of §5 by the same device used there; by passing to $X \wedge S^1$ if necessary, we can assume that X is constructed by the successive attachment of cones on G-spheres $(G/H \sqcup P) \wedge S^n$ with $N \cap H = 1$.

We now apply (8.2). By (8.1) the dual of $(G/H \sqcup P) \wedge S^n$ with respect to dimension n is $(G/H \sqcup P)$. By (5.5), all the stable attaching maps required to build up $D_G X$ can be realised by maps of spaces at the price of suspension S^{j^*W} .

Let X and $D_G X$ be as in (8.4), that is, N-free and G-S-dual with respect to a dimension j^*W .

Theorem 8.5. Then X/N and $(D_G X)/N$ are G/N-dual with respect to dimension W .

It seems that this was in doubt until recently. I owe the case $N = G$, $G/N = 1$ to letters from L.G. Lewis and J.P. May.

Proof. Let $\bar{G} = G/N$ and let Y run over the $\bar{G}$-Spanier-Whitehead category. Then we have the following (1-1) correspondences natural in Y .

$$\{Y \wedge X/N, S^W\}^{\bar{G}}$$

$$\longleftrightarrow \{j^*Y \wedge X, S^{j^*W}\}^G \qquad (5.3)$$

$$\longleftrightarrow \{j^*Y, D_G X\}^G \qquad \text{(G-S-duality)}$$

$$\longleftrightarrow \{Y, \frac{D_G X}{N}\}^{\bar{G}} \qquad (5.4)$$

This characterises $\frac{D_G X}{N}$ as the $\bar{G}$-S-dual of X/N with respect to dimension W .

I next recall that A. Ranicki [21] has given an "unconventional" treatment of G-S-duality, in which the group G need not be finite, but the G-complexes must be G-free (away from the base-point).

<u>Theorem 8.6</u>. If G is finite then the Ranicki dual of a G-free space X agrees with the conventional one.

Proof. Let Y run over the G-Spanier-Whitehead category; Y need not be G-free. Let X^* be a Ranicki n-dual of X. In view of (5.5), the Ranicki n-dual X^* is characterised by the first of the following two (1-1) correspondences which are natural in Y.

$$\{X^*, Y\}^G$$

$$\longleftrightarrow \{S^n, \frac{X \wedge Y}{G}^{G/G}\} \qquad \text{(Ranicki duality)}$$

$$\longleftrightarrow \{S^n, X \wedge Y\}^G \qquad (5.4)\ .$$

But this characterises X^* and X as conventional n-duals.

One of my correspondents suggests that the results presented above make it unnecessary for me (or Carlsson) to mention Ranicki duality. I take this point.

References.

[1] J.F. Adams, Stable homotopy and generalised homology, Chicago University Press 1974. MR 53 no. 6534.

[2] M.F. Atiyah and G.B. Segal, Equivariant K-theory and completion, Jour. of Differential Geometry 3 (1969) 1-18. MR 41 no. 4575.

[3] G.E. Bredon, Equivariant cohomology theories, Bull. Amer. Math. Soc. 73 (1967) 266-268. MR 34 no. 6762.

[4] G.E. Bredon, Equivariant stable stems, Bull. Amer. Math. Soc. 73 (1967) 269-273. MR 34 no. 6763.

[5] G.E. Bredon, Equivariant cohomology theories, Lecture Notes in Mathematics no. 34, Springer 1967. MR 35 no. 4914.

[6] G.E. Bredon, Equivariant homotopy, in Proceedings of the Conference on Transformation Groups, Springer 1968, pp 281-292. MR 40 no. 3542.

[7] T. Brocker and E.C. Hook, Stable equivariant bordism, Math. Zeit. 129 (1972) 269-277. MR 47 no. 9652.

[8] G. Carlsson, Equivariant stable homotopy and Segal's Burnside ring conjecture, preprint, University of California at San Diego, 1982.

[9] T. tom Dieck, Bordism of G-manifolds and integrality theorems, Preprint series 1968/69 no. 20, Aarhus University 1968.

[10] T. tom Dieck, Kobordismen-Theorie und Transformationgruppen, Preprint series 1968/69 no. 30, Aarhus University 1969.

[11] T. tom Dieck, Glättung äquivarianter Homotopiemengen, Arch. Math. 20 (1969) 288-295. MR 42 no. 2474.

[12] T. tom Dieck, Bordism of G-manifolds and integrality theorems, Topology 9 (1970) 345-358; MR 42 no. 1148.

[13] T. tom Dieck, Transformation groups and representation theory, Lecture Notes in Mathematics no. 766, Springer 1979. MR 82c 57025.

[14] H. Hauschild, Äquivariante Homotopie I, Arch. Math. 29 (1977) 158-165. MR 57 no. 7626.

[15] H. Hauschild, Review of "Equivariant stable homotopy theory", Mathematical Reviews 54 (1977) no. 11319 p1603.

[16] S. Illman, Equivariant algebraic topology, Thesis, Princeton 1972.

[17] C. Kosniowski, Equivariant cohomology and stable cohomotopy, Math. Ann. 210 (1974) 83-104. MR 54 no. 1202.

[18] L.G. Lewis and J.P. May, Equivariant Homotopy Theory, in preparation.

[19] L.G. Lewis, J.P. May and J. McClure, Ordinary RO(G)-graded cohomology, Bull. Amer. Math. Soc. 4 (1981) 208-212. MR 82 e 55008.

[20] T. Matumoto, On G-CW-complexes and a theorem of J.H.C. Whitehead, Jour. Fac. Sci. Univ. Tokyo 18 (1971) 363-374. MR 49 no. 9842.

[21] A. Ranicki, The algebraic theory of surgery II. Applications to topology, Proc. London Math. Soc. 40 (1980) 193-283. MR 82 f 57024b.

[22] G.B. Segal, Review of "Equivariant homotopy", Mathematical Reviews 40 (1970) no. 3542.

[23] G.B. Segal, Equivariant stable homotopy theory, in Proceedings of the International Congress of Mathematicians 1970, Gauthier-Villars 1971, vol. 2, 59-63. MR 54 no. 11319.

[24] E.H. Spanier, Algebraic Topology, McGraw-Hill 1966. MR 35 no. 1007.

[25] K. Wirthmüller, Equivariant homology and duality, Manuscripta Math. 11 (1974) 373-390. MR 49 no. 8004.

[26] K. Wirthmüller, Equivariant S-duality, Arch. Math. 26 (1975) 427-431. MR 51 no. 11493.

[27] A. Dold and D. Puppe, Duality, trace and transfer, in Proceedings of the International Conference on Geometric Topology, Polish Scientific Publishers, Warsaw 1980, pp 81-102.

[28] U. Namboodiri, Equivariant vector fields on spheres, Thesis, Chicago 1982.

[29] U. Namboodiri, Equivariant vector fields on spheres, Trans. Amer. Math. Soc., to appear.

[30] Genesis chapter 18 verse 32.