

**THE  $H\mathbb{F}_2$ -HOMOLOGY OF  $C_2$ -EQUIVARIANT  
EILENBERG-MACLANE SPACES**

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ABSTRACT. We extend Ravenel-Wilson Hopf ring techniques to  $C_2$ -equivariant homotopy theory. Our main application and motivation is a computation of the  $RO(C_2)$ -graded homology of  $C_2$ -equivariant Eilenberg-MacLane spaces. The result we obtain for some  $C_2$ -equivariant Eilenberg-MacLane spaces associated to the constant Mackey functor  $\mathbb{F}_2$  gives  $C_2$ -equivariant analogues of the classical computation due to Serre. We also investigate a twisted bar spectral sequence computing the homology of these equivariant Eilenberg-MacLane spaces and suggest the existence of another twisted bar spectral sequence with  $E^2$ -page given in terms of a twisted Tor functor.

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1. INTRODUCTION

Computations of invariants in equivariant homotopy theory have powerful applications contributing to solutions of outstanding classification problems in geometry, topology, and algebra. A primary example is Hill, Hopkins, and Ravenel's solution of the Kervaire invariant one problem [HHR16], which used computations in equivariant homotopy theory to answer the question of when a framed  $4k + 2$  dimensional manifold can be surgically converted into a sphere. Despite the success of numerous applications, many equivariant computations remain difficult to access

due to their rich structure. This is especially true for (unstable) equivariant spaces, for which many computations have not yet been completed, despite their analogous nonequivariant results being well known.

This paper extends Ravenel-Wilson Hopf ring techniques [RW77, RW80, Wil82] to  $C_2$ -equivariant homotopy theory. Our main application and motivation is to produce some computations of the  $RO(C_2)$ -graded homology of  $C_2$ -equivariant Eilenberg-MacLane spaces. The result (Theorem 5.2, Theorem 5.5) is some  $C_2$ -equivariant analogues of the classical cohomology computation completed by Serre in the 1950s [Ser53].

Non-equivariantly, Serre applied the Borel theorem [MT68, page 88, Theorem 1] to the path space fibration

$$K(\mathbb{F}_p, n) \simeq \Omega K(\mathbb{F}_p, n+1) \rightarrow P(K(\mathbb{F}_p, n+1)) \rightarrow K(\mathbb{F}_p, n+1)$$

to calculate the cohomology of  $K(\mathbb{F}_p, n+1)$  given  $H^*K(\mathbb{F}_p, n)$ . In  $C_2$ -equivariant homotopy theory, the constant Mackey functor  $\underline{\mathbb{F}}_2$  is the analogue of the group  $\mathbb{F}_2$  and the Eilenberg-MacLane spaces  $K_V = K(\underline{\mathbb{F}}_2, V)$  are graded on the real representations  $V$  of the group  $C_2$  rather than on the integers. Since the group  $C_2$  has two irreducible real representations, the trivial representation and the sign representation  $\sigma$ , the analogous equivariant computation would include computing the cohomology of  $K_{V+\sigma}$  from  $H^*K_V$  in addition to  $H^*K_{V+1}$  from  $H^*K_V$ . This would require a so called signed or twisted version of the Borel theorem. However, no such theorem is known to exist, making it difficult to study the spaces  $K_{V+\sigma}$  with these techniques. We call  $K_{V+\sigma}$  a signed delooping of  $K_V$  since the space of signed loops  $\Omega^\sigma K_{V+\sigma} \simeq K_V$ .

While direct extension of Serre's original argument does not allow for the computation of the cohomology of signed deloopings, it has been successfully applied to study trivial representation deloopings of  $K_\sigma$ , whose cohomology is known [HK01]. This approach is described in Ugur Yigit's thesis [Yig19], where it is noted that the  $RO(C_2)$ -graded cohomology of all  $C_2$ -equivariant Eilenberg-MacLane spaces  $K_{\sigma+*}$  can be computed using this method. Throughout, we use  $*$  to denote integer grading and reserve  $\star$  to denote grading by finite dimensional real representations.

A major reason to study Ravenel-Wilson Hopf ring techniques in  $C_2$ -equivariant homotopy theory is that they provide a way to study  $\sigma$ -deloopings. In particular, these techniques, which investigate multiplicative structures coming from  $H$ -space maps on spaces having a graded multiplication, lend additional structure that can be exploited to complete computations.

An important tool in classical applications of Ravenel-Wilson Hopf ring techniques is the classical bar construction  $B$ . This construction is significant because  $B$  is a trivial representation delooping functor with  $BK_V \simeq K_{V+1}$ . In the  $C_2$ -equivariant world, there is a twisted bar construction  $B^\sigma$ , which is a sign representation delooping functor with  $B^\sigma K_V \simeq K_{V+\sigma}$  [Liu20]. We use these two constructions to explicitly model multiplicative structures on the spaces  $K_V$  at the point set level, directly extending work by Ravenel and Wilson [RW80]. We describe the ensuing multiplicative structures on the  $C_2$ -equivariant Eilenberg-MacLane spaces  $K_V$

(Theorem 5.1) as well as our approach to using this structure to investigate signed and trivial representation deloopings (Theorem 5.2, Theorem 5.5) in Section 5.

Whereas Ravenel and Wilson use a collapsing integer-graded bar spectral sequence to inductively compute the homology of classical nonequivariant Eilenberg-MacLane spaces [Wil82], we deduce our equivariant computations from non-equivariant ones using a computational method introduced by Behrens-Wilson (Lemma 2.8 [BW18]). In particular, we use a graded multiplication on the spaces  $K_V$ , that is, Hopf ring structure coming from the  $\Omega$ -spectrum structure on  $H\mathbb{F}_2$ , to write down elements of the  $RO(C_2)$ -graded homology of signed Eilenberg-MacLane spaces  $K_{*\sigma}$  that are proposed as generators. We then use our point set level understanding of multiplicative structures on the spaces  $K_{*\sigma}$  coming from Theorem 5.1 to verify that these proposed generators in fact form a free basis for the homology.

Once we have computed  $H_*K_{*\sigma}$  (Theorem 5.2), we use  $RO(C_2)$ -graded bar spectral sequences to inductively compute  $H_*K_{\sigma+*}$  (Theorem 5.5) from  $H_*K_\sigma$ . In particular, we identify permanent cycles on the  $E^2$ -pages of these spectral sequences in terms of the Hopf ring structure and then use the Behrens-Wilson computational lemma to confirm that the identified circle products do in fact form a basis for homology. This approach more closely follows that of Ravenel-Wilson while also utilizing the equivariant computational tools of Behrens-Wilson, reducing much of the computation to understanding the nonequivariant case. We describe this in Section 5 and study the  $RO(C_2)$ -graded bar spectral sequence in more detail in Section 6.

One might expect a computation of the  $RO(C_2)$ -graded homology of the remaining equivariant Eilenberg-MacLane spaces  $K_V$  where  $2\sigma + 1 \subset V$  to follow in the same way, that is by iteratively applying  $RO(C_2)$ -graded bar spectral sequences to the  $RO(C_2)$ -graded homology of  $K_{n\sigma}$ ,  $n \geq 2$ . Instead, we discover a complication in this method of proof. Specifically, in contrast to the nonequivariant classical case, it is not possible to identify all cycles on the  $E^2$ -page with known permanent cycles. This complication is discussed in terms of the Hopf ring structure in Section 5 and in terms of differentials in the  $RO(C_2)$ -graded bar spectral sequence in Section 6.

One possible reason the computation has proven more difficult and is still in progress is that unlike in the classical case, there is a relation in the  $C_2$ -equivariant dual Steenrod algebra

$$\mathcal{A}_*^{C_2} = H\mathbb{F}_2[\tau_0, \tau_1, \dots, \xi_1, \xi_2, \dots] / (\tau_i^2 = (u + a\tau_0)\xi_{i+1} + a\tau_{i+1}),$$

and this must arise somewhere unstably. A conjecture concerning  $H_*K_V$  where  $2\sigma + 1 \subset V$ , as well as a potential strategy for proof are outlined in Section 7. We also describe what is known regarding how our unstable computations stabilize to give the  $C_2$ -equivariant dual Steenrod algebra in Section 7. This is fully known in nonequivariant case [Wil82].

Having amassed a large volume of information on the  $RO(C_2)$ -graded homology of the  $C_2$ -equivariant Eilenberg-MacLane spaces  $K_V$ , we turn to investigating the  $RO(C_2)$ -graded twisted bar spectral sequence and put our knowledge of the  $E^\infty$ -page in these cases to use. Much like the classical integer graded bar spectral sequence, the  $RO(C_2)$ -graded twisted bar spectral sequence arises from a filtered

complex. However, computations with this twisted spectral sequence are more complicated than in the classical case. For example, in contrast to the classical case where the integer-graded bar spectral sequence computing the non-equivariant homology of the classical Eilenberg-MacLane spaces  $K_* = K(\mathbb{F}_p, *)$  collapses on the  $E^2$ -page [Wil82], we find there are arbitrarily long equivariant degree shifting differentials, similar to those observed in Kronholm's study of the cellular spectral sequence [Kro10], in the  $RO(C_2)$ -graded twisted bar spectral sequence computing the homology of the signed representation spaces  $K_{n\sigma}$ ,  $n \geq 2$ .

While the  $RO(C_2)$ -graded twisted bar spectral sequence is quite complicated in general, the differentials and extensions appear to arise in an extremely structured way, largely governed by a norm structure. We use our knowledge of  $H_*K_{*\sigma}$  and the  $E^\infty$ -page to deduce information about these  $RO(C_2)$ -graded twisted bar spectral sequences. This allows us to write down conjectures concerning many of the differentials in Section 6. Here, our use of equivariant computations shows that, unlike the non-equivariant integer graded situation, the  $RO(C_2)$ -graded twisted bar spectral sequences computing  $H_*K_{n\sigma}$ ,  $n \geq 2$  have a rich structure quite distinct from the collapsing bar spectral sequence in the classical nonequivariant case [Wil82]. Further differences among integer graded and  $RO(C_2)$ -graded bar and twisted bar spectral sequences are discussed in Section 6.

In parallel with calculating the homology of a space, the corresponding computational tools are worth investigating in a purely algebraic setting. This study of the homological algebra involved produces tools which can also be applied in settings outside of topology. One example of this are Tor functors, the derived functors of the tensor product of modules over a ring. Besides playing a central role within algebraic topology theorems such as the Künneth Theorem and Coefficient Theorem, Tor functors can also be used to calculate the homology of groups, Lie algebras, and associative algebras. Within the context of the classical Ravenel-Wilson Hopf ring method, the identification of the  $E^2$ -page of the bar spectral sequence with Tor allows for the computations  $\text{Tor}^{E[x]}(\mathbb{F}_p, \mathbb{F}_p) \simeq \Gamma[sx]$  and  $\text{Tor}^{T[x]}(\mathbb{F}_p, \mathbb{F}_p) \simeq E[sx] \otimes \Gamma[\phi x]$ , where  $sx$  is the suspension of  $x$ ,  $\phi x$  is the transpotent, and  $T[x]$  is the truncated polynomial ring  $\mathbb{F}_p[x]/(x^p)$ , to be used inductively in the calculations of the mod  $p$  homology of Eilenberg-MacLane spaces [Wil82] and the Morava K-theory of Eilenberg-MacLane spaces [RW80].

In the  $C_2$ -equivariant setting, the  $RO(C_2)$ -graded homology of each signed delooping,  $K_{V+\sigma}$ , of an equivariant Eilenberg-MacLane space,  $K_V$ , also independently arises as the result of a  $C_2$ -equivariant twisted Tor computation. Thus under favorable circumstances, we believe it should be possible to formulate a twisted bar spectral sequence with  $E^2$ -page a twisted Tor functor arising as a derived functor of the twisted product of  $H\mathbb{F}_2$ -modules and use this to compute the  $E^2$ -page. However, we have not yet constructed such a spectral sequence.

Additionally, twisted Tor calculations are not yet well understood, with a complete lack of known examples. Theorems 5.2 and 5.5's computation of the  $RO(C_2)$ -graded homology of the equivariant Eilenberg-MacLanes spaces  $K_{*\sigma}$  and  $K_{\sigma+*}$  provide a countably infinite number of initial examples, which in turn lend insight on how such calculations might proceed in general. We discuss how the homology  $H_*K_{V+\sigma}$

arises as a result of twisted Tor and give evidence for  $Tor_{tw}^{E[x]}(H_*, H_*) \simeq E[\sigma x] \otimes \Gamma[\mathcal{N}_e^{C_2}(x)]$ , where  $\sigma x$  is the signed suspension of  $x$  and  $\mathcal{N}_e^{C_2}$  is the norm, under favorable circumstances in Section 7. We also discuss our hypothesis regarding twisted Tor and a twisted bar spectral sequence in Section 7.

**1.1. Statement of Theorems.** We state our main results. Recall that  $H\mathbb{F}_2$  has distinguished elements  $a \in H\mathbb{F}_2\{-\sigma\}$  and  $u \in H\mathbb{F}_2\{1-\sigma\}$ .

To describe our answer for  $H_*K_{*\sigma}$ , we need notation for  $H_*K_\sigma$ . Let

$$e_\sigma \in H_\sigma K_\sigma, \quad \bar{\alpha}_i \in H_{\rho i} K_\sigma, \quad (i \geq 0).$$

Then the homology,  $H_*K_\sigma$ , is exterior on generators

$$e_\sigma, \quad \bar{\alpha}_{(i)} = \bar{\alpha}_{2^i} \quad (i \geq 0)$$

with coproduct

$$\begin{aligned} \psi(e_\sigma) &= 1 \otimes e_\sigma + e_\sigma \otimes 1 + a(e_\sigma \otimes e_\sigma) \\ \psi(\bar{\alpha}_n) &= \sum_{i=0}^n \bar{\alpha}_{n-i} \otimes \bar{\alpha}_i + \sum_{i=0}^{n-1} u(e_\sigma \bar{\alpha}_{n-1-i} \otimes e_\sigma \bar{\alpha}_i). \end{aligned}$$

For finite sequences

$$J = (j_\sigma, j_0, j_1, \dots) \quad j_k \geq 0,$$

define

$$(e_\sigma \bar{\alpha})^J = e_\sigma^{\circ j_\sigma} \circ \bar{\alpha}_{(0)}^{\circ j_0} \circ \bar{\alpha}_{(1)}^{\circ j_1} \dots$$

where the  $\circ$ -product comes from the pairing  $\circ : K_V \wedge K_W \rightarrow K_{V+W}$ .

**Theorem** (Theorem 5.2). *Then*

$$H_*K_{*\sigma} \cong \otimes_J E[(e_\sigma \bar{\alpha})^J]$$

*As an algebra where the tensor product is over all  $J$  and the coproduct follows by Hopf ring properties from the  $\bar{\alpha}$ 's.*

Interestingly, this answer mirrors the classical non-equivariant answer at the prime 2 (Ravenel-Wilson: Theorem B, page 10).

From there, we use the  $RO(C_2)$ -graded bar spectral sequence to inductively compute  $H_*K_{\sigma+*}$  from  $H_*K_\sigma$ . To describe these answers, we need notation for  $H_*K_1$ ,  $H_*K_2$ , and  $H_*K_\rho$ . Let

$$e_1 \in H_1 K_1, \quad \alpha_i \in H_{2i} K_1, \quad \beta_i \in H_{2i} CP^\infty \quad i \geq 0.$$

This gives generators

$$e_1, \quad \alpha_{(i)} = \alpha_{p^i} \quad \beta_{(i)} = \beta_{p^i},$$

of  $H_*K_1$  and  $H_*K_2$  with coproducts

$$\psi(\alpha_n) = \sum_{i=0}^n \alpha_{n-i} \otimes \alpha_i, \quad \psi(\beta_n) = \sum_{i=0}^n \beta_{n-i} \otimes \beta_i.$$

Also let

$$\bar{\beta}_i \in H_{\rho i} K(\mathbb{Z}, \rho), \quad (i \geq 0).$$

This gives additional generators,

$$\bar{\beta}_{(i)} = \bar{\beta}_{2^i} \quad (i \geq 0),$$

of  $H_* K_\rho$  with coproduct

$$\psi(\bar{\beta}_n) = \sum_{i=0}^n \bar{\beta}_{n-i} \otimes \bar{\beta}_i.$$

Then for finite sequences

$$I = (i_1, i_2, \dots, i_k), \quad 0 \leq i_1 < i_2 < \dots,$$

$$W = (w_1, w_2, \dots, w_q), \quad 0 \leq w_1 < w_2 < \dots,$$

$$J = (j_{-1}, j_0, j_1, \dots, j_\ell), \quad \text{where } j_{-1} \in \{0, 1\} \text{ and all other } j_n \geq 0,$$

and

$$Y = (y_{-1}, y_0, y_1, \dots, y_r), \quad \text{where } y_{-1} \in \{0, 1\} \text{ and all other } y_n \geq 0,$$

define

$$(e_1 \alpha \beta)^{I,J} = e_1^{\circ j_{-1}} \circ \alpha_{(i_1)} \circ \alpha_{(i_2)} \circ \dots \circ \alpha_{(i_k)} \circ \beta_{(0)}^{\circ j_0} \circ \beta_{(1)}^{\circ j_1} \circ \dots \circ \beta_{(\ell)}^{\circ j_\ell},$$

$$(e_1 \alpha \beta)^{W,Y} = e_1^{\circ y_{-1}} \circ \alpha_{(w_1)} \circ \alpha_{(w_2)} \circ \dots \circ \alpha_{(w_q)} \circ \beta_{(0)}^{\circ y_0} \circ \beta_{(1)}^{\circ y_1} \circ \dots \circ \beta_{(r)}^{\circ y_r},$$

$$|I| = k, \quad |W| = q \quad ||J|| = \Sigma j_n, \quad \text{and } ||Y|| = \Sigma y_n.$$

**Theorem** (Theorem 5.5, page 20). *We have*

$$H_* K_{\sigma+i} \cong E[(e_1 \alpha \beta)^{I,J} \circ \bar{\alpha}_{(m)}, (e_1 \alpha \beta)^{W,Y} \circ \bar{\beta}_{(t)}]$$

where  $m > i_k$  and  $m \geq \ell$ ,  $t > w_q$  and  $t \geq y_r$ ,  $|I| + 2||J|| = i$  and  $|W| + 2||Y|| = i - 1$ , and the coproduct follows by Hopf ring properties from the  $\alpha_{(i)}$ 's,  $\beta_{(i)}$ 's,  $\bar{\alpha}_{(i)}$ 's and  $\bar{\beta}_{(i)}$ 's.

We observe that this equivariant answer mirrors the classical non-equivariant answer for odd primes (Ravenel-Wilson: Theorem A, page 10). For the reader's convenience, we explicitly write some low dimensional instances of the theorem. In particular, we have

$$H_* K_\rho \cong E[e_1 \circ \bar{\alpha}_{(i)}, \alpha_{(i_1)} \circ \bar{\alpha}_{(i_2)}, \bar{\beta}_{(i)}]$$

and

$$H_* K_{\sigma+2} \cong E[e_1 \circ \alpha_{(i_1)} \circ \bar{\alpha}_{(i_2)}, \alpha_{(i_1)} \circ \alpha_{(i_2)} \circ \bar{\alpha}_{(i_3)}, e_1 \circ \bar{\beta}_{(j_1)}, \beta_{(j_1)} \circ \bar{\alpha}_{(j_2)}, \alpha_{(i_1)} \circ \bar{\beta}_{(i_2)}]$$

where  $i_1 < i_2$ ,  $j_1 \leq j_2$ ; and the coproduct follows by Hopf ring properties from the  $\alpha_{(i)}$ 's,  $\beta_{(i)}$ 's,  $\bar{\alpha}_{(i)}$ 's and  $\bar{\beta}_{(i)}$ 's.

Having computed the homology of many  $C_2$ -equivariant Eilenberg-MacLane spaces  $K_V$ , we turn to using the results to investigate the twisted bar spectral sequence arising from the twisted bar construction. Unlike the non-equivariant bar spectral sequence, the twisted bar spectral sequence  $E^2$  page lacks an explicit homological description. This makes computations difficult in general. However, for the spaces  $B^\sigma \mathbb{F}_2 \simeq K_\sigma \simeq \mathbb{R}P_{tw}^\infty$ ,  $B^\sigma S^1 \simeq K(\mathbb{Z}, \rho) \simeq \mathbb{C}P_{tw}^\infty$ , and  $B^\sigma S^\sigma \simeq K(\mathbb{Z}, 2\sigma)$ , there is a gap in the spectral sequence forcing all differentials  $d^r$  for  $r > 1$  to be zero. Further for these spaces, if there were a non-zero  $d^1$  differential, we would end up killing

a known generator of the underlying non-equivariant integer graded homology and arrive at a contradiction. Thus we can calculate the additive  $RO(C_2)$ -graded homology of these spaces completely. The multiplicative structure can be deduced from the known cohomology [HK01].

**Examples** (Example 6.3, page 26). We have

$$\begin{aligned} H_*\mathbb{R}P_{tw}^\infty &= E[e_\sigma, \bar{\alpha}_{(0)}, \bar{\alpha}_{(1)}, \dots] = E[e_\sigma] \otimes \Gamma[\bar{\alpha}_{(0)}], |e_\sigma| = \sigma, |\bar{\alpha}_{(i)}| = \rho 2^i, \\ H_*\mathbb{C}P_{tw}^\infty &= E[\bar{\beta}_{(0)}, \bar{\beta}_{(1)}, \dots] = \Gamma[e_\rho] \text{ where } |\bar{\beta}_{(i)}| = \rho 2^i. \end{aligned}$$

**Theorem** (Theorem 6.1, page 27). *We have*

$$H_*K(\mathbb{Z}, 2\sigma) = E[e_{2\sigma}] \otimes \Gamma[\bar{x}_{(0)}] \text{ where } |e_{2\sigma}| = 2\sigma, |\bar{x}_{(0)}| = 2\rho.$$

In forthcoming work, we will use the homology of  $H_*K_V$  to deduce differentials in the twisted bar spectral sequence. The beginning stages of this work are described in Section 6.

**1.2. Paper structure.** This paper has two primary aims: (1) extending Ravenel-Wilson Hopf ring techniques [RW77, RW80, Wil82] to  $C_2$ -equivariant homotopy theory and (2) computing the  $RO(C_2)$ -graded homology of  $C_2$ -equivariant Eilenberg-MacLane spaces associated to the constant Mackey functor  $\mathbb{F}_2$ . These topics are investigated in several sections.

The first section consists of an introduction providing context for the main results, a description of the paper structure, and a list of notational conventions.

The second section recalls classical Ravenel-Wilson Hopf ring methods.

The third section recollects material from equivariant homotopy theory necessary for understanding our proof and computations.

The fourth section details the bar and twisted bar constructions, which are trivial and sign representation delooping functors respectively.

The fifth section applies the preliminaries of the previous sections to study multiplicative structures on  $C_2$ -equivariant Eilenberg-MacLane spaces. This section contains some primary extensions of Ravenel-Wilson Hopf ring methods to  $C_2$ -equivariant homotopy theory (Theorem 5.1). It also contains our calculation of the  $RO(C_2)$ -graded homology of many  $C_2$ -equivariant Eilenberg-MacLane spaces  $K_V$  associated to the constant Mackey functor  $\mathbb{F}_2$  (Theorem 5.2, Theorem 5.5).

The sixth section details a number of computations and observations regarding the  $RO(C_2)$ -graded bar and twisted bar spectral sequences. The examples we provide should be a useful stepping stone towards further computations.

The seventh section describes a few questions of immediate interest given the results of this paper.

**1.3. Notational conventions.**

The asterisk  $*$  denotes integer grading.

The star  $\star$  to denote representation grading.

By the classical or nonequivariant Eilenberg-MacLane space  $K_n$ , we mean the classical nonequivariant Eilenberg-MacLane space  $K_n = K(\mathbb{F}_p, n)$ , where  $p$  is prime.

$C_2$  is the cyclic group of order two with  $C_2 = \langle \gamma \rangle$ .

$\sigma$  denotes the one dimensional sign representation of  $C_2$ .

$\rho$  is the regular representation of  $C_2$ .

$S^V$  is the one point compactification of a finite dimensional real representation  $V$  where the point at infinity is given a trivial group action and taken as the base point.

$$\Sigma^V(-) = S^V \wedge -.$$

$\Omega^V(-)$  is the space of continuous based maps  $\text{Map}_*(S^V, -)$  where the group action is given by conjugation.

$\mathcal{S}$  is the category of spectra.

$\mathcal{S}^G$  is the category of  $G$ -spectra indexed on a complete universe.

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## 2. CLASSICAL RAVENEL-WILSON HOPF RING METHODS

Classically, one place Hopf rings arise in homotopy theory is in the study of  $\Omega$ -spectra. Consider an  $\Omega$ -spectrum

$$G = \{G_k\}$$

and a multiplicative homology theory  $E_*(-)$  with a Künneth isomorphism for the spaces  $G_k$ . The  $\Omega$ -spectrum  $G$  represents a generalized cohomology theory with

$$G^* X \simeq [X, G_*].$$

Since  $G^k X$  is an abelian group,  $G_k$  must be a homotopy commutative  $H$ -space (in fact  $G_k$  is an infinite loop space). This  $H$ -space structure

$$* : G_k \times G_k \rightarrow G_k$$

gives rise to a product in homology

$$* : E_* G_k \otimes E_* G_k \cong E_*(G_k \times G_k) \rightarrow E_* G_k$$

and the Künneth isomorphism implies the homology is in fact a Hopf algebra.



If  $G$  is a ring spectrum, then  $G^*X$  is a graded ring and the graded abelian group object  $G_*$  becomes a graded ring object in the homotopy category. The multiplication

$$G^k X \times G^n X \rightarrow G^{k+n} X$$

has a corresponding multiplication in  $G_*$  :

$$\circ : G_k \times G_n \rightarrow G_{k+n}$$

and applying  $E_*(-)$  we have

$$\circ : E_*G_k \otimes_{E_*} E_*G_n \rightarrow E_*G_{k+n}$$

turning  $E_*G$  into a graded ring object in the category of coalgebras.

As a ring,  $E_*G$  has a distributive law

$$(2.1) \quad x \circ (y * z) = \sum \pm (x' \circ y) * (x'' \circ z) \text{ where } \psi(x) = \sum x' \otimes x''$$

coming from the distributive law in  $G^*X$ .

Ravenel and Wilson pursued the idea that these two products could be used to construct many elements in homology from just a few. They successfully applied this approach to compute the Hopf ring for complex cobordism [RW77], the Morava K-theory of nonequivariant Eilenberg-MacLane spaces [RW80], and the mod  $p$  homology of classical Eilenberg-MacLane spaces [Wil82].

In the case of classical Eilenberg-MacLane spaces, the Eilenberg-MacLane spectrum

$$H\mathbb{F}_p = \{K(\mathbb{F}_p, n)\} = \{K_n\},$$

is a ring spectrum with  $\Omega K_{n+1} \simeq K_n$ . Further,  $H_*(-) := H_*(-; \mathbb{F}_p)$ , ordinary homology with mod  $p$  coefficients, has a Künneth isomorphism and thus the homology  $H_*K_*$  has the structure of a Hopf ring.

A key computational insight of Ravenel and Wilson was that the bar spectral sequence

$$E_{*,*}^2 \simeq \text{Tor}_{*,*}^{E_*G^k}(E_*, E_*) \Rightarrow E_*G_{k+1}$$

is in fact a spectral sequence of Hopf algebras. The additional structure of the  $\circ$  multiplication in the bar spectral sequence meant that they could inductively deduce the homology of Eilenberg-MacLane spaces using standard homological algebra. Starting with elements in  $H_*K_1$  and  $H_*\mathbb{C}P^\infty$  and identifying circle products in the bar spectral sequence, Ravenel and Wilson computed the Hopf ring associated to the mod  $p$  Eilenberg-MacLane spectrum [Wil82].

To describe their answer, let

$$e_1 \in H_1K_1, \quad \alpha_i \in H_{2i}K_1, \quad \beta_i \in H_{2i}\mathbb{C}P^\infty \quad i \geq 0.$$

The generators are

$$e_1, \quad \alpha_{(i)} = \alpha_{p^i} \quad \beta_{(i)} = \beta_{p^i}$$

with coproduct

$$\psi(\alpha_n) = \sum_{i=0}^n \alpha_{n-i} \otimes \alpha_i, \quad \psi(\beta_n) = \sum_{i=0}^n \beta_{n-i} \otimes \beta_i.$$

For finite sequences,

$$I = (i_1, i_2, \dots), \quad 0 \leq i_1 < i_2 < \dots,$$

and

$$J = (j_0, j_1, \dots), \quad j_k \geq 0,$$

define

$$\alpha_I = \alpha_{(i_1)} \circ \alpha_{(i_2)} \circ \dots, \beta^J = \beta_{(0)}^{\circ j_0} \circ \beta_{(1)}^{\circ j_1} \circ \dots,$$

and let  $T(x)$  denote the truncated polynomial algebra  $\mathbb{F}_p[x]/(x^p)$ .

**Theorem A** (Ravenel-Wilson [Wil82]). *We have*

$$H_*K_* \simeq \otimes_{I,J} E(e_1 \circ \alpha_I \circ \beta^J) \otimes_{I,J} T(\alpha_I \circ \beta^J)$$

as an algebra where the tensor product is over all  $I, J$  and the coproduct follows by Hopf ring properties from the  $\alpha$ 's and  $\beta$ 's.

When the prime  $p = 2$ , there are additional relations  $e_1 \circ e_1 = \beta_{(0)}$  and  $\alpha_{(i-1)} \circ \alpha_{(i-1)} = \beta_{(i)}$ . In this case, the theorem can be stated using only circle products of generators of  $\mathbb{R}P^\infty$ .

For finite sequences

$$I = (i_{(-1)}, i_0, i_1, i_2, \dots), \quad i_k \geq 0$$

define

$$(e_1\alpha)^I = e_1^{\circ i_{(-1)}} \circ \alpha_{(0)}^{\circ i_0} \circ \alpha_{(1)}^{\circ i_1} \circ \dots$$

**Theorem B** (Ravenel-Wilson [Wil82]). *Then*

$$H_*K_n \cong \otimes_I E[(e_1\alpha)^I]$$

where  $\sum i_k = n$ , and considering all spaces at once,

$$H_*K_* \simeq \otimes_I E[(e_1\alpha)^I]$$

as an algebra where the tensor product is over all  $I$  and the coproduct follows by Hopf ring properties from the  $\alpha$ 's.

Ravenel and Wilson also show that homology suspending  $\beta_{(i)}$  to define

$$\xi_i \in H_{2(p^i-1)}H,$$

and  $\alpha_{(i)}$  to define

$$\tau_i \in H_{2p^i-1}H,$$

Theorem A implies that stably,

$$H_*H \simeq E[\tau_0, \tau_1, \dots] \otimes P[\xi_1, \xi_2, \dots].$$

3. EQUIVARIANT PRELIMINARIES

We set notation and recall equivariant foundations. Throughout, the group  $G = C_2$ .

Given an orthogonal real  $G$ -representation  $V$ ,  $S^V$  denotes the representation sphere given by the one-point compactification of  $V$ . For a  $p$ -dimensional real  $C_2$  - representation  $V$ , we write

$$V \cong \mathbb{R}^{(p-q,0)} \oplus \mathbb{R}^{(q,q)}$$

where  $\mathbb{R}^{(1,0)}$  is the trivial 1-dimensional real representation of  $C_2$  and  $\mathbb{R}^{(1,1)}$  is the sign representation. We allow  $p$  and  $q$  to be integers, so  $V$  may be a virtual representation. The integer  $p$  is called the topological dimension while  $q$  is the weight or twisted dimension of  $V \cong \mathbb{R}^{(p,q)}$ .

The  $V$ -th graded component of the ordinary  $RO(C_2)$ -graded Bredon equivariant homology of a  $C_2$ -space  $X$  with coefficients in the constant Mackey functor  $\mathbb{F}_2$  is denoted  $H_V^{C_2}(X; \mathbb{F}_2) = H_{p,q}(X; \mathbb{F}_2)$ . To consider all representations at once we write  $H_*(X)$ , and when working nonequivariantly  $H_*(X^e)$  denotes the singular homology of the underlying topological space with  $\mathbb{F}_2$  coefficients.

It is often convenient to plot the bigraded homology in the plane. Our plots have the topological dimension  $p$  on the horizontal axis and the weight  $q$  on the vertical axis.

The homology of a point with coefficients in the constant Mackey functor  $\mathbb{F}_2$ , is the bi-graded ring

$$H_*(pt, \mathbb{F}_2) = \mathbb{F}_2[a, u] \oplus \frac{\mathbb{F}_2[a, u]}{(a^\infty, u^\infty)}\{\theta\}$$

where  $|a| = -\sigma$ ,  $|u| = 1 - \sigma$ , and  $|\theta| = 2\sigma - 2$ . A bi-graded plot of  $H_*(pt, \mathbb{F}_2)$  appears in Figure 3.1. The image on the left is more detailed with each lattice point within the two cones representing a copy of  $\mathbb{F}_2$ . The image on the right is a more succinct representation and appears in figures illustrating our spectral sequence computations.

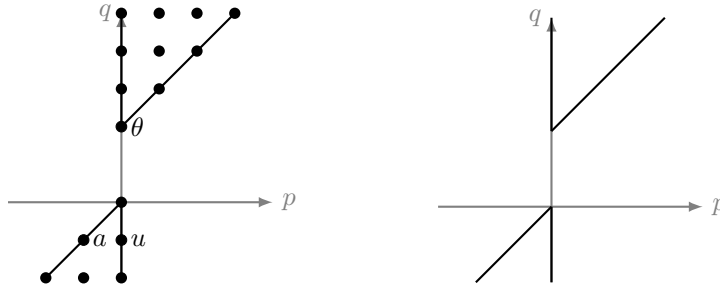


FIGURE 3.1.  $H_*(pt, \mathbb{F}_2)$  with axis gradings determined by  $V \simeq \mathbb{R}^{p-q} \oplus \mathbb{R}^{q\sigma}$ .

The genuine equivariant Eilenberg-MacLane spectrum representing  $H_*(-)$  is  $H\mathbb{F}_2$ , the Eilenberg-MacLane spectrum for the  $C_2$  constant Mackey functor  $\mathbb{F}_2$ . It has

underlying nonequivariant spectrum  $H\mathbb{F}_2$ . We denote the spaces of  $H\mathbb{F}_2$

$$H\mathbb{F}_2 = \{K(\mathbb{F}_2, V)\}_{V \cong k\sigma+l} = \{K_V\}_{V \cong k\sigma+l}.$$

Analogously to the nonequivariant case,  $H\mathbb{F}_2$  is characterized up to  $C_2$ -equivariant homotopy by  $H^V(X; \mathbb{F}_2) = [X, K_V]$  naturally for all  $C_2$ -spaces  $X$ .

We recall a computational lemma due to Behrens and Wilson [BW18], which allows us to check whether a set of elements in the  $RO(C_2)$ -homology in fact forms a free basis for  $H_*(X)$ , greatly simplifying our computations. To state this lemma, we first define two homomorphisms,  $\Phi^e$  and  $\Phi^{C_2}$ . Let  $Ca$  be the cofiber of the Euler class  $a \in \pi_{-\sigma}^{C_2} S$  given geometrically by the inclusion

$$S^0 \hookrightarrow S^\sigma.$$

Applying  $\pi_V^{C_2}$  to the map

$$H \wedge X \rightarrow H \wedge X \wedge Ca,$$

we get a homomorphism

$$\Phi^e : H_V(X) \rightarrow H_{|V|}(X^e).$$

Taking geometric fixed points of a map

$$S^V \rightarrow H \wedge X$$

gives a map

$$S^{V^{C_2}} \rightarrow H^{\Phi^{C_2}} \wedge X^{\Phi^{C_2}}.$$

Using the equivalence  $H_*^\Phi X \simeq H_*(X^{\Phi^{C_2}})[a^{-1}u]$  coming from  $H^{\Phi^{C_2}} \simeq \bigvee_{i \geq 0} \Sigma^i H\mathbb{F}_2$

and passing to the quotient by the ideal generated by  $a^{-1}u$  gives the homomorphism

$$\Phi^{C_2} : H_V(X) \rightarrow H_{|V^{C_2}|}(X^{\Phi^{C_2}}).$$

**Lemma 3.1** (Behrens-Wilson [BW18]). *Suppose  $X \in Sp^{C_2}$  and  $\{b_i\}$  is a set of elements of  $H_*(X)$  such that*

- (1)  $\{\Phi^e(b_i)\}$  is a basis of  $H_*(X^e)$  and
- (2)  $\{\Phi^{C_2}(b_i)\}$  is a basis of  $H_*(X^{\Phi^{C_2}})$ ,

*then  $H_*(X)$  is free over  $H_*$  and  $\{b_i\}$  is a basis.*

We will also use a property of Tambara functors over the constant Mackey functor  $\mathbb{F}_2$ . For a general Tambara functor, the map

$$a \mapsto \mathcal{N}_e^{C_2} \text{Res}_e^{C_2}(a)$$

is multiplicative but not additive. However, for Tambara functors over the constant Mackey functor  $\mathbb{F}_2$ , the failure of additivity is the class

$$[C_2](ab) = 2(ab) = 0,$$

so this map is in fact a ring homomorphism. This can be deduced from reciprocity of the sum formulas in [HM19].

We use the following notation for  $H^*K_\sigma$ .

**Theorem 3.1** (Hu-Kri[HK01]).  $H^*(\mathbb{R}P_{tw}^\infty) = H^*(pt)[\alpha, \beta]/(\alpha^2 = a\alpha + u\beta)$  where  $|\alpha| = \sigma$ ,  $|\beta| = \rho$ ,  $|a| = \sigma$ , and  $|u| = 1 - \sigma$ .

Since this cohomology is free, the homology  $H_*K_\sigma$  immediately follows. In our notation we have elements

$$e_\sigma \in H_\sigma K_\sigma, \quad \bar{\alpha}_i \in H_{\rho i} K_\sigma, \quad (i \geq 0).$$

The generators are

$$e_\sigma, \quad \bar{\alpha}_{(i)} = \bar{\alpha}_{2^i} \quad (i \geq 0)$$

with coproduct

$$\begin{aligned} \psi(e_\sigma) &= 1 \otimes e_\sigma + e_\sigma \otimes 1 + a(e_\sigma \otimes e_\sigma) \\ \psi(\bar{\alpha}_n) &= \sum_{i=0}^n \bar{\alpha}_{n-i} \otimes \bar{\alpha}_i + \sum_{i=0}^{n-1} u(e_\sigma \bar{\alpha}_{n-1-i} \otimes e_\sigma \bar{\alpha}_i). \end{aligned}$$

and ring structure  $H_*K_\sigma \simeq E[e_\sigma, \bar{\alpha}_{(i)}]$  which can be deduced from the underlying level using the Tambara functor structure guaranteed by freeness [Hil19].

We also require notation for  $H_*K(\mathbb{Z}, \rho)$ . This can be deduced by applying the  $RO(C_2)$ -graded bar spectral sequence to  $S^\sigma$ . Let

$$\bar{\beta}_i \in H_{\rho i} K(\mathbb{Z}, \rho) \quad (i \geq 0).$$

The generators are

$$\bar{\beta}_{(i)} = \bar{\beta}_{2^i} \quad (i \geq 0)$$

with coproduct

$$\psi(\bar{\beta}_n) = \sum_{i=0}^n \bar{\beta}_{n-i} \otimes \bar{\beta}_i$$

and ring structure

$$H_*K(\mathbb{Z}, \rho) \simeq E[\beta_{(i)}].$$

### 3.1. The fixed point spaces of $C_2$ -equivariant Eilenberg-MacLane spaces.

It is useful to understand the  $C_2$  fixed points of the  $C_2$ -equivariant Eilenberg-MacLane spaces  $K_V$  in applications of the Behrens-Wilson computational lemma. We state a proposition due to Caruso.

**Proposition 3.2** (Caruso [Car99]). *Let  $G = C_p$  and  $V$  be an  $n$ -dimensional fixed point free virtual representation of  $G$  with  $n > 0$  and  $m$  an integer. Then*

$$K(\mathbb{F}_p, m + V)^{C_p} \simeq K(\mathbb{F}_p, m) \times \cdots \times K(\mathbb{F}_p, m + n).$$

**3.2. Notation for underlying nonequivariant homology of  $K_V$ .** To use the Behrens-Wilson lemma, we also need to understand the homology of the fixed point spaces. Applying Theorem B to the nonequivariant homology of  $(K_{n\sigma})^{C_2}$  gives

$$H_*(K_{n\sigma}^{C_2}) \simeq E[e_0, a_{(i_1)}, a_{(i_1)} \circ a_{(i_2)}, \dots, a_{(i_1)} \circ \cdots \circ a_{(i_n)}]$$

where  $0 \leq i_1 \leq i_2 \leq \cdots \leq i_n$ ,  $|e_0| = 0$ , where  $|a_{(i)}| = 2^i$ .

## 4. BAR AND TWISTED BAR CONSTRUCTIONS

A first task in implementing the Ravenel-Wilson Hopf ring approach is to generalize the bar spectral sequence to the  $C_2$ -equivariant case. In the classical story, the bar spectral sequence is used to inductively compute the homology of  $K_n \simeq BK_{n-1}$  from  $H_*K_{n-1}$ . In the  $C_2$ -equivariant setting, our spaces  $K_V$  are bi-graded on the trivial and sign representations of  $C_2$ . Due to this new grading, we should now additionally compute the homology of  $K_{V+\sigma}$  inductively from  $H_*K_V$ . In order to do so, we need a good model of  $\sigma$ -delooping. We begin by reviewing the classical bar construction which is a trivial representation delooping functor.

**Construction 4.1. (The Classical Bar Construction)**

For a topological monoid  $A$ , the pointed space  $BA$  is defined as a quotient

$$BA = \coprod_n \Delta^n \times A^{\times n} / \sim$$

where the relation  $\sim$  is generated by

(1)

$$(t_1, \dots, t_n, a_1, \dots, a_n) \sim (t_1, \dots, \hat{t}_i, \dots, t_n, a_1, \dots, \hat{a}_i, (a_i a_{i+1}) \dots, a_n)$$

if  $t_i = t_{i+1}$  or  $x_i = *$ ,

(2) for  $i = n$ , delete the last coordinate if  $t_n = 1$  or  $a_n = *$ ,

for  $i = 0$ , delete the first coordinate if  $t_0 = -1$  or  $a_0 = *$ ,

and  $\Delta^n$  denotes the topological simplex

$$\Delta^n = \{(t_1, t_2, \dots, t_n) \in \mathbb{R}^n \mid -1 \leq t_1 \leq \dots \leq t_n \leq 1\}.$$

**Remark 4.2.** We use the slightly nonstandard topological  $n$ -simplex

$$\Delta^n = \{(t_1, t_2, \dots, t_n) \in \mathbb{R}^n \mid -1 \leq t_1 \leq \dots \leq t_n \leq 1\}$$

so that when we introduce a  $C_2$  action, the simplex rotates around the origin. This makes writing down a model for the  $H$ -space structure on the  $C_2$ -equivariant Eilenberg-MacLane spaces  $K_V$  more straightforward.

Given a commutative monoid  $A$ , we observe that  $BA$  is also a commutative monoid via the pairing

$$* : BX \times BX \rightarrow BX$$

defined by

$$\begin{aligned} (t_1, \dots, t_n, x_0, \dots, x_n) *_{\sigma} (t_{n+1}, \dots, t_{n+m}, x_{n+1}, \dots, x_{n+m}) \\ = (t_{\tau(1)}, \dots, t_{\tau(n+k)}, x_{\tau(1)}, \dots, x_{\tau(n+k)}), \end{aligned}$$

where  $\tau$  is any element of the symmetric group on  $n+k$  letters such that  $t_{\tau(i)} \leq t_{\tau(i+1)}$ . This pairing was first described by Milgram [Mil67].

**Definition 4.3** ([Liu20]). A  $C_2$ -space  $A$  is a twisted monoid if it is a topological monoid in the non-equivariant sense with the product satisfying  $\gamma(xy) = \gamma(y)\gamma(x)$  where  $C_2 \simeq \langle \gamma \rangle$ .

**Construction 4.4** ([Liu20]). *For any twisted monoid  $A$ , construct  $B_*^\sigma A$  in the same way as the non-equivariant bar construction, that is such that  $B_n^\sigma A = \Delta^n \times A^n$ . However, define a  $C_2$ -action on  $A^n$  by*

$$\gamma(a_1, a_2, \dots, a_n) = (\gamma a_n, \gamma a_{n-1}, \dots, \gamma a_1).$$

*Then the  $C_2$ -actions commute with the face and degeneracy maps as  $\gamma \circ s_i = s_{n-i} \circ \gamma$  and  $\gamma \circ d_i = d_{n-i} \circ \gamma$ . Further define the  $C_2$ -action on each*

$$\Delta^n = \{(t_1, t_2, \dots, t_n) \in \mathbb{R}^{t+1} \mid -1 \leq t_1 \leq \dots \leq t_n \leq 1\},$$

*by  $\gamma(t_1, t_2, \dots, t_n) = (-t_n, -t_{n-1}, \dots, -t_1)$ . Then define  $B^\sigma A$  to be the geometric realization*

$$\coprod \Delta^n \times A^n / \sim.$$

**Example 4.5.** *The space  $B^\sigma K_0 \simeq \mathbb{R}P_{tw}^\infty$  is the space of lines through the origin with conjugate action.*

We can inductively define an  $H$ -space pairing on  $B^l B^{k\sigma} \mathbb{F}_2$ , similar to the one given by Milgram in the non-equivariant case. Define a mapping

$$*_\sigma : B^\sigma X \times B^\sigma X \rightarrow B^\sigma X$$

by

$$\begin{aligned} (t_0, \dots, t_n, x_0, \dots, x_n) *_\sigma (t_{n+1}, \dots, t_{n+m}, x_{n+1}, \dots, x_{n+m}) \\ = (t_{\tau(1)}, \dots, t_{\tau(n+k)}, x_{\tau(1)}, \dots, x_{\tau(n+k)}), \end{aligned}$$

where  $\tau$  is any element of the symmetric group on  $n+k$  letters such that  $t_{\tau(i)} \leq t_{\tau(i+1)}$ . Then  $*_\sigma$  is well defined, continuous, and  $C_2$ -equivariant. Going forward, we suppress the  $\sigma$  notation in  $*_\sigma$ , using only  $*$  to denote the  $H$ -space pairing. The relevant  $C_2$ -action is deduced from context.

**Proposition 4.6** (Liu [Liu20]). *For any commutative monoid  $A$  in the category of based  $C_2$ -spaces, the  $V$ -degree bar construction  $B^V A$  is defined by applying the ordinary bar construction  $l$  times and the twisted bar construction  $m$  times for  $V = l + m\sigma$ . There exists a natural map  $A \rightarrow \Omega^V B^V A$ . When  $A$  is  $C_2$ -connected, this map is a  $C_2$ -equivalence.*

## 5. MULTIPLICATIVE STRUCTURES ON $C_2$ -EQUIVARIANT EILENBERG-MACLANE SPACES

We describe multiplicative structures on  $C_2$ -equivariant Eilenberg-MacLane spaces, extending Ravenel and Wilson's description of similar structures on classical non-equivariant Eilenberg-MacLane spaces. We use our understanding of these structures to compute the  $RO(C_2)$ -graded homology of many  $C_2$ -equivariant Eilenberg-MacLane spaces  $K_V$  associated to the constant Mackey functor  $\mathbb{F}_2$ . In particular, we compute the  $RO(C_2)$ -graded homology of all  $C_2$ -equivariant Eilenberg-MacLane spaces  $K_{*\sigma}$  and  $K_{\sigma+*}$ .

**5.1. Multiplicative structures on  $K_V$ .** The  $RO(C_2)$ -graded cup product is induced by a map

$$(5.1) \quad \circ = \circ_{V,W} : K_V \wedge K_W \rightarrow K_{V+W}.$$

We will construct  $\circ_{V,W}$  explicitly within the framework of trivial and  $\sigma$ -representation delooping given by  $B$  and  $B^\sigma$ . We will also discuss how  $\circ_{V,W}$  descends to a product on the fixed points.

Given a real  $C_2$  representation  $V \cong l + k\sigma$ , the Eilenberg-MacLane space  $K(\mathbb{F}_2, V \cong l + K\sigma)$  is a  $V$ -fold delooping of  $\mathbb{F}_2$  and therefore can be constructed iteratively by taking  $B^l B^{k\sigma} \mathbb{F}_2$  where  $l, k$  are non-negative integers. The following construction extends exposition by Ravenel and Wilson in their computation of the Morava  $K$  theory of Eilenberg-MacLane spaces [RW80].

We construct the map 5.1 inductively on  $V$ . Assuming  $\circ_{V,W}$  has been defined, we define  $\circ_{V+1,W}$  and  $\circ_{V+\sigma,W}$  by replacing  $K_{V+1}$ ,  $K_{V+W+1}$ , and  $K_{V+\sigma}$ ,  $K_{V+W+\sigma}$  with their bar and twisted bar constructions respectively. In both cases this is denoted as follows. There is a notationally suppressed  $C_2$ -action each case.

$$(5.2) \quad \left\{ \prod_n \Delta^n \times K_V^n / \sim \right\} \wedge K_W \rightarrow \left\{ \prod_n \Delta^n \times K_{V+W} / \sim \right\}$$

Let  $t \in \Delta^n$ ,  $x = (x_0, \dots, x_n) \in K_V$ , and  $y \in K_W$ . The image of  $x_i \wedge y \in K_V \wedge K_W$  under the map (5.1) is denoted  $x_i \circ y$ . We use the notation  $x \circ y$  to mean  $(x_0 \circ y, \dots, x_n \circ y)$ . Define (5.2) by

$$(5.3) \quad \{(t, x)\} \circ y = \{(t, x \circ y)\}.$$

**Theorem 5.1.** *The above construction is well defined and gives the cup product pairings  $\circ : K_{V+1} \wedge K_W \rightarrow K_{V+W+1}$  and  $\circ : K_{V+\sigma} \wedge K_W \rightarrow K_{V+W+\sigma}$ .*

**Lemma 5.4.** *The map  $\circ : K_0 \times K_V \rightarrow K_V$  is given by  $(q) \circ x = x^{*q}$  where  $q \in \mathbb{F}_2$ .*

*Proof.* This map multiplies  $\pi_V^{C_2} K_V \simeq \mathbb{F}_2$  by  $q$  which is what  $\circ$  should do restricted to  $(q) \times K_V \simeq K_V$ .  $\square$

*Proof. of Theorem 5.1* This proof is a direct extension of the nonequivariant argument in Ravenel-Wilson [RW80]. We prove our result by induction on  $i$  in the  $\sigma$  direction noting that the result also holds and is similar in the trivial representation direction (that is we assume the statement holds for  $V$ , show for  $V + \sigma$ ). Assume we have proved Theorem 1 for  $K_V \wedge K_W \rightarrow K_{V+W}$  with Lemma 1 beginning the induction. We need our construction to satisfy:

$$(z_1 * z_2) \circ y = (z_1 \circ y) * (z_2 \circ y).$$

For  $i = 0$ ,  $z_i = q_i \in \mathbb{F}_2 = K_0$ . So,

$$(q_1 * q_2) \circ y = (q_1 + q_2) \circ y = y^{q_1+q_2} = y^{*q_1} * y^{*q_2} = (q_1 \circ y) * (q_2 \circ y).$$



For  $i > 0$ ,

$$\begin{aligned}
 [z_1 * z_2] \circ y &= [(t, x) * (t_{n+1}, \dots, t_{n+k}; x_{n+1}, \dots, x_{n+k})] \circ y \\
 &= (t_{\tau(1)}, \dots, t_{\tau(n+k)}; x_{\tau(1)}, \dots, x_{\tau(n+k)}) \circ y \\
 &= (t_{\tau(1)}, \dots, t_{\tau(n+k)}; x_{\tau(1)} \circ y, \dots, x_{\tau(n+k)} \circ y) \\
 &= (t; x \circ y) * (t_{n+1}, \dots, t_{n+k}; x_{n+1} \circ y, \dots, x_{n+k} \circ y) \\
 &= (z_1 \circ y) * (z_2 \circ y)
 \end{aligned}$$

where the second line is due to the definition of  $*$ , the third is due to the induction hypothesis and (5.3), and the fourth is due to the definition of  $*$ .  $\square$

We must show 5.3 gives well defined maps  $K_{V+1} \wedge K_W \rightarrow K_{V+W+1}$  and  $K_{V+\sigma} \wedge K_W \rightarrow K_{V+W+\sigma}$ . The relations in the (twisted) bar construction make this the case. We show the main case, leaving the others to the reader. Assume  $0 \leq q < n$  with  $t_q = t_{q+1}$  or  $x_q = *$ . Then

$$\begin{aligned}
 (t, x) \circ y &= (t, x \circ y) \\
 &\sim (t_1, \dots, \hat{t}_q, \dots, t_n; x_1 \circ y, \dots, (x_q \circ y) * (x_{q+1} \circ y), \dots, x_n \circ y) \\
 &= (t_1, \dots, \hat{t}_q, \dots, t_n; x_1 \circ y, \dots, (x_q * x_{q+1}) \circ y, \dots, x_n \circ y) \\
 &= (t_1, \dots, \hat{t}_q, \dots, t_n; x_1, \dots, x_q * x_{q+1}, \dots, x_n) \circ y
 \end{aligned}$$

which is the necessary relation. That this map factors through the smash product is straightforward to verify using induction.

The remaining task is to show that this is the cup product pairing map. This follows by induction from the observation that  $\circ$  commutes with (signed) suspension on the first factor since  $B_1 K_V \simeq S^1 \wedge K_V$  and  $B_1^\sigma K_V \simeq S^\sigma \wedge K_V$ , and following diagrams commute.

$$\begin{array}{ccc}
 S^1 \wedge K_V \wedge K_W & \longrightarrow & S^1 \wedge K_{V+W} \\
 \downarrow & & \downarrow \\
 K_{V+1} \wedge K_W & \longrightarrow & K_{V+W+1} \\
 \\ 
 S^\sigma \wedge K_V \wedge K_W & \longrightarrow & S^\sigma \wedge K_{V+W} \\
 \downarrow & & \downarrow \\
 K_{V+\sigma} \wedge K_W & \longrightarrow & K_{V+W+\sigma}
 \end{array}$$

**5.2. Multiplicative structures on  $K_V^{C_2}$ .** We turn to understanding the  $\circ$ -product on the fixed points of the spaces  $K_V$ . Notice  $(B^\sigma A)^{C_2}$  consists of points of the form

$$(t_1, \dots, t_n, 0, -t_n, \dots, -t_1, a_1, \dots, a_n, a, \gamma(a_n), \dots, \gamma(a_1)) \in (B^\sigma A)^{[2n+1]}$$

where  $a \in A^{C^2}$  since for

$$(t_1, \dots, t_m, -t_m, \dots, -t_1, a_1, \dots, a_m, \gamma(a_m) \cdots \gamma(a_1)) \in (B^\sigma A)^{[2m]},$$

there is a degeneracy map inducing an equivalence to

$$(t_1, \dots, t_n, 0, -t_n, \dots, -t_1, a_1, \dots, a_n, *, \gamma(a_n), \dots, \gamma(a_1)) \in (B^\sigma A)^{[2n+1]}.$$

Taking the fixed points in the construction of map (5.2) we recover the classical nonequivariant  $\circ$  product on the fixed point spaces.

**5.3. Circle product generators for  $H_\star K_{n\sigma}$ .** Recall that  $H\mathbb{F}_2$  has generators  $a \in H\mathbb{F}_2\{-\sigma\}$  and  $u \in H\mathbb{F}_2\{1-\sigma\}$ . To describe our answer, we recall our notation for  $H_\star K_\sigma$ . Let

$$e_\sigma \in H_\sigma K_\sigma, \quad \bar{\alpha}_i \in H_{\rho i} K_\sigma, \quad (i \geq 0).$$

The homology,  $H_\star K_\sigma$ , is exterior on generators

$$e_\sigma, \quad \bar{\alpha}_{(i)} = \bar{\alpha}_{2^i} \quad (i \geq 0)$$

with coproduct

$$\begin{aligned} \psi(e_\sigma) &= 1 \otimes e_\sigma + e_\sigma \otimes 1 + a(e_\sigma \otimes e_\sigma) \\ \psi(\bar{\alpha}_n) &= \sum_{i=0}^n \bar{\alpha}_{n-i} \otimes \bar{\alpha}_i + \sum_{i=0}^{n-1} u(e_\sigma \bar{\alpha}_{n-1-i} \otimes e_\sigma \bar{\alpha}_i) \end{aligned}$$

For finite sequences

$$J = (j_\sigma, j_0, j_1, \dots) \quad j_k \geq 0,$$

define

$$(e_\sigma \bar{\alpha})^J = e_\sigma^{\circ j_\sigma} \circ \bar{\alpha}_{(0)}^{\circ j_0} \circ \bar{\alpha}_{(1)}^{\circ j_1} \dots$$

where the  $\circ$  product comes from the pairing  $\circ : K_V \wedge K_W \rightarrow K_{V+W}$ .

**Theorem 5.2.** *Then*

$$H_\star K_{*\sigma} \cong \otimes_J E[(e_\sigma \bar{\alpha})^J]$$

*As an algebra where the tensor product is over all  $J$  and the coproduct follows by Hopf ring properties from the  $\bar{\alpha}$ 's.*

*Proof.* For finite sequences

$$J = (j_\sigma, j_0, j_1, \dots) \quad j_k \geq 0,$$

define  $\|J\| = \sum j_k$  (including the  $\sigma$  subscript) and

$$(e_\sigma \bar{\alpha})^J = e_\sigma^{\circ j_\sigma} \circ \bar{\alpha}_{(0)}^{\circ j_0} \circ \bar{\alpha}_{(1)}^{\circ j_1} \dots$$

Consider elements

$$(e_\sigma \bar{\alpha})^J$$

with  $\|J\| = n$  in the homology of  $B^\sigma K_{(n-1)\sigma}$ .

To show these elements in fact form a free basis for the homology, we show that they satisfy the conditions of the Behrens-Wilson computational lemma. The map to the underlying homology,  $H_\star K_{n\sigma} \rightarrow H_\star K_n$ , the underlying homology of  $H_\star K_{n\sigma}$ , is given by

$$(e_\sigma \bar{\alpha})^J \mapsto (e_1 \alpha)^J.$$

The map on fixed points  $H_*K_{n\sigma} \rightarrow H_*K_{n\sigma}^{C_2}$ , is given by

$$(e_\sigma \bar{\alpha})^J \mapsto e_0^{\circ j_\sigma} \circ a_{(0)}^{\circ j_0} \circ a_{(1)}^{\circ j_1} \cdots .$$

Thus these elements form a free basis for  $H_*K_{n\sigma}$ .

We deduce the multiplicative ring structure using a Hopf ring argument due to Ravenel and Wilson [Wil82]. Each  $(e_\sigma \bar{\alpha})^J$  can be written as  $e_\sigma^{\circ j_\sigma} \circ \bar{\alpha}_{(0)}^{\circ j_0} \circ \bar{\alpha}_{(1)}^{\circ j_1} \cdots \bar{\alpha}_{(n)}^{\circ j_n}$  where  $n$  is some nonnegative integer or  $n = \sigma$ . By the distributive law (2.1),

$$(e_\sigma \bar{\alpha})^J * (e_\sigma \bar{\alpha})^J = e_\sigma^{\circ j_\sigma} \circ \bar{\alpha}_{(0)}^{\circ j_0} \circ \bar{\alpha}_{(1)}^{\circ j_1} \circ \cdots \circ (\bar{\alpha}_{(n)} * \bar{\alpha}_{(n)}) = 0.$$

The coproduct is induced by the map  $K_\sigma \times \cdots \times K_\sigma \rightarrow K_{n\sigma}$  which is a map of coalgebras on  $H_*$ .

□

**Remark 5.5.** Note that  $e_0^{\circ k} = e_0$  for  $k > 0$  by Lemma 5.4.

**5.4. Circle product generators for  $H_*K_{\sigma+i}$ .** We have computed  $H_*K_{n\sigma}$  and shown that it is free over  $H_*$ . To compute  $H_*K_V$  for general real representations  $V$ , we consider the  $RO(C_2)$ -graded bar spectral sequence of Hopf algebras,

$$E_{*,*}^2 \simeq \text{Tor}_{*,*}^{H_*K_V}(H_*, H_*) \Rightarrow H_*K_{V+1},$$

and observe that theorems of Thomason and Wilson, on Hopf rings in the bar spectral, sequence extend directly from the nonequivariant integer graded to the  $C_2$ -equivariant  $RO(C_2)$ -graded setting.

**Theorem 5.3** ([TW80]). *The  $\circ$  product factors as*

$$\begin{aligned} B_t K_V \times K_W &\rightarrow B_t K_{V+W} \\ \bigcap &\qquad \qquad \bigcap \\ \circ : BK_V \times K_W &\rightarrow BK_{V+W} \end{aligned}$$

and the map

$$\begin{aligned} B_t K_V / B_{t-1} K_V \times K_W &\rightarrow B_t K_{V+W} / B_{t-1} K_{V+W} \\ \wr &\qquad \qquad \wr \\ S^t \wedge K_V^{\wedge t} \times K_W &\rightarrow S^t \wedge K_{V+W}^{\wedge t} \end{aligned}$$

is described inductively as  $(k_1, \dots, k_t) \circ k = (k_1 \circ k, \dots, k_t \circ k)$ .

**Theorem 5.4** ([TW80]). *Let  $E_{*,*}^r(E_*K_V) \Longrightarrow E_*K_{V+\sigma}$  be the bar spectral sequence. Compatible with*

$$\circ : E_*K_{V+1} \otimes_{H_*} E_*K_W \rightarrow E_*K_{V+W+1}$$

is a pairing

$$(5.6) \quad E_{t,*}^r(E_*K_V) \otimes_{H_*} E_*K_W \rightarrow E_{t,*}^r(E_*K_{V+W})$$

with  $d^r(x) \circ y = d^r(x \circ y)$ . For  $r = 1$  this pairing is given by

$$(k_1 | \cdots | k_t) \circ k = \sum \pm (k_1 \circ k' | k_2 \circ k'' | \cdots | k_s \circ k^{(t)})$$

where  $k \rightarrow \sum k' \otimes k'' \otimes \cdots \otimes k^{(t)}$  is the iterated reduced coproduct.

We use this  $RO(C_2)$ -graded bar spectral sequence and the  $RO(C_2)$ -graded bar spectral sequence paring (5.6) to inductively compute  $H_*K_{\sigma+*}$  from  $H_*K_\sigma$ . We use the Behrens-Wilson computational lemma to confirm that the circle product elements on the  $E^2$ -page of this spectral sequence do in fact form a basis for the homology. To write our answers, we recall our notation for  $H_*K_\rho$ . Let

$$\bar{\beta}_i \in H_{\rho i}K(\mathbb{Z}, \rho), \quad (i \geq 0).$$

This gives additional generators,

$$\bar{\beta}_{(i)} = \bar{\beta}_{2^i} \quad (i \geq 0),$$

of  $H_*K_\rho$  with coproduct

$$\psi(\bar{\beta}_n) = \sum_{i=0}^n \bar{\beta}_{n-i} \otimes \bar{\beta}_i.$$

Then for finite sequences

$$I = (i_1, i_2, \dots, i_k), \quad 0 \leq i_1 < i_2 < \dots,$$

$$W = (w_1, w_2, \dots, w_q), \quad 0 \leq w_1 < w_2 < \dots,$$

$$J = (j_{-1}, j_0, j_1, \dots, j_\ell), \quad \text{where } j_{-1} \in \{0, 1\} \text{ and all other } j_n \geq 0,$$

and

$$Y = (y_{-1}, y_0, y_1, \dots, y_r), \quad \text{where } y_{-1} \in \{0, 1\} \text{ and all other } y_n \geq 0,$$

define

$$(e_1\alpha\beta)^{I,J} = e_1^{\circ j_{-1}} \circ \alpha_{(i_1)} \circ \alpha_{(i_2)} \circ \dots \circ \alpha_{(i_k)} \circ \beta_{(0)}^{\circ j_0} \circ \beta_{(1)}^{\circ j_1} \circ \dots \circ \beta_{(\ell)}^{\circ j_\ell},$$

$$(e_1\alpha\beta)^{W,Y} = e_1^{\circ y_{-1}} \circ \alpha_{(w_1)} \circ \alpha_{(w_2)} \circ \dots \circ \alpha_{(w_q)} \circ \beta_{(0)}^{\circ y_0} \circ \beta_{(1)}^{\circ y_1} \circ \dots \circ \beta_{(r)}^{\circ y_r},$$

$$|I| = k, \quad |W| = q \quad \|J\| = \Sigma j_n, \quad \text{and } \|Y\| = \Sigma y_n.$$

Then

**Theorem 5.5.** *We have*

$$H_*K_{\sigma+i} \cong E[(e_1\alpha\beta)^{I,J} \circ \bar{\alpha}_{(m)}, (e_1\alpha\beta)^{W,Y} \circ \bar{\beta}_{(t)}]$$

where  $m > i_k$  and  $m \geq l$ ,  $t > w_q$  and  $t \geq y_r$ ,  $|I| + 2\|J\| = i$  and  $|W| + 2\|Y\| = i - 1$ , and the coproduct follows by Hopf ring properties from the  $\alpha_{(i)}$ 's,  $\beta_{(i)}$ 's,  $\bar{\alpha}_{(i)}$ 's and  $\bar{\beta}_{(i)}$ 's.

*Proof.* Apply the Behrens-Wilson lemma to the generators  $(e_1\alpha\beta)^{I,J} \circ \bar{\alpha}_{(m)}$  and  $(e_1\alpha\beta)^{W,Y} \circ \bar{\beta}_{(t)}$  defined in the theorem. The map to the underlying homology is clear as the generators have no  $a$ -torsion. On fixed points,

$$(e_1\alpha\beta)^{I,J} \circ \bar{\alpha}_{(m)} \mapsto (e_1\alpha\beta)^{I,J} \circ a_{(m)}$$

and

$$(e_1\alpha\beta)^{W,Y} \circ \bar{\beta}_{(t)} \mapsto (e_1\alpha\beta)^{W,Y} \circ a_{(t)},$$

giving a basis for  $K_{\sigma+i}^{C_2} \simeq K_{i+1} \times K_i$  where the  $a_{(i)}$  are notation for the underlying nonequivariant homology of  $K_\sigma$  (see §3.2). The multiplicative and comultiplicative structures are deduced similarly to Theorem 5.2.  $\square$

**5.5. Complication when  $2\sigma+1 \subset V$ .** Given our calculations of  $H_*K_{n\sigma}$  (Theorem 5.2) and  $H_*K_{\sigma+i}$  (Theorem 5.5), it is natural to think we might be able to apply the same approach, iteratively applying the  $RO(C_2)$ -graded bar spectral sequence to compute  $H_*K_{n\sigma+i}$ , where  $n \geq 2$  and  $i \geq 1$ , from  $H_*K_{n\sigma}$ . This would require identifying all elements on the  $E^2$ -page of the bar spectral sequence in terms of circle products. We know all cycles representing non-trivial elements in  $\text{Tor}_2$  are products of elements in  $\text{Tor}_1$  or transpotence elements. Let  $\overline{\alpha_I\beta^J}$  denote a generic element in  $H_*K_{n\sigma+(i-1)}$ . Writing bar representatives for the cycles on the  $E^2$ -page we have

$$\overline{[\alpha_I\beta^J]} \quad \text{and} \quad \underbrace{\overline{[\alpha_I\beta^J]} \cdots \overline{[\alpha_I\beta^J]}}_{2^i \text{ - copies}}.$$

Let  $s\overline{\alpha_I\beta^J}$  denote  $\overline{[\alpha_I\beta^J]}$  as the suspension and  $\phi^i(\overline{\alpha_I\beta^J})$  denote the transpotence elements  $\underbrace{\overline{[\alpha_I\beta^J]} \cdots \overline{[\alpha_I\beta^J]}}_{2^i \text{ - copies}}$ , so  $\phi(\overline{\alpha_I\beta^J}) = \overline{[\alpha_I\beta^J]} \overline{[\alpha_I\beta^J]}$ .

For clarity of exposition, we specialize to the case of computing  $H_*BK_{2\sigma} \cong H_*K_{2\sigma+1}$  using the  $RO(C_2)$ -graded bar spectral sequence. All other remaining computations exhibit similar behavior. Recall

$$H_*K_{2\sigma} \cong E[e_\sigma \circ e_\sigma, e_\sigma \circ \bar{\alpha}_{(i)}, \bar{\alpha}_{(j_1)} \circ \bar{\alpha}_{(j_2)}]$$

where  $0 \leq i, 0 \leq j_1 \leq j_2$ ; and the coproduct follows by Hopf ring properties from the  $\bar{\alpha}_{(i)}$ 's and  $\bar{\beta}_{(i)}$ 's (Theorem 5.2). Thus in  $\text{Tor}_2$  we have cycles

$$\begin{aligned} [e_\sigma \circ e_\sigma], \quad [e_\sigma \circ \bar{\alpha}_{(i)}], \quad [\bar{\alpha}_{(j_1)} \circ \bar{\alpha}_{(j_2)}] \\ \phi^{(k)}(e_\sigma \circ e_\sigma), \quad \phi^{(k)}(e_\sigma \circ \bar{\alpha}_{(i)}), \quad \phi^{(k)}(\bar{\alpha}_{(j_1)} \circ \bar{\alpha}_{(j_2)}). \end{aligned}$$

Since suspension is the same as circling with  $e_1$ , we can identify  $[e_\sigma \circ e_\sigma]$  with  $e_1 \circ e_\sigma \circ e_\sigma$ , the bar representative  $[e_\sigma \circ \bar{\alpha}_{(i)}]$  with  $e_1 \circ e_\sigma \circ \bar{\alpha}_{(i)}$ , and the bar representative  $[\bar{\alpha}_{(j_1)} \circ \bar{\alpha}_{(j_2)}]$  with  $e_1 \circ \bar{\alpha}_{(j_1)} \circ \bar{\alpha}_{(j_2)}$ . Using the  $RO(C_2)$ -graded bar spectral sequence paring (5.6) for  $BK_\sigma \times K_{2\sigma-\sigma} \rightarrow K_{2\sigma}$  and  $BK_0 \times K_{2\sigma} \rightarrow BK_{2\sigma}$  we can identify  $\phi^{(k)}(e_\sigma \circ \bar{\alpha}_{(i)})$  with  $\bar{\beta}_{(j_1+1)} \circ \bar{\alpha}_{(j_2+1)}$  and the bar representative  $\phi^{(k)}(\bar{\alpha}_{(j_1)} \circ \bar{\alpha}_{(j_2)})$  with  $\alpha_{(i_1)} \circ \bar{\alpha}_{(i_2)} \circ \bar{\alpha}_{(j_3)}$ . However, for degree reasons, it is not possible to identify  $\phi^{(k)}(e_\sigma \circ e_\sigma)$  in terms of circle products and thus the Hopf ring circle product structure alone cannot be used to guarantee that these are in fact permanent cycles.

We shall return to this topic from the perspective of the  $RO(C_2)$ -graded bar spectral sequence in the Section 6 and state our conjectures regarding the homology of the spaces  $K_V$  where  $2\sigma+1 \subset V$  in Section 7.

## 6. BAR AND TWISTED BAR SPECTRAL SEQUENCE COMPUTATIONS

The first half of this section focuses on the  $RO(C_2)$ -graded bar spectral sequence. We begin our discussion with a description of the Tor term coinciding with the  $E^2$ -page. We then describe the  $d_1$ -differentials and make comparisons showing how it is considerably harder to use dimension arguments to compute higher differentials in the  $RO(C_2)$ -graded case than it in the integer graded case. We also make precise a gap in our argument that the  $RO(C_2)$ -graded bar spectral sequence computing the homology of  $HK_V$ , where  $2\sigma+1 \subset V$ , collapses on the  $E^2$ -page. While we

have not yet been able to prove that these spectral sequences collapse, we have no evidence to the contrary.

In the second half of this section, we study the analogous twisted spectral sequence, the  $RO(C_2)$ -graded twisted bar spectral sequence and give evidence of arbitrarily long equivariant degree shifting differentials appearing computations of the  $RO(C_2)$ -graded homology of the spaces  $K_{*\sigma}$ . We describe how these differentials appear to arise in an entirely structured way involving the norm.

**6.1. The  $RO(C_2)$ -graded bar spectral sequence.** The  $RO(C_2)$ -graded bar spectral sequence arises via a filtered complex in the same way as the ordinary integer graded version. The bar construction  $B$  on a topological monoid  $A$ , is filtered by

$$B^{[t]}A \simeq \coprod_{t \geq n \geq 0} \Delta^n \times A^n / \sim \subset BA$$

with associated graded pieces

$$B^{[t]}A/B^{[t-1]}A \simeq S^t \wedge A^{\wedge t}.$$

Applying  $H_*(-)$  to these filtered spaces gives the  $RO(C_2)$ -graded bar spectral sequence with  $E^1$ -page

$$E_{t,*}^1 = H(S^t) \otimes H_*(A)^{\otimes t},$$

computing  $H_*(BA)$ . This  $RO(C_2)$ -graded bar spectral sequence has

$$E_{*,*}^2 \simeq \mathrm{Tor}_{*,*}^{H_*K_V}(H\mathbb{F}_{2*}, H\mathbb{F}_{2*}) \implies H_*BK_V \cong H_*K_{V+1}.$$

At first glance, it may be unclear to what extent computations using this spectral sequence are readily accessible. Two examples where the  $RO(C_2)$ -graded bar spectral sequence collapses immediately are the spectral sequences

$$E_{*,*}^2 \simeq \mathrm{Tor}_{*,*}^{H_*S^1}(H\mathbb{F}_{2*}, H\mathbb{F}_{2*}) \implies H_*BS^1 \cong H_*\mathbb{C}P^\infty$$

and

$$E_{*,*}^2 \simeq \mathrm{Tor}_{*,*}^{H_*S^\sigma}(H\mathbb{F}_{2*}, H\mathbb{F}_{2*}) \implies H_*BS^\sigma \cong H_*\mathbb{C}P_{tw}^\infty$$

computing the  $RO(C_2)$ -graded homology of  $BS^1 \simeq \mathbb{C}P^\infty \simeq K(\mathbb{F}_2, 2)$  and  $BS^\sigma \simeq \mathbb{C}_{tw}^\infty \simeq K(\mathbb{F}_2, \rho)$ . These  $RO(C_2)$ -graded bar spectral sequence collapse on the  $E^1$ -page and we find

**Example 6.1.**

$$\begin{aligned} H_*\mathbb{C}P^\infty &= E[\beta_{(0)}, \beta_{(1)}, \dots] = \Gamma[e_2] \text{ where } |\beta_{(i)}| = 2^{i+1}, \\ H_*\mathbb{C}P_{tw}^\infty &= E[\bar{\beta}_{(0)}, \bar{\beta}_{(1)}, \dots] = \Gamma[e_\rho] \text{ where } |\bar{\beta}_{(i)}| = \rho 2^i. \end{aligned}$$

**6.2. The  $RO(C_2)$ -graded bar spectral sequence:  $d_1$ -differentials.** The classical bar construction does not introduce any group action hence the  $d_1$ -differentials in the  $RO(C_2)$ -graded bar spectral sequence behave in almost exactly the same way as those in the underlying integer graded spectral sequence. The difference is that the cycles supporting  $d_1$ -differentials in the  $RO(C_2)$ -graded spectral sequence are representation degree shifted copies of the  $RO(C_2)$ -graded homology of the point and their targets are the same. This is in contrast with the integer graded case where the differentials are maps of non-graded rings. For example, all  $d_1$  differentials in the  $RO(C_2)$ -graded case look behave like those shown in Figure 6.1.

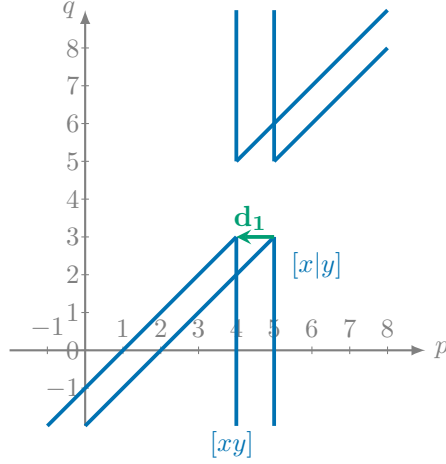


FIGURE 6.1. Example: A  $d_1$ -differential in the  $RO(C_2)$ -graded bar spectral sequence

In greater specificity, Figure 6.2 shows a  $d_1$  differential in the  $RO(C_2)$ -graded bar spectral sequence

$$E_{*,*}^2 \simeq \text{Tor}_{*,*}^{H_*K_\sigma}(H\mathbb{F}_{2*}, H\mathbb{F}_{2*}) \implies H_*BK_\sigma \cong H_*K_{\sigma+1}$$

computing the  $RO(C_2)$ -graded homology of  $K_\rho$ . In the figure,  $x := e_\sigma$  with  $|x| = \sigma$  and  $y := \bar{\alpha}_{(0)}$  with  $|y| = \rho$ . The two double cones shown are supported by the bar representatives  $[xy]$  and  $[x|y]$ . The  $d_1$ -differential maps from the unit of the infinite dimensional graded ring  $H\mathbb{F}_{2*}$  supported by  $[x|y]$  onto the unit of the  $RO(C_2)$ -graded homology of a point supported by the bar representative  $[xy]$ . Figure 6.2 depicts that this map of units in fact induces a map of graded rings surjecting onto the copy of the  $RO(C_2)$ -graded homology of a point supported by  $[xy]$ .

**6.3.  $RO(C_2)$ -graded bar spectral sequence computations that collapse on the  $E^2$ -page.** One  $RO(C_2)$ -graded bar spectral sequence computation known to collapse on the  $E^2$ -page is

$$E_{*,*}^2 \simeq \text{Tor}_{*,*}^{H_*\mathbb{F}_2}(H\mathbb{F}_{2*}, H\mathbb{F}_{2*}) \implies H_*B\mathbb{F}_2 \cong H_*K_1$$

**Example 6.2.**

$$H_*\mathbb{R}P^\infty = E[e_1, \alpha_{(0)}, \alpha_{(1)}, \dots] \text{ where } |e_1| = 1 \text{ and } |\alpha_{(i)}| = 2^{i+1},$$

Here, the computation is analogous to the underlying nonequivariant one as the  $d^1$  differentials have already been described. There are no other non-trivial differentials since all cycles are concentrated in trivial representation degree.

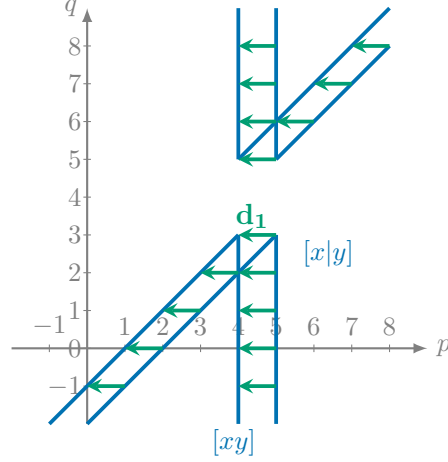


FIGURE 6.2. A more detailed picture of a  $d_1$ -differential in the  $RO(C_2)$ -graded bar spectral sequence

**6.4. Higher differentials: Comparisons between the  $RO(C_2)$  and integer grading.** Classically, Ravenel and Wilson showed that the underlying integer graded bar spectral sequence computing the homology of classical Eilenberg-MacLane spaces  $K_*$  collapses on the  $E^2$ -page [Wil82, Proof of 8.11, pgs 54-55]. This is surprising because many bar spectral sequence computations having arbitrarily long differentials. The Morava  $K$ -theory of classical Eilenberg-MacLane spaces is one example [RW80].

In order to prove the existence of these higher differentials, Ravenel and Wilson first used a degree argument in combination with the Hopf ring structure to describe the structure of possible non-trivial differentials [RW80, Lemma 6.10]. They then proved that these differentials do in fact exist in the integer graded bar spectral sequence computing the Morava  $K$ -theory of classical Eilenberg-MacLane spaces [RW80, Theorem 11.5(b)].

In the  $RO(C_2)$ -graded setting, we might expect that the bar spectral sequence computing the  $RO(C_2)$ -graded homology of the  $C_2$ -equivariant Eilenberg-MacLane spaces  $K_V$  also collapses on the  $E^2$ -page. In fact, we still hope to prove this. However, despite the  $RO(C_2)$ -graded bar spectral sequence once again being a spectral sequence of Hopf algebras, it is not possible to determine all possible nontrivial higher differentials using dimension and Hopf ring arguments similar to those of Ravenel and Wilson (compare, for instance, with [RW80] Lemma 6.10). The main reason for this is that  $RO(C_2)$ -graded homology of a point is an infinite dimensional graded ring as opposed to a ring concentrated in a single degree. The infinite number of graded dimensions complicates proofs relying on degree arguments, ultimately leaving open the possibility that some cycles on the  $E^2$ -page support a differential.



**6.5. Differentials in the  $RO(C_2)$ -graded bar spectral sequence computing  $H_*K_V$ ,  $2\sigma + 1 \subset V$ .** The lack of dimension arguments analogous to those in the nonequivariant case complicates our attempts to finish the computation of the  $RO(C_2)$ -graded homology of the spaces  $K_V$ , where  $2\sigma + 1 \subset V$ .

Consider the  $RO(C_2)$ -graded bar spectral sequence

$$E_{*,*}^2 \simeq \text{Tor}_{*,*}^{H_*K_{2\sigma}}(H\mathbb{F}_{2*}, H\mathbb{F}_{2*}) \implies H_*BK_{2\sigma} \cong H_*K_{2\sigma+1}$$

computing the  $RO(C_2)$ -graded homology of  $K_{2\sigma+1} \simeq BK_{2\sigma}$ . The indecomposable cycles on the  $E^2$ -page are

$$[e_\sigma \circ e_\sigma], \quad [e_\sigma \circ \bar{\alpha}_{(i)}], \quad [\bar{\alpha}_{(j_1)} \circ \bar{\alpha}_{(j_2)}]$$

$$\phi^{(k)}(e_\sigma \circ e_\sigma), \quad \phi^{(k)}(e_\sigma \circ \bar{\alpha}_{(i)}), \quad \phi^{(k)}(\bar{\alpha}_{(j_1)} \circ \bar{\alpha}_{(j_2)}),$$

and we have identified  $[e_\sigma \circ e_\sigma]$  with  $e_1 \circ e_\sigma \circ e_\sigma$ ,  $[e_\sigma \circ \bar{\alpha}_{(i)}]$  with  $e_1 \circ e_\sigma \circ \bar{\alpha}_{(i)}$ ,  $[\bar{\alpha}_{(j_1)} \circ \bar{\alpha}_{(j_2)}]$  with  $e_1 \circ \bar{\alpha}_{(j_1)} \circ \bar{\alpha}_{(j_2)}$ ,  $\phi^{(k)}(e_\sigma \circ \bar{\alpha}_{(i)})$  with  $\bar{\beta}_{(j_1+1)} \circ \bar{\alpha}_{(j_2+1)}$ , and  $\phi^{(k)}(\bar{\alpha}_{(j_1)} \circ \bar{\alpha}_{(j_2)})$  with  $\alpha_{(i_1)} \circ \bar{\alpha}_{(i_2)} \circ \bar{\alpha}_{(j_3)}$  (§5.5). Thus we know that these are all permanent cycles. However, we also saw that degree reasons make it impossible to identify  $\phi^{(k)}(e_\sigma \circ e_\sigma)$  in terms of circle products (there are too many sign representations) and thus it remains to determine if these cycles support any higher differentials.

We have not yet been able to eliminate the possibility of such higher differentials. Specifically, dimension arguments do not preclude the possibility that the elements  $\phi^{(k)}(e_\sigma \circ e_\sigma)$  support a higher differential hitting the negative cone of another permanent cycle (Figure 6.3).

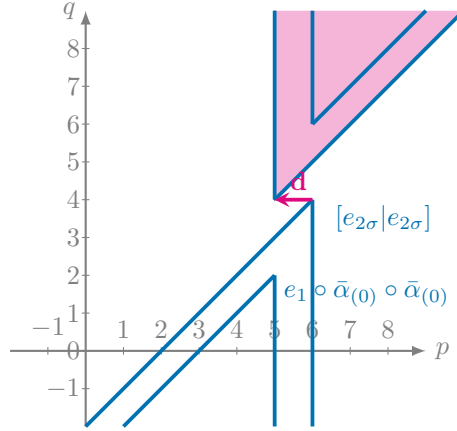


FIGURE 6.3. Possible higher differential in the bar spectral sequence computing  $H_*K_{2\sigma+1}$

However, while we have not yet been able to eliminate the possibility of such differentials hitting negative cones, we also have no evidence that such differentials do exist. The reader familiar with motivic homotopy theory might suggest working in that setting as there are no negative cones. However, sufficiently concrete models of  $\mathbb{G}_m$  delooping do not yet exist. The only one known to the author is Prop 5.2 of [EHK<sup>+</sup>21]. Additionally, what is known about the motivic homology

of motivic Eilenberg-MacLane spaces indicates that the answers are quite complicated ([Voe10], Remark 3.57). Thus we do not expect immediate results from this direction. An alternate potential proof strategy using the  $RO(C_2)$ -graded version of the Borel theorem [MT68, page 88, Theorem 1] is outlined in Section 7.

We have focused our discussion on  $H_*K_{2\sigma+1}$ , the lowest dimensional example where this complication shows up. The remaining cases  $H_*K_V$  where  $2\sigma + 1 \subset V$  are similar. In future work, we hope to show that the bar spectral sequences computing  $H_*K_V$ ,  $2\sigma + 1 \subset V$  collapse at the  $E^2$ -page. However, at the moment, we both have no evidence of higher differentials and no proof of their nonexistence for certain families of cycles on the  $E^2$ -page

**6.6. The  $RO(C_2)$ -graded twisted bar spectral sequence.** We now turn to the twisted analogue of the  $RO(C_2)$ -graded bar spectral sequence. Similar to the classical case, the twisted bar construction  $B^\sigma A$  is filtered by

$$(B^\sigma A)^{[t]} \simeq \coprod_{t \geq n \geq 0} \Delta^n \times A^n / \sim \subset B^\sigma A$$

with associated graded pieces

$$(B^\sigma A)^{[t]} / (B^\sigma A)^{[t-1]} \simeq S^{\lceil \frac{t}{2} \rceil \sigma + \lfloor \frac{t}{2} \rfloor} \wedge A^{\wedge t},$$

where the  $C_2$ -action on  $A^t$  is given by  $\gamma(a_1 \wedge \cdots \wedge a_n) = (\gamma a_n \wedge \cdots \wedge \gamma a_1)$ . Applying  $H_*(-)$  to these filtered spaces gives the twisted bar spectral sequence

$$E_{t,*}^1 = \tilde{H}_*(S^{\lceil \frac{t}{2} \rceil \sigma + \lfloor \frac{t}{2} \rfloor} \wedge A^t) \Rightarrow H_* B^\sigma A,$$

with differentials

$$d_r : E_{t,*}^r \rightarrow E_{t-r,*-1}^r,$$

computing  $H_*(B^\sigma A)$ .

In general, this spectral sequence lacks an explicit  $E^2$ -page and can be difficult to compute. We give some readily computable examples which collapse on the  $E^1$ -page and then turn to analyzing the structure of the twisted bar spectral sequence in examples computing the  $RO(C_2)$ -graded homology of  $C_2$ -equivariant Eilenberg-MacLane spaces.

**Example 6.3.** *The  $RO(C_2)$ -graded twisted bar spectral sequences computing the homology of  $B^\sigma \mathbb{F}_2 \simeq K(\mathbb{F}_2, \sigma) \simeq \mathbb{R}P_{tw}^\infty$  and  $B^\sigma S^1 \simeq K(\mathbb{Z}, \rho) \simeq \mathbb{C}P_{tw}^\infty$  collapse on the  $E^1$ -page. As rings,*

$$\begin{aligned} H_* \mathbb{R}P_{tw}^\infty &= E[e_\sigma, \bar{\alpha}_{(0)}, \bar{\alpha}_{(1)}, \cdots] = E[e_\sigma] \otimes \Gamma[\bar{\alpha}_{(0)}], \quad |e_\sigma| = \sigma, \quad |\bar{\alpha}_{(i)}| = \rho 2^i, \\ H_* \mathbb{C}P_{tw}^\infty &= E[\bar{\beta}_{(0)}, \bar{\beta}_{(1)}, \cdots] = \Gamma[e_\rho] \quad \text{where } |\bar{\beta}_{(i)}| = \rho 2^i. \end{aligned}$$

*We write the proof for  $H_* \mathbb{R}P_{tw}^\infty$  as the computation for  $H_* \mathbb{C}P_{tw}^\infty$  is similar.*

*Proof.* We first prove the additive statement that  $H_* \mathbb{R}P_{tw}^\infty$  is a free  $H_*$ -module with a single generator in each degree  $\lceil \frac{n}{2} \rceil \sigma + \lfloor \frac{n}{2} \rfloor$ . We then show  $H_* \mathbb{R}P_{tw}^\infty$  has ring structure  $E[e_\sigma, \bar{\alpha}_{(0)}, \bar{\alpha}_{(1)}, \cdots] = E[e_\sigma] \otimes \Gamma[\bar{\alpha}_{(0)}]$  where  $|e_\sigma| = \sigma$  and  $|\bar{\alpha}_{(i)}| = 2^i \rho$ . We start with the twisted bar spectral sequence

$$E_{t,*}^1 = \tilde{H}_*(S^{\lceil \frac{t}{2} \rceil \sigma + \lfloor \frac{t}{2} \rfloor} \wedge \mathbb{F}_2^t) \Rightarrow H_* B^\sigma \mathbb{F}_2.$$

Specifically,

$$\begin{aligned}
 E_{i,\star}^1 &\cong \tilde{H}_\star(B_i^\sigma \mathbb{F}_2 / B_{i-1}^\sigma \mathbb{F}_2) \\
 &\cong \tilde{H}_\star(S^{\lceil \frac{t}{2} \rceil \sigma + \lfloor \frac{t}{2} \rfloor} \wedge \mathbb{F}_2^{\wedge t}) && \text{(by definition)} \\
 &\cong \tilde{H}_\star(S^{\lceil \frac{t}{2} \rceil \sigma + \lfloor \frac{t}{2} \rfloor}) \otimes \tilde{H}_\star(\mathcal{N}_e^{C_2}(\mathbb{F}_2^{\wedge \lfloor \frac{t}{2} \rfloor} \wedge \mathbb{F}_2^\epsilon)) && \text{(freeness \& properties of } \mathcal{N}_e^{C_2}) \\
 &\cong \tilde{H}_\star(S^{\lceil \frac{t}{2} \rceil \sigma + \lfloor \frac{t}{2} \rfloor}) \otimes \tilde{H}_\star(\mathbb{F}_2)^{\wedge t} && \text{(homology of norm of underlying free space)}
 \end{aligned}$$

where in the last step, since the homology of  $\mathbb{F}_2$  splits as the homology of induced representation spheres, the homology of the norm is the norm of the homology of the underlying space [Hil19].

Because the filtration degree  $t$  corresponds to the topological degree  $p$  and differentials  $d^r$  shift topological degree down by one, there are no nonzero  $d^r$ ,  $r > 1$ . There can be no nonzero  $d^1$  because if there were, on passing to the nonequivariant homology of the underlying space,  $H_\star \mathbb{R}P^\infty$ , we would be killing a known generator which is a contradiction. Hence the homology is free with a single generator in each degree  $\lceil \frac{n}{2} \rceil \sigma + \lfloor \frac{n}{2} \rfloor$ . This  $E^1$ -page is depicted in Figure 6.4.

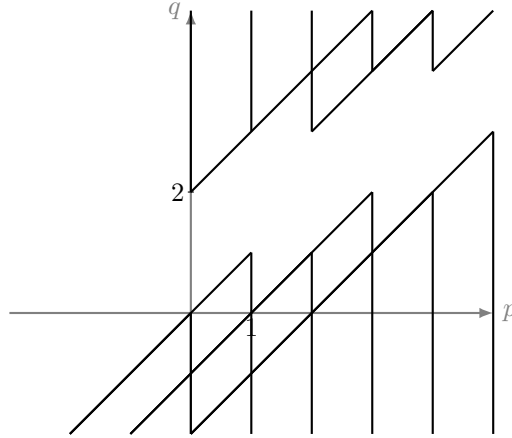


FIGURE 6.4.  $E^1$ -page of the twisted bar spectral sequence computing  $H_\star K_\sigma$

We deduce the multiplicative structure. There is no element in degree  $2\sigma$  so  $e_\sigma$  must be exterior. The remaining exterior structure can be deduced from  $H^\star K_\sigma$  being primitively generated.  $\square$

**Theorem 6.1.** *The  $RO(C_2)$ -graded twisted bar spectral sequence computing the homology of  $B^\sigma S^\sigma \simeq K(\mathbb{Z}, 2\sigma)$  collapses on the  $E^1$ -page. As a ring,*

$$H_\star K(\mathbb{Z}, 2\sigma) = E[e_{2\sigma}] \otimes \Gamma[\bar{x}_{(0)}] \text{ where } |e_{2\sigma}| = 2\sigma, |\bar{x}_{(0)}| = 2\rho.$$

The proof of Theorem 6.1 is analogous to the computation of  $H_\star \mathbb{R}P_{tw}^\infty$  given in Example 6.3.

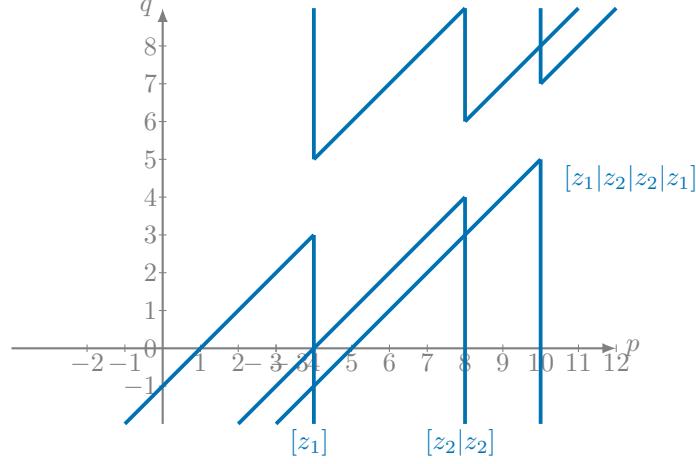


FIGURE 6.5. Twisted bar representatives fixed under the  $C_2$ -action support full double cones

**6.7. Higher differentials in the  $RO(C_2)$ -graded twisted bar spectral sequence.** In this section, we use our understanding of  $H_*K_{*\sigma}$  to analyze the structure of the twisted bar spectral sequence and find evidence of arbitrarily long equivariant degree shifting differentials.

Consider the  $RO(C_2)$ -graded twisted bar spectral sequence

$$E_{t,*}^1 = \tilde{H}_*(S^{\lceil \frac{t}{2} \rceil \sigma + \lfloor \frac{t}{2} \rfloor} \wedge K_\sigma^{\wedge t}) \Rightarrow H_*B^\sigma K_\sigma,$$

computing  $H_*K_{2\sigma}$ . There are two basic building blocks in this spectral sequence. Twisted bar representatives  $[z_1 | \cdots | z_n]$ ,  $z \in H_*K_\sigma$  that are fixed under the  $C_2$ -action of the twisted bar construction and those that possess non-trivial  $C_2$ -action. The twisted bar representatives which are fixed support a full double cone, that is an  $RO(C_2)$ -graded representation degree shifted copy of the homology of the point. An example where  $|z_1| = \sigma$  and  $|z_2| = \rho$  is shown in Figure 6.5. Let  $\gamma$  denote the generator of  $C_2$ . The remaining twisted bar representatives come in pairs  $[z_1 | \cdots | z_n]$  and  $\gamma \cdot [z_1 | \cdots | z_n]$ . Each pair gives a copy of  $C_{2+}$  and we choose a single twisted bar representative to represent each copy. In the twisted bar spectral sequence, the representatives  $[z_1 | \cdots | z_n]$  with non-trivial  $C_2$ -action support shifted degree copies of  $H_*C_{2+}$  as depicted in Figure 6.6.

A portion of the twisted bar spectral sequence computing  $H_*K_{2\sigma}$  appears in Figure 6.7, where  $x$  represents  $e_\sigma$  and  $y$  represents  $\bar{\alpha}_{(0)}$ . To compute the  $d_1$ -differential in this spectral sequence, consider the cofiber sequence

$$S^0 \xrightarrow{a} S^\sigma \longrightarrow Ca \simeq \Sigma C_{2+}.$$

This induces a long exact sequence in homology involving

$$H_*S^0 \xrightarrow{a} H_*S^\sigma \longrightarrow H_*(C_{2+})$$

as shown in Figure 6.8. The map

$$H_*(C_{2+}) \rightarrow H_*(S^{\sigma-1})$$

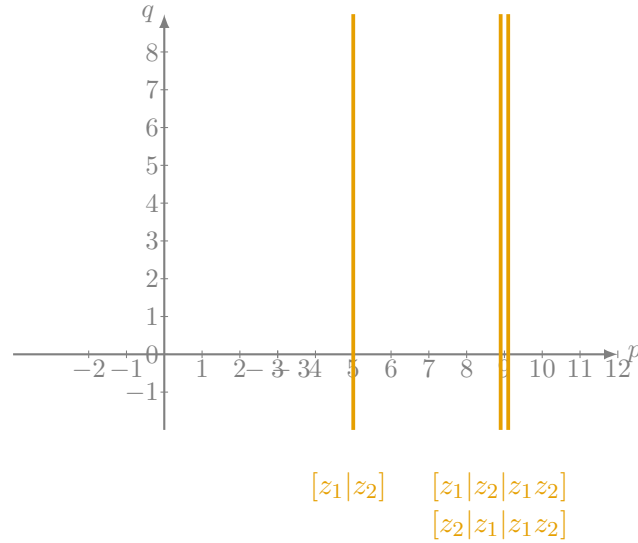


FIGURE 6.6. Twisted bar representatives with nontrivial  $C_2$  action support copies of  $H_*C_{2+}$ .

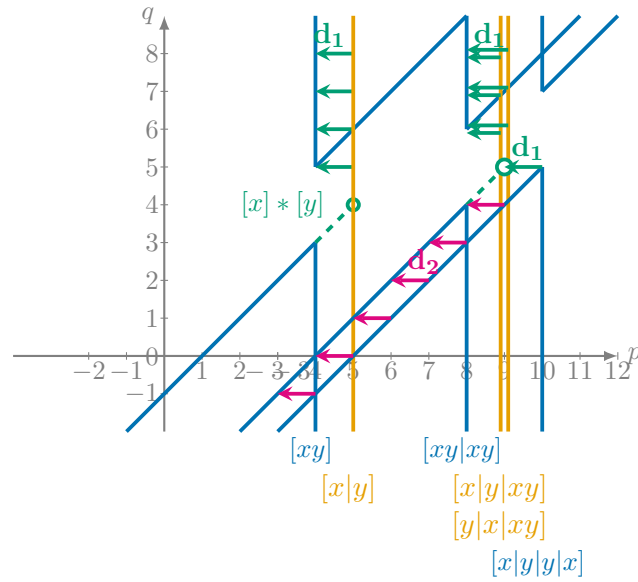


FIGURE 6.7. Differentials in the twisted bar spectral sequence computing  $H_*K_{2\sigma}$

is the map depicted in Figure 6.9.

We have shown that the  $d_1$ -differentials marked in green in Figure 6.7 both exist and have the behavior of the map in Figure 6.8. We also know from Theorem 5.2

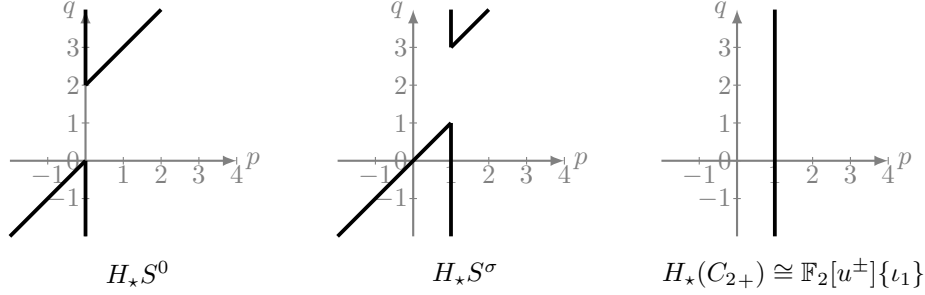


FIGURE 6.8. Computing a  $d_1$ -differential in the twisted bar spectral sequence

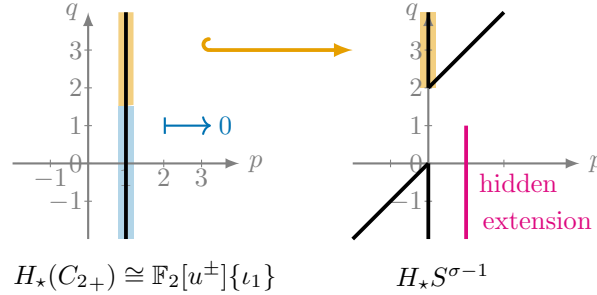


FIGURE 6.9.  $RO(C_2)$ -graded twisted bar spectral sequence  $d_1$ -differential with hidden extension

that

$$H_*K_{2\sigma} \cong E[e_{2\sigma}, \bar{\alpha}_{(j_1)} \circ \bar{\alpha}_{(j_2)}]$$

where  $j_1 \leq j_2$ .

Since the  $RO(C_2)$ -graded homology of  $K_{2\sigma}$  is free over the  $RO(C_2)$ -graded homology of a point, all copies of  $H_*C_{2+}$  appearing on the  $E^1$ -page must either be killed off or used in shifting the representation degree of the  $RO(C_2)$ -graded homology of a point, similar to the equivariant degree shifting differential  $d_1$  and hidden extension of Figure 6.9.

We also know the underlying integer graded homology of  $K_2$ , and have both the forgetful map  $H_*K_{2\sigma} \rightarrow H_2K_2$  and the fixed point map  $H_*K_{2\sigma} \rightarrow H_*(K_{2\sigma})^{C_2} \cong H_*(K_2 \times K_1 \times K_0)$ . Given that  $H_*K_{2\sigma}$  is free and in the underlying nonequivariant case  $[xy | xy]$  is killed by a  $d_1$  differential (all generators of  $H_*K_{2\sigma}$  have nontrivial underlying homology), the entire double cone supported by the twisted bar representative  $[xy | xy]$  must be hit by a differential.

There is a  $d_1$ -differential and hidden extension shifting the double cone supported by  $[xy | xy]$  up by representation degree  $\sigma$  so that by the  $E^2$ -page the double cone is in fact in representation degree  $\rho(|x| + |y|) + \rho + \sigma = \rho(\rho + \sigma) + \rho + \sigma = 4\rho + \sigma$ .

We hypothesize there is a  $d_2$ -differential induced by a  $d_1$ -differential supported by  $[x|y|y|x]$ . We notice that  $[x|y|y|x]$  is a norm of  $[xy|xy]$ . We expect such norms play an important role in governing the structure of all the higher nontrivial differentials.

As one goes farther along in the spectral sequence, considering cycles supported by twisted bar representatives such as  $[xy|xy|xy]$  and  $[xyz|xyz]$ , which must all be killed off in order to recover the correct underlying homology, we see that arbitrarily long equivariant degree shifting differentials are required in order to arrive at the answer given by Theorem 5.2. We conjecture all such cycles are killed by differentials induced by a norm structure on the twisted bar spectral sequence.

## 7. CONJECTURES AND ONGOING WORK

We describe a few questions of immediate interest given the results of this paper.

**7.1. Twisted Tor and the  $RO(C_2)$ -graded twisted bar spectral sequence.** In the  $C_2$ -equivariant setting, the  $RO(C_2)$ -graded homology of each signed delooping,  $K_{V+\sigma}$ , of an equivariant Eilenberg-MacLane space,  $K_V$ , also independently arises as the result of a  $C_2$ -equivariant twisted Tor computation. This can be seen by taking the model of  $\sigma$ -delooping defined in [Hil19]. In this model,  $A$  is an  $E_\sigma$ -algebra and

$$B^\sigma(A) = B(A, \text{Map}(C_2, A), \text{Map}(C_2, *)),$$

where the action of  $\text{Map}(C_2, A)$  on  $A$  is via the  $E_\sigma$ -structure ([Hil19], Definition 5.10). In the case that  $A$  has  $R$ -free homology, Hill constructs yet another twisted bar spectral sequence with  $E^2$ -page

$$E_2^{s, \mathbf{x}} = \text{Tor}_{-s}^{N_e^{C_2}(i_e^* R_*(i_e^* A))}(R_{\mathbf{x}}(\text{Map}(C_2, X)), R_{\mathbf{x}}(A)) \Rightarrow R_{\mathbf{x}-s}(B^\sigma(A))$$

([Hil19], Theorem 5.11). Computations with this spectral sequence are complicated and the literature lacks substantial examples. However, it does have a twisted Tor functor as its  $E^2$ -page and thus it would be interesting to compare with our computations.

One notable feature of the nonequivariant computation of  $H_*K(\mathbb{F}_p, *)$  is that the integer graded bar spectral sequences collapse on the  $E^2$ -page [Wil82]. In contrast, we saw that the  $RO(C_2)$ -graded twisted bar spectral sequences computing  $H_*K_{*\sigma}$  have arbitrarily long differentials in Section 6.7. Thus under favorable circumstances, we hope to formulate a twisted bar spectral sequence with  $E^2$ -page a twisted Tor functor arising as a derived functor of the twisted product of  $H\mathbb{E}_2$ -modules, which collapses in the relevant cases of  $H_*K_{*\sigma}$ .

Given our computation of  $H_*K_{*\sigma}$ , such a twisted Tor over an exterior algebra should have the property that

$$\text{Tor}_{tw}^{E[x]} \cong E[\sigma x] \otimes \Gamma[\mathcal{N}_e^{C_2}x].$$

## 7.2. Conjectures for $H_*K_V$ when $2\sigma + 1 \subset V$ .

**Conjecture 7.1.** The  $RO(C_2)$ -graded homology of  $K_V$ ,  $2\sigma + 1 \subset V$ , is exterior on generators given by the cycles on the  $E^2$ -page of the  $RO(C_2)$ -graded bar spectral sequence. Equivalently, the  $RO(C_2)$ -graded bar spectral sequences computing  $H_*K_V$  collapse on the  $E^2$ -page.

*Possible proof strategy:* Combine our knowledge of  $H_*K_{n\sigma}$  with the approach to studying trivial representation deloopings described in Yigit's thesis [Yig19]. Specifically, dualize the results of Theorem 5.2. Then rewrite that dual basis in terms of cohomology operations and apply the  $RO(C_2)$ -graded Borel theorem to show that the cohomology of  $K_{n\sigma+1}$  is in fact free on generators in the expected representation degrees. Use this information to show that the  $RO(C_2)$ -graded bar spectral sequence must collapse. Iterate this process to compute the homology of all the trivial representation deloopings of  $K_{n\sigma}$ .

**7.3. Global Hopf rings.** In their work computing the integer graded homology of classical nonequivariant Eilenberg-MacLane spaces, Ravenel and Wilson obtain a global statement. Specifically,

**Theorem C** (Ravenel-Wilson [Wil82]).  $H_*K_*$  is the free Hopf ring on  $H_*K_0 = H_*[\mathbb{F}_p]$ ,  $H_*K_1$ , and  $H_*\mathbb{C}P^\infty \subset H_*K_2$  subject to the relation that  $e_1 \circ e_1 = \beta_1$ .

It is natural to ask if a similar statement be obtained in the  $C_2$ -equivariant case, and in that case, what specifically, is the global structure of the Hopf rings that do arise. One may also ask how the Hopf rings here relate to Hill and Hopkins' work extending Ravenel and Wilson's construction of a universal Hopf ring over  $MU^*$  to  $C_2$ -equivariant homotopy theory [HH18].

**7.4. Stabilizing to the  $C_2$ -dual Steenrod algebra.** Besides understanding a global version of the unstable story, it also remains to fully understand how the the unstable answer for  $H_*K_V$  stabilizes to give the  $C_2$ -equivariant dual Steenrod algebra,

$$\mathcal{A}_*^{C_2} = H\underline{\mathbb{F}}_2[\tau_0, \tau_1, \dots, \xi_1, \xi_2, \dots] / (\tau_i^2 = (u + a\tau_0)\xi_{i+1} + a\tau_{i+1}).$$

By Hu-Kriz's construction of the  $C_2$ -equivariant dual Steenrod algebra [HK01], we should homology suspend  $\bar{\beta}_{(i)}$  to define

$$\xi_i \in H_{(2^i-1)\rho}H$$

and  $\bar{\alpha}_{(i)}$  to define

$$\tau_i \in H_{2^i\rho-\sigma}H.$$

However, it is not at all clear what an arbitrary element in  $H_*K_V$  should stabilize to in the the  $C_2$ -equivariant dual Steenrod algebra. Additionally, there is the interesting problem of understanding how the stable relation  $\tau_i^2 = (u + a\tau_0)\xi_{i+1} + a\tau_{i+1}$  arises unstably. We look forward to studying these questions in forthcoming work.



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