THE HOMOTOPY TYPE OF *MSU*

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0. Introduction. This paper examines the homotopy type of the Thom spectrum MSU associated with special unitary cobordism. For odd primes p, standard methods show that the p-localization $MSU_{(p)}$ is equivalent to a wedge of suspensions of the Brown-Peterson spectrum BP. For $p = 2$, however, this is not the case, and our work is devoted to determining the 2-primary homotopy type of MSU. This involves a new indecomposable spectrum, and our main results are the following.

There is an indecomposable 2-local spectrum, which we call BoP, such that $MSU_{(2)}$ is equivalent to a wedge of suspensions of BoP and BP. Under the equivalence, the Thom class lies in a BoP summand. As a comodule over the dual Steenrod algebra A [11], $H_*(BoP; \mathbf{Z}/2)$ is a sum of suspensions of $B = \mathbf{Z}/2[\zeta_1^4, \zeta_2^2, \ldots, \zeta_j^2, \ldots] \subset A$, where ζ_j is the conjugate of Milnor's generator ξ_j. There is one suspension of B beginning in each nonnegative dimension divisible by 8.

BoP bears strong similarities to BP and the (-1)-connected K-theory spectra bo and bu. In particular, in Section 6 we show there is a map $BoP \xrightarrow{\nu} bo_{(2)}$ inducing an epimorphism ν_* of homotopy groups. In fact, ν_* induces an isomorphism of torsion subgroups, and its torsion free kernel is concentrated in even dimensions.

A brief summary of our methods is as follows. In Sections 1 and 2 we describe the Adams spectral sequence for $\pi_*MSU_{(2)}$, including a computation of the differentials, with particular attention paid to the product structure. Anderson, Brown, and Peterson [4] gave a computation for these differentials, but their proof requires some correction, and in any case we will need the more extensive knowledge of the product structure.

In Sections 3 to 5, we construct BoP and show it is indecomposable. To produce BoP, first the Sullivan-Baas construction is applied to MSU, yielding a spectrum representing a bordism theory of SU-manifolds with

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certain singularities. Then we produce a map from the 2-localization of this spectrum to a wedge of *BP* suspensions, and the fibre is the desired spectrum *BoP*.

Sections 6 and 7 are devoted to producing a homotopy equivalence between $MSU_{(2)}$ and a wedge of *BoP* and *BP* suspensions. Maps from $MSU_{(2)}$ to a suspension of *BoP* are somewhat difficult to construct, so maps to a suspension of $bo_{(2)}$ are constructed first, using the Adams spectral sequence, and then lifted to *BoP* by obstruction theory.

Several of the indecomposable spectra which, like *BoP*, appear as summands in cobordism Thom spectra, have proven extremely useful in homotopy theory. The most notable are the Eilenberg-MacLane and Brown-Peterson spectra, upon which the Adams and Novikov spectral sequences are based. Hopefully, *BoP* too will have a useful role to play in homotopy theory. In particular, a generalized Adams-Novikov spectral sequence based on *BoP* has the advantage that the Hopf map $\eta \in \pi_1 S^0$ appears on the zero line. To apply *BoP* effectively, it would be useful to find a canonical description for it similar to Quillen's construction [1] of *BP*, and to know that *BoP* is a commutative ring spectrum. It must also be shown that *BoP* has better flatness properties than *bo*.

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1. The Mod Two Homology of MSU.

The Thom spectrum *MSU* is a commutative ring spectrum. Thus $H_*(MSU; \mathbf{Z}/2)$ is a graded left A comodule algebra [16], whose structure we will describe below.

Henceforth, 'A algebra' means 'graded left A comodule algebra', unstated coefficient groups are $\mathbf{Z}/2$, and $\otimes$ means $\otimes_{\mathbf{Z}/2}$. We give the polynomial algebra $C = \mathbf{Z}/2[x_8, x_{10}, \ldots, x_{2i}\ (i \neq 2^j - 1), \ldots]$ an A algebra structure by letting x_{2i} be in grade $2i$, and defining the coaction map by $\psi x_{4i} = \zeta_1^2 \otimes x_{4i-2} + 1 \otimes x_{4i}$ for $i \neq 2^j$, $\psi x_{4i-2} = 1 \otimes x_{4i-2}$, and $\psi x_{2^j} = 1 \otimes x_{2^j}$.

Since the subalgebra $B \subset A$ defined in the introduction is in fact a sub-A algebra of A, we can give $B \otimes C$ the natural A algebra structure of a tensor product. In previous work [13], we showed that

THEOREM 1.1. *There is an isomorphism $H_*MSU \cong B \otimes C$ of A algebras.*

We identify H_*MSU and $B \otimes C$ via such an isomorphism, and now proceed to analyze $B \otimes C$. Let $E(x)$ denote the primitive exterior Hopf algebra on x, and let E be the quotient Hopf subalgebra $E(\zeta_1^2)$ of A. Let P be the sub-A algebra $\mathbf{Z}/2[\zeta_1^2, \zeta_2^2, \ldots, \zeta_j^2, \ldots] \subset A$, which is isomorphic to H_*BP and dual to the quotient $A^*/A^*Sq^1A^*$ [7].

LEMMA 1.2. *There is an A algebra isomorphism $B \cong P\Box_E\mathbf{Z}/2$.*

Proof. The isomorphism $\mathbf{Z}/2[\zeta_1^2, \zeta_2, \zeta_3, \ldots] \cong A\Box_{E(\zeta_1)}\mathbf{Z}/2$ is well known [16, p. 511], and squaring provides a Hopf algebra isomorphism $A \cong P$. □

We will also need the following presumably well-known fact.

PROPOSITION 1.3. *If H is a connected graded commutative $\mathbf{Z}/2$ Hopf algebra, D is an H algebra, and E is a quotient Hopf algebra of H, then $(m \otimes 1) \circ (1 \otimes \chi \otimes 1) \circ (1 \otimes \psi_D): H\Box_E D \to (H\Box_E\mathbf{Z}/2) \otimes D$ is defined and is an H algebra isomorphism.*

Proof. After taking $\mathbf{Z}/2$ duals, the formula for the map and the fact that it is an isomorphism follow from the special case $N = \mathbf{Z}/2$ of Proposition 1.7 in [10] along with the commutativity of H. It is straightforward to check that it is an H algebra map. □

Together Lemma 1.2 and Propositon 1.3 imply

COROLLARY 1.4. $(m \otimes 1) \circ (1 \otimes \chi \otimes 1) \circ (1 \otimes \psi): P\Box_E C \to B \otimes C$ *is a P algebra isomorphism and hence an A algebra isomorphism.*

Next we examine $P\Box_E C$. By analyzing the E comodule structure of C, we will be able to express $P\Box_E C$ as a sum of cocyclic A comodules. Let $\tilde{\psi}$ be the quotient E-coaction map, and define $Sq^2: C \to C$ by $\tilde{\psi}c = \zeta_1^2 \otimes Sq^2c + 1 \otimes c$. Sq^2 is a differential and a derivation. Define

$$y_{8i} = \begin{cases} x_{8i} & \text{if} \quad i = 2^j \\ x_{4i}^2 & \text{if} \quad i \neq 2^j \quad \text{for} \quad i \geq 1, \end{cases}$$

and let $Y = \mathbf{Z}/2[y_8, \ldots, y_{8i}, \ldots] \subset C$. Note Sq^2 acts trivially on Y. For $i \neq 2^j$ let R_i be the subspace of C spanned by $\{x_{4i-2}^n, x_{4i}x_{4i-2}^n \mid n \geq 0\}$. R_i is closed under Sq^2. Let $R = \bigotimes_{i \neq 2^j} R_i$ with diagonal Sq^2 action. The natural map $R \otimes Y \to C$ is clearly an $E(Sq^2)$ module isomorphism.

Since $H_*(R; Sq^2) = \mathbf{Z}/2$, R is a sum of a single trivial summand in grade zero and a free E comodule with primitives Sq^2R. We will write R' for Sq^2R. The desired description of $P\Box_E C$ now follows:

PROPOSITION 1.5. *There is a sequence of A comodule isomorphisms*

$$P\Box_E C \cong (P\Box_E R) \otimes Y \cong (P\Box_E(\mathbf{Z}/2 \oplus (E \otimes R'))) \otimes Y$$
$$\cong (B \oplus (P \otimes R')) \otimes Y.$$

We will need to know something about the algebra structure in this description. Notice $P\Box_E C$ contains the A subalgebra $B \otimes Y$. The A algebra inclusion $B \subset P$ provides an obvious B module structure on $B \oplus (P \otimes R')$, and hence a $B \otimes Y$ module structure on $(B \oplus (P \otimes R')) \otimes Y$. The composite isomorphism of (1.5) is clearly a $B \otimes Y$ module map.

2. The Adams Spectral Sequence for $\pi_*\mathbf{MSU}_{(2)}$. The E_2 term of the 2-primary Adams spectral sequence [1] converging to $\pi_*MSU_{(2)}$ is given by $\mathrm{Ext}_A^{*,*}(\mathbf{Z}/2, H_*MSU)$. We abbreviate $\mathrm{Ext}_A^{*,*}(\mathbf{Z}/2, —)$ as $\mathrm{Ext}(—)$. By (1.5) we need only know $\mathrm{Ext}(B)$ and $\mathrm{Ext}(P)$ to describe E_2. They are given by

THEOREM 2.1. [9]. *There are isomorphisms*

$$\mathrm{Ext}(B) \cong \mathbf{Z}/2[q_0, h, q_1^2q_0, q_1^4, q_2, \ldots, q_k, \ldots]/(q_0h, h^3)$$

and

$$\mathrm{Ext}(P) \cong \mathbf{Z}/2[q_0, q_1, q_2, \ldots, q_k, \ldots],$$

where $h \in \mathrm{Ext}^{1,2}$ and $q_i \in \mathrm{Ext}^{1,2^{i+1}-1}$. Under the isomorphisms the map $B \to P$ induces the obvious algebra map.

Applying Theorem 2.1 to the description of H_*MSU provided by (1.5) immediately yields

THEOREM 2.2.

$$\mathrm{Ext}(H_*MSU) \cong (\mathrm{Ext}(B) \oplus (\mathrm{Ext}(P) \otimes R')) \otimes Y$$
$$\cong (\mathbf{Z}/2[q_0, h, q_1^2q_0, q_1^4, q_2, \ldots, q_k, \ldots]/(q_0h, h^3)$$
$$\oplus ((\mathbf{Z}/2[q_0, q_1, q_2, \ldots, q_k, \ldots]) \otimes R')) \otimes Y,$$

with the Ext(B) $\otimes$ Y *module structure induced on the tensor product in the obvious way using the algebra map* Ext(B) $\to$ Ext(P) *described in* (2.1).

We will now use this description of Ext(H_*MSU), along with a result of Conner and Floyd, to determine the differentials in the spectral sequence. The reader is urged to construct a picture of the E_2 term as described in (2.2). We will make frequent use of the following three lemmas, all proven in [4].

LEMMA 2.3. *h is a permanent cycle, and for each r, $E_r^{s,t} \xrightarrow{h\cdot} E_r^{s+1,t+2}$ is an epimorphism if $t - s$ is even, a monomorphism if $t - s$ is odd.*

LEMMA 2.4. *d_2 is zero on the summand* Ext(P) $\otimes R' \otimes Y$ *of* E_2.

LEMMA 2.5. *$q_1^2 q_0$ and q_1^4 are permanent cycles.*

Since d_2 is a derivation, it is completely determined by (2.4), (2.5), and the following theorem.

THEOREM 2.6. *There are elements*

$$q_j' \in E_2^{1,2^{j+1}-1}, \quad \text{for} \quad j \geq 2, \quad \text{and}$$

$$y_{8i}' \in E_2^{0,8i}, \quad \text{for} \quad i \geq 1,$$

of the form

$$q_j' = q_j + \text{decomposables in } \mathbf{Z}/2[q_2, \ldots, q_k, \ldots] \otimes Y,$$

and

$$y_{8i}' = y_{8i} + \text{decomposables in } Y,$$

such that

$$d_2 q_j' = 0,$$

and

$$d_2 y_{2^j}' = h q_{j-1}',$$

and

$$d_2 y_{8i}' = 0 \text{ if } 8i \text{ is not a power of two.}$$

Before proving the theorem, we will see how it immediately leads to the description of E_2 and d_2 that we desire. Let $\mathrm{Ext}(B)'$ denote the subalgebra $\mathbf{Z}/2[q_0, h, q_1^2q_0, q_1^4, q_2', \ldots, q_k', \ldots]$ of E_2 in which the element q_k in $\mathrm{Ext}(B)$ has been replaced by q_k' for $k \geq 2$. From (2.2) and (2.6) we have

COROLLARY 2.7. *There is an isomorphism* $E_2 \cong (\mathrm{Ext}(B)' \oplus (\mathrm{Ext}(P) \otimes R')) \otimes Y$ *with* $Y = \mathbf{Z}/2[y_8', \ldots, y_{8i}', \ldots]$, *and the differential* d_2 *is described explicitly by* (2.4), (2.5), and (2.6).

Proof of Theorem 2.6. Suppose inductively that appropriate q_j' and y_{8i}' have been found for all j such that $2^{j+1} < 8k$ and all i such that $8i < 8k$.

Let l be the largest integer such that $2^l < 8k$. Let $G_k^{*,*} = \mathbf{Z}/2[q_1^4, q_2', \ldots, q_{l-1}'] \otimes \mathbf{Z}/2[y_8', \ldots, y_{8(k-1)}'] \subset E_2^{*,*}$. Define a derivation $d: G_k^{s,t} \to G_k^{s+1,t-1}$ by letting $dy_{2^j}' = q_{j-1}'$ for $3 \leq j \leq l$ and letting $d = 0$ on all the other polynomial generators of G_k. Notice that on G_k, $d_2 = h \cdot d$ by the inductive hypothesis, and d is a differential. $H^{*,*}(G_k; d)$ is easily computed using the Künneth theorem, and we find that $H^{s,t}$ is nonzero only if $t - s \equiv 0 \pmod 8$.

Case I. *$8k$ is not a power of two.* Consider the 'column' $E_2^{s,t}$ with $t - s = 8k - 1$, and the map

$$h \cdot (\ker d)^{s-1,t-2}/h \cdot (\operatorname{im} d)^{s-1,t-2} \to (\ker d_2)^{s,t}/(\operatorname{im} d_2)^{s,t}.$$

Using (2.3) we see the numerators are equal. The left group is zero since $H^{s-1,t-2}(G_k; d) = 0$, and thus the two denominators are equal. So $d_2y_{8k} = d_2y$ for some $y \in G_k^{0,8k}$. Defining $y_{8k}' = y_{8k} + y$ completes the inductive step.

Case II. *$8k = 2^m$ for some m.* First we will examine the 'column' $E_2^{s,t}$ with $t - s = 2^m - 3$. The same argument as in Case I shows that $d_2q_{m-1} = d_2x$ for some $x \in G_k^{1,2^m-1}$. Define $q_{m-1}' = q_{m-1} + x$, so $d_2q_{m-1}' = 0$. The same argument also shows that $E_3^{s,t} = 0$ for $t - s = 2^m - 3$, so q_{m-1}' is a permanent cycle, and thus so is $h \cdot q_{m-1}' \in E_2^{2,2^m+1}$. From Conner and Floyd's work [8] we know $\pi_{2^m-1}MSU_{(2)} = 0$, so $h \cdot q_{m-1}'$ must be in the image of a differential. The only possibility is $d_2: E_2^{0,2^m} \to E_2^{2,2^m+1}$. Now $E_2^{0,2^m} \cong G_k^{0,2^m} \oplus \{y_{2^m}\} \oplus (R' \otimes Y)^{0,2^m}$. From (2.4) we know $d_2 = 0$ on the rightmost summand. Inductively d_2 has been determined on $G_k^{0,2^m}$, and $h \cdot q_{m-1}'$ is clearly not in the image. Thus there must be an element $y \in G_k^{0,2m}$ such that $d_2(y_{2^m} + y) = h \cdot q'_{m-1}$. Let $y'_{2^m} = y_{2^m} + y$. □

Finally, we will show that all higher differentials are zero. Let $G^{*,*} = \mathbf{Z}/2[q_1^4, q'_2, \ldots, q'_k, \ldots] \otimes \mathbf{Z}/2[y'_8, \ldots, y'_{8i}, \ldots] \subset E_2^{*,*}$, with d as in the proof of (2.6). $H^{*,*}(G; d)$ is easily computed, and $H^{s,t}(G; d) = 0$ unless $t - s \equiv 0 \pmod 8$. Now if $t - s$ is odd, $E_3^{s,t} \cong H^{s-1,t-2}(G; d)$, since the map

$$h \cdot (\ker d)^{s-1,t-2} / h \cdot (\operatorname{im} d)^{s-1,t-2} \to (\ker d_2)^{s,t} / (\operatorname{im} d_2)^{s,t}$$

is an isomorphism, and multiplication by h maps G monomorphically in E_2. Thus E_3 is rather sparse in the sense that $E_3^{s,t} = 0$ for $t - s \equiv 3$, 5, or 7 (mod 8). This will enable us to prove

PROPOSITION 2.8. *All the higher differentials* $d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}$, *for* $r \geq 3$, *are zero.*

Proof. By the sparseness of E_3 this is obvious except when $t - s \equiv 1$ or 2 (mod 8).

Case I. $t - s \equiv 1 \pmod 8$. Given $x \in E_r^{s,t}$, by (2.3) $x = h \cdot y$ for some y. Now $d_r y = 0$ by sparseness, so $d_r x = h \cdot d_r y = 0$.

Case II. $t - s \equiv 2 \pmod 8$. Given $x \in E_r^{s,t}$, by (2.3) we have $h \cdot d_r x = d_r(h \cdot x) = d_r(0) = 0$, so $d_r x = 0$, again by (2.3). □

3. The Sullivan-Baas Construction. In this section we will apply the Sullivan-Baas construction [6] to *MSU* to produce a spectrum whose 2-localization is closely related to the indecomposable spectrum *BoP* we seek.

First we describe a sequence of elements in $\pi_* MSU$ for use with the Sullivan-Baas construction. Define $z_{8i} \in Y$ for $i \geq 2$ by

$$z_{8i} = \begin{cases} y'_{8i} & \text{if} \quad i \neq 2^j \\ y'^2_{4i} & \text{if} \quad i = 2^j. \end{cases}$$

In our description of the Adams spectral sequence, $z_{8i} \in E_2^{0,8i}$ survives to $E_\infty^{0,8i}$ by (2.6) and (2.8), and is thus represented by an element $\hat{z}_{8i} \in \pi_{8i} MSU$. If $h : \pi_* MSU \to H_* MSU$ is the Hurewicz homomorphism, $h(\hat{z}_{8i}) = z_{8i}$.

LEMMA 3.1. *The sequence* $\{z_{8i}\}_{i\geq 2}$ *is a regular sequence in* H_*MSU.

Proof. We will follow $z_{8i} \in Y$ under the composite $S: Y \cong \mathbf{Z}/2 \otimes Y \subset P\square_E C \to B \otimes C$ involving the map $(m \otimes 1) \circ (1 \otimes \chi \otimes 1) \circ (1 \otimes \psi)$ of (1.4). Notice that

$$S(y_{8i}) = \begin{cases} S(x_{4i}^2) = \zeta_1^4 \otimes x_{4i-2}^2 + 1 \otimes x_{4i}^2 & \text{if} \quad i \neq 2^j \\ S(x_{8i}) = 1 \otimes x_{8i} \quad \text{if} \quad i = 2^j. \end{cases}$$

So if we write $B \otimes C = \mathbf{Z}/2[\zeta_1^4, \zeta_2^2, \ldots, \zeta_j^2, \ldots] \otimes \mathbf{Z}/2[x_8, x_{10}, \ldots, x_{2i} \; (i \neq 2^j - 1), \ldots]$ as $\mathbf{Z}/2[w_4, w_6, w_8, \ldots, w_{2i}, \ldots]$ in the obvious way, with w_{2i} in grade $2i$, we see that $S(Y) = \mathbf{Z}/2[w_{12}^2 + w_4 w_{10}^2, \ldots, w_{4i}^2 + w_4 w_{4i-2}^2 \; (i \neq 2^j), \ldots, w_8, w_{16}, \ldots, w_{2^j}, \ldots]$, and

$$S(z_{8i}) = \begin{cases} w_{4i}^2 + w_4 w_{4i-2}^2 + \text{decomposables in } S(Y), & \text{if} \quad i \neq 2^j \\ (w_{4i} + \text{decomposables in } S(Y))^2 \quad \text{if} \quad i = 2^j. \end{cases}$$

$\{S(z_{8i})\}_{i\geq 2}$ is clearly a regular sequence in $\mathbf{Z}/2[w_4, w_6, w_8, \ldots, w_{2i}, \ldots]$. □

Those aspects of the Sullivan-Baas construction relevant to our needs are summarized by

THEOREM 3.2. *Let* $\{*, [M_1], \ldots, [M_n], \ldots\}$ *be a sequence of elements in* $\Omega_*^{SU}(\text{point}) \cong \pi_* MSU$, *and let* $m_n = h([M_n]) \in H_*(MSU)$. *Suppose* $\{m_n\}_{n\geq 1}$ *is a regular sequence in the algebra* H_*MSU. *Then there are CW-spectra* $M(n)$ for $n \geq 0$ *and maps* $M(n) \xrightarrow{p_n} M(n+1)$ *with* $M(0) = MSU$, *such that the composite* $M(0) \xrightarrow{p_0} M(1) \xrightarrow{p_1} \cdots \xrightarrow{p_{n-1}} M(n)$ *is an epimorphism in* mod 2 *homology with kernel the ideal generated by* $\{m_1, \ldots, m_n\}$.

Proof. Let $S_n = \{*, [M_1], \ldots, [M_n]\}$ and let $MSU(S_n)_*(—)$ be the bordism theory of SU-manifolds with singularity set S_n (see Baas [6]). There are long exact sequences

(3.3)
$$\cdots \to \widetilde{MSU(S_n)_*(—)} \xrightarrow{\beta_n = \cdot[M_{n+1}]} \widetilde{MSU(S_n)_{*+\dim M_{n+1}}(—)} \xrightarrow{\gamma_n}$$
$$\widetilde{MSU(S_{n+1})_{*+\dim M_{n+1}}(—)} \xrightarrow{\delta_n} \widetilde{MSU(S_n)_{*-1}(—)} \to \cdots$$

relating the corresponding (reduced) generalized homology theories $\widetilde{MSU(S_n)}_*(-)$ and $\widetilde{MSU(S_{n+1})}_*(-)$ on the category of CW-spectra. It is clear from the bordism definition of $\widetilde{MSU(S_n)}_*(-)$ that it satisfies the direct limit axiom. Thus [3] it is represented by a CW-spectrum $M(n)$, and there is a map $M(n) \xrightarrow{p_n} M(n+1)$ inducing the natural transformation γ_n. Of course $M(0) \simeq MSU$.

If we apply the sequences (3.3) to the Eilenberg-MacLane spectrum $K(\mathbf{Z}/2)$, then β_n is just multiplication by $m_{n+1} = h([M_{n+1}])$, so we see inductively that it is a monomorphism, and conclude that the composite $M(0) \xrightarrow{p_0} M(1) \to \cdots \xrightarrow{p_{n-1}} M(n)$ has the desired property. □

Letting $M = \varinjlim_{p_n} M(n)$, we have

COROLLARY 3.4. *With the hypotheses of Theorem 3.2, there is a CW-spectrum M and a map $MSU \xrightarrow{p} M$, such that p is an epimorphism in* mod 2 *homology with kernel the ideal generated by* $\{m_1, \ldots, m_n, \ldots\}$.

Applying (3.4) to the sequence $\{\hat{z}_{8i}\}_{i \geq 2} \subset \pi_* MSU$ already described, using (3.1) and localizing at the prime 2, we obtain

PROPOSITION 3.5. *There is a 2-local CW-spectrum X, and a map $MSU_{(2)} \xrightarrow{p} X$, such that p is an epimorphism in* mod 2 *homology with kernel the ideal generated by* $\{z_{8i}\}_{i \geq 2}$.

4. The Adams Spectral Sequence for $\pi_* X$. We now examine the 2-primary Adams spectral sequence converging to $\pi_* X$ by studying the map of spectral sequences induced by $MSU_{(2)} \xrightarrow{p} X$. We begin by examining the map of E_2 terms.

Recall that Proposition 1.5 identified the A comodule $H_* MSU$ as $(B \oplus (P \otimes R')) \otimes Y$, with the obvious $B \otimes Y$ module structure from the subalgebra $B \otimes Y$. Let $Z = \mathbf{Z}/2[z_{16}, \ldots, z_{8i}, \ldots]$ be the polynomial subalgebra of Y generated by the regular sequence introduced in Section 3. It follows from (3.5) that in homology p induces the natural map

$$(B \oplus (P \otimes R')) \otimes Y \xrightarrow{p_*} (B \oplus (P \otimes R')) \otimes Y \| Z,$$

where $Y \| Z$ denotes the algebra quotient by the ideal generated by $\bar{Z}$. The induced map of E_2 terms is the natural projection

$$(\mathrm{Ext}(B) \oplus (\mathrm{Ext}(P) \otimes R')) \otimes Y \to (\mathrm{Ext}(B) \oplus (\mathrm{Ext}(P) \otimes R')) \otimes Y \| Z.$$

To grasp the behavior of the differentials d_2 we must first interpret the behavior of this map when $\mathrm{Ext}(H_*MSU)$ is identified as in (2.7). Clearly the Y module action on $(\mathrm{Ext}(B)' \oplus (\mathrm{Ext}(P) \otimes R')) \otimes Y$ induced from that on $\mathrm{Ext}(H_*MSU)$ is still just multiplication in the right factor, so in fact

PROPOSITION 4.1. *$p: MSU_{(2)} \to X$ induces a natural projection*

$$(\mathrm{Ext}(B)' \oplus (\mathrm{Ext}(P) \otimes R')) \otimes Y \xrightarrow{\mathrm{Ext}(p_*)} (\mathrm{Ext}(B)' \oplus (\mathrm{Ext}(P) \otimes R')) \otimes Y \| Z$$

of $\mathrm{Ext}_A(\mathbf{Z}/2, -)$ *groups.*

Since p induces a map of spectral sequences, the differentials d_2 on $\mathrm{Ext}(H_*X)$ are completely determined by those on $\mathrm{Ext}(H_*MSU)$ already described in Section 2. The description of $\mathrm{Ext}(H_*X)$ provided by (4.1) has a natural algebra structure respected by $\mathrm{Ext}(p_*)$, and d_2 is a derivation on $\mathrm{Ext}(H_*X)$.

PROPOSITION 4.2. *The map* $\mathrm{Ext}(p_*)$ *has a splitting which is a map of d_2 chain complexes.*

Proof. From the definition of the sequence $\{z_{8i}\}_{i \geq 2} \subset Y$ it is clear that $Y\|Z$ is an exterior algebra with generators represented by $\{y_{2j}'\}_{j \geq 3}$. So there is an obvious identification of $Y\|Z$ with the subspace of Y spanned by the monomials in the y_{2j}''s in which no y_{2j}' appears to a power greater than one. This splitting of $Y \to Y\|Z$ provides a splitting of $\mathrm{Ext}(p_*)$, and from the form of the differentials, as described in (2.7), it is clear the splitting commutes with d_2. □

PROPOSITION 4.3. *The map of E_3 terms induced by p is an epimorphism. All the differentials d_r for $r \geq 3$ in the spectral sequence for π_*X vanish.*

Proof. The splitting of (4.2) shows the map of E_3 terms is onto. The proposition now follows from (2.8). □

5. The Indecomposable Spectrum BoP. In this section we will produce BoP from X, examine the Adams spectral sequence converging to π_*BoP, and show BoP is indecomposable.

Recall from Section 4 that as an A comodule

$$H_*X \cong (B \oplus (P \otimes R')) \otimes Y\|Z \cong (B \otimes Y\|Z) \oplus (P \otimes R' \otimes Y\|Z).$$

From now on we identify H_*X with this direct sum. Since the A algebra P is isomorphic to H_*BP, it appears that X may decompose into two wedge summands, one a spectrum with homology $B \otimes Y||Z$, the other a wedge of BP summands. We will show that it lies in a fibration between two such spectra.

PROPOSITION 5.1. *Let W denote the graded vector space $R' \otimes Y||Z$, and let $\tilde{m}_*$ denote the canonical projection $H_*X \cong (B \otimes Y||Z) \oplus (P \otimes W) \to P \otimes W$. Then there is a map $X \xrightarrow{\tilde{m}} BP \wedge W$ inducing $\tilde{m}_*$ in* mod 2 *homology.*

Proof. The projection $H_*X \xrightarrow{\tilde{m}_*} P \otimes W \xrightarrow{\epsilon\otimes id} \mathbf{Z}/2 \otimes W \cong W$ corresponds naturally to a map $X \xrightarrow{m} K(\mathbf{Z}/2) \wedge W$, since W is of finite type and $H^*(X; \mathbf{Z})$ is concentrated in even dimensions and is torsion free. In mod 2 homology, m induces the natural map $H_*X \xrightarrow{\tilde{m}_*} P \otimes W \to A \otimes W$, since a map of graded bounded below comodules is uniquely determined by its composition with any projection of the target onto its A primitives. The obstructions to lifting m into $BP \wedge W$ lie in zero groups, so we obtain the lift $\tilde{m}$ we seek. □

Now define BoP to be the fibre of $\tilde{m}$, and let $i: BoP \to X$ be the inclusion of the fibre. We immediately have

PROPOSITION 5.2. *In* mod 2 *homology, i_* is a monomorphism onto the left summand $B \otimes Y||Z$* of $(B \otimes Y||Z) \oplus (P \otimes W) \cong H_*X$.

We identify H_*BoP with this summand, and remark that since $Y||Z$ is an exterior algebra on generators y'_{2j} with $j \geq 3$, H_*BoP is a sum of copies of the cocyclic A comodule B, with one copy beginning in each dimension divisible by 8.

Next we compute the Adams spectral sequence for π_*BoP. Not only does $\mathrm{Ext}(H_*X)$ split into the two summands $\mathrm{Ext}(B)' \otimes Y||Z$ and $\mathrm{Ext}(P) \otimes W$, but the form of the differentials, described by (4.1), (4.3), and (2.7), shows this is a splitting of d_2 chain complexes, and hence the entire spectral sequence for π_*X splits. Thus we have

PROPOSITION 5.3. *In the Adams spectral sequence converging to π_*BoP, with $E_2 \cong$ $\mathrm{Ext}(B)' \otimes Y||Z$, the differentials are as follows. d_2 is zero on $\{q_0, h, q_1^2 q_0, q_1^4, q'_2, \ldots, q'_k, \ldots\}$, $d_2 y'_{2j} = hq'_{j-1}$, and d_2 is a derivation with respect to the natural algebra structure of* $\mathrm{Ext}(B)' \otimes Y||Z$. *The higher differentials d_r for $r \geq 3$ are all zero.*

The indecomposability of BoP will follow from

THEOREM 5.4. *Let p be a prime, and suppose F is a CW-spectrum satisfying*

(1) *F is p-local;*

(2) *F is bounded below;*

(3) *$H_*(F; \mathbf{Z})$ is of finite type over $\mathbf{Z}_{(p)}$;*

(4) *The image of the $\mathbf{Z}/p$ Hurewicz homomorphism $\pi_*F \to H_*(F; \mathbf{Z}/p)$ has rank one.*

Then F has no nontrivial wedge decomposition.

Proof. Suppose $F \simeq F' \vee F''$. Since the Hurewicz homomorphism is additive on wedges, the Hurewicz isomorphism theorem along with (2) and (4) shows that one of the wedge summands, say F', has zero mod p homology. $H_*(F'; \mathbf{Z}) \cong H_*(F'; \mathbf{Z}_{(p)})$ is of finite type over $\mathbf{Z}_{(p)}$, so $H_*(F'; \mathbf{Z})$ is a sum of copies of $\mathbf{Z}_{(p)}$ and $\mathbf{Z}/p^n$ ($n \geq 1$). If it were nonzero, then $H_*(F'; \mathbf{Z}/p)$ would be nonzero. Thus $H_*(F'; \mathbf{Z}) = 0$, so by the Whitehead Theorem, F' is homotopy equivalent to a point. □

COROLLARY 5.5. *BoP has no nontrivial wedge decomposition.*

Proof. Clearly BoP satisfies (1), (2), (3) of (5.4). The image of the mod 2 Hurewicz homomorphism is given by the zero line $E_\infty^{0,*}$ in the Adams spectral sequence for π_*BoP. But the differentials, as described in (5.3), show this is of rank one. □

6. Maps from $\mathbf{MSU}_{(2)}$ to Suspensions of BoP. We ultimately intend to show $MSU_{(2)}$ is a wedge of BoP and BP suspensions by producing a map to such a wedge and showing it is a homotopy equivalence. In this section, we produce the necessary maps from $MSU_{(2)}$ to various suspensions of BoP. Such maps are difficult to produce directly. As an intermediate step, we first produce certain maps to suspensions of $bo_{(2)}$. It seems that the KO-theory techniques of [4, 17] would provide sufficient maps to $bo_{(2)}$, but we will use the Adams spectral sequence since this method will also produce a fundamental map we require from BoP to $bo_{(2)}$.

If M is a graded comodule or vector space, let M_n denote the n^{th} grade, M^n the n-skeleton. If a graded vector space is concentrated in even dimensions, we call it an evenly graded vector space (abbreviated *egvs*).

A priori the Adams spectral sequence with E_2 term isomorphic to $\mathrm{Ext}_A^{*,*}(H_*MSU, H_*bo)$ doesn't necessarily converge to $[MSU_{(2)}, bo_{(2)}]_*$, since MSU is not a finite complex. We intend to obtain elements of

$[MSU_{(2)}, bo_{(2)}]_*$ by examining $\varprojlim_n [MSU^{2n}_{(2)}, bo_{(2)}]_*$, where the spectra MSU^{2n} are an increasing sequence of finite subspectra of MSU (a choice of $2n$-skeleta for MSU) such that $MSU^{2n} \to MSU$ induces an inclusion onto $(H_*MSU)^{2n}$ in homology (such subspectra exist since $H_*(MSU; \mathbf{Z})$ is torsion free and even dimensional).

The E_2 term for the Adams spectral sequence converging to $[MSU^{2n}_{(2)}, bo_{(2)}]_*$ is given by $\mathrm{Ext}_A^{*,*}((H_*MSU)^{2n}, H_*bo)$, which we now examine.

Let A_1^* denote the subalgebra of A^* generated by Sq^1 and Sq^2. Then $H^*bo \cong A^* \otimes_{A_1^*} \mathbf{Z}/2$ [14; 15, Chapter XI], so $H_*bo \cong A \square_{A_1} \mathbf{Z}/2 \cong \mathbf{Z}/2[\zeta_1^4, \zeta_2^2, \zeta_3, \ldots, \zeta_j, \ldots]$ [15, p. 324], with $A_1 = \mathbf{Z}/2[\zeta_1, \zeta_2]/(\zeta_1^4, \zeta_2^2)$. Using the change of rings theorem [16, p. 498], we have

LEMMA 6.1.

$$\mathrm{Ext}_A^{*,*}((H_*MSU)^{2n}, H_*bo) \cong \mathrm{Ext}_{A_1}^{*,*}((H_*MSU)^{2n}, \mathbf{Z}/2).$$

To compute these groups we must analyze the structure of $H_*MSU \cong (B \otimes Y) \oplus (P \otimes R' \otimes Y)$ as a comodule over A_1. The coaction in fact makes H_*MSU a comodule over the Hopf subalgebra $E = E(\zeta_1^2)$ of A_1. So, as in Section 1, we need only understand the induced Sq^2 action on H_*MSU.

PROPOSITION 6.2. *As an A_1 comodule, $B \cong \mathbf{Z}/2 \oplus (E \otimes V')$, with V' an egvs.*

Proof. Sq^2 is a derivation and differential on B with $Sq^2(\zeta_j^2) = \zeta_{j-1}^4$. The Künneth theorem yields $H_*(B; Sq^2) = \mathbf{Z}/2$, so B is as described. □

Proposition 6.3. As an A_1 comodule, $P \cong E \otimes V''$, with V'' an egvs.

Proof. $H_*(P; Sq^2) = 0$. □

COROLLARY 6.4. *As an A_1 comodule, $H_*MSU \cong Y \oplus (E \otimes V)$, where V is an egvs.*

We immediately deduce that

COROLLARY 6.5. *As an A_1 comodule, $(H_*MSU)^{2n} \cong Y^{2n} \oplus V_{2n} \oplus (E \otimes V^{2n-2})$.*

To analyze $\varprojlim_n [MSU^{2n}_{(2)}, bo_{(2)}]_*$, we now consider, for each $k \geq 0$, the inclusion $r: MSU^{8k+6} \to MSU^{8(k+1)+6}$, and the induced map of Adams spectral sequences converging to

$$[MSU^{8(k+1)+6}_{(2)}, bo_{(2)}]_* \xrightarrow{r^*} [MSU^{8k+6}_{(2)}, bo_{(2)}]_*.$$

From (6.1) and (6.5) we see we can describe the E_2 terms of the two spectral sequences provided we know $\mathrm{Ext}_{A_1}^{*,*}(\mathbf{Z}/2, \mathbf{Z}/2)$ and $\mathrm{Ext}_{A_1}^{*,*}(E, \mathbf{Z}/2)$. It is well known [9] that

(6.6) $$\mathrm{Ext}_{A_1}^{*,*}(\mathbf{Z}/2, \mathbf{Z}/2) \cong \mathbf{Z}/2[q_0, h, q_1^2 q_0, q_1^4]/(q_0 h, h^3),$$

with the generators in the same bidegrees as in Section 2. Regarding $\mathrm{Ext}_{A_1}^{*,*}(E, \mathbf{Z}/2)$, it will suffice to know that

LEMMA 6.7. $\mathrm{Ext}_{A_1}^{s,t}(E, \mathbf{Z}/2) = 0$ *if* $t - s$ *is odd.*

Proof. Consider the unreduced, normalized cobar resolution [2] $A_1 \otimes F(A_1)$ for $\mathbf{Z}/2$ over A_1. The differential on the cobar resolution induces a differential on $\mathrm{Hom}_{A_1}^{*,*}(E, A_1 \otimes F(A_1)) \cong \mathrm{Hom}_{\mathbf{Z}/2}^{*,*}(E, F(A_1)) \cong E(Sq^2) \otimes F(A_1)$, with homology isomorphic to $\mathrm{Ext}_{A_1}^{*,*}(E, \mathbf{Z}/2)$. If we filter the chain complex $E(Sq)^2 \otimes F(A_1)$ by the skeletal filtration on $E(Sq)^2$, we obtain a spectral sequence converging to the desired Ext group, with E_1 term isomorphic to $E(Sq^2) \otimes \mathrm{Ext}_{A_1}^{*,*}(\mathbf{Z}/2, \mathbf{Z}/2)$. Moreover, it is easy to check that the only nonzero differential, d_2, is given by $d_2(1 \otimes x) = Sq^2 \otimes hx$ and $d_2(Sq^2 \otimes x) = 0$, for $x \in \mathrm{Ext}_{A_1}^{*,*}(\mathbf{Z}/2, \mathbf{Z}/2)$. The lemma now follows, since $\mathrm{Ext}_{A_1}^{s,t}(\mathbf{Z}/2, \mathbf{Z}/2) \xrightarrow{h\cdot} \mathrm{Ext}_{A_1}^{s+1,t+2}(\mathbf{Z}/2, \mathbf{Z}/2)$ is onto if $(t - s)$ is even, one-to-one if $(t - s)$ is odd. □

Now we can examine the induced map $\mathrm{Ext}(r_*)$ of E_2 terms.

PROPOSITION 6.8. $\mathrm{Ext}^{s,t}(r_*)$ *is onto if* $t - s$ *is congruent to* 7 *or* 0 mod 8.

Proof. Using the descriptions of $(H_*MSU)^{8(k+1)+6}$ and $(H_*MSU)^{8k+6}$ provided by (6.5), we first notice that the composite

$$Y^{8k+6} \oplus (E \otimes V^{8k+4}) \xrightarrow{i} (H_*MSU)^{8k+6} \xrightarrow{r_*} (H_*MSU)^{8(k+1)+6}$$

is a split A_1 comodule monomorphism, so $\mathrm{Ext}(r_*) \circ \mathrm{Ext}(i)$ is onto. But $\mathrm{Ext}(i)$ is an isomorphism for $t - s$ congruent to 7 or 0 mod 8 since V_{8k+6} is concentrated in grade $8k + 6$ and $\mathrm{Ext}_{A_1}^{s,t}(\mathbf{Z}/2, \mathbf{Z}/2) = 0$ for $t - s$ congruent to 5 or 6 mod 8. □

When (6.5), (6.6), and (6.7) are combined with the fact that Y is concentrated in dimensions divisible by 8, we find that

PROPOSITION 6.9. *For any* $k \geq 0$, $\mathrm{Ext}_{A_1}^{s,t}((H_*MSU)^{8k+6}, \mathbf{Z}/2) = 0$ *if* $t - s$ *is congruent to* 7 mod 8.

So the 'columns' with $t - s \equiv 0 \pmod 8$ in the two spectral sequences survive in their entirety to E_∞, and if we let $E_\infty^{s,t}(r)$ denote the induced map of E_∞ terms, it follows from (6.8) and (6.9) that

COROLLARY 6.10. *$E_\infty^{s,t}(r)$ is onto for $t - s \equiv 0 \pmod 8$.*

Now we need a technical lemma which ensures that the epimorphism at the E_∞ level really means the induced homomorphism of mapping groups is an epimorphism. We prove something slightly more general for use later in this section.

LEMMA 6.11. *Suppose U, V_1, V_2 (respectively U_1, U_2, V) are 2-local spectra with U (respectively U_1, U_2) finite and V_1, V_2 (respectively V) of finite type and bounded below. Consider the Adams spectral sequences for $[U, V_1]_*$ and $[U, V_2]_*$ (respectively $[U_1, V]_*$ and $[U_2, V]_*$). If $V_1 \xrightarrow{f} V_2$ (respectively $U_2 \xrightarrow{f} U_1$) induces an epimorphism of $E_\infty^{s,t}$ terms, for all groups with $t - s = i$ for some fixed i, then $[U, V_1]_i \xrightarrow{f\circ} [U, V_2]_i$ (respectively $[U_1, V]_i \xrightarrow{\circ f} [U_2, V]_i$) is onto.*

Proof. Let ${}^jF_i^s$, $s \geq 0$, denote the groups in the decreasing Adams filtration of $[U, V_j]_i$ (respectively $[U_j, V]_i$) for $j = 1, 2$. Since the spectral sequences converge, ${}^jE_\infty^{s,i+s} \cong {}^jF_i^s/{}^jF_i^{s+1}$ and $\bigcap_s {}^jF_i^s = 0$ for all i and j. Since the source spectra are finite dimensional and the targets are bounded below, the Adams and 2-adic filtrations induce equivalent topologies on the mapping groups [12, p. 189]. So in particular, for i fixed, we can choose s such that ${}^jF_i^s \subset 2\cdot{}^jF_i^0$ for $j = 1, 2$.

Consider the induced commutative square

$$\begin{array}{ccc} {}^1F_i^0/{}^1F_i^s & \longrightarrow & {}^1F_i^0/2\cdot{}^1F_i^0 \\ \downarrow & & \downarrow \\ {}^2F_i^0/{}^2F_i^s & \longrightarrow & {}^2F_i^0/2\cdot{}^2F_i^0. \end{array}$$

The lower horizontal is clearly onto. The left vertical is onto because the Adams filtrations induce finite filtrations on the two groups, and by assumption the map is onto when one passes to filtered quotients. Thus the right vertical is onto. It now follows from Nakayama's Lemma [5, Proposition 2.6] that ${}^1F_i^0 \to {}^2F_i^0$ is onto, as desired. □

Now we are equipped to produce the maps we desire from $MSU_{(2)}$ to suspensions of $bo_{(2)}$.

THEOREM 6.12. *If* $\bar{\lambda}: Y \to \Sigma^{8i}\mathbf{Z}/2$ *is a graded homomorphism from* Y *to the* $8i^{th}$ *suspension of* $\mathbf{Z}/2$, *then there is a map* $MSU_{(2)} \to \Sigma^{8i} bo_{(2)}$ *inducing the obvious composition*

$$\lambda : H_*MSU$$

$$\cong (B \otimes Y) \oplus (P \otimes R' \otimes Y) \xrightarrow{\pi_1} B \otimes Y \hookrightarrow \mathbf{Z}/2[\zeta_1^4, \zeta_2^2, \zeta_3, \ldots, \zeta_j, \ldots] \otimes Y$$

$$\cong H_*bo \otimes Y \xrightarrow{id \otimes \bar{\lambda}} H_*bo \otimes \Sigma^{8i}\mathbf{Z}/2 \cong \Sigma^{8i} H_*bo.$$

Proof. λ is clearly an element of $\mathrm{Hom}_A^{-8i}(H_*MSU, H_*bo) = \mathrm{Ext}_A^{0,-8i}(H_*MSU, H_*bo)$. The restrictions of λ to the skeleta $(H_*MSU)^{8k+6}$ for $k \geq 0$ are elements of $\mathrm{Ext}_A^{0,-8i}((H_*MSU)^{8k+6}, H_*bo)$, compatible with one another under restriction. Using (6.10), (6.11), and the fact that

$$\mathrm{Hom}_{A_1}^{0,-8i}((H_*MSU)^{8(k+1)+6}, \mathbf{Z}/2) \xrightarrow{or} \mathrm{Hom}_{A_1}^{0,-8i}((H_*MSU)^{8k+6}, \mathbf{Z}/2)$$

is clearly an isomorphism if $k \geq i$, we see there is a sequence of elements $\hat{\lambda}_k \in [MSU_{(2)}^{8k+6}, bo_{(2)}]_{-8i}$, compatible with each other under restriction, and each inducing the restriction of λ in mod 2 homology. Since $[MSU_{(2)}, bo_{(2)}]_{-8i} \to \varprojlim_r [MSU_{(2)}^{2n}, bo_{(2)}]_{-8i}$ *is onto, the theorem follows.* □

Recall that $H_*BoP \cong B \otimes Y \| Z$. Thus the BoP wedge summands in the desired decomposition of $MSU_{(2)}$ ought to be indexed by Z. As a first step towards a projection $MSU_{(2)} \to BoP \wedge Z$, we construct an appropriate map $MSU_{(2)} \to bo_{(2)} \wedge Z$ by fitting together maps made available by (6.12). Since Z is concentrated in dimensions divisible by 8, it follows from (6.12) that

THEOREM 6.13. *If* $Y \xrightarrow{\bar{\mu}} Z$ *is a* (*graded*) *projection of* Y *onto the subspace* Z, then there is a map $\mu : MSU_{(2)} \to bo_{(2)} \wedge Z$ *inducing the obvious composition*

$$H_*MSU$$

$$\cong (B \otimes Y) \oplus (P \otimes R' \otimes Y) \xrightarrow{\pi_1} B \otimes Y \hookrightarrow \mathbf{Z}/2[\zeta_1^4, \zeta_2^2, \zeta_3, \ldots, \zeta_j, \ldots] \otimes Y$$

$$\cong H_*bo \otimes Y \xrightarrow{id \otimes \bar{\mu}} H_*bo \otimes Z$$

in mod 2 *homology.*

Using a map of the type produced in (6.13) we wish to produce a map $MSU_{(2)} \to BoP \wedge Z$ which will serve as a projection to the BoP summands we claim exist in $MSU_{(2)}$. First we need a suitable map $BoP \to bo_{(2)}$.

THEOREM 6.14. *There is a map* $BoP \xrightarrow{\nu} bo_{(2)}$ *inducing the obvious map*

$$B \otimes Y||Z \xrightarrow{id \otimes \epsilon} B \otimes \mathbf{Z}/2 \cong B \hookrightarrow H_*bo$$

in mod 2 *homology.*

Proof. All the properties of $MSU_{(2)}$ used in the proof of (6.12) are also satisfied by BoP. So the same argument produces the desired map. □

THEOREM 6.15. $\nu_* : \pi_i BoP \to \pi_i bo_{(2)}$ *is an epimorphism for all i and an isomorphism for i odd.*

Proof. In (5.3) we computed the E_2 term of the Adams spectral sequence converging to $\pi_* BoP$, and all the differentials in the spectral sequence. The E_2 term of the Adams spectral sequence converging to $\pi_* bo_{(2)}$ is $\mathrm{Ext}_A^{*,*}(\mathbf{Z}/2, H_*bo) \cong \mathrm{Ext}_{A_1}^{*,*}(\mathbf{Z}/2, \mathbf{Z}/2)$, described in (6.6). There is no room for nonzero differentials in the spectral sequence. Combining (5.3), (6.6), and (6.14), we see that ν induces an epimorphism of E_∞ terms, and hence, by (6.11), of homotopy groups. From the description of the two spectral sequences it is clear that $\pi_i bo_{(2)}$ and $\pi_i BoP$ are zero for i odd unless $i \equiv 1$ (mod 8), in which case both groups are $\mathbf{Z}/2$. The theorem follows. □

Let F be the fibre of $\nu : BoP \to bo_{(2)}$. From (6.15) and the long exact homotopy sequence we immediately have

COROLLARY 6.16. $\pi_* F$ *is concentrated in even dimensions.*

In fact, Massey product and Toda bracket arguments can be used to show that ν_* induces an isomorphism of torsion subgroups, so $\pi_* F$ is torsion free. We will not prove this here.

Now we are prepared to produce the map $MSU_{(2)} \xrightarrow{f} BoP \wedge Z$ we seek. Note that for any such map, the induced map f_* in homology carries the primitives in $H_* MSU$ into the primitives $Y||Z \otimes Z$ in $H_*(BoP \wedge Z)$.

THEOREM 6.17. *There is a map* $f : MSU_{(2)} \to BoP \wedge Z$ *such that*

$$Z \xrightarrow{f_*|Z} (Y||Z) \otimes Z \xrightarrow{\epsilon \otimes id} \mathbf{Z}/2 \otimes Z \cong Z$$

is the identity.

Proof. Consider any map $\mu: MSU_{(2)} \to bo_{(2)} \wedge Z$ satisfying (6.13). We would like to lift μ to f making

$$\begin{array}{ccc} & \overset{f}{\nearrow} & BoP \wedge Z \\ MSU_{(2)} & & \downarrow{\scriptstyle \nu \wedge id} \\ & \underset{\mu}{\searrow} & bo_{(2)} \wedge Z \end{array}$$

commute. The obstructions to such a lift lie in the groups $H^p(MSU_{(2)}; \pi_{p-1}F)$. Since $H^*(MSU; \mathbf{Z})$ is torsion free and even dimensional, it follows from (6.16) that the obstruction groups are all zero. From (6.13) and (6.14) it is clear f_* has the desired property. □

7. The Decomposition of $\mathbf{MSU_{(2)}}$. In this section we will show $MSU_{(2)}$ is homotopy equivalent to a wedge of suspensions of BoP and BP.

The map $f: MSU_{(2)} \to BoP \wedge Z$ produced in (6.17) will serve as the projection to the BoP summands. But first we need to know more about the homology behavior of this map. So far we only know that the restriction of f_* to $Z \subset Y$ is such that $Z \xrightarrow{f_*} (Y||Z) \otimes Z \xrightarrow{\epsilon \otimes id} \mathbf{Z}/2 \otimes Z \cong Z$ is the identity. We will in fact need to know that $Y \xrightarrow{f_*} (Y||Z) \otimes Z$ is an isomorphism.

We will show this is forced by the differentials in the Adams spectral sequences for $\pi_* MSU$ and $\pi_*(BoP \wedge Z)$. The idea is roughly as follows. Since we know $f_*(1) = 1 \otimes 1$, it 'ought' to follow that $\mathrm{Ext}(f_*)(q_2') = q_2' \otimes 1$. Since d_2 commutes with $\mathrm{Ext}(f_*)$ and $d_2 y_8' = q_2'$ in both the spectral sequence for $\pi_* MSU_{(2)}$ and for $\pi_* BoP$, it follows that $\mathrm{Ext}(f_*)(y_8') = y_8' \otimes 1$, so $f_*(y_8') = y_8' \otimes 1$, as desired, etc. The details are rather technical, and we will relegate them to a lemma, from which our main result will follow easily.

To state the lemma, we need a few preliminaries. Recall that $Y||Z$ is isomorphic to the exterior algebra $E(y_8', \ldots, y_{2^j}', \ldots)$. Let $L: Y||Z \to Y$ be the obvious splitting of the projection $\rho: Y \to Y||Z$. In other words,

$$L\Big(\prod_k y_{2^{j_k}}'\Big) = \prod_k y_{2^{j_k}}' \quad \text{for} \quad j_1 < j_2 < \cdots.$$

Now consider $J: (Y||Z) \otimes Z \xrightarrow{L \otimes 1} Y \otimes Y \xrightarrow{m} Y$. Clearly J is an isomorphism and a map of right Z modules. Let I be the inverse of J. Specifically, any monomial in the y_{8i}''s can be written in the form

$$y = \left(\prod_k y'_{2^{j_k}}\right)\cdot z,$$

with $z \in Z$ and $j_k < j_{k+1}$ for all k. Then

$$I(y) = \left(\prod_k y'_{2^{j_k}}\right) \otimes z.$$

Let $F^{8r}Z$ denote $\bigoplus_{i \geq 8r} Z_i$ (and define $F^{8r}Y$ and $F^{8r}Y\|Z$ similarly). For r fixed, let $\pi : Z \to Z_{8r}$ be the natural projection to grade $8r$. So $id \wedge \pi$ maps $BoP \wedge Z$ onto $BoP \wedge Z_{8r}$. The technical lemma is as follows.

LEMMA 7.1. *The diagram*

$$\begin{array}{ccccc} & & Y \xrightarrow{f_*} (Y\|Z) \otimes Z & & \\ & \nearrow & & \searrow{\scriptstyle (id\wedge\pi)_*} & \\ Y\cdot F^{8r}Z & & & & (Y\|Z)\otimes Z_{8r} \\ & \searrow & & \nearrow{\scriptstyle (id\wedge\pi)_*} & \\ & & Y \xrightarrow{I} (Y\|Z)\otimes Z & & \end{array}$$

commutes.

Before proving the lemma, we will show how the decomposition of $MSU_{(2)}$ follows from it.

COROLLARY 7.2. *$f_* : Y \to (Y\|Z) \otimes Z$ is a monomorphism.*

Proof. Let $0 \neq y \in Y$. Since $Y = Y\cdot F^0Z$, and $\bigcap_r Y\cdot F^{8r}Z = 0$, there is some r with $y \in Y\cdot F^{8r}Z$ but $y \notin Y\cdot F^{8(r+1)}Z$. If $f_*(y) = 0$, then by (7.1), $(id \wedge \pi)_* \circ I(y) = 0$. But $I(Y\cdot F^{8r}Z) \subset (Y\|Z) \otimes F^{8r}Z$ since I is a Z module map, so

$$I(y) \in ((Y\|Z) \otimes F^{8r}Z) \cap \ker((id \wedge \pi)_*) = (Y\|Z) \otimes F^{8(r+1)}Z.$$

Thus $y = J(I(y)) \in Y\cdot F^{8(r+1)}Z$, a contradiction. □

LEMMA 7.3. *There is a map $g : MSU_{(2)} \to BP \wedge (R' \otimes Y)$ such that*

$$(B \otimes Y) \oplus (P \otimes R' \otimes Y) \cong H_*MSU_{(2)} \xrightarrow{g_*} H_*(BP \wedge (R' \otimes Y))$$

$$\cong P \otimes R' \otimes Y$$

is projection onto the second factor.

Proof. The proof of (5.1) carries over word for word to produce the desired map. □

We can combine the maps f from (6.17) and g from (7.3) to produce a map $h: MSU_{(2)} \to (BoP \wedge Z) \vee (BP \wedge (R' \otimes Y))$ that induces f and g when projected onto the left and right summands of the target.

THEOREM 7.4. *h is a homotopy equivalence.*

Proof. In homology, consider the restriction $h_*: Y \oplus (R' \otimes Y) \to ((Y||Z) \otimes Z) \oplus (R' \otimes Y)$ to the A primitives. Let $x \in Y \oplus (R' \otimes Y)$ with $x \neq 0$. If $x \notin Y$, then by (7.3), $g_*(x) \neq 0$, so $h_*(x) \neq 0$. If $x \in Y$, then $f_*(x) \neq 0$ by (7.2), so $h_*(x) \neq 0$. Thus h_* is a monomorphism on primitives, and hence a monomorphism. But the graded ranks of the source and target of h_* are the same, so h_* is an isomorphism. Since both source and target are even dimensional, $H_*(MSU_{(2)}; \mathbf{Z})$ and $H_*((BoP \wedge Z) \vee (BP \wedge (R' \otimes Y)); \mathbf{Z})$ are torsion free and h must induce an isomorphism in integral homology. Thus by Whitehead's Theorem h is a homotopy equivalence. □

Proof of Lemma 7.1. We will prove the diagram commutes when restricted to $Y \cdot Z_{8r+s}$ for any $s \geq 0$. For each s we will do this inductively by showing it commutes on $Y_{8i} \cdot Z_{8r+s}$ provided it commutes on $Y_{8n} \cdot Z_{8r+s}$ for $n < i$. To begin the induction we must show it commutes on Z_{8r+s}. In $\mathrm{Ext}^{*,*}(H_*MSU)$, $Z_{8r+s} \subset \ker(d_2)$. So

$$f_*(Z_{8r+s}) \subset \ker(d_2) \cap \mathrm{Ext}^{0,8r+s}(H_*(BoP \wedge Z)) = \mathbf{Z}/2 \otimes Z_{8r+s}.$$

Thus $(id \wedge \pi)_* \circ f_*(Z_{8r+s}) = 0$ if $s > 0$, and if $s = 0$, $(id \wedge \pi)_* \circ f_*: Z_{8r} \to (Y||Z) \otimes Z_{8r}$ is, by hypothesis, the natural inclusion. In either case, the diagram commutes, beginning the induction.

Define j, k by $2^j \leq 8i < 2^{j+1}$ and $2^k \leq 8i + s < 2^{k+1}$. Let $\bar{f} = (id \wedge \pi) \circ f: MSU_{(2)} \to BoP \wedge Z \to BoP \wedge Z_{8r}$. Consider the diagram

$$\begin{array}{ccccc}
 & & B \otimes [(Y||Z)/F^{8i-2^j+s}(Y||Z)] \otimes Z_{8r} & & \\
 & & \uparrow \alpha & & \\
(B \otimes Y) \oplus (P \otimes R' \otimes Y) & \xrightarrow{\bar{f}_*} & B \otimes (Y||Z) \otimes Z_{8r} & & \\
\beta \big\uparrow & & \big\uparrow & & \\
B \otimes [(F^{8i-2^j} Y) \cdot Z_{8r+s}] & \xrightarrow{\bar{f}_*} & B \otimes [F^{8i-2^k+s}(Y||Z)] \otimes Z_{8r} & \xrightarrow{\tau} & B \otimes (Y||Z)_{8i-2^k+s} \otimes Z_{8r}
\end{array}$$

involving the induced map $\bar{f}_*$ in homology, with τ the natural projection.

The composite $\alpha \circ \bar{f}_* \circ \beta$ is trivial, since the target has nonzero primitives only in grades less than $8i - 2^j + 8r + s$, the source is concentrated only in grades at least as large as this number, and an A comodule map (between bounded below graded A comodules) is determined by its composition with a projection of the target onto the A primitives. Thus $\bar{f}_*$ factors as shown because $j \leq k$.

Now consider the restriction of $\tau \circ \bar{f}_*$ to the primitives. This is obviously zero except (possibly) in grade $8i - 2^j + 8r + s$, and in this grade it equals $\tau \circ (id \wedge \pi)_* \circ (id \otimes I)$ by the inductive assumption. But $\tau \circ \bar{f}_*$ is uniquely determined by its restriction to the primitives, since the source is primitive in all grades less than or equal to $8i - 2^j + 8r + s$, and these are the only grades in which the target has nonzero primitives. Thus $\tau \circ \bar{f}_* = \tau \circ (id \wedge \pi)_* \circ (id \otimes I)$.

With this in hand let us consider (see Figure 1) the map induced by $\bar{f}$ between relevant portions of the E_2-terms of the Adams spectral sequences converging to $\pi_* MSU_{(2)}$ and $\pi_*(BoP \wedge Z_{8r})$. The commutative diagram in the figure requires some justification.

The vertical equalities are valid since $(F^{8i-2^j}Y)\cdot Z_{8r+s}$ is an ideal in Y, and $F^{8i+s-2^k}(Y\|Z)$ and $F^{8i+s-2^k+8}(Y\|Z)$ are ideals in $Y\|Z$. To see that d_2 and $d_2 \otimes id$ really land in the subgroups shown, recall that on Y, $d_2 = h \cdot d$, with the derivation d as described in Section 2 (this is true also on $Y\|Z$ in the spectral sequence for $\pi_* BoP$; note $dZ = 0$ so d passes naturally to a derivation on $Y\|Z$). We leave it to the reader to check that the definition of d on $\{y'_{8n}\}$, the fact that d is a derivation, and the fact that multiplication in Y adds filtrations F^*Y, ensure that $d(Y_{8i}) \subset \mathbf{Z}/2[q'_2, \ldots, q'_l, \ldots] \otimes F^{8i-2^j}Y$. Thus since $dZ = 0$, $d_2(Y_{8i}\cdot Z_{8r+s}) \subset \operatorname{Ext}(B)' \otimes (F^{8i-2^j}Y)\cdot Z_{8r+s}$, as claimed. By the same reasoning $d_2 \otimes id$ behaves as shown.

The lowest horizontal $\operatorname{Ext}^{0,*}(\bar{f}_*)$ is the map we wish to determine. We detect its behavior as follows.

$$
\begin{array}{ccccc}
\operatorname{Ext}(B) \otimes (F^{8i-2^j}Y)\cdot Z_{8r+s} & \xrightarrow{\operatorname{Ext}(\bar{f}_*)} & \operatorname{Ext}(B) \otimes F^{8i+s-2^k}(Y\|Z) \otimes Z_{8r} & \xrightarrow{\operatorname{Ext}(\tau)} & \operatorname{Ext}(B) \otimes (Y\|Z)_{8i+s-2^k} \otimes Z_{8r} \\
\| & & \| & & \| \\
\operatorname{Ext}(B)' \otimes (F^{8i-2^j}Y)\cdot Z_{8r+s} & \xrightarrow{\operatorname{Ext}(\bar{f}_*)} & \operatorname{Ext}(B)' \otimes F^{8i+s-2^k}(Y\|Z) \otimes Z_{8r} & \xrightarrow{\operatorname{Ext}(\tau)} & \operatorname{Ext}(B)' \otimes (Y\|Z)_{8i+s-2^k} \otimes Z_{8r} \\
\uparrow{\scriptstyle d_2} & & \uparrow{\scriptstyle d_2 \otimes id} & & \\
Y_{8i}\cdot Z_{8r+s} & \xrightarrow{\operatorname{Ext}^{0,*}(\bar{f}_*)} & (Y\|Z)_{8i+s} \otimes Z_{8r} & &
\end{array}
$$

Figure 1

Let $(j_1, \ldots, j_l)$ be the unique increasing integer sequence such that

$$8i + s = \sum_{m=1}^{l} 2^{j_m}.$$

Note $j_l = k$. Let

$$y = \prod_{m=1}^{l} y'_{2^{j_m}} \in Y_{8i+s}.$$

Of course $\rho(y)$ is the generator of $(Y\|Z)_{8i+s} = \mathbf{Z}/2$.

Now

$$\operatorname{Ext}(\tau) \circ (d_2 \otimes id)(\rho(y) \otimes z) = (\operatorname{Ext}(\tau))\left(h \otimes \sum_{n=1}^{l} q'_{j_n - 1} \otimes \rho\Big(\prod_{m \neq n} y'_{2^{j_m}}\Big) \otimes z\right)$$

$$= h \otimes q'_{k-1} \otimes \rho\left(\prod_{m=1}^{l-1} y'_{2^{j_m}}\right) \otimes z,$$

since $\operatorname{Ext}(\tau)$ kills all but one of the terms in the sum. Thus $\operatorname{Ext}(\tau) \circ (d_2 \otimes id)$ is a monomorphism. So any map Φ such that $\operatorname{Ext}(\tau) \circ (d_2 \otimes id) \circ \Phi = \operatorname{Ext}(\tau) \circ \operatorname{Ext}(\bar{f}_*) \circ d_2$ must equal the map $\operatorname{Ext}^{0,*}(\bar{f}_*)$ of interest. We now show $(id \wedge \pi)_* \circ I$ is such a Φ, which will complete the proof.

Earlier in the proof we showed that $\tau \circ \bar{f}_* = \tau \circ (id \wedge \pi)_* \circ (id \otimes I)$, so applying $\operatorname{Ext}(—)$ yields $\operatorname{Ext}(\tau) \circ \operatorname{Ext}(\bar{f}_*) = \operatorname{Ext}(\tau) \circ \operatorname{Ext}((id \wedge \pi)_*) \circ (id \otimes I)$. Now $(id \otimes I) \circ d = (d \otimes id) \circ I$ since $dZ = 0$, so $(id \otimes I) \circ d_2 = (d_2 \otimes id) \circ I$. Thus $\operatorname{Ext}(\tau) \circ \operatorname{Ext}(\bar{f}_*) \circ d_2 = \operatorname{Ext}(\tau) \circ \operatorname{Ext}((id \wedge \pi)_*) \circ (d_2 \otimes id) \circ I = \operatorname{Ext}(\tau) \circ (d_2 \otimes id) \circ (id \wedge \pi)_* \circ I$, as claimed. □

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