HOPF ALGEBRAS, WITT VECTORS, AND BROWN–GITLER SPECTRA

MORGAN OPIE
THURSDAY SEMINAR, HARVARD SC 507
3PM, 2/14/2019

§1. Introduction

This talk will focus on an alternative construction of Brown–Gitler spectra as those representing a certain functor from spectra to abelian groups. This functor is defined in terms of Hopf algebras and Witt vectors, and the "construction" is non-explicit, appealing to (an appropriate version of) Brown representability. The first part of the talk will be devoted to formulating and verifying the necessary hypotheses of Brown representability; the second to showing that the objects produced has the correct cohomology to fit our earlier definition of Brown–Gitler spectra.

This talk will be independent of other parts of the seminar in the sense that it has little dependance on or necessity for the rest of what we're doing. Nevertheless, let's recall a bit of general set-up: our goal is

1.1. Theorem (The Immersion Conjecture). *Every n-manifold immerses into $\mathbb{R}^{2n-\alpha(n)}$, where $\alpha(n)$ is the number of ones in the binary expansion of n.*

Recall that Brown–Gitler spectra entered the picture as follows. The existence of an immersion of M^n into $\mathbb{R}^{2n-\alpha(n)}$ can be reduced to exhibiting a factorization of the map $M \to BO$ classifying the stable normal bundle through the natural map $BO(n-\alpha(n)) \to BO$. This reduction via the Smale–Hirsch theorem (from Emily's talk). Rather than construct a map to $BO(n-\alpha(n))$ direction, the strategy is to produce an auxiliary spectrum $BO/I_n \to BO$ so that the classifying map of $-TM$ more obviously factors, and then show that $BO/I_n \to BO(n-\alpha(n))$ in a compatible way.

The Thomified version of BO/I_n, called Brown–Gitler spectra and denoted by $B(n)$ in Robert's talk, and are in fact by having a certain cohomology and satisfying some additional conditions. In this talk, we give another description.

We'll closely follows the paper of Goerss, Lannes, and Morel titled "Hopf algebras, Witt vectors, and Brown–Gitler spectra." Our main result of this talk will be the following result from that paper (functors involved to be defined in the next section):

1.2. Theorem. *For all n even, the functor $\mathrm{Sp} \to \mathrm{Ab}$ given by $X \mapsto D_n H_*(\Omega^\infty X; \mathbb{Z}/2)$ is is the n-th homology of a 2-local spectrum $B(n)$, whose cohomology is*

$$H^* B(n) \simeq A/A\{\chi(\mathrm{Sq}^i) : 2i > n\},$$

as a module over the Steenrod algebra A; and so that if $\iota\colon B(n) \to H\mathbb{Z}/2$ generates $H^0(B(n))$, then, for all CW complexes Z, i_* is surjective in dimension n:

$$B(n)_n(Z) \twoheadrightarrow H_n(Z).$$

That is, $B(n) = BO/I_n$.

1.3. Remark. Recall from Robert's talk that Brown–Gitler spectra are characterized as sequences of spectra $\{X_k\}$ having the cohomology as above, together with any of five equivalent conditions. It seems like the second fact above should imply (5) from Robert's talk, which asserted that for any pointed CW complex Z, the differential d_r in the Adams spectral sequence $E^r_{s,t}$ for $B(k/2) \otimes \Sigma^\infty Z$ the sequence vanished for $r \geq 1$, $t - s \leq 2k+1$. I'll try to clarify this relationship later. (NOTE k is off from Robert's talk – Goerss et al define $B(k) = B(k-1)$ for k odd. We'll just assume k is even.).

1.4. Remark. Goerss–Lannes–Morel prove an analogous result to 1.2 at all primes, under the hypothesis that $n \not\equiv \pm 1 \pmod{2p}$, but we'll specialize to $p = 2$. This is the relevant context for the seminar. The proof ideas go through for general primes, but (as usual) some algebraic assertions have different forms at odd p.

1.5. Notation. Following Goerss–Lannes–Morel, we write $K(G, n)$ for $\Sigma^n HG$, the n-th suspension of the Eilenberg–Mac Lane spectrum.

§2. Hopf algebras and Witt vectors

Our goal here is the define the functor which we seek to prove is representable. This necessitates a bit of background on Hopf algebras.

2.1. Definition. A Hopf algebra $(H, m, \eta, \Delta, \epsilon)$ over a field (or ring) k is a bialgebra H equipped an antipode map $S\colon H \to H$ such that

$$m \circ (1 \otimes S) \circ \Delta = m \circ (S \otimes 1) \circ \Delta = \eta \circ \epsilon.$$

2.2. Example. Hopf algebras as over k are equivalent to affine abelian group schemes over k under the usual equivalence

$$\{\text{Affine}\, k - \text{Schemes}\} \leftrightarrow \{k - \text{Algebras}\}.$$

2.3. Example. For an abelian group G, the group algebra $k[G]$ is a Hopf algebra over k with comultiplication induced by the assignment $g \mapsto g \otimes g$, counit induced by $g \mapsto 1_k$, and antipode induced by $g \mapsto g^{-1}$.

2.4. Definition. Let $\mathcal{H}$ the category of graded commutative and cocommutative Hopf algebras over k. In later sections, and as needed for simplification, we will assume $k = \mathbb{F}_2$.

The following is of primary interest to us:

2.5. Example. Given a spectrum X, $H_*(\Omega^\infty X; \mathbb{Z}/2)$ can be viewed as an element of $\mathcal{H}$: the underlying group is the usual graded abelian group structure on homology, multiplication is induced by composition of loops, and comultipliation comes from the space diagonal on $\Omega^\infty X$. The antipode map is probably induced by the antipode on the sphere. It's a fact that antipodes are unique if they exist, and so can generally be ignored: limits and colimits of Hopf algebras can be computed in bialgebras, Hopf agebras form a full subcategory, etc. etc.

2.6. Definition. A Hopf algebra $H \in \mathcal{H}$ is said to be connected if $H_0 \simeq k$. We write $\mathcal{H}_0$ for the full subcategory of $\mathcal{H}$ spanned by connected Hopf algebras.

The algebraic structure of $\mathcal{H}$ is, a priori, complicated and computations are difficult; however, our understanding can be simplified by $\mathcal{H}_0$. In order to do this, we begin by discussing certain particularly nice connected Hopf algebras, which are finitely generated projective and in fact generate $\mathcal{H}_0$ under quotients of coproducts in $\mathcal{H}$.

2.7. Remark. During the seminar, Haynes pointed out that the identification of $H(n)$ as finitely generated projective generators for $\mathcal{H}_0$, and the subsequent fact that $\hom_{\mathcal{H}}(H(n), -)$ establish an equivalence of categories between $\mathcal{H}_0$ and a more combinatorial module category is analogous to the study of spaces (say CW spaces) via simplicial sets and the functors $\hom_{\mathrm{Top}}(\Delta^n, -)$.

This construction requires a bit of work.

Construction/Definition:

Let $C(j)$ be the graded ring $\mathbb{Z}[x_0, \dots, x_n, \dots]$ where x_i has degree $2^{i+1}j$. Let

$$w_n := \sum_{i=0}^{n} p^i x_i^{p^{n-i}}.$$

Then Schoeller proves that there is a unique Hopf algebra structure making each w_i *primitive*, meaning that

$$\Delta(w_i) = w_i \otimes g + g \otimes w_i$$

for some element $g \in C(j)$ such that $\epsilon(g) = 1$ (meaning g is *grouplike*).

We then let $C(m, j)$ be the subring $\mathbb{Z}[x_0, \dots, x_m] \subset C(j)$. This is, in fact, a Hopf subalgebra: to see this, note that $w_i \in C(m, j)$ for $i \leq m$.

The next step is to get a Hopf algebra over $\mathbb{F}_2$; thus far, our Hopf algebras have been integral.

2.8. Definition. Let $\mathcal{V}$ be the category of graded $\mathbb{F}_2$-vector spaces. Define $\Phi\colon \mathcal{V} \to \mathcal{V}$ by $(\Phi V)_n = 0$ if n is odd and $(\Phi V)_{2m} = V_m$.

If $n = 2^m k$, where $(2, k) = 1$, $H(n)$ is determined by the formula

$$\Phi H(n) = \mathbb{F}_2 \otimes C(m, k).$$

Thus, $H(n) = \mathbb{F}_2[x_0, \dots, x_m]$ as an algebra, where x_i has degree $2^i k$.

2.9. Remark. Note that if $n = 2^m k$, $H(n) \hookrightarrow H(2n)$ as Hopf algebras, induced by the inclusion of polynomial rings. This simply corresponds to adding one additional algebra generator x_{m+1} of degree $2^{m+1}k$. For t odd, there is not an obvious relationship between $H(n)$ and $H(tn)$.

It's a fact, due to Schoeller, that $H(n)$ is a finitely generated and projective in $\mathcal{H}_0$ (where projectivity is defined, as usual, as lifting through surjections in $\mathcal{H}_0$).

For $H \in \mathcal{H}$, define a functor $D_n \colon \mathcal{H} \to \mathrm{Ab}$ by

$$D_n H := \hom_{\mathcal{H}}(H(n), H).$$

(Note that comultiplication on $H(n)$ gives $D_n H$ the structure of an abelian group.) Putting together the D_n's, we define a functor $D_* \colon \mathcal{H}_0 \to \mathcal{D}$, where $\mathcal{D}$ is the category of of Dieudonne modules: graded abelian groups M satisfying

- $M_n = 0$ if $n < 1$.
- $2^{m+1} M_n = 0$ if $n = 2^m k$ with k odd.

and equipped with *Verschiebung* and *Frobenius* maps $V \colon M_{2n} \to M_n$ and $F \colon M_n \to M_{2n}$ such that $FV = VF = p$, where p denotes multiplication by p.

In order to complete the definition of D_*, we need to define the action of V and F act on $D_*(H)$ functorially. For $n = 2^m k$, et $\phi \colon H(n) \to H(2n)$ be the inclusion $\mathbb{F}_2[x_0, \ldots, x_m] \hookrightarrow \mathbb{F}_2[x_0, \ldots, x_{m+1}]$. (See 2.9 above.)

Let $\psi \colon H(pn) = \mathbb{F}_2[x_0, \ldots, x_{m+1}] \to \mathbb{F}_2[x_0, \ldots, x_{m+1}]$ be the composition of the quotient map

$$\mathbb{F}_2[x_0, \ldots, x_{m+1}] \to \mathbb{F}_2[x_0, \ldots, x_{m+1}]/(x_0)$$

with the power/ shift map determined by $x_i \mapsto x_{i-1}^2$.

Then one checks that:

$$\begin{tikzcd} H(2n) \arrow[r, "\psi"] \arrow[dr, "{[2]}"'] & H(n) \arrow[d, "\phi"] \\ & H(2n) \end{tikzcd}$$

$$\begin{tikzcd} H(n) \arrow[r, "\phi"] \arrow[dr, "{[2]}"'] & H(2n) \arrow[d, "\psi"] \\ & H(n) \end{tikzcd}$$

commute. Given these, it's clear from the definition of the group structure of $D_n(H)$ that taking V and F to act by ϕ and ψ makes $D_*(-)$ a functor to $\mathcal{D}$.

§3. Representability

Classical Brown representability, as stated in Adams' his 1971 paper "A variant of E. H. Brown's representability theorem," asserts that

3.1. Theorem (Brown Representability). *Let $H \colon \mathrm{finCW} \to \mathrm{Ab}$ be a homotopy functor satisfying the wedge and Mayer-Vietoris axioms. Then H is representable.*

From this Adams deduces that

3.2. Theorem. *Generalised cohomology theories on finite CW complexes are representable by an Ω-spectrum.*

These two statements imply

3.3. Corollary. *If $B\colon \mathrm{Sp} \to \mathrm{Ab}$ are exact homotopy functor which takes arbitrary wedges to sums of groups and commutes with filtered colimits, then there is a spectrum E so that $B(X) = E_0(X)$ for all spectra X.*

Proof. Define a homology theory on finite spectra by

$$B_k(X) = B(\Sigma^{-k}X),$$

and a cohomology theory by $B^*\colon \mathrm{Spfin}^{\mathrm{op}} \to \mathrm{Ab}$ by

$$B^n(X) := B_{-n}((X)^*),$$

where $(-)^*$ denotes taking the Spanier–Whitehead dual. Note we need finite spectra for the dual to exist.

B^* gives a generalised cohomology theory on finite CW complexes by taking Σ^∞, and, by 3.2,we get that there exists a spectrum E representing it. Then it's clear that the generalised homology theory E_* on spectra agrees with B on finite suspension spectra. But any spectrum can be written as a filtered colimit of shifts of finite suspension spectra so in fact E_* gives B_* on all spectra.

□

We will verify that the functor $B = X \mapsto D_nH_*\Omega^\infty X$ from spectra to abelian groups satisfies the wedge and exactness axioms, and commutes with filtered colimits, so that by we can deduce:

3.4. Theorem. *There is a spectrum $B(n)$ so that for all $X \in \mathrm{Sp}$, $B(n)_n(X) = D_nH_*(\Omega^\infty X; \mathbb{Z}/2)$.*

Explicitly, the conditions to verify are:

(1) Given any filtered system $\{X_\alpha\}$ of spectra, the natural map
$$\operatorname{colim} B(X_\alpha) \to B(\operatorname{colim}_\alpha X)$$
is an isomorphism.

(2) For any coproduct $\vee_\alpha X_\alpha$ of spectra, the natural map
$$\oplus_\alpha B(X_\alpha) \to B(\vee_\alpha X_\alpha)$$
is an isomorphism.

(3) If $X \to Y \to Z$ is a (co)fiber sequence of spectra, then
$$B(X) \to B(Y) \to B(Z)$$
is exact.

The first two verifications are fairly straightforward:

Proof of 1. As a functor to abelian groups, homology commutes with filtered colimits (this is essential the fact that filtered colimits commute with finite limits, for example kernels, and all colimits commute). Up to weak equivalence, Ω^∞ also commutes with filtered colimits, which can be seen by taking homotopy groups. So, to show that the natural map $\operatorname{colim}_\alpha H_*(\Omega^\infty X_\alpha) \to H_*(\Omega^\infty \operatorname{colim}_\alpha X_\alpha) \simeq H_*(\operatorname{colim}_\alpha \Omega^\infty X_\alpha)$ is an isomorphism, we use the fact that the forgetful functor from the category BiAlg of cocommutative coalgebras in graded strictly commutative algebras to abelian groups commutes with sifted colimits, and the colimit in $\mathcal{H}$ is computed in BiAlg .

3.5. Remark. In more detail: objects of BiAlg are algebra objects in $\mathrm{grAlg}^{\mathrm{op}}$, and as a right adjoint the forgetful functor $\mathrm{Alg}(\mathrm{grAlg}^{\mathrm{op}}) \to \mathrm{grAlg}^{\mathrm{op}}$ preserves limits. Taking opposites into account, this shows that the colimit in bialgebras is the colimit of the underlying algebras. As a last step, note that the free commutative algebra monad on abelian groups preserves sifted colimits, and so the forgetful functor $\mathrm{Alg} \to \mathrm{Ab}$ preserves these too. Thus, the map in question is an isomorphism of the underlying abelian groups.

To finish, note that $D_n = \hom_{\mathcal{H}}(H(n), -)$, and $H(n)$ is a compact object of $\mathcal{H}$ since finitely generated (and in fact all colimits, since $H(n)$ are projective). $\square$

Proof of 2. By 1, we can assume our coproducts are finite and hence are also products of spectra. Then, as a right adjoint, Ω^∞ commutes with products:

$$\Omega^\infty \vee_\alpha X_\alpha \simeq \Pi \Omega^\infty X_\alpha,$$

where the product is in pointed spaces. Since we have field coefficients, we can apply the Künneth theorem to see that we have the graded tensor product of the constituent $H_*(\Omega^\infty X_\alpha)$, which is a coproduct in algebras and (as noted in 3.5 above) also the coproduct in Hopf algebras.

Now we again use that D_n preserve colimits. Hence, 2 is proved.

$\square$