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ScienceDirect
Advances in Mathematics 215 (2007) 569–600
ADVANCES IN Mathematics
www.elsevier.com/locate/aim

The stack of formal groups in stable homotopy theory

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Received 20 July 2005; accepted 12 April 2007

Available online 24 April 2007

Communicated by The Managing Editors

Abstract

We construct the algebraic stack of formal groups and use it to provide a new perspective onto a recent result of M. Hovey and N. Strickland on comodule categories for Landweber exact algebras. This leads to a geometric understanding of their results as well as to a generalisation.

Keywords: Stable homotopy theory; Algebraic stacks; Formal groups

1. Introduction

Ever since the fundamental work of S. Novikov and D. Quillen [30,32] the theory of formal groups is firmly rooted in stable homotopy theory. In particular, the simple geometric structure of the moduli space of formal groups has been a constant source of inspiration. This moduli space is stratified according to the height of the formal group. For many spaces X, $\mathrm{MU}_*(X)$ can canonically be considered as a flat sheaf on the moduli space and the stratification defines a resolution of $\mathrm{MU}_*(X)$, the Cousin-complex, which is well known to be the chromatic resolution of $\mathrm{MU}_*(X)$ and which is a central tool in the actual computation of the stable homotopy of X. J. Morava [28] was the first to realize the impact this has for the structure of $\mathrm{MU}_*\mathrm{MU}$-comodules, while the first explicit reference to the underlying geometry of the moduli space was made by M. Hopkins and B. Gross [18,19].

In fact, much deeper homotopy theoretic results have been suggested by this point of view and we mention two of them. All thick subcategories of the derived category of sheaves on the moduli space are rather easily determined by using the above stratification. This simple structure

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0001-8708/$ – see front matter
doi:10.1016/j.aim.2007.04.007

persists to determine all thick subcategories of the category of finite spectra, see [34, Theorem 3.4.3]. Similarly, every coherent sheaf on the moduli space can be reconstructed from its restriction to the various strata. Again, this result persists to homotopy theory as the chromatic convergence theorem [34, Theorem 7.5.7]. These are but specific aspects of the celebrated work of E. Devinatz, M. Hopkins and J. Smith on nilpotence in stable homotopy [7,20].

In conclusion, the derived category of sheaves on the moduli space of formal groups has turned out to be an excellent algebraic approximation to the homotopy category of (finite) spectra and the chief purpose of the present paper is to give a solid foundation for working with this and similar moduli spaces.

In fact, we start out more generally by making precise the relation between flat Hopf algebroids and a certain class of stacks. Roughly, the datum of a flat Hopf algebroid is equivalent to the datum of the stack with a specific presentation. Now, the category of comodules of the flat Hopf algebroid only depends on the stack. We will demonstrate the gain in conceptual clarity provided by this point of view by reconsidering the following remarkable recent result of M. Hovey and N. Strickland. For two Landweber exact BP_*-algebras R and S of the same height the categories of comodules of the flat Hopf algebroids $(R, \Gamma_R := R \otimes_{\mathrm{BP}_*} \mathrm{BP}_*\mathrm{BP} \otimes_{\mathrm{BP}_*} R)$ and $(S, \Gamma_S := S \otimes_{\mathrm{BP}_*} \mathrm{BP}_*\mathrm{BP} \otimes_{\mathrm{BP}_*} S)$ are equivalent. As an immediate consequence one obtains the computationally important change-of-rings isomorphism $\mathrm{Ext}^*_{\Gamma_R}(R, R) \simeq \mathrm{Ext}^*_{\Gamma_S}(S, S)$ which had been established previously by G. Laures [23, 4.3.3].

From our point of view, this result has the following simple explanation. Let $\mathfrak{X}$ be the stack associated with $(\mathrm{BP}_*, \mathrm{BP}_*\mathrm{BP})$ and $f : \mathrm{Spec}(R) \to \mathfrak{X}$ the canonical map. As we will explain, $\mathfrak{X}$ is closely related to the stack of formal groups and is thus stratified by closed substacks

$$\mathfrak{X} = \mathfrak{Z}^0 \supseteq \mathfrak{Z}^1 \supseteq \cdots.$$

We will show that the induced Hopf algebroid (R, Γ_R) is simply a presentation of the *stack*-theoretic image of f and that R being Landweber exact of height n implies that this image is $\mathfrak{X} - \mathfrak{Z}^{n+1}$. We conclude that (R, Γ_R) and (S, Γ_S) are presentations of *the same* stack which implies the main result of [15] but more is true: The comodule categories under consideration are in fact equivalent as *tensor* abelian categories ([15] treats their structure of abelian categories only) and we easily generalise the above proof to apply to all the stacks $\mathfrak{Z}^n - \mathfrak{Z}^{n+k}$ (with $n \geqslant 1$ allowed).

Returning to the stack of formal groups, we show that the stack associated with $(\mathrm{MU}_*, \mathrm{MU}_*\mathrm{MU})$ is closely related to this stack. Note, however, that this requires an a priori construction of the stack of formal groups, the problem being the following. The objects of a stack associated with a flat Hopf algebroid are only *flat locally* given in terms of the Hopf algebroid and it is in general difficult to decide what additional objects the stack contains. Given the central role of the stack of formal groups in stable homotopy theory, we believe that it is important to have a genuinely geometric understanding of it rather than just as the stack associated with some Hopf algebroid, so we solve this problem here. A different construction has recently been given in [35].

We review the individual sections in more detail. In Section 2 we collect the stack theoretic notions we will have to use in the following. In Section 3 we establish the relation between flat Hopf algebroids and algebraic stacks. In Section 4 we collect a number of technical results on algebraic stacks centring around the problem to relate the properties of a morphism between algebraic stacks with properties of the functors it induces on the categories of quasi-coherent sheaves. The main result is proved in Section 5. In the final Section 6 we construct the stack of formal groups and show that the algebraic stack associated with the flat Hopf algebroid $(\mathrm{MU}_*, \mathrm{MU}_*\mathrm{MU})$ is the stack of (one-dimensional, commutative, connected, formally smooth) formal groups together with a trivialization of the canonical line bundle and we explain its basic geometric properties.

To conclude the introduction we would like to acknowledge the profound influence of M. Hopkins on the present circle of ideas. We understand that he was the first to insist that numerous results on (comodules over) flat Hopf algebroids should be understood from a geometric, i.e. stack theoretic, point of view, cf. [17].

2. Preliminaries on algebraic stacks

In this section we will recall those concepts from the theory of stacks which will be used in the sequel.

Fix an affine scheme S and denote by Aff_S the category of affine S-schemes with some cardinality bound to make it small. We may write Aff for Aff_S if S is understood.

Definition 1. A category fibred in groupoids (understood: over Aff) is a category $\mathfrak{X}$ together with a functor $a : \mathfrak{X} \to \mathrm{Aff}$ such that

(i) ("existence of pull-backs") For every morphism $\phi : V \to U$ in Aff and $x \in \mathrm{Ob}(\mathfrak{X})$ with $a(x) = U$ there is a morphism $f : y \to x$ with $a(f) = \phi$.
(ii) ("uniqueness of pull-backs up to unique isomorphism") For every diagram in $\mathfrak{X}$

$$\begin{array}{ccc} & & z \\ & & \downarrow{\scriptstyle h} \\ y & \xrightarrow{f} & x \end{array}$$

lying via a over a diagram

$$\begin{array}{ccc} & & W \\ & \swarrow^{\psi} & \downarrow{\scriptstyle \chi} \\ V & \xrightarrow{\phi} & U \end{array}$$

in Aff there is a unique morphism $g : z \to y$ in $\mathfrak{X}$ such that $f \circ g = h$ and $a(g) = \psi$.

As an example, consider the category Ell of elliptic curves having objects E/U consisting of an affine S-scheme U and an elliptic curve E over U. Morphisms in Ell are cartesian diagrams

$$\begin{array}{ccc} E' & \longrightarrow & E \\ \downarrow & & \downarrow \\ U' & \xrightarrow{f} & U, \end{array} \tag{1}$$

equivalently isomorphisms of elliptic curves over U' from E' to $E \times_U U'$. For an explicit account of $\mathrm{Aut}_{\mathrm{Ell}}(E/U)$ see [37, Section 5].

There is a functor

$$a : \mathrm{Ell} \longrightarrow \mathrm{Aff}$$

sending E/U to U and a morphism in Ell as in (1) to f.

Checking that a makes Ell a category fibred in groupoids reveals that the main subtlety in Definition 1 lies in then non-uniqueness of cartesian products. A similar example can be given using vector bundles on topological spaces [12, Example B.2].

Let $a : \mathfrak{X} \to \mathrm{Aff}$ be a category fibred in groupoids. For $U \in \mathrm{Ob(Aff)}$ the fibre category $\mathfrak{X}_U \subseteq \mathfrak{X}$ is defined as the subcategory having objects $x \in \mathrm{Ob}(\mathfrak{X})$ with $a(x) = U$ and morphisms $f \in \mathrm{Mor}(\mathfrak{X})$ with $a(f) = \mathrm{id}_U$. The category $\mathfrak{X}_U$ is a groupoid. Choosing a pull-back as in Definition 1(i) for every $\phi : V \to U$ in Aff one can define functors $\phi^* : \mathfrak{X}_U \to \mathfrak{X}_V$ and, for composable $\phi, \psi \in \mathrm{Mor(Aff)}$, isomorphisms $\psi^* \circ \phi^* \simeq (\phi \circ \psi)^*$ satisfying a cocycle condition. Sometimes $\phi^*(x)$ will be denoted as $x|V$. This connects Definition 1 with the concept of fibred category as in [40, VI], as well as with the notion of lax/pseudo functor/presheaf on Aff with values in groupoids; see [12] and [43] for more details.

Categories fibred in groupoids constitute a 2-category in which 1-morphisms from $a : \mathfrak{X} \to \mathrm{Aff}$ to $b : \mathfrak{Y} \to \mathrm{Aff}$ are functors $f : \mathfrak{X} \to \mathfrak{Y}$ with $b \circ f = a$ (sic!) and 2-morphisms are isomorphisms between 1-morphisms. A 1-morphism $f : \mathfrak{X} \to \mathfrak{Y}$ is called a monomorphism (respectively isomorphism) if for all $U \in \mathrm{Ob(Aff)}$ the induced functor $f_U : \mathfrak{X}_U \to \mathfrak{Y}_U$ between fibre categories is fully faithful (respectively an equivalence of categories).

The next point is to explain what a sheaf, rather than a presheaf, of groupoids should be. This makes sense for any topology on Aff but we fix the *fpqc* topology for definiteness: It is the Grothendieck topology on Aff generated by the pretopology which as covers of an $U \in \mathrm{Aff}$ has the finite families of flat morphisms $U_i \to U$ in Aff such that $\coprod_i U_i \to U$ is faithfully flat, cf. [43, 2.3].

Definition 2. A stack (understood: over Aff for the *fpqc* topology) is a category fibred in groupoids $\mathfrak{X}$ such that

(i) ("descent of morphisms") For $U \in \mathrm{Ob(Aff)}$ and $x, y \in \mathrm{Ob}(\mathfrak{X}_U)$ the presheaf

$$\mathrm{Aff}/U \longrightarrow \mathrm{Sets}, \quad (V \xrightarrow{\phi} U) \longmapsto \mathrm{Hom}_{\mathfrak{X}_V}(x|V, y|V),$$

is a sheaf.

(ii) ("glueing of objects") If $\{U_i \xrightarrow{\phi_i} U\}$ is a covering in Aff, $x_i \in \mathrm{Ob}(\mathfrak{X}_{U_i})$ and $f_{ji} : (x_i|U_i \times_U U_j) \xrightarrow{\sim} (x_j|U_i \times_U U_j)$ are isomorphisms satisfying the cocycle condition then there are $x \in \mathrm{Ob}(\mathfrak{X}_U)$ and isomorphisms $f_i : (x|U_i) \xrightarrow{\sim} x_i$ such that $f_j|U_i \times_U U_j = f_{ji} \circ f_i|U_i \times_U U_j$.

The category fibred in groupoids Ell is a stack: Condition (i) of Definition 2 for Ell is a consequence of faithfully flat descent [2, 6.1, Theorem 6] and condition (ii) relies on the fact that elliptic curves canonically admit ample line bundles, see [43, 4.3.3].

Definition 3. Let $\mathfrak{X}$ be a stack. A substack of $\mathfrak{X}$ is a strictly full subcategory $\mathfrak{Y} \subseteq \mathfrak{X}$ such that

(i) For every $\phi : U \to V$ in Aff one has $\phi^*(\mathrm{Ob}(\mathfrak{Y}_V)) \subseteq \mathrm{Ob}(\mathfrak{Y}_U)$.

(ii) If $\{U_i \to U\}$ is a covering in Aff and $x \in \mathrm{Ob}(\mathfrak{X}_U)$ then we have $x \in \mathrm{Ob}(\mathfrak{Y}_U)$ if and only if $x|U_i \in \mathrm{Ob}(\mathfrak{Y}_{U_i})$ for all i.

As an example, consider the stack $\overline{\mathrm{Ell}}$ of generalised elliptic curves in the sense of [6]. Then $\mathrm{Ell} \subseteq \overline{\mathrm{Ell}}$ is a substack: Since a generalised elliptic curve is an elliptic curve if and only if it is smooth, condition (i) of Definition 3 holds because smoothness is stable under base change and condition (ii) holds because smoothness if *fpqc* local on the base.

Definition 4. A 1-morphism $f : \mathfrak{X} \to \mathfrak{Y}$ of stacks is an epimorphism if for every $U \in \mathrm{Ob}(\mathrm{Aff})$ and $y \in \mathrm{Ob}(\mathfrak{Y}_U)$ there exist a covering $\{U_i \to U\}$ in Aff and $x_i \in \mathrm{Ob}(\mathfrak{X}_{U_i})$ such that $f_{U_i}(x_i) \simeq y|U_i$ for all i.

A 1-morphism of stacks is an isomorphism if and only if it is both a monomorphism and an epimorphism [24], Corollaire 3.7.1. This fact can also be understood from a homotopy theoretic point of view [12, Corollary 8.16].

A fundamental insight is that many of the methods of algebraic geometry can be generalised to apply to a suitable class of stacks. In order to define this class, we first have to explain the concept of representable 1-morphisms of stacks which in turn needs the notion of algebraic spaces:

> Algebraic spaces are a generalisation of schemes. The reader unfamiliar with them can, for the purpose of reading this paper, safely replace algebraic spaces by schemes throughout. We have to mention them in order to confirm with our main technical reference [24]. Algebraic spaces were invented by M. Artin and we decided not to try to give any short account of the main ideas underlying this master piece of algebraic geometry but rather refer the reader to [1] for an introduction and to [21] as the standard technical reference.

We can now proceed on our way towards defining algebraic stacks.

Definition 5. A 1-morphism $f : \mathfrak{X} \to \mathfrak{Y}$ of stacks is representable if for every $U \in \mathrm{Aff}$ with a 1-morphism $U \to \mathfrak{Y}$ the fibre product $\mathfrak{X} \times_{\mathfrak{Y}} U$ is an algebraic space.

Here, we refer the reader to [24, 3.3] for the notion of finite limit for stacks.

Now let P be a suitable property of morphisms of algebraic spaces, e.g. being an open or closed immersion, being affine or being (faithfully) flat, see [24, 3.10] for a more exhaustive list. We say that a representable 1-morphism $f : \mathfrak{X} \to \mathfrak{Y}$ of stacks has the property P if for every $U \in \mathrm{Aff}$ with a 1-morphism $g : U \to \mathfrak{Y}$, forming the cartesian diagram

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{\;f\;} & \mathfrak{Y} \\ \uparrow & & \uparrow \\ \mathfrak{X} \times_{\mathfrak{Y}} U & \xrightarrow{\;f'\;} & U \end{array}$$

the resulting morphism f' between algebraic spaces has the property P.

As an example, let us check that the inclusion $\mathrm{Ell} \subseteq \overline{\mathrm{Ell}}$ is an open immersion: To give $U \in \mathrm{Aff}$ and a morphism $U \to \overline{\mathrm{Ell}}$ is the same as to give a generalised elliptic curve $\pi : E \to U$. Then

$\mathrm{Ell} \times_{\overline{\mathrm{Ell}}} U \to U$ is the inclusion of the complement of the image under π of the non-smooth locus of π and hence is an open subscheme of U.

Definition 6. A stack $\mathfrak{X}$ is algebraic if the diagonal 1-morphism $\mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is representable and affine and there is an affine scheme U and a faithfully flat 1-morphism $P : U \to \mathfrak{X}$.

See Section 3.2 for further discussion.

A convenient way of constructing stacks is by means of groupoid objects. Let (X_0, X_1) be a groupoid object in Aff, i.e. a Hopf algebroid, see Section 3. Then (X_0, X_1) determines a presheaf of groupoids on Aff and the corresponding category fibred in groupoids $\mathfrak{X}'$ is easily seen to satisfy condition (i) of Definition 2 for being a stack but not, in general, condition (ii). There is a canonical way to pass from $\mathfrak{X}'$ to a stack $\mathfrak{X}$ [24, Lemme 3.2] which can also be interpreted as a fibrant replacement in a suitable model structure on presheaves of groupoids [12].

We provisionally define the stack of formal groups $\mathfrak{X}_{FG}$ to be the stack associated with the Hopf algebroid $(\mathrm{MU}_*, \mathrm{MU}_*\mathrm{MU}[u^{\pm 1}])$. Then $\mathfrak{X}'_{FG,U}$ is the groupoid of formal group laws over U and their (not necessarily strict) isomorphisms. A priori, it is unclear what the fibre categories $\mathfrak{X}_{FG,U}$ are and in fact we will have to proceed differently in Section 6: We first construct a stack $\mathfrak{X}_{FG}$ directly and then prove that it is the stack associated with $(\mathrm{MU}_*, \mathrm{MU}_*\mathrm{MU}[u^{\pm 1}])$.

Note that there is a canonical 1-morphism $\mathrm{Spec}(\mathrm{MU}_*) \to \mathfrak{X}_{FG}$. The following is a special case of Proposition 27.

Proposition 7. *A* MU_**-algebra R is Landweber exact if and only if the composition* $\mathrm{Spec}(R) \to \mathrm{Spec}(\mathrm{MU}_*) \to \mathfrak{X}_{FG}$ *is flat.*

Useful accounts of Landweber exactness in this context include [26] and [38].

3. Algebraic stacks and flat Hopf algebroids

In this section we explain the relation between flat Hopf algebroids and their categories of comodules and a certain class of stacks and their categories of quasi-coherent sheaves of modules.

3.1. The 2-category of flat Hopf algebroids

We refer the reader to [33, Appendix A] for the notion of a (flat) Hopf algebroid. To give a Hopf algebroid (A, Γ) is equivalent to giving $(X_0 := \mathrm{Spec}(A), X_1 := \mathrm{Spec}(\Gamma))$ as a groupoid in affine schemes [24, 2.4.3] and we will formulate most results involving Hopf algebroids this way.

Recall that this means that X_0 and X_1 are affine schemes and that we are given morphisms $s, t : X_1 \to X_0$ (source and target), $\epsilon : X_0 \to X_1$ (identity), $\delta : X_1 \underset{s,X_0,t}{\times} X_1 \to X_1$ (composition) and $i : X_1 \to X_1$ (inverse) verifying suitable identities. The corresponding maps of rings are denoted η_L, η_R (left and right unit), ϵ (augmentation), Δ (comultiplication) and c (antipode).

The 2-category of flat Hopf algebroids $\mathcal{H}$ is defined as follows. Objects are Hopf algebroids (X_0, X_1) such that s and t are flat (and thus faithfully flat because they allow ϵ as a right inverse). A 1-morphism of flat Hopf algebroids from (X_0, X_1) to (Y_0, Y_1) is a pair of morphisms of affine schemes $f_i : X_i \to Y_i$ $(i = 0, 1)$ commuting with all the structure. The composition of 1-morphisms is component wise. Given two 1-morphisms $(f_0, f_1), (g_0, g_1) : (X_0, X_1) \to (Y_0, Y_1)$, a 2-morphism $c:(f_0,f_1)\to(g_0,g_1)$ is a morphism of affine schemes $c:X_0\to Y_1$ such that $sc=f_0$, $tc=g_0$ and the diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{(g_1,cs)} & Y_1 \underset{s,Y_0,t}{\times} Y_1 \\ {\scriptstyle (ct,f_1)}\big\downarrow & & \big\downarrow{\scriptstyle \delta} \\ Y_1 \underset{s,Y_0,t}{\times} Y_1 & \xrightarrow{\delta} & Y_1 \end{array}$$

commutes. For $(f_0,f_1)=(g_0,g_1)$ the identity 2-morphism is given by $c:=\epsilon f_0$. Given two 2-morphisms $(f_0,f_1)\xrightarrow{c}(g_0,g_1)\xrightarrow{c'}(h_0,h_1)$ their composition is defined as

$$c'\circ c:\ X_0 \xrightarrow{(c',c)} Y_1 \underset{s,Y_0,t}{\times} Y_1 \xrightarrow{\delta} Y_1.$$

One checks that the above definitions make $\mathcal{H}$ a 2-category which is in fact clear because, except for the flatness of s and t, they are merely a functorial way of stating the axioms of a groupoid, a functor and a natural transformation. For technical reasons we will sometimes consider Hopf algebroids for which s and t are not flat.

3.2. The 2-category of rigidified algebraic stacks

From Definition 2 one sees that every 1-morphism of algebraic stacks from an algebraic space to an algebraic stack is representable, cf. the proof of [24, Corollaire 3.13]. In particular, the condition in Definition 6 that P be faithfully flat makes sense. By definition, every algebraic stack is quasi-compact, hence so is every 1-morphism between algebraic stacks [24, Définition 4.16, Remarques 4.17]. One can check that finite limits and colimits of algebraic stacks, formed in the 2-category of stacks, are again algebraic stacks. If $\mathfrak{U}\overset{i}{\hookrightarrow}\mathfrak{X}$ is a quasi-compact open immersion of stacks and $\mathfrak{X}$ is algebraic then the stack $\mathfrak{U}$ is algebraic as one easily checks. In general, due to the quasi-compactness condition, an open substack of an algebraic stack need not be algebraic, see the introduction of Section 5.

A morphism P as in Definition 6 is called a presentation of $\mathfrak{X}$. As far as we are aware, the above definition of "algebraic" is due to P. Goerss [9] and is certainly motivated by the equivalence given in Section 3.3 below. We point out that the notion of "algebraic stack" well-establish in algebraic geometry [24, Définition 4.1] is different from the above. For example, the stack associated with $(\mathrm{BP}_*,\mathrm{BP}_*\mathrm{BP})$ in Section 5 is algebraic in the above sense but not in the sense of algebraic geometry because its diagonal is not of finite type [24, Lemme 4.2]. Of course, in the following we will use the term "algebraic stack" in the sense defined above.

The 2-category $\mathcal{S}$ of rigidified algebraic stacks is defined as follows. Objects are presentations $P : X_0 \to \mathfrak{X}$ as in Definition 6. A 1-morphism from $P : X_0 \to \mathfrak{X}$ to $Q : Y_0 \to \mathfrak{Y}$ is a pair consisting of $f_0 : X_0 \to Y_0$ in Aff and a 1-morphism of stacks $f : \mathfrak{X} \to \mathfrak{Y}$ such that the diagram

$$\begin{array}{ccc} X_0 & \xrightarrow{f_0} & Y_0 \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle Q} \\ \mathfrak{X} & \xrightarrow[f]{} & \mathfrak{Y} \end{array}$$

is 2-commutative. The composition of 1-morphisms is component wise. Given 1-morphisms $(f_0, f), (g_0, g) : (X_0 \to \mathfrak{X}) \to (Y_0 \to \mathfrak{Y})$ a 2-morphism in $\mathcal{S}$ from (f_0, f) to (g_0, g) is by definition a 2-morphism from f to g in the 2-category of stacks [24, 3].

3.3. *The equivalence of $\mathcal{H}$ and $\mathcal{S}$*

We now establish an equivalence of 2-categories between $\mathcal{H}$ and $\mathcal{S}$, see [13] for a generalisation. We define a functor $K : \mathcal{S} \to \mathcal{H}$ as follows:

$$K\big(X_0 \xrightarrow{P} \mathfrak{X} \big) := \Big(X_0, X_1 := X_0 \underset{P,\mathfrak{X},P}{\times} X_0\Big)$$

has a canonical structure of groupoid [24, Proposition 3.8], X_1 is affine because X_0 is affine and P is representable and affine and the projections $s, t : X_1 \rightrightarrows X_0$ are flat because P is. Thus (X_0, X_1) is a flat Hopf algebroid. If $(f_0, f) : (X_0 \xrightarrow{P} \mathfrak{X}) \to (Y_0 \xrightarrow{Q} \mathfrak{Y})$ is a 1-morphism in $\mathcal{S}$ we define $K((f_0, f)) := (f_0, f_0 \times f_0)$. If we have 1-morphisms $(f_0, f), (g_0, g) : (X_0 \xrightarrow{P} \mathfrak{X}) \to (Y_0 \xrightarrow{Q} \mathfrak{Y})$ in $\mathcal{S}$ and a 2-morphism $(f_0, f) \to (g_0, g)$ then we have by definition a 2-morphism $f \xrightarrow{\Theta} g : \mathfrak{X} \to \mathfrak{Y}$. In particular, we have $\Theta_{X_0} : \mathrm{Ob}(\mathfrak{X}_{X_0}) \to \mathrm{Mor}(\mathfrak{Y}_{X_0}) \supseteq \mathrm{Hom}_{\mathrm{Aff}}(X_0, Y_1)$ and we define $K(\Theta) := \Theta_{X_0}(\mathrm{id}_{X_0})$. One checks that $K : \mathcal{S} \to \mathcal{H}$ is a 2-functor.

We define a 2-functor $G : \mathcal{H} \to \mathcal{S}$ as follows. On objects we put $G((X_0, X_1)) := (X_0 \xrightarrow{can} \mathfrak{X} := [X_1 \rightrightarrows X_0])$, the stack associated with the groupoid (X_0, X_1) together with its canonical presentation [24, 3.4.3]; identify the X_i with the flat sheaves they represent to consider them as "S-espaces," see also Section 4.1. Then $G((X_0, X_1))$ is a rigidified algebraic stack: Saying that the diagonal of $\mathfrak{X}$ is representable and affine means that for every algebraic space X and morphisms $x_1, x_2 : X \to \mathfrak{X}$ the sheaf $\underline{\mathrm{Isom}}_X(x_1, x_2)$ on X is representable by an affine X-scheme. This problem is local in the *fpqc* topology on X because affine morphisms satisfy effective descent in the *fpqc* topology [40, exposé VIII, Théorème 2.1]. So we can assume that the x_i lift to X_0 and the assertion follows because $(s, t) : X_1 \to X_0 \times_S X_0$ is affine. A similar argument shows that $P : X_0 \to \mathfrak{X}$ is representable and faithfully flat because s and t are faithfully flat.

Given a 1-morphism $(f_0, f_1) : (X_0, X_1) \to (Y_0, Y_1)$ in $\mathcal{H}$ there is a unique 1-morphism $f : \mathfrak{X} \to \mathfrak{Y}$ making

$$\begin{array}{ccccc} X_1 & \rightrightarrows & X_0 & \xrightarrow{P} & \mathfrak{X} \\ {\scriptstyle f_1}\downarrow & & {\scriptstyle f_0}\downarrow & & \vdots\,{\scriptstyle f} \\ Y_1 & \rightrightarrows & Y_0 & \xrightarrow{Q} & \mathfrak{Y} \end{array}$$

2-commutative [24, proof of Proposition 4.18] and we define $G((f_0, f_1)) := f$.

Given a 2-morphism $c : X_0 \to Y_1$ from the 1-morphism $(f_0, f_1) : (X_0, X_1) \to (Y_0, Y_1)$ to the 1-morphism $(g_0, g_1) : (X_0, X_1) \to (Y_0, Y_1)$ in $\mathcal{H}$ we have a diagram

$$\begin{array}{ccccc} X_1 & \rightrightarrows & X_0 & \xrightarrow{P} & \mathfrak{X} \\ {\scriptstyle f_1}\downarrow\downarrow{\scriptstyle g_1} & & {\scriptstyle f_0}\downarrow\downarrow{\scriptstyle g_0} & & {\scriptstyle f}\downarrow\downarrow{\scriptstyle g} \\ Y_1 & \rightrightarrows & Y_0 & \xrightarrow{Q} & \mathfrak{Y} \end{array}$$

and need to construct a 2-morphism $\Theta = G(c) : f \to g$ in the 2-category of stacks. We will do this in some detail because we omit numerous similar arguments.

Fix $U \in \mathrm{Aff}$, $x \in \mathrm{Ob}(\mathfrak{X}_U)$ and a representation of x as in [24, proof of Lemme 3.2]

$$(U' \longrightarrow U,\ x' : U' \longrightarrow X_0,\ U'' := U' \times_U U' \xrightarrow{\sigma} X_1),$$

i.e. $U' \to U$ is a cover in Aff, $x' \in X_0(U') = \mathrm{Hom}_{\mathrm{Aff}}(U', X_0)$ and σ is a descent datum for x' with respect to the cover $U' \to U$. Hence, denoting by $\pi_1, \pi_2 : U'' \to U'$ and $\pi : U' \to U$ the projections, we have $\sigma : \pi_1^* x' \xrightarrow{\sim} \pi_2^* x'$ in $\mathfrak{X}_{U''}$, i.e. $x'\pi_1 = s\sigma$ and $x'\pi_2 = t\sigma$. Furthermore, σ satisfies a cocycle condition which we do not spell out.

We have to construct a morphism

$$\Theta_x : f(x) \longrightarrow g(x) \quad \text{in } \mathfrak{Y}_U$$

which we do by descent from U' as follows. We have a morphism

$$\pi^*\bigl(f(x)\bigr) = f\bigl(\pi^*(x) = x'\bigr) = f_0 x' \xrightarrow{\phi'} \pi^*\bigl(g(x)\bigr) = g_0 x' \quad \text{in } \mathfrak{Y}_{U'}$$

given by $\phi' := cx' : U' \to Y_1$. We also have a diagram

$$\begin{array}{ccc} \pi_1^*(\pi^*(f(x))) = f_0 x' \pi_1 & \xrightarrow{\pi_1^*(\phi')} & \pi_1^*(\pi^*(g(x))) = g_0 x' \pi_1 \\ {\scriptstyle \sigma_f}\downarrow & & \downarrow{\scriptstyle \sigma_g} \\ \pi_2^*(\pi^*(f(x))) = f_0 x' \pi_2 & \xrightarrow{\pi_2^*(\phi')} & \pi_2^*(\pi^*(g(x))) = g_0 x' \pi_2 \end{array}$$

in $\mathfrak{Y}_{U''}$ where σ_f and σ_g are descent isomorphisms for $f(x')$ and $g(x')$ given by $\sigma_f = f_1\sigma$ and $\sigma_g = g_1\sigma$. We check that this diagram commutes by computing in $\mathrm{Mor}(\mathfrak{Y}_{U''})$:

$$\begin{aligned} \sigma_g \circ \pi_1^*(\phi') &= \delta_Y(g_1\sigma, cx'\pi_1) = \delta_Y(g_1\sigma, cs\sigma) = \delta_Y(g_1, cs)\sigma \\ &\overset{(*)}{=} \delta_Y(ct, f_1)\sigma = \delta_Y(ct\sigma, f_1\sigma) = \delta_Y(cx'\pi_2, f_1\sigma) = \pi_2^*(\phi') \circ \sigma_f. \end{aligned}$$

Here δ_Y is the composition of (Y_0, Y_1) and in $(*)$ we used the commutative square in the definition of 2-morphisms in $\mathcal{H}$.

So ϕ' is compatible with descent data and thus descents to the desired $\Theta_x : f(x) \to g(x)$. We omit the verification that Θ_x is independent of the chosen representation of x and natural in x and U. One checks that $G : \mathcal{H} \to \mathcal{S}$ is a 2-functor.

Theorem 8. *The above* 2*-functors* $K : \mathcal{S} \to \mathcal{H}$ *and* $G : \mathcal{H} \to \mathcal{S}$ *are inverse equivalences.*

Proof. We have $G \circ K(X_0 \xrightarrow{P} \mathfrak{X}) = (X_0 \xrightarrow{can} [X_0 \times_{\mathfrak{X}} X_0 \rightrightarrows X_0])$ and there is a unique 1-isomorphism $\nu_P : [X_0 \times_{\mathfrak{X}} X_0 \rightrightarrows X_0] \to \mathfrak{X}$ with $\nu_p \circ can = P$ [24, Proposition 3.8]. One checks that this defines an isomorphism of 2-functors $G \circ K \xrightarrow{\simeq} \mathrm{id}_{\mathcal{S}}$.

Next we have $K \circ G(X_0, X_1) = (X_0, X_0 \underset{P,\mathfrak{X},P}{\times} X_0)$, where $(X_0 \xrightarrow{P} \mathfrak{X}) = G(X_0, X_1)$, and $X_1 \simeq X_0 \underset{P,\mathfrak{X},P}{\times} X_0$ [24, 3.4.3] and one checks that this defines an isomorphism of 2-functors $\mathrm{id}_{\mathcal{H}} \xrightarrow{\simeq} K \circ G$. $\square$

In the following, given a flat Hopf algebroid (X_0, X_1), we will refer to $G((X_0, X_1))$ simply as the (rigidified) algebraic stack associated with (X_0, X_1).

The forgetful functor from rigidified algebraic stacks to algebraic stacks is not full but we have the following.

Proposition 9. *If* (X_0, X_1) *and* (Y_0, Y_1) *are flat Hopf algebroids with associated rigidified algebraic stacks* $P : X_0 \to \mathfrak{X}$ *and* $Q : Y_0 \to \mathfrak{Y}$ *and* $\mathfrak{X}$ *and* $\mathfrak{Y}$ *are* 1*-isomorphic as stacks then there is a chain of* 1*-morphisms of flat Hopf algebroids from* (X_0, X_1) *to* (Y_0, Y_1) *such that every morphism in this chain induces a* 1*-isomorphism on the associated algebraic stacks.*

Remark 10. This result implies Theorem 6.5 of [15]: By Theorem 26 below, the assumptions of [15] imply that the flat Hopf algebroids (B, Γ_B) and $(B', \Gamma_{B'})$ considered there have the same open substack of the stack of formal groups as their associated stack. So they are connected by a chain of weak equivalences by Proposition 9, see Remark 14 for the notion of weak equivalence.

Proof of Proposition 9. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a 1-isomorphism of stacks and form the cartesian diagram

$$\begin{array}{ccc}
X_1' & \xrightarrow{f_1} & Y_1 \\
\downarrow\downarrow & & \downarrow\downarrow \\
X_0' & \xrightarrow{f_0} & Y_0 \\
{\scriptstyle P'}\downarrow & & \downarrow{\scriptstyle Q} \\
\mathfrak{X} & \xrightarrow{f} & \mathfrak{Y}.
\end{array}$$

To be precise, the upper square is cartesian for either both source or both target morphisms. Then (f_0, f_1) is a 1-isomorphism of flat Hopf algebroids. Next, $Z := X_0' \underset{P',\mathfrak{X},P}{\times} X_0$ is an affine scheme because X_0' is and P is representable and affine. The obvious 1-morphism $Z \to \mathfrak{X}$ is representable, affine and faithfully flat because P and P' are. Writing $W := Z \times_{\mathfrak{X}} Z \simeq X_1' \times_{\mathfrak{X}} X_1$

we have that $\mathfrak{X} \simeq [W \rightrightarrows Z]$ by the flat version of [24, Proposition 4.3.2]. There are obvious 1-morphisms of flat Hopf algebroids $(Z, W) \to (X_0', X_1')$ and $(Z, W) \to (X_0, X_1)$ covering $\mathrm{id}_{\mathfrak{X}}$ (in particular inducing an isomorphism on stacks) and we get the sought for chain as $(Y_0, Y_1) \leftarrow (X_0', X_1') \leftarrow (Z, W) \to (X_0, X_1)$. □

3.4. Comodules and quasi-coherent sheaves of modules

Let (A, Γ) be a flat Hopf algebroid with associated rigidified algebraic stack $X_0 = \mathrm{Spec}(A) \to \mathfrak{X}$. From Theorem 8 one would certainly expect that the category of Γ-comodules has a description in terms of $X_0 \to \mathfrak{X}$. In this section we prove the key observation that this category does in fact only depend on $\mathfrak{X}$ and not on the particular presentation $X_0 \to \mathfrak{X}$, cf. (2) below. Avoiding mentioning of stacks altogether, this is one of the main results of [16].

For basic results concerning the category $\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{X}})$ of quasi-coherent sheaves of modules on an algebraic stack $\mathfrak{X}$ we refer the reader to [24, Chapitre 13].

Fix a rigidified algebraic stack $X_0 \xrightarrow{P} \mathfrak{X}$ corresponding by Theorem 8 to the flat Hopf algebroid $(X_0 = \mathrm{Spec}(A), X_1 = \mathrm{Spec}(\Gamma))$ with structure morphisms $s, t : X_1 \to X_0$. As P is affine it is in particular quasi-compact, hence *fpqc*, and thus of effective cohomological descent for quasi-coherent modules [24, Théorème 13.5.5(i)]. In particular, P^* induces an equivalence

$$P^* : \mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{X}}) \xrightarrow{\simeq} \big\{F \in \mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{X_0}) + \text{ descent data}\big\},$$

cf. [2, Chapter 6] for similar examples of descent. A descent datum on $F \in \mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{X_0})$ is an isomorphism $\alpha : s^*F \to t^*F$ in $\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{X_1})$ satisfying a cocycle condition. Giving α is equivalent to giving either its adjoint $\psi_l : F \to s_*t^*F$ or the adjoint of α^{-1}, $\psi_r : F \to t_*s^*F$. Writing M for the A-module corresponding to F, α corresponds to an isomorphism $\Gamma \otimes_{\eta_L, A} M \to \Gamma \otimes_{\eta_R, A} M$ of Γ-modules and ψ_r and ψ_l correspond respectively to morphisms $M \to \Gamma \otimes_{\eta_R, A} M$ and $M \to M \otimes_{A, \eta_L} \Gamma$ of A-modules. One checks that this is a 1–1 correspondence between descent data on F and left- (respectively right-)Γ-comodule structures on M. For example, the cocycle condition for α corresponds to the coassociativity of the coaction. In the following we will work with left-Γ-comodules exclusively and simply call them Γ-comodules. The above construction then provides an explicit equivalence

$$\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{X}}) \xrightarrow{\simeq} \Gamma\text{-comodules}. \tag{2}$$

The identification of $\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{X}})$ with Γ-comodules allows to (re)understand a number of results on Γ-comodules from the stack theoretic point of view and we now give a short list of such applications which we will use later.

The adjunction $(P^*, P_*) : \mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{X}}) \to \mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{X_0})$ corresponds to the forgetful functor from Γ-comodules to A-modules, respectively to the functor "induced/extended comodule." The structure sheaf $\mathcal{O}_{\mathfrak{X}}$ corresponds to the trivial Γ-comodule A, hence taking the primitives of a Γ-comodule (i.e. the functor $\mathrm{Hom}_{\Gamma}(A, \cdot)$ from Γ-comodules to abelian groups) corresponds to $\mathrm{Hom}_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{O}_{\mathfrak{X}}, \cdot) = H^0(\mathfrak{X}, \cdot)$ and thus $\mathrm{Ext}^*_{\Gamma}(A, \cdot)$ corresponds to quasi-coherent cohomology $H^*(\mathfrak{X}, \cdot)$. Another application of (2) is the following correspondence between closed substacks and invariant ideals.

By [24, Application 14.2.7] there is a 1–1 correspondence between closed substacks $\mathfrak{Z} \subseteq \mathfrak{X}$ and quasi-coherent ideal sheaves $\mathcal{I} \subseteq \mathcal{O}_{\mathfrak{X}}$ under which $\mathcal{O}_{\mathfrak{Z}} \simeq \mathcal{O}_{\mathfrak{X}}/\mathcal{I}$ and by (2) these $\mathcal{I}$ correspond to Γ-subcomodules $I \subseteq A$, i.e. invariant ideals. In this situation, the diagram

$$\begin{array}{ccc} \mathrm{Spec}(\Gamma/I\Gamma) & \longrightarrow & \mathrm{Spec}(\Gamma) \\ \downarrow\downarrow & & \downarrow\downarrow \\ \mathrm{Spec}(A/I) & \longrightarrow & \mathrm{Spec}(A) \\ \downarrow & & \downarrow \\ \mathfrak{Z} & \longrightarrow & \mathfrak{X} \end{array}$$

is cartesian. Note that the Hopf algebroid $(A/I, \Gamma/I\Gamma)$ is induced from (A, Γ) by the map $A \to A/I$ because $A/I \otimes_A \Gamma \otimes_A A/I \simeq \Gamma/(\eta_L I + \eta_R I)\Gamma = \Gamma/I\Gamma$ since I is invariant.

We conclude this section by giving a finiteness result for quasi-coherent sheaves of modules. Let $\mathfrak{X}$ be an algebraic stack. We say that $\mathcal{F} \in \mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{X}})$ if *finitely generated* if there is a presentation $P : X_0 = \mathrm{Spec}(A) \to \mathfrak{X}$ such that the A-module corresponding to $P^*\mathcal{F}$ is finitely generated. If $\mathcal{F}$ is finitely generated then for every presentation $P : X_0' = \mathrm{Spec}(A') \to \mathfrak{X}$ the A'-module corresponding to $P'^*\mathcal{F}$ is finitely generated as one sees using [3, I, §3, Proposition 11].

Proposition 11. *Let (A, Γ) be a flat Hopf algebroid, M a Γ-comodule and $M' \subseteq M$ a finitely generated A-submodule. Then M' is contained in a Γ-subcomodule of M which is finitely generated as an A-module.*

Proof. See [42, Proposition 5.7]. □

Note that in this result, "finitely generated" cannot be strengthened to "coherent" as is shown by the example of the simple $\mathrm{BP}_*\mathrm{BP}$-comodule $\mathrm{BP}_*/(v_0, v_1, \ldots)$ which is not coherent as a BP_*-module.

Proposition 12. *Let $\mathfrak{X}$ be an algebraic stack. Then every $\mathcal{F} \in \mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{X}})$ is the filtering union of its finitely generated quasi-coherent subsheaves.*

Proof. Choose a presentation of $\mathfrak{X}$ and apply Proposition 11 to the resulting flat Hopf algebroid. □

This result may be compared with [24, Proposition 15.4].

4. Tannakian results

In [25], J. Lurie considers a Tannakian correspondence for "geometric" stacks which are exactly those stacks that are algebraic *both* in the sense of [24, Définition 4.1] and in the sense of Definition 6. He shows that associating with such a stack $\mathfrak{X}$ the category $\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{X}})$ is a fully faithful 2-functor. The recognition problem, i.e. giving an intrinsic characterisation of the categories $\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{X}})$, remains open but see [4] for a special case.

The usefulness of a Tannakian correspondence stems from being able to relate notions of linear algebra, pertaining to the categories $\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{X}})$ and their morphisms, to geometric notions, pertaining to the stacks and their morphisms. See [5, Propositions 2.20–29] for examples of this in the special case that $\mathfrak{X} = BG$ is the classifying stack of a linear algebraic group G. This relation can be studied without having solved the recognition problem and we do so in the present section, i.e. we relate properties of 1-morphisms (f_0, f_1) of flat Hopf algebroids to properties of the induced morphism $f : \mathfrak{X} \to \mathfrak{Y}$ of algebraic stacks and the adjoint pair $(f^*, f_*) : \mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{X}}) \to \mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{Y}})$ of functors.

4.1. The epi/monic factorisation

Every 1-morphism of stacks factors canonically into an epimorphism followed by a monomorphism and in this section we explain the analogous result for (flat) Hopf algebroids. In particular, this will explain the stack theoretic meaning of the construction of an induced Hopf algebroid, cf. [15], beginning of Section 2.

By a flat sheaf we will mean a set valued sheaf on the site Aff. The topology of Aff is subcanonical, i.e. every representable presheaf is a sheaf. We can thus identify the category underlying Aff with a full subcategory of the category of flat sheaves.

Every 1-morphism $f : \mathfrak{X} \to \mathfrak{Y}$ of stacks factors canonically $\mathfrak{X} \to \mathfrak{X}' \to \mathfrak{Y}$ into an epimorphism followed by a monomorphism [24, Proposition 3.7]. The stack $\mathfrak{X}'$ is determined up to unique 1-isomorphism and is called the image of f.

For a 1-morphism $(f_0, f_1) : (X_0, X_1) \to (Y_0, Y_1)$ of flat Hopf algebroids we introduce

$$\alpha := t\pi_2 : X_0 \underset{f_0, Y_0, s}{\times} Y_1 \longrightarrow Y_0 \quad \text{and}$$

$$\beta := (s, f_1, t) : X_1 \longrightarrow X_0 \underset{f_0, Y_0, s}{\times} Y_1 \underset{t, Y_0, f_0}{\times} X_0. \tag{3}$$

The 1-morphism $f : \mathfrak{X} \to \mathfrak{Y}$ induced by (f_0, f_1) on algebraic stacks is an epimorphism if and only if α is an epimorphism of flat sheaves as is clear from Definition 4. On the other hand, f is a monomorphism if and only if β is an isomorphism, as is easily checked.

Writing $X_1' := X_0 \underset{f_0, Y_0, s}{\times} Y_1 \underset{t, Y_0, f_0}{\times} X_0$, (f_0, f_1) factors as

$$\begin{array}{ccccc}
X_1 & \xrightarrow{f_1' := \beta} & X_1' & \xrightarrow{\pi_2} & Y_1 \\
\downarrow\downarrow & & \pi_1 \downarrow\downarrow \pi_3 & & \downarrow\downarrow \\
X_0 & \xrightarrow{f_0' := \mathrm{id}_{X_0}} & X_0 & \xrightarrow{f_0} & Y_0
\end{array}$$

and the factorisation of f induced by this is the epi/monic factorisation. Note that even if (X_0, X_1) and (Y_0, Y_1) are flat Hopf algebroids, (X_0, X_1') does not have to be flat.

We refer to (X_0, X_1') as the Hopf algebroid induced from (Y_0, Y_1) by f_0.

4.2. Flatness and isomorphisms

The proof of the next result will be given at the end of this section. The equivalence of (ii) and (iii) is equivalent to Theorem 6.2 of [15] but we will obtain refinements of it below, see Propositions 19 and 20.

Theorem 13. *Let $(f_0, f_1) : (X_0, X_1) \to (Y_0, Y_1)$ be a 1-morphism of flat Hopf algebroids with associated morphisms α and β as in (3) and inducing $f : \mathfrak{X} \to \mathfrak{Y}$ on algebraic stacks. Then the following are equivalent:*

(i) *f is a 1-isomorphism of stacks.*
(ii) *$f^* : \mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{X}}) \to \mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{Y}})$ is an equivalence.*
(iii) *α is faithfully flat and β is an isomorphism.*

Remark 14. This result shows that weak equivalences as defined in [14, Definition 1.1.4], are exactly those 1-morphisms of flat Hopf algebroids which induce 1-isomorphisms on the associated algebraic stacks.

We next give two results about the flatness of morphisms.

Proposition 15. *Let $(f_0, f_1) : (X_0, X_1) \to (Y_0, Y_1)$ be a 1-morphism of flat Hopf algebroids, $P : X_0 \to \mathfrak{X}$ and $Q : Y_0 \to \mathfrak{Y}$ the associated rigidified algebraic stacks and $f : \mathfrak{X} \to \mathfrak{Y}$ the induced 1-morphism of algebraic stacks. Then the following are equivalent:*

(i) *f is (faithfully) flat.*
(ii) *$f^* : \mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{Y}}) \to \mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{X}})$ is exact (and faithful).*
(iii) *$\alpha := t\pi_2 : X_0 \underset{f_0, Y_0, s}{\times} Y_1 \to Y_0$ is (faithfully) flat.*
(iv) *The composition $X_0 \xrightarrow{P} \mathfrak{X} \xrightarrow{f} \mathfrak{Y}$ is (faithfully) flat.*

Proof. The equivalence of (i) and (ii) holds by definition, the one of (i) and (iv) holds because P is *fpqc* and being (faithfully) flat is a local property for the *fpqc* topology. Abbreviating $Z := X_0 \underset{f_0, Y_0, s}{\times} Y_1$ we have a cartesian diagram

$$\begin{array}{ccccc} Z & & \xrightarrow{\quad\alpha\quad} & & Y_0 \\ {\scriptstyle \pi_1}\downarrow & & {\scriptstyle f_0}\nearrow & & \downarrow{\scriptstyle Q} \\ X_0 & \xrightarrow{P} & \mathfrak{X} & \xrightarrow{f} & \mathfrak{Y} \end{array}$$

which, as Q is *fpqc*, shows that (iv) and (iii) are equivalent. We check that this diagram is in fact cartesian by computing

$$X_0 \underset{fP, \mathfrak{Y}, Q}{\times} Y_0 = X_0 \underset{Qf_0, \mathfrak{Y}, Q}{\times} Y_0 \simeq X_0 \underset{f_0, Y_0, \mathrm{id}}{\times} Y_0 \underset{Q, \mathfrak{Y}, Q}{\times} Y_0 \simeq X_0 \underset{f_0, Y_0, s}{\times} Y_1 = Z,$$

and under this isomorphism the projection onto the second factor corresponds to α. □

Proposition 16. *Let (Y_0, Y_1) be a flat Hopf algebroid, $f_0 : X_0 \to Y_0$ a morphism in* Aff *and $(f_0, f_1) : (X_0, X_1 := X_0 \underset{f_0, Y_0, s}{\times} Y_1 \underset{t, Y_0, f_0}{\times} X_0) \to (Y_0, Y_1)$ the canonical 1-morphism of Hopf algebroids from the induced Hopf algebroid and $Q : Y_0 \to \mathfrak{Y}$ the rigidified algebraic stack associated with (Y_0, Y_1). Then the following are equivalent:*

(i) *The composition $X_0 \xrightarrow{f_0} Y_0 \xrightarrow{Q} \mathfrak{Y}$ is (faithfully) flat.*
(ii) *$\alpha := t\pi_2 : X_0 \underset{f_0, Y_0, s}{\times} Y_1 \to Y_0$ is (faithfully) flat.*

If either of this maps is flat, then (X_0, X_1) is a flat Hopf algebroid.

The last assertion of this proposition does not admit a converse: For $(Y_0, Y_1) = (\mathrm{Spec}(\mathrm{BP}_*), \mathrm{Spec}(\mathrm{BP}_*\mathrm{BP}))$ and $X_0 := \mathrm{Spec}(\mathrm{BP}_*/I_n) \to Y_0$, the induced Hopf algebroid is flat but $X_0 \to \mathfrak{Y}$ is not, cf. Section 5.1.

Proof. The proof of the equivalence of (i) and (ii) is the same as in Proposition 15, using that Q is *fpqc*. Again denoting $Z := X_0 \underset{f_0, Y_0, s}{\times} Y_1$ one checks that the diagram

$$\begin{array}{ccc} Z & \xrightarrow{\alpha} & Y_0 \\ \uparrow & & \uparrow f_0 \\ X_1 & \xrightarrow{t} & X_0 \end{array}$$

is cartesian which implies the final assertion of the proposition because flatness is stable under base change. □

Proposition 17. *Let (Y_0, Y_1) be a flat Hopf algebroid, $f_0 : X_0 \to Y_0$ a morphism in* Aff *such that the composition $X_0 \xrightarrow{f_0} Y_0 \xrightarrow{Q} \mathfrak{Y}$ is faithfully flat, where $Q : Y_0 \to \mathfrak{Y}$ is the rigidified algebraic stack associated with (Y_0, Y_1). Let $(f_0, f_1) : (X_0, X_1) \to (Y_0, Y_1)$ be the canonical 1-morphism with (X_0, X_1) the Hopf algebroid induced from (Y_0, Y_1) by f_0. Then (X_0, X_1) is a flat Hopf algebroid and (f_0, f_1) induces a 1-isomorphism on the associated algebraic stacks.*

Proof. The 1-morphism f induced on the associated algebraic stacks is a monomorphism as explained in Section 4.1. Proposition 16 shows that (X_0, X_1) is a flat Hopf algebroid and that α is faithfully flat, hence an epimorphism of flat sheaves. Thus f is an epimorphism of stacks as noted in Section 4.1 and, finally, f is a 1-isomorphism by [24, Corollaire 3.7.1]. □

We now start to take the module categories into consideration. Given $f : X \to Y$ in Aff we have an adjunction $\psi_f : \mathrm{id}_{\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_Y)} \to f_* f^*$. We recognise the epimorphisms of representable flat sheaves as follows.

Proposition 18. *Let $f : X \to Y$ be a morphism in* Aff. *Then the following are equivalent:*

(i) *f is an epimorphism of flat sheaves.*
(ii) *There is some $\phi : Z \to X$ in* Aff *such that $f\phi$ is faithfully flat.*

If (i) *and* (ii) *hold, then* ψ_f *is injective.*

If f *is flat, the conditions* (i) *and* (ii) *are equivalent to* f *being faithfully flat.*

As an example of a morphism satisfying the conditions of Proposition 18 without being flat one may take the unique morphism $\mathrm{Spec}(\mathbb{Z}) \sqcup \mathrm{Spec}(\mathbb{F}_p) \to \mathrm{Spec}(\mathbb{Z})$.

Proof of Proposition 18. That (i) implies (ii) is seen by lifting $\mathrm{id}_Y \in Y(Y)$ after a suitable faithfully flat cover $Z \to Y$ to some $\phi \in X(Z)$.

To see that (ii) implies (i), fix some $U \in \mathrm{Aff}$ and $u \in Y(U)$ and form the cartesian diagram

$$\begin{array}{ccccc} Z & \xrightarrow{\phi} & X & \xrightarrow{f} & Y \\ \uparrow{\scriptstyle v} & & & & \uparrow{\scriptstyle u} \\ W & & \longrightarrow & & U. \end{array}$$

Then $W \to U$ is faithfully flat and u lifts to $v \in Z(W)$ and hence to $\phi v \in X(W)$.

To see the assertion about flat f, note first that a faithfully flat map is trivially an epimorphism of flat sheaves. Secondly, if f is flat and an epimorphism of flat sheaves, then there is some $\phi : Z \to X$ as in (ii) and the composition $f\phi$ is surjective (on the topological spaces underlying these affine schemes), hence so is f, i.e. f is faithfully flat [3, Chapter II, §2, no 5, Corollary 4(ii)]. The injectivity of ψ_f is a special case of [3, I, §3, Proposition 8(i)]. □

We have a similar result for epimorphisms of algebraic stacks.

Proposition 19. *Let* $(f_0, f_1) : (X_0, X_1) \to (Y_0, Y_1)$ *be a* 1*-morphism of flat Hopf algebroids inducing* $f : \mathfrak{X} \to \mathfrak{Y}$ *on associated algebraic stacks and write* $\alpha := t\pi_2 : X_0 \underset{f_0, Y_0, s}{\times} Y_1 \to Y_0$. *Then the following are equivalent*:

- (i) f *is an epimorphism.*
- (ii) α *is an epimorphism of flat sheaves.*
- (iii) *There is some* $\phi : Z \to X_0 \underset{f_0, Y_0, s}{\times} Y_1$ *in* Aff *such that* $\alpha\phi$ *is faithfully flat.*

If these conditions hold then $\mathrm{id}_{\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{Y}})} \to f_* f^*$ *is injective.*

Proof. The equivalence of (i) and (ii) is "mise pour memoire," the one of (ii) and (iii) has been proved in Proposition 18. Assume that these conditions hold and let $g : \mathfrak{X}' \to \mathfrak{X}$ be any morphism of algebraic stacks. Assume that $\mathrm{id}_{\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{Y}})} \to (fg)_*(fg)^*$ is injective. Then the composition $\mathrm{id}_{\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{Y}})} \to f_* f^* \to f_* g_* g^* f^* = (fg)_*(fg)^*$ is injective and hence so is $\mathrm{id}_{\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{Y}})} \to f_* f^*$. Taking $g := P : X_0 \to \mathfrak{X}$ to be the canonical presentation we see that we can assume that

$\mathfrak{X} = X_0$, in particular $f : X_0 \to \mathfrak{Y}$ is representable and affine (and an epimorphism). Now let $Q : Y_0 \to \mathfrak{Y}$ be the canonical presentation and form the cartesian diagram

$$\begin{array}{ccc} Z_0 & \xrightarrow{g_0} & Y_0 \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle Q} \\ X_0 & \xrightarrow{f} & \mathfrak{Y}. \end{array} \tag{4}$$

As Q is *fpqc* we know that $\mathrm{id}_{\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{Y}})} \to f_* f^*$ is injective if and only if $Q^* \to Q^* f_* f^* \simeq g_{0,*} P^* f^* \simeq g_{0,*} g_0^* Q^*$ is injective, we used flat base change, [24, Proposition 13.1.9], and this will follow from the injectivity of $\mathrm{id}_{\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{Y_0})} \to g_{0,*} g_0^*$ because Q is flat.

As f is representable and affine, Z_0 is an affine scheme hence, by Proposition 18, we are done because g_0 is an epimorphism of flat sheaves [24, Proposition 3.8.1]. □

There is an analogous result for monomorphisms of algebraic stacks.

Proposition 20. *Let* $(f_0, f_1) : (X_0, X_1) \to (Y_0, Y_1)$ *be a* 1*-morphism of flat Hopf algebroids,* $P : X_0 \to \mathfrak{X}$ *the rigidified algebraic stack associated with* (X_0, X_1)*,* $f : \mathfrak{X} \to \mathfrak{Y}$ *the associated* 1*-morphism of algebraic stacks,* $\Theta : f^* f_* \to \mathrm{id}_{\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{X}})}$ *the adjunction and* $\beta = (s, f_1, t) : X_1 \to X_0 \underset{f_0, Y_0, s}{\times} Y_1 \underset{t, Y_0, f_0}{\times} X_0$*. Then the following are equivalent*:

(i) *f is a monomorphism.*
(ii) *β is an isomorphism.*
(iii) *$\Theta_{P_* \mathcal{O}_{X_0}}$ is an isomorphism.*

If f is representable then these conditions are equivalent to:

(iiia) *Θ is an isomorphism.*
(iiib) *f_* is fully faithful.*

Remark 21. This result may be compared to the first assertion of Theorem 2.5 of [15]. There it is proved that Θ is an isomorphism if f is a *flat* monomorphism.

In the situation of Proposition 20(iiib) it is natural to ask for the essential image of f_*, see Proposition 22.

I do not know whether every monomorphism of algebraic stacks is representable, cf. [24, Corollaire 8.1.3].

Proof of Proposition 20. We already know that (i) and (ii) are equivalent. Consider the diagram

$$\begin{array}{ccccc} X_0 & \xrightarrow{P} & \mathfrak{X} & \xrightarrow{f} & \mathfrak{Y} \\ {\scriptstyle \Delta'}\downarrow\uparrow{\scriptstyle \pi_1'} & & {\scriptstyle \Delta_f}\downarrow\uparrow{\scriptstyle \pi_1} & & \uparrow{\scriptstyle f} \\ \pi : \mathfrak{Z} & \xrightarrow[P']{} & \mathfrak{X} \underset{f, \mathfrak{Y}, f}{\times} \mathfrak{X} & \xrightarrow[\pi_2]{} & \mathfrak{X} \end{array}$$

in which the squares made of straight arrows are cartesian. As fP is representable and affine, we have $fP = \underline{\mathrm{Spec}}(f_* P_* \mathcal{O}_{X_0})$, cf. [24, 14.2], and $\pi = \underline{\mathrm{Spec}}(f^* f_* P_* \mathcal{O}_{X_0})$. We know that (i) is equivalent to the diagonal of f, Δ_f, being an isomorphism [24, Remarque 2.3.1]. As Δ_f is a section of π_1 this is equivalent to π_1 being an isomorphism. As P is an epimorphism, this is equivalent to π_1' being an isomorphism by [24, Proposition 3.8.1]. Of course, π_1' admits $\Delta' := (\mathrm{id}_{X_0}, \Delta_f P)$ as a section so, finally, (i) is equivalent to Δ' being an isomorphism. One checks that $\Delta' = \underline{\mathrm{Spec}}(\Theta_{P_* \mathcal{O}_{X_0}})$ and this proves the equivalence of (i) and (iii).

Now assume that f is representable and a monomorphism. We will show that (iiia) holds. Consider the cartesian diagram

$$\begin{array}{ccc} Z & \xrightarrow{f'} & Y_0 \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle Q} \\ \mathfrak{X} & \xrightarrow{f} & \mathfrak{Y}. \end{array}$$

We have

$$P^* f^* f_* \simeq f'^* Q^* f_* \simeq f'^* f'_* P^*.$$

As P^* reflects isomorphism, (iiia) will hold if the adjunction $f'^* f'_* \to \mathrm{id}_{\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_Z)}$ is an isomorphism. As f is representable, this can be checked at the stalks of $z \in Z$, and we can replace f' by the induced morphism $\mathrm{Spec}(\mathcal{O}_{Z,z}) \to \mathrm{Spec}(\mathcal{O}_{Y_0,y})$ ($y := f'(z)$) which is a monomorphism. In particular, we have reduced the proof of (iiia) to the case of affine schemes, i.e. the following assertion: If $\phi : A \to B$ is a ring homomorphism such that $\mathrm{Spec}(\phi)$ is a monomorphism, i.e. the ring homomorphism corresponding to the diagonal $B \otimes_A B \to B$, $b_1 \otimes b_2 \mapsto b_1 b_2$, is an isomorphism, then, for every B-module M, the canonical homomorphism of B-modules $M \otimes_A B \to M$ is an isomorphism. This is however easy:

$$M \otimes_A B \simeq (M \otimes_B B) \otimes_A B \simeq M \otimes_B (B \otimes_A B) \simeq M \otimes_B B \simeq M,$$

and we leave it to the reader to check that the composition of these isomorphisms is the natural map $M \otimes_A B \to M$.

Finally, the proof that (iiia) and (iiib) are equivalent is a formal manipulation with adjunctions which we leave to the reader, and trivially (iiia) implies (iii). □

Proposition 22. *In the situation of Proposition* 20 *assume that f is representable and a monomorphism, let $Q : Y_0 \to \mathfrak{Y}$ be the rigidified algebraic stack associated with (Y_0, Y_1) and form the cartesian diagram*

$$\begin{array}{ccc} Z_0 & \xrightarrow{g_0} & Y_0 \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle Q} \\ \mathfrak{X} & \xrightarrow{f} & \mathfrak{Y}. \end{array} \tag{5}$$

Then Z_0 is an algebraic space and a given $\mathcal{F} \in \mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{Y}})$ is in the essential image of f_* if and only if $Q^*\mathcal{F}$ is in the essential image of $g_{0,*}$. Consequently, f_* induces an equivalence between $\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{X}})$ and the full subcategory of $\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{Y}})$ consisting of such $\mathcal{F}$.

Proof. Firstly, Z_0 is an algebraic space because f is representable. We know that f_* is fully faithful by Proposition 20(iiib) and need to show that the above description of its essential image is correct. If $\mathcal{F} \simeq f_*\mathcal{G}$ then $Q^*\mathcal{F} \simeq Q^* f_*\mathcal{G} \simeq g_{0,*}P^*\mathcal{G}$ so $Q^*\mathcal{F}$ lies in the essential image of $g_{0,*}$. To see the converse, extend (5) to a cartesian diagram

$$\begin{array}{ccc}
Z_1 & \xrightarrow{g_1} & Y_1 \\
\downarrow\downarrow & & \downarrow\downarrow \\
Z_0 & \xrightarrow{g_0} & Y_0 \\
{\scriptstyle P}\downarrow & & \downarrow{\scriptstyle Q} \\
\mathfrak{X} & \xrightarrow{f} & \mathfrak{Y}.
\end{array}$$

Note that $\mathfrak{X} \simeq [Z_1 \rightrightarrows Z_0]$, hence (Z_0, Z_1) is a flat groupoid (in algebraic spaces) representing $\mathfrak{X}$. Now let there be given $\mathcal{F} \in \mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{Y}})$ and $G \in \mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{Z_0})$ with $Q^*\mathcal{F} \simeq g_{0,*}G$. We define σ to make the following diagram commutative:

$$\begin{array}{ccc}
s^*Q^*\mathcal{F} & \xrightarrow[\sim]{can} & t^*Q^*\mathcal{F} \\
\sim\downarrow & & \sim\downarrow \\
s^*g_{0,*}G & & t^*g_{0,*}G \\
\sim\downarrow & & \sim\downarrow \\
g_{1,*}s^*G & \xrightarrow[\sigma]{\sim} & g_{1,*}t^*G.
\end{array}$$

As f is representable and a monomorphism, so is g_1 and thus $g_1^*g_{1,*} \xrightarrow{\sim} \mathrm{id}_{\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{Z_1})}$ and $g_{1,*}$ is fully faithful by Proposition 20(iiia), (iiib). We define τ to make the following diagram commutative:

$$\begin{array}{ccc}
g_1^*g_{1,*}s^*G & \xrightarrow[\sim]{g_1^*(\sigma)} & g_1^*g_{1,*}t^*G \\
\sim\downarrow & & \sim\downarrow \\
s^*G & \xrightarrow{\tau} & t^*G.
\end{array}$$

Then τ satisfies the cocycle condition because it does so after applying the faithful functor $g_{1,*}$. So τ is a descent datum on G, and G descents to $\mathcal{G} \in \mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{X}})$ with $P^*\mathcal{G} \simeq G$ and we have $Q^* f_*\mathcal{G} \simeq g_{0,*}P^*\mathcal{G} \simeq Q^*\mathcal{F}$, hence $f_*\mathcal{G} \simeq \mathcal{F}$, i.e. $\mathcal{F}$ lies in the essential image of f_* as was to be shown. □

To conclude this section we give the proof of Theorem 13 the notations and assumptions of which we now resume.

Proof of Theorem 13. If (iii) holds then f is an epimorphism and a monomorphism by Proposition 19(iii) $\Rightarrow$ (i) and Proposition 20(ii) $\Rightarrow$ (i) hence (i) holds by [24, Corollaire 3.7.1]. The proof that (i) implies (ii) is left to the reader and we assume that (ii) holds. Since (f^*, f_*) is an adjoint pair of functors, f_* is a quasi-inverse for f^* and $\Theta : f^* f_* \to \mathrm{id}_{\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{X}})}$ is an isomorphism so β is an isomorphism by Proposition 20(iii) $\Rightarrow$ (ii). As f^* is in particular exact and faithful, α is faithfully flat by Proposition 15(ii) $\Rightarrow$ (iii) and (iii) holds. □

5. Landweber exactness and change of rings

In this section we will use the techniques from Section 4 to give a short and conceptional proof of the fact that Landweber exact BP_*-algebras of the same height have equivalent categories of comodules. In fact, we will show that the relevant algebraic stacks are 1-isomorphic.

Let p be a prime number. We will study the algebraic stack associated with the flat Hopf algebroid $(\mathrm{BP}_*, \mathrm{BP}_*\mathrm{BP})$ where BP denotes Brown–Peterson homology at p.

We will work over $S := \mathrm{Spec}(\mathbb{Z}_{(p)})$, i.e. Aff will be the category of $\mathbb{Z}_{(p)}$-algebras with its *fpqc* topology. We refer the reader to [33, Chapter 4] for basic facts about BP, e.g. $\mathrm{BP}_* = \mathbb{Z}_{(p)}[v_1, \ldots]$ where the v_i denote either the Hazewinkel- or the Araki-generators, it does not matter but the reader is free to make a definite choice at this point if she feels like doing so.

Now, $(V := \mathrm{Spec}(\mathrm{BP}_*), W := \mathrm{Spec}(\mathrm{BP}_*\mathrm{BP}))$ is a flat Hopf algebroid and we denote by $P : V \to \mathfrak{X}_{FG}$ the corresponding rigidified algebraic stack. We refer the reader to Section 6 for an intrinsic description of the stack $\mathfrak{X}_{FG}$.

For $n \geqslant 1$ the ideal $I_n := (v_0, \ldots, v_{n-1}) \subseteq \mathrm{BP}_*$ is an invariant prime ideal where we agree that $v_0 := p$, $I_0 := (0)$ and $I_\infty := (v_0, v_1, \ldots)$.

As explained in Section 3.4, corresponding to these invariant ideals there is a sequence of closed substacks

$$\mathfrak{X}_{FG} = \mathfrak{Z}^0 \supseteq \mathfrak{Z}^1 \supseteq \cdots \supseteq \mathfrak{Z}^\infty.$$

We denote by $\mathfrak{U}^n := \mathfrak{X}_{FG} - \mathfrak{Z}^n$ $(0 \leqslant n \leqslant \infty)$ the open substack complementary to $\mathfrak{Z}^n$ and have an ascending chain

$$\emptyset = \mathfrak{U}^0 \subseteq \mathfrak{U}^1 \subseteq \cdots \subseteq \mathfrak{U}^\infty \subseteq \mathfrak{X}_{FG}.$$

For $0 \leqslant n < \infty$, I_n is finitely generated, hence the open immersion $\mathfrak{U}^n \subseteq \mathfrak{X}_{FG}$ is quasi-compact and $\mathfrak{U}^n$ is an algebraic stack. However, $\mathfrak{U}^\infty$ is not algebraic: If it was, it could be covered by an affine (hence quasi-compact) scheme and the open covering $\mathfrak{U}^\infty = \bigcup_{n \geqslant 0,\, n \neq \infty} \mathfrak{U}^n$ would allow a finite subcover, which it does not.

5.1. *The algebraic stacks associated with Landweber exact* BP_**-algebras*

In this section we prove our main result, Theorem 26, which determines the stack theoretic image of a morphism $X_0 \to \mathfrak{X}_{FG}$ corresponding to a Landweber exact BP_*-algebra. It turns out that the same arguments apply more generally to morphisms $X_0 \to \mathfrak{Z}^n$ for every $n \geqslant 0$ and we work in this generality from the very beginning.

Fix some $0 \leqslant n < \infty$. The stack $\mathfrak{Z}^n$ is associated with the flat Hopf algebroid (V_n, W_n) where $V_n := \operatorname{Spec}(\mathrm{BP}_*/I_n)$ and $W_n := \operatorname{Spec}(\mathrm{BP}_*\mathrm{BP}/I_n\mathrm{BP}_*\mathrm{BP})$, the flatness of this Hopf algebroid is established by direct inspection, and we have a cartesian diagram

$$\begin{array}{ccc} W_n & \hookrightarrow & W = W_0 \\ \downarrow\downarrow & & \downarrow\downarrow \\ V_n & \overset{i_n}{\hookrightarrow} & V = V_0 \\ {\scriptstyle Q_n}\downarrow & & \downarrow{\scriptstyle Q} \\ \mathfrak{Z}^n & \hookrightarrow & \mathfrak{X}_{FG} \end{array} \tag{6}$$

in which the horizontal arrows are closed immersions.

We have an ascending chain of open substacks

$$\emptyset = \mathfrak{Z}^n \cap \mathfrak{U}^n \subseteq \mathfrak{Z}^n \cap \mathfrak{U}^{n+1} \subseteq \cdots \subseteq \mathfrak{Z}^n \cap \mathfrak{U}^\infty \subseteq \mathfrak{Z}^n.$$

Let $X_0 \xrightarrow{\phi} V_n$ be a morphism in Aff corresponding to a morphism of rings $\mathrm{BP}_*/I_n \to R := \Gamma(X_0, \mathcal{O}_{X_0})$. Slightly generalising Definition 4.1 of [15] we define the height of ϕ to be

$$\operatorname{ht}(\phi) := \max\{N \geqslant 0 \mid R/I_N R \neq 0\}$$

which may be ∞ and we agree to put $\operatorname{ht}(\phi) := -1$ in case $R = 0$, i.e. $X_0 = \emptyset$. Recall that a geometric point of X_0 is a morphism $\Omega \xrightarrow{\alpha} X_0$ in Aff where $\Omega = \operatorname{Spec}(K)$ is the spectrum of an algebraically closed field K. The composition $\Omega \xrightarrow{\alpha} X_0 \xrightarrow{\phi} V_n \overset{i_n}{\hookrightarrow} V$ specifies a p-typical formal group law over K and $\operatorname{ht}(i_n\phi\alpha)$ is the height of this formal group law. The relation between $\operatorname{ht}(\phi)$ and the height of formal group laws is the following.

Proposition 23. *In the above situation we have*

$$\operatorname{ht}(\phi) = \max\bigl\{\operatorname{ht}(i_n\phi\alpha) \bigm| \alpha : \Omega \longrightarrow X_0 \textit{ a geometric point}\bigr\},$$

with the convention that $\max\emptyset = -1$.

This proposition means that $\operatorname{ht}(\phi)$ is the maximum height in a geometric fibre of the formal group law over X_0 parametrised by $i_n\phi$.

Proof. Clearly, $\operatorname{ht}(i_n\phi\psi) \leqslant \operatorname{ht}(\phi)$ for every morphism $\psi : Y \to X_0$ in Aff. For every $0 \leqslant N' \leqslant \operatorname{ht}(\phi)$ we have $I_{N'}R \neq R$ so there is a maximal ideal of R containing $I_{N'}R$, and a geometric point α of X_0 supported at this maximal ideal will satisfy $\operatorname{ht}(i_n\phi\alpha) \geqslant N'$. □

Another geometric interpretation of $\operatorname{ht}(\phi)$ is given by considering the composition $f : X_0 \xrightarrow{\phi} V_n \xrightarrow{Q_n} \mathfrak{Z}^n$.

Proposition 24. *In this situation we have*

$$\mathrm{ht}(\phi)+1=\min\{N\geqslant 0 \mid f \text{ factors through } \mathfrak{Z}^n\cap\mathfrak{U}^N\hookrightarrow\mathfrak{Z}^n\}$$

with the convention that $\min\emptyset=\infty$ *and* $\infty+1=\infty$.

Proof. For every $\infty>N\geqslant n$ we have a cartesian square

$$\begin{array}{ccc} V_n^N & \xrightarrow{\;j\;} & V_n \\ \Big\downarrow & & \Big\downarrow Q_n \\ \mathfrak{Z}^n\cap\mathfrak{U}^N & \xrightarrow{\;i\;} & \mathfrak{Z}^n \end{array} \tag{7}$$

where $V_n^N=V_n-\mathrm{Spec}(\mathrm{BP}_*/I_N)=\bigcup_{i=n}^{N-1}\mathrm{Spec}((\mathrm{BP}_*/I_n)[v_i^{-1}])$ hence f factors through i if and only if $\phi:X_0\to V_n$ factors through j. As j is an open immersion, this is equivalent to $|\phi|(|X_0|)\subseteq|V_n^N|\subseteq|V_n|$ where $|\cdot|$ denotes the topological space underlying a scheme. But this condition can be checked using geometric points and the rest is easy, using Proposition 23. □

Recall from [15, Definition 2.1] that, if (A,Γ) is a flat Hopf algebroid, an A-algebra $f:A\to B$ is said to be *Landweber exact* over (A,Γ) if the functor $M\mapsto M\otimes_A B$ from Γ-comodules to B-modules is exact. For $(X_0:=\mathrm{Spec}(A), X_1:=\mathrm{Spec}(\Gamma))$, $\phi:=\mathrm{Spec}(f):Y_0:=\mathrm{Spec}(B)\to X_0$ and $P:X_0\to\mathfrak{X}$ the rigidified algebraic stack associated with (X_0,X_1) this exactness is equivalent to the flatness of the composition $Y_0\xrightarrow{\phi}X_0\xrightarrow{P}\mathfrak{X}$ because the following square of functors commutes up to natural isomorphism

$$\begin{array}{ccc} (P\phi)^*:\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{X}}) & \longrightarrow & \mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{Y_0}) \\ \Big\downarrow\simeq & & \Big\downarrow\simeq \\ \Gamma\text{-comodules} & \xrightarrow{M\mapsto M\otimes_A B} & B\text{-modules,} \end{array}$$

where the horizontal equivalences are those given by (2).

In case $\mathfrak{X}=\mathfrak{Z}^n$ this flatness has the following decisive consequence which paraphrases the fact that the image of a flat morphism is stable under generalisation.

Proposition 25. *Assume that* $n\geqslant 0$ *and that* $\phi:\emptyset\neq X_0\to V_n$ *is Landweber exact of height* $N:=\mathrm{ht}(\phi)$*, hence* $n\leqslant N\leqslant\infty$*. Then for every* $n\leqslant j\leqslant N$ *there is a geometric point* $\alpha:\Omega\to X_0$ *such that* $\mathrm{ht}(i_n\phi\alpha)=j$.

Proof. Let ϕ correspond to $\mathrm{BP}_*/I_n\to R$. We first note that $v_n,v_{n+1},\ldots\in R$ is a regular sequence by Proposition 27 below. Now assume that $N<\infty$ and fix $n\leqslant j\leqslant N$. Then $v_j\in R/I_{j-1}R\neq 0$ is not a zero divisor and thus there is a minimal prime ideal of $R/I_{j-1}R$ not containing v_j. A geometric point supported at this prime ideal solves the problem. In the remaining case $j=N=\infty$ we have $R/I_\infty R\neq 0$ and every geometric point of this ring solves the problem. □

The main result of this paper is the following.

Theorem 26. *Assume that $n \geqslant 0$ and that $\emptyset \neq X_0 \to V_n$ is Landweber exact of height N, hence $n \leqslant N \leqslant \infty$. Let (X_0, X_1) be the Hopf algebroid induced from (V, W) by the composition $X_0 \xrightarrow{\phi} V_n \overset{i_n}{\hookrightarrow} V$. Then (X_0, X_1) is a flat Hopf algebroid and its associated algebraic stack is given as*

$$[X_1 \rightrightarrows X_0] \simeq \mathfrak{Z}^n \cap \mathfrak{U}^{N+1} \quad \text{if } N \neq \infty \quad \text{and}$$

$$[X_1 \rightrightarrows X_0] \simeq \mathfrak{Z}^n \quad \text{if } N = \infty.$$

Proof. Note that (X_0, X_1) is also induced from the flat Hopf algebroid (V_n, W_n) along ϕ and thus is a flat Hopf algebroid using the final statement of Proposition 16 and the Landweber exactness of ϕ. We first assume that $N \neq \infty$. Then by Proposition 24 the composition $X_0 \xrightarrow{\phi} V_n \to \mathfrak{Z}^n$ factors as $X_0 \xrightarrow{\psi} \mathfrak{Z}^n \cap \mathfrak{U}^{N+1} \xrightarrow{i} \mathfrak{Z}^n$ and ψ is flat because i is an open immersion and $X_0 \to \mathfrak{Z}^n$ is flat by assumption. By Proposition 17 we will be done if we can show that ψ is in fact faithfully flat. For this we consider the presentation $\mathfrak{Z}^n \cap \mathfrak{U}^{N+1} \simeq [W_n^{N+1} \rightrightarrows V_n^{N+1}]$ given by the cartesian diagram

$$\begin{array}{ccc}
W_n^{N+1} & \longrightarrow & W_n \\
\downarrow\downarrow & & \downarrow\downarrow \\
V_n^{N+1} & \longrightarrow & V_n \\
\downarrow & & \downarrow{\scriptstyle Q_n} \\
\mathfrak{Z}^n \cap \mathfrak{U}^{N+1} & \longrightarrow & \mathfrak{Z}^n
\end{array}$$

and note that ψ lifts to $\rho : X_0 \to V_n^{N+1}$ and induces $\alpha := t\pi_2 : X_0 \underset{\rho, V_n^{N+1}, s}{\times} W_n^{N+1} \to V_n^{N+1}$ which is flat and we need it to be faithfully flat to apply Proposition 15(iii) $\Rightarrow$ (iv) and conclude that ψ is faithfully flat. So we have to prove that α is surjective on the topological spaces underlying the schemes involved.

This surjectivity can be checked on geometric points and for any such geometric point $\Omega \xrightarrow{\mu} V_n^{N+1}$ we know that $j := \mathrm{ht}(\Omega \xrightarrow{\mu} V_n^{N+1} \to V_n \overset{i_n}{\hookrightarrow} V)$ satisfies $n \leqslant j \leqslant N$. By Proposition 25 there is a geometric point $\Omega' \xrightarrow{\nu} X_0$ with $\mathrm{ht}(\Omega' \xrightarrow{\nu} X_0 \to V_n \overset{i_n}{\hookrightarrow} V) = j$ and we can assume that $\Omega = \Omega'$ because the corresponding fields have the same characteristic, namely 0 if $j = 0$ and p otherwise. As any two formal group laws over an algebraically closed field having the same height are isomorphic we find some $\sigma : \Omega \to W_n^{N+1}$ fitting into a commutative diagram

$$\begin{array}{ccc}
X_0 \underset{\rho, V_n^{N+1}, s}{\times} W_n^{N+1} & \xrightarrow{\alpha} & V_n^{N+1} \\
\uparrow{\scriptstyle (\nu,\sigma)} & \nearrow{\scriptstyle \mu} & \\
\Omega. & &
\end{array}$$

As μ was arbitrary this shows that α is surjective. We leave the obvious modifications for the case $N=\infty$ to the reader. $\square$

To conclude this section we explain the relation of Landweber exactness and Landweber's regularity condition. This has in fact been worked out in detail in [8, Section 3, Theorem 8] but we include it here anyway. Fix some $n \geqslant 0$ and let $\phi: \mathrm{BP}_*/I_n \to R$ be a BP_*/I_n-algebra. Then Landweber's condition is

$$\text{The sequence } \phi(v_n), \phi(v_{n+1}), \ldots \in R \text{ is regular.} \tag{8}$$

Proposition 27. *In the above situation,* (8) *holds if and only if the composition* $\mathrm{Spec}(R) \to \mathrm{Spec}(\mathrm{BP}/I_n) \to \mathfrak{Z}^n$ *is flat.*

Proof. From [27, Proposition 2.2] we know that the restriction of

$$f^*: \mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{Z}^n}) \longrightarrow \mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathrm{Spec}(R)})$$

to finitely presented comodules is exact if and only if (8) holds. But f^* itself is exact, and hence f is flat, if and only if its above restriction is exact because every $\mathrm{BP}_*\mathrm{BP}/I_n$-comodule is the filtering direct limit of finitely presented comodules. This was pointed out to me by N. Strickland. In case $n=0$ this result is [27, Lemma 2.11] and the general case follows from [14, Proposition 1.4.1(e), Proposition 1.4.4, Lemma 1.4.6 and Proposition 1.4.8]. $\square$

5.2. Equivalence of comodule categories and change of rings

In this section we will spell out some consequences of the above results in the language of comodules but we need some elementary preliminaries first.

Let A be a ring, $I=(f_1,\ldots,f_n) \subseteq A$ ($n \geqslant 1$) a finitely generated ideal and M an A-module. We have a canonical map

$$\bigoplus_i M_{f_i} \longrightarrow \bigoplus_{i<j} M_{f_i f_j}, \quad (x_i)_i \longmapsto \left(\frac{x_i}{1} - \frac{x_j}{1}\right)_{i,j},$$

and a canonical map

$$\alpha_M : M \longrightarrow \ker\left(\bigoplus_i M_{f_i} \longrightarrow \bigoplus_{i<j} M_{f_i f_j}\right).$$

For $X := \mathrm{Spec}(A)$, $Z := \mathrm{Spec}(A/I)$, $j: U := X - Z \hookrightarrow X$ the open immersion and $\mathcal{F}$ the quasi-coherent $\mathcal{O}_X$-module corresponding to M, α_M corresponds to the adjunction $\mathcal{F} \to j_* j^* \mathcal{F}$. Note that $\ker(\alpha_M)$ is the I-torsion submodule of M. The cokernel of α_M corresponds to the local cohomology $H^1_Z(X,\mathcal{F})$, cf. [11]. We say that M is I-local if α_M is an isomorphism. A quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is in the essential image of j_* if and only if $\mathcal{F} \to j_* j^* \mathcal{F}$ is an isomorphism if and only if the A-module corresponding to $\mathcal{F}$ is I-local. If $n=1$ then M is $I=(f_1)$-local if and only if f_1 acts invertibly on M.

We now formulate a special case of Proposition 22 in terms of comodules.

Proposition 28.

(i) *For every $n \geqslant 0$ the category* $\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{Z}^n})$ *is equivalent to the full subcategory of* $\mathrm{BP}_*\mathrm{BP}$-*comodules M such that $I_n M = 0$.*

(ii) *For every $0 \leqslant n \leqslant N < \infty$ the category* $\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{Z}^n \cap \mathfrak{U}^{N+1}})$ *is equivalent to the full subcategory of* $\mathrm{BP}_*\mathrm{BP}$-*comodules M such that $I_n M = 0$ and M is I_{N+1}/I_n-local as a* BP_*/I_n-*module.*

Remark 29. We know from (2) that $\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{Z}^n})$ is equivalent to the category of $\mathrm{BP}_*\mathrm{BP}/I_n$-comodules. The alert reader will have noticed that we have not yet mentioned any graded comodules. This is not sloppy terminology, we really mean comodules without any grading even though the flat Hopf algebroids are all graded. However, it is easy to take the grading into account, in particular all results of this section have analogues for graded comodules, cf. Remark 34.

Proof of Proposition 28. For the proof of part (i), fix $0 \leqslant n < \infty$. The 1-morphism $\mathfrak{Z}^n \hookrightarrow \mathfrak{X}_{FG}$ is representable and a closed immersion (in particular a monomorphism) because its base change along $V \to \mathfrak{X}_{FG}$ is a closed immersion and being a closed immersion is *fpqc*-local on the base [10, 2.7.1, (xii)]. Proposition 22 identifies $\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{Z}^n})$ with the full subcategory of $\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{X}_{FG}})$ consisting of those $\mathcal{F}$ such that $Q^*\mathcal{F} \simeq i_{n,*}G$ for some $G \in \mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{V_n})$ (with notations as in (6)). Identifying, as in Section 3.4, $\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{X}_{FG}})$ with the category of $\mathrm{BP}_*\mathrm{BP}$-comodules, $\mathcal{F}$ corresponds to some $\mathrm{BP}_*\mathrm{BP}$-comodule M and $Q^*\mathcal{F}$ corresponds to the BP_*-module underlying M. So the condition of Proposition 22 is that the BP_*-module M is in the essential image of $i_{n,*}$, i.e. M is an BP_*/I_n-module, i.e. $I_n M = 0$.

We now prove part (ii): Fix $0 \leqslant n \leqslant N < \infty$. We apply Proposition 22 to $i : \mathfrak{Z}^n \cap \mathfrak{U}^{N+1} \to \mathfrak{X}_{FG}$ which is representable and a quasi-compact immersion (in particular a monomorphism) because it sits in a cartesian diagram

$$\begin{array}{ccc} V_n^{N+1} & \xrightarrow{\ j\ } & V \\ \downarrow & & \downarrow{\scriptstyle Q} \\ \mathfrak{Z}^n \cap \mathfrak{U}^{N+1} & \xrightarrow{\ i\ } & \mathfrak{X}_{FG}, \end{array}$$

cf. (7), in which j is a quasi-compact immersion and one uses [10, 2.7.1, (xi)] as above. Arguing as above, we are left with identifying the essential image of j_* which, as explained at the beginning of this section, corresponds to the BP_*-modules M such that $I_n M = 0$ and M is I_{N+1}/I_n-local as a BP_*/I_n-module. □

Corollary 30. *Let $n \geqslant 0$ and let $\mathrm{BP}_*/I_n \to R \neq 0$ be Landweber exact of height N, hence $n \leqslant N \leqslant \infty$. Then $(R, \Gamma) := (R, R \otimes_{\mathrm{BP}_*} \mathrm{BP}_*\mathrm{BP} \otimes_{\mathrm{BP}_*} R)$ is a flat Hopf algebroid and its category of comodules is equivalent to the full subcategory of* $\mathrm{BP}_*\mathrm{BP}$-*comodules M such that $I_n M = 0$ and M is I_{N+1}/I_n-local as a* BP_*/I_n-*module. The last condition is to be ignored in case $N = \infty$.*

Proof. By Theorem 26, (R, Γ) is a flat Hopf algebroid with associated algebraic stack $\mathfrak{Z}^n \cap \mathfrak{U}^{N+1}$ (respectively $\mathfrak{Z}^n$ if $N = \infty$). So the category of (R, Γ)-comodules is equivalent to $\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{Z}^n \cap \mathfrak{U}^{N+1}})$ (respectively $\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{Z}^n})$). Now use Proposition 28. □

Remark 31. The case $n = 0$ of Corollary 30 corresponds to the situation treated in [15] where, translated into the present terminology, $\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{U}^{N+1}})$ is identified as a *localisation* of $\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{X}_{FG}})$. This can be done because $f : \mathfrak{U}^{N+1} \to \mathfrak{X}_{FG}$ is flat, hence f^* exact. To relate more generally $\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{Z}^n \cap \mathfrak{U}^{N+1}})$ to $\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{X}_{FG}})$ it seems more appropriate to identify the former as a full subcategory of the latter as we did above. However, using Proposition 1.4 of [15] and Proposition 20 one sees that $\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{Z}^n \cap \mathfrak{U}^{N+1}})$ is equivalent to the localisation of $\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{X}_{FG}})$ with respect to all morphisms α such that $f^*(\alpha)$ is an isomorphism where $f : \mathfrak{Z}^n \cap \mathfrak{U}^{N+1} \to \mathfrak{X}_{FG}$ is the immersion. As f is not flat for $n \geqslant 1$ this condition seems less tractable than the one in Corollary 30.

Of course, equivalences of comodule categories give rise to change of rings theorems and we refer to [15] for numerous examples (in the case $n = 0$) and only point out the following, cf. [34, Theorem B.8.8] for the notation and a special case: If $n \geqslant 1$ and M is a $\mathrm{BP}_*\mathrm{BP}$-comodule such that $I_n M = 0$ and v_n acts invertibly on M then

$$\mathrm{Ext}^*_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, M) \simeq \mathrm{Ext}^*_{\Sigma(n)}\big(\mathbb{F}_p\big[v_n, v_n^{-1}\big], M \otimes_{\mathrm{BP}_*} \mathbb{F}_p\big[v_n, v_n^{-1}\big]\big).$$

In fact, this is clear from the case $n = N$ of Corollary 30 applied to the obvious map $\mathrm{BP}_*/I_n \to \mathbb{F}_p[v_n, v_n^{-1}]$ which is Landweber exact of height n.

To make a final point, in [15] we also find many of the fundamental results of [22] generalised to Landweber exact algebras whose induced Hopf algebroids are presentations of our $\mathfrak{U}^{N+1}$. One may generalise these results further to the present case, i.e. to $\mathfrak{Z}^n \cap \mathfrak{U}^{N+1}$ for $n \geqslant 1$, but again we leave this to the reader and only point out an example: In the situation of Corollary 30 every non-zero graded (R, Γ)-comodule has a non-zero primitive.

To prove this, consider the comodule as a quasi-coherent sheaf $\mathcal{F}$ on $\mathfrak{Z}^n \cap \mathfrak{U}^{N+1}$ and use that the primitives we are looking at are $H^0(\mathfrak{Z}^n \cap \mathfrak{U}^{N+1}, \mathcal{F}) \simeq H^0(\mathfrak{X}_{FG}, f_*\mathcal{F}) \neq 0$ because f_* is faithful and using the result of P. Landweber that every non-zero graded $\mathrm{BP}_*\mathrm{BP}$-comodule has a non-zero primitive.

6. The stack of formal groups

In this section we take a closer look at the algebraic stacks associated with the flat Hopf algebroids $(\mathrm{MU}_*, \mathrm{MU}_*\mathrm{MU})$ and $(\mathrm{BP}_*, \mathrm{BP}_*\mathrm{BP})$.

A priori, these stacks are given by the abstract procedure of stackification and in many instances one can work with this definition directly, the results of the previous sections are an example of this. For future investigations, e.g. those initiated in [9], it might be useful to have the genuinely geometric description of these stacks which we propose to establish in this section.

For this, we require a good notion of formal scheme over an arbitrary affine base as given by N. Strickland [36] and we quickly recall some of his results now.

The category $X_{fs,\mathbb{Z}}$ of formal schemes over $\mathrm{Spec}(\mathbb{Z})$ is defined to be the ind-category of $\mathrm{Aff}_{\mathbb{Z}}$ which we consider as usual as a full subcategory of the functor category $C := \underline{\mathrm{Hom}}(\mathrm{Aff}_{\mathbb{Z}}^{\mathrm{op}}, \mathrm{Sets})$, cf. [36, Definition 4.1] and [41, exposé I, 8]. A formal ring is by definition a linearly topologised Hausdorff and complete ring and FRings denotes the category of formal rings with continuous ring homomorphisms. Every ring can be considered as a formal ring by giving it the discrete topology. There is a fully faithful functor $\mathrm{Spf} : \mathrm{FRings}^{\mathrm{op}} \to X_{fg,\mathbb{Z}} \subset C$ [36, Section 4.2] given by

$$\mathrm{Spf}(R)(S) := \mathrm{Hom}_{\mathrm{FRings}}(R, S) = \mathrm{colim}_I \, \mathrm{Hom}_{\mathrm{Rings}}(R/I, S),$$

the limit being taken over the directed set of open ideals $I \subseteq R$.

In particular, every ring R can be considered as a formal scheme over $\mathbb{Z}$ and we thus get the category $X_{fs,R} := X_{fs,\mathbb{Z}}/\mathrm{Spf}(R)$ of formal schemes over R. For varying R, these categories assemble into an *fpqc*-stack X_{fs} over $\mathrm{Spec}(\mathbb{Z})$ which we call the stack of formal schemes [36], Remark 2.58, Proposition 4.51 and Remark 4.52.

Define X_{fgr} to be the category of commutative group objects in X_{fs}. Then X_{fgr} is canonically fibred over $\mathrm{Aff}_{\mathbb{Z}}$ and is in fact an *fpqc*-stack over $\mathrm{Spec}(\mathbb{Z})$ because being a commutative group object can be expressed by the existence of suitable structure morphisms making appropriate diagrams commute. Finally, define $X \subseteq X_{fgr}$ to be the substack of those objects which are *fpqc*-locally isomorphic to $(\hat{\mathbb{A}}^1, 0)$ as *pointed formal schemes* (of course, a formal group is considered as a pointed formal schemes via its zero section). It is clear that $X \subseteq X_{fgr}$ is in fact a substack and in particular is itself an *fpqc*-stack over $\mathrm{Spec}(\mathbb{Z})$ which we will call the stack of formal groups. We will see in a minute that X (unlike X_{fgr}) is in fact an algebraic stack.

Our first task will be to determine what formal schemes occur in the fibre category X_R for a given ring R. This requires some notation:

For a locally free R-module V of rank one we denote by $\hat{SV}$ the symmetric algebra of V over R completed with respect to its augmentation ideal. This $\hat{SV}$ is a formal ring. The diagonal morphism $V \to V \oplus V$ induces a structure of formal group on $\mathrm{Spf}(\hat{SV})$. Indeed, for any faithfully flat extension $R \to R'$ with $V \otimes_R R' \simeq R'$ we have $\mathrm{Spf}(\hat{SV}) \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R') \simeq \hat{\mathbb{G}}_{a,R'}$ in $X_{R'}$. On the other hand, denote by $\Sigma(R)$ the set of isomorphism classes of pointed formal schemes in X_R. We have a map $\rho_R : \mathrm{Pic}(R) \to \Sigma(R)$, $[V] \mapsto [\mathrm{Spf}(\hat{SV})]$.

Proposition 32. *For every ring R, the map $\rho_R : \mathrm{Pic}(R) \to \Sigma(R)$ is bijective.*

Proof. By definition, $\Sigma(R)$ is the set of *fpqc*-forms of the pointed formal scheme $(\hat{\mathbb{A}}^1, 0)$ over R. We thus have a Čech-cohomological description

$$\Sigma(R) \simeq \check{H}^1\big(R, \underline{\mathrm{Aut}}(\hat{\mathbb{A}}^1, 0)\big) = \mathrm{colim}_{R \to R'} \check{H}^1\big(R'/R, \underline{\mathrm{Aut}}(\hat{\mathbb{A}}^1, 0)\big),$$

where $G^0 := \underline{\mathrm{Aut}}(\hat{\mathbb{A}}^1, 0)$ is the sheaf of automorphisms of the pointed formal scheme $(\hat{\mathbb{A}}^1, 0)$ over R and the limit is taken over all faithfully flat extensions $R \to R'$. For an arbitrary R-algebra R' we can identify

$$G^0(R') = \big\{ f \in R'[[t]] \bigm| f(0) = 0,\ f'(0) \in R^* \big\}$$

with the multiplication of the right-hand side being substitution of power series. We have a split epimorphism $\pi : G^0 \to \mathbb{G}_m$ given on points by $\pi(f) := f'(0)$ with kernel $G^1 := \ker(\pi)$ and we define more generally for every $n \geqslant 1$, $G^n(R') := \{f \in G^0(R') \mid f = 1 + O(t^n)\}$. For every $n \geqslant 1$ we have an epimorphism $G^n \to \mathbb{G}_a$, $f = 1 + \alpha t^n + O(t^{n+1}) \mapsto \alpha$, with kernel G^{n+1}. One checks that the G^n are a descending chain of normal subgroups in G^0 defining for every R-algebra R' a structure of complete Hausdorff topological group on $G^0(R')$.

Using $\check{H}^1(R'/R, \mathbb{G}_a) = 0$ and an approximation argument shows that

$$\check{H}^1\big(R'/R, G^1\big) = 0$$

for every R-algebra R', hence the map $\phi : \check{H}^1(R, G^0) \to \check{H}^1(R, \mathbb{G}_m)$ induced by π is injective, and as π is split we see that ϕ is a bijection. As $\check{H}^1(R, \mathbb{G}_m) \simeq \mathrm{Pic}(R)$ we have obtained a bijection $\Sigma(R) \simeq \mathrm{Pic}(R)$ and unwinding the definitions shows that it coincides with ρ_R. □

The stack X carries a canonical line bundle:

> For every ring R and $G \in X_R$ we can construct the locally free rank one R-module $\omega_{G/R}$ as usual [36, Definition 7.1] and as its formation is compatible with base change it defines a line bundle ω on X. We remark without proof that $\mathrm{Pic}(X) \simeq \mathbb{Z}$, generated by the class of ω.

We define a $\mathbb{G}_m$-torsor $\pi : \mathfrak{X} := \underline{\mathrm{Spec}}(\bigoplus_{\nu \in \mathbb{Z}} \omega^{\otimes \nu}) \to X$, compare [24, 14.2] and now check that $\mathfrak{X}$ is the algebraic stack associated with the flat Hopf algebroid $(\mathrm{MU}_*, \mathrm{MU}_*\mathrm{MU})$.

For every ring R, the category $\mathfrak{X}_R$ is the groupoid of pairs $(G/R, \omega_{G/R} \xrightarrow{\simeq} R)$ consisting of a formal group G/R together with a trivialization of the R-module $\omega_{G/R}$. The morphisms in $\mathfrak{X}_R$ are the isomorphisms of formal groups which respect the trivializations in an obvious sense. Since $\omega_{\mathrm{Spf}(\hat{SV})/R} \simeq V$ we see from Proposition 32 that every $G \in \mathfrak{X}_R$ is isomorphic to $(\hat{\mathbb{A}}^1, 0)$ as a pointed formal scheme over R. This easily implies that the diagonal of $\mathfrak{X}$ is representable and affine. Now recall the affine scheme $\mathrm{FGL} \simeq \mathrm{Spec}(\mathrm{MU}_*)$ [36, Example 2.6] parametrising formal group laws. We define $f : \mathrm{FGL} \to \mathfrak{X}$ by specifying the corresponding object of $\mathfrak{X}_{\mathrm{FGL}}$ as follows: We take $G := \hat{\mathbb{A}}^1_{\mathrm{FGL}} = \mathrm{Spf}(\mathrm{MU}_*[[x]])$ with the group structure induced by a fixed choice of universal formal group law over MU_* together with the trivialization $\omega_{G/\mathrm{MU}_*} = (x)/(x^2) \xrightarrow{\simeq} \mathrm{MU}_*$ determined by $x \mapsto 1$. We then claim that f is faithfully flat and thus $\mathfrak{X}$ is an algebraic stack with presentation f (this will also imply that X is an algebraic stack):

> Given a 1-morphism $\mathrm{Spec}(R) \to \mathfrak{X}$ we can assume that the corresponding object of $\mathfrak{X}_R$ is given as $(\hat{\mathbb{A}}^1_R, (x)/(x^2) \xrightarrow{\simeq} R, x \mapsto u)$ with the group structure on $(\hat{\mathbb{A}}^1_R, 0)$ defined by some formal group law over R and with some unit $u \in R^*$. Then $\mathrm{Spec}(R) \times_{\mathfrak{X}} \mathrm{FGL}$ parametrises isomorphisms of formal group laws with leading term u. This is well known to be representable by a polynomial ring over R, hence it is faithfully flat.

The same argument shows that $\mathrm{FGL} \times_{\mathfrak{X}} \mathrm{FGL} \simeq \mathrm{FGL} \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{SI} \simeq \mathrm{Spec}(\mathrm{MU}_*\mathrm{MU})$ where SI parametrises strict isomorphisms of formal group laws [33, Appendix A 2.1.4] and this establishes the first half of the following result.

Theorem 33.

(i) *The algebraic stack $\mathfrak{X}$ is associated with the flat Hopf algebroid* $(\mathrm{MU}_*, \mathrm{MU}_*\mathrm{MU})$.
(ii) *For every prime p, $\mathfrak{X} \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathbb{Z}_{(p)})$ is the algebraic stack associated with the flat Hopf algebroid* $(\mathrm{BP}_*, \mathrm{BP}_*\mathrm{BP})$.

Proof. The proof of (ii) is identical to the proof of (i) given above except that to see that the obvious 1-morphism $\mathrm{Spec}(\mathrm{BP}_*) \to \mathfrak{X} \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathbb{Z}_{(p)})$ is faithfully flat one has to use Cartier's theorem saying that every formal group law over a $\mathbb{Z}_{(p)}$-algebra is strictly isomorphic to a p-typical one, see for example [33, Appendix A 2.1.18]. □

Remark 34.

(i) We explain how the grading of MU_* fits into the above result. The stack $\mathfrak{X}$ carries a $\mathbb{G}_m$-action given on points by

$$\alpha \cdot (G/R, \phi : \omega_{G/R} \xrightarrow{\simeq} R) := (G/R, \phi : \omega_{G/R} \xrightarrow{\simeq} R \xrightarrow{\cdot\alpha} R) \quad \text{for } \alpha \in R^*.$$

This action can be lifted to the Hopf algebroid (FGL, FGL $\times$ SI) as in [36, Example 2.97] and thus determines a grading of the flat Hopf algebroid $(\mathrm{MU}_*, \mathrm{MU}_*\mathrm{MU})$. As observed in [36] this is the usual (topological) grading except that all degrees are divided by 2.

(ii) We know from Section 3 and Theorem 33(i) that for every $n \geqslant 0$

$$\mathrm{Ext}^n_{\mathrm{MU}_*\mathrm{MU}}(\mathrm{MU}_*, \mathrm{MU}_*) \simeq \mathrm{H}^n(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}).$$

As $\pi : \mathfrak{X} \to X$ is affine its Leray spectral sequence collapses to an isomorphism $\mathrm{H}^n(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \simeq \mathrm{H}^n(X, \pi_*\mathcal{O}_{\mathfrak{X}}) \simeq \bigoplus_{k\in\mathbb{Z}} \mathrm{H}^n(X, \omega^{\otimes k})$. The comparison of gradings given in (i) implies that this isomorphism restricts, for every $k \in \mathbb{Z}$, to an isomorphism

$$\mathrm{Ext}^{n,2k}_{\mathrm{MU}_*\mathrm{MU}}(\mathrm{MU}_*, \mathrm{MU}_*) \simeq \mathrm{H}^n\big(X, \omega^{\otimes k}\big).$$

In particular, we have $\mathrm{H}^*(X, \omega^{\otimes k}) = 0$ for all $k < 0$.

(iii) As $\pi : \mathfrak{X} \to X$ is *fpqc*, the pull back π^* establishes an equivalence between $\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_X)$ and the category of quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-modules equipped with a descent datum with respect to π, cf. the beginning of Section 3.4. One checks that a descent datum on a given $\mathcal{F} \in \mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{X}})$ with respect to π is the same as a $\mathbb{G}_m$-action on $\mathcal{F}$ compatible with the action on $\mathfrak{X}$ given in (i). Hence π^* gives an equivalence between $\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_X)$ and the category of *evenly graded* $\mathrm{MU}_*\mathrm{MU}$-comodules.

(iv) The referee suggest a different way of looking at (iii): Since $\mathfrak{X} \to X$ is a $\mathbb{G}_m$-torsor it is in particular *fpqc* and hence the composition $\mathrm{Spec}(\mathrm{MU}_*) \to \mathfrak{X} \to X$ is a presentation of X and one checks that the corresponding flat Hopf algebroid is $(\mathrm{MU}_*, \mathrm{MU}_*\mathrm{MU}[u^{\pm 1}])$ thereby justifying our ad hoc definition of X in Section 2. This again shows that $\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_X)$ is equivalent to the category of evenly graded $\mathrm{MU}_*\mathrm{MU}$-comodules, this time the grading being accounted for by the coaction of u.

(v) The analogues of (i)–(iv) above with X (respectively MU) replaced by $X \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathbb{Z}_{(p)})$ (respectively BP) hold true.

The last issue we would like to address is the stratification of X by the height of formal groups. For every prime p we put $Z^1_p := X \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathbb{F}_p) \subseteq X$.

The universal formal group G over Z^1_p comes equipped with a relative Frobenius $F : G \to G^{(p)}$ which can be iterated to $F^{(h)} : G \to G^{(p^h)}$ for all $h \geqslant 1$.

For $h \geqslant 1$ we define $Z^h_p \subseteq Z^1_p$ to be the locus over which the p-multiplication of G factors through $F^{(h)}$. Clearly, $Z^h_p \subseteq X$ is a closed substack, hence Z^h_p is the stack of formal groups over $\mathrm{Spec}(\mathbb{F}_p)$ which have height at least h. The stacks labeled $\mathfrak{Z}^n$ ($n \geqslant 1$) in Section 5 are the preimages of Z^n_p under $\pi \times \mathrm{id} : \mathfrak{X} \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathbb{Z}_{(p)}) \to X \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathbb{Z}_{(p)})$.

For every $n \geqslant 1$ we define the (non-closed) substack $Z^n := \bigcup_{p\,\mathrm{prime}} Z^n_p \subseteq X$ with complement $U^n := X - Z^n$.

If $\mathrm{MU}_* \to B$ is a Landweber exact MU_*-algebra which has height $n \geqslant 1$ at every prime as in [15, Section 7] then the stack theoretic image of $\mathrm{Spec}(B) \to \mathrm{Spec}(\mathrm{MU}_*) \to \mathfrak{X}$ is the preimage of U^n under $\pi : \mathfrak{X} \to X$ which we will write as $\mathfrak{U}^n := \pi^{-1}(U^n) \subseteq \mathfrak{X}$. This can be checked as in Section 5 and shows that the equivalences of comodule categories proved in [15] are again a consequence of the fact that the relevant algebraic stacks are 1-isomorphic. We leave the details to the reader. To conclude we would like to point out the following curiosity:

As complex K-theory is Landweber exact of height 1 over MU_* we know that the flat Hopf algebroid $(\mathrm{K}_*, \mathrm{K}_*\mathrm{K})$ has $\mathfrak{U}^1$ as its associated algebraic stack. So J. Adams' computation of $\mathrm{Ext}^1_{\mathrm{K}_*\mathrm{K}}(\mathrm{K}_*, \mathrm{K}_*)$ implies that for every integer $k \geqslant 2$ we have

$$\left|\mathrm{H}^1\left(U^1, \omega^{\otimes k}\right)\right| = 2 \cdot \text{denominator}\left(\zeta(1-k)\right),$$

where ζ is the Riemann zeta function and we declare the denominator of 0 to be 1. To check this one uses Remark 34(ii) with X replaced by U^1, [39, Proposition 19.22] and [29, VII, Theorem 1.8].

Unfortunately, the orders of the (known) groups $\mathrm{H}^2(U^1, \omega^{\otimes k})$ have nothing to do with the nominators of Bernoulli-numbers.

Acknowledgments

I would like to thank E. Pribble for making a copy of [31] available to me, J. Heinloth, J. Hornbostel, M. Hovey, G. Kings, S. Rairat and N. Strickland for useful conversation, one referee for suggesting substantial improvements of the exposition and the other for his help in putting the present results into proper perspective.

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