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Forms of K-Theory

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The results of this paper were suggested by three integer-valued cobordism invariants (for complex-oriented manifolds) with striking number-theoretic properties. It seems best to begin with these examples, before describing the general theory in which they are embedded.

Example A. The Ramanujan numbers are integers τ_n defined by the generating function

$$\sum_{n\geq 1} \tau_n q^n = q \prod_{n\geq 1} (1-q^n)^{24};$$

the first few are given by $\tau_1=1$, $\tau_2=-2^3\cdot 3$, $\tau_3=2^2\cdot 3^2\cdot 7$, $\tau_5=2\cdot 3\cdot 5\cdot 7\cdot 23$, $\tau_7=-2^3\cdot 7\cdot 13\cdot 23$, $\tau_{11}=2^2\cdot 3\cdot 13\cdot 34\cdot 27$.

Proposition A. *There is a cobordism invariant $\tau\colon \mathscr{U}^*\to\mathbb{Z}$ (i.e. a homomorphism of ungraded groups, where $\mathscr{U}^*$ denotes the complex cobordism ring) such that*

1) $\tau(\mathbb{CP}(n-1))=\tau_n$,
2) $\tau(M\times N)=\tau(M)\cdot\tau(N)$.

Here $\mathbb{CP}(k)$ denotes $2k$-dimensional complex projective space.

Note that the projective spaces do *not* generate the ring $\mathscr{U}^*(pt)$ over $\mathbb{Z}$, although they do over $\mathbb{Q}$. Hence the existence of such a ring homomorphism as τ is nontrivial.

In fact we can do better than this. If h^* is a cohomology theory with a multiplicative structure, in which complex vector bundles are orientable, then a standard construction in topology defines a ring homomorphism t-ind h: $\mathscr{U}^*=\mathscr{U}^*(pt)\to h^*(pt)$, which we call the topological index of h^*; we are motivated by the example of $h^*=$complex K-theory; the associated topological index is just the Todd genus.

Theorem A. *There is a cohomology theory h^*_τ with the properties described above, taking values in the category of $\mathbb{Z}_{(11)}$-modules, whose associated topological index*

$$t\text{-ind}\, h\colon\ \mathscr{U}^*\to\mathbb{Z}_{(11)}$$

satisfies t-ind$(M)=\tau(M)$.

Example B. Let D be a square-free integer; for simplicity let us take D odd. We recall the definition of the quadratic symbol [Lang IV §2; Serre VI §1.3]: if p is a prime dividing D, we set $\left(\frac{D}{p}\right)=0$; otherwise, we set $\left(\frac{D}{p}\right)=\pm 1$ according to whether the congruence $X^2 \equiv D$ modulo p has (or has not) a solution in the integers. For arbitrary n, we write $\left(\frac{D}{n}\right)=\prod_{p|n}\left(\frac{D}{p}\right)$.

Proposition B. *There is an integer-valued genus of complex-oriented manifolds* $\rho_D: \mathscr{U}^* \to \mathbb{Z}$ *such that* $\rho_D(\mathbb{C}\mathbb{P}(n-1))=\left(\frac{D}{p}\right)$.

Theorem B. *There exists a cohomology theory* h_D^* *as above taking values in the category of* $\mathbb{Z}[D^{-1}]$*-modules, whose index is the genus* ρ_D *just described.*

In this case we can do even better:

Theorem B Analytic. *There exists a universal elliptic differential operator for almost-complex manifolds with* $\mathbb{Z}/D\mathbb{Z}$*-action whose index (on manifolds with trivial* $\mathbb{Z}/D\mathbb{Z}$*-action) is the genus* ρ_D *(up to a factor dependent only on* $\dim M$*);* cf. 6.2.6.

Example C. Let Δ be a *fourth*-power-free integer. We recall the definition of the *bi*quadratic symbol [Hasse p. 161]: Let $v=v_0+iv_1$ be an *odd* Gaussian integer; we interpret this to mean that v_0 is odd, and v_1 is even, or thus that $v \equiv 1 \bmod 2+2i$. If μ is another Gaussian integer, we set $\left(\frac{\mu}{v}\right)_4=0$ if μ and v are not coprime; but if μ is prime to v, we set $\left(\frac{\mu}{v}\right)_4=i^\alpha$, where α is chosen so that $i^\alpha \equiv \mu^{\frac{|v|^2-1}{4}}$ modulo v. Evidently

$$\left(\frac{\mu}{v_1 v_2}\right)_4=\left(\frac{\mu}{v_1}\right)_4\left(\frac{\mu}{v_2}\right)_4.$$

Proposition C. *There is an integer-valued genus* ρ_Δ *of complex-oriented manifolds such that*

$$\rho_\Delta(\mathbb{C}\mathbb{P}(n-1))=\sum_{\sigma\,\mathrm{odd},\,|\sigma|^2=n}\left(\frac{\Delta}{\sigma}\right)_4\bar{\sigma}.$$

Theorem C. *Let* A_Δ *denote the localization of the integers obtained by inverting* 2Δ*, as well as all primes congruent to* 3 *modulo* 4*. Then there is a cohomology theory* h_Δ^* *as above, taking values in the category of* A_Δ*-modules, whose topological index satisfies* t-ind $h_\Delta(M)=\rho_\Delta(M)$.

The cohomology functors of Theorems A, B, C can be easily defined: if $\rho: \mathscr{U}^* \to A_\rho$ is any one of the genera defined above, then a theorem of Landweber implies that $\mathscr{U}^*(X)\otimes A_\rho$ is a cohomology theory.

Underlying these results is a basic fact. $\mathbb{BU}$ is not very rigid as a ring-spectrum. In fact, the complex K-functor can be *continuously* deformed as a multiplicative cohomology theory, as can be seen from Theorems 1 and 2 below; the parameter space can be the p-adic units or even the punctured unit disk in the complex plane.

The main objective of this paper is to classify those cohomology theories which arise as deformations of complex K-theory. We call such theories *ordinary*; Theorem 1 attaches to each ordinary theory h^* over $\hat{\mathbb{Z}}_p$, an invariant $\mathrm{inv}(h) \in \mathbb{Z}_p^*$ which characterizes the theory up to isomorphism. There is a global version of this theorem over $\mathbb{Z}$ but in some ways Theorem 2 is more interesting.

Notation. $\hat{\mathbb{Z}}_p$ is the ring of p-adic integers, $\mathbb{Z}_{(p)}$ the ring of rational numbers with denominators prime to p. We write W for the ring obtained by adjoining to $\hat{\mathbb{Z}}_p$, all primitive n^{th} roots of unity, where $p \nmid n$.

Theorem 1. *There is a family $K_\alpha^*(-)$ of multiplicative cohomology theories, taking values in the category of $\hat{\mathbb{Z}}_p$-modules, with the following properties*

1) $K_\alpha^q(S^n) = \hat{\mathbb{Z}}_p$ *when* $q \equiv n \bmod 2$
$= 0$ *otherwise; the K_α^* form a periodic cohomology theory.*

2) *When $\alpha = 1 + p$, K_α^* is canonically equivalent to complex K-theory, completed at p.*

3) *If α, $\beta \in \hat{\mathbb{Z}}_p^*$ are distinct, then K_α, K_β are not isomorphic (as multiplicative theories) but*

4) *$K_\alpha(-) \bigotimes_{\mathbb{Z}_p} W$ and $K_\beta(-) \bigotimes_{\mathbb{Z}_p} W$ are isomorphic as multiplicative theories (though not canonically).*

Finally, if h^* is any *ordinary* cohomology theory satisfying 1), there is a unique, computable $\mathrm{inv}(h) \in \hat{\mathbb{Z}}_p^*$ such that h^* is isomorphic to K_α^*, $\alpha = \mathrm{inv}(h)$. For example, the invariant of the theory over the 11-adic integers defined by Example A is $B^{-1} + 11 \cdot B$, where B is the unique solution of

$$B^2 - \tau(11)\, B + 11^{11} = 0$$

which is an 11-adic unit; [cf. S19, pp. 498–512].

The functors K_α^* are obtained as above; we define a genus $\rho_\alpha \colon \mathcal{U}^* \to \hat{\mathbb{Z}}_p$, whose zeta-function (see §1) is $\sum \rho_\alpha(\mathbb{CP}(n-1))\, n^{-s} = (1 - \alpha p^{-s} + p^{1-2s})^{-1}$. Then $K_\alpha^*(X) \equiv \mathcal{U}^*(X) \bigotimes_{\rho_\alpha} \hat{\mathbb{Z}}_p$.

Note that if $\alpha \in \mathbb{Z}_{(p)}^\times$, i.e. $\alpha = a/b$ with a, b prime to p, then $K_\alpha^*(X)$ actually takes values in the category of $\mathbb{Z}_{(p)}$-modules. To state Theorem 2, let $q \in \mathbb{C}$ be a complex number in the punctured unit disc, $0 < |q| < 1$; let

$$E_4(q) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)\, q^n, \qquad E_6(q) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)\, q^n$$

be the normalized Eisenstein series, where $\sigma_k(n) = \sum_{d \mid n, d \geq 1} d^k$.

Theorem 2. *There is a multiplicative genus $\rho_q \colon \mathcal{U}^* \to \mathbb{Z}[E_4(q), E_6(q)]$ of almost-complex manifolds, and a corresponding cohomology theory $K_q^*(-)$ over a certain* *localization of* $\mathbb{Z}[E_4, E_6]$ *(depending on* q*). Two values* q, q' *of the parameter give inequivalent functors as long as the modular invariants* $j(q)$, $j(q')$ *are different; where* $j(q)=q^{-1}+744+196,844\,q+\ldots$.

The formal group law of $K_q^*(-)$ is the formal completion of the group law on the elliptic curve $\mathbb{C}^*/\{q^n | n\in\mathbb{Z}\}$; on adjoining q to the coefficient ring, all these theories become isomorphic to complex K-theory.

The index ρ_q can be interpreted as the index of a certain formal family of differential operators parameterized by q.

For example, C above is essentially a special case of this theorem, with $j(q)=1728$. Similarly, example A is essentially an elliptic cohomology theory for $\Gamma_0(11)$, cf. [S 19, p. 504].

The paper is in six sections, with an appendix [and a supplementary bibliography added recently.]

Section 1 is concerned with topological and algebraic preliminaries; Sects. 2 and 3 contain the real results of this paper (including the main theorem of § 3, which equates the category of ordinary K-theories to a certain category of formal groups) while Sects. 4 and 5 summarize results from the algebraic literature which provide the concrete examples. Section 6 sketches a formalism of universal elliptic operators, which is the K-theoretic analogue of Hirzebruch's multiplicative sequences, and the appendix is appended to explain the title.

The results of this paper give some indication of the incredible arithmetic richness of the cobordism ring. I want to draw attention to our complete ignorance of any explanation for this richness. There seems to be a level of structure on differentiable manifolds which is as deep as that revealed for algebraic varieties by the Weil conjectures; but the nature of this structure is a mystery.

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§ 1. Genera and Formal Group Laws

Let C denote a field of characteristic 0; it will usually be $\mathbb{C}$. Let $A\subset C$ be a subring, and let B be the quotient field of A.

1.1. **Definition.** An A-valued genus ρ is a ring-homomorphism $\rho: \mathscr{U}^* \to A$. In general, we call a ring-homomorphism $\mathscr{U}^* \to R$ a genus provided R is torsion-free. Since $\mathscr{U}^* \otimes_{\mathbb{Z}} \mathbb{Q}$ is generated by the projective spaces, ρ is determined by the elements $\rho_n = \rho(\mathbb{CP}(n)) \in A$. Consequently ρ is completely specified by either of the generating functions

$$\ell_\rho(X) = \sum_{n \geqq 1} \rho_{n-1} \frac{X^n}{n} \in B[\![X]\!] \qquad \text{(the } \textit{logarithm} \text{ of } \rho\text{)},$$

$$\zeta_\rho(s) = \sum_{n \geqq 1} \rho_{n-1}\, n^{-s} \qquad \text{(the } \textit{zeta-function} \text{ of } \rho\text{)}.$$

The latter formula defines a formal Dirichlet series with coefficients in A. We will call ζ_ρ the Mellin transform of ℓ_ρ, and vice versa.

1.2. *Examples.* Let $A = \mathbb{Z}$; $t: \mathscr{U}^* \to \mathbb{Z}$ is the Todd genus. Then $t(\mathbb{CP}(n)) = 1$, so

$$\ell_t(X) = \sum_{n \geqq 1} \frac{X^n}{n} = -\log(1-X),$$

and

$$\zeta_t(X) = \sum_{n \geqq 1} n^{-s} = \prod_{p \in \text{primes}} (1-p^{-s})^{-1} = \zeta(s)$$

is the classical Riemann ζ-function.

More generally, if $y \in \mathbb{C}$, we can define Hirzebruch's χ_y-*genus* for a Kähler manifold M in terms of the Hodge cohomology:

$$\chi_y(M) = \sum_{p,q \geqq 0} (-1)^p y^q \dim H^{p,q}(M).$$

The Atiyah-Singer index theorem shows this number to be a cobordism invariant; thus $\chi_y: \mathscr{U}^* \to A$ is defined, when A is the ring generated over $\mathbb{Z}$ by y. This genus is defined by the logarithm

$$\ell_y(X) = (1+y)^{-1} \log\left(\frac{1+yX}{1-X}\right).$$

1.2.1. *Example.* When $y = +1$, we recover Hirzebruch's L-genus.

The general significance of ℓ_ρ comes from the following:

1.3. **Theorem** (Miščenko). *The series $F_\rho(X, Y) = \ell_\rho^{-1}(\ell_\rho(X) + \ell_\rho(Y)) \in B[\![X, Y]\!]$ actually lies in $A[\![X, Y]\!]$.*

1.3.1. *Example.* Applying this to ℓ_y, we find

$$F_y(X, Y) = \frac{X + Y + (y-1)XY}{1 + yXY} \in \mathbb{Z}[y][\![X, Y]\!].$$

When $y=0$, we get $X+Y-XY$; when $y=1$, $\frac{X+Y}{1-XY}$. It follows that an A-genus ρ determines a *formal group law* F_ρ over A. This is a formal series $F(X, Y)$ such that

1) $F(X, Y)=F(Y, X)$,
2) $F(X, Y)=X+Y+$higher order terms,
3) $F(X, F(Y, Z))=F(F(X, Y), Z)$.

The converse is also clear, for given $F(X, Y)$ over A, we define $F_2(X, 0)$ to be the value of $\frac{\partial F}{\partial Y}$ at $Y=0$; then $d\ell_\rho(X)=\frac{dX}{F_2(X, 0)}$ allows us to recover ℓ_ρ. Thus F_ρ, ℓ_ρ, ζ_ρ, and ρ determine one another over rings such as A. In this language we can restate Quillen's fundamental theorem:

1.4. **Theorem** (Quillen). *There is a canonical* 1–1 *correspondence between formal groups over any ring R, any ring homomorphisms $\mathscr{U}^* \to R$.*

Thus a logarithm (or its Mellin transform, the ζ-function) is a simple way of defining a formal group law over a ring without torsion.

Note that a formal group law on a ring R is nothing but a Hopf R-algebra structure on $R[\![T]\!]$: define a diagonal Δ_F by $\Delta_F(T)=F(T\otimes 1, 1\otimes T)$. It is convenient to refer to a Hopf R-algebra which is isomorphic to $R[\![T]\!]$ for *some* T as a formal group; when an explicit generator T has been chosen, we speak of a formal group *law* over R.

The function ζ_ρ is very useful for constructing formal group laws, as can be seen from the following

1.5. **Theorem** (Honda). *Suppose ρ: $\mathscr{U}^* \to \mathbb{Q}$ is a genus whose ζ-function has an Euler product expansion*

$$(*) \qquad \zeta_\rho(s)=\prod_{p\in\text{primes}} (1-b_{1,p}p^{-s}-\ldots-b_{n,p}p^{n(1-s)-1}-\ldots)^{-1},$$

in which $b_{ij}\in\mathbb{Z}$. Then the formal group law defined by the Mellin transform $\ell_\rho(X)$ is actually a formal group over $\mathbb{Z}$; that is, $F_\rho(X, Y)\in\mathbb{Z}[\![X, Y]\!]$; and any formal group law over $\mathbb{Z}$ is isomorphic to some such law [14, p. 239].

Remark. There is a local variant of this result: If ρ: $\mathscr{U}^* \to \mathbb{Q}_p$ (where $\mathbb{Q}_p$ is the p-adic field) is a genus with Euler expansion as above, such that $b_{i,j}\in\hat{\mathbb{Z}}_p$ (or $\mathbb{Z}_{(p)}$) for all primes p, then $F_\rho(X, Y)\in\hat{\mathbb{Z}}_p[\![X, Y]\!]$ (or $\mathbb{Z}_{(p)}[\![X, Y]\!]$).

However, the result holds only (so far as I know) over the prime fields; the analogue for arbitrary fields of char. 0 may be false.

1.5.1. **Corollary.** *Any Euler product of the form* $(*)$, *with $b_{i,j}\in\mathbb{Z}$, determines a formal group law and an integral genus of almost complex manifolds.*

Let us return to the examples of the Introduction.

Example A. The function $\Delta(q)=q\cdot\prod_{n\geq 1}(1-q^n)^{24}$ is a well-known automorphic form; it is a so-called *cusp form* of weight 12. If we write $\Phi(s)=\sum_{n\geq 1}\tau_n n^{-s}$ for its Mellin transform, then the theory of Hecke operators [Serre VII, 5.4–5] shows that $\Phi(s)$ has an Euler product expansion

$$\Phi(s)=\prod_{p\in\text{primes}}(1-\tau_p p^{-s}+p^{11-2s})^{-1},$$

which is of the form $(*)$.

Example B. In this example, the ζ-function of ρ_D is

$$\zeta_D(s)=\sum_{n\geqq 1}\left(\frac{D}{n}\right)n^{-s}=\prod_{p\in\text{primes}}\left(1-\left(\frac{D}{p}\right)p^{-s}\right)^{-1}$$

which is of the form $(*)$. Note that $\zeta_D(s)\cdot\zeta(s)$ is the Dirichlet ζ-function of the number-field $\mathbb{Q}(\sqrt{D})$.

Example C. The ζ-function of this genus is

$$\zeta_\Delta(s)=\sum_{\sigma\,\text{odd}}\left(\frac{\Delta}{\sigma}\right)_4\bar{\sigma}\cdot|\sigma|^{-2s}=\prod_{\pi\in\text{odd Gaussian primes}}\left(1-\left(\frac{\Delta}{\pi}\right)_4\bar{\pi}\cdot|\pi|^{-2s}\right)^{-1}.$$

It can be shown (cf. Birch and Swinnerton-Dyer) that this ζ-function can be rewritten as

$$\zeta_\Delta(s)=\prod_{p\nmid 2\Delta}(1+(N_p-p)p^{-s}+p^{1-2s})^{-1}$$

where N_p is the number of integral solutions of the congruence $y^2=x^3-\Delta x$ modulo p. In fact $\dfrac{\zeta(s)\,\zeta(1-s)}{\zeta_\Delta(s)}$ is the Weil ζ-function of the curve $y^2=x^3-\Delta x$.

Remark. The combined theorems of Honda and Quillen provide an enormous supply of multiplicative genera with interesting arithmetic properties. In the next section we will show that among these genera, there are many with interesting *topological* properties. (See also [S 10, S 15].)

§ 2. Ordinary K-Theories

2.1. *Conventions.* Cohomology theories treated in this paper are assumed to be multiplicative, i.e. to possess a natural product map $h^*(X)\otimes_{\mathbb{Z}}h^*(Y)\to h^*(X\times Y)$, such that $h^*(pt)$ is a ring with unit. We will usually refer to $h^0(pt)$ as the ground ring of the theory.

The following result is fundamental and very useful:

2.2. **Adam's Lemma.** *For a multiplicative cohomology theory h^*, the following are equivalent:*

1) *The inclusion $S^2\hookrightarrow\mathbb{CP}(\infty)$ induces a surjection $h^*(\mathbb{CP}(\infty))\to h^*(S^2)$.*
2) *$h^*(\mathbb{CP}(\infty))$ is a formal group over $h^*(pt)$.*

3) *For any complex vector bundle E, there exists a Thom class* $u_E \in h^*(T(E))$.

4) *There exists a multiplicative natural transformation* $\mathscr{U}^*(-) \to h^*(-)$ *which preserves Thom classes.*

An element $T \in h^*(\mathbb{CP}(\infty))$ whose image generates $h^*(S^2)$ as an $h^*(pt)$-module is called an *orientation* for the theory h; it canonically determines a formal group *law* on $h^*(\mathbb{CP}(\infty))$, a canonical Thom class on each complex vector bundle, and a particular natural transformation $t\text{-ind}\, h$: $\mathscr{U}^*(-) \to h^*(-)$. Applying this to $X = pt$ yields a genus $t\text{-ind}\, h$: $\mathscr{U}^*(pt) \to h^*(pt)$. [If M is a complex-oriented manifold of dimension n, then there is a canonical "fundamental class" for the theory h^*, i.e. a map $h^*(M) \to h^{*+n}(pt)$; applying this map to $1 \in h^0(M)$, we obtain $t\text{-ind}\, h(M) \in h^n(pt)$. We call $t\text{-ind}\, h$ the *genus* of the (oriented) theory h.

2.3. **Definition.** Let ρ: $\mathscr{U}^* \to A$ be a genus (where A is a ring without torsion) and let p be a prime number. We say that p is an *ordinary* prime for ρ if either

i) p is invertible in A, i.e. p is trivial, or

ii) $\rho(\mathbb{CP}(p-1))$ becomes a unit in A/pA.

We say that ρ is ordinary if all primes are ordinary for ρ. Since any oriented cohomology theory determines a genus, we will call an oriented cohomology theory an ordinary K-theory if its genus is ordinary; we will see below that this property is independent of the choice of an orientation.

2.3.1. *Example.* Complex K-theory, with its canonical orientation, determines the Todd genus t, and $t(\mathbb{CP}(n)) = 1$ for all n, so classical K-theory is an ordinary K-theory. Note that $H^*(-;\mathbb{Q})$ is an ordinary K-theory for trivial reasons.

2.4. **Lemma.** 1) *If* ρ: $\mathscr{U}^* \to A$ *is an ordinary genus, and* ϕ: $A \to B$ *is any ring homomorphism, then* $\phi\rho$ *is ordinary.*

2) *If* ρ: $\mathscr{U}^* \to A$ *is ordinary at p, and A is complete in the p-adic topology, then* $\rho(\mathbb{CP}(p-1))$ *is a unit in A.*

Proof. The first assertion is trivial, and the second is a corollary of Hensel's lemma.

2.5. **Definition.** Let ρ: $\mathscr{U}^*(pt) \to A$ be an ordinary genus. We define the associated graded genus ρ_*: $\mathscr{U}^*(pt) \to A[t, t^{-1}]$, where t is an indeterminate of dimension 2, by $\rho_*(M) = \rho(M)\, t^{-\dim_{\mathbb{C}} M}$ on homogeneous elements. We define the associated K-theory to be $K_\rho^*(X) = \mathscr{U}^*(X) \bigotimes_\rho A^*[t, t^{-1}]$, given the induced total grading.

Evidently $K_\rho^*(-)$ has a multiplicative structure, and has a canonical element satisfying the hypotheses of Adams' Lemma; so K_ρ is an ordinary K-theory if it is a cohomology theory at all. But this is a corollary of

2.6. **Landweber's Exactness Theorem.** *Given a prime p, let* v_{p-1}, $v_{p^2-1}, \dots, v_{p^n-1} \in \mathscr{U}^*(pt)$ *denote Milnor generators for* $\mathscr{U}^*(pt) \otimes \mathbb{Z}_{(p)}$ *of dimension* $2(p-1)$, $2(p^2-1), \dots, 2(p^n-1)$. *If* A^* *is a graded* $\mathscr{U}^*(pt)$*-algebra, then* $\mathscr{U}^*(X) \bigotimes_{\mathscr{U}^*(pt)} A^*$ *is a cohomology theory if the following conditions are satisfied: For any prime p,*

0) *p-multiplication:* $A^* \to A^*$ *is injective.*

1) v_{p-1}*-multiplication:* $A^*/pA^* \to A^*/pA^*$ *is injective.*

2) v_{p^2-1}*-multiplication:* $A^*/(p, v_{p-1})\, A_* \to A^*/(p, v_{p-1})\, A_*$ *is injective, etc.*

2.6.1. **Corollary.** *If A^* is a $\mathscr{U}^*(pt)$-algebra without torsion such that either*

1) *p is a unit in A^*, or*

2) *v_{p-1} is a unit in A^*/pA^*, then the conditions hold. Since v_{p-1} can be taken to be $\mathbb{CP}(p-1)$, K_ρ is a cohomology theory.*

Remark. Landweber's theorem follows very elegantly from decomposition theory [every cobordism module M admits a filtration $M_0 \supset M_1 \supset \ldots$ such that $M_i/M_{i+1} = \mathscr{U}^*(pt)/\mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal, invariant under the Landweber-Novikov operators, and a classification theorem for such prime ideals: all are of the form $(p, v_{p-1}, v_{p^2-1}, \ldots, v_{p^n-1})$ for some prime p and integer n.

We will sketch a quite different proof of the corollary above in § 4.

2.7. *Examples.* We can now prove Theorems A, B, C of the Introduction. All we need to do is compute the ordinary primes of τ, ρ_D, ρ_Δ; we can make each genus ordinary by trivializing (i.e. inverting) the non-ordinary primes in $\mathbb{Z}$. Now these three genera come from ζ-functions with Euler products; for such genera, ordinariness is easy to check.

2.7.1. **Lemma.** *If ρ: $\mathscr{U}^*(pt) \to \mathbb{Z}$ is defined by $\zeta_\rho(s) = \prod_{p \in \text{primes}} (1 - b_{1,p} p^{-s} - \ldots)^{-1}$ then ρ is ordinary at p iff p does not divide $b_{1,p}$.*

Proof. By direct computation: $\rho(\mathbb{CP}(p-1)) = b_{1,p}$.

Remark. If $\gamma_\rho(s) = (1 - b_{1,p} p^{-s} - \ldots)^{-1}$ is nontrivial only at the prime p, then explicit formulae of Hazewinkel can be used to compute the value of ρ on Milnor generators; thus $\rho(v_{p-1}) = b_{1,p}$,

$$\rho(v_{p^2-1}) = b_{2,p} + b_{1,p}^2 \cdot \frac{1 - b_{1,p}^{p-1}}{p}.$$

2.7.2. **Corollary.** A) *The Ramanujan genus is ordinary at $p = 11$, for $\tau_{11} = 2^3 \cdot 3 \cdot 13 \cdot 3427 \equiv 1$ modulo 11.*[1]

B) *ρ_D is ordinary except when $p | D$.*

C) *ρ_Δ is ordinary except when $p | 2\Delta$, or when the number of solutions to $y^2 \equiv x^3 - \Delta x$ modulo p is divisible by p. This happens only if $p \equiv 3 \bmod 4$; in that case there are no solutions.*

Example D. If y is a p-adic integer, then the Hirzebruch χ_y-genus is ordinary at p; but if $-y$ is (sufficiently close, in the p-adic metric, to) a primitive p^{th} root of unity, then χ_y is *not* ordinary at p.

In fact let $\phi_\alpha(X) = 1 - \left(1 - \alpha^{-1} \dfrac{X}{1-X}\right)^\alpha$, where $(1+Z)^\alpha$ is the formal power series in $\mathbb{Z}$ defined by the binomial theorem (when α is a p-adic integer, this series has p-adic integer coefficients.) It is easy to see (using the logarithms) that when $\alpha = -(1+y)^{-1}$, $\phi_\alpha G_m(X, Y) = F_y(\phi_\alpha(X), \phi_\alpha(Y))$; i.e., if y is a p-adic integer, F_y is isomorphic to the multiplicative group. Since being ordinary at

[1] In fact the only known primes for which τ is *not* ordinary are 2, 3, 5, 7, and 2411, cf. [S19, p. 723]

a prime is an isomorphism invariant of formal groups, and since the multiplicative group is ordinary at every prime, we are through.

The second statement follows from the observation that $\chi_{-y}(\mathbb{CP}(n-1)) = \Phi_{n-1}(y)$ is the cyclotomic polynomial $y^{n-1}+\ldots+y+1$. For example, the prime 2 is not ordinary for the L-genus: a well-known fact in surgery theory [cf. also 56].

§ 3. The Classification Theorem

Let h^* be an ordinary K-theory, and let B^* be a ($\mathbb{Z}$-torsion-free) $h^*(pt)$-algebra. Then by Lemma 2.4 and Landweber's theorem, $h^*(-)\bigotimes_{h^*(pt)} B^*$ is also an ordinary K-theory.

3.1. **Definition.** A *normalization* of h^* is a graded isomorphism $h^*(pt) \cong h^0(pt)[t, t^{-1}]$, with t of grade 2. Of course h^* may not have a normalization, but we can always extend the ground ring by some B^* (as above) to get a normalized theory without losing any generality; for example, we can take $B^* = h^*(pt)[t, t^{-1}]$ with the induced grading. Note that a normalized theory is periodic of order 2, while an arbitrary ordinary K-theory need not be periodic at all.

In this category we characterize the category of normalized K-theories.

3.2. **Definition.** $\mathscr{K}$ is the category whose objects are normalized K-theories, over rings without torsion. If $K_0, K_1 \in \mathscr{K}$, a morphism ϕ: $K_0^* \to K_1^*$ in $\mathscr{K}$ is a multiplicative transformation of graded cohomology theories.

3.2.1. *Example.* Let K_0 be classical complex K-theory, and let $K_1 = H^*(-) \otimes \mathbb{Q}$ be rational cohomology, given the even-odd grading. The Chern character ch: $K^* \to H^* \otimes \mathbb{Q}$ is a morphism in $\mathscr{K}$.

3.3. **Definition.** Let $K_0^* \in \mathscr{K}$, with $K_0^*(pt) = A[t, t^{-1}]$. By definition, K_0 is oriented, so there exists a canonical T_0 such that $K_0^*(\mathbb{CP}(\infty) = A[t, t^{-1}][\![T_0]\!]$, with diagonal $\Delta_0(T_0) = F_0(T_0 \otimes 1, 1 \otimes T_0)$, where F_0 is a formal group law with coefficients in $A[t, t^{-1}]$. We write $\bar{F}_0$ for the formal group law on A induced by $t \mapsto 1$; $\bar{F}_0$ is precisely the formal group induced by the genus

$$\mathscr{U}^*(pt) \xrightarrow{t\text{-ind}\, K_0} A[t, t^{-1}] \to A.$$

3.3.1. *Example.* If K_0 is classical K-theory, its formal group law is the multiplicative group $\bar{F}_0(X, Y) = X + Y - XY$; if $K_1 = H^* \otimes \mathbb{Q}$, then $\bar{F}_1(X, Y) = X + Y$ is the additive group.

3.4. **Main Theorem.** *The map $\check{M}$: $\mathscr{K} \to \mathscr{F}\mathscr{G}$ sending K_0 to its associated formal group law extends to give a (contravariant) faithful functor.*

The image of this functor will be described below [3.4.8], giving a complete description of $\mathscr{K}$. The proof breaks into a succession of lemmas.

3.4.1. **Lemma.** *Any morphism ϕ in $\mathscr{K}$ can be canonically factored in the form $K_0^* \xrightarrow{\phi''} K_{0,1}^* \xrightarrow{\phi'} K_1^*$, in which ϕ' induces the identity map from $K_{0,1}^0(pt)$ to $K_1^0(pt)$, and ϕ'' is an extension of scalars.*

Proof. Define $K_{0,1}^*(X) = K_0^*(X) \otimes_{K_0^0(pt)} K_1^0(pt)$, where $K_1^0(pt)$ is a $K_0^0(pt)$-module via ϕ.

3.4.2. **Definition.** A morphism $\phi\colon K_0^* \to K_1^*$ will be called *strict* if $\phi = \phi'$, i.e. if $\phi\colon K_0^0(pt) \to K_1^0(pt)$ is the identity. Similarly, a morphism $\psi\colon (A[\![T_0]\!], \Delta_0) \to (B[\![T_1]\!]\,\Delta_1)$ of Hopf algebras can be factored:

$$(A[\![T_0]\!], \Delta_0) \xrightarrow{\psi''} (B[\![T_0]\!], \Delta_0) \xrightarrow{\psi'} (B[\![T_1]\!], \Delta_1),$$

where ψ' is a morphism of formal groups over B. Then there is a unique formal power series $\tilde{\psi}$ such that $\tilde{\psi}(T_1) = \psi'(T_0)$.

3.4.3. **Lemma.** *Any morphism $\phi\colon K_0^* \to K_1^*$ in $\mathscr{K}$ induces a morphism $\tilde{\phi}\colon \bar{F}_1 \to \bar{F}_0$ in $\mathscr{F}\mathscr{G}$.*

Proof. Since morphisms in $\mathscr{K}$ and $\mathscr{F}\mathscr{G}$ have canonical factorizations as above it suffices to show that a strict morphism $\phi\colon K_0^* \to K_1^*$ (i.e. $\phi\colon K_0^0(pt) \to K_1^0(pt) = A$ is the identity) induces a morphism $\tilde{\phi}\colon \bar{F}_1 \to \bar{F}_0$ of formal group laws over A.

This is essentially trivial: There is a commutative diagram

$$\begin{array}{ccc}
K_0^*(\mathbb{CP}(\infty)) & \xrightarrow{\Delta_0} & K_0^*(\mathbb{CP}(\infty)) \hat{\otimes} K_0^*(\mathbb{CP}(\infty)) \\
\Big\downarrow{\scriptstyle \phi} & & \Big\downarrow{\scriptstyle \phi \otimes \phi} \\
K_1^*(\mathbb{CP}(\infty)) & \xrightarrow{\Delta_1} & K_1^*(\mathbb{CP}(\infty)) \hat{\otimes} K_1^*(\mathbb{CP}(\infty))
\end{array}$$

where $\Delta_0(T_0) = F_0(T_0 \otimes 1, 1 \otimes T_0)$, $\Delta_1(T_1) = F_1(T_1 \otimes 1, 1 \otimes T_1)$. ($T_0$, T_1 are the appropriate orientation classes) and F_0, F_1 are formal group laws on $A[t, t^{-1}]$. Define $\tilde{\phi}'$ as the formal power series with coefficients in $A[t, t^{-1}]$ such that $\tilde{\phi}'(T_1) = \phi(T_0)$; then according to the diagram above,

$$\begin{aligned}(\phi \otimes \phi)\,\Delta_0(T_0) &= (\phi \otimes \phi)\,F_0(T_0 \otimes 1, 1 \otimes T_0) = F_0(\phi(T_0) \otimes 1, 1 \otimes \phi(T_0)) \\ &= F_0(\tilde{\phi}'(T_1) \otimes 1, 1 \otimes \tilde{\phi}'(T_1))\end{aligned}$$

while $\Delta_1\,\phi(T_0) = \Delta_1\,\tilde{\phi}'(T_1) = \tilde{\phi}'\,\Delta_1(T_1) = \tilde{\phi}'\,F_1(T_1 \otimes 1, 1 \otimes T_1)$. In other words, $\tilde{\phi}'\,F_1(X, Y) = F_0(\tilde{\phi}'(X), \tilde{\phi}'(Y))$, so $\tilde{\phi}'\colon F_1 \to F_0$ is a morphism of formal group laws over $A[t, t^{-1}]$. Setting $t = 1$ then gives a morphism $\tilde{\phi}\colon \bar{F}_1 \to \bar{F}_0$ of formal group laws over A.

3.4.4. *Example.* If $K_0 = K^*(-)$, $K_1 = H^*(-;\mathbb{Q})$ as above, then the Chern character factors $K \xrightarrow{ch''} K \otimes \mathbb{Q} \xrightarrow{ch'} H \otimes \mathbb{Q}$, with ch' a strict morphism in $\mathscr{K}$. If T_H,

T_K denote the canonical orientation classes, then $ch(T_K)=1-e^{-T_H}$; so $c\tilde{h}(X)=1-e^{-X}$, which defines a morphism $c\tilde{h}$: $\mathbb{G}_a \to \mathbb{G}_m$ of the additive to the multiplicative group over $\mathbb{Q}$.

3.4.5. **Lemma.** *Let ϕ: $K_0^* \to K_1^*$ be a strict morphism $\mathscr{K}$; let A be the common groundring. There is a unit $\chi(\phi) \in A^\times$ such that $\phi(\sigma_0^n)=\chi(\phi)^n \cdot \sigma_1^n$, where $\sigma_0^n \in K_0^{2a}(S^{2n})$ is the canonical generator (similarly for σ_1).*

Moreover if ϕ, ψ are composable strict morphisms in $\mathscr{K}$, then $\chi(\phi \cdot \psi)=\chi(\phi)\,\chi(\psi)$.

Proof. ϕ induces a map ϕ: $K_0^*(pt)=A[t, t^{-1}] \to A[t, t^{-1}]=K_1^*(pt)$, consequently $\phi(t) \in K_1^2(pt)=At$, $\phi(t^{-1}) \in K_1^{-2}(pt)=At^{-1}$; hence $\phi(t)=\chi(\phi)\,t$, $\phi(t^{-1})=\chi'(\phi)\,t^{-1}$ for some $\chi(\phi)$, $\chi'(\phi) \in A$. But since $\phi(1)=\phi(t \cdot t^{-1})=1$, we find that $\chi(\phi)\,\chi'(\phi)=1$, so $\chi(\phi)$ is a unit. Then we just suspend.

3.4.6. **Lemma.** *If $\phi \in \mathscr{K}$ is a strict morphism, then $\tilde{\phi}(T)=\chi(\phi)\,T+$ higher order terms. In particular, $\tilde{\phi}$ is an invertible morphism of formal group laws.*

Proof. Consider the diagram

$$\begin{array}{ccc} K_0^*(\mathbb{CP}(\infty)) & \xrightarrow{i_0^*} & K_0^*(S^2) \\ \Big\downarrow{\scriptstyle \phi} & & \Big\downarrow{\scriptstyle \phi} \\ K_1^*(\mathbb{CP}(\infty)) & \xrightarrow{i_1^*} & K_1^*(S^2). \end{array}$$

Clearly $\phi\, i^*(T_0)=\phi(\sigma_0)=\chi(\phi)\,\sigma_1$, $i^*\,\phi(T_1)=i^*\,\tilde{\phi}(T_1)=\tilde{\phi}_1\,\sigma_1$, where $\tilde{\phi}(X)=\sum_{i \geq 1} \tilde{\phi}_i X^i$.

Since $\chi(\phi)=\tilde{\phi}_1$ is a unit, $\tilde{\phi}$ is an invertible formal series over A.

3.4.7. *Example.* Let $L \in \mathscr{K}$, with $L^0(pt)=A$; let $a \in A^\times$ be a unit, and let $T \in L^*(\mathbb{CP}(\infty))$ be the orientation. Then a $T \in L^*(\mathbb{CP}(\infty))$ is also an orientation; we define a new object $L_a \in \mathscr{K}$ by giving K this new orientation. If $F(X, Y)$ is the formal group of L, then $F^a(X, Y)=a^{-1}\,F(aX, aY)$ is the formal group of L_a. The obvious morphism $[a]$: $F^a \to F$, $[a](X)=aX$, comes from a strict morphism $[a]$: $L^* \to L_a^*$: we simply send $t \in L^*(pt)$ to $at \in L_a^*(pt)$, and extend this in the obvious way to a morphism of functors.

3.4.8. **Main Theorem** (continued). *If K_0, $K_1 \in \mathscr{K}$, with associated formal group $\bar{F}_0$, $\bar{F}_1$, then a morphism $\bar{F}_1 \to \bar{F}_0$ is in the image of $\mathscr{K}$ iff its strict component is an isomorphism.*

Corollary 1. *Two (normalized) K-theories are equivalent as multiplicative cohomology functors iff their formal group laws are isomorphic.*

Corollary 2. *Let ξ: $K^* \to K_1^*$ be a morphism in $\mathscr{K}$. Then the normalized K-Theory $K_{(\xi)}^*(-)=K^*(-)\bigotimes_{K^*(pt)} K_1^*(pt)$, whose orientation class is $\tilde{\xi}^{-1}(T_K)$ is equivalent as an oriented multiplicative cohomology theory, to K_1^*.*

Proof. We may assume that ξ is a *strict* morphism, thus $\tilde{\xi}$ is an invertible power series, and $\tilde{\xi}^{-1}(T_K)$ is an orientation class for K. Now the obvious map ξ': $K_{(\xi)}^* \to K_1^*$ takes the orientation class $\tilde{\xi}^{-1}(T_K)$ to $\xi'(\tilde{\xi}^{-1}(T_K))=\tilde{\xi}^{-1}(\tilde{\xi}(T_1))=T_1$.

Proof of the Main Theorem. We will show first that if K_0^*, K_1^* are ordinary K-theories, and $\phi\colon \bar{F}_1 \to \bar{F}_0$ is an isomorphism of their formal group laws, then ϕ lifts to an isomorphism $\phi\colon K_0^* \to K_1^*$ of cohomology theories.

We may assume $\chi(\phi)=1$, by replacing $\phi\colon \bar{F}_1 \to \bar{F}_0$ with

$$[\chi(\phi)^{-1}]\cdot\phi\colon \bar{F}_1 \to \bar{F}_0^{[\chi(\phi^{-1})]},$$

with $[\chi(\phi^{-1})]$ as in the example above. The proof then follows from

3.4.9. **Main Lemma.** *For any $f(T)=T+$higher order terms$\in A[\![T]\!]$, there is a natural (ungraded) ring homomorphism $[f]\colon \mathscr{U}^*(-)\otimes A \to \mathscr{U}^*(-)\otimes A$. In fact there is a natural action of the group $\Gamma_0(A)$ of all formal power series such as the above on $\mathscr{U}^*(-)\otimes A$.*

Proof. This is simply a reformulation of the theory of Landweber-Novikov operations [16]. Let S^* denote the algebra of Landweber-Novikov operations, and let S_* denote the dual Hopf algebra: it is a polynomial algebra over $\mathbb{Z}$, and there is a coaction map $\psi_X\colon \mathscr{U}^*(X)\to\mathscr{U}^*(X)\otimes S_*$. Now if $\alpha\colon S_*\to\mathbb{Z}$ is an (ungraded) ring homomorphism, then the composition

$$\mathscr{U}^*(X)\xrightarrow{\psi_X}\mathscr{U}^*(X)\bigotimes_{\mathbb{Z}} S_*\xrightarrow{1\otimes\alpha}\mathscr{U}^*(X)\bigotimes_{\mathbb{Z}}\mathbb{Z}\longrightarrow\mathscr{U}^*(X)$$

is a ring homomorphism; and it is well-known [Milnor-Moore] that the set $S_*(\mathbb{Z})$ of ring homomorphisms has a natural group structure, making $\mathscr{U}^*(X)$ into a representation of $S_*(\mathbb{Z})$. Nothing changes if we replace $\mathbb{Z}$ with an arbitrary ring A; if $S_*(A)$ denotes the group of ring homomorphisms from S_* to A, then $\mathscr{U}^*(X)\bigotimes_{\mathbb{Z}} A$ has a natural $S_*(A)$-module structure.

This correspondence is discussed further by Landweber [17] (and in [S 13]); here we simply sketch the definition of the operators $[f]$.

Thus let $f(T)=T+$higher order terms$\in A[\![T]\!]$ be given; by regarding T as the canonical orientation class, we regard $f(T)$ as an element of $\mathscr{U}^*(\mathbb{CP}(\infty))\otimes A$. Then

$$\prod_{i=1}^{n}\frac{f(T_i)}{T_i}\in\mathscr{U}^*(\mathbb{CP}(\infty)\times\ldots\times\mathbb{CP}(\infty))\bigotimes_{\mathbb{Z}} A \quad (n\text{ copies})$$

is defined, and is symmetric under permutations of the factors; by Borel's theorem, we recover an element of $\mathscr{U}^*(BU(n))$. Now $f_n\to f_{n-1}$ under the map induced by $BU(n-1)\hookrightarrow BU(n)$, since $T^{-1}f(T)=1$ at $T=0$; hence these elements define a class $f_\infty\in\mathscr{U}^*(BU(\infty))\otimes A$. Clearly $\Delta f_\infty=f_\infty\hat{\otimes}f_\infty$ in the natural Hopf algebra structure on $\mathscr{U}^*(BU(\infty))\otimes A$.

Now the Thom isomorphism induces a coalgebra isomorphism

$$\varphi\colon \mathscr{U}^*(BU(\infty))\otimes A\to\mathscr{U}^*(\mathbb{MU})\otimes A=\mathscr{U}^*(pt)\bigotimes_{\mathbb{Z}} S^*\otimes A;$$

it is not hard to see that $\varphi(f_\infty)\in S^*\otimes A$, and is thus a multiplicative operation on $\mathscr{U}^*(-)\bigotimes_{\mathbb{Z}} A$, which we denote $[f]$. From the construction it is clear that $[f](T)=f(T)$, where $T\in\mathscr{U}^*(\mathbb{CP}(\infty))\otimes A$.

3.4.10. **Lemma.** *Let $\rho_0, \rho_1\colon \mathscr{U}^* \to A$ be genera defining F_0, F_1, and let $f\colon F_1 \to F_0$ be an isomorphism as above. Then the diagram*

$$\begin{tikzcd} \mathscr{U}^* \arrow[rr, "{[f]}"] \arrow[dr, "\rho_0"'] & & \mathscr{U}^* \arrow[dl, "\rho_1"] \\ & A & \end{tikzcd}$$

commutes.

Proof. Consider the composition $\rho_1[f]\colon \mathscr{U}^*(\mathbb{CP}(\infty)) \to A[[T]]$. If Δ is the diagonal on $\mathscr{U}^*(\mathbb{CP}(\infty))$, Δ' the induced diagonal on $A[[T]]$, then evidently

$$\rho_1[f]\,\Delta = \Delta'\,\rho_1[f]$$

consequently

$$\Delta'\,\rho_1[f](T) = \Delta'\,\rho_1 f(T) = \rho_1\,\Delta f(T) = \rho_1 f\,F(T\otimes 1, 1\otimes T) = f\,F_1(T\otimes 1, 1\otimes T)$$

while

$$\rho_1[f]\,\Delta(T) = \rho_1[f]\,F(T\otimes 1, 1\otimes T) = F'(f(T)\otimes 1, 1\otimes f(T)).$$

Hence $F'(f(X), f(Y)) = f\,F_1(X, Y)$, so $f\colon F_1 \to F'$ is an isomorphism. But then $F' = F_0$. Hence $\rho_1[f]$ induces F_0, and so equals ρ_0.

We can now prove the corollary to 3.4.8: Given normalized K-theories K_0, K_1 and an isomorphism $f\colon \bar{F}_1 \to \bar{F}_0$, we will construct an isomorphism $f^*\colon K_0^* \to K_1^*$. We first reduce to the case in which $f(T) = T +$ higher order terms; then there is a unique f^* making the diagram

$$\begin{tikzcd} \mathscr{U}^*(X) \bigotimes_{\mathbb{Z}} A \arrow[r, "{[f]}"] \arrow[d, "\rho_0"] & \mathscr{U}^*(X) \bigotimes_{\mathbb{Z}} A \arrow[d, "\rho_1"] \\ K_0^*(X) \arrow[r, "f^*"] & K_1^*(X) \end{tikzcd}$$

commutative: since ρ_0, ρ_1 are surjective, it suffices to see that $[f]$ maps $\ker\rho_0$ to $\ker\rho_1$; but $\ker\rho_0$ consists of elements $x v \otimes a - x \otimes \rho_0(v)\,a$, $x \in \mathscr{U}^*(X)$, $v \in \mathscr{U}^*(pt)$, $a \in A$; evidently

$$\begin{aligned} &[f](x v \otimes a - x \otimes \rho_0(v)\,a) \\ &\quad = [f](x)\cdot[f](v) \otimes a - [f](x) \otimes \rho_0(v)\,a \\ &\quad = [f](x)\cdot[f](v) \otimes a - [f](x) \otimes \rho_1([f](v))\,a \in \ker\rho_1. \end{aligned}$$

Finally, to complete the proof of 3.4.8, we must see that a strict morphism in $\mathscr{K}$ which induces the identity map of formal group laws is itself the identity. However, this is clear; suppose $\phi\colon K_0^* \to K_0^*$ is such a morphism. If $\mu_0\colon \mathscr{U}^*(-) \to K_0^*(-)$ is the canonical morphism defined by the orientation, then $\phi\,\mu_0$

is a another morphism inducing the same orientation. Both μ_0, $\phi\,\mu_0 \in K_0^*(\mathbb{MU})$; since this is isomorphic to $K_0^*(BU)$ by the Thom isomorphism and since μ_0, $\phi\,\mu_0$ have the same image in $K_0^*(\mathbb{CP}(\infty))$, we find that $\mu_0 = \phi\,\mu_0$. Since μ_0 is surjective, $\phi = 1$.

3.4.11. *Remark.* The main theorem can be considerably strengthened by studying unstable cobordism operations; these are now accessible through the results of Steve Wilson.

To any normalized K-Theory L^*, we can associate a $\mathbb{Z}/2$-graded theory $L^\#$, with $L^\# = L^0$ when $\# = 0$, $L^\# = L^{-1}$ when $\# = +1$; a boundary map is defined by the composition

$$L^{-1}(X, A) \xrightarrow{\delta} L^{-2}(A) \xrightarrow{t\text{-mult}} L^0(A).$$

Such a $\mathbb{Z}/2$-graded theory also has an associated formal group law.

Theorem. *The functor $\check{M}$: ($\mathbb{Z}/2$-graded ordinary K-theories) $\to \mathscr{F}\mathscr{G}$ is fully faithful.*

The point is that in the category of $\mathbb{Z}$-graded ordinary K-theories, the only nontrivial morphisms are isomorphisms, but the example of the Adams operations on $\mathbb{Z}/2$-graded complex K-theory shows that there are plenty of non-invertible morphisms in the $\mathbb{Z}/2$-graded category. Since I hope to study unstable operations elsewhere, with Wilson's help, I omit the proof [See S 1, S 13].

§ 4. Classification of Some Formal Group Laws

The Main Theorem of the preceding section reduces the study of the category $\mathscr{K}$ to the purely algebraic problem of classifying certain formal group laws. Quite a bit is known about this problem now, thanks to results of Cartier, Honda, and Hill, and this section is a summary of the simplest cases of their results.

4.1.1. **Definition.** Let F be a formal group law over the ring R. We define formal series $[n]_F(T)$ inductively by $[0]_F(T) = 0$, $[n]_F(T) = F([n-1]_F(T), T)$. Note that if R is torsion-free, then

$$[n]_F(T) = \ell_F^{-1}(n\,\ell_F(T)),$$

with $\ell_F(T)$ the canonical logarithm of F.

4.1.2. **Lemma.** *If R is of characteristic p, then there exists a formal power series φ such that $[p]_F(X) = \varphi(X^p)$.*

Proof. Let $F(T_1, T_2, \ldots, T_n)$ be defined inductively by $F(T_1, \ldots, T_{n-1}, T_n) = F(F(T_1, \ldots, T_{n-1}), T_n)$; then evidently $F(T, \ldots, T) = [n]_F(T)$, where T is repeated n times. Now $F(T_1, \ldots, T_n)$ is symmetric in $T_1, \ldots, T_n$, and thus can be rewritten in the form $F_{\mathrm{sym}}(\sigma_1, \ldots, \sigma_n)$, where $\sigma_1, \ldots, \sigma_n$ are the elementary symmetric functions of $T_1, \ldots, T_n$. If now $T_1 = \ldots = T_n$, then $\sigma_1 = nT$, $\sigma_2 = \frac{1}{2}n(n-1)\,T^2, \ldots$, $\sigma_n = T^n$; in particular, if $n = p$, then $\sigma_1 \equiv 0$, $\sigma_2 \equiv 0, \ldots, \sigma_{p-1} \equiv 0$, $\sigma_p \equiv T^p$ modulo p. Thus over a ring of char. p, $[p]_F(T) = F_{\mathrm{sym}}(0, 0, \ldots, T^p)$.

4.1.3. **Definition.** Over a ring of char. p, a formal group law F has height *one* if $[p]_F(T) = u\,T^p +$ higher order terms, with u a unit of R. Over an arbitrary ring R we say that F has height one at p if the induced formal group law has height one over R/pR.

4.1.4. **Proposition.** *Being of height one is an isomorphism invariant.*

Proof. Suppose f: $F \to G$ is an isomorphism of formal groups over R (of char. p), with F of height one. Then $G(X, Y) = f^{-1} F(f(X), f(Y))$; it follows that $[p]_G(X) = f^{-1}[p]_F(f(X))$. Since f is an isomorphism, we have $f(X) = f_1 X + \ldots$, $f^{-1}(X) = f_1^{-1} X + \ldots$; hence if $[p]_F(X) = uX^p$, then $[p]_G(X) = f_1^{p-1} uX^p + \ldots$. Thus G is also of height one.

Example. If $F(X, Y) = X + Y - XY$ is the multiplicative formal group, then $[h]_F(X) = 1 - (1 - X)^n$. Consequently

$$\begin{aligned}[p]_F(X) &\equiv 1 - (1 - X^p) \bmod p\\ &\equiv X^p \text{ modulo } p,\end{aligned}$$

and F is of height one.

In fact, a formal group of height one is the most natural generalization of the notion of multiplicative group.

4.1.5. **Proposition.** *There is a class $u \in \mathscr{U}^{2(p-1)}(pt)$ with the following property: a formal group F over R is of height one at p iff the classifying map ρ: $\mathbb{U}^* \to R$ sends u to a unit modulo p.*

Proof. Let $[p]_{\mathscr{U}}(X) = pX + \ldots + uX^p + \ldots$ be the series defined above for the *universal* formal group law. Then F is of height one iff $\rho(u)$ is a unit, mod p. That u is of dimension $2(p-1)$ is immediate.

4.1.6. **Lemma.** *We may take $u = \mathbb{C}\mathbb{P}(p-1)$.*

Proof. An elementary formal group argument shows that the u defined above must be a generator for $\mathscr{U}^* \bigotimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ in dimension $2(p-1)$. It is well-known that $\mathbb{C}\mathbb{P}(p-1)$ is such a (Milnor) generator.

4.1.7. **Corollary.** *A formal group law is of height one iff its genus is ordinary.*

4.2. **Theorem.** 1) *Two formal group laws of height one over a complete discrete valuation ring R are isomorphic iff the induced laws over the residue field k are isomorphic.*

2) *Two formal group laws of the same height over an algebraically closed field are isomorphic.*

3) *If F is of height one over a field of* char. p, *then the ring* End(F) *of endomorphisms of F is canonically isomorphic to $\hat{\mathbb{Z}}_p$.*

Proof (By references). 2) is a result of Lazard; see Fröhlich [p. 72]; for a proof of 3, see Fröhlich, loc. cit. 1) is a result of Lubin-Tate [19]. The canonical isomorphism of 3) arises as follows: there is an obvious map $\mathbb{Z} \ni n \mapsto [n]_F \in \mathrm{End}(F)$, which can easily be seen to be injective. Further $\mathrm{End}(F)$ is complete in the p-adic topology, so the map extends to an injection $\mathbb{Z}_p \to \mathrm{End}(F)$; one then shows this map to be onto as well.

4.2.1. **Definition.** Suppose F is a formal group over $\mathbb{F}_p$ of height one. If $F(X, Y) = \sum_{i,j \geq 0} a_{ij} X^i Y^j$ with $a_{ij} \in \mathbb{F}_p$, then $a_{ij}^p = a_{ij}$; so $F(X^p, Y^p) = F(X, Y)^p$. Consequently $\mathscr{F}(X) = X^p$ is an endomorphism (the Frobenius endomorphism) of F. Under the valuation on $\mathrm{End}(F)$ induced from the standard formal power series valuation, $\mathscr{F}$ has valuation one; hence there exists a unit $u \in \hat{\mathbb{Z}}_p \cong \mathrm{End}(F)$ such that $u\,p = \mathscr{F}$.

This unit is an isomorphism invariant for F over $\mathbb{F}_p$. (If f is an invertible series over $\mathbb{F}_p$, then there is a canonical isomorphism of $\mathrm{End}(F)$ with $\mathrm{End}(F^f)$, where $F^f(X, Y) = f^{-1} F(f(X), f(Y))$. Thus we can compare $\mathscr{F}$ with the Frobenius endomorphism $\mathscr{F}_f$ of F^f; but $\mathscr{F}_f(X) = f^{-1} \mathscr{F} f(X) = f^{-1}(f(X)^p)$. However, since f has coefficients in $\mathbb{F}_p$, $f(X)^p = f(X^p)$; so $\mathscr{F}_f = \mathscr{F}$.

Example. The multiplicative group law has invariant $\mathscr{F} = 1$, since $[p](X) = X^p$.

4.2.2. **Theorem** [Cartier, Honda, Hill]. *Two formal group laws of height one over $\mathbb{F}_p$ ($\hat{\mathbb{Z}}_p$) are isomorphic iff they have the same invariant. Further, any invariant can arise; if $F_\alpha(X, Y)$ is the formal group law over $\hat{\mathbb{Z}}_p$ defined by the ζ-function $\zeta_\alpha(s) = (1 - \alpha p^{-s} + p^{1-2s})^{-1}$, then $\alpha = u^{-1} + p u$, where $u(F_\alpha)$ is the invariant $\in \hat{\mathbb{Z}}_p^*$ defined above.*

Proof [11, 14]. In both references it is shown that $\mathscr{F} = \mathscr{F}(F_\alpha)$ satisfies the equation $\mathscr{F}^2 - \alpha \mathscr{F} + p = 0$ in $\mathrm{End}(F)$. That this invariant classifies formal groups over $\hat{\mathbb{Z}}_p$ as well as $\mathbb{F}_p$ follows from 4.2, part 1: given F over $\hat{\mathbb{Z}}_p$, we define its invariant to be that of the induced law over $\mathbb{F}_p$.

4.2.3. **Corollary.** *A formal group law F of height one over $\hat{\mathbb{Z}}_p$ is specified by the invariant u, which is the unique p-adic unit such that*

$$\ell_F^{-1}(u\,p\,\ell_F(X)) \equiv X^p \text{ modulo } p.$$

4.2.4. Let $\mathscr{W}$ be the field obtained by adjoining to $\mathbb{Q}$ all primitive n^{th} roots of unity, with $p \nmid n$; let $\hat{\mathscr{W}}$ be its p-adic completion. The ring W of integers of $\hat{\mathscr{W}}$ is a complete, discrete valuation ring, whose residue field is algebraically closed; W can be obtained from $\hat{\mathbb{Z}}_p$ by adjoining the n^{th} roots of unity, $p \nmid n$.

Corollary. *Two formal group laws over W of height one are isomorphic.*

Proof. Apply 4.2, parts 1 and 2.

4.3. We can now prove Theorem 1 of the Introduction.

According to Honda's theorem (§ 1) the ζ-function $\zeta_\alpha(s) = (1 - \alpha p^{-s} + p^{1-2s})^{-1}$ defines a formal group law over $\hat{\mathbb{Z}}_p$ whenever α is a p-adic integer; by 2.7.1, this genus is *ordinary* if α is a p-adic unit. Then by Landweber's theorem, $K_\alpha^*(X) = \mathscr{U}^*(X) \bigotimes_{\rho_\alpha^*} \hat{\mathbb{Z}}_p[t, t^{-1}]$ is an ordinary, normalized K-theory.

We can now apply the main theorem of § 3; none of the resulting cohomology theories are isomorphic over $\hat{\mathbb{Z}}_p$, since the invariants of their formal group laws differ (4.2.2); when $\alpha = 1+p$, $u=1$, so the associated K-theory has formal group law isomorphic to the multiplicative group, and the functor is equivalent to classical K-theory. When we extend the ground ring from $\hat{\mathbb{Z}}_p$ to W, all formal groups of height one become isomorphic (4.2.4) so all the ordinary normalized K-theories over W are isomorphic, by the main theorem.

4.4. We end by sketching a "geometric" proof of Corollary 2.6.1.

We first observe that W is *faithfully flat* over $\mathbb{Z}_{(p)}$; this means that if $0 \to E' \to E \to E'' \to 0$ is a sequence of $\mathbb{Z}_{(p)}$-modules which becomes exact upon tensoring with W, then $Q \to E' \to E \to E'' \to 0$ was exact in the first place. This means that to show that K_ρ is a cohomology theory, where ρ is a $\hat{\mathbb{Z}}_p$ or $\mathbb{Z}_{(p)}$-valued genus, it suffices to show that $K_\rho \otimes W$ is a cohomology theory [cf. S4].

Thus we restrict attention to W-valued genera; an ordinary such is a ring homomorphism $\mathscr{U}^*(pt)[\mathbb{CP}(p-1)^{-1}] \to W$, and we denote by $\Lambda_1(W)$ the set of all these homomorphisms; it carries a natural topology. (If $M \in \mathscr{U}^* \otimes W$, a subbasic open set D_M is the set of all $\rho \in \Lambda_1(W)$ such that $\rho(M)$ is a unit of W.) This is the smallest topology making the genera continuous; it is just the Zariski topology.

Now recall (§ 3) that the group $\Gamma(W)$ of formal power series $\{f \in W[\![T]\!] \mid f(T) = uT + \text{higher order terms}, u \in W^\times\}$, with composition being the group operation, acts on $\Lambda_1(W)$; (if $\rho\colon \mathscr{U}^*(pt)[\mathbb{CP}(p-1)^{-1}] \to W$ and $f \in \Gamma(W)$, then $f(\rho) = [f] \cdot \rho$ is the translate of ρ by f.) According to 3.4.10, we can interpret this action as follows: a point of $\Lambda_1(W)$ is a formal group F of height one over W. If $f \in \Gamma(W)$, then f sends $F(X,Y)$ to the formal group law $f^{-1} F(f(X), f(Y))$.

4.2.4. **Corollary,** *restated. The group $\Gamma(W)$ acts transitively on the space $\Lambda_1(W)$.*

By the main theorem of § 3, the points of $\Lambda_1(W)$ can be identified with ordinary K-theories; but it is most natural to view the ordinary K-theories as a *sheaf* $\mathscr{K}$ of *cohomology theories* over $\Lambda_1(W)$, in an obvious sense; if $\rho \in \Lambda_1(W)$ is a genus, then the *stalk* of this sheaf over ρ is just the theory K_ρ^*. In general, if $D_{M_1,\ldots,M_n} \subset \Lambda_1(W)$ is the open set of genera ρ in $\Lambda_1(W)$ such that $\rho(M_1), \ldots, \rho(M_n)$ are units in W, we define the module of sections of the sheaf over $D_{M_1,\ldots,M_n}$ to be $\mathscr{U}^*(-)[\mathbb{CP}(p-1)^{-1}, M_1^{-1}, \ldots, M_n^{-1}]$. Since localization is an exact functor, $X \mapsto \Gamma_D \mathscr{K}(X)$ is a cohomology theory for any open $D \subset \Lambda_1(M)$.

There is a canonical lifting of the action of $\Gamma(W)$ on $\Lambda_1(W)$ defined above, to the sheaf $\mathscr{K}(X)$: if $D \subset \Lambda_1(W)$ is open, then $f \in \Gamma(W)$ defines a map $f_D\colon f^{-1} D \to D$, and a map

$$[f]_D\colon \Gamma_{f^{-1}D}\, \mathscr{K}(X) \to \Gamma_D\, \mathscr{K}(X)$$

of modules of sections, satisfying the apparent axioms.

The following is the most intuitive way I know of understanding Landweber's theorem:

Proposition. *An equivariant sheaf $\mathscr{E}$ (with respect to a group G acting transitively on a base X) is locally trivial.*

That is, X can be covered with open sets $\{U_i\}$, such that $\mathscr{E}|U_i$ is isomorphic to the sheaf of functions from U_i to a *fixed* model E. Note that this is almost the definition of an (algebraic) vector bundle over X, with a G-action; but in that case one would require that E be a free W-module. Of course the analogous statement over a *field* is actually true: an equivariant sheaf over a space with a transitive group action actually is a vector bundle (under suitable finiteness conditions, which are also needed here). I have proved such a theorem elsewhere and will not give a proof of the above proposition here; but note that under the local triviality which is the conclusion of this theorem, the fact that formation of the theory K_ρ is an exact functor becomes a triviality.

§ 5. Examples of Ordinary K-Theories from the Theory of Elliptic Curves

Let $q \in \mathbb{C}$ be a complex number in the punctured unit disk, $0<|q|<1$. Let $\Gamma_q = \{q^n | n \in \mathbb{Z}\}$ be the subgroup of $\mathbb{C}^*$ generated by q. Then $\mathbb{C}^*/\Gamma_q = E_q$ is a compact complex-analytic manifold, diffeomorphic to the 2-torus, with a natural group structure.

If z_0, z_1 are points on E_q sufficiently near the identity, then there is a convergent power series $F_q(z_1, z_2) = \sum a_{ij}(q)\, z_1^q\, z_2^q$ with $a_{ij}(q) \in \mathbb{C}$, such that $F_q(z_1, z_2)$ is the sum of z_1, z_2 in the group structure of E_q. Thus we have a family of formal group laws over $\mathbb{C}$, depending continuously on the parameter q.

In fact we can do better. Let $E_k(q) = 1 + (-1)^{k/2} \dfrac{2k}{b_k} \sum_{n \geqq 0} \sigma_{k-1}(n)\, q^n$ denote the normalized Eisenstein series, where b_k is a Bernoulli number, $k \geqq 4$ is even and $\sigma_{k-1}(n) = \sum_{d|n, d \geqq 1} d^{k-1}$. Let A_q denote the ring of integers of the field $\mathbb{Q}(E_4(q), E_6(q))$.

5.1. **Theorem** (Tate-Jacobi; for a proof, see Roquette). *The group E_q determines a formal group law F_q with coefficients $a_{ij}(q) \in A_q$. If we adjoin q to the ring A_q, then we can construct an isomorphism of F_q with the multiplicative group.* [See S 3, S 8 (§ 8.8), or S 11.]

Remark. One can always find a formal group law associated to E_q whose coefficients lie in the field $\mathbb{Q}(j(q))$, where j is the elliptic modular function, but the law is not canonical.

Sketch of the Construction. The first step is to construct a canonical embedding of E_q into the complex projective plane. Suppose $t \in \mathbb{C}^* - \{1\}$; send it to the point $[x(t), y(t), 0] \in \mathbb{CP}(2)$ (in homogeneous coordinates) via the equations

$$x(t) = \sum_{k \in Z} \frac{q^k t}{(1-q^k t)^2} - 2 \sum_{k=1}^{\infty} \frac{q^k}{1-q^k},$$

$$y(t) = \sum_{k \in Z} \frac{(q^k t)^2}{(1-q^k t)^3} + \sum_{k=1}^{\infty} \frac{q^k}{1-q^k}.$$

Now $x(t) = \dfrac{t}{(1-t)^2} + x_0(t)$, $y(t) = \dfrac{t^2}{(1-t)^3} + y_0(t)$, with $x_0(t)$, $y_0(t) \in \mathbb{Z}[t, t^{-1}]\llbracket q \rrbracket$, and $x_0(t)$, $y_0(t) \to 0$ as $t \to 1$. Thus as $t \to 1$, $[x(t), y(t), 0] \to [1-t, t, 0]$, so the map defined above extends to give a map of $\mathbb{C}^*$ into $\mathbb{CP}(2)$, sending 1 to $[0, 1, 0]$. Moreover, evidently $x(q^t)=x(t)$, $y(qt)=y(t)$, so the map factors through E_q, and the resulting map $E_q \to \mathbb{CP}(2)$ is an embedding.

Now by Chow's lemma, E_q is an *algebraic* subvariety of $\mathbb{CP}(2)$, and it can be shown that E_q is embedded as the locus of solutions of the equation

$$y^2 z + xyz = x^3 + b(q)\,x z^2 + c(q)\,z^3,$$

where

$$b(q) = 3^{-1}\cdot 2^{-4}\cdot(1 - E_4(q))$$

$$c(q) = 3^{-3}\cdot 2^{-6}(1 - 3E_4(q) + 2E_6(q))$$

can be shown to lie in $\mathbb{Z}[\![q]\!]$. Thus E_q becomes an algebraic variety defined over the ring A_q.

Now $\omega = \frac{x}{y}$ defines a uniformizing parameter for E_q near the identity. If we expand it at $1-T$, we obtain a formal series $\omega(T) = T +$ higher order terms, where $\omega \in \mathbb{Z}[\![q, T]\!]$. Thus ω defines a formal isomorphism ω: $\mathbb{C}^*$ (near 1) $\to E_q$ (near $[0, 1, 0]$) of algebraic group varieties; in other words, ω: $\mathbb{G}_m \to E_q$ is a formal isomorphism, defined over $\mathbb{Z}[\![q]\!]$.

5.2. *Remarks*. Now Theorem 2 of the Introduction follows by applying the main theorem of § 3, as Theorem 1 was proved. It is necessary to have some idea of the "bad primes" for the formal group E_q; these are primes at which E_q is singular (specifically where it has a cusp) or supersingular (i.e. has Hasse invariant zero). The first is easy to decide; the second is decided by a recent theorem of Deligne: The curve E_q is supersingular at p iff p divides $E_{p-1}(q)$, at least if $p \geqq 2$ or 3; cf. Katz, or [S 2].

I should perhaps point out that an elliptic curve is *ordinary* at a prime p iff its formal group is of height one there; this suggested my "ordinary K-theory" terminology.

If the modular invariant $j(E)$ of an elliptic curve E is *rational*, then results of Neron imply that the curve has an essentially unique defining relation with *integral* coefficients; thus the associated formal group law is itself defined over $\mathbb{Z}$ and results of Cartier, Honda, and Hill connect that formal group law with the ζ-function; this occurs in Example C.

§ 6. Differential Operators

In this section we show that the genus ρ_D of the Introduction can be interpreted as the index of a differential operator; there is a similar interpretation for the genus ρ_q constructed from the elliptic curves of the preceding section.

[These interpretations are in no way *explanations* of the integrality properties of these genera; rather, they imply that certain differential operators, a priori rather mysterious, have some sort of topological significance. I don't know what that might be.]

We first describe the notion of a universal elliptic operator (for almost-complex manifolds).

6.1. **Definition.** Let $T=\{z\in\mathbb{C} \mid |z|=1\}$ be the circle group, acting as usual on the complex plane $\mathbb{C}$: $z\in T$, $w\in\mathbb{C}\mapsto z\,w\in\mathbb{C}$; we will denote this representation by L.

An element α of the equivariant K-group $K_T(L)$ (formed with compact supports) is (provisionally) a universal elliptic operator for almost-complex manifolds. In particular, the Thom isomorphism φ_L: $K_T(pt)\to K_T(L)$ sends $1\in K_T(pt)$ to a class $\varphi_L(1)$ which we call the universal Dolbeault operator. Our first object is to justify this terminology.

6.1.1. Let $U(n)$ denote the unitary group, acting in the usual way on L^n; let T^n denote a maximal torus. The Borel map

$$K_{U(n)}(L^n)\to K_{T^n}(L^n)=K_T(L)\bigotimes_{K(pt)}\cdots\bigotimes_{K(pt)}K_T(L)\quad (n\text{ factors})$$

is an injection onto the symmetric elements. Consequently $\alpha\otimes\ldots\otimes\alpha$ defines an element $\alpha^{\otimes n}\in K_{U(n)}(L^n)$.

Now let M be a compact closed almost-complex manifold, of real dimension $2n$. Then there exists a principal $U(n)$-bundle P over M, and an isomorphism of $P\times_{U(n)}L^n$ with the complex cotangent bundle of M, inducing a map

$$K_{U(n)}(L^n)\xrightarrow{\theta_M}K_{U(n)}(P\times L^n)=K(P\times_{U(n)}L^n)=K(T^*M).$$

Applying this to the element $\alpha^{\otimes n}$ above, we obtain an element $\alpha_M\in K^*(TM)$. Thus we can construct in a canonical way (the symbol of) an elliptic operator on any such M. In particular this construction constricts to $\varphi_L(1)$ the standard Dolbeault complex of M.

6.1.2. **Definition.** α-ind(M) is the index of α_M; it is a well defined element of $K(pt)$.

Proposition. α-ind: $\mathcal{U}^*(pt)\to K(pt)$ *is a ring homomorphism.*

Proof. That α-ind is a cobordism invariant follows from the index theorem. That the resulting map is a ring homomorphism is immediate.

6.1.3. **Corollary.** *Every* $\alpha\in K_T(L)$ *determines a formal group law on* $K(pt)=\mathbb{Z}$.

We are interested in modifying this construction, which yields rather simple formal group laws.

6.1.4. **Definition.** Let $I_T\subset K_T(pt)$ be the kernel of the natural map $K_T(pt)\to K(pt)$. We write $\hat{K}_T(L)$ for the completion of $K_T(L)$ in the I_T-adic topology. To any $\alpha\in\hat{K}_T(L)$ we can associate an element $\alpha^{\otimes n}$ of $K_{U(n)}(L^n)$ as above; the bundling construction then associates to any closed compact almost-complex M, an element in $K(TM)$. Note that no completion occurs in this last group.

Consequently we can extend our definition, and regard elements of $\hat{K}_T(L)$ as universal elliptic operators. To each class $\alpha\in\hat{K}_T(L)$, we again associate a map α-ind: $\mathcal{U}^*(pt)\to K(pt)=\mathbb{Z}$; this gives a much larger family of formal group laws. We can identify these laws with the help of

6.1.5. **Theorem** (Atiyah-Segal). *The natural map* θ: $K_T(pt)\to K(BT)=K(\mathbb{CP}(\infty))$ *becomes an isomorphism on* I_T*-adic completion.*

Now a generator of $K(\mathbb{CP}(\infty))$ determines an orientation for complex K-theory; if X is the standard orientation class, then the units of $\hat{K}_T(pt)$ are put in 1–1 correspondence with orientations on K by the map $\alpha \mapsto \theta(\alpha) X$.

6.1.6. **Proposition.** *The formal group law determined by the orientation $\theta(\alpha)X$ agrees with the formal group law defined by the operator $\alpha\,\varphi_L(1) \in \hat{K}_T(L)$. ($\alpha$ is a unit of $\hat{K}_T(pt)$.)*

Proof. The operator $\alpha\,\varphi_L(1)$ is also a Thom class; by Adams' lemma, these are in 1–1 correspondence with orientations.

6.1.7. **Corollary.** *Let ξ: $K^* \to K_1^*$ be a morphism in $\mathscr{K}$. Then t-$\mathrm{ind}_1(M) = \varphi_L(X^{-1}\,\bar{\xi}^{-1}(X)) - \mathrm{ind}(M)$.*

Proof. Here $\bar{\xi}$: $\bar{F}_1 \to \bar{F}_K$ is the induced map of formal group laws, and $\bar{\xi}^{-1}$ is its formal inverse. Note that $\varphi_L(X^{-1}\,\bar{\xi}^{-1}(X)) \in \hat{K}_T(L) \bigotimes_{K(pt)} K_1(pt)$; we have extended the index map $K(T^*M) \to K(pt)$ linearly to a map $K(T^*M) \bigotimes_{K(pt)} K_1(pt) \to K_1(pt)$. By the preceding proposition, the formal group induced by $\varphi_L(X^{-1}\,\bar{\xi}^{-1}(X))$ is the same as that induced by the orientation $\bar{\xi}^{-1}(X)$ on $K^*(-) \bigotimes_{K^*(pt)} K_1^*(pt)$. By Corollary 2 to the main theorem, this theory is equivalent, as an oriented theory, to K_1^*. Then the two formal group laws coincide, and so do the associated genera.

6.2. *An Application.* In Example B of the Introduction, we introduced a cohomology theory h_D^* over the groundring $\mathbb{Z}[D^{-1}]$, whose associated formal group law has logarithm $\sum_{n \geq 1} \left(\frac{D}{n}\right) \frac{X^n}{n}$, and whose associated genus ρ_D sends $\mathbb{CP}(n-1)$ to the quadratic symbol $\left(\frac{D}{n}\right)$. We use the corollary above to construct a $\mathbb{Z}/D\mathbb{Z}$-equivariant elliptic operator, whose index is the invariant ρ_D.

We will use K_D^*-theory, which is the K-theory of the category of vector bundles with an action of the cyclic group $\mathbb{Z}/D\mathbb{Z}$; we will apply this only to spaces with a *trivial* $\mathbb{Z}/D\mathbb{Z}$-action, in which case $K_D^*(X) \cong K^*(X) \otimes K_D^*(pt) = K^*(X) \otimes R(D)$, where $R(D)$ is the representation ring of $\mathbb{Z}/D\mathbb{Z}$; we can identify it with $\mathbb{Z}[\omega]$, where ω is a primitive D^{th} root of unity.

Now the formal group law F_χ with logarithm $\ell_\chi(X) = \sum_{n \geq 1} \chi(n) \frac{X^n}{n}$ is actually defined over $\mathbb{Z}$. We prove

6.2.1. **Proposition.** *There is a morphism ξ: $F_\chi \to \mathbb{G}_m$ of formal groups over $R(D)$; it becomes an isomorphism when D is inverted.*

Proof. This is actually a result of Honda [13, Theorem 4] but we have translated it over $R(D)$.

6.2.2. We construct a virtual character of the group $\mathbb{Z}/D\mathbb{Z}$, which is central to the proposition above.

Let D^* denote the group of units in the ring $\mathbb{Z}/D\mathbb{Z}$; it is cyclic of order $\varphi(D)$, where φ is Euler's function, and its elements can be identified with the mod D residue classes of integers which are relatively prime to D. The quadratic symbol determines a homomorphism χ: $D^* \to \{\pm 1\}$.

Let $\underline{\mathfrak{a}}$ denote the algebra of complex-valued functions on D^*. If ω is a primitive D^{th} root of unity, then we define a map $T\colon \mathfrak{a}\to\underline{\mathfrak{a}}$ by $(Tf)(n)=\omega^n f(n)$, for $n\in D^*$, $f\in\underline{\mathfrak{a}}$. Clearly $T^D=1$, so $\underline{\mathfrak{a}}$ becomes a representation of $\mathbb{Z}/D\mathbb{Z}$. Now let $\varepsilon_\chi\colon \mathfrak{a}\to\underline{\mathfrak{a}}$ be the map defined by $\varepsilon_\chi(f)(n)=\chi(n)f(n)$, $n\in D^*$. Evidently $\varepsilon_\chi^2=1$, and ε_χ commutes with T, so the representation $\underline{\mathfrak{a}}$ splits into $\underline{\mathfrak{a}}_+$, $\underline{\mathfrak{a}}_-$, which are the appropriate eigenspaces of ε_χ. We are interested in the virtual representation $\underline{\mathfrak{a}}_\chi=[\mathfrak{a}_+]-[\mathfrak{a}_-]\in R(D)$.

6.2.3. **Lemma.** *The character* $\underline{\mathfrak{a}}_\chi(n)=\chi(n)\,T_\chi$, *where we have extended* χ *to a function from* $\mathbb{Z}/D\mathbb{Z}$ *to* $\mathbb{C}$, *and* T_χ *satisfies* $T_\chi^2=\chi(-1)\,D$.

Proof. This amounts to evaluating a Gauss sum. The representation a decomposes into one-dimensional representations; if b is the standard such representation, in which $1\in\mathbb{Z}/D\mathbb{Z}$ acts by ω-multiplication, then $\underline{\mathfrak{a}}=\bigoplus_{(k,D)=1,\,k \bmod D} b^{\otimes k}$, and evidently $\underline{\mathfrak{a}}_\pm=\bigoplus_{\chi(k)=\pm 1,\,k \bmod D} b^{\otimes k}$. The trace of $n\in\mathbb{Z}/D\mathbb{Z}$ on b is just ω^n, so $\underline{\mathfrak{a}}_\pm(n)=\sum_{\chi(k)=\pm 1}\omega^{nk}$; hence $\underline{\mathfrak{a}}_\chi(n)=\sum_{k \bmod D}\chi(k)\,\omega^{nk}$, and we can refer to Lang, [IV § 3]; but beware typos. Thus $\underline{\mathfrak{a}}_\chi(n)=\chi(n)\,\tau_\chi$, $\tau_\chi=\sum\chi(k)\,\omega^k$ being a square root of $\chi(-1)\,D$.

6.2.4. **Corollary.** $\underline{\mathfrak{a}}_\chi$ *is an eigenfunction of the Adams operations;* $\psi^n\,\mathfrak{a}_\chi=\chi(n)\,\underline{\mathfrak{a}}_\chi$.

6.2.5. **Definition.** $\xi(X)=1-\lambda_{-X}(\mathfrak{a}_\chi)\in R(D)[\![X]\!]$. (Here $\lambda_T(a)=\sum_{n\geq 0}\lambda^n(a)\,T^n$ is the total λ-operation on $a\in R(D)$).

Evidently $\xi(X)=\mathfrak{a}_\chi X+$higher order terms becomes invertible when D is inverted. (It suffices to observe that the map $R(D)\to\mathbb{Z}[\omega]$ evaluating a character at 1 is an isomorphism; hence $\mathfrak{a}_\chi$ is sent to τ_χ, which is a square root of D, invertible if D is.)

We have to see that ξ maps F_χ to $\mathbb{G}_m$; it is easier to show that $\xi_0(X)=\underline{\mathfrak{a}}_\chi^{-1}(1-\lambda_{-X}(\underline{\mathfrak{a}}_\chi))=X+\ldots$ maps F_χ to $\mathbb{G}_{m,\chi}(X,Y)=X+Y-\mathfrak{a}_\chi XY$; this statement is equivalent, due to the isomorphism $[\mathfrak{a}_\chi](X)=\mathfrak{a}_\chi X$, $[\underline{\mathfrak{a}}_\chi]\colon \mathbb{G}_{m,\chi}\to\mathbb{G}_m$.

Thus it suffices to see that ξ_0 takes the logarithm of $\mathbb{G}_{m,\chi}$ to the logarithm of F_χ, or equivalently, that it does so on the differentials. Now the logarithm of $\mathbb{G}_{m,\chi}$ is $\sum_{n\geq 1}\mathfrak{a}_\chi^{n-1}\dfrac{X^n}{n}$, so its differential is $\dfrac{dX}{1-\mathfrak{a}_\chi X}$. We compute

$$\frac{d\,\xi_0(X)}{1-\underline{\mathfrak{a}}_\chi\,\xi_0(X)}=-\mathfrak{a}_\chi^{-1}\,\frac{d\,\lambda_{-X}(\mathfrak{a}_\chi)}{\lambda_{-X}(\mathfrak{a}_\chi)}=-\underline{\mathfrak{a}}_\chi^{-1}\,d\log\lambda_{-X}(\underline{\mathfrak{a}}_\chi)).$$

Using Adams' original definition of the ψ^n, we have

$$\begin{aligned}\frac{d\,\xi_0(X)}{1-\mathfrak{a}_\chi\,\xi_0(X)}&=\mathfrak{a}_\chi^{-1}\sum_{n\geq 1}\psi^n(\underline{\mathfrak{a}}_\chi)\,X^{n-1}\,dX=\underline{\mathfrak{a}}_\chi^{-1}\sum_{n\geq 1}\underline{\mathfrak{a}}_\chi\cdot\chi(n)\,X^{n-1}\,dX\\&=\sum_{n\geq 1}\chi(n)\,X^{n-1}\,dX=d\,\ell_\chi(X).\end{aligned}$$

6.2.6. **Corollary.** *Let $\delta \in \hat{K}_{\mathbb{Z}/D\mathbb{Z}\times T}(pt)$ be the unique element such that $\lambda_{-\delta}(\mathfrak{a}_\chi)=1-\mathfrak{a}_\chi X$, with X the standard generator. Then $\varphi_L(\delta)\in K_{\mathbb{Z}/D\mathbb{Z}\times T}(L)$ is a universal $\mathbb{Z}/D\mathbb{Z}$-equivariant operator; its index on a closed almost-complex manifold of real dimension $2n$, with trivial $\mathbb{Z}/D\mathbb{Z}$-action is $\rho_D(M)\,\mathfrak{a}_\chi^n$.*

We can give a similar construction for the genera $\rho_q\colon \mathscr{U}^* \to A_q$ of § 5. The function $\omega(t)$, $t\in\mathbb{C}$ was defined there; it is holomorphic in the unit disk around $t=1$, with a simple zero there, and is periodic: $\omega(qt)=\omega(t)$. Expanding about $X=1-t$, we obtain $\omega=\sum_{n\geq 1}\omega_n(q)\,X^n\in\mathbb{Z}[\![q]\!][\![X]\!]$.

6.2.7. **Proposition.** *The element $\varphi_L\,\omega\in\hat{K}_T(L)\bigotimes_{\mathbb{Z}}\mathbb{Z}[\![q]\!]$ defines a formal family of elliptic operators in an obvious sense; if M is almost-complex, and closed, then $\varphi_L\,\omega$-ind$(M)=\rho_q(M)$.*

Remark. One expects the $\omega_n(q)$ to be closely related to the theory of modular forms.

Appendix: On Galois Phenomena

I want to explain briefly how the phenomena in Theorem 1 are rather reasonable from the point of view of Galois theory. To simplify the exposition, I will discuss cohomology theories taking values in the category of k-algebras, where k is a field of characteristic p. (Of course any cohomology has such a "mod p reduction" defined by smashing with a Moore space, aside from incidental difficulties when $p=2$.)

Definition A1. Let h_0^*, h_1^* be multiplicative cohomology theories in k-vector spaces. We say that h_0^* (resp. h_1^*) is a *form* of h_1^* (resp. h_0^*) if there is an isomorphism $\alpha\colon h_0^*\bigotimes_k\bar{k}\xrightarrow{\sim} h_1^*\bigotimes_k\bar{k}$ of cohomology theories, where $\bar{k}$ is the algebraic closure of k. (We say that h_0 is a nontrivial form of h_1 if the two theories aren't isomorphic over k). Given h over k, we write Forms(h) for the set of k-isomorphism classes of forms of h.

Theorem. *There is an injective map* Forms$(h)\to H^1(\mathrm{Gal}(\bar{K}/K);\bar{A}(h))$.

The target of i is just the first cohomology of the Galois group of the closure of k, with coefficients in a certain (in general nonabelian) group $\bar{A}(h)$ with a Gal$(\bar{k}/k)$-action.

Definition A2. Let h be a cohomology theory over a field k; let $\bar{A}(h)$ denote the group of $\bar{k}$-algebra of automorphisms of the cohomology theory $h\bigotimes_k\bar{k}$ (we don't require that they preserve gradings). If $g\in\bar{A}(h)$, and $\sigma\in\mathrm{Gal}(\bar{k}/k)$, we define g^σ to be the composition

$$h\bigotimes_k\bar{k}\xrightarrow{1\otimes\sigma}h\bigotimes_k\bar{k}\xrightarrow{g}h\bigotimes_k\bar{k}\xrightarrow{1\otimes\sigma^{-1}}h\bigotimes_k\bar{k}.$$

Thus $\bar{A}(h)$ has a natural Gal$(\bar{k}/k)$-structure.

The map i is defined as follows. Let h_1 be a form of h; there then exists an identification α: $h \otimes_k \bar{k} \xrightarrow{\vee} h_1 \otimes_k \bar{k}$. The map i_α: $\mathrm{Gal}(\bar{k}/k) \to \bar{A}(h)$ defined by $i_\alpha(\sigma) = \alpha^{-1} \cdot \alpha^\sigma$ satisfies $i_\alpha(\sigma\tau) = i_\alpha(\tau)(i_\alpha(\sigma))^\tau$, i.e. it lies in the set $Z^1(\mathrm{Gal}(\bar{k}_i/k); \bar{A}(h))$ of cocycles. The set $H^1(\mathrm{Gal}(\bar{k}/k); \bar{A}(h))$ is the quotient of this set under the equivalence relation $c \sim \gamma^{-1} c \gamma^\sigma$, for $\gamma \in \bar{A}(h)$. Thus i_α maps to the trivial class in H^1 iff there exists an automorphism γ of $h \otimes_k \bar{k}$ such that $\alpha\gamma$: $h \otimes_k \bar{k} \to h_1 \otimes_k \bar{k}$ is defined over k; in that case, $h \simeq h_1$ over k, and the result follows.

To classify the forms of a given theory, we need to know $\bar{A}(h)$.

Example H. Let $h = H^*(-; \mathbb{F}_p)$ be ordinary cohomology, $\bar{k} = \bar{\mathbb{F}}_p$ be the union of the finite fields of char. p.

Let F be an indeterminate; the ring $k\langle\langle F \rangle\rangle$ is a formal power series ring, except that F doesn't commute with k; if $\lambda \in k$, then $\lambda^p F = F^\lambda$. It can be shown (cf. Atiyah-Hirzebruch) that $\bar{A}(H)$ is the group of units of the ring $k\langle\langle F \rangle\rangle$. The powers of the maximal ideal in $k\langle\langle F \rangle\rangle$ give a family of normal subgroups of $\bar{A}(H)$, such that the quotient of one subgroup in the filtration by the next is either a copy of k, or k^*; the galois group $\mathrm{Gal}(\bar{k}/k)$ acts in the apparent way on $k\langle\langle F \rangle\rangle$. It follows by induction that $H^1(\mathrm{Gal}(\bar{k}/k); \bar{A}(H)) = \{0\}$; ordinary cohomology has no nontrivial forms.

It's easier to prove this by obstruction theory.

Example U. Let $h^* = \mathscr{U}^*(-; \mathbb{F}_p)$. Then $\bar{A}(\mathscr{U}) = \Gamma(k)$ is the group of formal series $\sum_{i \geqq 1} a_i T^i$, with $a_i \in k$, $a_1 \neq 0$; the group operation is composition. The filtration of power series by order gives a filtration of $\Gamma(k)$, whose quotients are as above: either copies of k or k^*. Thus complex cobordism has no nontrivial forms.

Example K. Let $h^* = K^*(-; \mathbb{F}_p)$. Then the automorphism group $\bar{A}(K) \cong \mathbb{Z}_p^*$, the p-adic units; if $\alpha \in \hat{\mathbb{Z}}_p^*$, the induced map on K^{2n} is $\alpha^{-n} \psi^\alpha$, where ψ^α is the p-adic limit of usual Adams operations, cf. Atiyah-Tall [5]. Now $\mathrm{Gal}(\bar{k}/k) = \hat{\mathbb{Z}}$ acts *trivially* on this group; there are no new automorphisms if we go to $\bar{\mathbb{F}}_p$. Moreover $\bar{A}(K)$ is abelian, so $H^1(\mathrm{Gal}(\bar{k}/k); \bar{A}(K)) = \mathrm{Hom}_{\mathrm{cont.}}(\hat{\mathbb{Z}}; \hat{\mathbb{Z}}_p^*) = \hat{\mathbb{Z}}_p^*$. Thus, a priori, there are lots of possible forms of K-theory.

Corollary to Theorem 1. *All the possible forms of* mod p *K-theory actually exist.*

Remark. When studying Galois cohomology in general, it is always easy to see that an invariant analogous to i gives an injective map from the forms to some H^1; in many cases the map is actually a bijection, but it takes some construction techniques to prove this. These techniques are lacking for spectra, but the examples of this paper suggest that they might exist [cf. S 17].

Historical Remarks

This paper was written in 1973, but it has had to wait for the discovery of a geometric interpretation [by Ed Witten, with the aid of Peter Landweber, Serge Ochianine, the Chudnovsky brothers, Doug Ravenel, Larry Smith, and Bob Stong, cf. [S9]] to find an audience. In this supplementary bibliography I have tried to compile a sampling of references for relevant work that has appeared in the meantime.

The original motivation for this paper came from classical questions about integrality properties of characteristic classes of smooth manifolds; since that subject has been greatly deepened by the discovery that many classical cobordism invariants are but the constant terms of new invariants (of physical interest, e.g. the index of a Dirac operator on loopspaces) taking values in rings of modular forms, it may be worth making some remarks of a general nature. Thom showed that a cobordism class is in some sense determined by characteristic numbers, which are geometric invariants of a type (roughly, integrals of curvature tensors) familiar to differential geometers. In particular the (Chern – Hurewicz – Thom – Boardman) monomorphism

$$\pi_*(\mathbf{MU}) \to H_*(\mathbf{MU}, \mathbb{Z})$$

defines an *isomorphism* of the complex cobordism ring with an algebra of Chern numbers (which, as an algebra of symmetric functions, is relatively well – understood) *after* tensoring with the rational numbers. In the dual language of algebraic geometry we can reformulate the fact locally: the induced map

$$\operatorname{Proj} H_*(\mathbf{MU}, \mathbb{Z}) \to \operatorname{Proj} \pi_* \mathbf{MU}$$

of schemes over $\operatorname{Spec} \mathbb{Z}$ is an isomorphism above the *generic* point, but otherwise has quite complicated behavior. Integrality theorems are interesting precisely because of the (arithmetic) difficulty of understanding this map over the finite primes, e.g. in terms of differential geometry. The most compelling interpretation of these integrality theorems comes from formulae expressing the indices of differential operators as characteristic numbers; by providing us with new classes of interesting differential operators (suggested by analysis on the free loopspace), elliptic cohomology has revitalized this classical subject.

It would probably also help to make some historical remarks. This paper was written after Deligne had reduced the Ramanujan – Petersson conjectures to those of Weil, but before he proved the latter. It is also a product of the days when the existence of a product in singularity cobordism was a controversial subject, and it was roughly contemporary with the revolution in the theory of foliations which introduced topologists to notions of continuous variation in homotopy theory. Nowadays we know much more about homotopy theory [S16, S21]; in particular we know that the stable homotopy category has *infinite* Krull dimension, in a certain sense [S5], and can thus support 'modular' behavior of arbitrary depth. [Thus elliptic cohomology gets us down to codimension two [S14].] Fifteen years ago this idea was perhaps less easy to believe; this paper was written in the hope that its simple constructions, based on nothing more complicated than Landweber's exact functor theorem, would convince its readers that homotopy theory might be at least as deep as arithmetic. In the end, the editors of journals to which this paper was submitted did the author the favor of forgetting that he might have been capable of so regrettable a lapse [S 18].

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