The Kervaire Invariant

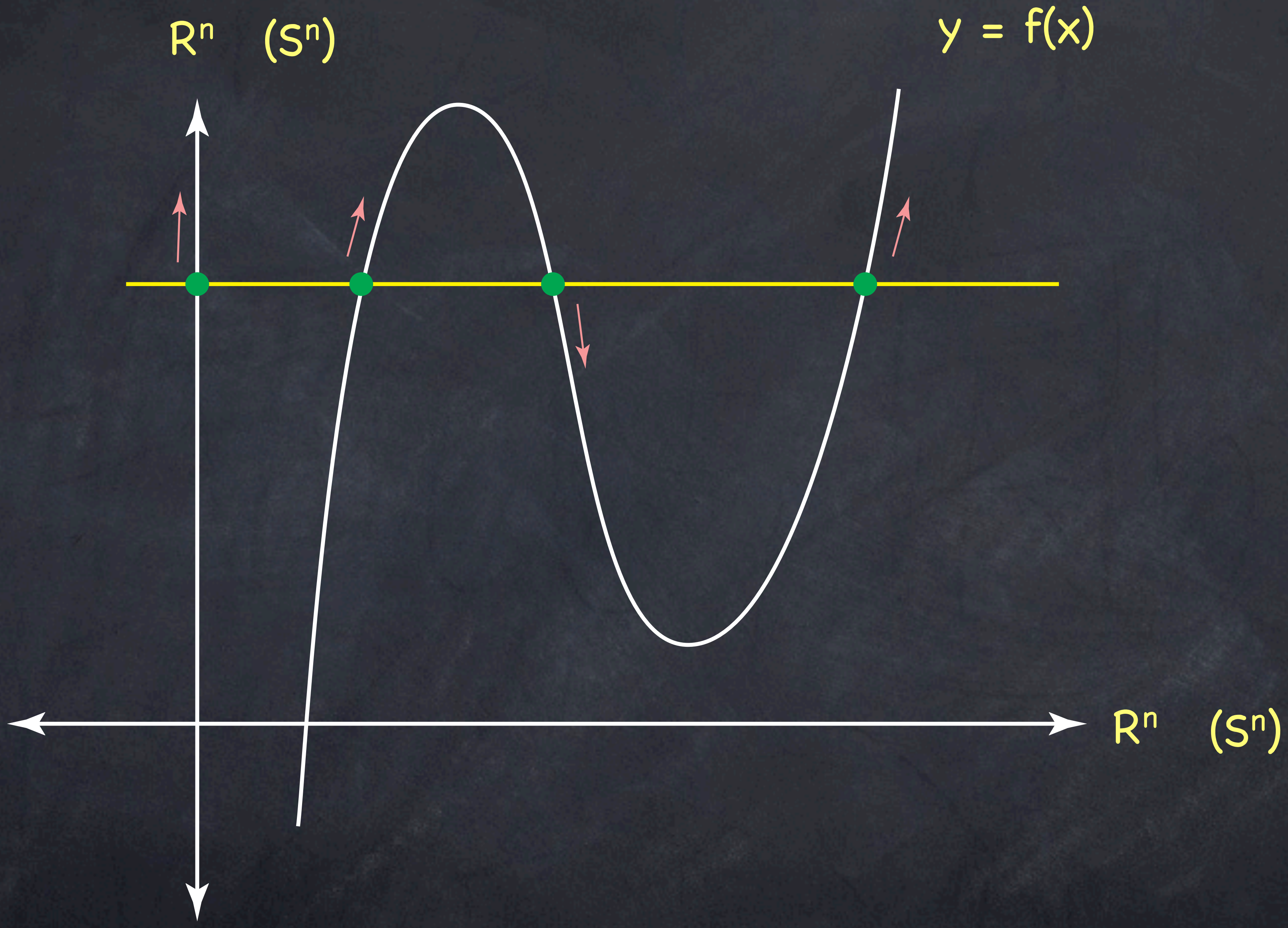
(joint work with Mike Hill and Doug Ravenel)

Theorem (Hill, H., Ravenel)

If M is a smooth, stably framed manifold of Kervaire invariant one, then the dimension of M is one of

$$2, 6, 14, 30, 62, \text{ or } 126$$

Topology before the 1930's



counting the solutions to n equations in n unknowns

Topology before the 1930's (Brouwer, Hopf)

Definition of the degree of a map

$$f : M \longrightarrow S^n \qquad (\dim M = n)$$

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Theorem: Two maps of the same degree are homotopic.

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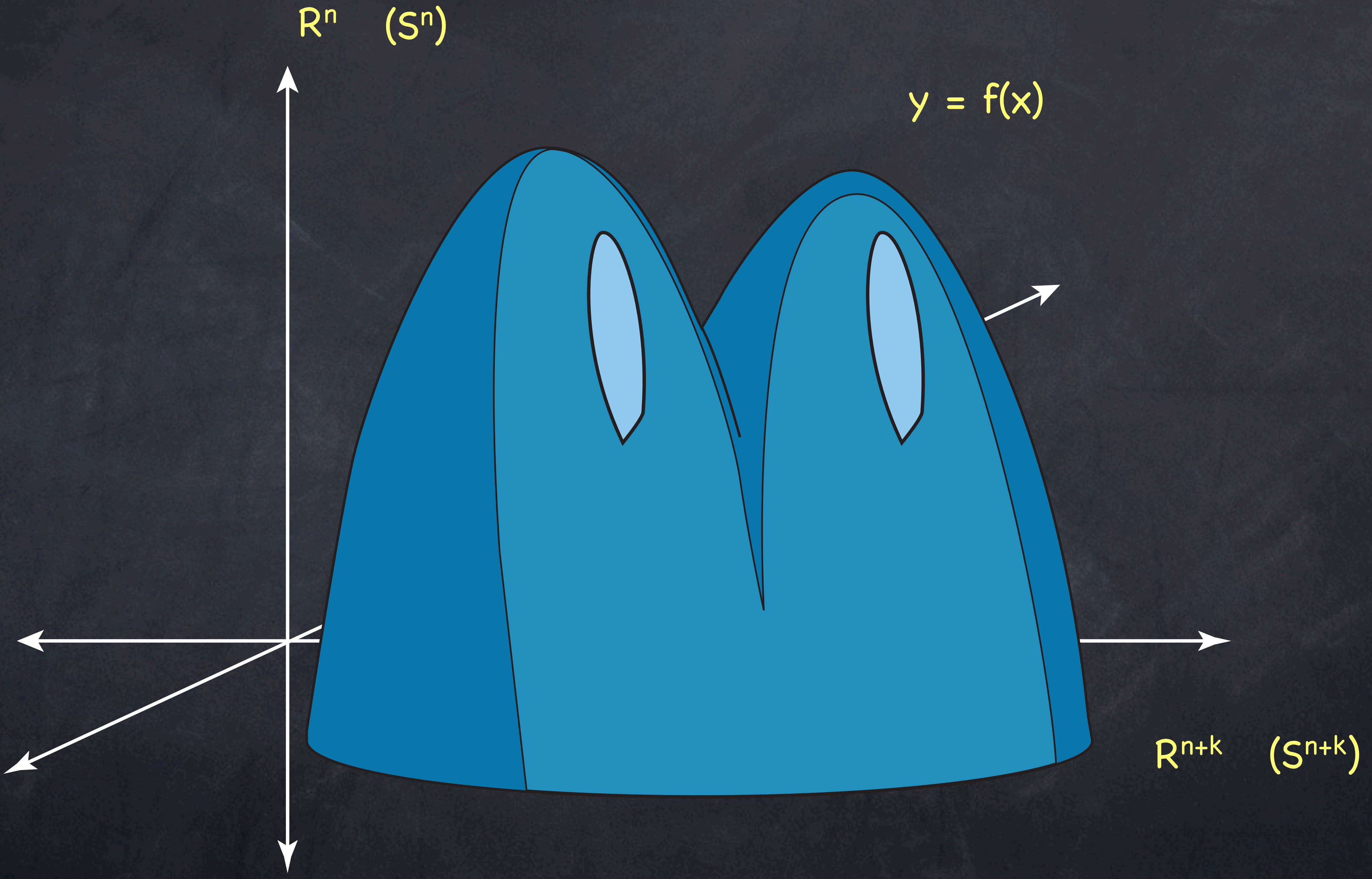
$$f : M \longrightarrow S^n \qquad (\dim M = n)$$

Theorem: Two maps of the same degree are homotopic.

Key tool: Cohomology $H^*(M)$.

Pontryagin (1930's)



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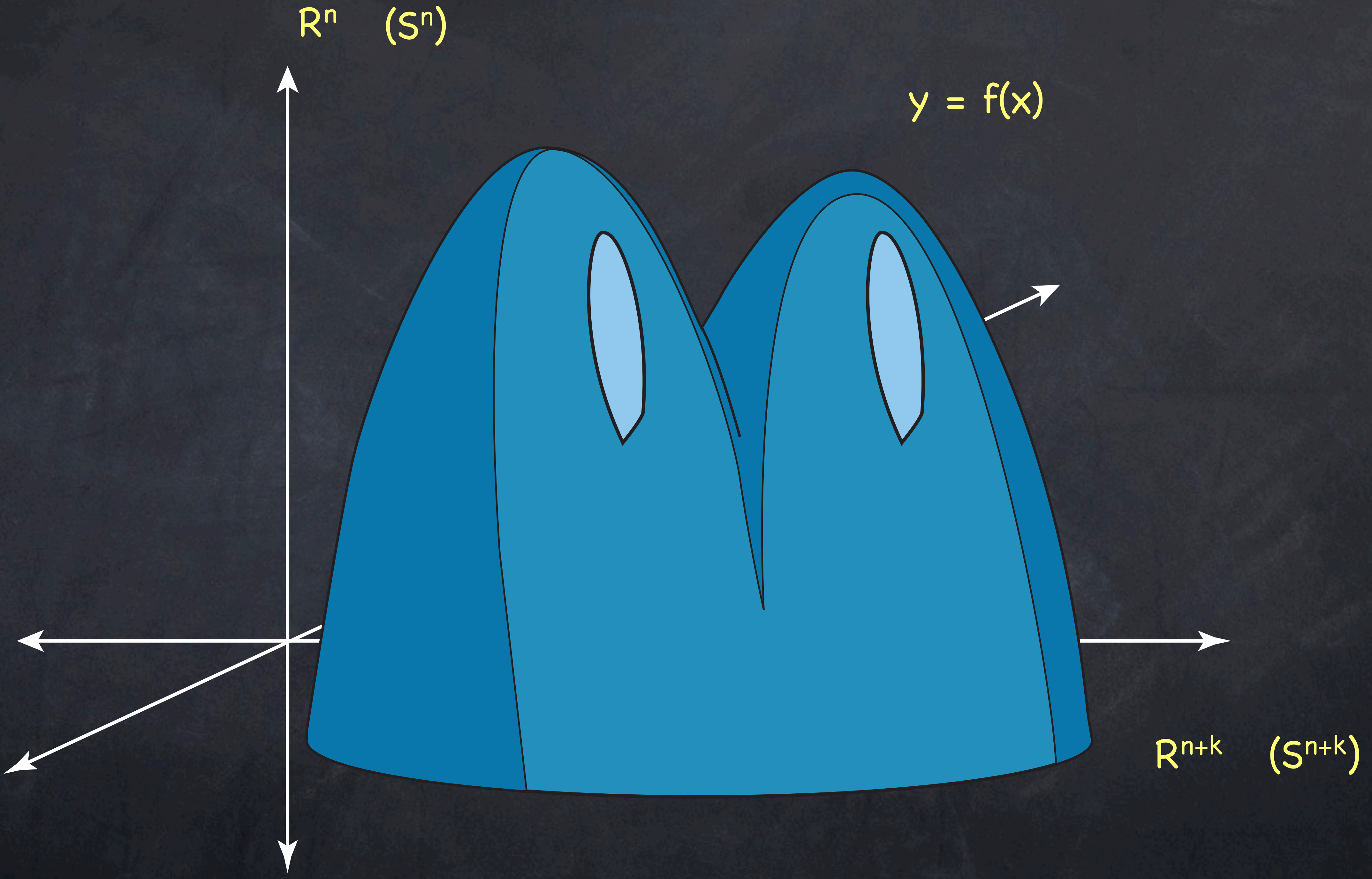
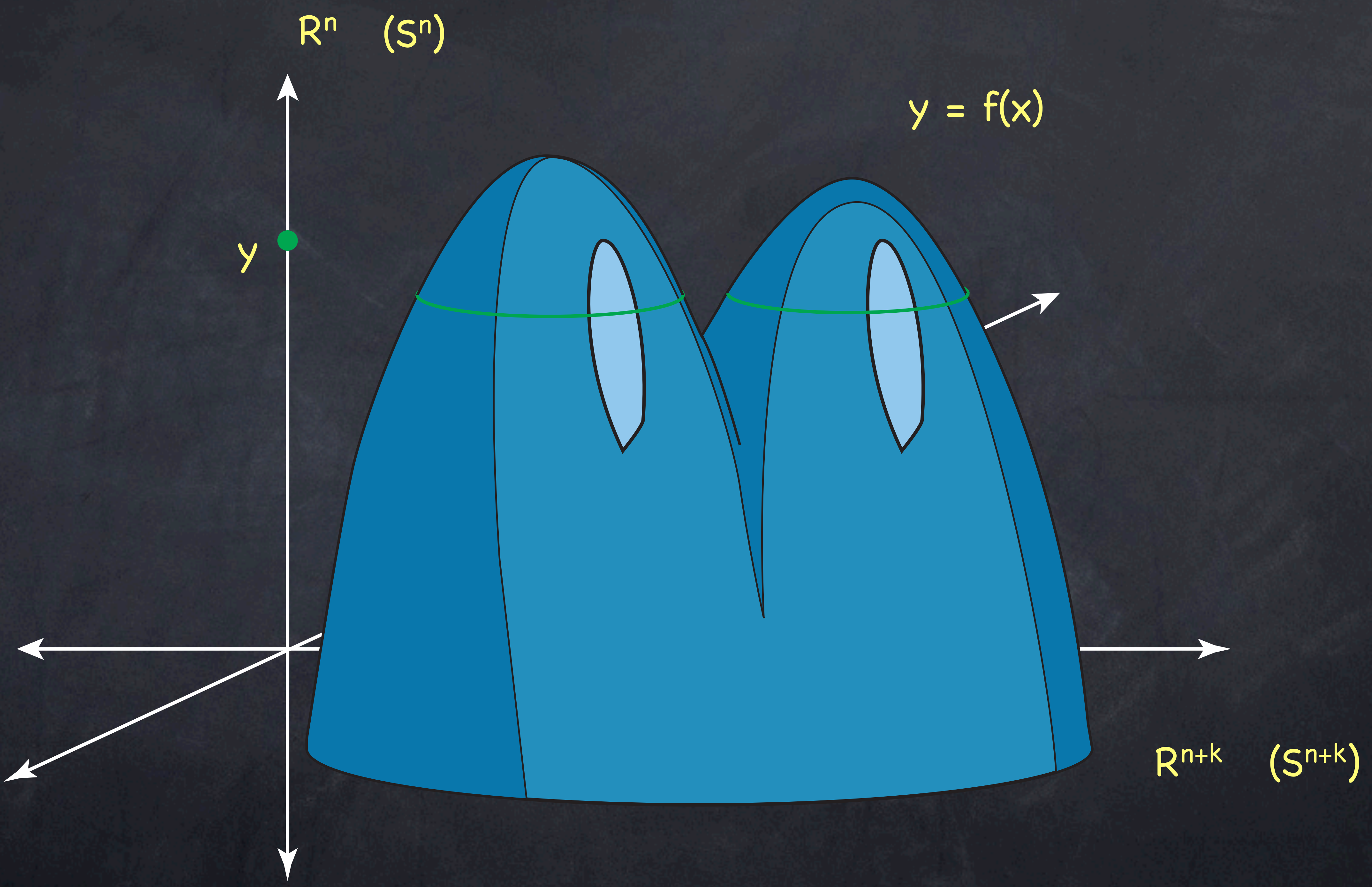


Problem: Count the solutions to systems of n equations in (n+k) unknowns

Pontryagin (1930's)



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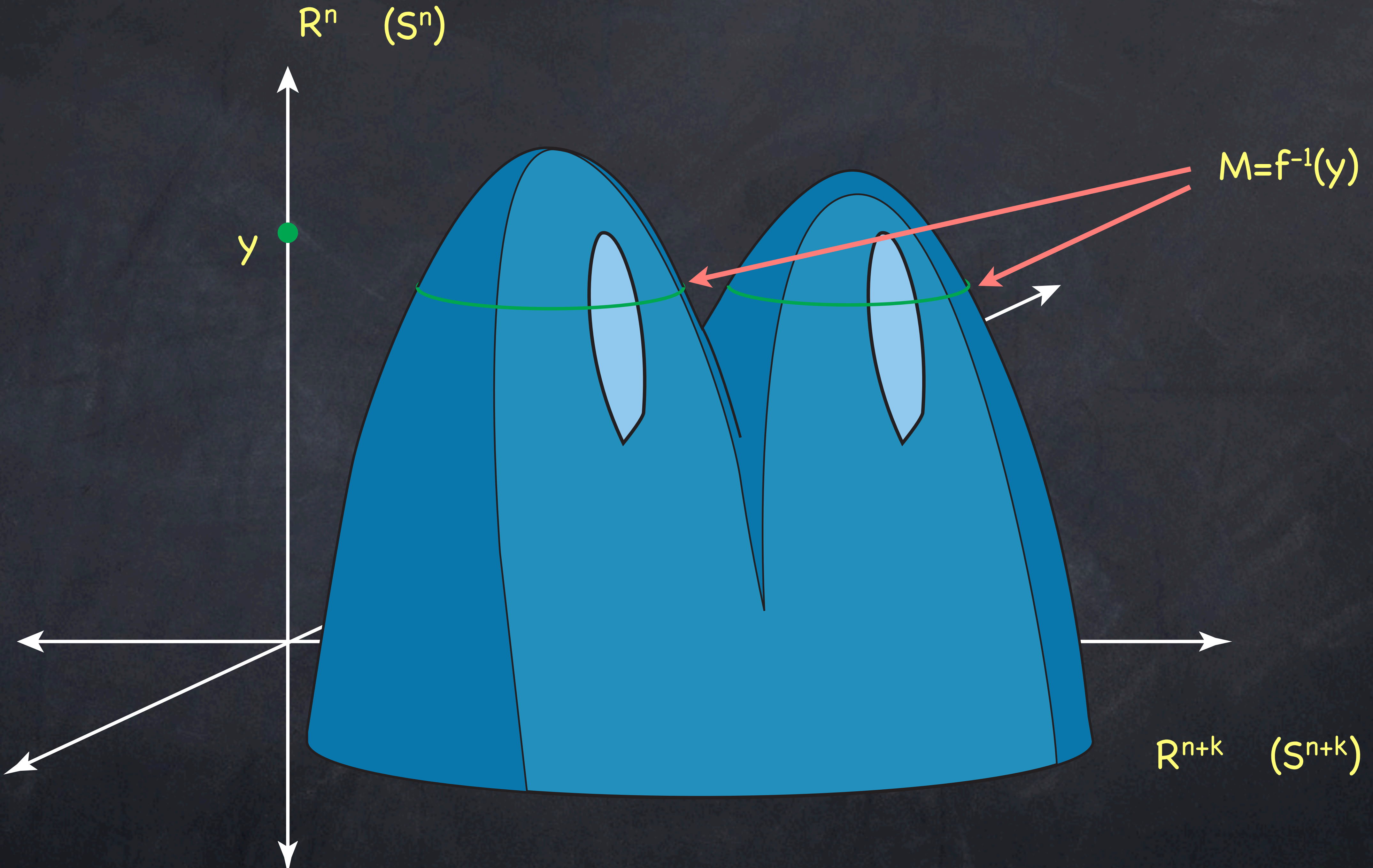

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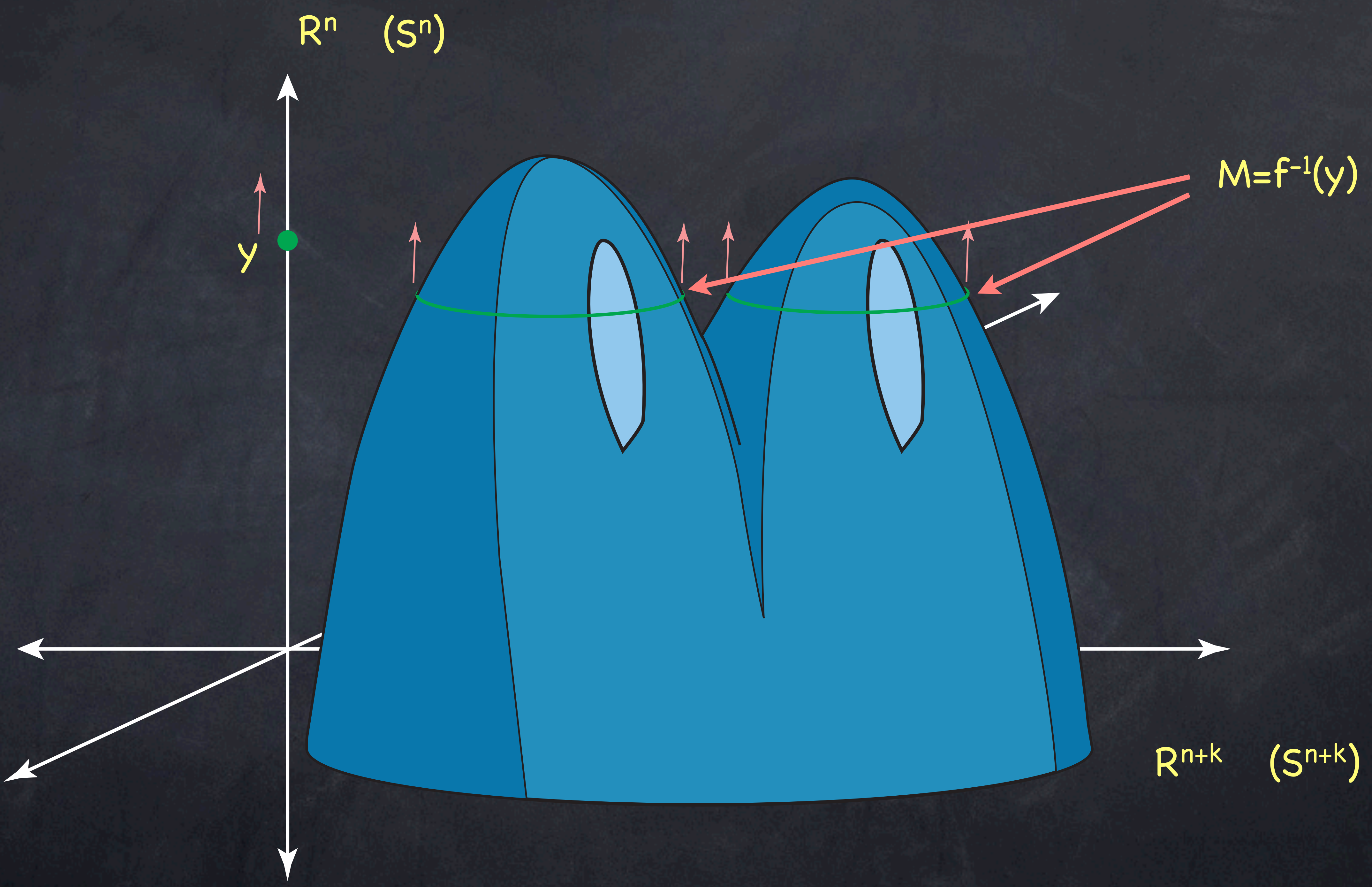

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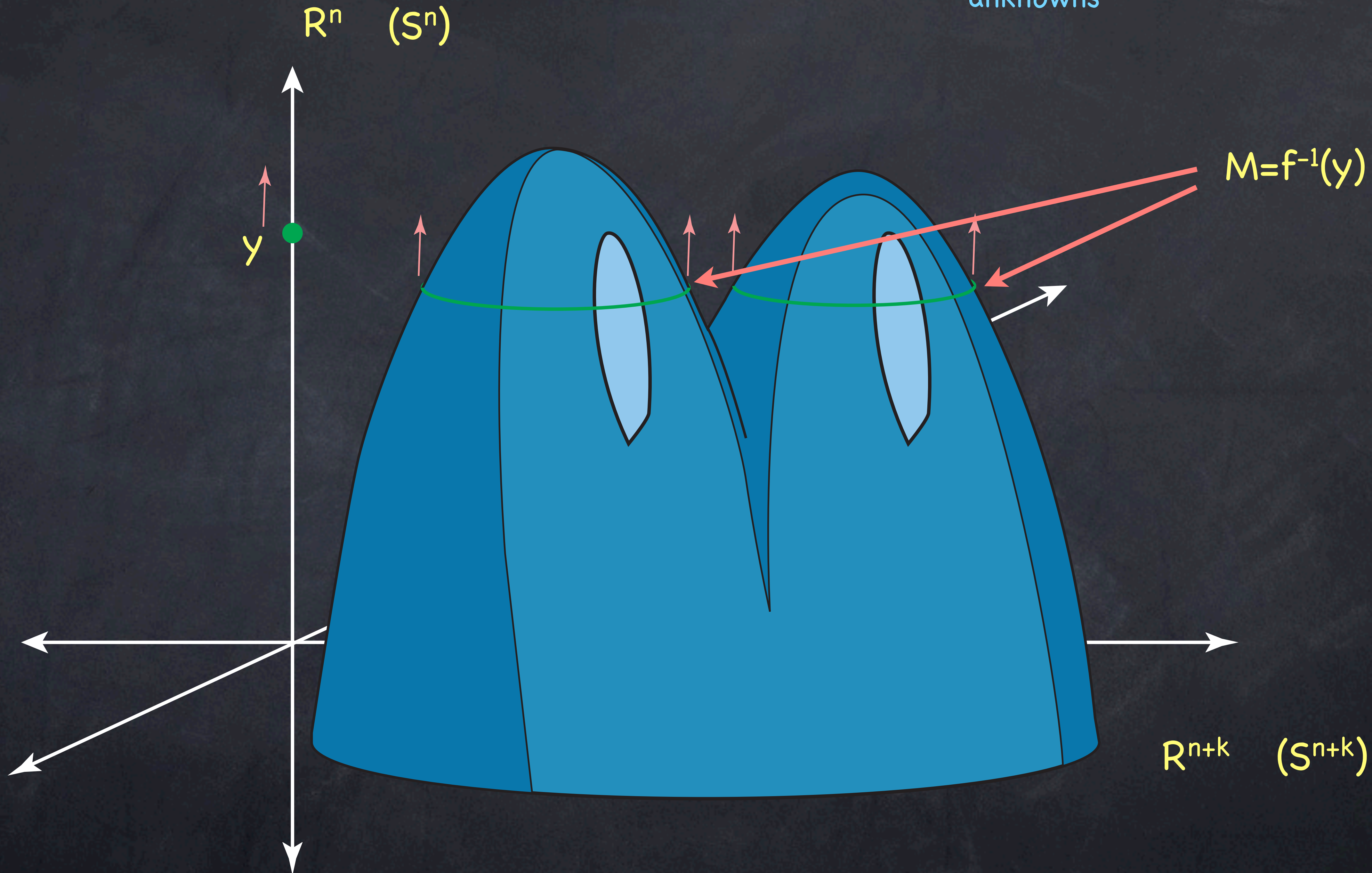

Answer: A stably framed manifold of dimension k.

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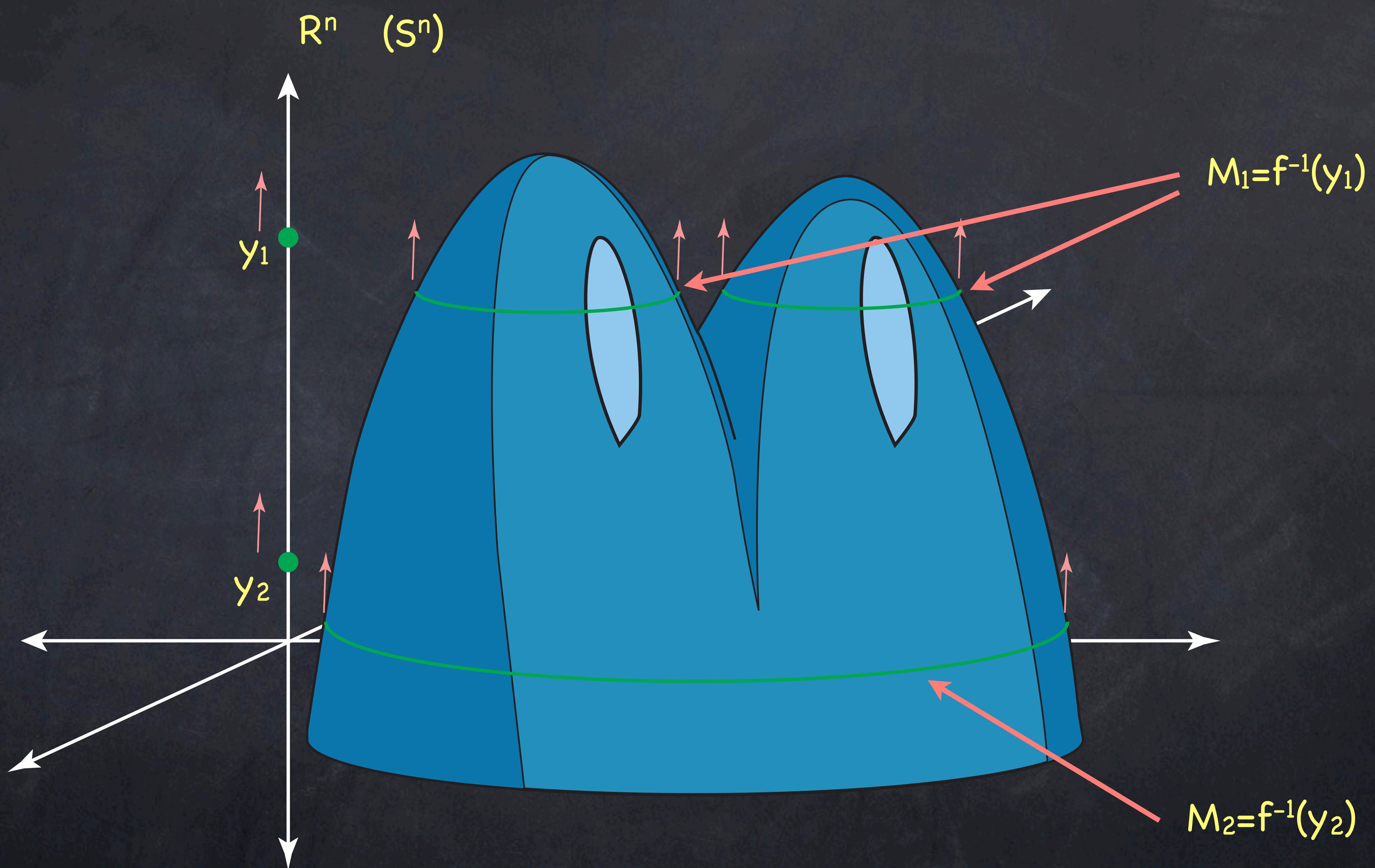

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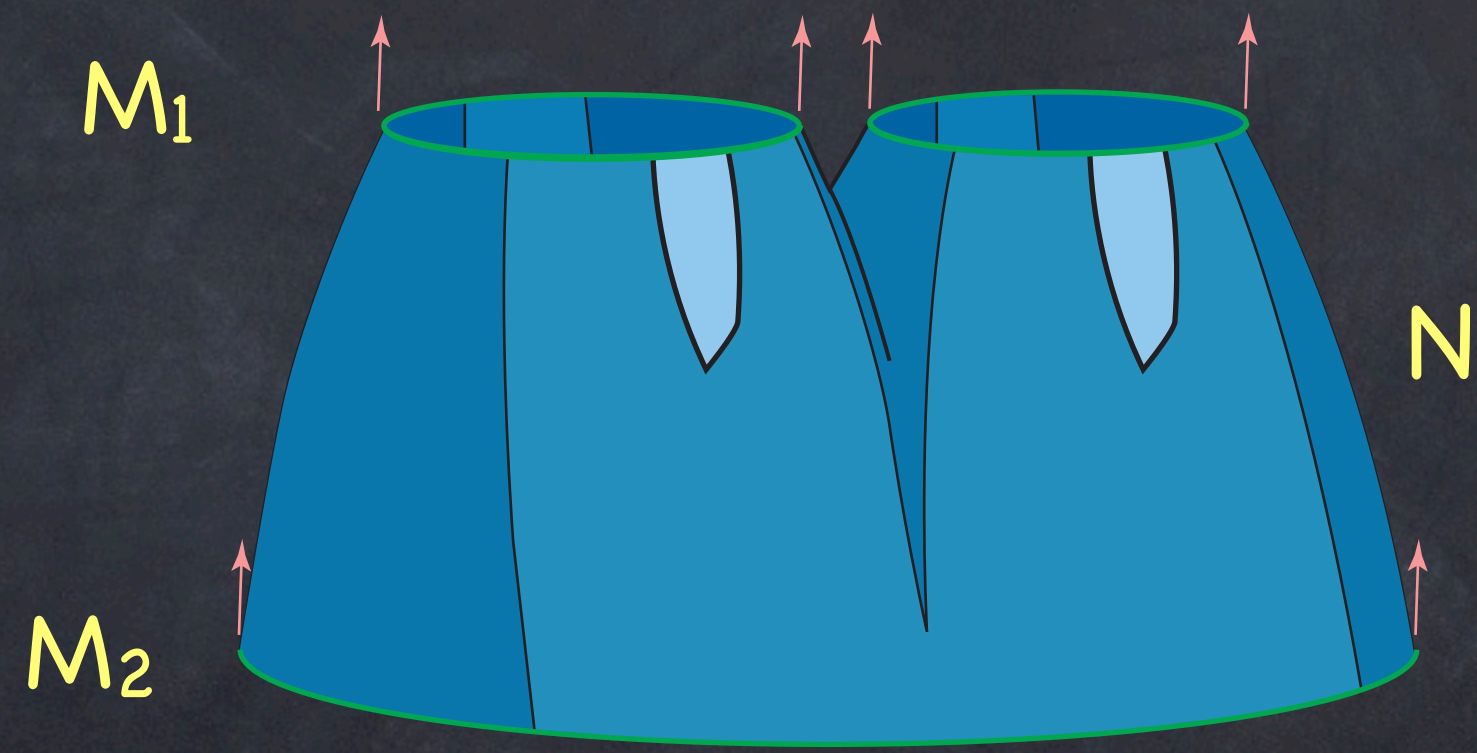
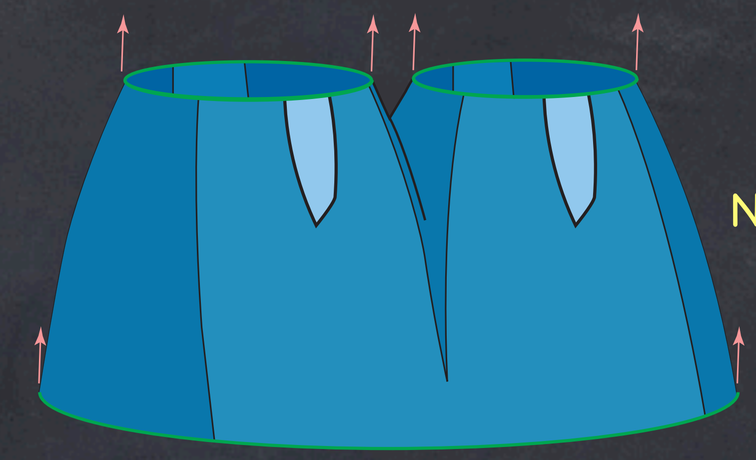


Framed cobordism

Pontryagin (1930's)



Ω_k := {stably framed k-manifolds}/ cobordism

Theorem: The above construction gives a bijection

$$\pi_{n+k}(S^n) \approx \Omega_k$$

where

$$\pi_{n+k}(S^n) := \{\text{maps } S^{n+k} \to S^n\}/_{\text{homotopy}}$$

Pontryagin (1930's)

$k=0$

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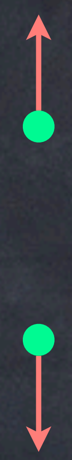

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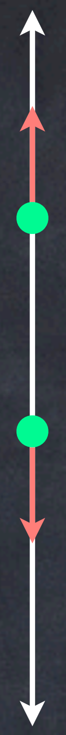

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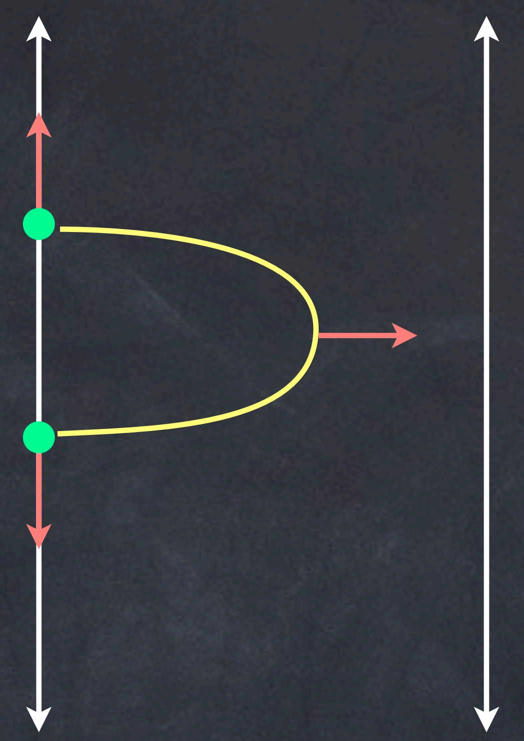

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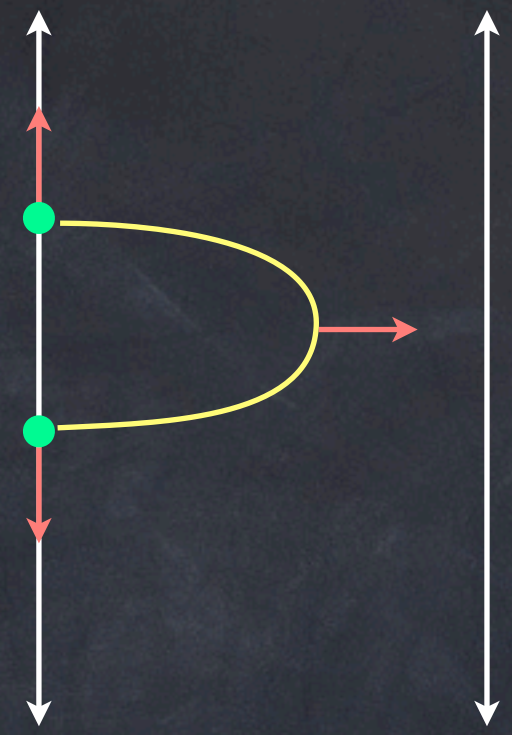

$$\pi_n(S^n) = \mathbb{Z}$$

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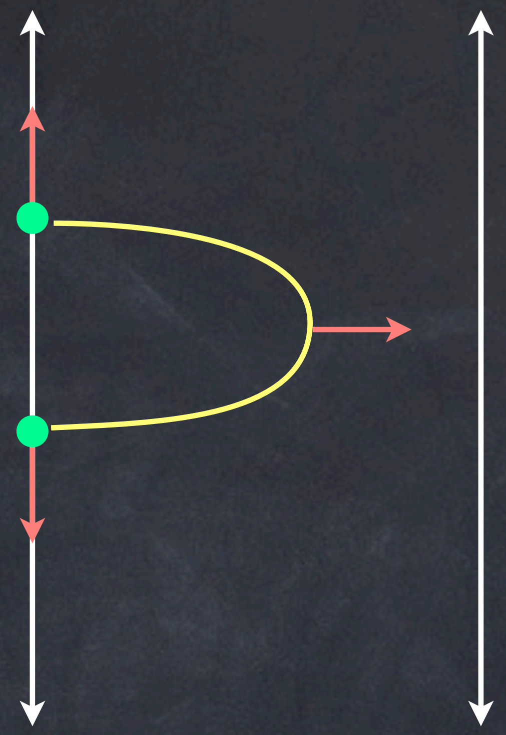

$\pi_n(S^n) = \mathbb{Z}$

(the degree)

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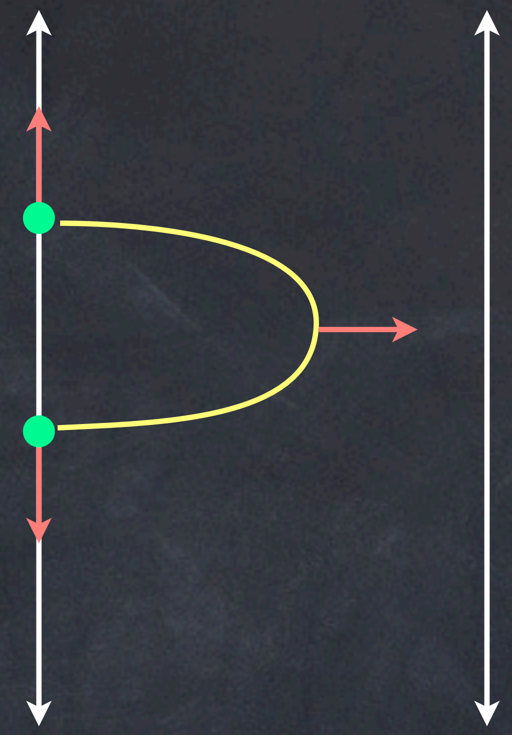

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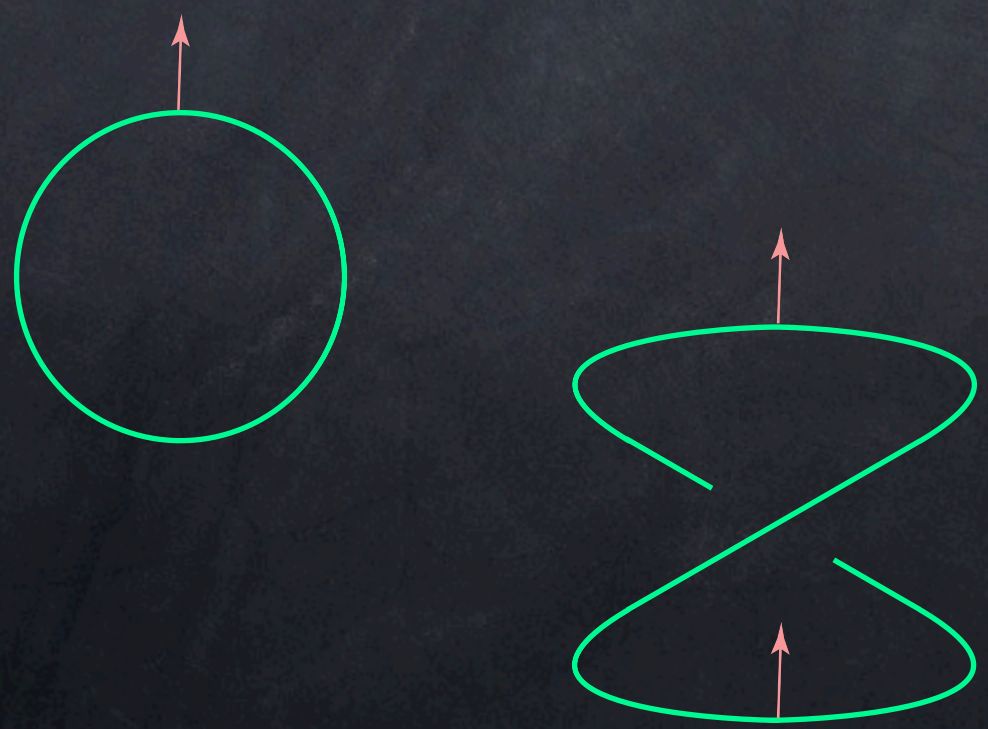

Pontryagin (1930's)

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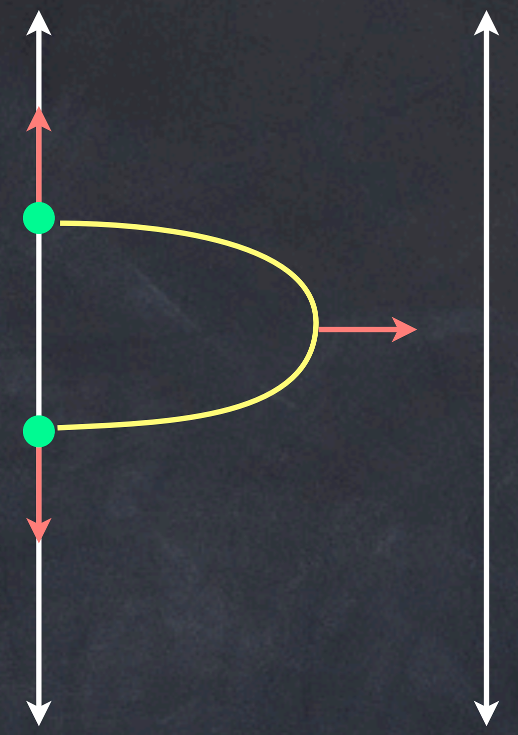
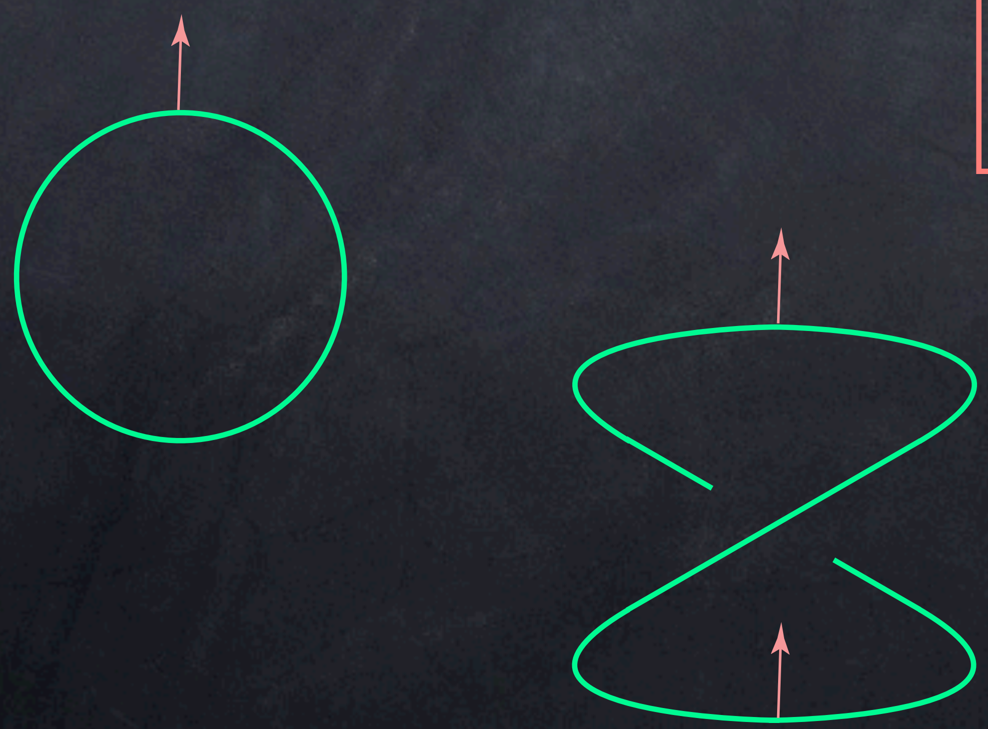


$\pi_n(S^n) = \mathbb{Z}$

(the degree)

$\pi_{n+1}(S^n) = \mathbb{Z}/2$

k=1



Pontryagin (1930's)

$k=2$

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$k=2$ genus $M = 0$ $\Rightarrow$ M is a boundary

(since S^2 bounds a disk and $\pi_2(GL_n(\mathbf{R}))=0$)

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k=2 genus M = 0 $\Rightarrow$ M is a boundary

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Suppose the genus of M is greater than 0.

Pontryagin (1930's)

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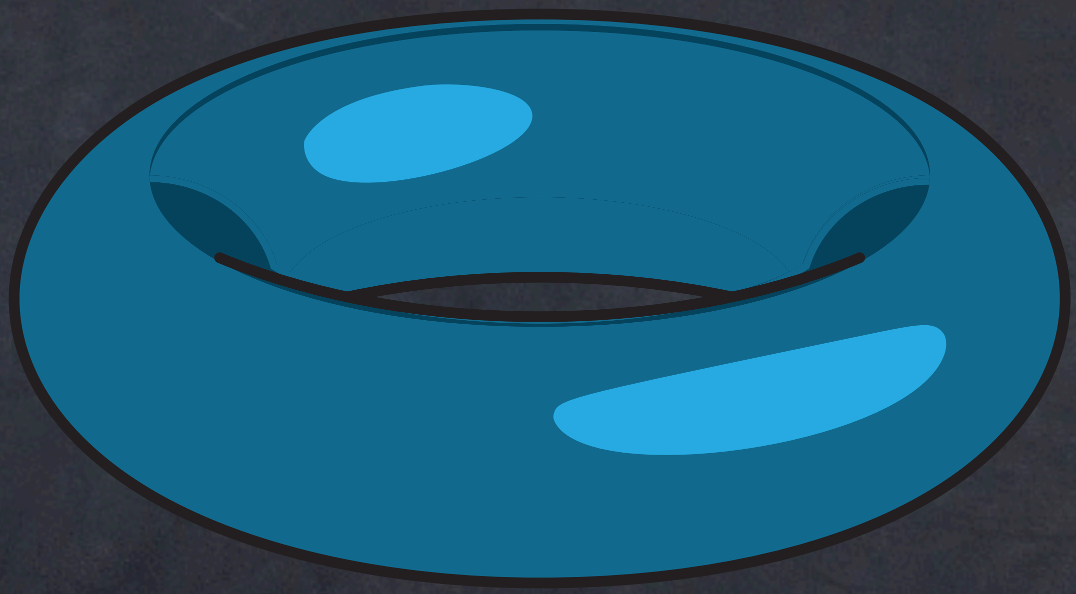

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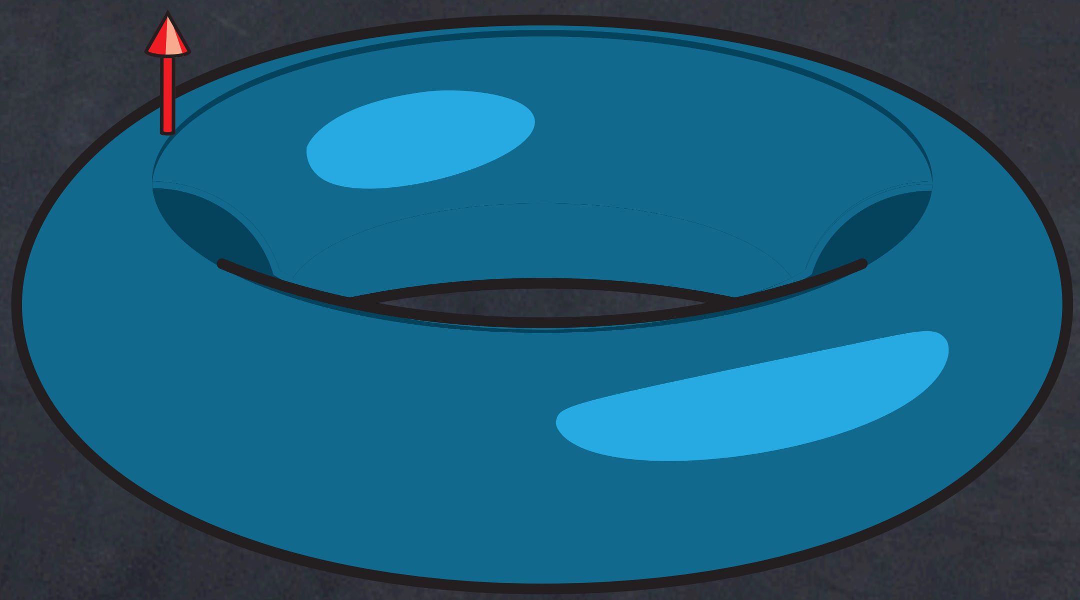

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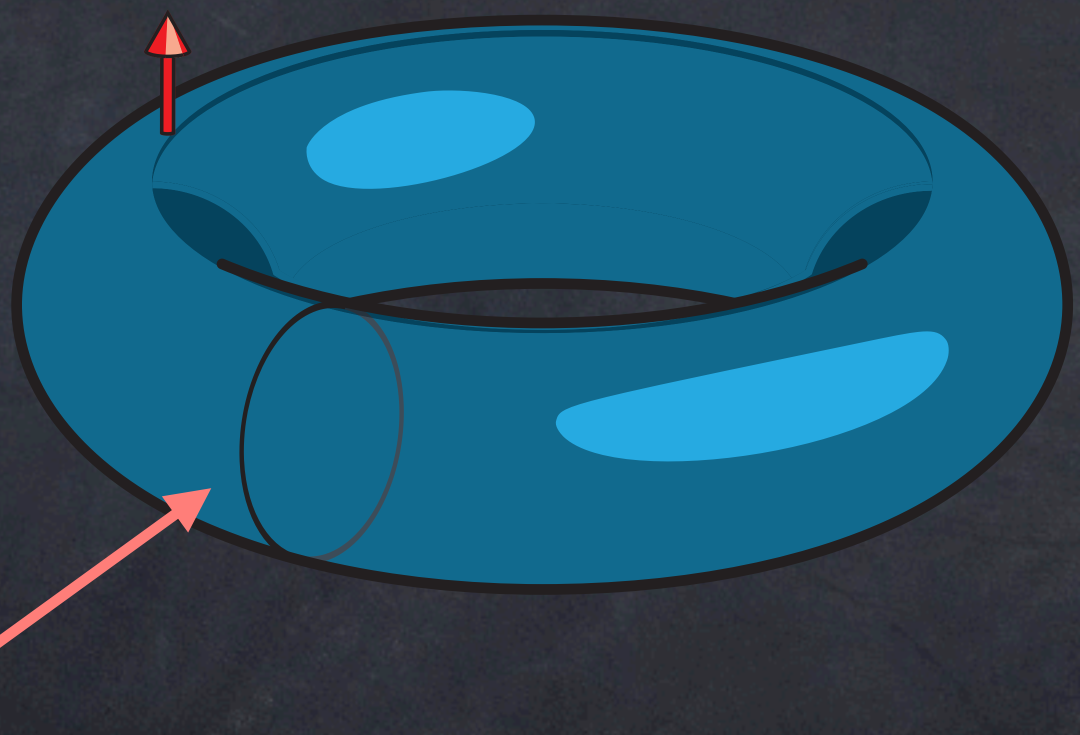

choose an
embedded arc

Pontryagin (1930's)

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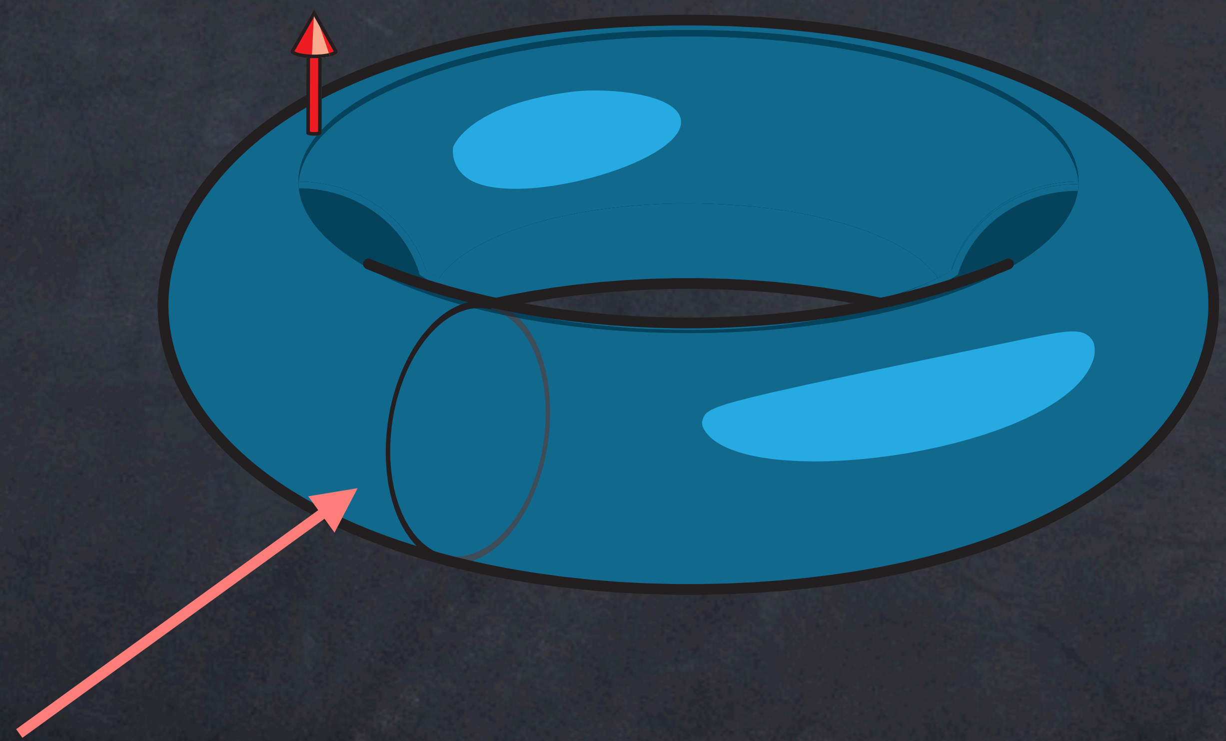
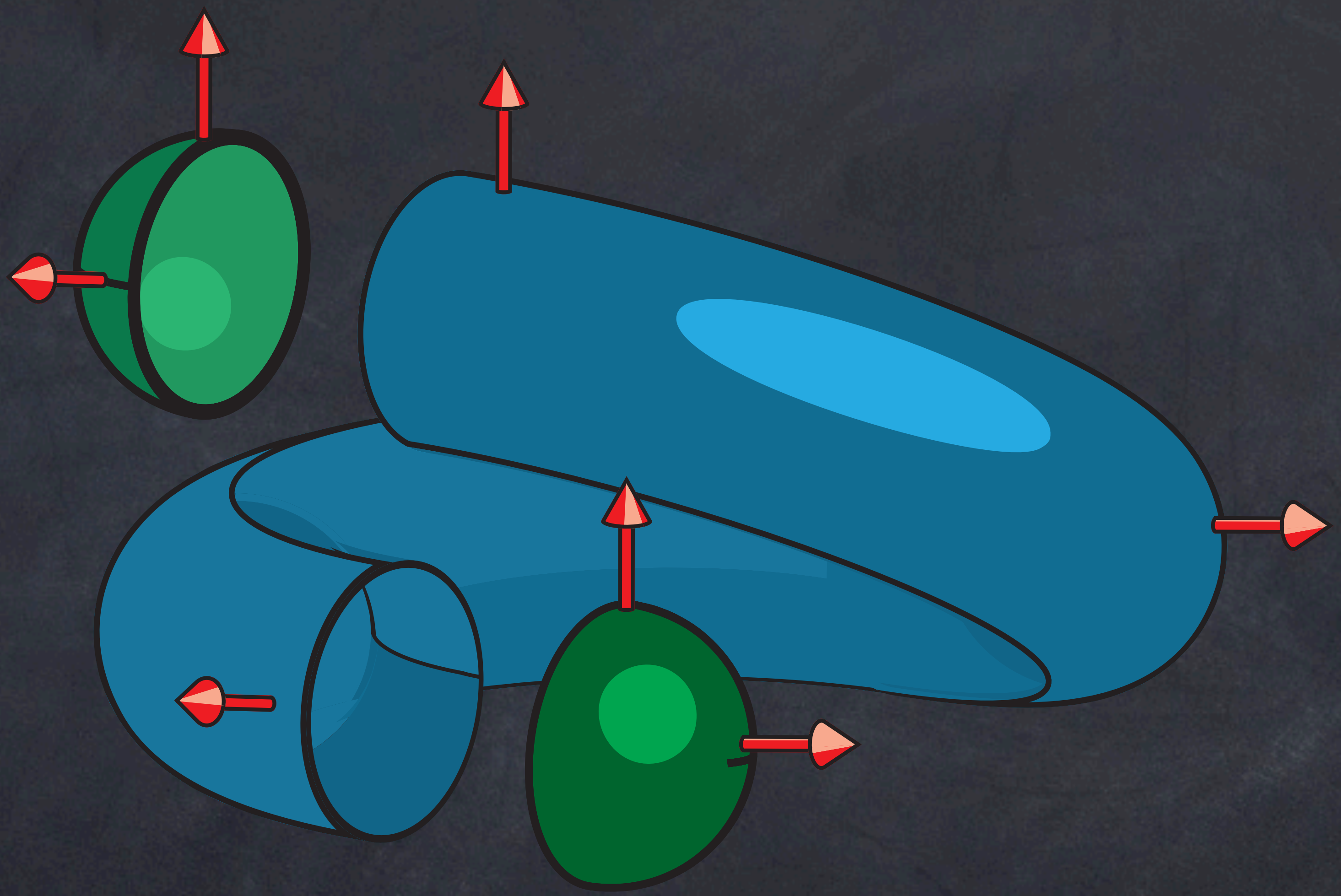


choose an
embedded arc

cut the surface open
and glue in disks

Pontryagin (1930's)

k=2



framed surgery

Pontryagin (1930's)

Obstruction: $\varphi : H_1(M; \mathbb{Z}/2) \to \mathbb{Z}/2$

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Argument: Since the dimension of $H_1(M; \mathbb{Z}/2)$ is even, there is always a non-zero element in the kernel of φ, and so surgery can be performed.

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Conclusion: $\Omega_2 = \pi_{n+2}(S^n) = 0$.

Pontryagin (1930's)

Error: The function φ is not linear:

$$\varphi(x+y) - \varphi(x) - \varphi(y) = \int_M x\, y$$

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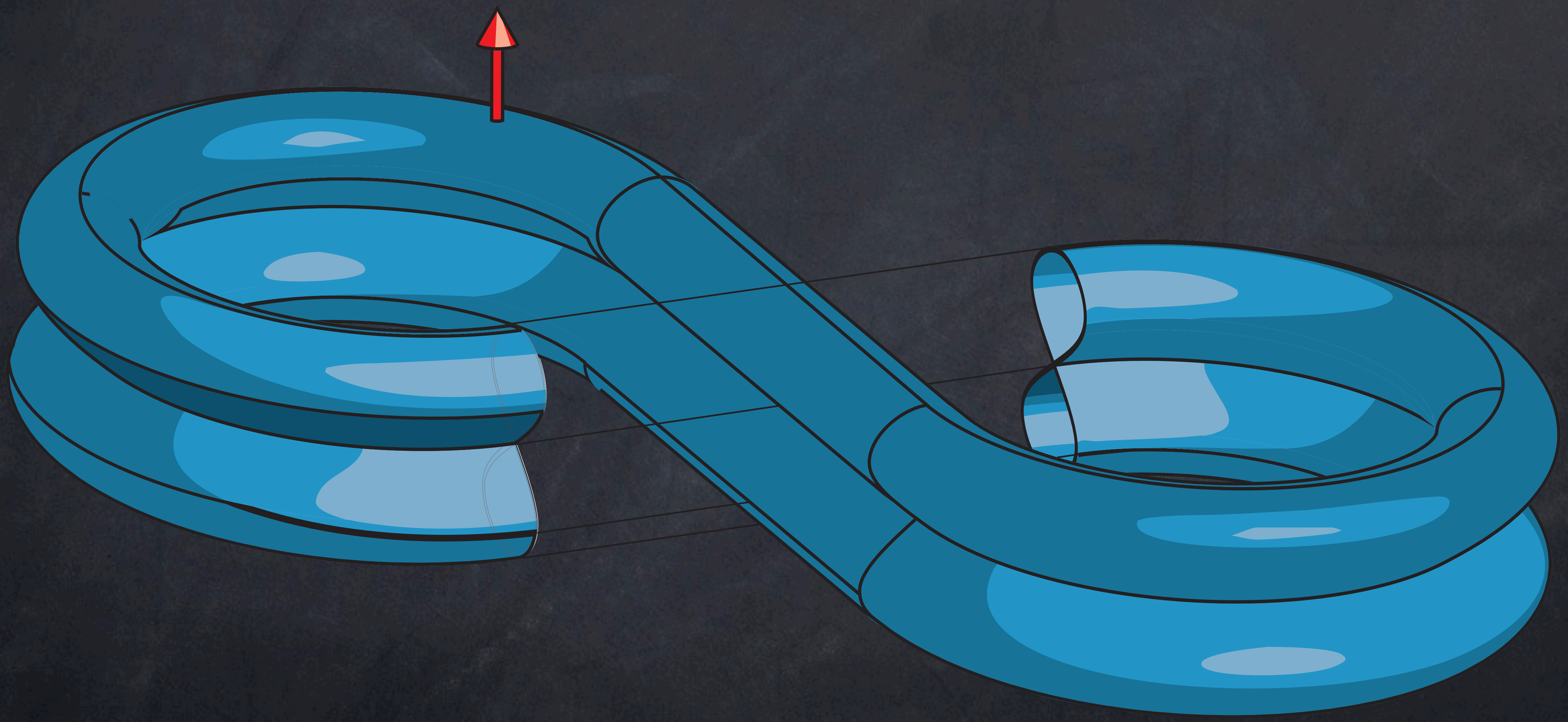
The Arf invariant of φ gives an isomorphism

$$\Omega_2 = \pi_{n+2}(S^n) = \mathbb{Z}/2$$

Pontryagin (1930's)



Pontryagin (1930's)

Question: In which dimensions every stably framed manifold cobordant to a (homotopy) sphere?

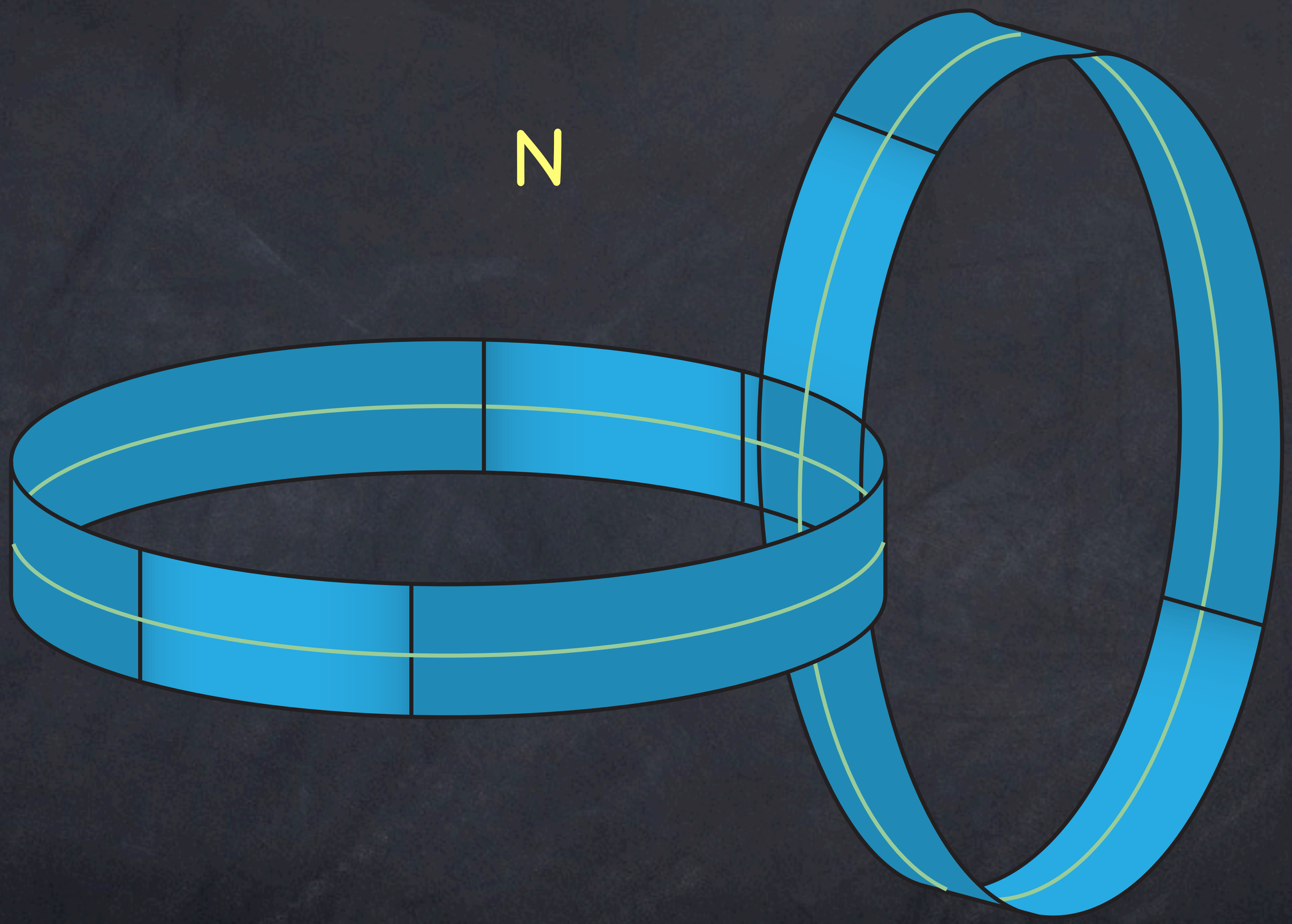
Question: In which dimensions would Pontryagin's construction have worked?

Topology circa 1960: Milnor's spheres

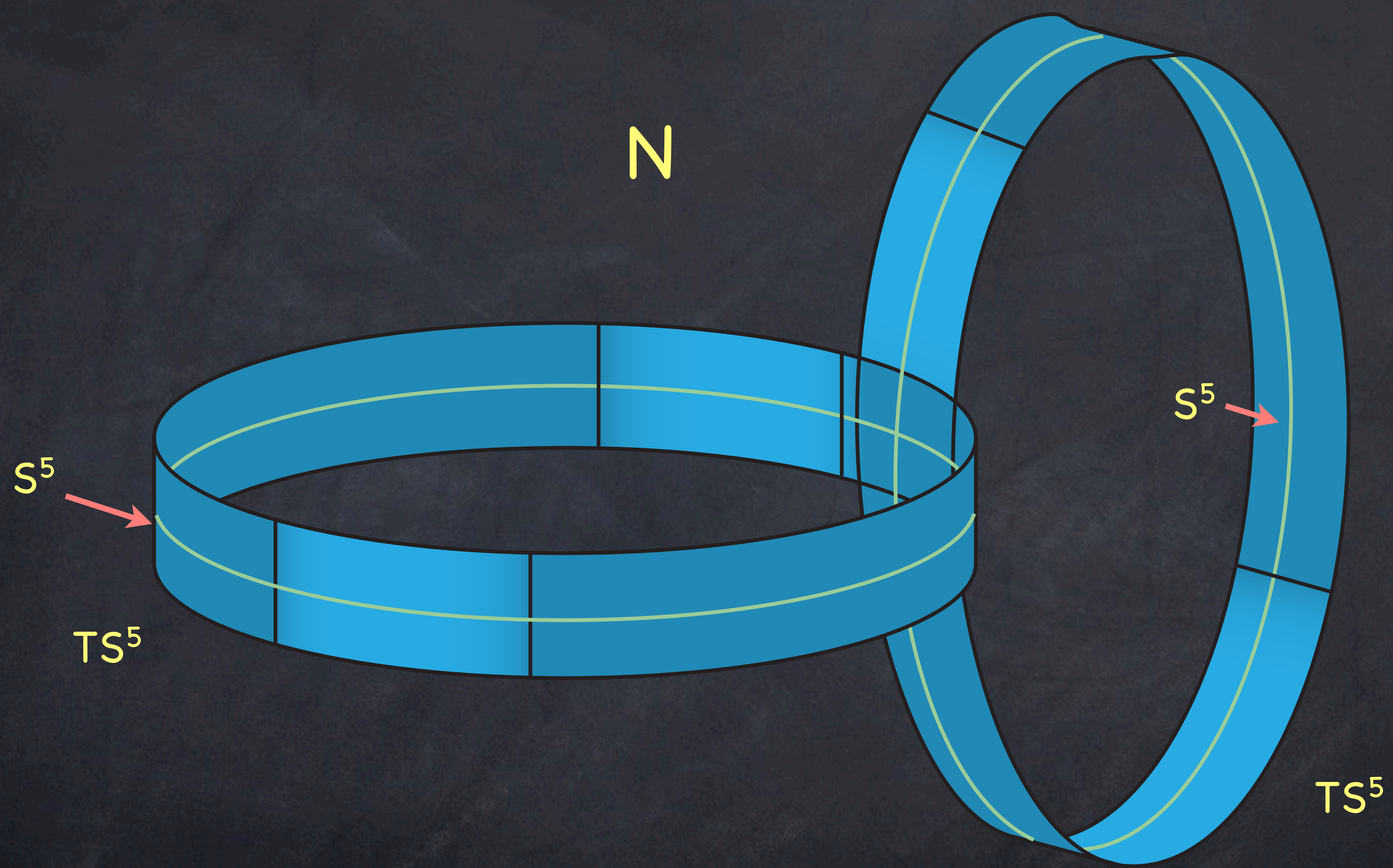
1956: Milnor gave an example of a manifold of dimension 7, homeomorphic but not diffeomorphic to the 7-sphere.

1961: Milnor introduced a generalization of Pontryagin's "surgery" maneuver, and initiated a scheme for studying differentiable structures on manifolds in other dimensions.

Topology circa 1960: Kervaire's example



Topology circa 1960: Kervaire's example



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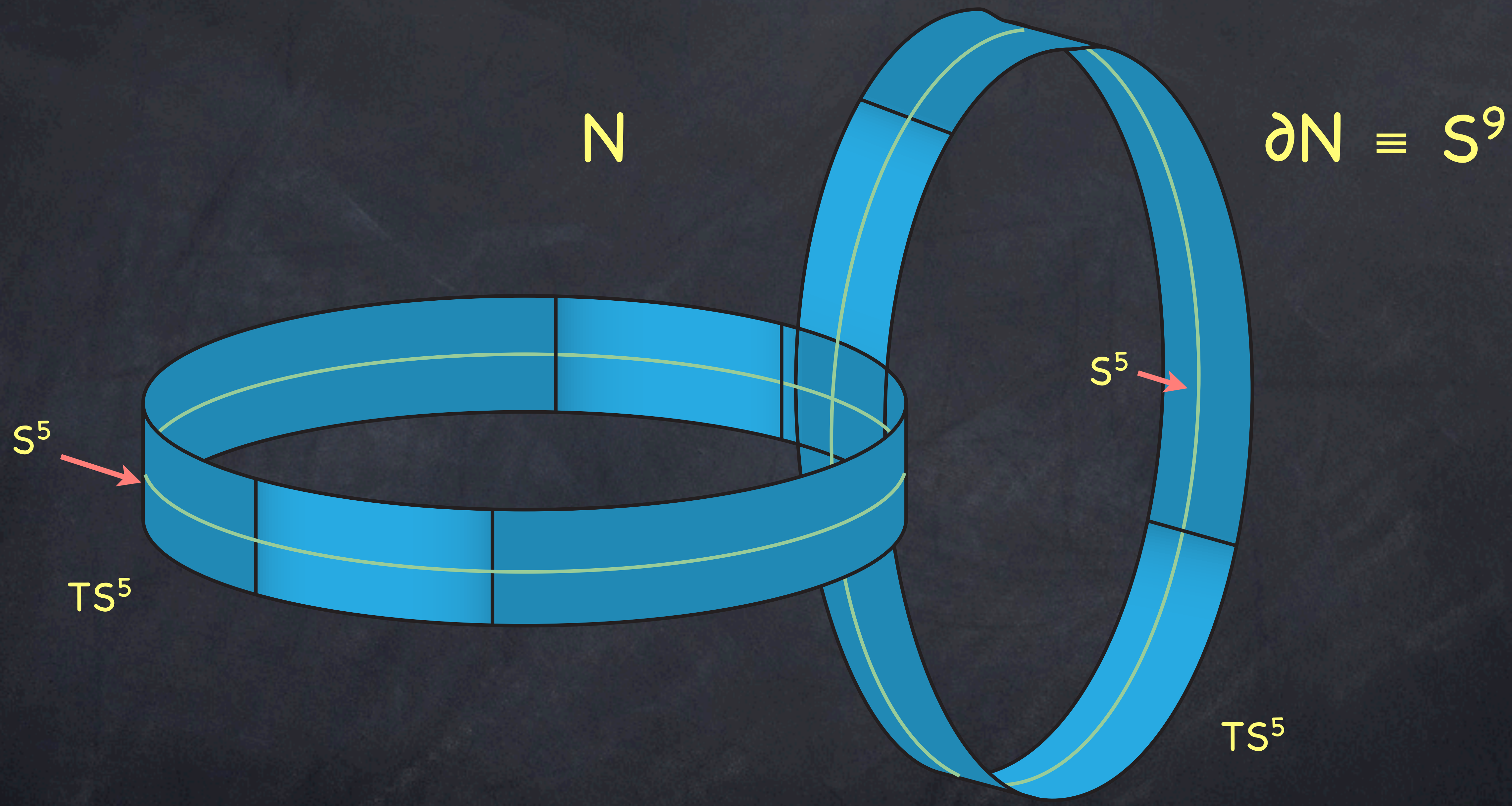

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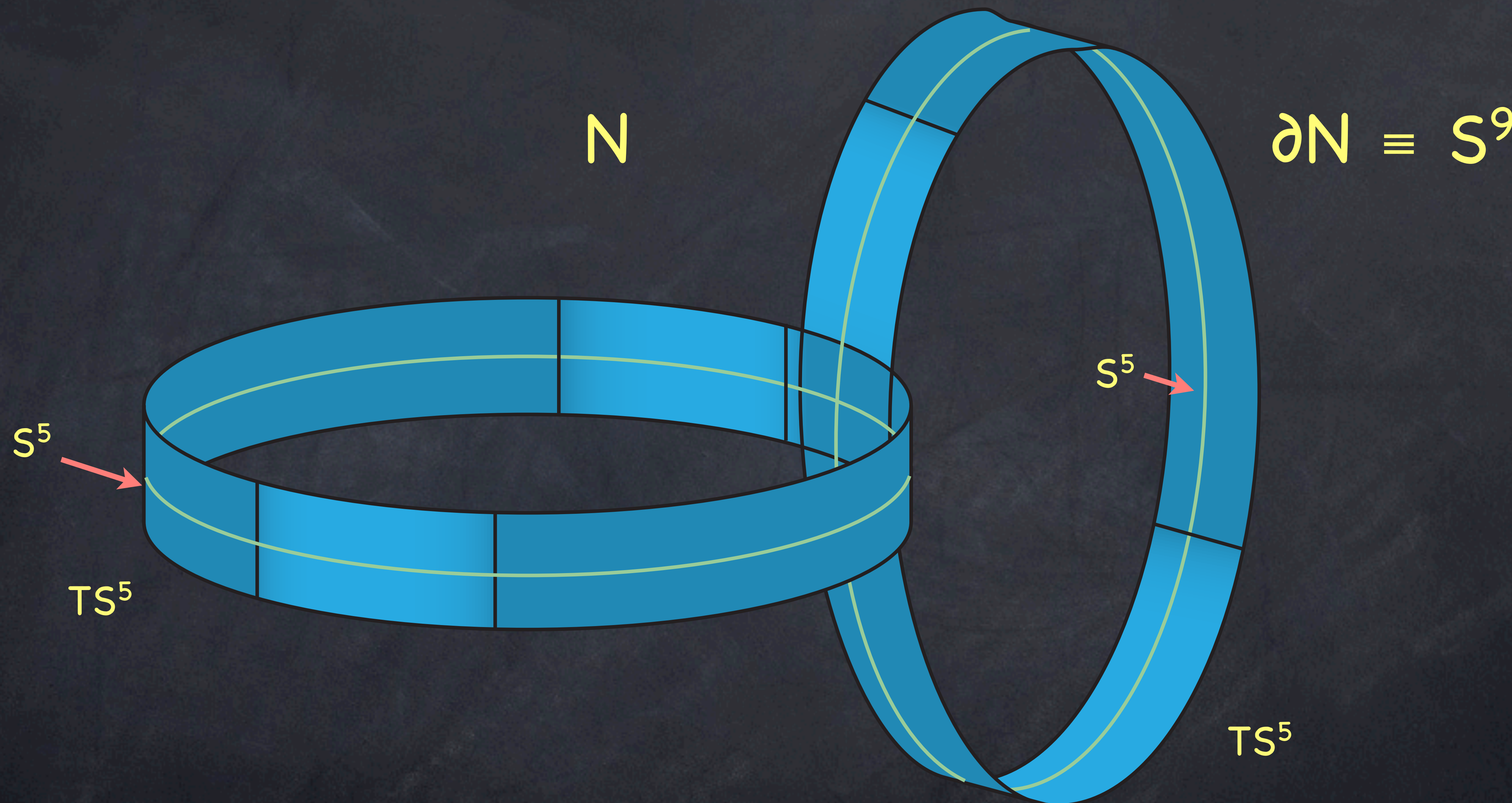

$X = N/\partial N$

(a triangulable manifold)

Topology circa 1960: Kervaire's example

1960: Kervaire defined for certain manifolds M of dimension (4k+2)

$$\varphi : H^{2k+1}(M) \to \mathbb{Z}/2$$

satisfying

$$\varphi(x+y) - \varphi(x) - \varphi(y) = \int_M x\, y$$

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He set

$$\boxed{\Phi(M) = \mathrm{Arf}(\varphi)}$$

Topology circa 1960: Kervaire's example

Theorem (Kervaire): If M is a smooth, stably framed manifold of dimensions 10 or 18, then

$$\Phi(M) = 0.$$

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Theorem (Kervaire): If X can be smoothed then $\Phi(X) = 1$.

Topology circa 1960: Kervaire's example

Theorem (Kervaire): If M is a smooth, stably framed manifold of dimensions 10 or 18, then

$$\Phi(M) = 0.$$

Theorem (Kervaire): If X can be smoothed then $\Phi(X) = 1$.

Corollary (Kervaire): The triangulable manifold X has no smooth structure.

Topology circa 1960

The number $\Phi(M)$ is called the Kervaire invariant of M.

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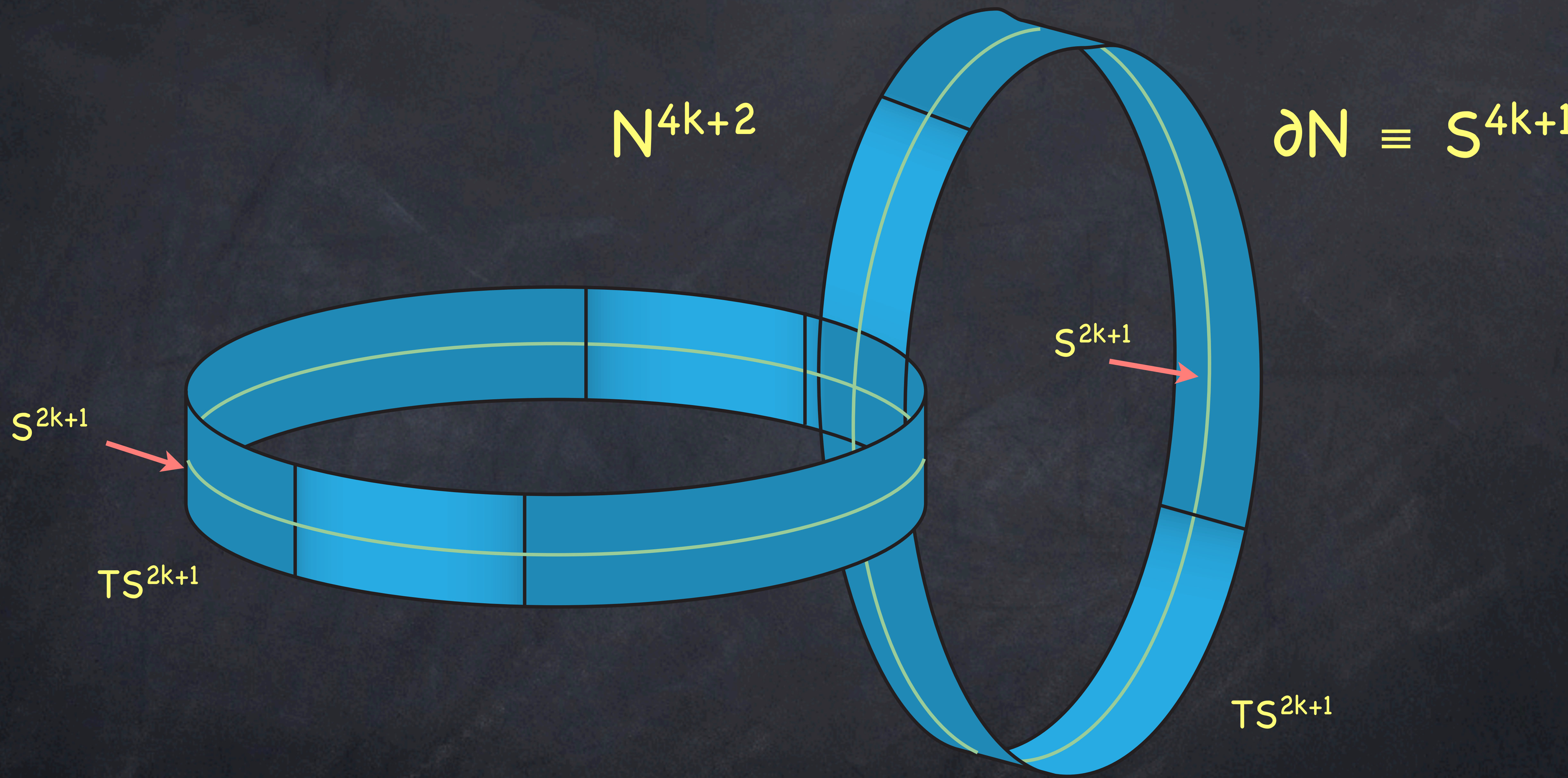
The Kervaire invariant problem

Topology circa 1960: Kervaire's example



$$X^{4k+2} = N/\partial N$$

Topology circa 1960

Question: In which dimensions can X^{4k+2} be given a smooth structure? In which dimensions is ∂N^{4k+1} diffeomorphic to the sphere S^{4k+1}?

Topology circa 1960

Question: In which dimensions can X^{4k+2} be given a smooth structure? In which dimensions is ∂N^{4k+1} diffeomorphic to the sphere S^{4k+1}?

Remark: If X^{4k+2} can be given a smooth structure, it then becomes a stably framed manifold which is not cobordant to a homotopy sphere.

Kervaire and Milnor (1958, 1963)

Exotic spheres

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Definition: The group Θ_n is the group of homotopy n-spheres, up to h-cobordism.

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Exotic spheres

Definition: The group Θ_n is the group of homotopy n-spheres, up to h-cobordism.

The group structure is connected sum.

Kervaire and Milnor (1958, 1963)

dim > 4 $\Rightarrow$

h-cobordism = diffeomorphism (Smale)

and (Smale, Stallings, Zeeman)

homotopy sphere $\Leftrightarrow$ topological sphere.

Kervaire and Milnor (1958, 1963)

dim > 4 $\Rightarrow$

h-cobordism = diffeomorphism (Smale)

and (Smale, Stallings, Zeeman)

homotopy sphere $\Leftrightarrow$ topological sphere.

Θ_n = the group of diffeomorphism classes of smooth structures on the n-sphere.

Kervaire and Milnor (1958, 1963)

Theorem: The order of Θ_{4m-1} is given by

$$|\Theta_{4m-1}| = a_m \, |\pi_{4m-1+n} S^n| \, 2^{2m-4} (2^{2m-1}-1) B_m/m$$

with

B_m = m^{th} Bernoulli number

a_m = 1 if m is even, 2 if m is odd

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They were unable to settle a factor of 2 in the order of Θ_{4m+1} and Θ_{4m+2} (Kervaire's ∂N and X).

Kervaire and Milnor (1958, 1963)

Question: What are the orders of the groups

Θ_{4m+1} and Θ_{4m+2}?

Methods of homotopy theory

1966: The state of the art

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Theorem (Kervaire ’60): If the dimension of M is 10 or 18 then $\Theta(M) = 0$

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Theorem (Kervaire '60): If the dimension of M is 10 or 18 then $\Theta(M) = 0$

$\Theta(M)$ can be 1 for $M = S^1 \times S^1, S^3 \times S^3, S^7 \times S^7$ (dimensions 2, 6, 14).

Methods of homotopy theory

1966: The state of the art

Theorem (Kervaire '60): If the dimension of M is 10 or 18 then $\Theta(M) = 0$

$\Theta(M)$ can be 1 for $M = S^1 \times S^1$, $S^3 \times S^3$, $S^7 \times S^7$ (dimensions 2, 6, 14).

Kervaire and Milnor speculated that $\Theta(M)$ can be non-zero only in these three dimensions.

Methods of homotopy theory

Theorem (Brown-Peterson '65, '66): If M is a stably framed manifold of dimension (8k+2) then the Kervaire invariant of M is zero.

Methods of homotopy theory

Theorem (Brown-Peterson ’65, ’66): If M is a stably framed manifold of dimension $(8k+2)$ then the Kervaire invariant of M is zero.

(This extends Kervaire’s sequence of 10 and 18.)

Methods of homotopy theory

Theorem (Browder '69): If $\Phi(M)$ is non-zero then the dimension of M is of the form $(2^{j+1} - 2)$.

Methods of homotopy theory

Theorem (Browder '69): If $\Phi(M)$ is non-zero then the dimension of M is of the form $(2^{j+1} - 2)$.

In this dimension a manifold M with Kervaire invariant 1 exists if and only if there is an element ϑ_j in $\pi_{2^{j+1}-2} S^0$ represented at the E_2-term of the classical Adams spectral sequence by h_j^2.

Methods of homotopy theory

Theorem (Barratt, Mahowald, Tangora, Jones '70-'84):

The elements ϑ_j exist for j ≤ 5.

Methods of homotopy theory

Theorem (Barratt, Mahowald, Tangora, Jones '70-'84):

The elements ϑ_j exist for j ≤ 5.

(the overwhelming belief was that all ϑ_j exist)

Four Questions

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Question: In which dimensions can there exist a (stably framed) manifold of Kervaire invariant one?

Answer: In all dimensions dimensions except possibly

2, 6, 14, 30, 62, and 126.

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Question: What are $|\Theta_{4m+1}|$ and $|\Theta_{4m+2}|$?

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Question: What are $|\Theta_{4m+1}|$ and $|\Theta_{4m+2}|$?

Answer: Unless (4m+2) is of the form

2, 6, 14, 30, 62, or 126,

the group Θ_{4m+1} is twice as large as it might have been, while Θ_{4m+2} is half as large as it might have been.

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Kervaire's X^{4k+2} has no smooth structure, and ∂N^{4k+2} is an exotic sphere.

Four Questions

Question: For which j does ϑ_j exist?

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Answer: For j = 1, 2, 3, 4, 5 and possibly 6.

A finite set of things

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The universe was created in 6 days.

Day 1: ϑ_1 Day 2: ϑ_2 Day 3: ϑ_3

Day 4: ϑ_4 Day 5: ϑ_5 Day 6: ϑ_6

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Birthdays. Hill: almost 30 Ravenel: 62

Hopkins+Browder: 126

A finite set of things

Possible connections (Bökstedt and others)

$$E_6 \leftrightarrow \vartheta_4$$

$$E_7 \leftrightarrow \vartheta_5$$

$$E_8 \leftrightarrow \vartheta_6$$

Outline of the proof

Cohomology theory: Ω

(like in the study of the degree)

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(like in the study of the degree)

General properties:

contravariant functor $X \mapsto \Omega^*(X)$

suspension ismorphism $\Omega^{*+n}(\Sigma^n X) \approx \Omega^*(X)$

Outline of the proof

The Ω degree:

$$
\begin{array}{ccc}
\Omega^n(S^{n+2^{j+1}-2}) & \xleftarrow{\vartheta_j} & \Omega^n(S^n) \\
\big\downarrow & & \big\uparrow \\
\Omega^{2-2^{j+1}}(pt) & & \Omega^0(pt)
\end{array}
$$

Outline of the proof

The Ω degree:

$$\begin{array}{ccc}
\Omega^n(S^{n+2^{j+1}-2}) & \xleftarrow{\vartheta_j} & \Omega^n(S^n) \\
\downarrow & & \uparrow \\
\Omega^*(\vartheta_j) \in \Omega^{2-2^{j+1}}(pt) & & \Omega^0(pt) \ni 1
\end{array}$$

Outline of the proof

Detection Theorem: If ϑ_j exists then $\Omega^*(\vartheta_j)$ is a non-zero element of $\Omega^{2-2^{j+1}}(pt)$.

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Periodicity Theorem: The cohomology theory Ω is periodic: for any X

$$\Omega^{*+256}(X) \approx \Omega^*(X).$$

Outline of the proof

Detection Theorem: If ϑ_j exists then $\Omega^*(\vartheta_j)$ is a non-zero element of $\Omega^{2-2^{j+1}}(pt)$.

Periodicity Theorem: The cohomology theory Ω is periodic: for any X

$$\Omega^{*+256}(X) \approx \Omega^*(X).$$

Gap Theorem: The groups $\Omega^*(pt)$ are zero for

$$0 < * < 4$$

Outline of the proof

Detection Theorem: Inventory a table of long-known computations.

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A large class of naturally constructed cohomology theories satisfy periodicity and gap theorems (the gap is always the same, the period varies). We choose Ω to be one with the smallest period but large enough to satisfy the Detection Theorem.

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