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Author(s): Norihiko Minami

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ON THE CHROMATIC TOWER

By Norihiko Minami

Dedicated to Professor Yosimura for his 60th birthday

Abstract. We fix a prime p and work in the p-local stable homotopy category. Then, Hopkins' chromatic splitting conjecture essentially predicts the information of a p-completed finite spectrum, obtained using the first $(n+1)$ Morava K-theories $K(0),\ K(1), \dots, K(n)$, may be obtained using a single higher Morava K-theory $K(m)$. $(m \geq n+1)$. However, in spite of its importance, this conjecture is very difficult and subtle. Actually, Devinatz noted the conjecture is false, as soon as we omit the finiteness assumption to include such a nice infinite spectrum as the p-completed BP spectrum. In this paper, we prove a result which reconciles Hopkins' chromatic splitting conjecture and Devinatz' observation about the p-completed BP spectrum. For the p-completion of "nice" spectra, including finite spectra and the BP-spectrum, our result essentially claims that the information obtained using the first $(n+1)$ Morava K-theories $K(0),\ K(1), \dots, K(n)$, may be obtained using any $m-k$ consecutive higher Morava K-theories $K(k+1),\ K(k+2), \dots, K(m-1),\ K(m)$ with $m-k \geq n+s_0+1$. Here, n_0 is the Hopkins-Ravenel (Hovey-Sadofsky) uniform horizontal vanishing line for the $E(n)$-based standard Adams-Novikov spectral sequence.

1. Introduction. In this paper, we fix a prime p, and we work in the p-local stable homotopy category.

In mid 70's, Miller-Ravenel-Wilson [MRW77] introduced the chromatic spectral sequence to compute the E_2-term of the Adams-Novikov spectral sequence for the stable homotopy groups of the sphere. Although this was originally algebraic, its geometric interpretation and realizations were soon given [JY80, Rav84, Rav87] in terms of the localization of spectra with respect to homology, invented by Bousfield [Bou79b]. This point of view was central in the success of the chromatic technology [Rav84, DHS88, HS98, Rav92]. To review necessary results, we recall some standard notations of Bousfield's localization of spectra:

$$\begin{aligned} L_E F &= \text{Bousfield localization of a spectrum } F \text{ with respect to} \\ &\quad\ \text{the homology theory } E_*, \text{ defined by a spectrum } E, \\ L_n F &= L_{E(n)} F \\ &= L_{K(0)\vee K(1)\vee\cdots\vee K(n)} F \qquad \text{[Rav84, 2.1.(d)]}, \end{aligned}$$

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where $E(n)$ is the Johnson-Wilson spectrum [JW73] with $E(n)_* = \mathbb{Z}_{(p)}[v_1, \ldots, v_{n-1}][v_n, v_n^{-1}]$ and $K(n)$ is the nth Morava K-theory spectrum [Mor89] with $K(n)_* = \mathbb{F}_p[v_n, v_n^{-1}]$ for $n > 0$ and $K(0) = H_{\mathbb{Q}}$, the rational Eilenberg-MacLane spectrum.

Then the natural maps [JY80, Rav84]

$$\begin{array}{ccccc} F & \longrightarrow & L_nF & \longrightarrow & L_{n-1}F \\ & & \| & & \| \\ & & L_{K(0)\vee K(1)\vee\cdots\vee K(n-1)\vee K(n)}F & \longrightarrow & L_{K(0)\vee K(1)\vee\cdots\vee K(n-1)}F \end{array}$$

induce the map from F to the natural inverse system

$$\begin{array}{ccccccc} & & & F & & & \\ & & \swarrow & \downarrow & \searrow & & \\ \cdots \longrightarrow & L_{n+1}F & \longrightarrow & L_nF & \longrightarrow & L_{n-1}F & \longrightarrow \cdots, \end{array}$$

which further induces $F \to \operatorname{holim}_n L_nF$. This tower, yielding $\operatorname{holim}_n L_nF$, is called the chromatic tower. Here and after, we will not explicitly name those maps induced by the natural transformations of Bousfield localizations like

$$\begin{array}{ccc} & X & \\ \swarrow & & \searrow \\ L_{E_1}X & \longrightarrow & L_{E_2}X \end{array}$$

with $\langle E_1 \rangle \geq \langle E_2 \rangle$ [Bou79a]. Now, Hopkins-Ravenel [Rav92] showed the following important theorem:

HOPKINS-RAVENEL CHROMATIC CONVERGENCE THEOREM. *When F is finite, the natural map $F \to \operatorname{holim}_n L_nF$ is an equivalence.*

Since the fiber of the p-adic completion is always $H_{\mathbb{Q}} = E(0) = K(0)$ local, the equivalence $F \xrightarrow{\simeq} \operatorname{holim}_n L_nF$ also holds for any p-completion of a finite spectrum F. The Hopkins-Ravenel chromatic convergence theorem is the essence of the chromatic philosophy.

Hopkins [Hov95] went further to propose the following conjecture:

HOPKINS' CHROMATIC SPLITTING CONJECTURE. *Let F be a p-completed finite spectrum. Then the canonical map $L_{n+1}F \to L_nF$ factors through the canonical* *map* $L_{n+1}F \longrightarrow L_{K(n+1)}F$.

$$\begin{array}{ccc} L_{n+1}F & \longrightarrow & L_{K(n+1)}F \\ \downarrow & \swarrow & \\ L_nF & & \end{array}$$

Remark 1.1. (i) Hopkins and Hovey [Hov95] constructed the following commutative diagram of cofiber sequences:

$$\begin{array}{ccccc} F(L_{n-1}S^0, L_nF) & \longrightarrow & L_nF & \longrightarrow & L_{K(n)}F \\ \| & & \downarrow & & \downarrow \\ F(L_{n-1}S^0, L_nF) & \longrightarrow & L_{n-1}F & \longrightarrow & L_{n-1}L_{K(n)}F. \end{array}$$

Thus, Hopkins' Chromatic Splitting Conjecture is equivalent to any one of the following:

- $F(L_{n-1}S^0, L_nF) \to L_nF \to L_{n-1}F$ is null-homotopic.
- $L_{n-1}F \to L_{n-1}L_{K(n)}F$ is a split injection. (This is where the name "chromatic splitting" comes from.)

(ii) Together with the Hopkins-Ravenel Chromatic convergence theorem $F \xrightarrow{\simeq} \operatorname{holim}_n L_nF$, Hopkins' chromatic splitting conjecture claims that the canonical map

$$F \to \prod_n L_{K(n)}F$$

is a split injection [Hov95] and that there is an equivalence

$$F \xrightarrow{\simeq} \operatorname{holim}_n L_{K(n)}F,$$

where the inverse system is given by the composite

$$L_{K(n+1)}F \to L_nF \to L_{K(n)}F$$

with the first map being the one predicted to exist in Hopkins' chromatic splitting conjecture.

(iii) Conceptually, this conjecture claims that the information of a p-completed finite spectrum, obtained by using the first $(n+1)$ Morava K-theories

$$K(0),\ K(1), \ldots, K(n),$$

may be obtained by using a single higher Morava K-theory $K(m)$. $(m \geq n+1)$.

However, in spite of its obvious importance and some advances in analogous problems in the unstable homotopy theory [Bou99, Wil99, Min1], Hopkins' chromatic splitting conjecture has rejected various attempts, except some computational evidences for small n by Shimomura and his collaborators (e.g. [SY95]). Although there is a realted conjecture in [Hov95] concerning the structure of $F(L_{n-1}S^0, L_nF)$, which was meant to be a part of a program to prove the Hopkins Chromatic Splitting Conjecture in our (restricted) sense, Shimomura and Wang [SW] recently disproved it for the case $n = 2, p = 3$. To make the situation worse, even for the Hopkins Chromatic Splitting Conjecture in our (restricted) sense, Devinatz [Dev98] noted the conjecture is false, as soon as we omit the finiteness assumption to include such a nice infinite spectrum as the p-completed BP spectrum.

The purpose of this paper is to prove a general result, valid for a large class of spectra, which contains p-completed finite spectra, focused in the Hopkins' chromatic splitting conjecture, and the p-completed BP-spectrum, found to yield Devinatz' counter-example. Naturally, our result does not claim so much for p-completed finite spectra as Hopkins' chromatic splitting conjecture, but reconciles with Devinatz' counter-example.

To specify what kind of spectra we can deal with, we prepare a definition.

Definition 1.2. A spectrum X is called *robust with type* τ, if the following conditions are satisfied:

(1) bounded below;
(2) for each d, $BP_d(X)$ is a finitely generated $\mathbb{Z}_p^\wedge$-module;
(3) there exists some $\tau \geq 0$ (τ may stand for "type"), such that
    (a) $X = \Sigma^{-\tau} N_\tau X$ (for $\tau > 0$, this condition is the same as $L_{\tau-1}X = *$);
    (b) for each $k \geq \tau$, the cofiber sequence

$$N_k X \longrightarrow M_k X \longrightarrow N_{k+1} X$$

induces a short exact sequence

$$0 \longrightarrow BP_*(N_k X) \longrightarrow BP_*(M_k X) \longrightarrow BP_*(N_{k+1} X) \longrightarrow 0.$$

Now our main theorem states:

Main Theorem. (i) *Given n, let s_0 be the Hopkins-Ravenel uniform horizontal vanishing line for the standard $E(n)$-based Adams-Novikov spectral sequence* (*see e.g.* [Rav92]), *and m and k be nonnegative integers m, k with*

$$m - k \geq n + s_0 + 1.$$

Then, for any spectrum T, which is the smash product of a robust spectrum and a finite spectrum, the canonical map $L_mT \to L_nT$ factors through the canonical map $L_mT \to L_{K(k+1)\vee K(k+2)\vee\cdots\vee K(m-1)\vee K(m)}T$.

$$\begin{array}{ccc} L_mT & \longrightarrow & L_{K(k+1)\vee K(k+2)\vee\cdots\vee K(m-1)\vee K(m)}T \\ \downarrow & \swarrow & \\ L_nT & & \end{array}$$

(ii) *Suppose further that $k \geq n$, then the following horizontal maps are split injections*:

$$\begin{array}{ccc} L_nT & \longrightarrow & L_nL_{K(k+1)\vee K(k+2)\vee\cdots\vee K(m-1)\vee K(m)}T \\ \| & & \| \\ L_n(L_mT) & \longrightarrow & L_n(L_{K(k+1)\vee K(k+2)\vee\cdots\vee K(m-1)\vee K(m)}T). \end{array}$$

Since both the p-completed sphere and the p-completed BP-spectrum are robust [Rav84, 6.1], p-completed finite spectra, focused in the Hopkins' chromatic splitting conjecture, and the p-completed BP-spectrum, found to yield Devinatz' counter-example, both satisfy the assumption of our Main Theorem.

Conceptually, the Main Theorem claims the information of T, obtained by using the first $(n+1)$ Morava K-theories

$$K(0),\ K(1), \ldots, K(n),$$

may be obtained by using any $m-k$ consecutive higher Morava K-theories

$$K(k+1),\ K(k+2), \ldots, K(m-1),\ K(m)$$

with $m - k \geq n + s_0 + 1$ (cf. Remark 1.1. (iii)).

Note that Main Theorem (ii) follows from Main Theorem (i), for we have the following diagram

$$\begin{array}{ccccc} F(L_kS^0, L_mT) & \longrightarrow & L_mT & \longrightarrow & L_{K(k+1)\vee K(k+2)\vee\cdots\vee K(m-1)\vee K(m)}T \\ & & \downarrow & & \\ & & L_nT, & & \end{array}$$

where the top sequence is a cofiber sequence with $F(L_kS^0, L_mT) = L_kF(L_kS^0, L_mT)$.

On the other hand, setting $T = X \wedge F$ with X robust and F finite, we find Main Theorem (i) follows from the following with $Y = F(L_k S^0, L_m T) \wedge DF$ by the S-duality.

THEOREM 1.3. *Given n, let s_0 be the Hopkins-Ravenel uniform horizontal vanishing line for the standard $E(n)$-based Adams-Novikov spectral sequence. Then, for any nonnegative integers m, k with*

$$m - k \geq n + s_0 + 1$$

and for any robust spectrum X, any map of the form

$$f\colon L_k Y \to L_m X$$

always yields the null composite

$$L_k Y \xrightarrow{f} L_m X \to L_n X.$$

Whereas the assumptions on X (and so T) are rather technical, we will discuss some related qualitative properties of general bounded below harmonic spectra in a sequel [Min3].

This paper is organized as follows:

(1) Introduction.
(2) The modified Adams-Novikov spectral sequence.
(3) The spectral sequence for $[Y, L_m(X_l)]$.
(4) Proof of Theorem 1.3.

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2. The modified Adams-Novikov spectral sequence. In this section, we review the modified Adams-Novikov spectral sequence of Devinatz-Hopkins [Dev97] and Franke [Franke]. We mostly follow Devinatz-Hopkins [Dev97].

Definition 2.1. (i) A spectrum I is said to be *E-injective*, if the following two conditions are satisfied:

(1) E_*I is an injective E_*E-comodule;

(2) the natural transformation

$$[X,I]_* \to \mathrm{Hom}^*_{E_*E}(E_*X, E_*I)$$

is an isomorphism for any X.

(ii) Given a spectrum X, its *geometric E-injective embedding* is a spectra map

$$j\colon X \to I$$

such that the target I is E-injective and that the induced map

$$E_*(j)\colon E_*(X) \to E_*(I)$$

is an embedding of E_*X in an E_*E-injective comodule $E_*(I)$.

MODIFIED ANSS. *Let E represent a Landweber exact cohomology theory with E_* concentrated in even dimensions. Then, for any spectra Y and X, we may construct a spectral sequence abutting to $[Y,X]$ as follows:*

(1) *Starting with $F_0 = X$, construct a sequence of spectra $\{F_l\}_{l\geq 0}$ and spectra maps $\{p_l\colon F_l \to F_{l-1}\}_{l\geq 1}$ induced from cofiber sequences*

$$F_{l+1} \xrightarrow{p_{l+1}} F_l \xrightarrow{q_l} \Sigma^{-l}J_l \xrightarrow{\Sigma^{-l}r_l} \Sigma F_{l+1},$$

where $F_l \xrightarrow{q_l} \Sigma^{-l}J_l$ is a geometric E-injective embedding. Then, maps

$$e := q_o\colon X \to J_0$$

$$d_l := \Sigma^l q_l \circ r_l\colon J_l \xrightarrow{r_l} \Sigma^{l+1}F_{l+1} \xrightarrow{\Sigma^{l+1}q_{l+1}} J_{l+1}$$

induce

$$X \xrightarrow{e} J_0 \xrightarrow{d_0} J_1 \xrightarrow{d_1} J_2 \xrightarrow{d_2} \cdots,$$

which we call a geometric E-injective resolution of X.

(2) *A geometric E-injective resolution induces an algebraic E_*E-injective resolution of $E_*(X)$:*

$$0 \to E_*(X) \xrightarrow{E_*(e)} E_*(J_0) \xrightarrow{E_*(d_0)} E_*(J_1) \xrightarrow{E_*(d_1)} E_*(J_2) \xrightarrow{E_*(d_2)} \cdots$$

such that the cofiber sequences

$$\Sigma^s F_s \xrightarrow{\Sigma^s q_s} J_s \xrightarrow{r_s} \Sigma^{s+1}F_{s+1}$$

realize its splices:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Ker} E_*(d_s) & \longrightarrow & E_*(J_s) & \longrightarrow & \operatorname{Im} E_*(d_s) & \longrightarrow & 0 \\
 & & \cong\downarrow & & \| & & \downarrow\cong & & \\
0 & \longrightarrow & E_*(\Sigma^s F_s) & \xrightarrow{E_*(\Sigma^s q_s)} & E_*(J_s) & \xrightarrow{E_*(r_s)} & E_*(\Sigma^{s+1} F_{s+1}) & \longrightarrow & 0.
\end{array}
$$

(3) *For another spectrum Y, impose a filtration on*

$$[Y, X]$$

so that $f\colon Y \to X$ has a filtration equal to or larger than s, iff f has a lift $\tilde{f}\colon Y \to F_s$ (which of course satisfies $f = (p_1 \circ \cdots p_s) \circ \tilde{f}$).

$$
\begin{array}{ccccccccc}
 & & & & & & & & Y \\
 & & & & \tilde{f} \nearrow & & & & \downarrow f \\
\cdots & \xrightarrow[p_{s+2}]{} & F_{s+1} & \xrightarrow[p_{s+1}]{} & F_s & \xrightarrow[p_s]{} \cdots \longrightarrow & F_1 & \xrightarrow[p_1]{} & F_0 = X \\
 & & \downarrow q_{s+1} & & \downarrow q_s & & \downarrow q_1 & & \downarrow e \\
 & & \Sigma^{-s+1} J_{s+1} & & \Sigma^{-s} J_s & & \Sigma^{-1} J_1 & & J_0
\end{array}
$$

Then the resulting spectral sequence enjoys the following properties:

(a) *The spectral sequence is independent of any particular geometric E-injective resolution of X from the E_2-term on, with*

$$E_2^{s,t} = \operatorname{Ext}_{E_*E}^{s,t}(E_*Y, E_*X).$$

(b) *If $E_*(Y)$ is E_*-projective, then the spectral sequence may be identified with the ordinary Adams-Novikov spectral sequence with the canonical relative injective resolution* [Rav84].

(c) *The filtration works well with respect to the composition, and the composition of maps may be studied by the Yoneda pairing at the E_2-term:*

$$\operatorname{Ext}_{E_*E}^{s_1,t_1}(E_*Z, E_*Y) \otimes \operatorname{Ext}_{E_*E}^{s_2,t_2}(E_*Y, E_*X) \to \operatorname{Ext}_{E_*E}^{s_1+s_2,t_1+t_2}(E_*Z, E_*X)$$

The spectral sequence constructed above is called *E-based modified Adams-Novikov spectral sequence* abutting to $[Y, X]$. In practice, the following proposition is very useful:

PROPOSITION 2.2. (i) *Any sequence of spectra*

$$X \xrightarrow{e} J_0 \xrightarrow{d_0} J_1 \xrightarrow{d_1} J_2 \xrightarrow{d_2} \cdots,$$

inducing an algebraic E_*E-injective resolution of $E_*(X)$

$$0 \to E_*(X) \xrightarrow{E_*(e)} E_*(J_0) \xrightarrow{E_*(d_0)} E_*(J_1) \xrightarrow{E_*(d_1)} E_*(J_2) \xrightarrow{E_*(d_2)} \cdots,$$

is a geometric E-injective resolution of X.

(ii) *The association of a geometric E-injective resolution as in* (i) *is natural with respect to maps between such sequences of spectra:*

$$\begin{array}{ccccccccc} X & \xrightarrow{e} & J_0 & \xrightarrow{d_0} & J_1 & \xrightarrow{d_1} & J_2 & \xrightarrow{d_2} & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X' & \xrightarrow{e'} & J'_0 & \xrightarrow{d'_0} & J'_1 & \xrightarrow{d'_1} & J'_2 & \xrightarrow{d'_2} & \cdots. \end{array}$$

Proof. (i) Starting with $F_0 = X$, we shall construct a sequence of spectra $\{F_l\}_{l \geq 0}$ and spectra maps $\{p_l\colon F_l \to F_{l-1}\}_{l \geq 1}$ such that the cofiber sequences

$$\Sigma^s F_{s+1} \xrightarrow{\Sigma^s p_{s+1}} \Sigma^s F_s \xrightarrow{\Sigma^s q_s} J_s \xrightarrow{r_s} \Sigma^{s+1} F_{s+1} \xrightarrow{\Sigma^{s+1} p_{s+1}} \Sigma^{s+1} F_s$$

realize its splices:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Ker} E_*(d_s) & \longrightarrow & E_*(J_s) & \longrightarrow & \operatorname{Im} E_*(d_s) & \longrightarrow & 0 \\ & & \cong\downarrow & & \| & & \downarrow\cong & & \\ 0 & \longrightarrow & E_*(\Sigma^s F_s) & \xrightarrow{E_*(\Sigma^s q_s)} & E_*(J_s) & \xrightarrow{E_*(r_s)} & E_*(\Sigma^{s+1} F_{s+1}) & \longrightarrow & 0, \end{array}$$

by induction on l.

Suppose F_l's have been constructed satisfying these conditions for $l \leq s$. We will define $q_s\colon F_s \to \Sigma^{-s} J_s$ and construct F_{s+1} as the cofiber of $\Sigma^{-1} q_s$, where q_s is required to factorize as in the following diagram:

$$\begin{array}{ccccccc} J_{s-2} & \xrightarrow{d_{s-2}} & J_{s-1} & \xrightarrow{d_{s-1}} & J_s & \xrightarrow{d_s} & J_{s+1} \\ & {\scriptstyle r_{s-2}}\searrow \quad \nearrow{\scriptstyle \Sigma^{s-1} q_{s-1}} & & {\scriptstyle r_{s-1}}\searrow \quad \nearrow{\scriptstyle \Sigma^s q_s} & & {\scriptstyle r_s}\searrow & \\ & \Sigma^{s-1} F_{s-1} & & \Sigma^s F_s & & & \Sigma^{s+1} F_{s+1}. \end{array}$$

Since $E_*(J_s)$ is E_*E-injective, the short exact sequence

$$0 \to E_*(\Sigma^{s-1}F_{s-1}) \xrightarrow{E_*(\Sigma^{s-1}q_{s-1})} E_*(J_{s-1}) \xrightarrow{E_*(r_{s-1})} E_*(\Sigma^s F_s) \to 0,$$

induces another short exact sequence

$$\begin{array}{ccccccccc} & & [\Sigma^s F_s, J_s] & \xrightarrow{(r_{s-1})^*} & [J_{s-1}, J_s] & \xrightarrow{(\Sigma^{s-1}q_{s-1})^*} & [\Sigma^{s-1}F_{s-1}, J_s] & & \\ & & \| & & \| & & \| & & \\ 0 & \longrightarrow & H_E(\Sigma^s F_s, J_s) & \xrightarrow{(r_{s-1})^*} & H_E(J_{s-1}, J_s) & \xrightarrow{(\Sigma^{s-1}q_{s-1})^*} & H_E(\Sigma^{s-1}F_{s-1}, J_s) & \longrightarrow & 0, \end{array}$$

where we have abbreviated as $H_E(X, Y) = \mathrm{Hom}_{E_*E}(E_*(X), E_*(Y))$.

Consider the element

$$[d_{s-1}] \in [J_{s-1}, J_s] = H_E(J_{s-1}, J_s) = \mathrm{Hom}_{E_*E}(E_*(J_{s-1}), E_*(J_s)).$$

Since

$$\begin{aligned} (d_{s-2})^*[d_{s-1}] &= 0 \in \mathrm{Hom}_{E_*E}(E_*(J_{s-2}), E_*(J_s)), \\ (d_{s-2})^* &= (r_{s-2})^* \circ (\Sigma^{s-1}q_{s-1})^*, \end{aligned}$$

and since

$$(r_{s-2})^*\colon\ \mathrm{Hom}_{E_*E}(E_*(\Sigma^{s-1}F_{s-1}), E_*(J_s)) \to \mathrm{Hom}_{E_*E}(E_*(J_{s-2}), E_*(J_s))$$

is injective, we see

$$(\Sigma^{s-1}q_{s-1})^*[d_{s-1}] = 0 \in \mathrm{Hom}_{E_*E}(E_*(\Sigma^{s-1}F_{s-1}), E_*(J_s)).$$

Thus, from the exact sequence, there is a unique element

$$[\Sigma^s q_s] \in [\Sigma^s F_s, J_s]$$

such that

$$(r_{s-1})^*[\Sigma^s q_s] = [d_{s-1}] \in [J_{s-1}, J_s].$$

Then, $r_s\colon J_s \to \Sigma^{s+1}F_{s+1}$ is defined as the cofiber of $\Sigma^s q_s\colon \Sigma^s F_s \to J_s$, and it is easy to see that this cofiber sequence induces a short exact sequence of E_*E-comodules, as desired.

(ii) This may be shown in a straightforward manner, just like (i). □

It is also standard and useful to interpret the spectral sequence in terms of a tower under X. For this purpose, we define a sequence of spectra $\{X_l\}_{l\geq 0}$ by cofiber sequences

$$F_{l+1} \xrightarrow{p_1\circ\cdots\circ p_{l+1}} X \xrightarrow{\pi_l} X_l,$$

and the spectra maps $\{\bar{p}_l\colon X_l \to X_{l-1}\}_{l\geq 1}$ by the following commutative diagram of cofiber sequences:

$$\begin{array}{ccccccccccc}
F_{l+1} & \xrightarrow{p_{l+1}} & F_l & \xrightarrow{q_l} & \Sigma^{-l}J_l & \xrightarrow{\Sigma^{-l}r_l} & \Sigma F_{l+1} & \xrightarrow{\Sigma p_{l+1}} & \Sigma F_l & \xrightarrow{\Sigma q_l} & \Sigma^{-l+1}J_l \\
\downarrow{\scriptstyle p_1\circ\cdots\circ p_{l+1}} & & \downarrow{\scriptstyle p_1\circ\cdots\circ p_l} & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
X & = & X & \longrightarrow & * & \longrightarrow & \Sigma X & = & \Sigma X & \longrightarrow & * \\
\downarrow{\scriptstyle \pi_l} & & \downarrow{\scriptstyle \pi_{l-1}} & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
X_l & \xrightarrow{\bar{p}_l} & X_{l-1} & \xrightarrow{\bar{q}_{l-1}} & \Sigma^{-l+1}J_l & \xrightarrow{\Sigma^{-l+1}\bar{r}_l} & \Sigma X_l & \xrightarrow{\Sigma\bar{p}_l} & \Sigma X_{l-1} & \xrightarrow{\Sigma\bar{q}_{l-1}} & \Sigma^{-l+2}J_l \\
\downarrow & & \downarrow & & \| & & \downarrow & & \downarrow & & \| \\
\Sigma F_{l+1} & \xrightarrow{\Sigma p_{l+1}} & \Sigma F_l & \xrightarrow{\Sigma q_l} & \Sigma^{-l+1}J_l & \xrightarrow{\Sigma^{-l+1}r_l} & \Sigma^2 F_{l+1} & \xrightarrow{\Sigma^2 p_{l+1}} & \Sigma^2 F_l & \xrightarrow{\Sigma^2 q_l} & \Sigma^{-l+2}J_l.
\end{array}$$

PROPOSITION 2.3. *For a given geometric E-injective resolution of X*

$$X \xrightarrow{e} J_0 \xrightarrow{d_0} J_1 \xrightarrow{d_1} J_2 \xrightarrow{d_2} \cdots,$$

there is a map from X to a tower of spectra and spectra maps $\{\bar{p}_l\colon X_l \to X_{l-1}\}_{l\geq 1}$ *such that:*

(1) $X_0 = J_0$*;*

(2) *There are cofiber sequences*

$$X_l \xrightarrow{\bar{p}_l} X_{l-1} \xrightarrow{\bar{q}_{l-1}} \Sigma^{-(l-1)}J_l \xrightarrow{\Sigma^{-(l-1)}\bar{r}_l} \Sigma X_l$$

such that

$$d_l = \Sigma^l\bar{q}_l \circ \bar{r}_l\colon\ J_l \xrightarrow{\bar{r}_l} \Sigma^l X_l \xrightarrow{\Sigma^l\bar{q}_l} J_{l+1};$$

(3) *For any spectrum Y, the spectral sequence for*

$$[Y, \underset{\bar{p}_l}{\operatorname{holim}} X_l],$$

defined by the tower $\{\bar{p}_l\colon X_l \to X_{l-1}\}_{l\geq 1}$, *may be identified with the modified Adams-Novikov spectral sequence which computes* $[Y,X]$.

We now recall the main theorem of [Min2].

THEOREM 2.4. *There does exist a uniform horizontal vanishing line for the* E_∞*-term of the* $E(n)$*-based modified Adams-Novikov spectral sequence. More precisely, there is some height* s_1 *and a function* ϕ, *independent of* Y, X, *such that*

$$E_\infty^{s,*}(Y,X) = 0 \quad \text{for} \quad s > s_1,$$
$$E_r^{s,*}(Y,X) = E_\infty^{s,*}(Y,X) \quad \text{for} \quad r > \phi(s)$$

for the $E(n)$*-based modified Adams-Novikov spectral sequence which computes*

$$[Y, L_nX] = [L_nY, L_nX].$$

Actually, we may take $s_1 = s_0+n$, *where* s_0 *is the Hopkins-Ravenel (Hovey-Sadofsky) height of the uniform horizontal vanishing line for the ordinary* $E(n)$*-based Adams-Novikov spectral sequence.*

We thank the referee for pointing out that such a uniform horizontal vanishing line for the E_∞-term of the $E(n)$-based modified Adams-Novikov spectral sequence was already established in [HSt99, Prop. 6.5]. However, the horizontal vanishing line height established in [HSt99, Prop. 6.5] is $(n+1)s_0$, which is much larger than $s_0 + n$ in general.

Now, since the proof of Theorem 2.4 in [Min2] showed the iterated composite

$$p_1 \circ p_2 \circ \cdots \circ p_{s_1+1}\colon F_{s_1+1} \to X$$

is trivial, we immediately get the following corollary:

COROLLARY 2.5. $\pi_{s_1}\colon X \to X_{s_1}$ *is a split injection.*

We now make the following definition in the spirit of [JLY81] (see also [Yos84], where a uniform approach is given for the main theorems of [JY80] and [JLY81]):

Definition 2.6. (1) Denote by $\mathcal{E}(n)$ the category defined by

$$Obj\mathcal{E}(n) = E(n)_*E(n)\text{-comodules}$$
$$Mor\mathcal{E}(n) = E(n)_*\text{-module}$$

(2) An $E(n)_*$-module M is called *$\mathcal{E}(n)$-injective*, if, for any $i > 0$ and $C \in Obj\mathcal{E}(n)$,

$$\mathrm{Ext}^i_{E(n)_*}(C, M) = 0.$$

(3) $w.inj - \dim_{\mathcal{E}(n)} M$ is defined so as to be less than $d+1$ if, for any $j > d$ and $C \in Obj\mathcal{E}(n)$,

$$\mathrm{Ext}^j_{E(n)_*}(C, M) = 0.$$

Using this concept, we see the above horizontal line results may be slightly improved for some special type of spectra:

THEOREM 2.7. *Let X be a spectrum such that the following conditions are satisfied for some integer τ with $0 \le \tau \le n$ (τ may stand for "type");*

(1) $X = \Sigma^{-\tau} N_\tau X$ *(for $\tau > 0$, this condition is the same as $L_{\tau-1}X = *$);*

(2) *for each $k \ge \tau$, the cofiber sequence*

$$N_k X \to M_k X \to N_{k+1} X$$

induces a short exact sequence

$$0 \to BP_*(N_k X) \to BP_*(M_k X) \to BP_*(N_{k+1} X) \to 0.$$

Then we may slightly lower the uniform horizontal vanishing line for the $E(n)$-based modified Adams-Novikov spectral sequence, computing $[-, L_n X]$, to $s_0 + n - \tau$. Here s_0 is the Hopkins-Ravenel (Hovey-Sadofsky) height of the uniform horizontal vanishing line for the ordinary $E(n)$-based Adams-Novikov spectral sequence.

In particular,

$$L_n(\pi_{s_0+n-\tau}) \colon L_n X \to L_n X_{s_0+n-\tau}$$

is a split injection.

Proof. We start with the canonical geometric $E(n)$-injective resolution of X [Min2, Corollary 4.2] up to the stage $n - \tau - 1$:

$$(1) \qquad X \xrightarrow{e} J_0 \xrightarrow{d_0} J_1 \xrightarrow{d_1} \cdots \xrightarrow{d_{n-\tau-2}} J_{n-\tau-1} \xrightarrow{r_{n-\tau-1}} \Sigma^{n-\tau} F_{n-\tau}.$$

We now wish to show $E(n)_*(\Sigma^{n-\tau}F_{n-\tau})$ is $\mathcal{E}(n)$-injective. First, since an $E(n)_*$-injective module is $\mathcal{E}(n)$-injective, [Min2, Lemma 4.1] implies $E(n)_*(J_l)$ $(0 \leq l \leq n-\tau-1)$ are all $\mathcal{E}(n)$-injective. This immediately implies, for any $i > 0$ and an $E(n)_*E(n)$-comodule C,

$$\begin{aligned}\mathrm{Ext}^i_{E(n)_*}(C, E(n)_*(\Sigma^{n-\tau}F_{n-\tau})) &= \mathrm{Ext}^{i+1}_{E(n)_*}(C, E(n)_*(\Sigma^{n-\tau-1}F_{n-\tau-1}))\\ &= \cdots = \mathrm{Ext}^{i+n-\tau-1}_{E(n)_*}(C, E(n)_*(\Sigma^1 F_1))\\ &= \mathrm{Ext}^{i+n-\tau}_{E(n)_*}(C, E(n)_*(F_0))\\ &= \mathrm{Ext}^{i+n-\tau}_{E(n)_*}(C, E(n)_*(X)).\end{aligned}$$

Thus, it suffices to show

$$w.inj - \dim_{\mathcal{E}(n)} E(n)_*(\Sigma^{n-\tau}F_{n-\tau}) < n - \tau + 1.$$

However, this immediately follows from the following long exact sequence

$$\begin{aligned}0 \to E(n)_*X \to E(n)_*M_\tau X \to E(n)_*M_{\tau+1}\\ \to \cdots \to E(n)_*M_{n-1}X \to E(n)_*M_nX \to 0,\end{aligned}$$

where $E(n)_*(M_lX)$ $(\tau \leq l \leq n)$ are all $\mathcal{E}(n)$-injective by [JLY81], as was remarked in [Min2, Theorem 3.1].

Now that we know $E(n)_*(\Sigma^{n-\tau}F_{n-\tau})$ is $\mathcal{E}(n)$-injective, the rest of the proof runs exactly as "Completion of the proof of Theorem 1.1" in [Min2]. □

3. The spectral sequence for $[Y, L_m(X_l)]$. Let X be a type τ robust spectrum (cf. Definition 1.2) and E be a Landweber exact cohomology theory with E_* concentrated in even dimensions, as in the Modified ANSS in §2. Then, we may construct a tractable geometric E-injective resolution of X, as follows:

$$X \to J_0 \to J_1 \cdots \tag{2}$$

$$J_k = \bigvee_{0 \leq i \leq k} E \wedge \bar{E}^{\wedge i} \wedge M_{\tau+k-i}X, \tag{3}$$

which is constructed by applying $\Sigma^{-\tau}$ to the "total complex" of the following:

$$
(4)\qquad
\begin{array}{ccccccc}
\Sigma^t X = N_\tau X & \longrightarrow & M_\tau X & \longrightarrow & M_{\tau+1} X & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
E \wedge N_\tau X & \longrightarrow & E \wedge M_\tau X & \longrightarrow & E \wedge M_{\tau+1} X & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
E \wedge \bar{E} \wedge N_\tau X & \longrightarrow & E \wedge \bar{E} \wedge M_\tau X & \longrightarrow & E \wedge \bar{E} \wedge M_{\tau+1} X & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
E \wedge \bar{E}^{\wedge 2} \wedge N_\tau X & \longrightarrow & E \wedge \bar{E}^{\wedge 2} \wedge M_\tau X & \longrightarrow & E \wedge \bar{E}^{\wedge 2} \wedge M_{\tau+1} X & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\vdots & & \vdots & & \vdots & &
\end{array}
$$

Here, the horizontal arrows are the usual chromatic sequence, the vertical arrows are the canonical BP-based relative injective Adams-Novikov resolution of X, and (2) does become a geometric BP-injective resolution of X by [Min2, Cor. 4.2] and [JLY81]. Note that (2) gives us a tower over X

$$\cdots \to F_2 \to F_1 \to F_0 = X$$

and a tower under X

$$
\begin{array}{ccccccccc}
 & & & \swarrow & X & \searrow & & & \\
 & & & & \downarrow & & & & \\
\cdots & \longrightarrow & X_{l+1} & \longrightarrow & X_l & \longrightarrow & X_{l-1} & \longrightarrow & \cdots,
\end{array}
$$

by Proposition 2.2 and Proposition 2.3.

Fix an integer m with $m \geq \tau$. We first construct a spectral sequence to compute $[Y, L_m X]$ for arbitrary Y. We do this simply by applying L_m to the geometric BP-injective resolution and the corresponding towers over and lower

X constructed above for the case $E = BP$:

$$\begin{array}{ccccccccc}
 & & & & & & & & Y \\
 & & & \overset{\tilde{f}}{\swarrow\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} & & & & & \downarrow f \\
\cdots \xrightarrow[L_m p_{s+2}]{} & L_m F_{s+1} & \xrightarrow[L_m p_{s+1}]{} & L_m F_s & \xrightarrow[L_m p_s]{} \cdots & \longrightarrow & L_m F_1 & \xrightarrow[L_m p_1]{} & L_m F_0 = L_m X \\
 & \downarrow L_m q_{s+1} & & \downarrow L_m q_s & & & \downarrow L_m q_1 & & \downarrow L_m e \\
 & \Sigma^{-s+1} L_m J_{s+1} & & \Sigma^{-s} L_m J_s & & & \Sigma^{-1} L_m J_1 & & L_m J_0.
\end{array}$$

PROPOSITION 3.1. *Let $E_2^{s,t}(Y, L_m X)$ be the E_2-term of the spectral sequence constructed above for a type τ robust spectrum X. Then there is a natural map from this spectral sequence to the $E(m)$-based modified Adams-Novikov spectral sequence, both computing $[Y, L_m X]$, which becomes an isomorphism after the E_2-term on:*

$$E_2^{s,t}(Y, L_m X) \xrightarrow{\sim} \mathrm{Ext}^{s,t}_{E(m)_*E(m)}(Y, L_m X).$$

Before its proof, we recall the following useful reformulation of Ohkawa's theorem [Ohk93], which generalized [Yos88], due to [Nee97][CS98, Prop 4.6].

LEMMA 3.2. *For any spectrum Y, there is a cofiber sequence*

$$P \longrightarrow Q \longrightarrow Y,$$

which enjoys the following properties:

(1) *Q is a wedge sum of finite spectra;*

(2) *P is a direct summand of a wedge sum of finite spectra;*

(3) *For any homology theory h_*, the induced long exact sequence is reduced to short exact sequences*

$$0 \longrightarrow h_*P \longrightarrow h_*Q \longrightarrow h_*Y \longrightarrow 0.$$

Proof of Proposition 3.1. Recall that L_m is smashing [Rav92] and that

$$L_m M_l X = \begin{cases} M_l X & \text{for } l \leq m; \\ * & \text{for } l > m. \end{cases}$$

Then, substituting the map

$$L_m BP \longrightarrow E(m)$$

for E in (4), we have a canonical map from

$$
\begin{array}{ccccccc}
\Sigma^{\tau} L_m X = L_m N_{\tau} X & \longrightarrow & M_{\tau} X & \longrightarrow & \cdots & \longrightarrow & M_m X \\
\downarrow & & \downarrow & & & & \downarrow \\
BP \wedge L_m N_{\tau} X & \longrightarrow & BP \wedge M_{\tau} X & \longrightarrow & \cdots & \longrightarrow & BP \wedge M_m X \\
\downarrow & & \downarrow & & & & \downarrow \\
BP \wedge \overline{BP} \wedge L_m N_{\tau} X & \longrightarrow & BP \wedge \overline{BP} \wedge M_{\tau} X & \longrightarrow & \cdots & \longrightarrow & BP \wedge \overline{BP} \wedge M_m X \\
\downarrow & & \downarrow & & & & \downarrow \\
BP \wedge \overline{BP}^{\wedge 2} \wedge L_m N_{\tau} X & \longrightarrow & BP \wedge \overline{BP}^{\wedge 2} \wedge M_{\tau} X & \longrightarrow & \cdots & \longrightarrow & BP \wedge \overline{BP}^{\wedge 2} \wedge M_m X \\
\downarrow & & \downarrow & & & & \downarrow \\
\vdots & & \vdots & & \vdots & &
\end{array}
$$

to

$$
\begin{array}{ccccccc}
\Sigma^{\tau} L_m X = L_m N_{\tau} X & \longrightarrow & M_{\tau} X & \longrightarrow & \cdots & \longrightarrow & M_m X \\
\downarrow & & \downarrow & & & & \downarrow \\
E(m) \wedge L_m N_{\tau} X & \longrightarrow & E(m) \wedge M_{\tau} X & \longrightarrow & \cdots & \longrightarrow & E(m) \wedge M_m X \\
\downarrow & & \downarrow & & & & \downarrow \\
E(m) \wedge \overline{E(m)} \wedge L_m N_{\tau} X & \longrightarrow & E(m) \wedge \overline{E(m)} \wedge M_{\tau} X & \longrightarrow & \cdots & \longrightarrow & E(m) \wedge \overline{E(m)} \wedge M_m X \\
\downarrow & & \downarrow & & & & \downarrow \\
E(m) \wedge \overline{E(m)}^{\wedge 2} \wedge L_m N_{\tau} X & \longrightarrow & E(m) \wedge \overline{E(m)}^{\wedge 2} \wedge M_{\tau} X & \longrightarrow & \cdots & \longrightarrow & E(m) \wedge \overline{E(m)}^{\wedge 2} \wedge M_m X \\
\downarrow & & \downarrow & & & & \downarrow \\
\vdots & & \vdots & & \vdots & & .
\end{array}
$$

This induces the desired map between the spectral sequences.

To see that this induces an isomorphism of the E_2-terms, it suffices to show the following map

$$\mathrm{Ext}^{s,t}_{BP_*BP}(BP_*Y, BP_*M_lX) \to \mathrm{Ext}^{s,t}_{E(m)_*E(m)}(E(m)_*Y, E(m)_*M_lX)$$

is an isomorphism for any Y and X by the usual double complex spectral sequence.

But this may be done in the following order:

(1) By Lemma 3.2, we may assume Y is a finite spectrum. Then, by the Landweber filtration theorem [Lan76], it suffices to show

$$\mathrm{Ext}^{s,t}_{BP_*BP}(BP_*/I_i, BP_*M_lX) \to \mathrm{Ext}^{s,t}_{E(m)_*E(m)}(E(m)_*/I_i, E(m)_*M_lX)$$

is an isomorphism for $0 \leq i \leq m$.

(2) By the obvious Bockstein long exact sequences, we may assume $i = 0$ in 1.

(3) Then the map is an isomorphism by the Hovey-Sadofsky change of rings theorem [HS99].

Now the proof is complete. □

THEOREM 3.3. *Let X be a type τ robust spectrum. Then, the cofiber sequence*

$$\Sigma^{-m-1}N_{m+1}X \to X \to L_mX$$

induces a long exact sequence of the E_2-terms:

$$\begin{aligned} \cdots &\to \mathrm{Ext}^{s-m+\tau-1,t+\tau}_{BP_*BP}(BP_*Y, BP_*N_{m+1}X) \to \mathrm{Ext}^{s,t}_{BP_*BP}(BP_*Y, BP_*X) \\ &\to \mathrm{Ext}^{s,t}_{E(m)_*E(m)}(E(m)_*Y, E(m)_*X) \to \mathrm{Ext}^{s-m+\tau,t+\tau}_{BP_*BP}(BP_*Y, BP_*N_{m+1}X) \\ &\to \mathrm{Ext}^{s+1,t}_{BP_*BP}(BP_*Y, BP_*X) \to \mathrm{Ext}^{s+1,t}_{E(m)_*E(m)}(E(m)_*Y, E(m)_*X) \to \cdots. \end{aligned}$$

In particular, the Thom reduction

$$\mathrm{Ext}^{s,*}_{BP_*BP}(BP_*Y, BP_*X) \to \mathrm{Ext}^{s,*}_{E(m)_*E(m)}(E(m)_*Y, E(m)_*X)$$

is injective for $s \leq m - \tau$ and bijective for $s < m - \tau$.

Proof. By the affirmative proof of the smashing conjecture [Rav92], smashing

$$\Sigma^{-m-1}N_{m+1}S^0 \to S^0 \to L_mS^0. \tag{5}$$

with (2) gives us the following:

$$\begin{array}{ccccccc}
\Sigma^{-m-1}N_{m+1}X & \longrightarrow & J'_0 & \longrightarrow & J'_1 & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
X & \longrightarrow & J_0 & \longrightarrow & J_1 & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
L_mX & \longrightarrow & J''_0 & \longrightarrow & J''_1 & \longrightarrow & \cdots,
\end{array}$$

where

$$J'_k = \begin{cases} \bigvee_{0\le i<k-(m-\tau)} BP \wedge \overline{BP}^{\wedge i} \wedge M_{\tau+k-i}X & \text{if } k > m-\tau \\ * & \text{if } k \le m-\tau \end{cases}$$

$$J_k = \bigvee_{0\le i\le k} BP \wedge \overline{BP}^{\wedge i} \wedge M_{\tau+k-i}X$$

$$J''_k = \begin{cases} \bigvee_{k-(m-\tau)\le i\le k} BP \wedge \overline{BP}^{\wedge i} \wedge M_{\tau+k-i}X & \text{if } k > m-\tau \\ \bigvee_{0\le i\le k} BP \wedge \overline{BP}^{\wedge i} \wedge M_{\tau+k-i}X & \text{if } k \le m-\tau \end{cases}$$

and the vertical maps are canonical inclusions and projections of direct summands. This gives us the following short exact sequence of chain complexes

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \cdots \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}_{BP_*BP}(BP_*Y, BP_*J'_0) & \longrightarrow & \mathrm{Hom}_{BP_*BP}(BP_*Y, BP_*J'_1) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}_{BP_*BP}(BP_*Y, BP_*J_0) & \longrightarrow & \mathrm{Hom}_{BP_*BP}(BP_*Y, BP_*J_1) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}_{BP_*BP}(BP_*Y, BP_*J''_0) & \longrightarrow & \mathrm{Hom}_{BP_*BP}(BP_*Y, BP_*J''_1) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & \cdots,
\end{array}$$

whose resulting long exact sequence is easily seen to be the desired one by Proposition 3.1. □

Remark 3.4. (i) Applying the functor $[Y, -]$ to (4) with $E = BP$, we get a double complex. Then the corresponding double complex spectral sequence looks like

$$\mathrm{Ext}^{s,t}_{BP_*BP}(BP_*Y, BP_*M_uX) \Rightarrow \mathrm{Ext}^{s+u-\tau,t}_{BP_*BP}(BP_*Y, BP_*X),$$

where X is a type τ robust spectrum. Note that the special case $Y = X = S^0$ with $\tau = 0$ is nothing but the original Miller-Ravenel-Wilson chromatic spectral sequence [MRW77].

(ii) Hikida-Shimomura [HiShi94] first stated Theorem 3.3 for the special case $Y = S^0$. Unfortunately, their proof contains a fatal mistake. Actually, in the proofs of Lemma 3.15 (p. 653) and Proposition 3.13 (p. 652), they had to use a "Hopf algebroid map" $(B, \Sigma) \to (K_i, \Sigma_i)$, where $(B, \Sigma) = (E(n)_*, E(n)_*E(n))$ and $(K_i, \Sigma_i) = (K(i)_*, K(i)_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} K(i)_*)$. Of course, this does not make sense, unless $n = i$. However, as our proof indicates, Theorem 3.3 for the special case $Y = S^0$ more or less follows from [HS99].

LEMMA 3.5. *Let X be a bounded below spectrum such that $BP_d(X)$ is a finitely generated $\mathbb{Z}_p^\wedge$-module for each d. Then, for any spectrum Y and nonnegative integer i,*

$$\mathrm{Ext}^{s,t}_{BP_*BP}(BP_*M_iY, BP_*X) = 0 \qquad \text{(for any } s, t.).$$

Proof. By the smashing conjecture [Rav92], we see $BP_*M_iY = (BP \wedge M_iS^0)_*Y$. Thus, by Lemma 3.2 with $h = BP \wedge M_iS^0$, we may assume Y is a finite spectrum.

Now, let us call a sub BP_*BP-comodule of $BP_*M_iS^0$ *vanishing*, if it is simultaneously a sub $v_i^{-1}BP_*$-module and

$$\mathrm{Ext}^{s,t}_{BP_*BP}(C, BP_*X) = 0 \qquad \text{(for any } s, t,)$$

for its arbitrary sub-quotient BP_*BP-comodule C, which is simultaneously a sub $v_i^{-1}BP_*$-module. Then, if we could show $BP_*M_iS^0$ itself is vanishing, then we can prove the claim by induction on the number of cells in Y.

Applying the Milnor sequence, we can apply Zorn's lemma to get a maximal sub vanishing BP_*BP-comodule M inside $BP_*M_iS^0$.

It now suffices to show $M = BP_*M_iS^0$. Suppose not. Then, by the Landweber filtration theorem [Lan76], there is a sub BP_*BP-comodule M' of $BP_*M_iS^0$ with a short exact sequence of BP_*BP-comodules:

$$0 \to M \to M' \to v_i^{-1}BP_*/I_i \to 0.$$

If we could show

$$\text{(6)} \qquad \operatorname{Ext}^{s,t}_{BP_*BP}(v_i^{-1}BP_*/I_i, BP_*X) = 0 \qquad \text{(for any } s, t.),$$

then we can easily see that M' is also vanishing, which is a contradiction.

Thus, it only suffices to prove (6). But this can be easily shown in the following order:

(1) Observe $\operatorname{Ext}^{s,t}_{BP_*BP}(BP_*, BP_*X)$ is a finitely generated $\mathbb{Z}_p^\wedge$-module, because $BP_d(X)$ is so for each d, and vanishes when $t-s$ is sufficiently small, because of the bounded below assumption on X.

(2) By the Bockstein long exact sequence, coming from

$$0 \to BP_*/I_{i-1} \xrightarrow{\times v_{i-1}} BP_*/I_{i-1} \to BP_*/I_i \to 0,$$

observe that the properties in 1 also hold for $\operatorname{Ext}^{s,t}_{BP_*BP}(BP_*/I_i, BP_*X)$.

(3) In the Milnor sequence

$$\begin{aligned} 0 \to \lim{}^1 \operatorname{Ext}^{s-1,*}_{BP_*BP}(BP_*/I_i, BP_*X) &\to \operatorname{Ext}^{s,*}_{BP_*BP}(v_i^{-1}BP_*/I_i, BP_*X) \\ &\to \lim{}^0 \operatorname{Ext}^{s,*}_{BP_*BP}(BP_*/I_i, BP_*X) \to 0, \end{aligned}$$

observe that both $\lim^0$ and $\lim^1$ vanish by 2.

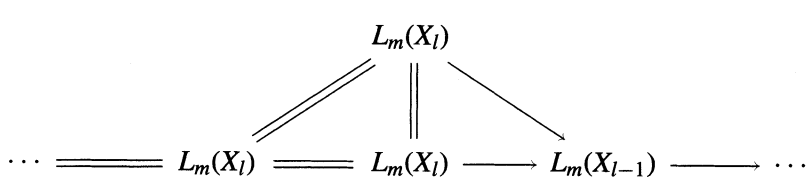
Now the proof is complete. □

COROLLARY 3.6. *Let X be a bounded below spectrum such that $BP_d(X)$ is a finitely generated $\mathbb{Z}_p^\wedge$-module for each d. Then, for any spectrum Y and nonnegative integer i, m,*

$$\operatorname{Ext}^{s,t}_{E(m)_*E(m)}(E(m)_*M_iY, E(m)_*X) = 0 \qquad \textit{(for any } t \textit{ and } s \textit{ with } s < m - \tau.).$$

Proof. This is an immediate consequence of Theorem 3.3 and Lemma 3.5. □

Actually, we should really understand not only X and L_mX but also $L_m(X_l)$. To compute $[Y, L_m(X_l)]$, we construct its spectral sequence, arising from the truncated tower under $L_m(X_l)$:



PROPOSITION 3.7. *Let $E_2^{s,t}(Y, L_m(X_l))$ be the E_2-term of the spectral sequence constructed above. If X is a robust spectrum (of type τ), then the natural map*

$$\mathrm{Ext}^{s,t}_{E(m)_*E(m)}(E(m)_*Y, E(m)_*X) \to E_2^{s,t}(Y, L_m(X_l)),$$

induced by the canonical map from the tower

$$\begin{array}{ccccccccc} & & & & L_mX & & & & \\ & & & \swarrow & \downarrow & \searrow & & & \\ \cdots & \longrightarrow & L_m(X_{l+1}) & \longrightarrow & L_m(X_l) & \longrightarrow & L_m(X_{l-1}) & \longrightarrow & \cdots \end{array}$$

to the truncated tower

$$\begin{array}{ccccccccc} & & & & L_m(X_l) & & & & \\ & & & /\!\!/ & \| & \searrow & & & \\ \cdots & = & L_m(X_l) & = & L_m(X_l) & \longrightarrow & L_m(X_{l-1}) & \longrightarrow & \cdots, \end{array}$$

is

$$\begin{cases} \textit{bijective} & \textit{if} \quad s < l; \\ \textit{injective} & \textit{if} \quad s = l, \end{cases}$$

and, for $s > l$,

$$E_2^{s,t}(Y, L_m(X_l)) = 0.$$

Proof. This is simply because

$$\mathrm{Ext}^{s,t}_{E(m)_*E(m)}(E(m)_*Y, E(m)_*X) \to E_2^{s,t}(Y, L_m(X_l))$$

is the induced homology map of the chain complexes, which is obtained by applying the functor $[Y, -]$ to the following commutative diagram:

$$\begin{array}{ccccccccccccc} * & \longrightarrow & J''_0 & \longrightarrow & \cdots & \longrightarrow & J''_l & \longrightarrow & J''_{l+1} & \longrightarrow & J''_{l+2} & \longrightarrow & \cdots \\ \| & & \| & & & & \| & & \downarrow & & \downarrow & & \\ * & \longrightarrow & J''_0 & \longrightarrow & \cdots & \longrightarrow & J''_l & \longrightarrow & * & \longrightarrow & * & \longrightarrow & \cdots. \end{array}$$

Here

$$J''_k = \begin{cases} \bigvee_{k-(m-\tau)\le i\le k} BP \wedge \overline{BP}^{\wedge i} \wedge M_{\tau+k-i}X & \text{if } k > m-\tau; \\ \bigvee_{0\le i\le k} BP \wedge \overline{BP}^{\wedge i} \wedge M_{\tau+k-i}X & \text{if } k \le m-\tau, \end{cases}$$

as in the proof of Theorem 3.3. □

4. Proof of Theorem 1.3. We start with the following important lemma:

LEMMA 4.1. *Let X be a type τ robust spectrum. Then, for any map $M_iY \to L_m(X_l)$, the composite*

$$M_iY \to L_m(X_l) \to L_m(X_{l-1}),$$

is trivial for $l \le m-\tau$.

Proof. This is an immediate consequence of Corollary 3.6 and Proposition 3.7. □

COROLLARY 4.2. *Let X be a type τ robust spectrum. Then for any map $f\colon L_kY \to L_mX$, the composite*

$$L_kY \xrightarrow{f} L_mX \to L_m(X_{m-\tau-1-k})$$

is trivial.

Proof. Since the composite

$$\Sigma^{-k}M_kY \to L_kY \xrightarrow{f} L_mX \to L_m(X_{m-\tau-1})$$

is null by Lemma 4.1, we have the following commutative diagram:

$$\begin{array}{ccccc} \Sigma^{-k}M_kY & \longrightarrow & L_kY & \longrightarrow & L_{k-1}Y \\ & & {\scriptstyle f}\downarrow & & \vdots\,{\scriptstyle f'} \\ & & L_mX & \longrightarrow & L_m(X_{m-\tau-1}). \end{array}$$

Replacing f by f' and so on, we iterate the same argument to obtain the following commutative diagram:

$$\begin{array}{ccccccccc} L_kY & \longrightarrow & L_{k-1}Y & \longrightarrow & \cdots & \longrightarrow & L_0Y & \longrightarrow & * \\ {\scriptstyle f}\downarrow & & \vdots\,{\scriptstyle f'} & & & & \vdots\,{\scriptstyle f''} & & \vdots \\ L_mX & \longrightarrow & L_m(X_{m-\tau-1}) & \longrightarrow & \cdots & \longrightarrow & L_m(X_{m-\tau-k}) & \longrightarrow & L_m(X_{m-\tau-1-k}). \end{array}$$

Now the proof is complete. □

Proof of Theorem 1.3. Since the assumption $m - k \geq n + s_0 + 1$ holds if and only if $m - \tau - 1 - k \geq s_0 + n - \tau$, we have the following commutative diagram:

$$\begin{array}{ccccc} L_kY & \xrightarrow{\quad f \quad} & L_mX & \longrightarrow & L_m(X_{m-\tau-1-k}) \\ & & \Big\downarrow & & \Big\downarrow \\ & & L_nX & \longrightarrow & L_n(X_{s_0+n-\tau}). \end{array}$$

Since the top horizontal composite is null by Corollary 4.2 and the bottom horizontal arrow is a split injection by Theorem 2.7, we see the composite

$$L_kY \xrightarrow{f} L_mX \to L_nX$$

is null, as desired. □

DEPARTMENT OF MATHEMATICS, NAGOYA INSTITUTE OF TECHNOLOGY, GOKISO, SHOWA-KU, NAGOYA 466-8555, JAPAN
E-mail: norihiko@math.kyy.nitech.ac.jp

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