The Adams Spectral Sequence and the Triple Transfer
Author(s): Norihiko Minami
Source: *American Journal of Mathematics*, Vol. 117, No. 4 (Aug., 1995), pp. 965-985
Published by: The Johns Hopkins University Press
Stable URL: https://www.jstor.org/stable/2374955
Accessed: 07-04-2019 21:30 UTC

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THE ADAMS SPECTRAL SEQUENCE AND THE TRIPLE TRANSFER

By NORIHIKO MINAMI

Dedicated to Professor Shôro Araki

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0. Introduction. Recently, we [M1][M2] discovered some very strong restrictions on the mod-p Hurewicz image of $\Sigma^\infty B\Sigma_p \wedge \Sigma_p$, using BP-Adams operations [N][Ar2].

In this paper, we apply this BP-Adams operation technique to the case of BV_3, where $V_3 = (\mathbb{Z}/2)^3$. Just as before [M1][M2], the calculation of $BP_*(\wedge^n P)$ by Johnson-Wilson-Yang [JW][JWY], based upon the affirmative solution of the Conner-Floyd conjecture [RW][Mt], is used in an essential way. But the new ingredient here is the determination of $PH_*(BV_3)$ $\big(:= \mathbb{Z}/2 \otimes_{\mathcal{A}_*} H_*(BV_3)\big)$ due to Kameko and others [K][ACH][B2]. (Of course, only those elements in $BP_*(BV_3)$, whose Thom reduction image is contained in $PH_*(BV_3)$, are relevant for our purpose.) Furthermore, Boardman [B2] gave a complete analysis of the composite

$$PH_n(BV_3) \twoheadrightarrow \mathbb{Z}/2 \otimes_{GL_3(\mathbb{F}_2)} PH_n(BV_3) \longrightarrow \mathrm{Ext}_{\mathcal{A}_*}^{3,n+3}(\mathbb{Z}/2, \mathbb{Z}/2)$$

and showed this latter map, defined by Singer [S], is an isomorphism. Notice that this composite is induced by the triple transfer

$$BV_{3+} \to S^0,$$

which is the stable adjoint to the composite

$$BV_3 \xrightarrow{Breg} B\Sigma_8 \xrightarrow{D-L} Q_8S^0,$$

where $reg : V_3 \to \Sigma_{|V_3|} = \Sigma_8$ is the regular representation, and $D-L : B\Sigma_8 \to Q_8S^0$ is the Dyer Lashof map. Anyway, this allows us to interpret our calculation in the context of the Adams spectral sequence [Ad1][Wa].

Now, the following is our main result.

Manuscript received December 23, 1993; revised June 23, 1994.
Research supported in part by a University of Alabama Research Grant.
American Journal of Mathematics 117 (1995), 965–985.

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THEOREM 0.1. *Suppose an element in* $\mathrm{Ext}_{\mathcal{A}_*}^{3,n+3}(\mathbb{Z}/2,\mathbb{Z}/2)$ *is a permanent cycle and detects an element of* $\pi_n^s(S^0)$ *which factors through the triple transfer* $BV_{3+} \to S^0$. *Then it is one of the following*

$$\begin{cases} h_u h_{t+u} h_{s+t+u} & \text{such that } s \geq 2, t \geq 2, u \leq 6 \\ c_u & \text{such that } s = 2, t = 1, u \leq 6 \\ h_{1+u} h_{s+u-1}^2 & \text{such that } s \geq 5, t = 0, u \leq 5 \\ h_u^2 h_{s+u},\ h_{1+u} h_{s+u-1}^2 + h_u^2 h_{s+u} & \text{such that } s \geq 5, t = 0, u \leq 6 \\ h_u^2 h_{s+u} & \text{such that } 3 \leq s \leq 4, t = 0, u \leq 6 \\ h_{u+1}^3 = h_u^2 h_{2+u} & \text{such that } s = 2, t = 0, u \leq 5 \\ h_0 h_t^2 & \text{such that } s = 0, t \geq 3 \\ h_0^3 & \end{cases}$$

COROLLARY 0.2. *No element in the image of*

$$(Sq^0)^7 : \mathrm{Ext}_{\mathcal{A}_*}^{3,n+3}(\mathbb{Z}/2,\mathbb{Z}/2) \to \mathrm{Ext}_{\mathcal{A}_*}^{3,2^7(n+3)}(\mathbb{Z}/2,\mathbb{Z}/2)$$

detects an element of $\pi_{2^7(n+3)-3}^s(S^0)$ *which factors through the triple transfer* $BV_{3+} \to S^0$.

These, together with our previous results [M1][M2], tempt us to propose:

NEW DOOMSDAY CONJECTURE. *For each s, there exists some integer n(s) such that no element in the image of*

$$(\mathcal{P}^0)^{n(s)}\left(\mathrm{Ext}_{\mathcal{A}_*}^{s,*}(\mathbb{Z}/p,\mathbb{Z}/p)\right) \subseteqq \left(\mathrm{Ext}_{\mathcal{A}_*}^{s,p^{n(s)}*}(\mathbb{Z}/p,\mathbb{Z}/p)\right)$$

is a nontrivial permanent cycle.

Following a suggestion of Mark Mahowald, we can also formulate an analogous speculation in terms of the root invariant:

R.I. DOOMSDAY CONJECTURE. *For each s, there exists some integer m(s) such that, for any element* $f \in \pi_*^s(S^0)$ *of Adams filtration s, its m(s)-fold iterated root invariant* $R^{m(s)}(f)$ *has Adams filtration strictly higher than s.*

This paper is organized as follows: In §1, we use the Adams spectral sequence to show that those relevant elements in $BP_*(\wedge^3 P)$ have gigantic order. In §2, we study the action of the BP-Adams operations on $BP_*(\wedge^3 P)$. This forces elements in the BP-Hurewicz image $\pi_*^s(\wedge^3 P) \to BP_*(\wedge^3 P)$ to have relatively low order. In §3, we recall the results of Kameko [K], Ali-Crabb-Hubbuck [ACH],

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and Boardman [B1, B2]; then these results allow us to apply our studies in §1 and §2 to prove Theorem 0.1, whose statement involves the third line elements in the Adams spectral sequence of the sphere and the triple transfer. Finally, in §4, we pose the aforementioned conjectures, and discuss their background. Philosophically, this is the core of this paper.

Acknowledgments. The main result in this paper was announced during the Čech centennial homotopy theory conference at Northeastern University in June 1993. The author would like to thank Mike Boardman, Mark Mahowald, Haynes Miller, Doug Ravenel, and Steve Wilson for their suggestions and constant encouragement. The referee's suggestions greatly helped us to improve our presentation of this paper. Also, the author would like to express his gratitude to mathematicians at the Massachusetts Institute of Technology for their hospitality during the author's visits to MIT during the summer of 1993. Neil Strickland offered the author significant help with Emacs. The author's visit to MIT would not have been made possible without the generous support of Dan and Nora Kan, to whom the author expresses his highest gratitude. The travel support to MIT was provided by Alabama EPSCoR.

Notation and conventions.

$\mathbb{Z}/2\{g\}$ stands for $\mathbb{Z}/2$ with g as its generator.

H_* stands for the mod-2 homology

$$P = \Sigma^\infty \mathbb{R}P^\infty$$

$x_i \in H_i(P)$ is the generator.

$$BP_* = \mathbb{Z}_{(2)}[v_1, v_2, \ldots].$$

$$\mathcal{A}_* = P(\xi_1, \xi_2, \ldots) \qquad \text{where } |\xi_n| = 2^n - 1$$

$$\mathrm{Ext}^{*,*}_{\mathcal{A}_*}\left(\mathbb{Z}/2, H_*(BP)\right) \cong \mathrm{Ext}^{*,*}_{E_*}(\mathbb{Z}/2, \mathbb{Z}/2) = \mathbb{Z}/2[u_0, u_1, \ldots],$$

where $u_i \in \mathrm{Ext}^{1,2^{i+1}-1}_{E_*}(\mathbb{Z}/2, \mathbb{Z}/2)$ is expressed as $[\xi_{i+1}]$ in the cobar complex, and corresponds to the usual (Hazewinkel [H] or Araki [Ar1], whichever) generator $v_i \in BP_{2^{i+1}-2}$ (*resp.* p) when $i \geqq 1$ (*resp.* $i = 0$). $E\langle k\rangle$ is the exterior quotient Hopf algebra of $\mathcal{A}_*$, generated by $\xi_1, \ldots, \xi_{k+1}$, whose notation is intended to suggest

$$\mathrm{Ext}^{*,*}_{\mathcal{A}_*}\left(\mathbb{Z}/2, H_*\left(BP\langle k\rangle\right)\right) \cong \mathrm{Ext}^{*,*}_{E\langle k\rangle_*}(\mathbb{Z}/2, \mathbb{Z}/2).$$

1. The Adams spectral sequence of $BP_*(\wedge^3 P)$. For our purpose, we need to know $BP_*(BV_3)$. But the canonical projection

$$BV_3 = (B\mathbb{Z}/2) \times (B\mathbb{Z}/2) \times (B\mathbb{Z}/2) \to (B\mathbb{Z}/2) \wedge (B\mathbb{Z}/2) \wedge (B\mathbb{Z}/2) = \wedge^3 P$$

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stably splits so that $\wedge^3 P$ constitutes the essential part of BV_3: $x_i \otimes x_j \otimes x_k \in H_*(BV_{3+})$ $0 \leq i,j,k$ is contained in $H_*(\wedge^3 P)$ iff. $1 \leq i,j,k$.

Therefore, we are going to study $BP_*(\wedge^3 P)$, which still has a very complicated additive structure. Fortunately, as was noticed by [JWY][JW], the affirmative solution [RW][Mt] of the Connor-Floyd conjecture implies that its classical Adams spectral sequence collapses at its E_2-term, and this allows us to use more tractable $\mathrm{Ext}^{*,*}_{E\langle 3\rangle_*}(\mathbb{Z}/2, H_*(\wedge^3 P))$, instead of $BP_*(\wedge^3 P)$, to evaluate the BP_*-order of those elements we are interested in. (Note that the multiplication by 2 in $BP_*(\wedge^3 P)$ corresponds to the multiplication by $u_0 \in \mathrm{Ext}^{1,1}_{E\langle 3\rangle_{**}}(\mathbb{Z}/2, \mathbb{Z}/2)$ in $\mathrm{Ext}^{*,*}_{E\langle 3\rangle_*}(\mathbb{Z}/2, H_*(\wedge^3 P))$.) We begin with a summary of known results, which are necessary in our approach:

Proposition A. (a) *The Adams spectral sequence*

$$\mathrm{Ext}^{*,*}_{\mathcal{A}_*}\left(\mathbb{Z}/2, H_*\left(BP \wedge \wedge^3 P\right)\right) \Rightarrow BP_*(\wedge^3 P)$$

collapses.

(b) *As* $\mathbb{Z}/2[u_3]$*-modules,*

$$\mathrm{Ext}^{*,*}_{E\langle 3\rangle_*}\left(\mathbb{Z}/2, H_*\left(\wedge^3 P\right)\right) = \mathrm{Ext}^{*,*}_{E\langle 2\rangle_*}\left(\mathbb{Z}/2, H_*\left(\wedge^3 P\right)\right) \otimes \mathbb{Z}/2[u_3].$$

(c) $\mathrm{Ext}^{*,*+odd}_{E\langle 3\rangle_*}(\mathbb{Z}/2, H_*(\wedge^3 P))$ *contains a sub* $\mathbb{Z}/2[u_0, u_1, u_2, u_3]$*-module*

$$\oplus_{k,l,m\geq 1}\mathbb{Z}/2\{x_{2k-1} \otimes x_{2l-1} \otimes x_{2m-1}\} \otimes \mathbb{Z}/2[u_3],$$

where $x_{2k-1} \otimes x_{2l-1} \otimes x_{2m-1} \in \mathrm{Ext}^{0,2(k+l+m)-3}_{E\langle 3\rangle_*}(\mathbb{Z}/2, H_*(\wedge^3 P))$.

(d) *There is a canonical identification of the above* $\mathbb{Z}/2[u_0, u_1, u_2, u_3]$*-submodules of* $\mathrm{Ext}^{*,*+odd}_{E\langle 3\rangle_*}(\mathbb{Z}/2, H_*(\wedge^3 P))$

$$\begin{array}{cccc} i: \oplus_{k,l,m\geq 1}\mathbb{Z}/2\{x_{2k-1} \otimes x_{2l-1} \otimes x_{2m-1}\} \otimes \mathbb{Z}/2[u_3], & \to & \otimes^3_{\mathbb{Z}/2[u_0,u_1,u_2,u_3]} \mathrm{Ext}^{*,*}_{E\langle 3\rangle_*}\left(\mathbb{Z}/2, H_*(P)\right) \\ x_{2k-1} \otimes x_{2l-1} \otimes x_{2m-1} & \mapsto & x_{2k-1} \otimes x_{2l-1} \otimes x_{2m-1} \end{array}$$

Proof. For (a)(b), see [JWY][JW]. (c) follows from (a)(b) and the solution of the Conner-Floyd conjecture [RW][Mt], which was also used in (a) [JWY].

For (d), we first notice that $\otimes^3_{\mathbb{Z}/2[u_0,u_1,u_2,u_3]} \mathrm{Ext}^{*,*}_{E\langle 3\rangle_*}(\mathbb{Z}/2, H_*(P))$ is generated by $\oplus_{k,l,m\geq 1}\mathbb{Z}/2\{x_{2k-1} \otimes x_{2l-1} \otimes x_{2m-1}\}$, as a $\mathbb{Z}/2[u_0, u_1, u_2, u_3]$-module. This is because such is the case for each tensor factor $\mathrm{Ext}^{*,*}_{E\langle 3\rangle_*}(\mathbb{Z}/2, H_*(P))$ [D]. So the claim would follow if we could canonically embed

$$\otimes^3_{\mathbb{Z}/2[u_0,u_1,u_2,u_3]} \mathrm{Ext}^{*,*}_{E\langle 3\rangle_*}(\mathbb{Z}/2, H_*(P))$$

in $\mathrm{Ext}^{*,*}_{E\langle 3\rangle_*}(\mathbb{Z}/2, H_*(\wedge^3 P))$, thanks to (c). Thus, it is enough to establish a short

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exact sequence

$$\text{(K)}\quad\begin{aligned}&0 \to \otimes^3_{\mathbb{Z}/2[u_0,u_1,u_2,u_3]} \operatorname{Ext}^{*,*}_{E\langle 3\rangle_*}(\mathbb{Z}/2, H_*(P)) \to \operatorname{Ext}^{*,*+odd}_{E\langle 3\rangle_*}(\mathbb{Z}/2, H_*(\wedge^3 P))\\ &\to \operatorname{Tor}_{\mathbb{Z}/2[u_0,u_1,u_2,u_3]}(\operatorname{Ext}^{*,*}_{E\langle 3\rangle_*}(\mathbb{Z}/2, H_*(P)),\\ &\operatorname{Tor}_{\mathbb{Z}/2[u_0,u_1,u_2,u_3]}(\operatorname{Ext}^{*,*}_{E\langle 3\rangle_*}(\mathbb{Z}/2, H_*(P)), \operatorname{Ext}^{*,*}_{E\langle 3\rangle_*}(\mathbb{Z}/2, H_*(P\wedge P)))) \to 0.\end{aligned}$$

But, (K) is an immediate consequence of the Ext-analogue of the Landweber's bordism Künneth theorem [L]: The point is that the cofiber sequence

$$\mathbb{R}P^\infty \xrightarrow{\ i\ } \mathbb{C}P^\infty \longrightarrow T,$$

where i is induced by the inclusion $\mathbb{Z}/2 \hookrightarrow S^1$ and T is the cofiber of i (actually a Thom complex), induces the short exact sequence (here we use the notation $P = \Sigma^\infty \mathbb{R}P^\infty$)

$$\begin{aligned}0 &\to \operatorname{Ext}^{*-1,*-1}_{E\langle 3\rangle_*}\left(\mathbb{Z}/2, H_*\left(\Sigma^{-1}\mathbb{C}P^\infty\right)\right)\\ &\to \operatorname{Ext}^{*,*}_{E\langle 3\rangle_*}\left(\mathbb{Z}/2, H_*\left(\Sigma^{-1}T\right)\right)\\ &\to \operatorname{Ext}^{*,*}_{E\langle 3\rangle_*}(\mathbb{Z}/2, H_*(P)) \to 0,\end{aligned}$$

where $\operatorname{Ext}^{*,*}_{E\langle 3\rangle_*}(\mathbb{Z}/2, H_*(\Sigma\mathbb{C}P^\infty)) \cong \operatorname{Ext}^{*,*}_{E\langle 3\rangle_*}(\mathbb{Z}/2, H_*(\Sigma^{-1}T))$ is a free $\mathbb{Z}/2[u_0, u_1, u_2, u_3]$-module. Actually, the usual Landweber's bordism Künneth theorem [L][JW] is based on the fact that $BP_*(\Sigma^{-1}\mathbb{C}P^\infty)$ and $BP_*(\Sigma^{-1}T) \cong BP_*(\Sigma\mathbb{C}P^\infty_+)$ are both free over BP_*, and the following short exact sequence

$$0 \to BP_*(\Sigma^{-1}\mathbb{C}P^\infty) \to BP_*(\Sigma^{-1}T) \to BP_*(P) \to 0.$$

Thus, we get (K) just as in [JW] (see also our discussion after Lemma 2.1). □

For us to understand the order of those relevant elements in $BP_*(\wedge^3 P)$, we need to know the formula for the u_0-action on

$$\oplus_{k,l,m\geq 1}\mathbb{Z}/2\{x_{2k-1} \otimes x_{2l-1} \otimes x_{2m-1}\} \otimes \mathbb{Z}/2[u_3],$$

regarded as a $\mathbb{Z}/2[u_0, u_1, u_2, u_3]$-submodules of $\operatorname{Ext}^{*,*+odd}_{E\langle 3\rangle_*}(\mathbb{Z}/2, H_*(\wedge^3 P))$.

For a technical reason, we can minimize our task by writing down the actions of u_1 and u_2 simultaneously. To express such actions concisely, we let $u_jX^rY^sZ^t$ stand for the $\mathbb{Z}/2[u_3]$-module map

$$x_{2k-1} \otimes x_{2l-1} \otimes x_{2m-1} \longmapsto u_j x_{2(k-r)-1} \otimes x_{2(l-s)-1} \otimes x_{2(m-t)-1}.$$

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Now, the following is the action formula:

LEMMA 1.1. *The* $\mathbb{Z}/2[u_0, u_1, u_2, u_3]$*-module structure on*

$$\oplus_{k,l,m\geq 1}\mathbb{Z}/2\{x_{2k-1}\otimes x_{2l-1}\otimes x_{2m-1}\}\otimes \mathbb{Z}/2[u_3]$$

is determined by the following equalities as $\mathbb{Z}/2[u_3]$*-module self-maps on it:*

$$\begin{aligned} u_0 &= u_3 \sum_{i,j,k\in\Lambda, i+j+k=7} X^i Y^j Z^k \\ u_1 &= u_3 \sum_{i,j,k\in\Lambda, i+j+k=6} X^i Y^j Z^k \\ u_2 &= u_3 \sum_{i,j,k\in\Lambda, i+j+k=4} X^i Y^j Z^k, \end{aligned}$$

where $\Lambda = \{0, 1, 2, 4\}$.

Proof. As these actions of u_0, u_1, u_2, and u_3 clearly commute, they together define a $\mathbb{Z}/2[u_0, u_1, u_2, u_3]$-module structure. Also, we can see very easily

$$\text{(C)} \qquad \begin{cases} u_0 + u_1X + u_2X^3 + u_3X^7 = 0 \\ u_0 + u_1Y + u_2Y^3 + u_3Y^7 = 0 \\ u_0 + u_1Z + u_2Z^3 + u_3Z^7 = 0. \end{cases}$$

Actually,

$$\begin{aligned} u_0+u_1X+u_2X^3+u_3X^7 &= u_3(X^4Y^2Z+X^4YZ^2+X^2Y^4Z+X^2YZ^4+XY^4Z^2+XY^2Z^4) \\ &\quad + u_3X(X^4Y^2 + X^4YZ + X^4Z^2 + X^2Y^4 + X^2Y^2Z^2 \\ &\quad + X^2Z^4 + XY^4Z + XYZ^4 + Y^4Z^2 + Y^2Z^4) \\ &\quad + u_3X^3(X^4 + X^2Y^2 + X^2YZ + X^2Z^2 + XY^2Z + XYZ^2 \\ &\quad + Y^4 + Y^2Z^2 + Z^4) \\ &\quad + u_3X^7 \\ &= 0, \end{aligned}$$

for example.

Notice that (C) is nothing but the characterizing property of the $\mathbb{Z}/2[u_0, u_1, u_2, u_3]$-module structure on $\otimes^3_{\mathbb{Z}/2[u_0,u_1,u_2,u_3]} \mathrm{Ext}^{*,*}_{E\langle 3\rangle_*}(\mathbb{Z}/2, H_*(P))$. This is because $\mathrm{Ext}^{*,*}_{E\langle 3\rangle_*}(\mathbb{Z}/2, H_*(P))$ is generated by $\{x_{2i-1}\}_{i\in\mathbb{N}}$ with the relations $\{u_0x_{2k-1} + u_1x_{2(k-1)-1} + u_2x_{2(k-3)-1} + u_3x_{2(k-7)-1}\}_{k\geq 8}$ (i.e. $u_0 + u_1X + u_2X^3 + u_3X^7 = 0$), as a

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$\mathbb{Z}/2[u_0, u_1, u_2, u_3]$-module [D]. So, $\otimes^3_{\mathbb{Z}/2[u_0,u_1,u_2,u_3]} \mathrm{Ext}^{*,*}_{E\langle 3\rangle_*}(\mathbb{Z}/2, H_*(P))$ is generated as a $\mathbb{Z}/2[u_0, u_1, u_2, u_3]$-module by $\{x_{2k-1} \otimes x_{2l-1} \otimes x_{2m-1}\}_{k,l,m\geq 1}$ subject to (C). Therefore, when we denote $\left(\oplus_{k,l,m\geq}\mathbb{Z}/2\{x_{2k-1} \otimes x_{2l-1} \otimes x_{2m-1}\} \otimes \mathbb{Z}/2[u_3]\right)'$ by $\oplus_{k,l,m\geq 1}\mathbb{Z}/2\{x_{2k-1} \otimes x_{2l-1} \otimes x_{2m-1}\} \otimes \mathbb{Z}/2[u_3]$ equipped with the proposed $\mathbb{Z}/2[u_0, u_1, u_2, u_3]$-module structure, we can define a $\mathbb{Z}/2[u_0, u_1, u_2, u_3]$-module homomorphism

$$h : \otimes^3_{\mathbb{Z}/2[u_0,u_1,u_2,u_3]} \mathrm{Ext}^{*,*}_{E\langle 3\rangle_*}(\mathbb{Z}/2, H_*(P)) \\ \to \left(\oplus_{k,l,m\geq 1}\mathbb{Z}/2\{x_{2k-1} \otimes x_{2l-1} \otimes x_{2m-1}\} \otimes \mathbb{Z}/2[u_3]\right)'$$

so that $h(x_{2k-1} \otimes x_{2l-1} \otimes x_{2m-1}) = x_{2k-1} \otimes x_{2l-1} \otimes x_{2m-1}$ for any $k, l, m \geq 1$.

On the other hand, in Proposition A (d), we constructed a $\mathbb{Z}/2[u_0, u_1, u_2, u_3]$-module homomorphism

$$i : \oplus_{k,l,m\geq 1}\mathbb{Z}/2\{x_{2k-1} \otimes x_{2l-1} \otimes x_{2m-1}\} \otimes \mathbb{Z}/2[u_3] \\ \to \otimes^3_{\mathbb{Z}/2[u_0,u_1,u_2,u_3]} \mathrm{Ext}^{*,*}_{E\langle 3\rangle_*}(\mathbb{Z}/2, H_*(P))$$

such that $i(x_{2k-1} \otimes x_{2l-1} \otimes x_{2m-1}) = x_{2k-1} \otimes x_{2l-1} \otimes x_{2m-1}$ for any $k, l, m \geq 1$. Then, clearly,

$$h \circ i : \oplus_{k,l,m\geq 1} \mathbb{Z}/2\{x_{2k-1} \otimes x_{2l-1} \otimes x_{2m-1}\} \otimes \mathbb{Z}/2[u_3] \\ \longrightarrow \left(\oplus_{k,l,m\geq 1}\mathbb{Z}/2\{x_{2k-1} \otimes x_{2l-1} \otimes x_{2m-1}\} \otimes \mathbb{Z}/2[u_3]\right)'$$

is a $\mathbb{Z}/2[u_0, u_1, u_2, u_3]$-module isomorphism which induces the identity map on the underlying set. This immediately implies that the proposed $\mathbb{Z}/2[u_0, u_1, u_2, u_3]$-module structure on $\oplus_{k,l,m\geq 1}\mathbb{Z}/2\{x_{2k-1} \otimes x_{2l-1} \otimes x_{2m-1}\} \otimes \mathbb{Z}/2[u_3]$ is the right one. □

Remark 1.2. Using the similar method, we can write down explicitly the $\mathbb{Z}/p[u_0, \ldots, u_n]$-module structure of

$$\otimes^n_{\mathbb{Z}/p[u_0,\ldots,u_n]} \mathrm{Ext}^{*,*}_{E\langle n\rangle}(\mathbb{Z}/p, H_*(P)),$$

for any n and p. For this, see [M3].

We now define $Sq^0 : H_*(BV_3) \to H_{2*+3}(BV_3)$ by $Sq^0(x_i \otimes x_j \otimes x_k) = x_{2i+1} \otimes x_{2j+1} \otimes x_{2k+1}$.

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PROPOSITION 1.3. *The order of any element in* $BP_{2^u(*+3)-3}(BV_3)$, *whose Thom reduction image is contained in* $(Sq^0)^u H_*(BV_3)$, *is divisible by*

$$2^{\left[\frac{2^u-1}{8}\right]+1} = 2^{(2^{u-3})} \qquad \text{if} \quad u \geqq 3.$$

Proof. We begin by introducing the usual lexicographical order among monomials in $H_*(BV_3)$:

$$x_a \otimes x_b \otimes x_c \prec x_{a'} \otimes x_{b'} \otimes x_{c'}$$
$$\iff \quad a < a';\ \text{or } a = a', b < b';\ \text{or } a = a', b = b', c < c'.$$

Then, we may assume that any element in $(Sq^0)^u H_*(BV_3)$ is of the form

$$x_{i_0} \otimes x_{j_0} \otimes x_{k_0} + \text{sum of monomials with higher order}$$

with $2^u - 1 \leq i_0 \leq j_0 \leq k_0$, by changing the order of the factors if necessary.

Now, suppose that the Thom reduction of an element $\Theta \in BP_{2^u(*+3)-3)}(BV_3)$ is in $(Sq^0)^u H_*(BV_3)$. Then, as $Sq^0\,(H_*\,(BV_3)) \subseteq H_{2*+3}(\wedge^3 P)$ and $\wedge^3 P$ is a stable summand of BV_3, we may assume $\Theta \in BP_{2^u(*+3)-3)}(\wedge^3 P)$ and it is enough to show $2^{\left[\frac{2^u-1}{8}\right]}\Theta \neq 0$ for such Θ. By Proposition A, what we have to show is

$$u_0^{\left[\frac{2^u-1}{8}\right]}\left(x_{i_0} \otimes x_{j_0} \otimes x_{k_0} + \text{sum of monomials with higher order}\right)$$
$$\neq\ 0 \in \oplus_{k,l,m \geq 1} \mathbb{Z}/2\{x_{2k-1} \otimes x_{2l-1} \otimes x_{2m-1}\} \otimes \mathbb{Z}/2[u_3]$$

But this is certainly so, for Lemma 1.1 implies the left-hand side is

$$u_3^{\left[\frac{2^u-1}{8}\right]}\left(x_{i_0-8\left[\frac{2^u-1}{8}\right]} \otimes x_{j_0-4\left[\frac{2^u-1}{8}\right]} \otimes x_{k_0-2\left[\frac{2^u-1}{8}\right]} + \begin{array}{c}\text{sum of monomials}\\ \text{with higher order}\end{array}\right),$$

which is clearly nonzero, as $2^u - 1 \leq i_0 \leq j_0 \leq k_0$. □

2. Adams operations on $BP_{odd}(\wedge^3 P)$**.** Here we study the action of the BP-Adams operation ψ^3 [N][Ad2] on $BP_{odd}(\wedge^3 P)$. For this purpose, we need to describe the ψ^3 action on $BP_*(X \wedge P)$ for general X, in the context of the Landweber bordism Künneth theorem [L]:

$$0 \to BP_*(P) \otimes BP_*(X) \to BP_*(X \wedge P) \to \mathrm{Tor}\,(BP_*\,(P)\,, BP_*\,(X)) \to 0.$$

We now list necessary elementary results about the BP-Adams operation ψ^3.

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LEMMA B. (a) $\psi^3 : BP_{2k-1}(P) \to BP_{2k-1}(P)$ *acts as the multiplication by* 3^k.

(b) *Let* $\alpha_3 : \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ *be the map induced by* $S^1 \to S^1$, *regarding* $\mathbb{C}P^\infty = BS^1$. *Then* $\psi^3\alpha_{3*} ; BP_{2l}(\mathbb{C}P^\infty) \to BP_{2l}(\mathbb{C}P^\infty)$ *acts as the multiplication by* 3^l.

(c) ψ^3 *is a map of ring spectrum, and so commutes with the pairings: The diagram*

$$\begin{array}{ccc} BP_*(P) \otimes BP_*(X) & \xrightarrow{\wedge} & BP_*(P \wedge X) \\ \downarrow{\scriptstyle \psi^3\otimes\psi^3} & & \downarrow{\scriptstyle \psi^3} \\ BP_*(P) \otimes BP_*(X) & \xrightarrow{\wedge} & BP_*(P \wedge X) \end{array}$$

commutes.

(d) *The diagram*

$$\begin{array}{ccc} BP_*(\mathbb{C}P^\infty) \otimes BP_*(X) & \underset{\cong}{\xrightarrow{\wedge}} & BP_*(\mathbb{C}P^\infty \wedge X) \\ \downarrow{\scriptstyle \psi^3\alpha_{3*}\otimes\psi^3} & & \downarrow{\scriptstyle \psi^3\alpha_{3*}} \\ BP_*(\mathbb{C}P^\infty) \otimes BP_*(X) & \underset{\cong}{\xrightarrow{\wedge}} & BP_*(\mathbb{C}P^\infty \wedge X) \end{array}$$

commutes.

Proof. (a)(b) This is well-known; see for instance [H].

(c) This is also well-known: see [Ad2] or [W].

(d) This follows from the following commutative diagram:

$$\begin{array}{ccc} (BP \wedge X) \wedge (BP \wedge \mathbb{C}P^\infty) & \longrightarrow & BP \wedge (X \wedge \mathbb{C}P^\infty) \\ {\scriptstyle (\psi^3\wedge 1)\wedge(\psi^3\wedge 1)}\downarrow & & \downarrow{\scriptstyle \psi^3\wedge(1\wedge 1)} \\ (BP \wedge X) \wedge (BP \wedge \mathbb{C}P^\infty) & \longrightarrow & BP \wedge (X \wedge \mathbb{C}P^\infty) \\ {\scriptstyle (1\wedge 1)\wedge(1\wedge\alpha_3)}\downarrow & & \downarrow{\scriptstyle 1\wedge(1\wedge\alpha_3)} \\ (BP \wedge X) \wedge (BP \wedge \mathbb{C}P^\infty) & \longrightarrow & BP \wedge (X \wedge \mathbb{C}P^\infty), \end{array}$$

where the commutativity of the upper diagram follows from the fact that ψ^3 is a map of ring spectra. □

For our purpose, it is more convenient to express the above results in the following form:

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Lemma 2.1. *We have the following commutative diagrams:*

$$
(1)\qquad
\begin{array}{ccccc}
\oplus_k BP_{2k-1}(P)\otimes BP_*(X) & \twoheadrightarrow & BP_*(P)\otimes BP_*(X) & \rightarrowtail & BP_*(P\wedge X)\\
{\scriptstyle \oplus_k 3^k\otimes\psi^3}\downarrow & & {\scriptstyle \psi^3}\downarrow & & {\scriptstyle \psi^3}\downarrow\\
\oplus_k BP_{2k-1}(P)\otimes BP_*(X) & \twoheadrightarrow & BP_*(P)\otimes BP_*(X) & \rightarrowtail & BP_*(P\wedge X)
\end{array}
$$

$$
(2)\qquad
\begin{array}{ccccc}
BP_*(P\wedge X) & \twoheadrightarrow & \mathrm{Tor}_{BP_*}(BP_*P, BP_*X) & \rightarrowtail & \oplus_l BP_{2l}(\mathbb{C}P^\infty)\otimes BP_*(X)\\
{\scriptstyle \psi^3}\downarrow & & {\scriptstyle \psi^3}\downarrow & & {\scriptstyle \oplus_l 3^l\otimes\psi^3}\downarrow\\
BP_*(P\wedge X) & \twoheadrightarrow & \mathrm{Tor}_{BP_*}(BP_*P, BP_*X) & \rightarrowtail & \oplus_l BP_{2l}(\mathbb{C}P^\infty)\otimes BP_*(X)
\end{array}
$$

Proof. (1) This follows from Lemma B(a)(c).

(2) For this, just recall that the bordism Künneth formula is induced by the cofibration sequence associated with the canonical nontrivial map $P \to \mathbb{C}P^\infty$, which accompanies the homotopy commutative diagram:

$$
\begin{array}{ccc}
P & \longrightarrow & \mathbb{C}P^\infty\\
\| & & \downarrow{\scriptstyle \alpha_3}\\
P & \longrightarrow & \mathbb{C}P^\infty.
\end{array}
$$

Then the claim is an immediate consequence of Lemma B(b)(d). □

Now, we are ready to write down the ψ^3 action on $BP_{odd}(\wedge^3 P)$ almost completely. For this, we recall the presentation of $BP_{odd}(\wedge^3 P)$ via the bordism Künneth theorem [L][JW]: First, by putting $X = P$ in the bordism Künneth theorem,

$$
\begin{aligned}
BP_{even}(P\wedge P) &= BP_*(P)\otimes_{BP_*} BP_*(P)\\
BP_{odd}(P\wedge P) &= \mathrm{Tor}_{BP_*}(BP_*(P), BP_*(P)).
\end{aligned}
$$

Then, apply the bordism Künneth theorem of [L] again, by putting $P\wedge P$ in the above forms,

$$
\begin{aligned}
0 &\to \otimes^3_{BP_*} BP_*(P) \to BP_{odd}(\wedge^3 P)\\
&\to \mathrm{Tor}_{BP_*}\left(BP_*(P), \mathrm{Tor}_{BP_*}(BP_*(P), BP_*(P))\right) \to 0.
\end{aligned}
$$

(cf. [JW] and our proof of Prop. A (d).)

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COROLLARY 2.2. *The following diagram is commutative:*

$$\begin{array}{ccccc}
[\otimes^3_{BP_*} BP_*(P)]_{2n-3} & \rightarrowtail & BP_{2n-3}(\wedge^3 P) & \twoheadrightarrow & [\mathrm{Tor}_{BP_*}(BP_*P, \mathrm{Tor}_{BP_*}(BP_*P, BP_*P))]_{2n-3} \\
\downarrow{\times 3^n} & & \downarrow{\psi^3} & & \downarrow{\times 3^{n-1}} \\
[\otimes^3_{BP_*} BP_*(P)]_{2n-3} & \rightarrowtail & BP_{2n-3}(\wedge^3 P) & \twoheadrightarrow & [\mathrm{Tor}_{BP_*}(BP_*P, \mathrm{Tor}_{BP_*}(BP_*P, BP_*P))]_{2n-3}
\end{array}$$

Proof. We just have to use Lemma 2.1 (1) (*resp.* (2)) repeatedly twice for the commutativity of the left (*resp.* right) hand side diagram. □

We now recall the usual result in the elementary number theory [AA][MM][H][M1]:

LEMMA C. *Write $n = 2^r m$, with m odd. Then*

$$\nu_2(3^n - 1) = \begin{cases} r+2 & \text{if } r \geqq 1 \\ 1 & \text{if } r = 0. \end{cases}$$

Proof. This is well-known and quite easy to show: The key is to prove

$$(1+2)^{2^r} \equiv 1 + 2^{r+2} \pmod{2^{r+3}}$$

for all $r \geqq 1$, by mathematical induction. □

Now, Corollary 2.2 and Lemma C immediately imply the following:

PROPOSITION 2.3. *The order of any element in*

$$\mathrm{Ker}\ (\psi^3 - 1)\ |_{BP_{4d-3}(\wedge^3 P)}$$

divides $2^{\nu_2(d)+4}$.

Proof. Let $x \in \mathrm{Ker}\ (\psi^3 - 1)\ |_{BP_{4d-3}(\wedge^3 P)}$. We first claim $2x \in [\otimes^3_{BP_*} BP_*(P)]_{4d-3}$. But, this immediately follows from Corollary 2.2, as $\nu_2(3^{2d-1} - 1) = 1$ by Lemma C. Now, using Corollary 2.2 and Lemma C again, we see that the order of $2x$ divides $2^{\nu_2(3^{2d}-1)} = 2^{\nu_2(d)+3}$. So the claim follows. □

3. The main theorem and its proof. Our main result focuses upon the composite

$$\pi^s_n(BV_3) \to PH_n(BV_3) \twoheadrightarrow \mathbb{Z}/2 \otimes_{GL_3(\mathbb{F}_2)} PH_n(BV_3) \longrightarrow \mathrm{Ext}^{3,n+3}_{\mathcal{A}_*}(\mathbb{Z}/2, \mathbb{Z}/2),$$

where the last map, defined by [S], was shown to be an isomorphism by [B2][Wa].

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To state known results related to this composite, we need different types of Steenrod operations which we now review. A general reference here is [My]. (For simplicity, we consider only the 2 primary case):

Let C_2 be the cyclic group of order 2 generated by t. Then one of the definitions of the Steenrod operations on a topological space X, the one given as the cup-r products, begins with a map of augmented chain complexes of $\mathbb{Z}/2[C_2]$-modules

$$\mathbf{W} \otimes S_*(X) \to S_*(X) \otimes S_*(X),$$

where $\mathbf{W}$ is the standard free resolution of $\mathbb{Z}/2$ over $\mathbb{Z}/2[C_2]$ ($\mathbf{W}_i = \mathbb{Z}/2[C_2]\{e_i\}$ $d(e_{i+1}) = (1+t)e_i$ for $i \geq 0$), $S_*(X)$ is the singular chain complex of X tensored with $\mathbb{Z}/2$, and $\mathbb{Z}/2$ acts on the source (*resp*. the target) through its action on $\mathbf{W}$ (*resp*. by interchanging the factors). Such a map is obtained by the method of the acyclic models, and induces a map of chain complexes

$$\theta : \mathbf{W} \otimes_{\mathbb{Z}/2[C_2]} \left(S^*(X) \otimes S^*(X)\right) \to S^*(X).$$

Here $S^*(X) = \mathrm{Hom}_{\mathbb{Z}/2}\left(S_*(X), \mathbb{Z}/2\right)$, and the $\mathbb{Z}/2[C_2]$-module structure of $(S^*(X) \otimes S^*(X))$ is induced from the C_2-action given by interchanging the factors. θ may be chosen to extend the square $S^*(X) \otimes S^*(X) \to S^*(X)$ induced by the Alexander-Whitney map, and defines the Steenrod squares $\{Sq^i\}_{i\geq 0}$ by

$$Sq^i([x]) = H(\theta)(e_{n-i} \otimes x \otimes x) \in H^{n+i}(X, \mathbb{Z}/2)$$

for $[x] \in H^n(X, \mathbb{Z}/2)$.

Now, the Steenrod squares are also defined on the cohomology of the cocommutative Hopf algebras over $\mathbb{Z}/p$, in particular, the Ext of the Steenrod algebras [Ll][My]: Let A be a cocommutative Hopf algebra over $\mathbb{Z}/2$, e.g. the mod 2 Steenrod algebra. Let $B_*(A)$ be the inhomogeneous bar resolution of A, and let $C_*(A^*) = \mathrm{Hom}_A\left(B_*(A), \mathbb{Z}/2\right)$ be the corresponding cobar complex for A^*, the dual Hopf algebra of A. Recall that $\mathrm{Ext}_A^s(\mathbb{Z}/2.\mathbb{Z}/2) = H_s(C_*(A^*))$. Then there is an appropriate map of augmented chain complexes of C_2-equivariant A-modules

$$\mathbf{W} \otimes B_*(A) \to B_*(A) \otimes B_*(A),$$

from which we get a map of augmented chain complexes of $\mathbb{Z}/2$-modules

$$\theta : \mathbf{W} \otimes_{\mathbb{Z}/2[C_2]} \left(C_*(A^*) \otimes C_*(A^*)\right) \to C_*(A^*).$$

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Then, as before, the Steenrod squares $\{Sq^i\}_{i\geq 0}$ on the Ext are defined by

$$Sq^i([\omega]) = H(\theta)(e_{n-i}\otimes\omega\otimes\omega) \in \mathrm{Ext}_A^{n+i}(\mathbb{Z}/2,\mathbb{Z}/2)$$

for $[\omega]\in \mathrm{Ext}_A^n(\mathbb{Z}/2,\mathbb{Z}/2)$.

Even though the Sq^i's on Ext shares most properties with the Sq^i's on $H^*(X)$, Sq^0 is not necessarily identity. In fact, when A is graded,

$$\begin{aligned} Sq^0 : \mathrm{Ext}_A^{s,t}(\mathbb{Z}/2,\mathbb{Z}/2) &\to \mathrm{Ext}_A^{s,2t}(\mathbb{Z}/2,\mathbb{Z}/2) \\ [\zeta_1|\zeta_2|\cdots|\zeta_s] &\mapsto [\zeta_1^2|\zeta_2^2|\cdots|\zeta_s^2], \end{aligned}$$

in terms of the cobar complex [My].

Recently, Kameko [K] defined

$$Sq^0 : PH_*(BV_s) \to PH_{2*+s}(BV_s)$$

by $x_{i_1}\otimes x_{i_2}\otimes\cdots\otimes x_{i_s}\mapsto x_{2i_1+1}\otimes x_{2i_2+1}\otimes\cdots\otimes x_{2i_s+1}$, and made full use of it in [K]. This usage of the same notation is more than accidental. Actually, Kameko's Sq^0 is shown to commute with the Sq^0 on $\mathrm{Ext}^{s,s+*}$ of the mod 2 Steenrod algebra through the Singer homomorphism by [B2] for $s=3$ and by [M3] for general s.

Now we are ready to state a known result, where we suppose n can be uniquely written as

$$n = (2^{s+t+u}-1)+(2^{t+u}-1)+(2^u-1) = 2^{s+t+u}+2^{t+u}+2^u-3,$$

for some $s,t,u\geq 0$ such that $s=0$ only if $u=0$. Actually, if n is not of this form, $PH_n(BV_3)=0$ by the theorem of Wood [Wo] (formally Peterson's conjecture [P]), and the uniqueness of such an expression of n is not difficult to see [B1].

Proposition D. (a) [Wa]

$$\mathrm{Ext}_{\mathcal{A}_*}^{3,n+3}(\mathbb{Z}/2,\mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2\{h_u h_{t+u} h_{s+t+u}\} & \text{if } s\geq 2, t\geq 2 \\ \mathbb{Z}/2\{c_u = \langle h_{u+1}, h_u, h_{u+2}^2\rangle\} & \text{if } s=2, t=1 \\ \mathbb{Z}/2\{h_u^2 h_{s+u}\}\oplus\mathbb{Z}/2\{h_{1+u}h_{s+u-1}^2\} & \text{if } s\geq 5, t=0 \\ \mathbb{Z}/2\{h_u^2 h_{s+u}\} & \text{if } 3\leq s\leq 4, t=0 \\ \mathbb{Z}/2\{h_u^2 h_{2+u} = h_{u+1}^3\} & \text{if } s=2, t=0 \\ \mathbb{Z}/2\{h_0 h_t^2\} & \text{if } s=0, t\geq 3 \\ \mathbb{Z}/2\{h_0^3\} & \text{if } s=0, t=0 \\ 0 & \text{if otherwise.} \end{cases}$$

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(b) [K][ACH][B2]

$$PH_n(BV_3) = \begin{cases} \omega\{h_u h_{t+u} h_{s+t+u}\} & \textit{if } s \geq 2, t \geq 2 \\ \pi\{c_u\} & \textit{if } s = 2, t = 1 \\ \omega\{h_u^2 h_{s+u}\} \oplus \omega\{h_{1+u} h_{s+u-1}^2\} & \textit{if } s \geq 5, t = 0 \\ \omega\{h_u^2 h_{4+u}\} \oplus \lambda' & \textit{if } s = 4, t = 0 \\ \omega\{h_u^2 h_{3+u}\} \oplus \mu' & \textit{if } s = 3, t = 0 \\ \omega\{h_u^2 h_{2+u}\} & \textit{if } s = 2, t = 0 \\ \omega\{h_0 h_t^2\} & \textit{if } s = 0, t \geq 3 \\ \omega\{h_0^3\} & \textit{if } s = 0, t = 0 \\ 0 & \textit{if otherwise.} \end{cases}$$

Here $\omega\{h_i h_j h_k\}$ is the $GL_3(\mathbb{F}_2)$ representation inside $PH_n(BV_3)$, generated by $h_i h_j h_k$ (such a representation is said to be an $h-$ orbit [B2]); $\pi\{c_u\}$ is the $GL_3(\mathbb{F}_2)$ representation inside $PH_n(BV_3)$, generated by $c_u = (Sq^0)^u c_0$, where $c_0 = x_3 \otimes x_3 \otimes x_2 + x_3 \otimes x_4 \otimes x_1 + x_5 \otimes x_2 \otimes x_1 + x_6 \otimes x_1 \otimes x_1 \in PH_8(BV_3)$; λ' and μ' are some h-orbits (see [B2] for more details).

(c) *In (b), the $GL_3(\mathbb{F}_2)$-coinvariant of all the nontrivial representations is isomorphic to $\mathbb{Z}/2$, except for λ' and μ', which have the trivial $GL_3(\mathbb{F}_2)$-coinvariant.*

(d) *Under the map*

$$PH_n(BV_3) \twoheadrightarrow \mathbb{Z}/2 \otimes_{GL_3(\mathbb{F}_2)} PH_n(BV_3) \longrightarrow \mathrm{Ext}_{\mathcal{A}_*}^{3,n+3}(\mathbb{Z}/2, \mathbb{Z}/2),$$

an element in the list of (b) is mapped to the element in the list of (a) with the same name. Furthermore, this map commutes with the Squaring operations Sq^0's; i.e. i) Kameko's [K] $Sq^0 : PH_n(BV_3) \to PH_{2n+3}(BV_3)$, which is induced by $x_i \otimes x_j \otimes x_k \mapsto x_{2i+1} \otimes x_{2j+1} \otimes x_{2k+1}$ (cf. Corollary 1.3), and ii) Sq^0 defined in $\mathrm{Ext}_{\mathcal{A}_*}^{3,*}(\mathbb{Z}/2, \mathbb{Z}/2)$, using the cobar complex [Ll][Ad1][My].

(e) *Kameko's Sq^0 [K] acts as*

$$Sq^0(h_i h_j h_k) = h_{i+1} h_{j+1} h_{k+1}, \qquad Sq^0(c_i) = c_{i+1},$$

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and its iterations induce isomorphisms

$$
\begin{array}{ll}
(Sq^0)^u : \omega\{h_0h_th_{s+t}\} \xrightarrow{\cong} \omega\{h_uh_{t+u}h_{s+t+u}\} & \textit{if } s \geq 2, t \geq 2 \\
(Sq^0)^u : \pi\{c_0\} \xrightarrow{\cong} \pi\{c_u\} & \textit{if } s = 2, t = 1 \\
(Sq^0)^u : \omega\{h_0^2h_s\} \xrightarrow{\cong} \omega\{h_u^2h_{s+u}\} & \textit{if } s \geq 5, t = 0 \\
(Sq^0)^{u+1} : \omega\{h_0h_{s-2}^2\} \xrightarrow{\cong} \omega\{h_{1+u}h_{s+u-1}^2\} & \textit{if } s \geq 5, t = 0 \\
(Sq^0)^u : \omega\{h_0^2h_4\} \xrightarrow{\cong} \omega\{h_u^2h_{4+u}\} & \textit{if } s = 4, t = 0 \\
(Sq^0)^u : \omega\{h_0^2h_3\} \xrightarrow{\cong} \omega\{h_u^2h_{3+u}\} & \textit{if } s = 3, t = 0 \\
(Sq^0)^u : \omega\{h_0^2h_2\} \xrightarrow{\cong} \omega\{h_u^2h_{2+u}\} & \textit{if } s = 2, t = 0
\end{array}
$$

(f) *In (b),* $\omega\{h_ih_jh_k\}^{GL_3(\mathbb{F}_2)} = \mathbb{Z}/2$, *and the* $GL_3(\mathbb{F}_2)$*-invariant of the rest of the representations are trivial. As a special case, we have*

$$\omega\{h_u^2h_{2+u}\}^{GL_3(\mathbb{F}_2)} = \mathbb{Z}/2\{h_{u+1}^3\} = \mathbb{Z}/2\{(Sq^0)^{u+1}(h_0^3)\}.$$

(g) *Suppose* $h_ih_jh_k \in \mathrm{Ext}_{\mathcal{A}_*}^{3,n+3}(\mathbb{Z}/2, \mathbb{Z}/2)$ *(or a linear combination of such) is the image of an element* $\theta \in \pi_n^s(BV_3)$*, then we may modify* θ *(if necessary) so that its stable mod-2 Hurewicz image is invariant under the action of* $GL_3(\mathbb{F}_2)$*.*

Proof. For these claims from (a) to (f), see [B2] and the references there.

For (g), we first recall from [B2] that, if $s \geq 2, t \geq 2$, $\omega\{h_uh_{t+u}h_{s+t+u}\}$ is the permutation module of dimension 21 corresponding to the upper triangular subgroup, and any h-orbit is a quotient module of this permutation representation. In particular, the nontrivial $GL_3(\mathbb{F}_2)$-invariant elements in those h-orbits are the images of such an element in this 21 dimensional permutation representation under the quotient map. Now, notice that $GL_3(\mathbb{F}_2)$ has a subgroup H of order 21. Actually, $GL_3(\mathbb{F}_2) \cong PSL_2(\mathbb{F}_7)$, which acts transitively on $P_1(\mathbb{F}_7)$, a finite set consisting of 8 elements. So we simply take H to be the isotropy subgroup under this action (through the isomorphism). Then, it is immediate that the element $\sum_{g\in H} g_*\theta \in \pi_n^s(BV_3)$ satisfies the desired property. □

We now prove our main theorem. Of course, this would follow from Proposition 1.3, Proposition 2.3, and Proposition D: We may suppose $u \geq 3$. First, by

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Proposition 1.3 and Proposition D (especially (e)(f)), we see if

$$\left\{\begin{array}{cl} h_u h_{t+u} h_{s+t+u} & \text{such that } s \geq 2, t \geq 2 \\ c_u & \text{such that } s = 2, t = 1 \\ h_{1+u} h_{s+u-1}^2 & \text{such that } s \geq 5, t = 0 \\ h_u^2 h_{s+u},\ h_{1+u} h_{s+u-1}^2 + h_u^2 h_{s+u} & \text{such that } s \geq 5, t = 0 \\ h_u^2 h_{s+u} & \text{such that } 3 \leq s \leq 4, t = 0 \\ h_{u+1}^3 = h_u^2 h_{2+u} & \text{such that } s = 2, t = 0 \end{array}\right\}$$

comes from the triple transfer, then

$$\left\{\begin{array}{c} 2^{(2^{u-3})} \\ 2^{(2^{u-3})} \\ 2^{(2^{u-2})} \\ 2^{(2^{u-3})} \\ 2^{(2^{u-3})} \\ 2^{(2^{u-2})} \end{array}\right\}$$ divides the order of the BP_*-Hurewicz image of an appropriate triple transfer lift.

On the other hand, let us set d by

$$\begin{aligned} n &= (2^{s+t+u} - 1) + (2^{t+u} - 1) + (2^u - 1) = 2^{s+t+u} + 2^{t+u} + 2^u - 3 \\ &= 4d - 3, \end{aligned}$$

as in Proposition 2.3. Then we can easily see that $\nu_2(d)$ is given by

$$\begin{cases} u-2 & \text{if } t \geq 1 \\ u-1 & \text{if } s \geq 2, t = 0 \end{cases}$$

Thus, by Proposition 2.3, if

$$\left\{\begin{array}{cl} h_u h_{t+u} h_{s+t+u} & \text{such that } s \geq 2, t \geq 2 \\ c_u & \text{such that } s = 2, t = 1 \\ h_{1+u} h_{s+u-1}^2 & \text{such that } s \geq 5, t = 0 \\ h_u^2 h_{s+u},\ h_{1+u} h_{s+u-1}^2 + h_u^2 h_{s+u} & \text{such that } s \geq 5, t = 0 \\ h_u^2 h_{s+u} & \text{such that } 3 \leq s \leq 4, t = 0 \\ h_{u+1}^3 = h_u^2 h_{2+u} & \text{such that } s = 2, t = 0 \end{array}\right\}$$

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comes from the triple transfer, then the order of the BP_*-Hurewicz image of the above triple transfer lift divides $\left\{\begin{array}{c} 2^{u+2} \\ 2^{u+2} \\ 2^{u+3} \\ 2^{u+3} \\ 2^{u+3} \\ 2^{u+3} \end{array}\right\}$

Therefore, combining these two arguments, we see if

$$\left\{\begin{array}{cl} h_u h_{t+u} h_{s+t+u} & \text{such that } s \geq 2, t \geq 2 \\ c_u & \text{such that } s = 2, t = 1 \\ h_{1+u} h_{s+u-1}^2 & \text{such that } s \geq 5, t = 0 \\ h_u^2 h_{s+u},\ h_{1+u} h_{s+u-1}^2 + h_u^2 h_{s+u} & \text{such that } s \geq 5, t = 0 \\ h_u^2 h_{s+u} & \text{such that } 3 \leq s \leq 4, t = 0 \\ h_{u+1}^3 = h_u^2 h_{2+u} & \text{such that } s = 2, t = 0 \end{array}\right\}$$

comes from the triple transfer, then $\left\{\begin{array}{c} 2^{u-3} \leq u+2 \\ 2^{u-3} \leq u+2 \\ 2^{u-2} \leq u+3 \\ 2^{u-3} \leq u+3 \\ 2^{u-3} \leq u+3 \\ 2^{u-2} \leq u+3 \end{array}\right\}$.

Now the claim follows immediately. □

Remark 3.1. Even before the appearance of our [M1], Mahowald conjectured that the higher h_j^3's are not permanent cycles, as early as the summer of 1988 (during the Toda's conference, Kinosaki, Japan). So the special case of our Theorem 3.1 can be regarded as a supporting evidence of this conjecture of Mahowald.

4. Speculations. The following was made by Joel Cohen, named by Michael Barratt, and appears in *Algebraic Topology, Proceedings of Symposia in Pure Mathematics, XXII*, page 199, Conjecture 73 (we thank the referee for supplying this information):

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DOOMSDAY CONJECTURE. *Fix a prime number p, and consider the Adams spectral sequence of the stable homotopy groups of the sphere at p. Then* $E_\infty^{s,*}$ *is finite for each s.*

But the doomsday conjecture was doomed: Mark Mahowald's η_j family [Ma2] was the first counter-example. Soon after that, Ralph Cohen's $h_0 b_j$ family [C] was added to the list.

Now, to fix the situation, we are tempted to propose the following in view of Corollary 3.2 and our previous results [M0][M1][M2]:

NEW DOOMSDAY CONJECTURE. *For each s, there exists some integer n(s) such that no element in the image of*

$$(\mathcal{P}^0)^{n(s)}\left(\mathrm{Ext}_{\mathcal{A}_*}^{s,*}(\mathbb{Z}/p,\mathbb{Z}/p)\right) \quad \left(\subseteqq \mathrm{Ext}_{\mathcal{A}_*}^{s,p^{n(s)}*}(\mathbb{Z}/p,\mathbb{Z}/p)\right)$$

is a permanent cycle. Here $\mathcal{P}^0 = Sq^0$, *when* $p = 2$.

Notice that, unlike the case of the classical doomsday conjecture, neither Mark Mahowald's η_j family [Ma2] nor Ralph Cohen's $h_0 b_j$ family [C] are counter-examples of this new doomsday conjecture. But this conjecture is extremely difficult to attack whether it is right or wrong, as its first nontrivial unsolved case corresponds to the Kervaire invariant one problem. On the other hand, recall that many known differentials in the (classical) Adams spectral sequence are consequences of either the Hopf invariant one differentials or the Toda's differentials strengthened significantly by Ravenel [R0]), both of which represent the new doomsday phenomena. Therefore, it might be the case that most of the differentials of the classical Adams spectral sequence are consequences of this N.D. (which stands for New Doomsday) phenomenon. Of course, this might be true, even if the new doomsday conjecture does not hold.

Recently, Mark Mahowald suggested us to relate the N.D. philosophy with the root invariant [Ma1][MR]. Mahowald's suggestion was based on the fact that Sq^0 becomes quite frequently the (algebraic) root invariant [Ka][Mg][J][MR], which we now summarize. Roughly speaking, the connection between Sq^0 and the root invariant is given as follows: Recall the map

$$\theta : \mathbf{W} \otimes_{\mathbb{Z}/2[C_2]} \left(C_*(A^*) \otimes C_*(A^*)\right) \to C_*(A^*),$$

considered at the beginning of §3. This is clearly a reminiscence of the extended power map

$$EC_{2+} \wedge_{C_2} \left(S^0 \wedge S^0\right) \to S^0,$$

induced by the H^∞-ring structure of the sphere. The precise connection between

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these two points of view was clarified by [Ka][Mg]. In this way, Sq^i's on Ext, defined as the algebraic cup-r construction, are related to the geometric cup-r construction given by the extended power map. But unlike the Ext situation, the geometric cup product $\alpha \cup_r \alpha \in \pi^s_{2n+r}(S^0)$ for $\alpha \in \pi^s_n(S^0)$ and its stable analogue (see [J]) are defined only when $\alpha \cup_i \alpha$ contains 0 for any $0 \leq i < r$. Then the Jones' key observation [J] claims that the first nontrivial geometric stable cup-r product is none other than the root invariant. (To be precise, both are well-defined only as cosets.)

Now the following iteration of the root invariant is very suggestive:

$$2 \overset{R}{\mapsto} \eta \overset{R}{\mapsto} \nu \overset{R}{\mapsto} \sigma \overset{R}{\mapsto} \theta_3 = \sigma^2 \overset{R}{\mapsto} \theta_4 \overset{R}{\mapsto} \theta_5?$$

where $a \overset{R}{\mapsto} b$ indicates that the root invariant of a contains b. Notice that the corresponding Adams spectral sequence E_2-term elements are connected by Sq^0 as follows

$$\begin{gathered} h_0 \overset{Sq^0}{\mapsto} h_1 \overset{Sq^0}{\mapsto} h_2 \overset{Sq^0}{\mapsto} h_3 \\ h_3^2 \overset{Sq^0}{\mapsto} h_4^2 \overset{Sq^0}{\mapsto} h_5^2, \end{gathered}$$

but Sq^0 does not connect between h_3 and h_3^2. Of course, the Adams filtration jumps by 1 here and the right map is Sq^1. But a more fundamental reason is that h_4 is not a permanent cycle because of the Hopf invariant-1 differential, which we regard as an example of N.D. phenomena. Furthermore, N.D. philosophy predicts that the higher Kervaire invariant elements θ_j's do not exist, which in turn suggests that some iteration of the root invariant of θ_4 to have Adams filtration strictly higher than 2. In this way, we are led to speculate the following:

R.I. DOOMSDAY CONJECTURE. *For each s, there exists some integer $m(s)$ such that, for any element of the stable homotopy groups of the sphere $f \in \pi_*(S^0)$ with Adams filtration s, its $m(s)$-fold iterated root invariant $R^{m(s)}(f)$ has Adams filtration strictly higher than s.*

A nice thing about this formulation is that it is stated solely in terms of permanent cycles. We thank Mark Mahowald for his substantial help with the Mahowald invariant (alias root invariant).

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