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ON RELATIONS BETWEEN ADAMS SPECTRAL SEQUENCES, WITH AN APPLICATION TO THE STABLE HOMOTOPY OF A MOORE SPACE

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0. Introduction

A ring-spectrum B determines an Adams spectral sequence

$$E_2(X;\ B) \Rightarrow \pi_*(X)$$

abutting to the stable homotopy of X. It has long been recognized that a map $A \to B$ of ring-spectra gives rise to information about the differentials in this spectral sequence. The main purpose of this paper is to prove a systematic theorem in this direction, and give some applications.

To fix ideas, let p be a prime number, and take B to be the mod p Eilenberg–MacLane spectrum H and A to be the Brown–Peterson spectrum BP at p. For p odd, and X torsion-free (or for example X a Moore-space $V = S^0 \cup_p e^1$), the classical Adams E_2-term $E_2(X; H)$ may be *tri*graded; and as such it is E_2 of a spectral sequence (which we call the *May spectral sequence*) converging to the Adams–Novikov E_2-term $E_2(X; \mathrm{BP})$. One may say that the classical Adams spectral sequence has been broken in half, with all the "BP-primary" differentials evaluated first. There is in fact a precise relationship between the May spectral sequence and *the* H-Adams spectral sequence. In a certain sense, the May differentials are the Adams differentials modulo higher BP-filtration. One may say the same for $p = 2$, but in a more attenuated sense. In this paper we restrict attention to d_2, although I believe that the machinery developed here sheds light on the higher differentials as well.

Assertions similar to these, in case X is torsion-free, have been made by Novikov [24], who however provided only the barest hint of a proof. I have attempted to provide in Section 1 a convenient account of part of the abstract theory of spectral sequences of Adams type, and in Sections 3, 5, and 6, I construct the May spectral sequence and prove the theorem outlined above. The constructions here are

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reminiscent of some of P. Deligne's work on mixed Hodge structures (compare (4.1) with [7, (1.4.9.2)], for example), but are more elaborate in that we must work directly in a triangulated category, while Deligne can work on the level of chain complexes.

The major application of this result included here is a determination of the differentials in the "metastable" part of the Adams spectral sequence for the Moore space $V=S^0\cup_p e^1$ when p is odd. In this range $E_3=E_\infty$, and the calculation has the following corollary. Let $\varphi:\Sigma^{2p-2}V\to V$ denote the Adams self-map, $\delta:\Sigma^{-1}V\to V$ the Bockstein, and let composition with the inclusion $S^0\to V$ of the bottom cell be understood when necessary. It is then well known that

$$\mathbb{F}_p[\varphi]\otimes E[\delta\varphi]\subseteq\pi_*(V).$$

Theorem 4.11. *For any $x\in\pi_*(V)$ there exists $n\geq 0$ such that $\varphi^n x\in\mathbb{F}_p[\varphi]\otimes E[\delta\varphi]$.*

This theorem has been conjectured by a number of people, and Adams and Baird, Bousfield, and Dwyer, have independently obtained interesting homotopy-theoretic consequences of it. One reason for conjecturing it is that

$$\varphi^{-1}E_2(V;\,\mathrm{BP})=\mathbb{F}_p[\varphi,\varphi^{-1}]\otimes E[\delta\varphi].$$

However, the localized BP-Adams spectral sequence may fail to converge. This difficulty has frustrated all attempts at a proof using BP alone, and has necessitated the present approach.

The calculation of the differential in $E_2(V;H)$ is carried out in Section 9 by use of the formal-group theoretic description of BP-operations. This section also contains a new (historically the first) proof of the localization theorem of [21]. To apply formal group techniques, we reconstruct the May spectral sequence algebraically in Section 8. Section 7 provides the machinery necessary to identify the May E_2-term. This section also contains a proof of the presumably well-known fact that the MU-Adams E_2-term for X is isomorphic to the cohomology of the Landweber–Novikov algebra with coefficients in $\mathrm{MU}_*(X)$. These three sections are grouped into "Part II" and are entirely algebraic and independent of Part I.

PART I

1. Adams resolutions and Adams spectral sequences

We shall begin by collecting some standard material on Adams resolutions and the associated spectral sequences. We shall work in the homotopy category of CW spectra [5, III] $\mathscr{S}$, although the reader sensitive to generalizations will recognize that a spectral sequence of Adams type always arises from an injective class [11] in a triangulated category [28]. Some of this work is indebted to [23].

A *ring-spectrum* is a spectrum A together with morphisms $\eta: S \to A$ and $\mu: A \wedge A \to A$ such that the diagram

$$\begin{tikzcd} S\wedge A \arrow[r, "\eta\wedge A"] \arrow[dr, "="'] & A\wedge A \arrow[d, "\mu"] & A\wedge S \arrow[l, "A\wedge\eta"'] \arrow[dl, "="] \\ & A & \end{tikzcd} \tag{1.1}$$

commutes. Here of course S denotes the sphere spectrum. A spectrum X is *A-injective* if it is a retract of $A \wedge Y$ for some spectrum Y. It is equivalent to require that $X = S \wedge X \to A \wedge X$ split. A *sequence* $X' \to X \to X''$ is a pair of morphisms with trivial composition. A sequence $X' \to X \to X''$ is *A-exact* if

$$[X', I] \leftarrow [X, I] \leftarrow [X'', I] \tag{1.2}$$

is exact for every A-injective I. A longer sequence is *A-exact* if every two-term subsequence is. A morphism $f: X' \to X$ is *A-monic* if $* \to X' \to X$ is A-exact. The following useful facts are easily verified.

Lemma 1.3. *$f: X' \to X$ is A-monic if and only if $A \wedge f: A \wedge X' \to A \wedge X$ is split-monic.*

Lemma 1.4. (a) *If I is A-injective, so is $I \wedge Y$.*
(b) *If $f: X' \to X$ is A-monic, so is $f \wedge Y: X' \wedge Y \to X \wedge Y$.*

Lemma 1.5. *If $X' \to X$ is A-monic and $X' \to X \to X''$ is a cofibration sequence, then $* \to X' \to X \to X'' \to *$ is A-exact.*

It is elementary to show that these notions define an injective class [11] in $\mathcal{S}$, which is *stable* in the sense that I is A-injective if and only if its suspension ΣI is. Thus, given X one may construct an A-exact sequence

$$* \to X \to I^0 \to I^1 \to \cdots \tag{1.6}$$

such that I^s is A-injective for all s. This is an *A-resolution* of X. Given A-resolutions $X \to I^*$ and $Y \to J^*$, any $f: X \to Y$ lifts to a chain-map $f^*: I^* \to J^*$ which is unique up to a chain-homotopy.

Given an A-resolution (1.6), one can inductively construct a sequence of exact triangles

$$\begin{tikzcd} X = X^0 \arrow[dr, "j"'] & & X^1 \arrow[ll, "i"'] \arrow[dr, "j"'] & & X^2 \arrow[ll, "i"'] & \cdots, \\ & I^0 \arrow[ur, dashed, "k"'] & & \Sigma I^1 \arrow[ur, dashed, "k"'] & & \end{tikzcd} \tag{1.7}$$

where the dotted morphisms have degree -1, such that $jk=d$. The sequence

$$*\to X^s\to\Sigma^s I^s\to\Sigma X^{s+1}\to *$$

is then A-exact for all s. Given the A-resolution I^*, the associated *A-Adams resolution* (1.7) is unique up to an isomorphism. Furthermore, given $f: X\to Y$, any chain-map of A-resolutions covering f lifts to a map of Adams resolutions.

Remark 1.8. It follows from Corollary 1.4 that given an Adams resolution (1.7) for X, an Adams resolution for $X\wedge Y$ may be obtained simply by smashing (1.7) with Y. In particular, an Adams resolution for the sphere spectrum S defines an Adams resolution for any spectrum X.

Remark 1.9. The most frequently encountered example of an Adams resolution is the *canonical resolution*, in which $j: X^s\to\Sigma^s I^s$ is the map

$$X^s=S\wedge X^s\xrightarrow{\eta\wedge X} A\wedge X^s=\Sigma^s I^s$$

for each $s\geq 0$. It is unique up to a (non-canonical) isomorphism. Note that it is obtained by smashing X with the canonical resolution for the sphere spectrum.

Remark 1.10. By use of suitable mapping telescopes, we may always assume that X^{s+1} is a subspectrum of X^s, with quotient $\Sigma^s I^s$.

Applying stable homotopy π_* to the Adams resolution (1.7) of X results in an exact couple and hence a spectral sequence $E_r(X;A)$, functorial from E_2, called the *A-Adams spectral sequence.* Observe that $E_2^{s,s+u}(-;A)$ is the sth right-derived functor of π_u with respect to the A-injective class:

$$E_2^{s,s+u}(X;A)=R_A^s\pi_u(X). \tag{1.11}$$

Here and throughout this paper u is reserved for the topological dimension. We also have the associated functorial filtration of stable homotopy:

$$F_A^s\pi_*(X)=\operatorname{im}(\pi_*(X^s)\to\pi_*(X)). \tag{1.12}$$

Notice that $d_r: E_r^{s,s+u}\to E_r^{s+r,s+u+r-1}$. Thus $E_{s+1}^{s,*}\supseteq E_{s+2}^{s,*}\cdots$; let $E_\infty^{s,*}$ denote the intersection of this system:

$$E_\infty^{s,*}=\varprojlim E_r^{s,*}.$$

If $x\in F_A^s\pi_*(X)$ let $\bar{x}\in\pi_*(X^s)$ be a lifting. Then $j\bar{x}\in\pi_*(I^s)=E_1^{s,*}(X;A)$ is a permanent cycle and thus defines an element of $E_\infty^{s,*}$. We claim this element depends only on $x \bmod F_A^{s+1}\pi_*(X)$. For suppose in fact $x\in F_A^{s+1}\pi_*(X)$. Let $\bar{y}\in\pi_*(X^{s+1})$ be a lifting, and let r be the minimal integer such that $0=i^r(\bar{x}-i\bar{y})\in\pi_*(X^{s-r})$. Then there exists $z\in\pi_*(I^{s-r})$ such that $kz=i^{r-1}(\bar{x}-i\bar{y})$. By construction, then, $z\in E_1^{s-r,*}$ survives to E_r, and $d_r\{z\}=\{j\bar{x}\}$; so $j\bar{x}$ maps to $0\in E_\infty^{s,*}$. Thus j induces a natural homomorphism, the "generalized Hopf invariant",

$$\varrho : \mathrm{gr}_A^s \pi_*(X) \to E_\infty^{s,*}(X; A). \tag{1.13}$$

We shall say that the spectral sequence *converges* provided that ϱ is an isomorphism and that the filtration of $\pi_*(X)$ is such that

$$\pi_*(X) \to \varprojlim \pi_*(X)/F^s\pi_*(X)$$

is an isomorphism. This agrees with Adams' use of the word "convergent" ([5, III 8.2]).

2. The Mahowald spectral sequence

In his work on the order of the image of the J-homomorphism, Mahowald uses a bo-Adams resolution to produce a spectral sequence converging to the classical mod 2 Adams E_2-term. We recall that construction here, in greater generality.

We begin with an easy observation.

Lemma 2.1. *Let A and B be ring-spectra, and suppose that B is A-injective. Then B-injectives are A-injective, and A-exact sequences are B-exact.*

For example, let $\theta : A \to B$ be a morphism of ring-spectra. That is, θ is a morphism of spectra such that $\theta\eta_A = \eta_B$ and $\theta\mu_A = \mu_B(\theta \wedge \theta)$. Then $\mu_B(\theta \wedge 1)$ splits $B = S \wedge B \to A \wedge B$, so B is A-injective.

Now let (1.7) be an A-Adams resolution for X. Then each sequence

$$* \to X^s \to \Sigma^s I^s \to \Sigma X^{s+1} \to *$$

is B-exact and so gives rise to a long exact sequence in $R_B^*\pi_* = E_2^{**}(-; B)$, with connecting homomorphisms

$$\partial : E_2^{t,t+u-1}(X^{s+1}; B) \to E_2^{t+1,t+u}(X^s; B).$$

These link together into an exact couple, giving rise to a spectral sequence converging to $E_2(X; B)$. Its E_1-term is

$$E_1^{s,t,u} = E_2^{t,t+u}(\Sigma^s I^s; B) = R_B^t \pi_{u-s}(I^s)$$

so

$$E_2^{s,t,u} = R_A^s R_B^t \pi_{u-s}(X). \tag{2.2}$$

Note that $d_r : E_r^{s,t,u} \to E_r^{s-r+1,t+1,u-1}$, and that the associated filtration on $E_2^*(X; B)$ is given by

$$F_A^s E_2^{s+t}(X; B) = \mathrm{im}(\partial^s : E_2^t(X^s; B) \to E_2^{s+t}(X; B)). \tag{2.3}$$

One may regard this as the Grothendieck spectral sequence for the composite $\pi_* \circ \mathrm{id}$, where $\mathrm{id} : \mathscr{S} \to \mathscr{S}$ is the identity functor; only the injective class changes. We call it the *Mahowald spectral sequence*. It is independent of the A-resolution used in its construction from E_2 on. The associated filtration of $E_2(X;B)$ is functorial.

Example 2.4. Let B be the mod 2 Eilenberg–MacLane spectrum and let A be the spectrum bo of orthogonal connective K-theory. The resulting Mahowald spectral sequence agrees with the spectral sequence of [14]. It is algebraically determined by $H_*(X)$, as explained in Remark 8.15 below.

3. A geometric May spectral sequence

Let A and B be ring-spectra. Let $X \to I^*$ be an A-resolution and let $F_B^t\pi_*$ be the B-filtration of homotopy. Define a filtration of the complex $\pi_*(I^*)$ by

$$F^{s+t}\pi_*(I^s) = F_B^t\pi_*(I^s). \tag{3.1}$$

Then $d_*F^{s+t} \subseteq F^{s+t+1}$, so that

$$E_1^{s+t,s,u} = E_0^{s+t,s,u} = F_B^t\pi_{u-s}(I^s)/F_B^{t+1}\pi_{u-s}(I^s) \tag{3.2}$$

in the resulting spectral sequence abutting to $E_2(X;A)$. Note that $d_r : E_r^{s+t,s,u} \to E_r^{s+t+r,s+1,u-1}$. To further identify (3.2), we introduce a definition.

Definition 3.3. A spectrum X is *(A,B)-primary* if there exists an A-resolution $X \to I^*$ such that for all s, $E_r(I^s;B)$ converges (as in Section 1 above) and collapses at E_2.

If $X \to I^*$ is such a resolution, then our spectral sequence has

$$E_1^{s+t,s,u} = E_2^{t,s+t+u}(I^s;B) = R_B^t\pi_{s+u}(I^s)$$

and

$$E_2^{s+t,s,u} = R_A^sR_B^t\pi_{s+u}(X). \tag{3.4}$$

The spectral sequence converges and is functorial in the (A,B)-primary spectrum X from E_2 on. We shall call this the *May spectral sequence*. It is analogous to the spectral sequence considered in J.P. May's thesis [16] in that it arises from a filtration of a resolution. Novikov considered a special case of it in [24].

Example 3.5. By a theorem of Mahowald and Milgram [18], S is (bo,H)-primary where bo and H are as in Example 2.3.

Example 3.6. Let p be prime and let V be the Moore-spectrum $S^0 \cup_p e^1$. Let $H\mathbb{Z}_{(p)}$ (resp. H) denote the $\mathbb{Z}_{(p)}$ (resp. $\mathbb{F}_p$) Eilenberg–MacLane spectrum. Then $V \wedge X$ is $(H\mathbb{Z}_{(p)},H)$-primary for any spectrum X. For one can smash any connective $H\mathbb{Z}_{(p)}$-resolution of X with V to obtain a resolution of $V \wedge X$ with the desired properties. This observation is related to [23].

Example 3.7. Suppose $A \wedge A$ splits as a wedge of suspensions of A. Then X is (A,B)-primary whenever $E_r(A \wedge X;B)$ converges and collapses at E_2.

4. Statement and application of the main theorem

Now suppose that B is A-injective and that X is (A,B)-primary. Comparing (2.2) with (3.4), we see that the E_2-term of the May spectral sequence coincides with the E_2-term of the Mahowald spectral sequence. Schematically,

$$\begin{tikzcd}[column sep=small] & R_A R_B \pi_*(X) \arrow[dl, Rightarrow, "\mathrm{May}"'] \arrow[dr, Rightarrow, "\mathrm{Mahowald}"] & \\ R_A \pi_*(X) \arrow[dr, Rightarrow, "A\text{-Adams}"'] & & R_B \pi_*(X) \arrow[dl, Rightarrow, "B\text{-Adams}"] \\ & \pi_*(X) & \end{tikzcd} \tag{4.1}$$

In the following theorem $F_A^s E_2(X;B)$ denotes the filtration associated with the Mahowald spectral sequence (2.3), and d_r^B denotes a differential in the B-Adams spectral sequence.

Theorem 4.2. *Let A and B be ring-spectra such that B is A-injective, and suppose that X is an (A,B)-primary spectrum. Then the differential d_2^B carries $F_A^s E_2^{s+t}(X;B)$ into $F_A^{s+1} E_2^{s+t+2}(X;B)$. Suppose that the Mahowald spectral sequence collapses at E_2. Let $z \in F_A^s E_2^{s+t}(X;B)$ project to $a = z + F_A^{s+1} E_2^{s+t}(X;B) \in R_A^s R_B^t \pi_*(X)$. Then*

$$-d_2^B z \in d_2^{\mathrm{May}} a. \tag{4.3}$$

This theorem is a special case of Theorem 6.1, which will be proved in Section 6.

The remainder of this section is devoted to applications of this theorem in case A is the Brown–Peterson spectrum BP at the odd prime p and B is the mod p Eilenberg–MacLane spectrum H. In Section 8 we shall see that the Mahowald spectral sequence often collapses in this case — for example, when X is p-torsion-free, or when X is the mod p Moore spectrum $V = S^0 \cup_p e^1$.

We shall begin by recovering the differentials on $E_2^1(S;H)$ solving the mod p Hopf invariant problem [1, 13, 27]. Here we allow $p=2$. Recall that $E_2^1(S;H)$ is generated by elements $h_{1,j}$ and that there are elements $b_{1,j} \in E_2^2(S;H)$; if $p=2$, $b_{1,j} = h_{1,j}^2$. See (8.11), (8.12).

Theorem 4.4. (Adams, Liulevicius, Shimada–Yamanoshita). *In $E_2(S;H)$,*

$$d_2 h_{1,j} = q_0 b_{1,j-1}, \quad j \geq 1.$$

Proof. We shall see in Lemma 8.13 that this is true (with sign changed) in the May spectral sequence. Hence by Theorem 4.2 (or Theorem 6.1 for $p=2$) it holds in the Adams spectral sequence modulo higher filtration. But for degree reasons the higher filtration is trivial.

Remark 4.5. At $p=2$, q_0 is commonly called h_0 and $h_{1,j}$ is h_{j+1}. Thus h_2 and h_3 survive because $h_0 h_1^2 = 0 = h_0 h_2^2$, and $d_2 h_j = h_0 h_{j-1}^2$ for $j > 3$.

Remark 4.6. It seems likely that Milgram's results [17] relating d_2 to Steenrod operations in the E_2-term can be recovered *modulo higher filtration* by these methods.

We turn now to $V=S^0\cup_p e^1$, with p odd. Adams [3] constructs a map

$$\varphi:\Sigma^{2(p-1)}V\to V$$

detected by the Steenrod operation Q_1. It follows that $\varphi\in\pi_{2(p-1)}V$ (where we are neglecting to indicate composition with the inclusion $S^0\to V$ of the bottom cell) is represented in the Adams spectral sequence by

$$q_1=\{[\tau_1]\}\in E_2^{1,2p-1}(V;H).$$

In [20] we constructed an algebra homomorphism

$$E_2(V;H)\to\mathbb{F}_p[q_1]\otimes E[h_{n,0}:n\geq 1]\otimes P[b_{n,0}:n\geq 1] \tag{4.7}$$

where $|h_{n,0}|=(1,2(p^n-1))$ and $|b_{n,0}|=(2,2p(p^n-1))$, which is bijective in bidegrees (s,t) for which $t-s\leq(p^2-p-1)(s+1)$. In the spectral sequence, $h_{1,0}$ survives to $\delta\varphi$ where $\delta:\Sigma^{-1}V\to V$ is the Bockstein. Since q_1^n survives for all n, we may localize the Adams spectral sequence with respect to the multiplicative system of powers of q_1, thus "inverting" q_1. Then (4.7) identifies the localization $q_1^{-1}E_s(V;H)$.

Theorem 4.8. *In the localized Adams spectral sequence*

$$q_1^{-1}E_3(V;H)=\mathbb{F}_p[q_1,q_1^{-1}]\otimes E[h_{1,0}].$$

Proof. We shall see in Section 9 that

$$d_2h_{n,0}=-q_1b_{n-1,0},\qquad n>1, \tag{4.9}$$

in the May spectral sequence, so by Theorem 4.2 this is true (with sign changed) modulo higher filtration in the Adams spectral sequence. But this is enough, by an obvious spectral sequence argument.

Corollary 4.10. *For* $p>2$

$$E_\infty(V;H)=\mathbb{F}_p[q_1]\otimes E[h_{1,0}]$$

in bidegrees (s,t) *for which* $t-s\leq(p^2-p-1)(s+1)$.

This Corollary is the analogue for p odd of a result of M. Mahowald [14]. It plays an important role in an approach to mod p homotopy theory analogous to Mahowald's [15], which we will treat in joint work with J. Harper. Here we merely indicate some immediate corollaries.

Theorem 4.11. *Let p be an odd prime. For any $x\in\pi_*(V)$ there exists $n\geq 0$ such that*

$\varphi^n x \in \mathbb{F}_p[\varphi] \otimes E[\delta\varphi]$. *That is,*

$$\varphi^{-1}\pi_*(V) = \mathbb{F}_p[\varphi, \varphi^{-1}] \otimes E[\delta\varphi].$$

Proof. Suppose $\varphi^n x \neq 0$ for all $n \geq 0$. Since q_1-multiplication acts *parallel* to the vanishing edge ([2]; see also [20, Proposition 4.7b]), there exists $k \geq 0$ such that if $\varphi^k x$ is represented by $y \in E_2(V)$ then $\varphi^{k+n} x$ is represented by $q_1^n y \in E_2(V)$. Thus the result follows from Corollary 4.10.

Corollary 4.12. *Let $\varphi^{-1}V$ denote the mapping telescope of the direct system*

$$V \xrightarrow{\Sigma^{-q}\varphi} \Sigma^{-q}V \xrightarrow{\Sigma^{-2q}\varphi} \Sigma^{-2q}V \longrightarrow \cdots.$$

Then

$$\pi_*(\varphi^{-1}V) = \mathbb{F}_p[\varphi, \varphi^{-1}] \otimes E[\delta\varphi].$$

Proof. $\pi_*(\varphi^{-1}V) = \varphi^{-1}\pi_*(V)$.

These results have been applied to the study of K-theoretic localization by Adams and Baird, Bousfield, Dwyer, and probably others.

5. Smash-products of Adams resolutions

In this section we show how to embed all the spectral sequences considered above into a unified construction. This will form the basis of the proof of the main theorem in Section 6. We begin with a lemma.

Lemma 5.1. *Let $X = X^0 \supseteq X^1 \supseteq \cdots$ and $Y = Y^0 \supseteq Y^1 \supseteq \cdots$ be decreasing filtrations of CW spectra X and Y by subspectra, with quotients $X^s/X^{s+1} = \Sigma^s I^s$ and $Y^t/Y^{t+1} = \Sigma^t J^t$. Let*

$$Z^n = \bigcup_{s+t=n} X^s \wedge Y^t \subseteq X \wedge Y = Z$$

and

$$K^n = \bigvee_{s+t=n} I^s \wedge J^t.$$

Then $Z = Z^0 \supseteq Z^1 \supseteq \cdots$ is a filtration by subspectra, with quotients $Z^n/Z^{n+1} = \Sigma^n K^n$.

The maps involved are the obvious ones, and the proof is routine. It may help the reader in visualizing this arrangement to imagine the various subquotients of Z as subsets of the first quadrant of the (s,t)-plane. For example $X^s \wedge Y^t$ is represented by $\{(x,y) : s \leq x, t \leq y\}$, $X^s \wedge \Sigma^t J^t$ by $\{(x,y) : s \leq x, t \leq y \leq t+1\}$, and $\Sigma^s I^s \wedge \Sigma^t J^t$ by $\{(x,y) : s \leq x \leq s+1, t \leq y \leq t+1\}$. The diagram for $\Sigma^n K^n$ is inaccurate in that the intersections of the $\Sigma^n I^s \wedge J^t$ summands should be identified to a single point. Define filtrations of these objects by

$$F^sZ^n = \bigcup_{\substack{\sigma\geq s,\tau\geq 0\\ \sigma+\tau=n}} X^\sigma \wedge Y^\tau$$

$$F^sK^n = \bigvee_{\substack{\sigma\geq s,\tau\geq 0\\ \sigma+\tau=n}} I^\sigma \wedge J^\tau.$$

Remark 5.2. For each t,

$$\begin{tikzcd}
Z^t = F^0Z^t \arrow[d] & F^1Z^t \arrow[l] \arrow[d] & \cdots \arrow[l] & F^tZ^t \arrow[l] \arrow[d] & F^{t+1}Z^t \arrow[l] \arrow[d] & \cdots \arrow[l] \\
I^0 \wedge Y^t & \Sigma I^1 \wedge Y^{t-1} & & \Sigma^t I^t \wedge Y^0 & \Sigma^{t+1} I^{t+1} \wedge Y^0 &
\end{tikzcd}$$

is an A-Adams resolution, by Corollary 1.4.

Remark 5.3. For each s,

$$\begin{tikzcd}
F^sZ = F^sZ^s \arrow[d] & F^sZ^{s+1} \arrow[l] \arrow[d] & \cdots \arrow[l] \\
\Sigma^s F^s K^s & \Sigma^{s+1} F^s K^{s+1} &
\end{tikzcd} \tag{5.4}$$

is a B-Adams resolution. We must show that $k : F^sZ^n \to \Sigma^n F^s K^n$ is B-monic for all $n \geq s \geq 0$. This is proved by downward induction on s. For $s = n$, $k : X^s \wedge Y^0 \to \Sigma^s I^s \wedge J^0$ is the composite of $X^s \wedge Y^0 \to \Sigma^s I^s \wedge Y^0$ (which is A-monic by Corollary 1.4, hence B-monic by Lemma 2.1) and $I^s \wedge Y^0 \to I^s \wedge J^0$ (which is B-monic by Corollary 1.4), and so is B-monic. For the inductive step, chase the diagram obtained by mapping

$$\begin{tikzcd}
F^{s+1}Z^n \arrow[r] \arrow[d] & \Sigma^n F^{s+1} K^n \arrow[d] \\
F^sZ^n \arrow[r] \arrow[d] & \Sigma^n F^s K^n \arrow[d] \\
\Sigma^s I^s \wedge Y^t \arrow[r] & \Sigma^n I^s \wedge J^t
\end{tikzcd} \qquad (s+t=n)$$

into a B-injective; the top and bottom rows give epimorphisms, and the right column is split short exact, so the middle row is epic. Cf. [8].

Note that this B-Adams resolution maps to the B-Adams resolution

$$\begin{tikzcd}
Z = Z^0 \arrow[d] & Z^1 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
K^0 & \Sigma K^1 &
\end{tikzcd}$$

by a map of "degree" s (in the obvious sense).

Now define a filtration of $E_r(Z; B)$ by

$$F_A^s E_r^{s+t}(Z; B) = \mathrm{im}(E_r^t(F^s Z; B) \to E_r^{s+t}(Z; B)). \tag{5.5}$$

This filtration has the following properties:

(5.6) $F_A^{s+t+1} E_r^{s+t}(Z; B) = 0.$

(5.7) $F_A^0 E_r^{s+t}(Z; B) = E_r^{s+t}(Z; B).$

(5.8) The isomorphism

$$E_{r+1}(Z; B) \to H(E_r(Z; B))$$

is filtration-preserving.

(5.9) If $F_B^{s+t}\pi_*(Z)$ is filtered by

$$F_A^s F_B^{s+t}\pi_*(Z) = \mathrm{im}(\pi_*(F^s Z^{s+t}) \to \pi_*(Z))$$

and $\mathrm{gr}_B^{s+t}\pi_*(Z)$ is given the induced filtration, then the map

$$\varrho : \mathrm{gr}_B \pi_*(Z) \to E_\infty(Z; B)$$

of (1.9) is filtration-preserving.

(5.10) At $r = 1$

$$F_A^s E_1^{s+t}(Z; B) = \pi_*(F^s K^{s+t}),$$

and the spectral sequence defined by this filtration agrees with the Mahowald spectral sequence of Section 2. In particular, the filtration on E_2 is independent of choice of A-Adams resolution $\{X^s, I^s\}$. Also, in the Mahowald spectral sequence

$$E_1^{s,t} = \pi_*(I^s \wedge J^t) \tag{5.11}$$

since the filtration of K^{s+t} splits.

Remark 5.12. The May spectral sequence is obtained from the filtration

$$F_B^t \pi_*(I^s) = \mathrm{im}(\pi_*(I^s \wedge Y^t) \to \pi_*(I^s \wedge Y^0)).$$

In case X is (A,B)-primary, and $E_2^t(I^s; B) = \mathrm{gr}_B^t(I^s)$, the isomorphism between the Mahowald and May E_2-terms appears as follows. Let $z_0 \in \pi_*(I^s \wedge J^t)$ in $E_1^{s,t}$ of the Mahowald spectral sequence survive to E_2; so it is a cycle: $(1 \wedge d_J)z_0 = 0$. Thus it lives to $E_2^t(I^s; B)$, and hence lifts to $x \in \pi_*(I^s \wedge Y^t)$. The image of x in $F_B^t \pi_*(I^s)$ represents the element in the May E_2-term corresponding to z_0.

6. Proof of the main theorem

In this section we shall complete the proof of the following theorem.

Theorem 6.1. *Let A and B be ring-spectra such that B is A-injective, and suppose that X is an (A,B)-primary spectrum. Then the differential d_2^B carries $F_A^s E_2^{s+t}(X;B)$ into $F_A^{s+1}E_2^{s+t+2}(X;B)$. Furthermore, every $\bar{z}\in F_A^s E_s^{s+t}(X;B)$ has a representative $a\in R_A^s R_B^t \pi_*(X)$ in the Mahowald spectral sequence whose differential $d_2^{\mathrm{May}}a\in R_A^{s+1}R_B^{t+1}\pi_*(X)$ in the May spectral sequence represents $-d_2^B\bar{z}$ in the Mahowald spectral sequence.*

For the proof, we shall work with the double complex of Section 5 under the additional assumptions

(i) that Y is equivalent to the sphere spectrum S, and

(ii) that $X\to I^*$ is an A-resolution such that $E_2(I^s;B)=\mathrm{gr}_B\pi_*(I^s)$ (which is possible since X is (A,B)-primary). Thus $Z=X\wedge Y\simeq X$. In this section we shall allow maps to have nonzero dimension.

Let $\bar{z}\in F_A^s E_2^{s+t}(Z;B)$ be represented by $z\in E_1^t(F^sZ;B)=\pi_*(F^sK^{s+t})$. Then $d_K z=0$ where d_K is the differential in the chain-complex K^*. Since

$$F^sK^{s+t}\simeq \bigvee_{i=0}^{t} I^{s+i}\wedge J^{t-i}$$

we can write

$$z=\sum_{i=0}^{t} z_i,\qquad z_i\in\pi_*(I^{s+i}\wedge J^{t-i}). \tag{6.2}$$

Then $0=(1\wedge d_J)z_0\in\pi_*(I^s\wedge J^{t+1})$, so $z_0\in E_1^t(I^s;B)$ survives to E_2 and hence, by (ii), to $E_\infty^{s,t}=\mathrm{gr}_B^t\pi_*(I^s)$. Thus $z_0\in\mathrm{im}(\pi_*(I^s\wedge Y^t)\to\pi_*(I^s\wedge J^t))$ and so

$$0=(1\wedge k)z_0\in\pi_*(I^s\wedge Y^{t+1}). \tag{6.3}$$

This fact enables us to apply Lemma 6.7, proved at the end of this section, to $z\in\pi_*(F^sK^{s+t})$ in the diagram

$$\begin{tikzcd}
F^{s+1}Z^{s+t} \arrow[r] \arrow[d] & F^{s+1}K^{s+t} \arrow[r] \arrow[d, "g"] & F^{s+1}Z^{s+t+1} \arrow[d, "h"] \arrow[ll, dashed, bend right, "i"'] \\
F^sZ^{s+t} \arrow[r] \arrow[d] & F^sK^{s+t} \arrow[r, "k"] \arrow[d] & F^sZ^{s+t+1} \arrow[d] \\
I^s\wedge Y^t \arrow[r, "1\wedge j"] \arrow[uu, dashed, bend left, "\partial"] & I^s\wedge J^t \arrow[r] & I^s\wedge Y^{t+1}.
\end{tikzcd}$$

Let $x\in\pi_*(I^s\wedge Y^t)$ and $y\in\pi_*(F^{s+1}Z^{s+t+1})$ be the resulting elements, so that

$$(1\wedge j)x=z_0,\qquad kz=hy,\qquad iy=-\partial x. \tag{6.4}$$

Claim 6.5. $d^2\bar{z} \in F_A^{s+1}E_2^{s+t+2}(Z;B)$.

Indeed, we claim that y lifts to an element $\tilde{y} \in \pi_*(F^{s+1}Z^{s+t+2})$. Consider the commutative diagram

$$\begin{tikzcd}
F^{s+1}Z^{s+t+2} \arrow[r] & F^{s+1}Z^{s+t+1} \arrow[r, "j"] \arrow[d, "h"] & F^{s+1}K^{s+t+1} \arrow[d, "g"] \\
 & F^sZ^{s+t+1} \arrow[r, "j"] & F^sK^{s+t+1} \\
 & F^sK^{s+t} \arrow[u, "k"] \arrow[ur, "d_K"'] &
\end{tikzcd}$$

Now $gjy = jhy = jkz = d_Kz = 0$; but g splits, so $jy = 0$ and y lifts as claimed.

We turn now to the second assertion of the theorem. To construct the representative $a \in R_A^sR_B^t\pi_*(X)$, notice that this is $E_2^{s+t,s}$ of the May spectral sequence. We have $x \in \pi_*(I^s \wedge Y^t) = E_1^{s+t,s}$, and

Claim 6.6. x survives to E_2 in the May spectral sequence.

We are asserting the existence of an element $\tilde{x} \in \pi_*(I^{s+1} \wedge Y^{t+1})$ such that

$$(1 \wedge i^{t+1})\tilde{x} = (1 \wedge i^t)(d_I \wedge 1)x \in \pi_*(I^s \wedge Y^0).$$

Let $\tilde{x} = -k\tilde{y}$. Then (6.4) together with a chase of the diagram

$$\begin{tikzcd}
F^{s+1}Z^{s+t+2} \arrow[r, "i^2"] \arrow[d, "k"] & F^{s+1}Z^{s+t} \arrow[d] & I^s \wedge Y^t \arrow[l, "\partial"'] \arrow[d, "d_I \wedge 1"] \\
I^{s+1} \wedge Y^{t+1} \arrow[r] & I^{s+1} \wedge Y^{t-1} & I^{s+1} \wedge Y^t \arrow[l]
\end{tikzcd}$$

shows that in fact

$$(1 \wedge i^2)\tilde{x} = (1 \wedge i)(d_I \wedge 1)x \in \pi_*(I^s \wedge Y^{t-1}).$$

So let $a = \{\tilde{x}\} \in R_A^sR_B^t\pi_*(X)$. In the May spectral sequence d_2a is represented at E_1 by $(1 \wedge j)\tilde{x} \in \pi_*(I^{s+1} \wedge J^{t+1})$. But this also represents $-j\tilde{y}$ modulo im $\pi_*(F^{s+2}K^{s+t+2})$, as a chase of the following diagram shows.

$$\begin{tikzcd}
 & F^{s+2}K^{s+t+2} \arrow[d] \\
F^{s+1}Z^{s+t+2} \arrow[r, "j"] \arrow[d] & F^{s+1}K^{s+t+2} \arrow[d] \\
I^{s+1} \wedge Y^{t+1} \arrow[r, "1\wedge j"] & I^{s+1} \wedge J^{t+1}
\end{tikzcd}$$

This completes the proof of Theorem 6.1. We return now to the lemma we needed.

Lemma 6.7. *Suppose that*

$$\begin{tikzcd}
\bullet \arrow[r, "i"] \arrow[d, "p"'] & \bullet \arrow[r, "j"] \arrow[d] & \bullet \arrow[d] \\
\bullet \arrow[r] \arrow[d, "q"'] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet \arrow[r] & \bullet
\end{tikzcd} \tag{6.8}$$

is a commutative diagram of cofibration sequences in which each row has arrows i, j, k, and each column has arrows p, q, r. If z satisfies $jqz=0$, then there exist x and y such that

$$ix=qz, \qquad jz=py, \qquad ky+rx=0.$$

Proof. By means of mapping cylinders we may suppose that the upper left corner has the form

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & Y \arrow[d] \\
X \arrow[r] & Z
\end{tikzcd}$$

where all four arrows are inclusions of subcomplexes and $A=X\cap Y$. Then the diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Y/A \arrow[d] \\
X \arrow[r] \arrow[d] & X \cup Y \arrow[r] \arrow[d] & X \cup Y/X \arrow[d] \\
X/A \arrow[r] & X \cup Y/Y \arrow[r] & *
\end{tikzcd}$$

maps into (6.8), and $jqz=0$ implies that z lifts to $\bar{z} \in \pi_*(X \cup Y)$. Then there are unique elements $\bar{x} \in \pi_*(X/A)$ and $\bar{y} \in \pi_*(Y/A)$ satisfying

$$i\bar{x} = q\bar{z}, \qquad j\bar{z} = p\bar{y},$$

and we claim that

$$k\bar{y} + r\bar{x} = 0 \tag{6.9}$$

as well. The lemma then follows by taking x and y to be the images of $\bar{x}$ and $\bar{y}$.

Let $C_\pm$ denote the cone functor, and form the commutative diagram

$$\begin{tikzcd}
X \cup Y \arrow[r] \arrow[d] & X \cup C_-Y \arrow[d] \\
C_+X \cup Y \arrow[r] & C_+X \cup C_-Y.
\end{tikzcd} \tag{6.10}$$

In virtue of the cofibration sequence

$$A \to C_+X \vee C_-Y \to C_+X \cup C_-Y,$$

$C_+X \cup C_-Y \simeq \Sigma A$ since $C_+X \vee C_-Y$ is contractible. In fact, (6.10) is equivalent to the diagram

$$\begin{tikzcd}
X \cup Y \arrow[r] \arrow[d] & X/A \arrow[d, "-r"] \\
Y/A \arrow[r, "k"] & \Sigma A,
\end{tikzcd}$$

and (6.9) follows.

PART II

In this Part we shall gather together some elementary observations about "Hopf algebroids" and their cohomology. We construct a "May spectral sequence" by filtering a resolution by powers of an invariant ideal, and derive some corollaries, notably a vanishing line. These results were contained in [19]. Then in Section 9 the homology localization theorem of [20] is brought into play, and the BP localization theorem of [21] is proved as a corollary. Finally, we compute the May differential needed in Section 4.

Throughout Part II, R will denote a commutative ring, "R-module" will mean graded left R-module and "R-algebra" will mean commutative graded R-algebra.

7. Split Hopf algebroids

In Section 8 we shall give an algebraic construction of the May spectral sequence under suitable conditions. Here we prepare the way to identifying the resulting E_2-term.

Recall [4, 21] that if E is a commutative associative ring-spectrum such that $\Gamma = E_*(E)$ is flat over $A = E_*$, then (A, Γ) is a *Hopf algebroid,* i.e., a cogroupoid object in the category of commutative graded algebras. Furthermore, $E_*(X)$ is naturally a Γ-comodule, and

$$E_2(X; E) = \mathrm{Ext}_\Gamma(A, E_*(X)) \tag{7.1}$$

where $\mathrm{Ext}_\Gamma(A, M)$ is defined (for example — see also [21]) as the homology of a suitable cobar construction $\Omega(\Gamma; M)$. We refer the reader to [22] for a description of this complex.

To motivate the next construction, take note of the following class of groupoids. Suppose a group G acts from the right on a set X. Define a groupoid $X \tilde{\times} G$ with object set X, and, for $x, y \in X$,

$$\mathrm{Hom}_{X \tilde{\times} G}(x, y) = \{g \in G : xg = y\}.$$

Composition comes from multiplication in G. A groupoid is *split* if it is isomorphic to one of this form.

We shall mimic the dual of this construction in the category of commutative graded R-algebras. Thus let S be a commutative Hopf algebra over R with involution c, and let A be a right S-comodule-algebra. That is, (A, ψ) is an S-comodule, and the multiplication $\mu : A \otimes A \to A$ is an S-comodule map when $A \otimes A$ is given the diagonal S-coaction. Thus S is a cogroup object in the category of R-algebras, coacting on A from the right. This situation has been studied by P.S. Landweber [12].

Now define a Hopf algebroid $(A, A\tilde{\otimes}S)$ with cooperation algebra $A\otimes S$ and structure maps

$$\eta_L = A\otimes\eta : A \cong A\otimes R \to A\otimes S$$

$$\eta_R = \psi : A \to A\otimes S$$

$$\varepsilon = A\otimes\varepsilon : A\otimes S \to A\otimes R \cong A$$

$$\Delta = A\otimes\Delta : A\otimes S \to A\otimes S\otimes S \cong (A\otimes S)\otimes_A (A\otimes S)$$

$$c = (A\otimes\mu)(\psi\otimes c) : A\otimes S \to A\otimes S\otimes S \to A\otimes S.$$

A Hopf algebroid is *split* if it is isomorphic to one of this form.

Landweber goes on to define an A-module *over* S as a triple (M, φ, ψ) where (M, ψ) is an S-comodule, and (M, φ) is an A-module for which $\varphi : A\otimes M \to M$ is an S-comodule map when $A\otimes M$ is given the diagonal S-coaction. Such objects, together with the obvious morphisms, form a category (A-mod/S), which we claim is equivalent to ($A\tilde{\otimes}S$-comod). To see this, define, for any R-module M, two R-module maps, f and g:

$$g : M\otimes S \to (A\tilde{\otimes}S)\otimes_A M \tag{7.2}$$

by

$$g(m\otimes s) = (-1)^{|m||s|} 1\otimes c(s)\otimes m,$$

$$f : (A\tilde{\otimes}S)\otimes_A M \to M\otimes S \tag{7.3}$$

by

$$f(a\otimes s\otimes m) = \Sigma(-1)^{|m|(|a''|+|s|)} a'm\otimes a''c(s)$$

where $\psi : A \to A\otimes S$ by $\psi(a) = \Sigma a'\otimes a''$. The reader may check that f actually factors through the tensor product over A. Now it is easy to verify the

Lemma 7.4. *The correspondences*

$$F : (M, \Psi) \mapsto (M, f\Psi),$$

$$G : (M, \psi) \mapsto (M, g\psi)$$

define inverse functors

$$(A\tilde{\otimes}S\text{-comod}) \underset{G}{\overset{F}{\rightleftarrows}} (A\text{-mod}/S).$$

Let $\Omega_r(S; M)$ denote the unnormalized cobar construction of S with coefficients in the right S-comodule M. If $M \in (A\text{-mod}/S)$, then we have a natural differential isomorphism

$$\Omega(A\tilde{\otimes}S; M) \cong \Omega_r(S; M) \tag{7.5}$$

(omitting $G(-)$). Thus

Proposition 7.6. *For $M \in (A\text{-mod}/S)$,*

$$\mathrm{Ext}_{A \otimes S}(A, M) \cong \mathrm{Ext}_S(R, M).$$

Example 7.7. Let B be the unitary Thom spectrum MU. Then $\Gamma = \mathrm{MU}_*(\mathrm{MU})$ is free over $A = \mathrm{MU}_*(S)$, so (A, Γ) is a Hopf algebroid. It is furthermore well known [12] that Γ *splits* as $A \tilde{\otimes} S$, where S is the Hopf algebra dual to the algebra of Landweber–Novikov operations. Therefore,

Corollary 7.8. $E_2(X; \mathrm{MU}) \cong \mathrm{Ext}_S(\mathbb{Z}, \mathrm{MU}_*(X))$.

8. An algebraic May spectral sequence

Let (A, Γ) be a Hopf algebroid. An ideal $I \subset A$ is *invariant* if $\eta_L(I)\Gamma = \eta_R(I)\Gamma$. It follows that $\eta_L(I^t)\Gamma = \eta_R(I^t)\Gamma$: i.e., the "left" and "right" I-adic filtrations on Γ coincide. Thus all possible I-adic filtrations on $\Gamma \otimes_A \Gamma$ coincide. Furthermore, all the structure maps preserve the I-adic filtration. Thus $(E_0 A, E_0 \Gamma)$ is a (bigraded) Hopf algebroid. A Γ-comodule M has a natural I-adic filtration which is respected by $\psi : M \to \Gamma \otimes_A M$, and $E_0 M$ is an $E_0\Gamma$-comodule.

The I-adic filtrations on Γ and M define a tensor-product filtration $F^*_\otimes$ on $\Omega(\Gamma; M)$. We modify this filtration by setting

$$F^{s+t}\Omega^s(\Gamma; M) = F^t_\otimes \Omega^s(\Gamma; M). \tag{8.1}$$

Then $dF^{s+t} \subseteq F^{s+t+1}$, so $E_0 = E_1$ in the resulting spectral sequence. It is easy to check that

$$E_1 = \Omega(E_0\Gamma; E_0 M)$$

differentially, so

$$E_2 = \mathrm{Ext}_{E_0\Gamma}(E_0 A, E_0 M). \tag{8.2}$$

We call this the *May spectral sequence;* it is analogous to the principal spectral sequence of [16], in that it is obtained from the filtration on a (co)bar construction induced from a filtration on the (co)algebra.

Remark 8.3. On indexing: Give $E_0 M$ bidegree (t, i) where t is the filtration degree and i is the degree of the corresponding element in M. Write

$$E_2^{s+t,s,u} = \mathrm{Ext}_{E_0\Gamma}^{s,t,s+u}(E_0 A, E_0 M),$$

where s is the homological degree and $(t, s+u)$ is the internal bidegree. Then the indexing coincides with the indexing in the spectral sequence of Section 3 under the correspondence described in Remark 8.14.

Remark 8.4. On convergence: The *I-adic completion* of an A-module M is

$$\hat{M} = \varprojlim M/I^n M,$$

and M is *complete at I* provided that $M \to \hat{M}$ is an isomorphism. Assume that Γ is of finite type over A. Then $\Omega(\Gamma; M)$ is complete at I whenever M is; and then the associated spectral sequence converges in the sense of [5, III 8.2]. If $I = JA$, where J is an ideal in R, then the I-adic and J-adic filtrations (and hence completions) of M coincide. If furthermore M is of finite type over R, then $\hat{M} = \hat{R} \otimes_R M$; and if finally A is of finite type over R, then

$$\Omega(\Gamma; \hat{M}) = \hat{R} \otimes_R \Omega(\Gamma; M).$$

Since $\hat{R}$ is R-flat, we then have

$$\mathrm{Ext}_\Gamma(A, \hat{M}) = \hat{R} \otimes_R \mathrm{Ext}_\Gamma(A, M).$$

Under these assumptions, then, the spectral sequence for M converges to this module.

Our main application of this spectral sequence will involve the Hopf algebroid of cooperations of the Brown–Peterson spectrum BP associated to the prime p. Recall [5, II; 10; 6] that

$$A = \mathrm{BP}_* = \mathbb{Z}_{(p)}[v_1, v_2, \dots], \qquad |v_i| = 2(p^i - 1)$$
$$\Gamma = \mathrm{BP}_*(\mathrm{BP}) = A[t_1, t_2, \dots], \qquad |t_i| = 2(p^i - 1).$$

Thus Γ is A-flat, so (A, Γ) is a Hopf algebroid over $\mathbb{Z}_{(p)}$. For a description of the structure maps see [22]. Let $I = (p, v_1, \dots) \subset A$. I is an invariant maximal ideal with quotient field $\mathbb{F}_p$. Passing to associated graded objects, we obtain a Hopf algebroid with

$$\begin{aligned} Q = E_0 A = \mathbb{F}_p[q_0, q_1, q_2, \dots], &\quad |q_i| = (1, 2(p^i - 1)), \\ E_0 \Gamma = Q[t_1, t_2, \dots], &\quad |t_i| = (0, 2(p^i - 1)) \end{aligned} \tag{8.5}$$

(using the indexing of Remark 8.3) where q_i is the class of v_i, and $v_0 = p$. In [22, Lemma 4.5] we showed that in $E_0\Gamma \otimes_Q E_0\Gamma$,

$$\Delta t_n = \sum_{i=0}^{n} t_i \otimes t_{n-i}^{p^i}. \tag{8.6}$$

Similarly, the right unit $\eta_R : Q \to E_0\Gamma$ is given by

$$\eta_R q_n = \sum_{i=0}^{n} q_i t_{n-i}^{p^i}. \tag{8.7}$$

This implies that $E_0\Gamma$ splits as $Q \tilde{\otimes} P$, where P is the Hopf algebra $\mathbb{F}_p[t_1, t_2, \dots]$ with diagonal given by (8.6), and Q is given the right P-coaction

$$\psi q_n = \sum_{i=0}^{n} q_i \otimes t_{n-i}^{p^i}. \tag{8.8}$$

Consequently, by Proposition 7.6, the May E_2-term is

$$E_2 = \mathrm{Ext}_P(\mathbb{F}_p, E_0M). \tag{8.9}$$

Note that P is just the dual of the Hopf algebra of Steenrod reduced powers; t_i is conjugate to Milnor's ξ_i (ξ_i^2 for $p=2$).

We shall now give several applications of this spectral sequence. First we have a vanishing line, which has also been obtained by R. Zahler [29], for the BP-Adams E_2-term. Let $q=2(p-1)$.

Theorem 8.10. *If M is an $(m-1)$-connected Γ-comodule, then* $\mathrm{Ext}_\Gamma^s(A,M)$ *is $(T(s)+m-1)$-connected, where if $p=2$ $T(s)=s$ and if p is odd*

$$T(2r) = pqr$$

$$T(2r+1) = (pr+1)q.$$

Proof. If N is an $(n-1)$-connected P-comodule, then by [2] $\mathrm{Ext}_P^s(\mathbb{F}_p, N)$ is $(T(s)+n-1)$-connected. Since E_0^tM is $(m-1)$-connected for all $t \geq 0$, $E_2^{s+t,s}$ in the May spectral sequence is $(T(s)+m-1)$-connected. For M of finite type the result then follows by (8.3) and (8.4); but any comodule is a union of comodules of finite type ([21, Corollary 2.13]) so we are done.

We offer the following as a sample calculation; it is used in Theorem 4.4 above. We shall compute d_2 on

$$E_2^{1,1} = \mathrm{Ext}_P^1(\mathbb{F}_p, \mathbb{F}_p),$$

which by [1] and [13] is additively generated by

$$h_{1,j} = \{[t_1^{p^j}]\}, \quad j \geq 0. \tag{8.11}$$

Recall the classes

$$b_{1,j} = \left\{ \sum_{k=1}^{p-1} \frac{\binom{p}{k}}{p} [t_1^{kp^j} | t_1^{(p-k)p^j}] \right\} \tag{8.12}$$

in $E_2^{2,2}$; thus $b_{1,j} = h_{1,j}^2$ for $p=2$.

Lemma 8.13. *In the May spectral sequence for the comodule A,*

$$d_2 h_{1,j} = -q_0 b_{1,j-1}, \quad j \geq 1.$$

Proof. Compute in $\Omega(\Gamma; A)$:

$$d[t_1^{p^j}] = -\sum_{k=1}^{p^j-1} \binom{p^j}{k} [t_1^k | t_1^{p^j-k}].$$

The result follows since mod p^2,

$$\binom{p^j}{k} \equiv \begin{cases} 0 & \text{if } p^{j-1} \nmid k, \\ \binom{p}{l} & \text{if } k = p^{j-1}l. \end{cases}$$

Remark 8.14. Let H be the mod p Eilenberg–MacLane spectrum and let $\theta : \mathrm{BP} \to H$ denote the Thom reduction. Then $I = \ker(\theta_* : \mathrm{BP}_* \to H_*)$. If X is connective and $H_*(X; \mathbb{Z}_{(p)})$ is free, then $\mathrm{BP} \wedge X$ splits as a wedge of suspensions of BP; so X is (BP, H)-primary, and furthermore the I-adic filtration on $\mathrm{BP}_*(X) = \pi_*(\mathrm{BP} \wedge X)$ agrees with the H-Adams filtration. It follows that the May spectral sequence of Section 3 coincides with the spectral sequence of this Section. This coincidence holds for certain other spectra as well: for the Moore spectrum $V = S^0 \cup_p e^1$, for example.

Remark 8.15. To complete this circle of ideas, we construct algebraically a spectral sequence coinciding in certain cases with the Mahowald spectral sequence of Section 2. Let (A, Γ) be a Hopf algebroid, and let

$$0 \to A \to N \to \bar{N} \to 0$$

be a short exact sequence of Γ-comodules, with N flat over A. Then

$$0 \to \bar{N}^{\otimes s} \otimes M \to N \otimes \bar{N}^{\otimes s} \otimes M \to \bar{N}^{\otimes (s+1)} \otimes M \to 0$$

(with tensor-products over A) is exact for all s. Splicing these together, we have a long exact sequence

$$0 \to M \to N \otimes M \to N \otimes \bar{N} \otimes M \to \cdots \tag{8.16}$$

and applying $\mathrm{Ext}_\Gamma(A, -)$ we get a spectral sequence converging to $\mathrm{Ext}_\Gamma(A, M)$.

This situation occurs geometrically when C is an associative commutative ring-spectrum such that $\Gamma = C_*(C)$ is flat over $A = C_*$, and B is a ring-spectrum such that C is B-injective and $N = C_*(B)$ is flat over C_*. Then the spectral sequence agrees with the Mahowald spectral sequence, whose E_2-term thus depends algebraically on $C_*(X)$.

Suppose for example that $B = BP$ and $C = H$ is the mod p Eilenberg–MacLane spectrum; so Γ is the dual Steenrod algebra. Then it is easy to see that the spectral sequence associated to (8.16) is the Cartan–Eilenberg spectral sequence of the coalgebra extension

$$P \to \Gamma \to E$$

where P is as above and E is an exterior algebra. For *p odd,* this spectral sequence is well known to collapse if X is torsion-free or if $X = V$ (see e.g. [20, Section 4]).

9. Localization of the May spectral sequence

In this section we shall use the spectral sequence of Section 8 to combine the homology localization theorem of [20] with the theory of formal groups associated with the Brown–Peterson spectrum. We obtain a new proof of the BP localization theorem of [21], and a computation of the differential required for Theorem 4.8.

We begin by recalling the statement of the BP localization theorem. Fix a prime p and for $n \geq 0$ define

$$K(n) = \begin{cases} \mathbb{Q} & n = 0, \\ \mathbb{F}_p[v_n, v_n^{-1}] & n > 0, \end{cases}$$

and

$$\Gamma(n) = K(n) \otimes_A \Gamma \otimes_A K(n)$$

where $A = \mathrm{BP}_*$ and $\Gamma = \mathrm{BP}_*\mathrm{BP}$. $\Gamma(n)$ is then a $K(n)$-Hopf algebra, and by [25],

$$\Gamma(n) = K(n)[t_1, t_2, \ldots]/(t_i^{p^n} = v_n^{p^i - 1} t_i) \tag{9.1}$$

as algebras.

Let $I_n = (p, v_1, \ldots, v_{n-1}) \subset A$. We shall say that a Γ-comodule M is of *height* n provided that $I_n M = 0$ and $v_n | M$ is bijective.

Theorem 9.2 [21]. *The natural map*

$$\mathrm{Ext}_\Gamma(A, M) \to \mathrm{Ext}_{\Gamma(n)}(K(n), K(n) \otimes_A M)$$

is an isomorphism if M is of height n.

Proof. As in [21], we begin by noting that it suffices to prove this for $M = B(n) = v_n^{-1} A / I_n$. For $n = 0$ this is trivial, so suppose $n > 0$.

Give A and Γ the I-adic filtration, and give $K(n)$ and $\Gamma(n)$ the resulting (doubly infinite) filtrations. Then as Hopf algebras,

$$E_0 \Gamma(n) = E_0 K(n) \otimes P(n) \tag{9.3}$$

where

$$P(n) = P/(t_1^{p^n}, t_2^{p^n}, \ldots).$$

This is true as algebras by (9.1), and the result follows since $E_0^0 \Gamma = P$ as Hopf algebras. Thus we have maps of May spectral sequences

$$\begin{array}{ccc} \mathrm{Ext}_P(\mathbb{F}_p, E_0 B(n)) & \longrightarrow & \mathrm{Ext}_{P(n)}(\mathbb{F}_p, E_0 K(n)) \\ \Big\Downarrow & & \Big\Downarrow \\ \mathrm{Ext}_\Gamma(A, B(n)) & \longrightarrow & \mathrm{Ext}_{\Gamma(n)}(K(n), K(n)). \end{array} \tag{9.4}$$

Here we have used the obvious isomorphism

$$\mathrm{Ext}_{E_0 K(n) \otimes P(n)}(E_0 K(n), E_0 K(n)) = \mathrm{Ext}_{P(n)}(\mathbb{F}_p, E_0 K(n)).$$

Now the main theorem of [20] implies that the top arrow is an isomorphism. But the filtrations defining these spectral sequences are bicomplete, so the map of abutments is also an isomorphism.

We turn now to a computation of the differential in the localized May spectral sequence

$$\mathrm{Ext}_{P(1)}(\mathbb{F}_p, E_0K(1)) \Rightarrow \mathrm{Ext}_{\Gamma(1)}(K(1), K(1)). \tag{9.5}$$

As noted in Section 4,

$$\mathrm{Ext}_{P(1)}(\mathbb{F}_p, E_0K(1)) = E_0K(1) \otimes E[h_{n,0} : n \geq 1] \otimes P[b_{n,0} : n \geq 1]$$

where in the cobar construction $\Omega(P(1))$

$$[t_n] \in h_{n,0}, \tag{9.6}$$

$$\sum_{i=1}^{p-1} \frac{\binom{p}{i}}{p} [t_n^{p-i} | t_n^i] \in b_{n,0}. \tag{9.7}$$

We will compute $d_2 h_{n,0}$, and to do this we must study Δt_n in $\Gamma(1)$. For this we will need a lemma from the theory of p-typical formal groups; an excellent reference is [6]. For any collection $\mathscr{X} = \{X_i\}$ of indeterminates, write

$$C_n(\mathscr{X}) = \frac{1}{\varepsilon_n} ((\Sigma X_i)^n - \Sigma(X_i^n))$$

where

$$\varepsilon_n = \begin{cases} p & \text{if } n = p^i \text{ for some } i, \\ 1 & \text{otherwise.} \end{cases}$$

Let A denote a commutative ring, let $G_a(X, Y) = X + Y$ be the additive formal group, and let $\gamma_0(T) = T$ denote the standard curve.

Proposition 9.8. *Let F be a formal group over A. Suppose $J \subset A$ is an ideal such that $J^2 = 0$, and that $F \equiv G_a$* mod *J.*

(a) (Lazard). *Then*

$$F(X, Y) = X + Y + \sum_{n \geq 1} u_n C_{n+1}(X, Y) \tag{9.9}$$

for unique elements $u_n \in J$.

(b) *The Frobenius operator $\mathfrak{F}_q$ satisfies*

$$\mathfrak{F}_q \gamma_0(T) = -\sum_{k \geq 1} \frac{q}{\varepsilon_{kq}} u_{kq-1} T^k \tag{9.10}$$

for any $q \geq 2$.

(c) *Suppose that p is a prime number and that A is a $\mathbb{Z}_{(p)}$-algebra. Then F is p-typical if and only if $u_n = 0$ for all n not of the form $p^i - 1$. If F is p-typical, and*

$$\mathfrak{F}_p\gamma_0(T) = \sum_{i\geq 1}{}^F v_i T^{p^{i-1}}, \tag{9.11}$$

then

$$F(X,Y) = X + Y - \sum_{i\geq 1} v_i C_{p^i}(X,Y). \tag{9.12}$$

Proof. Part (a) is Lazard's main lemma; see [5, II Section 7]. (b) is a straightforward calculations using the definition of $\mathfrak{F}_q$, and (c) follows from (b).

Remark 9.13. Equation (9.11) is Araki's characterization [6] of the Hazewinkel generators v_i for BP_*. Note that (9.12) describes the universal p-typical formal group G modulo J^2, where $J = (v_1, v_2, \ldots)$. It follows from this that $v_i \in \mathrm{BP}_*$ is indecomposable for all i (cf. [10]).

Remark 9.14. It follows from Proposition 9.8(c) that

$$\sum{}^G \gamma_i \equiv \sum \gamma_i - \sum_{n\geq 1} v_n C_{p^n}(\gamma) \mod J^2$$

where $\gamma = \{\gamma_i\}$ is any collection of curves.

Now [5, II Theorem 16.1] the diagonal in Γ is determined by the identity in $\Gamma[\![T]\!]$

$$\sum_{n\geq 0}{}^G (\Delta t_n) T^{p^n} = \sum_{i,j\geq 0}{}^G (t_i \otimes t_j^{p^i}) T^{p^{i+j}}. \tag{9.15}$$

Here we have written G for $\eta_L{*}G$. This can be computed modulo $(\eta_L(J)\Gamma)^2$ ("modulo J^2" for short) using Remark 9.14, and if we work furthermore modulo $(v_1^2, v_2, v_3, \ldots)$,

$$\begin{aligned} &\sum_{n\geq 0} (\Delta t_n) T^{p^n} - v_1 C_p(T, (\Delta t_1) T^p, \ldots) \\ &\quad \equiv \sum_{i,j\geq 0} (t_i \otimes t_j^{p^i}) T^{p^{i+j}} - v_1 C_p((t_i \otimes t_j^{p^i}) T^{p^{i+j}} : i,j \geq 0). \end{aligned}$$

Now pass to $\Gamma(1)$, and remember that $t_n^p = v_1^{p^n-1} t_n$ lies in higher filtration than t_n does. So working modulo I-adic filtration 2, for $p > 2$:

$$\begin{aligned} &\sum_{n\geq 0} (\Delta t_n) T^{p^n} - v_1 C_p(T, (\Delta t_1) T^p, \ldots) \equiv \\ &\quad \equiv T + \sum_{n\geq 1} (t_n \otimes 1 + 1 \otimes t_n) T^{p^n} - v_1 C_p(T; (t_i \otimes 1) T^{p^i}, (1 \otimes t_i) T^{p^i} : i \geq 1). \end{aligned} \tag{9.16}$$

For $p = 2$ we must add

$$(v_1 t_1 \otimes t_1) T^4$$

to the right-hand side. We are interested in the coefficient of T^{p^n}. Note that it is exactly $\Delta t_n \bmod I^2$ on the left-hand side of (9.16). On the right, we can achieve T^{p^n} in the obvious way in the first sum, and for $n \geq 2$, as

$$-\frac{v_1}{p}[((t_{n-1}\otimes 1)T^{p^{n-1}}+(1\otimes t_{n-1})T^{p^{n-1}})^p$$

$$-((t_{n-1}\otimes 1)^pT^{p^n}+(1\otimes t_{n-1})^pT^{p^n})]$$

in the second. Thus for $n\geq 2$,

$$\Delta t_n \equiv t_n\otimes 1+1\otimes t_n - v_1\sum_{i=1}^{p-1}\frac{\binom{p}{i}}{p}t_{n-1}^i\otimes t_{n-1}^{p-i} \mod I^2 \tag{9.17}$$

unless $n=2$ and $p=2$, when

$$\Delta t_2 \equiv t_2\otimes 1+1\otimes t_2 \mod I^2. \tag{9.18}$$

It follows that in the localized May spectral sequence (9.5),

$$d_2h_{n,0}=-q_1b_{n-1,0}, \quad n\geq 2 \tag{9.19}$$

for $p>2$. This is the assertion required to complete the proof of Theorem 4.8. It shows also that in (9.5)

$$E_\infty = E_3 = \mathbb{F}_p[q_1,q_1^{-1}]\otimes E[h_{1,0}].$$

Since this is a free commutative algebra, no extensions are possible, and we have reobtained $H^*(\Gamma(1))$ [26]. Similarly, for $p=2$,

$$d_2h_{n,0}=-q_1h_{n-1,0}^2, \quad n\geq 3 \tag{9.20}$$

and

$$E_3=\mathbb{F}_2[q_1,q_1^{-1}]\otimes\mathbb{F}_2[h_{1,0}]\otimes E[h_{2,0}].$$

$h_{2,0}$ lifts to $v_1^3\varrho_1=(t_2-t_1^3)+v_1^{-1}v_2t_1$, so no further differentials are possible, and we have computed $H^*(\Gamma(1))$ for $p=2$ as well (cf. [26]).

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