THE BURNSIDE BICATEGORY OF GROUPOIDS

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Dedicated to the memory of Gaunce Lewis, 1950–2006

Sometime in the 1980's Gaunce Lewis described to me a "Burnside category" of groupoids. It provides an additive completion of the Burnside category of groups. In these notes we offer two quite different models for this construction, arising from two different but equivalent bicategories [5]. Morphisms from H to G in the Burnside category of groups are provided by sets with commuting right and left actions of H and G, which are free and finite over G. The first model extends this construction to groupoids by considering a G-action on a set S as a homomorphism $G \to \mathrm{Aut}(S)$. This model has alternative expressions in terms of "bi-actions" or in terms of covering groupoids. The second model is a theory of correspondences of groupoids. In both cases the morphism categories are symmetric monoidal, and the morphism groups in the Burnside category are obtained by adjoining inverses to the commutative monoids of isomorphism classes.

In each case the composition laws are "bilinear," and presumably satisfy the axioms described by Bert Guillou [3]. Guillou shows that such pairings can be rigidified to provide the input required by Tony Elmendorf and Mike Mandell [2] to produce a spectral category. The functor $G \mapsto \Sigma^\infty_+ BG$ will then extend to a spectral functor to spectra. On the subcategory of finite groupoids, the maps on morphism spectra are appropriate completions; this is the content of the "Segal conjecture." Moreover, the (enriched) category of spectrally enriched functors from this category to spectra is a model for "global" equivariant homotopy theory.

In fact the disjoint union of groupoids plays the role of both coproduct and product in these bicategories. In this sense we have a "semi-additive bicategory." We have not attempted to write down a proper definition of this, but presumably: there is such a notion; it implies Guillou's axioms (for a "pre-additive bicategory"); and examples such as the ones presented hear are additive bicategories. In any case, passing to the associated category and group-completing the hom-monoids provides a "Burnside category" of groupoids – an additive category with compatible functors from the category of groupoids and equivalence classes of functors and from the opposite of the subcategory in which morphisms are equivalence classes of “finite weak covers.”

These models have different virtues. The bi-set idea can be copied easily to give a “Morita category” of rings, in which the morphism symmetric monoidal categories are the groupoids of bimodules which are finitely generated and projective over the source ring, and bimodule isomorphisms. This leads to an extension of the construction of the algebraic K-theory of a ring to a spectral functor from the “spectral Morita category.” The bi-action model is a variant of the discrete version of a topological construction which provides a convenient account of the theory of orbifolds. The correspondance model makes clear the universal property of the Burnside bicategory: it is a bicategorical Q-construction.

My interest in making these constructions explicit was re-ignited by a lecture by Clark Barwick in April, 2009, in which he carried out the analogue for schemes, and showed that algebraic K-theory extends to an enriched functor on the resulting spectral category. I claimed at the time that the construction for groupoids was standard and well known, but was not able to back up that claim when Dustin Clausen asked what I meant. These notes are my attempt to make good on that claim. My objective is to be self-contained and explicit. Much deeper and far reaching work is being done in this area by many people, among them Clark Barwick, Anna Marie Bohmann, David Gepner, Bert Guillou, Rune Haugseng, Peter May, and Stefan Schwede.

The first section below describes various equivalent notions of an action by a groupoid. This is used in Section 2 to set up the “Burnside bicategory” of groupoids, **B**, using the notion of bi-sets or equivalently of bi-actions. Section 3 describes the correspondence bicategory of groupoids, **C**, and Section 4 carries out the comparison between the bicategories **B** and **C**.

I am grateful to Chris Schommer-Pries for pointing out the bi-action model (see [4, 6]). Conversations with Jacob Lurie, Clark Barwick, Matthew Gelvin, Angélica Osorno, and Martin Frankland have also been helpful.

These notes are dedicated to the memory of Gaunce Lewis, who would have done a much better job writing up his idea than I have done here.

1. G-sets, G-actions, and covers

We begin by establishing several equivalent ways to view an "action" of a groupoid on a set, and note how the conditions of being finite or free appear from these various perspectives.

1.1. G-sets.

Write G_0 for the set of objects of the groupoid G, and G_1 for the set of morphisms. Call a functor $T : G \to \mathbf{Set}$ from a groupoid G to the category of sets a "left G-set" (or just a "G-set") and a functor $S : G^{\mathrm{op}} \to \mathbf{Set}$ a "right G-set." Write T_γ and S^γ for the values at $\gamma \in G_0$. If $g : \gamma' \to \gamma$ is a morphism in G, write $gx \in T_\gamma$ for the image of $x \in T_{\gamma'}$ under γ, and $yg \in S^{\gamma'}$ for the image of $y \in S^\gamma$. Left G-sets form a category G-$\mathbf{Set}$ in which morphisms are natural transformations.

Let's begin by noting some categorical features of G-$\mathbf{Set}$.

The forgetful functor from G-sets to G_0-sets (where we regard G_0 as the discrete subgroupoid of G consisting of the identity morphisms) preserves limits and colimits.

Lemma 1.1. *The forgetful functor G-$\mathbf{Set} \to G_0$-$\mathbf{Set}$ creates coequalizers.*

Proof. Let $u, v : X \to Y$, and for each $\gamma \in G_0$ let Z_γ be the coequalizer of $u_\gamma, v_\gamma : X_\gamma \to Y_\gamma$. We need to see that Z extends uniquely to a functor from G, so let $g : \gamma' \to \gamma$. Since u and v are natural transformations, there is a unique map $g : Z_{\gamma'} \to Z_\gamma$ compatible with $g : Y_{\gamma'} \to Y_\gamma$ under the projection map $Y \to Z$. Uniqueness implies that with this structure Z is a G-set. To check that it is the coequalizer, let $w : Y \to W$ be a map of G-sets such that $wu = wv$. For each γ, there is a unique map $Z_\gamma \to W_\gamma$ factorizing f_γ. These maps are compatible with g because they are after composing with the surjection $Y_\gamma \to Z_\gamma$. □

Corollary 1.2. *A morphism in G-$\mathbf{Set}$ is an effective epimorphism if and only if it is surjective on objects.*

Proof. Suppose that $f : X \to Y$ is such that f_γ is surjective for each $\gamma \in G_0$. Surjections of sets are effective epimorphisms; that is,

$$X_\gamma \times_{Y_\gamma} X_\gamma \rightrightarrows X_\gamma \to Y_\gamma$$

is a coequalizer diagram. By the lemma this extends to a coequalizer diagram in G-$\mathbf{Set}$. □

It's easy to see that any epimorphism in G-$\mathbf{Set}$ is effective.

Definition 1.3. *A G-set T is* finite *if its colimit (written $G\backslash T$) is finite.*

Definition 1.4. *A G-set T is* free *if for all $\gamma, \gamma' \in G_0$ and all $x \in T_\gamma$ the map $G(\gamma', \gamma) \to T_{\gamma'}$ sending g to gx is injective.*

The *empty groupoid* $\varnothing$ has no objects (and hence no morphisms). It is initial in the category of categories. So there is a single $\varnothing$-set (namely the empty one). Its colimit is empty (and hence finite), and the freeness condition is satisfied vacuously.

Lemma 1.5. *The following are equivalent conditions on a G-set T.*
(1) *T is a coproduct of co-representable G-sets.*
(2) *T is free.*
(3) *T is projective in the category G-**Set**.*

Proof. Let T be any left G-set. The evident natural map

$$\coprod_{\gamma \in G_0} T_\gamma \to G\backslash T$$

is surjective, so by the axiom of choice it admits a section. This means that there are subsets $X_\gamma \subseteq T_\gamma$ such that the composite map

$$\coprod_{\gamma \in G_0} X_\gamma \to G\backslash T$$

is bijective. The evident natural map

$$i : \coprod_{\gamma \in G_0} \coprod_{x \in X_\gamma} G(\gamma, -) \to T \tag{1}$$

is then surjective on objects and hence an effective epimorphism.

Consider the images of orbit maps $G(\gamma', \gamma) \to T_\gamma$ and $G(\gamma'', \gamma) \to T_\gamma$, given by $x \in X_{\gamma'}$ and $y \in X_{\gamma''}$. If $f : \gamma' \to \gamma$ and $g : \gamma'' \to \gamma$ have $fx = gy$, then $g^{-1}f : \gamma' \to \gamma''$ sends x to y, and so $x \in X'_\gamma$ and $y \in X_{\gamma''}$ have the same image in $G\backslash T$. Therefore $\gamma' = \gamma''$ and $x = y$.

If we now assume that T is free, this implies that $f = g$, and the map i is injective as well as surjective on objects, and hence an isomorphism: so any free G-set is a coproduct of co-representables.

For any $\gamma_0 \in G_0$, the co-representable functor $G(\gamma_0, -)$ is projective, and coproducts of co-representables are too.

For any $\gamma_0 \in G_0$, the co-representable functor $G(\gamma_0, -)$ is free: Given $\gamma, \gamma' \in G_0$ and $x \in G(\gamma_0, \gamma')$, $G(\gamma', \gamma) \to G(\gamma_0, \gamma)$ by $f \mapsto f \circ x$ is not just monic but bijective. So any coproduct of co-representable left G-sets is free.

Finally, if T is projective then the map in (1) is split-epi, so T is a subobject of a coproduct of co-representables and hence of a free object. But clearly any subobject of a free G-set is free. $\square$

1.2. **G-actions.** The notion of a G-set follows the image of a G-action on X as a homomorphism $G \to \mathrm{Aut}(X)$. Normally however one thinks of an action of a group G on a set X as a map $\alpha : G \times X \to X$ satisfying certain properties. The groupoid story can be developed following this model as well.

A *left action* of a groupoid $G = (G_0, G_1)$ is a set P together with a map $\pi : P \to G_0$ and an "action map"

$$\alpha : G_1{}^s\!\times_{G_0} P \to P$$

over G_0 which is unital and associative. Here the prescript indicates that G_1 is to be regarded as a set over G_0 via the source map s. The fiber product is regarded as a set over G_0 via t. So, in formulas, if we write $\alpha(g, x) = gx$, we are requiring $\pi(gx) = tg, (g'g)x = g'(gx), 1x = x$. For $\gamma \in G_0$, write $P_\gamma = \pi^{-1}(\gamma)$. A morphism of left G-actions is defined in the evident way.

A G-set $X : G \to \mathbf{Set}$ determines an action of G on the set

$$P_X = \coprod_{\gamma \in G_0} X_\gamma$$

by defining gx to be $X_g(x)$ (which was written gx above). Conversely, an action of G on P determines a G-set X_P by $(X_P)_\gamma = p^{-1}(\gamma) = P_\gamma$, with functoriality determined in the evident way by the action map. These constructions provide an equivalence of categories between G-sets and G-actions.

Under this correspondence, a G-set is finite precisely when the "orbit set" $P/\sim$ of the corresponding action is finite. The equivalence relation on P is the one given by $x \sim gx$.

A G-set is free precisely when the corresponding G-action satisfies the following property: the shear map

$$\sigma : G_1{}^s\!\times_{G_0} P \to P \times P \quad , \quad (g, x) \mapsto (x, gx)$$

is injective.

1.3. **Covering maps of groupoids.** A G-set X determines a new groupoid, the *translation groupoid* GX with $(GX)_0 = X$ and $GX(x, y) = \{g \in G_1 : gx = y\}$. The map of G-sets $X \to *$ induces a map $GX \to G$, which is a "covering map" according to the following definition.

Definition 1.6. *A map of groupoids $p : H \to G$ is a* covering map *provided that for any $\eta' \in H_0$ and any $g : p\eta' \to \gamma$ there is a unique $h : \eta' \to \eta$ in H such that $ph = g$.*

Conversely, a covering map $p : H \to G$ determines a G-set X with $X_\gamma = p^{-1}(\gamma)$ and $g : X_{\gamma'} \to X_\gamma$ defined using the unique morphism lifting. This establishes and equivalence of categories between G-sets and covering maps to G.

The orbit set or colimit $G\backslash X$ of a G-set is precisely the set of components of the translation groupoid GX, so X is a finite G-set if and only if GX has finitely many components. The notion for covering maps corresponding to freeness for G-sets is this: A covering map $p : H \to G$ is free if for every $\eta \in H_0$ and $\gamma \in G_0$, the map $G(p\eta, \gamma) \to H_0$, sending g to the target of its unique lift with source η, is injective.

2. Bi-sets and bi-actions

Here we discuss the bivariant form of G-sets and G-actions, and show how they form the objects of the morphism categories in a bicategory structure on groupoids.

2.1. **Bi-sets.** The *bi-set category* $\mathbf{B}(H, G)$ determined by a pair of groupoids G, H is defined using the following construction. If G and H are groupoids, call a functor $H^{\mathrm{op}} \times G \to \mathbf{Set}$ an "(H, G)-bi-set." An object of $\mathbf{B}(H, G)$ is an (H, G)-bi-set X such that X^η is free and finite as G-set for all $\eta \in H_0$. Morphism are the natural *isomorphisms.* $\mathbf{B}(H, G)$ is a symmetric monoidal groupoid, with the tensor product given by disjoint union.

This definition provides us with the morphism categories for a bicategory [5] in which the objects are groupoids. The composition functor is defined using the coend construction. The coend of functors $S : G^{\mathrm{op}} \to \mathbf{Set}$ and $T : G \to \mathbf{Set}$ is the set

$$S \times_G T = \coprod_{\gamma \in G_0} S^\gamma \times T_\gamma / \sim$$

where the equivalence relation is given by $(sg, t) \sim (s, gt)$. Note that $G\backslash T = * \times_G T$ (where $*$ denotes the constant functor with singleton values). Note also the natural isomorphisms

$$S \times_G G(\gamma, -) \xrightarrow{\cong} S^\gamma \quad , \quad G(-, \gamma) \times_G T \xrightarrow{\cong} T_\gamma \tag{2}$$

Define the composition functor $\mathbf{B}(G,F)\times\mathbf{B}(H,G)\to\mathbf{B}(H,F)$ by sending X,Y to $X\times_G Y$ with

$$(X\times_G Y)^\eta_\varphi = X_\varphi \times_G Y^\eta$$

We must check that $X\times_G Y^\eta$ is free and finite. First freeness: Since $X\times_G -$ commutes with coproducts, we may assume by Lemma 1.5 that $Y^\eta = G(\gamma,-)$ for some $\gamma\in G_0$. Then by (2) $X\times_G Y^\eta = X^\gamma$, which is free by assumption.

Now finiteness: Note that

$$F\backslash(X\times_G Y^\eta) = *\times_F (X\times_G Y^\eta)\cong(*\times_F X)\times_G Y^\eta = (F\backslash X)\times_G Y^\eta$$

Then use the facts that $F\backslash X^\gamma$ is finite for every γ and that $G\backslash Y^\eta$ is finite, and the following observation.

Let T be a left G-set, and let $\{t_i\in T_{\gamma_i}\}$ represent the elements of $G\backslash T$. Then for any right G set S the map

$$\coprod_i S^{\gamma_i}\to S\times_G T$$

sending $x\in S^{\gamma_i}$ to (x,γ_i,t_i) is surjective. To see this, let $(s,\gamma,t)\in S\times_G T$. There exists i and $g:\gamma_i\to\gamma$ such that $gt_i=t$, so $(s,\gamma,t) = (s,\gamma,gt_i) = (sg,\gamma_i,t_i)$.

The identity object in $\mathbf{B}(G,G)$ is given by

$$(1_G)^{\gamma'}_\gamma = G(\gamma',\gamma)$$

The isomorphisms (2) provide the unitors

$$\rho: X\times_G 1_G\to X \quad , \quad \lambda: 1_G\times_G Y\to Y$$

The associator

$$\alpha: X\times_G(Y\times_H Z)\to(X\times_G Y)\times_H Z$$

sends a list $(\varphi, x\in X^\gamma_\varphi,\gamma,y\in Y^\eta_\gamma,\eta,z\in Z^\kappa_\eta,\kappa)$ to the same list, differently bracketed. The triangle identity for the unitors and the pentagon identity for the associator are easily checked.

Let $\mathbf{G}$ denote the bicategory of groupoids, functors, and natural transformations. There is a functor

$$S:\mathbf{G}\to\mathbf{B}$$

which sends a groupoid to itself, a functor $q: H\to G$ to $S(q)\in\mathbf{B}(H,G)$ where

$$S(q)^\eta_\gamma = G(q\eta,\gamma)\,,$$

and a natural transformation $\alpha : q' \to q$ to the isomorphism of (H, G)-bi-sets given by $(\alpha^{-1})^* : G(q\eta, \gamma) \to G(q'\eta, \gamma)$. The required compatibilities with the associators (a hexagon) and unitors are easily verified.

2.2. **Burnside category.** By passing to the commutative monoids of isomorphism classes of objects in the morphism symmetric monoidal categories and the group-completing, we receive the *Burnside category* of groupoids. It is preadditive by construction: abelian group structures on the morphism sets are given, and composition is bilinear.

2.3. **Additivity.** One of the most pleasing aspects of the bi-set bicategory $\mathbf{B}$ is that it is *additive*: the coproduct groupoid serves as both coproduct and product in $\mathbf{B}$.

Let G' and G'' be two groupoids. The inclusion functor $\mathrm{in}_1 : G' \to G' \coprod G''$ induces the $(G', G' \coprod G'')$-bi-set $X(1) = S(\mathrm{in}_1)$ with

$$X(1)_{\gamma_1}^{\gamma_1'} = G'(\gamma_1', \gamma_1) \quad , \quad X(1)_{\gamma_1}^{\gamma_2} = \varnothing \quad , \quad \gamma_1, \gamma_1' \in G_0' \, , \gamma_2 \in G_0'' \, .$$

The bi-set $X(2)$ has a similar description. These morphisms induce an equivalence of categories

$$\mathbf{B}(G' {\textstyle\coprod} G'', K) \to \mathbf{B}(G', K) \times \mathbf{B}(G'', K)$$

which establishes $G' \coprod G''$ as the coproduct in the bicategory $\mathbf{B}$.

On the other hand, define the $(G' \coprod G'', G')$-bi-set $Y(1)$ by

$$Y(1)^{\gamma_1} = G'(\gamma_1, -) \quad , \quad Y(1)^{\gamma_2} = \varnothing \quad , \quad \gamma_1 \in G_0' \, , \gamma_2 \in G_0'' \, ,$$

and define $Y(2)$ similarly. Since $G' \backslash G'(\gamma_1, -) = *$, $Y(1)$ is finite; and it is clearly free. Thus we have morphisms in the category $\mathbf{B}$, which induce a functor

$$\mathbf{B}(K, G' {\textstyle\coprod} G'') \to \mathbf{B}(K, G') \times \mathbf{B}(K, G'')$$

This functor is also an equivalence: An object Z on the left is a pair of functors $Z(1) : K^{\mathrm{op}} \times G' \to \mathbf{Set}$, $Z(2) : K^{\mathrm{op}} \times G'' \to \mathbf{Set}$, and

$$Y(1) \times_{G' \coprod G''} Z = Z(1) \quad , \quad Y(2) \times_{G' \coprod G''} Z = Z(2)$$

The groupoid $G' \coprod G''$, together with these structure morphisms, serves as a product in the bicategory $\mathbf{B}$.

Consequently, the Burnside category is *additive*: it is pointed, the coproduct and product of pairs of objects coincide under the natural map, and the induced commutative monoid structures on morphism sets in fact render them abelian groups.

2.4. **Bi-actions.** We now indicate the "bi-action" analogue of the bi-set bicategory.

Given a right G-action on Q and a left G-action on P, the "balanced product" is defined as the set

$$Q \times_G P = Q \times_{G_0} P/\sim$$

where $(qg, p) \sim (q, gp)$ for $q \in Q^{\gamma'}$, $p \in P_\gamma$, $g : \gamma' \to \gamma$.

A *bi-action* of a pair of groupoids G, H is a set P with a left action of G (with projection $p : P \to G_0$) and a right action of H (with projection $q : P \to H_0$), such that $p(xh) = p(x)$, $q(gx) = q(x)$, and $(gx)h = g(xh)$.

This is the same thing as a left action by $H^{\mathrm{op}} \times G$, so bi-actions are equivalent to functors $H^{\mathrm{op}} \times G \to \mathbf{Set}$, i.e. to bi-sets. The shear map for a bi-action is compatible with projections to H_0:

$$\begin{tikzcd} {G_1}^s\!\times_{G_0} P \arrow[rr, "\sigma"] \arrow[dr] & & P \times_{H_0} P \arrow[dl] \\ & H_0 & \end{tikzcd}$$

Requiring X^η to be free for every $\eta \in H_0$ is equivalent to requiring that σ be injective. Requiring X^η to be finite for $\eta \in H_0$ is equivalent to requiring that in the corresponding bi-action P, the G-action on the fiber of P over η is finite. Write $\mathbf{A}(H, G)$ for the category of bi-actions of (G, H) satisfying these conditions. This is category is equivalent to $\mathbf{B}(H, G)$.

Given another groupoid F, there is "composition" functor $\mathbf{A}(G, F) \times \mathbf{A}(H, G) \to \mathbf{A}(H, F)$ given by sending (P, Q) to the balanced product $Q \times_G P$, with the evident actions of F and H.

3. Correspondences of Groupoids

In addition to the functor $S : \mathbf{B} \to \mathbf{G}$ from the bicategory of groupoids to the Burnside category, described above, there is a contravariant functor T from a certain subcategory of $\mathbf{B}$, which is related to S via a "double coset formula." In fact $\mathbf{B}$ is characterized up to equivalence as the universal bicategory accepting such a pair of functors. As such it admits a natural description in terms of correspondences or "spans," diagrams of groupoids of the form $H \leftarrow K \to G$ with conditions on $H \leftarrow K$ making it analogous to a fibration whose fibers are homotopy equivalent to finite sets. One should think of $\mathbf{B}$

as the stabilization of $\mathbf{G}$; S plays the role of suspension, and T that of the transfer.

We begin with some considerations about the "slice" or "overcategory" in the bicategorical context. This is quite general, but we will continue to speak of groupoids.

Suppose G is a groupoid. An object of the over-bicategory $\mathbf{G}/G$ is a map $p : L \to G$ in $\mathbf{G}$. A morphism from $p' : L' \to G$ to $p : L \to G$ is given by a functor $t : L' \to L$ together with a natural transformation $\theta : p' \to pt$. The composition $(t'', \theta'') \circ (t', \theta')$ is given by $(t'' \circ t', \theta''_{t'} \circ \theta')$. The identity morphism on $p : L \to G$ is $(1, 1)$.

The morphisms from p' to p form the objects of a category $\mathbf{G}/G(p', p)$, in which a morphism $(\overline{t}, \overline{\theta}) \to (t, \theta)$ consists in a natural transformation $\psi : \overline{t} \to t$ such that $p\psi \circ \overline{\theta} = \theta : p' \to pt$. Composition is given by composition of natural transformations.

The composition law $(t'', \theta'') \circ (t', \theta')$ extends to a functor. Let $\psi' : (\overline{t}', \overline{\theta}') \to (t', \theta')$, $\psi'' : (\overline{t}'', \overline{\theta}'') \to (t'', \theta'')$ be morphisms. The composed morphism $\psi : (\overline{t}''\overline{t}', \overline{\theta}''_{\overline{t}'}\overline{\theta}') \to (t''t', \theta''_{t'}\theta')$ is given by the diagonal in the commutative diagram

$$\begin{tikzcd}
\overline{t}''\overline{t}' \arrow[r, "\overline{t}''\psi'"] \arrow[d, "\psi''_{\overline{t}'}"'] \arrow[dr, "\psi"] & \overline{t}''t' \arrow[d, "\psi''_{t'}"] \\
t''\overline{t}' \arrow[r, "t''\psi'"] & t''t'
\end{tikzcd}$$

The required digram commutes by virtue of

$$\begin{tikzcd}
p' \arrow[r, "\overline{\theta}'"] \arrow[dd, "\theta'"'] & p\overline{t}' \arrow[r, "\overline{\theta}''_{\overline{t}'}"] \arrow[ddl, "p\psi'"'] & p''\overline{t}''\overline{t}' \arrow[d, "p''\overline{t}''\psi'"] \\
 & & p''\overline{t}''t' \arrow[d, "p''\psi''_{t'}"] \\
pt' \arrow[urr, "\overline{\theta}''_{t'}"] \arrow[rr, "\theta''_{t'}"] & & p''t''t'
\end{tikzcd}$$

The "covering" conditions will be described using the bicategorical pullback, and we turn to a *précis* of this construction.

So let $L \xrightarrow{p} G \xleftarrow{m} K$ be a diagram of groupoids. The bicategorical *pullback* is the groupoid $K \times_G L$ with

$$(K \times_G L)_0 = \{(\kappa, g, \lambda) : \kappa \in K_0, g : p\lambda \to m\kappa, \lambda \in L_0\}$$

and $(K \times_G L)((\overline{\kappa}, \overline{g}, \overline{\lambda}), (\kappa, g, \lambda))$ given by

$$\left\{ \begin{array}{l} k : \overline{\kappa} \to \kappa \\ l : \overline{\lambda} \to \lambda \end{array} : \begin{array}{ccc} p\overline{\lambda} & \xrightarrow{pl} & p\lambda \\ \downarrow_{\overline{g}} & & \downarrow_{g} \\ m\overline{\kappa} & \xrightarrow{mk} & m\kappa \end{array} \text{ commutes} \right\}$$

It is equipped with evident functors $\overline{m} : K \times_G L \to L$ and $\overline{p} : K \times_G L \to K$ and a natural transformation $p \circ \overline{m} \to m \circ \overline{p}$, and is the terminal example of such structure.

Formation of bicategorical pullbacks is natural. Suppose given morphisms $(s, \sigma) : (K', n') \to (K, n)$ and $(t, \theta) : (L', p') \to (L, p)$. They induce a morphism $K' \times_G L' \to K \times_G L$ which sends (κ', g, λ') to $(s\kappa', \theta_{\kappa'} \circ g \circ \sigma_{\lambda'}^{-1}, t\lambda')$. It sends the morphism $(\overline{\kappa}', \overline{g}, \overline{\lambda}') \to (\kappa', g, \lambda')$ given by $(k' : \overline{\kappa}' \to \kappa', l' : \overline{\lambda}' \to \lambda')$ to (sk', tl'). The required diagram commutes by naturality of σ and θ, and the fact that $g \circ pl' = n'k' \circ \overline{g}$.

For example, the *homotopy fiber* of $m : K \to G$ over $\gamma \in G_0$ is the bicategorical pullback along the functor from the singleton category into G with value γ; that is, it is the under-category γ/m, or, what is the same, the translation category $KG(\gamma, m(-))$. Its set of components is thus $K \backslash G(\gamma, m(-))$.

Formation of bicategorical pullbacks is associative. Given

$$K \xrightarrow{m} G \xleftarrow{p} L \xrightarrow{n} H \xleftarrow{q} M$$

there is a bijection $((K \times_G L) \times_H M)_0 \to (K \times_G (L \times_H M))_0$ given by $((\kappa, g, \lambda), h, \mu) \mapsto (\kappa, g, (\lambda, h, \mu))$. One checks that this extends to an isomorphism of groupoids.

Finally, we need to understand the pullback of $n : K \to G$ along the identity map $1 : G \to G$. The objects are triples (κ, g, γ) where $g : \gamma \to n\kappa$. A morphism $(\overline{\kappa}, \overline{g}, \overline{\kappa})$ is given by $(k : \overline{\kappa} \to \kappa, g' : \overline{\gamma} \to \gamma)$ such that $g \circ g' = nk \circ \overline{g}$. There is a functor $s : K \to K \times_G G$ given by sending κ to $(\kappa, 1, n\kappa)$ and $k : \overline{\kappa} \to \kappa$ to (k, nk). It is a quasi-inverse to the projection $p_1 : K \times_G G \to K$. Indeed $p_1 \circ s = 1$, while $(1, g) : (\kappa, g, \gamma) \to (\kappa, 1, n\kappa)$ provides a natural isomorphism $1 \to s \circ p_1$.

Consequently given $K \longrightarrow G \longleftarrow L \longleftarrow M$,

$$(K \times_G L) \times_L M = K \times_G (L \times_L M) \simeq K \times_G M$$

We can now state our "free and finite" conditions.

Definition 3.1. *A groupoid is* discrete *if there is at most one morphism between any two objects—that is, each component is unicursal. A discrete groupoid is* finite *if it has finitely many components. A map of groupoids is a* weak cover *if all of its homotopy fibers are discrete, and a* finite weak cover *if all of its homotopy fibers are discrete and finite.*

Notice that the condition of being discrete or discrete and finite is invariant under equivalence of groupoids.

Weak covers and finite weak covers pull back to the same. It's also easy to see that compositions of finite weak covers are finite weak covers, and that if $K' \to G$ and $K'' \to G$ are both finite weak covers then so is $K' \coprod K'' \to G$.

An object in the *correspondence groupoid* $\mathbf{C}(H,G)$ associated to a pair of groupoids G, H is a diagram $H \xleftarrow{q} L \xrightarrow{p} G$ where q is a finite weak cover. A morphism from (q', L', p') to (q, L, p) is an isomorphism $t : L' \to L$ together with a pair of natural transformations $\theta : p' \to pt$ and $\varphi : q' \to qt$. These data give maps in the categories of groupoids over G and over H, and composition is defined accordingly.

These groupoids form the morphism categories in a bicategory $\mathbf{C}$, the *correspondence bicategory* of groupoids. The composition functor $\mathbf{C}(G,F) \times \mathbf{C}(H,G) \to \mathbf{C}(H,F)$ is defined using the pullback:

$$\begin{tikzcd}
 & & K \times_G L \arrow[dl, "\overline{n}"'] \arrow[dr, "\overline{p}"] & & \\
 & L \arrow[dl, "q"'] \arrow[dr, "p"] & & K \arrow[dl, "n"'] \arrow[dr, "m"] & \\
H & & G & & F
\end{tikzcd}$$

The pulled back functor $\overline{n}$ is a finite weak cover because n is, and so the composite $q \circ \overline{n}$ is again a finite weak cover. The behavior on morphisms follows from the functoriality of the fiber product described above.

We leave a discussion of the associator and unitors to the interested reader. There is a functor $S : \mathbf{G} \to \mathbf{C}$ which sends a groupoid to itself, a functor $p : H \to G$ to the object $H \xleftarrow{1} H \xrightarrow{p} G$ of $\mathbf{C}(H,G)$, and a natural transformation $\alpha : p' \to p$ to $(1, 1, \alpha)$.

There is another functor to $\mathbf{C}$ from the opposite of the bicategory of groupoids and finite weak covers, which we denote by T. It sends a groupoid to itself, a finite weak cover $q : H \to G$ to the object

$G \xleftarrow{q} H \xrightarrow{1} H$ of $\mathbf{C}(G, H)$, and a natural transformation $\beta : q' \to q$ to $(\beta, 1, 1)$.

These two functors are related by a "double coset formula": Let $p : H \to G$ be a functor and $q : F \to G$ a finite weak cover, and form the pullback $H \xleftarrow{\overline{q}} F \times_G H \xrightarrow{\overline{p}} F$. Using the natural equivalence between $G \times_G G$ and G noted above, it is easy to construct a natural equivalence

$$T(q) \times_G S(p) \simeq S(\overline{p}) \times_{F \times_G H} T(\overline{q})$$

in $\mathbf{C}(H, F)$. The pair of functors S, T, together with this natural equivalence, is universal among such.

4. Comparison

Now we will see how the bicategories $\mathbf{B}$ and $\mathbf{C}$ are related.

It may help to look at the case of groups, first. So let G be a group and K a subgroup. The inclusion $p : K \to G$ expresses K as a groupoid over G; but there is another groupoid over G which is expressed in terms of the transitive G set G/K, namely the translation groupoid $G(G/K)$. These two groupoids over G are in a suitable sense equivalent.

There is a natural map $s : K \to G(G/K)$, which sends the unique object in K to $K \in (G(G/K))_0$. Note that in $G(G/K)$, $\mathrm{Aut}(K) = \{g \in G : gK = K\} = K$, and this isomorphism defines the functor on morphisms. This functor commutes strictly with the projection maps to G.

There is a non-natural quasi-inverse, obtained by picking coset representatives g_i, so that $G = \coprod_i g_i K$. In terms of these, define $t : G(G/K) \to K$ by sending $g_i K$ to the unique object of K, and $g : g_i K \to g_j K$ to $g_j^{-1} g g_i \in K$. This is functorial, but it does not commute with the projections to G. However, there is a natural transformation $\theta : q \to pt$ (where $q : G(G/K) \to G$ is the projection which forgets the objects). It is defined by $\theta_{g_i K} = g_i^{-1}$.

The composite $K \to G(G/K) \to K$ sends the morphism k first to the morphism $k : g_0 K \to k g_0 K = g_0 K$ (where g_0 is the coset representative of K itself, $g_0 \in K$), and then on to the morphism $g_0^{-1} k g_0 \in K$. This functor is accompanied by the natural transformation given by $g_0^{-1} \in K$. The composite is not the identity (unless g_0 happens to be central in K), but it is an isomorphism.

The composite $G(G/K) \to K \to G(G/K)$ sends every object $g_i K$ to $g_0 K$, and $g : g_i K \to g_j K$ to $g_j^{-1} g g_i$. The natural transformation $q \to stq$ is given by $\theta_{g_i K} = g_i^{-1}$.

This morphism in $\mathbf{G}/G$ is far from an isomorphism, but there is an isomorphism from it to the identity in the category of endomorphisms of $G(G/K) \to G$ in the bicategory $\mathbf{G}/G$, given by $\psi_{g_iK} = g_i^{-1}$.

Now we describe the analogous constructions for groupoids.

Let G and H be groupoids and X an (H, G)-bi-set. The "double translation groupoid" GXH has objects

$$(GXH)_0 = \coprod_{(\gamma,\eta)\in G_0\times H_0} X_\gamma^\eta$$

and morphisms $(\overline{\gamma}, \overline{x}, \overline{\eta}) \to (\gamma, x, \eta)$ given by pairs $g : \overline{\gamma} \to \gamma, h : \overline{\eta} \to \eta$ such that $g\overline{x} = xh \in X_\gamma^{\overline{\eta}}$. Composition is evident: Given also $(g', h') : (\gamma, x, \eta) \to (\hat{\gamma}, \hat{x}, \hat{\eta})$, the composite is given by $(g'g, h'h)$.

The double translation groupoid comes equipped with functors $p : GXH \to G$ and $q : GXH \to H$, both covariant.

Lemma 4.1. *If X^η is free and finite for all $\eta \in H_0$ then $q : GXH \to H$ is a finite weak cover.*

Proof. Fix $\eta_0 \in H_0$ and consider the homotopy fiber $GXH \times_H *$ of q over η_0. An object is (γ, x, η, h) where $x \in X_\gamma^\eta$ and $h : \eta_0 \to \eta$. A morphism $(\overline{\gamma}, \overline{x}, \overline{\eta}, \overline{h}) \to (\gamma, x, \eta, h)$ is given by $(g' : \overline{\gamma} \to \gamma, h' : \overline{\eta} \to \eta)$ such that $g'\overline{x} = xh'$ and $h'h = \overline{h}$. The second equation shows that h' is determined by h and $\overline{h}$, and g' is determined by its value on $\overline{x}$, by freeness of X^η. So the homotopy fiber is discrete.

To understand $\pi_0(GXH \times_H *)$ it is useful to re-express things in terms of the Grothendieck construction: Let $F : H^{\mathrm{op}} \to \mathbf{Cat}$ be a functor. The *Grothendieck construction* of F is the category FH with objects

$$(FH)_0 = \coprod_{\eta\in H_0} F_0^\eta$$

and morphisms $(\eta', \varphi' \in F_0^{\eta'}) \to (\eta, \varphi \in F_0^\eta)$ given by $(h : \eta' \to \eta, f : \varphi' \to \varphi h)$. There is a covariant functor $FH \to H$ sending (η, φ) to η. If F^η has only identity morphisms, for each η, then the Grothendieck construction coincides with the translation category.

Fix $\eta_0 \in H_0$ and consider the homotopy fiber $FH \times_H *$ over it. An object is $(\eta, \varphi \in F_0^\eta, h : \eta_0 \to \eta)$. A morphism $(\eta', \varphi', h') \to (\gamma, \varphi, h)$ is $(\overline{h} : \eta' \to \eta, \overline{f} : \varphi' \to \varphi\overline{h})$ such that $\overline{h}h' = h$.

There is a functor

$$F^{\eta_0} \to FH \times_H *$$

sending $\varphi \in F_0^{\eta_0}$ to $(\eta_0, \varphi, 1)$ and $f : \varphi' \to \varphi$ to $(1, f)$. It is fully faithful, and if H is a groupoid it is representative.

We apply this to the functor $GX : H^{\mathrm{op}} \to \mathbf{Cat}$:

$$GX^{\eta_0} = (GX)^{\eta_0} \xrightarrow{\simeq} GXH \times_H *$$

Therefore

$$\operatorname*{colim}_G X^{\eta_0} = \pi_0(GX^{\eta_0}) \cong \pi_0(GXH \times_H *)$$

so the condition that the homotopy fibers of $GXH \to H$ should be finite is the same as the condition that the X^{η_0}'s should be finite. $\square$

So far we have verified that the double translation groupoid gives a map on objects from $\mathbf{B}(H, G)$ to $\mathbf{C}(H, G)$. Let $f : X' \to X$ be a map of (H, G)-bi-sets. It induces a functor $f : GX'H \to GXH$ sending the object (γ, x', η) to (γ, fx', η) and the morphism $(g, h) : (\overline{\gamma}, \overline{x}', \overline{\eta}) \to (\gamma, x', \eta)$ to the morphism $(g, h) : (\overline{\gamma}, f\overline{x}', \overline{\eta}) \to (\gamma, fx', \eta)$: $g(f\overline{x}') = f(g\overline{x}') = f(x'h) = (fx')h$.

We have now constructed a functor $\mathbf{B}(H, G) \to \mathbf{C}(H, G)$ given by the double translation category. This construction participates in a functor of bicategories [5]: There is a natural equivalence

$$\phi : (GXH) \times_H (HYK) \to G(X \times_H Y)K$$

in $\mathbf{C}(H, K)$ where on the left "$\times_H$" denotes the pullback over the groupoid H while on the right it indicates the coend. This map is given on objects by sending

$$((\gamma, x \in X^{\eta}_{\gamma}, \eta), h : \eta' \to \eta, (\eta', y \in Y^{\kappa}_{\eta'}, \kappa))$$

to

$$(\gamma, xh \in X^{\eta'}_{\gamma}, y, \kappa) = (\gamma, x, hy \in Y^{\kappa}_{\eta}, \kappa)\,.$$

A morphism

$$((\overline{\gamma}, \overline{x}, \overline{\eta}), \overline{h}, (\overline{\eta}', \overline{y}, \overline{\kappa})) \to ((\gamma, x, \eta), h, (\eta', y, \kappa))$$

in the pullback $(GXH) \times_H (HYK)$ consists of morphisms $g : \overline{\gamma} \to \gamma$, $\hat{h} : \overline{\eta} \to \eta$, $h' : \overline{\eta}' \to \eta'$, $k : \overline{\kappa} \to \kappa$ such that $g\overline{x} = x\hat{h}$, $h'\overline{y} = yk$, and $hh' = \hat{h}\overline{h}$. This triple determines a morphism

$$(\overline{\gamma}, \overline{x}\overline{h}, \overline{y}, \overline{\kappa}) \to (\gamma, xh, y, \kappa)$$

in $G(X \times_H Y)K$ given by (g, k). The required equality $(g\overline{x}\overline{h}, \overline{y}) = (xh, yk)$ in $X \times_H Y$ is established by the morphism h', since $g\overline{x}\overline{h} = x\hat{h}\overline{h} = xhh'$ and $h'\overline{y} = yk$.

We leave the verification of the compatibility diagrams to the reader.

The next step is to construct a quasi-inverse of the functor $\mathbf{B}(H, G) \to \mathbf{C}(H, G)$. In the case of groups, with $G = 1$, you want to send a finite index subgroup K of H to the H-set H/K. When $G \neq 1$ we get the

(H,G)-set $H \times_K G$. The groupoid version sends $H \xleftarrow{q} K \xrightarrow{p} G$ to the (H,G)-set X with X_γ^η defined as the coend

$$X_\gamma^\eta = G(p(-),\gamma) \times_K H(\eta, q(-))$$

Proposition 4.2. *If q is a finite weak cover then X^η is free and finite for all $\eta \in H_0$.*

Proof. To see that X^η is free we must show that for all $x = (g : p\kappa \to \gamma, \kappa, h : \eta \to \eta') \in X_\gamma^\eta$, the map $G(\gamma,\gamma') \to X_{\gamma'}^\eta$ sending $\overline{g}$ to $\overline{g}x = (\overline{g}g, \kappa, h)$ is injective. Suppose that $\overline{g}'x = \overline{x} \in X_{\gamma'}^\eta$. By the relation in the definition of the coend, there exists $k : \kappa \to \kappa$ such that $q(k)h = h$ and $\overline{g}'gp(k) = \overline{g}g$. The first identity implies that $k = 1$ since by hypothesis η/q is discrete. Then the second relation implies that $\overline{g} = \overline{g}'$.

To see that X^η is finite note that $* \times_G G(\gamma_0, -) = \pi_0(GG(\gamma_0,-))$ is a singleton because the translation category is unicursal, so

$$* \times_{\gamma\in G} X_\gamma^\eta = (* \times_{\gamma\in G} (G(p(-),\gamma)) \times_K H(\eta, q(-)) = * \times_K H(\eta, q(-))$$

Now $KH(\eta, q(-))$ is the homotopy fiber of q over η, and its set of components $* \times_K H(\eta, q(-))$ is finite since q is a finite weak cover. □

This construction extends to a functor $\mathbf{C}(H,G) \to \mathbf{B}(H,G)$. Let $(t : K' \to K,\ \theta : p' \to pt,\ \varphi : q' \to qt)$ be a morphism $(q', K', p') \to (q, K, p)$. To define the induced map $G(p'(-),\gamma) \times_{K'} H(\eta, q'(-)) \to G(p(-),\gamma) \times_K H(\eta, q(-))$, pick $\kappa' \in K_0'$ and look at the natural map $\mathrm{in}_{\kappa'} : G(p'\kappa',\gamma) \times H(\eta, q'\kappa') \to G(p'(-),\gamma) \times_{K'} H(\eta, q'(-))$. The map we are looking for is induced by $\mathrm{in}_{t\kappa'} \circ (\theta_{\kappa'}^{-1} \times \varphi_{\kappa'}) : G(p'\kappa',\gamma) \times H(\eta, q'\kappa') \to G(p(-),\gamma) \times_K H(\eta, q(-))$. It is straightforward to check that these maps are compatible under morphisms in K'.

Proposition 4.3. *These two constructions form an adjoint equivalence between $\mathbf{B}(G,H)$ and $\mathbf{C}(G,H)$.*

Proof. Given an (H,G)-bi-set, there is a map

$$\beta : G(p(-),\gamma) \times_{GXH} H(\eta, q(-)) \to X_\gamma^\eta$$

given as follows. An object of GXH is given as $(\overline{\gamma}, x \in X_{\overline{\gamma}}^{\overline{\eta}}, \overline{\eta})$. The map is

$$(g : \overline{\gamma} \to \gamma, (\overline{\gamma}, x, \overline{\eta}), h : \overline{\eta} \to \eta) \mapsto gxh \in X_\gamma^\eta$$

This map is an isomorphism. It's surjective since $(1, (\gamma, x, \eta), 1) \mapsto x \in X_\gamma^\eta$. To check that it's injective, let $(g' : \overline{\gamma}' \to \gamma, (\overline{\gamma}', x', \overline{\eta}'), h' : \eta \to \overline{\eta}')$ map to the same element of X_γ^η: $g'x'h' = gxh$. We seek a morphism in GXH—$(\hat{g} : \overline{\gamma}' \to \overline{\gamma}, \hat{h} : \overline{\eta}' \to \overline{\eta})$ with $\hat{g}x' = x\hat{h} \in X_{\overline{\gamma}}^{\overline{\eta}'}$—such that

$g\hat{g} = g'$ and $\hat{h}h' = h$. There's a unique solution, $\hat{g} = g^{-1}g', \hat{h} = hh'^{-1}$. Then we calculate that $\hat{g}x' = g^{-1}g'x' = g^{-1}gxhh'^{-1} = x\hat{h}$.

And there is a map

$$\alpha : K \to G(G(p, -) \times_K H(-, q))H$$

given as follows. An element in the double translation category is determined by $(\gamma \in G_0, g : p\kappa \to \gamma, \kappa \in K_0, h : \eta \to q\kappa, \eta \in H_0)$. We send $\kappa \in K_0$ to $(p\kappa, 1, \kappa, 1, q\kappa)$. On morphisms, $k : \overline{\kappa} \to \kappa$ is sent to the pair (pk, qk); the coherence morphism is provided by k itself. This functor α is a morphism in the category of groupoids over G and over H; in fact the natural transformation allowed in a morphism is the identity map. It is not an isomorphism of groupoids, though, and this is why we need to allow non-invertible morphisms in $\mathbf{C}(H, G)$. $\square$

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