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PAIRINGS OF CATEGORIES AND SPECTRA

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For Saunders, with gratitude and admiration

One flourishing branch of category theory, namely coherence theory, lies at the heart of algebraic K-theory. Coherence theory was initiated in MacLane's paper [13]. There is an analogous coherence theory of higher homotopies, and the classifying space construction transports categorical coherence to homotopical coherence. When applied to interesting discrete categories, this process leads to the products and pairings (and deeper internal structure) of algebraic K-theory.

In much of the literature on algebraic K-theory, the underlying coherence theory is tacitly assumed (as indeed it is throughout mathematics). However, the details of coherence theory are crucial for rigor. For one thing, they explain which diagrams can, and which cannot, be made simultaneously to commute. For example, a symmetric monoidal category is one with a coherently unital, associative, and commutative product. It can be replaced by an equivalent permutative category, namely one with a strictly unital and associative but coherently commutative product. One cannot achieve strict commutativity except in trivial cases.

Thomason [26] has given an amusing illustration of the sort of mistake that can arise from a too cavalier attitude towards this kind of categorical distinction when studying pairings of categories, and one of my concerns is to correct a similar mistake of my own.

In [17], I developed a coherence theory of higher homotopies for ring spaces up to homotopy and for pairings of H-spaces. That theory is entirely correct. I also discussed the analogous categorical coherence, proving some results and asserting others. That theory too is entirely correct, my unproven assertions having been carefully proven by Laplaza [unpublished]. However, my translations from the categorical to the homotopical theories in [17], that of course being the part I thought to be obvious, are quite wrong.

The moral is that to treat the transition from categorical coherence to homotopical coherence smoothly and rigorously, one should take advantage of the definitional framework established by the category theorists. Given the work of MacLane, Kelly, Street, Laplaza, and others [9, 10, 24], this transition is really quite easy. One can handle the simplest coherence situations satisfactorily without it, as in Segal's original passage from permutative categories to Γ-spaces [22] or my original passage from permutative categories to $\mathscr{I}$-spaces [16], but these procedures are inadequate to handle the full generality of morphisms between permutative categories and inappropriate for the study of more complicated types of categorical input. In particular, neither cited approach works to handle pairings, at least not with the very simple topological notion of pairing that I shall introduce here. I should have learned this philosophy from MacLane. In fact, I learned it from Thomason.

This analysis of the categorical input is half of the remedy needed to retrieve and extend the results of [17]. The other half is a generalization of the homotopical coherence theory needed to make it accept the space level categorical output as input, and this generalization should be of independent interest. While I talked about the generalized E_∞ ring theory at Aspen, that theory will be presented in a sequel. Here I shall restrict myself to the simpler theory of pairings. Only the pairings on the relevant spectra, and not their deeper infrastructure, are of present use in algebraic K-theory, and this separation of material allows at least an attempt to make the exposition accessible to algebraists. The two theories have a quite different flavor, and there is a real need for a full treatment of pairings since nothing in the literature is adequate for spectrum level theoretical work. It is essential to study naturality, associativity, etc., up to natural isomorphism (or even natural transformation) on the category level and up to homotopy on the spectrum level. Such a theory has not been worked out before. I shall illustrate by proving the "projection formula" relating K_*R to K_*S when given a ring homomorphism $S\to R$ such that R is a finitely generated projective S-module. While that formula was known, the corresponding formula in mod q K-theory was not. With our topological proof, the latter is no more difficult than the former. I should admit that this formula actually could be obtained without much difficulty from alternative approaches, but the much deeper fact that the formula comes from a commutative diagram of spectra could not.

A quick review of additive infinite loop space theory will establish notations and set the stage for the present theory.

The idea of homotopical coherence theory on H-spaces is to specify enough higher homotopies for the product on an H-space Y to ensure that Y has a classifying space, or is an n-fold or infinite loop space. It would be horrendous to specify the required homotopies explicitly, so one incorporates them in some abstract framework. See Adams [2] for a nice intuitive discussion. There are two main ways of doing this, either by use of parameter spaces $\mathscr{C}(j)$ for j-fold products or by use of sequences $\{X_n\}$ which look formally and homotopically as if they were sequences $\{Y^n\}$ of powers of a based space Y.

In the former approach, the spaces $\mathscr{C}(j)$ are so related as to comprise an operad (as described in section one below), and an action of $\mathscr{C}$ on Y is just a suitably related collection of maps $\mathscr{C}(j)\times Y^j\to Y$. See [15, Section 1].

In the latter approach, due to Segal [22], one starts with the category Π with objects the finite based sets $\mathbf{n}=\{0,1,\dots,n\}$ and morphisms $\phi:\mathbf{m}\to\mathbf{n}$ such that

$\phi^{-1}(j)$ has at most one element for $1 \leq j \leq n$. Thus Π consists of injections, projections, and permutations. One defines a Π-space to be a functor $\Pi \rightarrow \mathcal{T}$, where $\mathcal{T}$ is the category of (well-behaved) based spaces, such that the n projections $\mathbf{n} \rightarrow \mathbf{1}$ induce an equivalence $\delta : X_n \rightarrow X_1^n$ (and a technical cofibration condition is satisfied). One lets $\mathcal{F}$ be the category of sets $\mathbf{n}$ and all based functions. An $\mathcal{F}$-space (or Γ-space) is a functor $\mathcal{F} \rightarrow \mathcal{T}$ whose restriction to Π is a Π-space.

In our axiomatization of infinite loop space theory [20], Thomason and I developed a common generalization of these two notions. A category of operators is a topological category $\hat{\mathscr{C}}$ with object set $\{\mathbf{n} \mid n \geq 0\}$ such that $\hat{\mathscr{C}}$ contains Π and is augmented over $\mathcal{F}$ by a functor which restricts to the inclusion on Π. A $\hat{\mathscr{C}}$-space is a continuous functor $\hat{\mathscr{C}} \rightarrow \mathcal{T}$ whose restriction to Π is a Π-space. When $\hat{\mathscr{C}} = \mathcal{F}$, this is Segal's notion. When $X = \{Y^n\}$ for a space Y, this is essentially an operad action. To make the last assertion precise, we associate a category of operators $\hat{\mathscr{C}}$ to an operad $\mathscr{C}$ in such a way that a $\mathscr{C}$-action on Y determines and is determined by a $\hat{\mathscr{C}}$-action on $\{Y^n\}$. For that operad $\mathcal{N}$ such that each $\mathcal{N}(j)$ is a single point, $\hat{\mathcal{N}}$ is precisely $\mathcal{F}$.

See [20, Sections 1, 4] for details of these definitions; the cited sections are short and are independent of the rest of that paper.

Let us say that an operad $\mathscr{C}$ is spacewise contractible if each $\mathscr{C}(j)$ is contractible. For such $\mathscr{C}$, there is an essentially unique functor from $\hat{\mathscr{C}}$-spaces to spectra. It now seems perfectly clear that the notion of $\hat{\mathscr{C}}$-space is definitively the right one for the study of coherence homotopies on (additive) H-spaces.

This sketch makes it very natural to seek a development of homotopical coherence theory for pairings of H-spaces in which the underlying additive coherence theory is based on the use of $\hat{\mathscr{C}}$-spaces. While the categorical applications are based on the use of $\mathcal{F}$-spaces, our passage from $\mathcal{F}$-spaces to spectra will exploit the extra generality, and the extra generality is bound to have other applications.

We introduce the notion of a pairing of $\hat{\mathscr{C}}$-spaces and state our main theorems on the passage from space level to spectrum level information in Section 1. We recall the notion of a pairing of permutative categories, state our main theorems on the passage from category level to space level information, and prove the projection formula in Section 2. We review the passage from permutative categories to $\mathcal{F}$-spaces in Section 3 and prove the theorems stated in Section 2 in Section 4.

Our theorems in Section 1 are stated in terms of maps $D \wedge E \rightarrow F$ of spectra in the stable category. If we were willing to settle for pairings of spectra in the crude old-fashioned sense of maps $D_i \wedge E_j \rightarrow F_{i+j}$ such that appropriate diagrams commute up to homotopy, then we could simply elaborate the proofs I gave in [17, IX Section 2] via the generalization of the additive theory given in [20, Section 6]. However, at this late date, no self-respecting homotopy theorist could be satisfied with such an imprecise treatment. The extra precision requires the introduction of a notion of pairings of $\mathcal{I}_*$-prespectra, a review of how smash products are constructed in the stable category, and a study of the passage from pairings of $\mathcal{I}_*$-prespectra to pairings of spectra, all of which is given in Section 5. Since the problem of constructing pairings in the stable category arises very often in stable homotopy theory and the general prescription we shall give is adequate for many applications far removed from our present concerns, this material should be of independent interest.

We prove the theorems of Section 1 in Section 6. We shall use the "May machine", but I have little doubt that, upon restriction to $\mathscr{F}$-spaces, the results could also be proven by use of the "Segal machine". Note that nothing in the earlier sections is bound to any particular choice of machinery. There is probably also a uniqueness theorem for pairings along the lines of the uniqueness theorem in [20], but at this writing there are unresolved technical obstructions to a proof. Of course, one can simply translate the pairings here along the equivalence of additive machines to introduce pairings in the Segal or any other machine.

Our notion of a pairing $(X,X)\rightarrow X$ on an $\mathscr{F}$-space X is simpler than Segal's notion in [22, Section 5] of a multiplication on X. The extra complication is unnecessary for the known applications starting from categorical input but is necessary for applications in étale homotopy theory. In an appendix, we explain how to generalize our theory of pairings to accept the more complicated input data. After writing the body of this paper, but before writing the appendix, I learned that Robinson [30] has recently used the Segal machine to construct pairings of spectra from (generalized) pairings of $\mathscr{F}$-spaces; he has not considered commutativity and associativity diagrams (or naturality on the up to homotopy morphisms our theory accepts).

Loday [12] gave the first systematic study of products in algebraic K-theory. It is immediate from the diagram following Corollary 6.5 below and a direct comparison of definitions that the appropriate specializations of our pairings agree with his pairings. The basic difference is that he obtains space level diagrams which only commute up to weak homotopy. There is one $\lim^1$ ambiguity obstructing their commutativity up to space level homotopy and another $\lim^1$ ambiguity obstructing their commutativity up to spectrum level homotopy. Our theory circumvents these ambiguities. The extra precision is irrelevant if all one cares about are the actual K-groups but is essential to the more sophisticated spectrum level analysis (which can lead to powerful calculational consequences, as in recent work of Thomason for example).

Waldhausen [29, II Section 9] used pairings of Q-constructions on exact categories to obtain pairings in algebraic K-theory, the point being that connectivity allows direct use of induced pairings of classifying spaces. This gets around the first, but not the second, $\lim^1$ ambiguity mentioned above. It is intuitively clear, although I have not checked the details, that his result [29, 9.26] can be used to show that his pairings in algebraic K-theory agree with ours.

By the axiomatization of the spectra of algebraic K-theory given by Fiedorowicz [16] (but see also Thomason [26]), the present theory of pairings directly implies that the machine-built spectra of algebraic K-theory are equivalent to those obtained ring theoretically by Gersten [7] and Wagoner [27].

I am much indebted to Steinberger for finding the mistakes in [17] and to Thomason for a number of very useful discussions of this material. The mod q

projection formula is proven here at Thomason's request, with a view towards applications in his work. Also, at the end of Section 3, I use an argument given to me by Thomason to generalize my uniqueness theorem for the passage from permutative categories to spectra [18] so as to allow its application to lax rather than strict morphisms. I am very grateful to Steiner for his paper [23], which vastly improves the geometric theory of [17] and thus of section 6 below. The appendix is included at Friedlander's request, with a view towards applications in étale homotopy theory.

1. Pairings of $\hat{\mathscr{C}}$-spaces

Wedges and smash products of finite based sets induce functors $\vee : \Pi\times\Pi\rightarrow\Pi$ and $\wedge : \Pi\times\Pi\rightarrow\Pi$, and similarly on the larger category $\mathscr{F}$. To be precise, we identify $\mathbf{m}\vee\mathbf{n}$ with $\mathbf{m}+\mathbf{n}$ in blocks and identify $\mathbf{m}\wedge\mathbf{n}$ with $\mathbf{mn}$ via lexicographic ordering of pairs.

A pairing of Π-spaces $f : (X, Y)\rightarrow Z$ is a natural transformation of functors $f : X\wedge Y\rightarrow Z\circ\wedge$. That is, we require maps $f_{mn} : X_m\wedge Y_n\rightarrow Z_{mn}$ such that the following diagrams commute for morphisms $\phi : \mathbf{m}\rightarrow\mathbf{p}$ and $\psi : \mathbf{n}\rightarrow\mathbf{q}$ in Π:

$$\begin{tikzcd}
X_m\wedge Y_n \arrow[r, "f_{mn}"] \arrow[d, "\phi\wedge\psi"'] & Z_{mn} \arrow[d, "\phi\wedge\psi"] \\
X_p\wedge Y_q \arrow[r, "f_{pq}"] & Z_{pq}
\end{tikzcd} \tag{$*$}$$

The simplest example occurs when X, Y, and Z arise from powers of based spaces U, V, and W. Here we are given a pairing of based spaces, that is a map $f : U\wedge V\rightarrow W$, and the maps $f_{mn} : U^m\wedge V^n\rightarrow W^{mn}$ can and must be defined to have (i,j)th coordinate the given map applied to the ith coordinate in U^m and the jth coordinate in V^n. Because $\phi^{-1}(r)$ and $\psi^{-1}(s)$ have at most one element for $1\leq r\leq p$ and $1\leq s\leq q$, the commutativity of $(*)$ is automatic.

We regard pairings of Π-spaces as underlying space-level scaffolding, and we want to elaborate to take account of products and richer internal structure on X, Y, and Z. For example, X, Y, and Z could be $\mathscr{F}$-spaces rather than just Π-spaces in the definition above. If they arose from spaces U, V, and W, then these spaces would be Abelian monoids and the diagrams $(*)$ for ϕ and ψ in $\mathscr{F}$ would be equivalent to bilinearity of the original map $f : U\wedge V\rightarrow W$. We have the following simple and natural generalization of this notion of a pairing of $\mathscr{F}$-spaces.

Definition 1.1. Let $\hat{\mathscr{C}}$, $\hat{\mathscr{D}}$, and $\hat{\mathscr{E}}$ be categories of operators. A pairing $\wedge : \hat{\mathscr{C}}\times\hat{\mathscr{D}}\rightarrow\hat{\mathscr{E}}$ is a functor such that the following diagram commutes:

$$\begin{tikzcd}
\Pi\times\Pi \arrow[r] \arrow[d, "\wedge"'] & \hat{\mathscr{C}}\times\hat{\mathscr{D}} \arrow[r] \arrow[d, "\wedge"] & \mathscr{F}\times\mathscr{F} \arrow[d, "\wedge"] \\
\Pi \arrow[r] & \hat{\mathscr{E}} \arrow[r] & \mathscr{F}
\end{tikzcd}$$

Let X, Y, and Z be a $\hat{\mathscr{C}}$-space, a $\hat{\mathscr{D}}$-space, and an $\hat{\mathscr{E}}$-space, respectively. A pairing $f : (X, Y) \rightarrow Z$ is a natural transformation of functors $X \wedge Y \rightarrow Z \circ \wedge$. That is, f consists of maps $f_{mn} : X_m \wedge Y_n \rightarrow Z_{mn}$ such that the following parametrized version of the diagram (*) commutes:

$$\begin{tikzcd}
\hat{\mathscr{C}}(\mathbf{m}, \mathbf{p}) \times X_m \times \hat{\mathscr{D}}(\mathbf{n}, \mathbf{q}) \times Y_n \arrow[r, "(\wedge \times f_{mn})(1 \times \tau \times 1)"] \arrow[d] & \hat{\mathscr{E}}(\mathbf{mn}, \mathbf{pq}) \times Z_{mn}. \arrow[d] \\
X_p \times Y_q \arrow[r, "f_{pq}"] & Z_{pq}
\end{tikzcd}$$

Here the vertical arrows are evaluation maps (and we have suppressed the evident quotient maps to smash products). A morphism $f \rightarrow f'$ of such pairings is a triple (α, β, γ) consisting of morphisms of $\hat{\mathscr{C}}$-, $\hat{\mathscr{D}}$-, and $\hat{\mathscr{E}}$-spaces such that the following diagram of functors commutes up to homotopy:

$$\begin{tikzcd}
X \wedge Y \arrow[r, "f"] \arrow[d, "\alpha \wedge \beta"'] & Z \circ \wedge \arrow[d, "\gamma \wedge"] \\
X' \wedge Y' \arrow[r, "f'"] & Z' \circ \wedge
\end{tikzcd}$$

While morphisms of $\hat{\mathscr{C}}$-spaces are just natural transformations, with no homotopies allowed, we emphasize that it is not sufficient to require the last diagram to commute strictly. The following general definition makes the phrase "up to homotopy" precise.

Definition 1.2. Let $\mathscr{G}$ be a topological category and let d and d' be natural transformations between continuous functors X and Y from $\mathscr{G}$ to $\mathscr{T}$. A homotopy $h : d \simeq d'$ consists of homotopies

$$h_n : X_n \wedge I^+ \rightarrow Y_n, \qquad h_n : d_n \simeq d'_n,$$

for objects $n \in \mathscr{G}$ such that the following diagrams commute for morphisms $\phi : m \rightarrow n$ in $\mathscr{G}$:

$$\begin{tikzcd}
X_m \wedge I^+ \arrow[r, "h_m"] \arrow[d, "\phi \wedge 1"'] & Y_m \arrow[d, "\phi"] \\
X_n \wedge I^+ \arrow[r, "h_n"] & Y_n
\end{tikzcd}$$

Here $X \wedge I^+ = X \times I / * \times I$, and use of this reduced cylinder amounts to restriction to homotopies through based maps. Thus a homotopy $h : d \simeq d'$ is a homotopy through natural transformations $X \rightarrow Y$.

We next write down unit, associativity, and commutativity specifications. While this could be done in the general context above, we restrict attention to ring type pairings for notational simplicity. However, we remark that (left or right) module objects over ring objects have obvious definitions for which all of our results throughout the paper remain valid (where "objects" are $\hat{\mathscr{C}}$-spaces, permutative categories, $\mathscr{I}_*$-prespectra, or spectra).

Note that Π and $\mathscr{F}$ are permutative categories under $\wedge$. That is, $\wedge$ is associative and unital with unit $\mathbf{1}$ and is commutative up to the natural isomorphism $\tau : \wedge\to\wedge\circ t$ specified on the object $(\mathbf{m},\mathbf{n})$ as the transposition permutation

$$\tau(m,n) : \mathbf{mn}=\mathbf{m}\wedge\mathbf{n}\to\mathbf{n}\wedge\mathbf{m}=\mathbf{nm},$$

the left and right equalities being lexicographical identifications. As here, we shall write t for transposition functors and τ for transposition isomorphisms throughout the paper.

In particular, for $\hat{\mathscr{C}}$-spaces X and X', the transposition homeomorphisms $X_i\wedge X'_j\to X'_j\wedge X_i$ specify a natural isomorphism $\tau : X\wedge X'\to(X'\wedge X)\circ t$ of functors $\hat{\mathscr{C}}\times\hat{\mathscr{C}}\to\mathscr{T}$.

Note too that there is an obvious definition of the smash product $Y\wedge X : \hat{\mathscr{C}}\to\mathscr{T}$ of a space Y and a functor $X : \hat{\mathscr{C}}\to\mathscr{T}$.

Definition 1.3. A permutative category of operators is a category of operators $\hat{\mathscr{C}}$ which is a permutative category under a pairing $\wedge: \hat{\mathscr{C}}\times\hat{\mathscr{C}}\to\hat{\mathscr{C}}$ whose unit is $\mathbf{1}$ and whose commutativity isomorphism $\tau : \wedge\to\wedge\circ t$ is given by the permutations $\tau(m,n)$ in $\Pi\subset\hat{\mathscr{C}}$. Thus $\Pi\to\hat{\mathscr{C}}$ and $\hat{\mathscr{C}}\to\mathscr{F}$ are morphisms of permutative categories.

A $\hat{\mathscr{C}}$-space X is said to be a ring $\hat{\mathscr{C}}$-space if it has a unit $e : S^0\to X_1$ (that is, a second basepoint $1\in X_1$) and a pairing $f : (X,X)\to X$ such that the following diagrams of functors commute up to homotopy:

$$\begin{tikzcd}
S^0\wedge X \arrow[r, "e\wedge 1"] \arrow[dr, equal] & X_1\wedge X, \arrow[d, "f"] & X\wedge X_1 \arrow[d, "f"'] & X\wedge S^0 \arrow[l, "1\wedge e"'] \arrow[dl, equal] \\
 & X & X &
\end{tikzcd}$$

and

$$\begin{tikzcd}
X\wedge X\wedge X \arrow[r, "f\wedge 1"] \arrow[d, "1\wedge f"'] & (X\circ\wedge)\wedge X=(X\wedge X)\circ(\wedge\times 1) \arrow[d, "f(\wedge\times 1)"] \\
X\wedge(X\circ\wedge)=(X\wedge X)\circ(1\times\wedge) \arrow[r, "f(1\times\wedge)"] & X\circ\wedge\circ(1\times\wedge)=X\circ\wedge\circ(\wedge\times 1)
\end{tikzcd}$$

X is said to be commutative if the following diagram of functors commutes up to homotopy:

$$\begin{tikzcd}
X\wedge X \arrow[r, "f"] \arrow[d, "\tau"'] & X\circ\wedge \arrow[d, "X\tau"] \\
(X\wedge X)t \arrow[r, "ft"] & X\circ\wedge\circ t
\end{tikzcd}$$

A map $\alpha : X \to X'$ of ring $\hat{\mathscr{C}}$-spaces is a map of $\hat{\mathscr{C}}$-spaces such that there is a path connecting 1 to $\alpha(1)$ in X'_1 and the triple (α,α,α) is a morphism of pairings of $\hat{\mathscr{C}}$-spaces.

Our passage from pairings of $\hat{\mathscr{C}}$-spaces to pairings of spectra will depend on the use of operads. Briefly, an operad $\mathscr{C}$ is a sequence of spaces $\mathscr{C}(j)$ such that $\mathscr{C}(0) = \{*\}$, there is a unit $1 \in \mathscr{C}(1)$, the symmetric group Σ_j acts from the right on $\mathscr{C}(j)$, and there is a suitably associative, unital, and equivariant family of maps

$$\gamma : \mathscr{C}(k) \times \mathscr{C}(j_1) \times \cdots \times \mathscr{C}(j_k) \to \mathscr{C}(j_1 + \cdots + j_k).$$

See [15, p. 1]. The associated category of operators $\hat{\mathscr{C}}$ has morphism spaces

$$\hat{\mathscr{C}}(\mathbf{m},\mathbf{n}) = \coprod_{\phi \in \mathscr{F}(\mathbf{m},\mathbf{n})} \prod_{1 \leq j \leq n} \mathscr{C}(|\phi^{-1}(j)|).$$

Its composition is specified on [20, p. 215]. All useful categories of operators seem to be of this form. The following is the operad level precursor of the pairing data we have assumed on categories of operators.

Definition 1.4. A pairing $\wedge : (\mathscr{C}, \mathscr{D}) \to \mathscr{E}$ of operads consists of maps

$$\wedge : \mathscr{C}(j) \times \mathscr{D}(k) \to \mathscr{E}(jk)$$

such that the following properties hold, where $c \in \mathscr{C}(j)$ and $d \in \mathscr{D}(k)$.

(i) If $\mu \in \Sigma_j$ and $\nu \in \Sigma_k$, then

$$c\mu \wedge d\nu = (c \wedge d)(\mu \wedge \nu),$$

where $\mu \wedge \nu$ is regarded as a permutation in Σ_{jk}.

(ii) If $c_q \in \mathscr{C}(h_q)$ for $1 \leq q \leq j$ and $d_r \in \mathscr{D}(i_r)$ for $1 \leq r \leq k$, then

$$\gamma\left(c \wedge d;\ \mathop{\times}_{(q,r)} (c_q \wedge d_r)\right)\omega = \gamma\left(c;\ \mathop{\times}_{q} c_q\right) \wedge \gamma\left(d;\ \mathop{\times}_{r} d_r\right),$$

where ω is the natural distributivity isomorphism

$$\bigvee_{(q,r)} \mathbf{h}_q \wedge \mathbf{i}_r \to \left(\bigvee_q \mathbf{h}_q\right) \wedge \left(\bigvee_r \mathbf{i}_r\right)$$

regarded as a permutation (via block and lexicographic identifications of the source and target). A permutative operad is one equipped with a unital (with unit 1) and associative pairing $\wedge : (\mathscr{C}, \mathscr{C}) \to \mathscr{C}$ which is commutative in the sense that

$$c \wedge d = (d \wedge c)\tau(j,k).$$

An elementary inspection of definitions gives the following result, which only asserts the correctness of the preceding definition.

Lemma 1.5. *A pairing $\wedge : (\mathscr{C}, \mathscr{D}) \to \mathscr{E}$ determines and is determined by a pairing*

$\wedge : (\hat{\mathscr{C}}, \hat{\mathscr{D}}) \to \hat{\mathscr{E}}$. An operad $\mathscr{C}$ is permutative if and only if the associated category of operators $\hat{\mathscr{C}}$ is permutative.

Examples appear in [15, p. 72], [17, p. 250], and Section 6 below. However, the example to concentrate on is $\mathscr{C} = \mathscr{N}$, with $\hat{\mathscr{C}} = \mathscr{F}$. If $\mathscr{C}$ is spacewise contractible, then a $\hat{\mathscr{C}}$-space X determines a spectrum EX whose zero-th space E_0X is a "group completion" of X_1 and there is a unique such functor E; see [20]. With this definitional apparatus, our main theorems read as follows. We shall recall what the "stable category" is in Section 5.

Theorem 1.6. *Let $\wedge : (\mathscr{C}, \mathscr{D}) \to \mathscr{E}$ be a pairing of spacewise contractible operads. Then pairings $f : (X, Y) \to Z$ of a $\hat{\mathscr{C}}$-space X and $\hat{\mathscr{D}}$-space Y to an $\hat{\mathscr{E}}$-space Z functorially determine pairings $Ef : EX \wedge EY \to EZ$ in the stable category.*

Theorem 1.7. *Let $\mathscr{C}$ be a permutative spacewise contractible operad and let X be a ring $\hat{\mathscr{C}}$-space. Then EX is functorially a ring spectrum. That is, the following diagrams commute, where S is the sphere spectrum and $e : S \to EX$ is induced by $e : S^0 \to X_1$:*

$$\begin{tikzcd}
S \wedge EX \arrow[r, "e \wedge 1"] \arrow[dr, equal] & EX \wedge EX \arrow[d, "Ef"] & EX \wedge S \arrow[l, "1 \wedge e"'] \arrow[dl, equal] \\
& EX &
\end{tikzcd}$$

and

$$\begin{tikzcd}
EX \wedge EX \wedge EX \arrow[r, "Ef \wedge 1"] \arrow[d, "1 \wedge Ef"'] & EX \wedge EX \arrow[d, "Ef"] \\
EX \wedge EX \arrow[r, "Ef"] & EX
\end{tikzcd}$$

If X is commutative, then EX is commutative; that is, the following diagram commutes:

$$\begin{tikzcd}
EX \wedge EX \arrow[dr, "Ef"] \arrow[dd, "\tau"'] & \\
& EX \\
EX \wedge EX \arrow[ur, "Ef"'] &
\end{tikzcd}$$

Of course, these diagrams must be interpreted in the stable category, and the proof will require an understanding of the coherence isomorphisms for the unity, associativity, and commutativity of the smash product in that category. We reiterate that these results are much stronger than mere assertions about pairings of spectra in the classical sense.

Our proof of Theorem 1.6 will have as a byproduct an analogous result on pairings of machine-built m-fold and n-fold loop spaces to machine-built $(m+n)$-fold loop spaces. Of course, the use of three different operads is essential to any such result.

The module theoretic version of Theorem 1.7 is perhaps more interesting than the version stated. In practice, machine-built commutative ring spectra have a great deal more internal structure. That will be the subject of the sequel, but the basic idea will become apparent in Section 5.

As a matter for amusement, our proofs of Theorems 1.6 and 1.7 will actually work without the spacewise contractibility hypothesis on the operads in question.

2. Pairings of permutative categories

Write $\mathcal{O}\mathcal{A}$ and $\mathcal{M}\mathcal{A}$ for the object and morphism spaces of a small topological category $\mathcal{A}$; we require the identity function $\mathcal{O}\mathcal{A}\rightarrow\mathcal{M}\mathcal{A}$ to be a cofibration. Let Cat denote the category of small topological categories and continuous functors (and suppress the adjectives henceforward). In the applications to algebraic K-theory, everything will be discrete.

A pairing of categories is simply a functor $\otimes : \mathcal{A}\times\mathcal{B}\rightarrow\mathcal{C}$. If $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are symmetric monoidal categories (under $\oplus$ and 0), we obtain the notion of a pairing by requiring the zero objects to act as strict zeros, $a\otimes 0=0$ and $0\otimes b=0$, and requiring a coherent natural bidistributivity isomorphism

$$(a\oplus a')\otimes(b\oplus b')\cong((a\otimes b)\oplus(a\otimes b'))\oplus((a'\otimes b)\oplus(a'\otimes b')).$$

The meaning of coherence here has been made precise by LaPlaza [unpublished, but see 9 and 10]. We insist on strict zeros since we want to arrive at smash products on the topological level; with a bit of extra verbiage, we could manage just as well with nullity of zero coherence isomorphisms.

To avoid excess parentheses and other more substantial annoyances, it is convenient to restrict attention to permutative categories. Then one way to make coherence precise is to require prescribed subsets $\mathcal{G}\mathcal{A}$ of $\mathcal{O}\mathcal{A}$ and $\mathcal{G}\mathcal{B}$ of $\mathcal{O}\mathcal{B}$ which generate $\mathcal{O}\mathcal{A}$ and $\mathcal{O}\mathcal{B}$ under $\oplus$. One requires strict zeros and strict equality

$$\left(\sum_{i=1}^{m} a_i\right)\otimes\left(\sum_{j=1}^{n} b_j\right)=\sum_{(i,j)} a_i\otimes b_j$$

for all sequences $(a_1,\ldots,a_m)$ and $(b_1,\ldots,b_n)$ of objects in $\mathcal{G}\mathcal{A}$ and $\mathcal{G}\mathcal{B}$ and of morphisms between objects of $\mathcal{G}\mathcal{A}$ and $\mathcal{G}\mathcal{B}$. Here the right-hand sum is taken in lexicographic order. One requires these equalities for different orderings of the a_i and b_j to be compatible with the commutativity isomorphisms for $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ [see 17, p. 247 (where various $\otimes$'s should be $\oplus$'s in (**))].

The use of permutative rather than symmetric monoidal categories results in no loss of generality. As shown in [17, IX.1.2], passage from symmetric monoidal to permutative categories preserves pairings, the proof there showing that the use of generating sets of objects in formulating coherence is not as artificial as it might appear at first sight. While we no longer have any pressing mathematical need to replace symmetric monoidal by permutative categories, since this step is not essential for the passage to $\mathscr{F}$-spaces given in the next section, strictly as a matter of taste I prefer to eliminate all coherence isomorphisms that can be eliminated.

Thus our objects of study will be permutative categories and their pairings. However, as observed by Thomason [25], the notion of morphism used in [17] and implicitly in [22] is unnecessarily strict. A morphism of symmetric monoidal categories should be a functor $F : \mathscr{A} \to \mathscr{A}'$ together with coherent natural transformations $\lambda : 0 \to F(0)$ and $\Phi : Fa \oplus Fb \to F(a+b)$, the coherence diagrams being those of Lewis [9; see also 17, p. 153]. There is no need for λ and Φ to be isomorphisms, hence no possibility for them to become the identity on passage to permutative categories. Thus a morphism of permutative categories should be the same thing as a morphism of underlying symmetric monoidal categories.

Similarly, we define a morphism of pairings of symmetric monoidal or permutative categories to be a triple (F, G, H) of morphisms of symmetric monoidal or permutative categories such that the following diagram commutes up to a coherent natural transformation $\Psi : F \otimes G \to H \circ \otimes$:

$$\begin{tikzcd}
\mathscr{A} \times \mathscr{B} \arrow[r, "\otimes"] \arrow[d, "F \times G"'] & \mathscr{C} \arrow[d, "H"] \\
\mathscr{A}' \times \mathscr{B}' \arrow[r, "\otimes"] & \mathscr{C}'
\end{tikzcd}$$

The precise meaning of coherence in this particular situation is probably not in the literature, but can be extracted by the methods of [9, 10]. Certainly passage from pairings of symmetric monoidal to pairings of permutative categories is functorial.

We define a ring permutative category to be a permutative category $\mathscr{A}$ with a unit object 1 (and resulting unit injection $e : * \to \mathscr{A}$) and a pairing $\otimes : \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ such that the following diagrams of categories commute up to coherent natural isomorphism:

$$\begin{tikzcd}
* \times \mathscr{A} \arrow[r, "e \times 1"] \arrow[dr, equal] & \mathscr{A} \times \mathscr{A}, \arrow[d, "\otimes"] & \mathscr{A} \times \mathscr{A} \arrow[d, "\otimes"'] & \mathscr{A} \times * \arrow[l, "1 \times e"'] \arrow[dl, equal] \\
& \mathscr{A} & \mathscr{A} &
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathscr{A} \times \mathscr{A} \times \mathscr{A} \arrow[r, "\otimes \times 1"] \arrow[d, "1 \times \otimes"'] & \mathscr{A} \times \mathscr{A} \arrow[d, "\otimes"] \\
\mathscr{A} \times \mathscr{A} \arrow[r, "\otimes"] & \mathscr{A}
\end{tikzcd}$$

We say that $\mathscr{A}$ is commutative if the following diagram commutes up to coherent natural isomorphism:

$$\begin{tikzcd}
\mathscr{A}\times\mathscr{A} \arrow[dr, "\otimes"] \arrow[dd, "t"'] & \\
& \mathscr{A} \\
\mathscr{A}\times\mathscr{A} \arrow[ur, "\otimes"'] &
\end{tikzcd}$$

Coherence here has been made precise by LaPlaza [9 and unpublished]; see also Kelly [10]. A morphism $F:\mathscr{A}\rightarrow\mathscr{A}'$ of ring permutative categories is a morphism of permutative categories such that there is a morphism $1\rightarrow F(1)$ in $\mathscr{A}'$ and the triple (F,F,F) is a morphism of pairings of permutative categories (compare Definition 1.3).

Passage from symmetric monoidal to permutative categories by the usual procedures [e.g., 16, Section 4] would not reduce any of the natural isomorphisms above to identities, as a glance at the proof of [17, IX.1.2] will make clear. It is for this reason that use of homotopies was essential to the definitions in the preceding section. If we concentrate on commutative ring theory and resolutely ignore the possibility of pairing different categories, the situation changes completely (compare [17, VI Section 3]), and much sharper results than those to follow emerge. These will be studied in the sequel.

Henceforward, we leave all unspecified details of coherence to the interested category theorist, but with the warning that this means that all substantive work in the proofs of the following theorems is also being left to the category theorist.

As will be recalled in Section 4, there is a functor which associates an $\mathscr{F}$-space $B\tilde{\mathscr{A}}$ to a permutative category $\mathscr{A}$. Its first space $B\tilde{\mathscr{A}}_1$ is equivalent to the classifying space $B\mathscr{A}$. We shall prove the following results in Section 4.

Theorem 2.1. *Pairings $\mathscr{A}\times\mathscr{B}\rightarrow\mathscr{C}$ of permutative categories functorially determine pairings $B\tilde{\mathscr{A}}\times B\tilde{\mathscr{B}}\rightarrow B\tilde{\mathscr{C}}$ of $\mathscr{F}$-spaces.*

Theorem 2.2. *If $\mathscr{A}$ is a ring permutative category, then $B\tilde{\mathscr{A}}$ is functorially a ring $\mathscr{F}$-space. If $\mathscr{A}$ is commutative, then $B\tilde{\mathscr{A}}$ is commutative.*

Write $E\mathscr{A}=EB\tilde{\mathscr{A}}$. This passage from permutative categories to spectra was axiomatized in [18] (see Remarks 4.1 below). The previous theorems feed directly into those of the first section to yield pairings of spectra from pairings of categories. We apply this to prove the "projection formula" for higher algebraic K-theory.

Corollary 2.3. *Let $f:S\rightarrow R$ be a homomorphism of commutative rings (with unit) such that R is a finitely generated projective S-module via f. Let $\mathscr{P}(R)$ and $\mathscr{P}(S)$ be the categories of finitely generated projective R and S-modules (made permutative). Let $f_*:\mathscr{P}(R)\rightarrow\mathscr{P}(S)$ be the forgetful functor which sends an R-module P to P*

regarded as an S-module by pullback along f. Let $f^* : \mathscr{P}(S) \to \mathscr{P}(R)$ *be the extension of scalars functor which sends an S-module Q to the R-module* $R \otimes_S Q$. *By passage to spectra and then to homotopy groups,* f_* *and* f^* *induce homomorphisms*

$$f_* : K_*R \to K_*S \quad \text{and} \quad f^* : K_*S \to K_*R$$

such that f^* *is a morphism of commutative graded rings and*

$$f_*(xf^*(y)) = f_*(x)y$$

in $K_{q+r}Y$ *for* $x \in K_qR$ *and* $y \in K_rS$.

Proof. Since $\mathscr{P}(R)$ is a commutative ring permutative category, $E\mathscr{P}(R)$ is a commutative ring spectrum. By definition, or by a standard argument if one prefers another definition, $K_*R = \pi_*E\mathscr{P}(R)$. The product on K_*R is obtained by composing the smash product between maps from sphere spectra to $E\mathscr{P}(R)$ with the product of $E\mathscr{P}(R)$, hence K_*R is a commutative graded ring. Since $f^* : \mathscr{P}(S) \to \mathscr{P}(R)$ is a morphism of ring permutative categories by virtue of the coherent natural isomorphism

$$(R \otimes_S Q) \otimes_R (R \otimes_S Q') \cong R \otimes_S (Q \otimes_S Q'),$$

$f^* : E\mathscr{P}(S) \to E\mathscr{P}(R)$ is a map of ring spectra and $f^* : K_*S \to K_*R$ is a ring homomorphism. The coherent natural isomorphism

$$P \otimes_R (R \otimes_S Q) \cong P \otimes_S Q$$

of S-modules gives the commutativity of the following diagram up to coherent natural isomorphism:

$$\begin{tikzcd}
\mathscr{P}(R) \times \mathscr{P}(R) \arrow[d, "\otimes"'] & \mathscr{P}(R) \times \mathscr{P}(S) \arrow[l, "1 \times f^*"'] \arrow[r, "f_* \times 1"] & \mathscr{P}(S) \times \mathscr{P}(S) \arrow[d, no head, shift right] \arrow[d, "\otimes", shift left] \\
\mathscr{P}(R) \arrow[rr, "f_*"] & & \mathscr{P}(S)
\end{tikzcd}$$

This may be viewed as a morphism of pairings of permutative categories, hence induces a similar commutative diagram on passage to spectra, and the projection formula follows.

The special case $q=0$ of the projection formula was proven by Quillen [21, Section 7, 2.10]. The general case is implicit in Loday [12] and is given a different proof in Gillet [8, 2.9].

The advantage of our proof is that one can easily apply standard topological constructions to it. For example, Browder [4] has shown the efficacy of introducing coefficients into algebraic K-theory. Let M be the Moore spectrum with 0th homology group Z_q for some prime power $q=p^n$ and with no other non-zero homology groups. Then

$$K_*(R; Z_q) = \pi_*(E\mathscr{P}(R) \wedge M).$$

If $p \neq 2$, M has a product. If also $p \neq 3$, then M is a ring spectrum and therefore so is $E \wedge M$ for any ring spectrum E. Now the projection formula in mod q K-theory is immediate: one need only replace $E\mathscr{P}(R)$ and $E\mathscr{P}(S)$ by their smash products with M in the proof just given.

3. Street's first construction

Our passage from pairings of categories to pairings of $\mathscr{F}$-spaces is based on use of Street's first construction in [24]. Since we need facts about this that are most simply verified just by looking at it, we review the relevant definitions. While the work here is due to Street, understanding of its relevance to infinite loop space theory is due to Thomason [25].

The category theorist will know that the following three definitions specify the 0-cells, 1-cells, and 2-cells of a 2-category [14, p. 44], but we eschew all avoidable categorical terminology (*pace* Saunders).

Definition 3.1. Let $\mathscr{G} \in \mathrm{Cat}$. A lax functor $A : \mathscr{G} \to \mathrm{Cat}$ is a pair of functions which assign a category $A(n)$ to each object n of $\mathscr{G}$ and a functor $A(\phi) : A(m) \to A(n)$ to each morphism $\phi : m \to n$ of $\mathscr{G}$ together with natural transformations

$$\varrho(n) : A(1) \to \mathrm{id} \quad \text{and} \quad \sigma(\psi,\phi) : A(\psi\phi) \to A(\psi)A(\phi)$$

for each identity morphism $1 : n \to n$ and each composable pair of morphisms (ψ, ϕ) such that the following diagrams of functors commute:

$$\begin{tikzcd}
A(\phi)A(1) \arrow[dr, "A(\phi)\varrho(m)"'] & A(\phi) \arrow[l, "\sigma(\phi,1)"'] \arrow[r, "\sigma(1,\phi)"] \arrow[d, equal] & A(1)A(\phi) \arrow[dl, "\varrho(n)A(\phi)"] \\
& A(\phi) &
\end{tikzcd}$$

and

$$\begin{tikzcd}
A(\omega\psi\phi) \arrow[r, "\sigma(\omega\psi,\phi)"] \arrow[d, "\sigma(\omega,\psi\phi)"'] & A(\omega\psi)A(\phi) \arrow[d, "\sigma(\omega,\psi)A(\phi)"] \\
A(\omega)A(\psi\phi) \arrow[r, "A(\psi)\sigma(\psi,\phi)"] & A(\omega)A(\psi)A(\phi)
\end{tikzcd}$$

In our applications, the $\varrho(n)$ will be identities and the $\sigma(\psi,\phi)$ will be isomorphisms. This is not Street's definition but its opposite, called an op-lax functor by Thomason [25].

Definition 3.2. Let $A, B : \mathscr{G} \to \mathrm{Cat}$ be lax functors. A (left) lax natural transformation $d : A \to B$ is a pair of functions which assign a functor $d(n) : A(n) \to B(n)$ to each object n of $\mathscr{G}$ and a natural transformation

$$d(\phi) : B(\phi)d(m) \to d(n)A(\phi)$$

to each morphism $\phi : m \to n$ of $\mathscr{G}$ such that the following diagrams of functors commute for $1 : n \to n$ and $\psi : n \to p$:

$$\begin{tikzcd}
B(1)d(n) \arrow[rr, "d(1)"] \arrow[dr, "\varrho(n)d(n)"'] & & d(n)A(1) \arrow[dl, "d(n)\varrho(n)"] \\
& d(n) &
\end{tikzcd}$$

and

$$\begin{tikzcd}
B(\psi\phi)d(m) \arrow[rr, "d(\psi\phi)"] \arrow[d, "\sigma(\psi,\phi)d(m)"'] & & d(p)A(\psi\phi) \arrow[d, "d(p)\sigma(\psi,\phi)"] \\
B(\psi)B(\phi)d(m) \arrow[r, "B(\psi)D(\phi)"] & B(\psi)d(n)A(\phi) \arrow[r, "d(\psi)A(\phi)"] & d(p)A(\psi)A(\phi)
\end{tikzcd}$$

The composite of $d : A \to B$ and $e : B \to C$ is specified by $(ed)(n) = e(n)d(n)$ on objects and by the composite

$$(ed)(\phi) : C(\phi)e(m)d(m) \xrightarrow{e(\phi)d(m)} e(n)B(\phi)d(m) \xrightarrow{e(n)d(\phi)} e(n)d(n)A(\phi)$$

on morphisms. There results a category of lax functors and lax natural transformations.

In our applications, the $d(\phi)$ will usually be isomorphisms. The name adopted in the following definition is non-standard.

Definition 3.3. Let $d, d' : A \to B$ be lax natural transformations of lax functors $\mathscr{G} \to \mathrm{Cat}$. A natural homotopy $\delta : d \to d'$ consists of natural transformations $\delta(n) : d(n) \to d'(n)$ such that the following diagram of functors commutes for $\phi : m \to n$:

$$\begin{tikzcd}
B(\phi)d(m) \arrow[r, "B(\phi)\delta(m)"] \arrow[d, "d(\phi)"'] & B(\phi)d'(m) \arrow[d, "d'(\phi)"] \\
d(n)A(\phi) \arrow[r, "\delta(n)A(\phi)"] & d'(n)A(\phi)
\end{tikzcd}$$

If $\delta' : d' \to d''$ is another natural homotopy of lax natural transformations $A \to B$, then $\delta'\delta : d \to d''$ is specified by $(\delta'\delta)(n) = \delta'(n)\delta(n)$. If $e, e' : B \to C$ are lax natural transformations and $\varepsilon : e \to e'$ is a natural homotopy thereof, then $\varepsilon\delta : ed \to e'd'$ is the natural homotopy with $(\varepsilon\delta)(n)$ the common composite, "$\varepsilon(n)\delta(n)$", in the diagram

$$\begin{tikzcd}
& e(n)d'(n) \arrow[dr, "\varepsilon(n)d'(n)"] & \\
e(n)d(n) \arrow[ur, "e(n)\delta(n)"] \arrow[dr, "\varepsilon(n)d(n)"'] & & e'(n)d'(n) \\
& e'(n)d(n) \arrow[ur, "e'(n)\delta(n)"'] &
\end{tikzcd}$$

Of course, all this is utterly familiar to the category theorist, who will immediately see the standard 2-category condition $(\varepsilon'\delta')(\varepsilon\delta)=(\varepsilon'\varepsilon)(\delta'\delta)$:

$$\begin{tikzcd}[column sep=huge]
A \arrow[r, bend left=70, "d", ""{name=D, below}] \arrow[r, "d'" description, ""{name=D1}] \arrow[r, bend right=70, "d''"', ""{name=D2, above}] & B \arrow[r, bend left=70, "e", ""{name=E, below}] \arrow[r, "e'" description, ""{name=E1}] \arrow[r, bend right=70, "e''"', ""{name=E2, above}] & C
\arrow[from=D, to=D1, "\delta"]
\arrow[from=D1, to=D2, "\delta'"]
\arrow[from=E, to=E1, "\varepsilon"]
\arrow[from=E1, to=E2, "\varepsilon'"]
\end{tikzcd}$$

Homotopy theorists may be appalled by this definitional apparatus, but it is unquestionably right for the purposes at hand.

In nature, lax functors $\mathscr{G}\rightarrow$Cat are ubiquitous but actual functors are rare. Street [24] introduced the following rectification of lax functors to genuine functors. Much more can be said about its categorical properties, but we restrict attention to what we shall use.

Theorem 3.4. *There is a functor, written $A\rightarrow\tilde{A}$ on objects and $d\rightarrow\tilde{d}$ on morphisms, from the category of lax functors $\mathscr{G}\rightarrow$Cat and lax natural transformations to the category of genuine functors $\mathscr{G}\rightarrow$Cat and genuine natural transformations. For each object n of $\mathscr{G}$, there is an adjoint pair of functors $\varepsilon:\tilde{A}(n)\rightarrow A(n)$ and $\eta:A(n)\rightarrow\tilde{A}(n)$, and the η are the functors of a lax natural transformation $A\rightarrow\tilde{A}$. If A is a genuine functor, the ε specify a genuine natural transformation $\tilde{A}\rightarrow A$. If $\delta:d\rightarrow d'$ is a natural homotopy of lax natural transformations between lax functors A and B, then there are induced natural transformations $\tilde{\delta}(n):\tilde{d}(n)\rightarrow\tilde{d}'(n)$ such that the following diagram of functors commutes for $\phi:m\rightarrow n$:*

$$\begin{tikzcd}[column sep=large]
\tilde{B}(\phi)\tilde{d}(m) \arrow[r, "\tilde{B}(\phi)\tilde{\delta}(m)"] \arrow[d, equal] & \tilde{B}(\phi)\tilde{d}'(m) \arrow[d, equal] \\
\tilde{d}(n)\tilde{A}(\phi) \arrow[r, "\tilde{\delta}(n)\tilde{A}(\phi)"] & \tilde{d}'(n)\tilde{A}(\phi)
\end{tikzcd}$$

That is, $\tilde{\delta}$ is a natural homotopy of genuine natural transformations. Passage from δ to $\tilde{\delta}$ preserves both compositions of natural homotopies.

For the benefit of homotopy theorists lost in the notation, we explain what this says homotopically before proceeding to the proof. Let $\mathscr{I}$ be the category with objects 0 and 1 and one non-identity morphism $\iota:0\rightarrow 1$. Recall that a natural transformation $\chi:F\rightarrow G$ between functors $\mathscr{A}\rightarrow\mathscr{B}$ determines and is determined by the functor $\chi:\mathscr{A}\times\mathscr{I}\rightarrow\mathscr{B}$ which restricts to F and G on $\mathscr{A}\times\{0\}$ and $\mathscr{A}\times\{1\}$ and is the common composite $\chi F(\alpha)=G(\alpha)\chi$ on morphisms (α,ι). Recall too that the classifying space functor B preserves products and carries $\mathscr{I}$ to I, hence carries categories, functors, and natural transformations to spaces, maps, and homotopies.

In particular, it carries adjoint pairs of functors to inverse homotopy equivalences.

Now restrict attention to based categories and consider the theorem. $B\tilde{A} : \mathscr{G} \to \mathscr{T}$ is a functor with $B\tilde{A}(n)$ equivalent to $BA(n)$, $B\tilde{d} : B\tilde{A} \to B\tilde{B}$ is a natural transformation, and, the heart of the matter for our purposes, considering $\tilde{\delta}(n)$ as a functor $\tilde{A}(n) \times \mathscr{I} \to \tilde{B}(n)$, $B\tilde{\delta}$ is precisely a homotopy between natural transformations in the sense of Definition 1.2. Thus the theorem serves to convert the lax notions to which categorical coherence theory naturally gives rise to exactly the sort of space level data one needs to apply our homotopical coherence theory.

We give the constructions, since we need the details, but we omit all verifications in the following outline proof of the theorem. Write $\chi(a)$ for the value of a natural transformation χ on an object a.

Proof. For an object $n \in \mathscr{G}$, $\tilde{A}(n)$ is the category whose objects are pairs $(\phi; a)$, where $\phi : m \to n$ is a morphism of $\mathscr{G}$ and a is an object of $A(m)$, and whose morphisms $(\phi; a) \to (\phi'; a')$, $\phi' : m' \to n$, are pairs $(\psi; \alpha)$, where $\psi : m \to m'$ satisfies $\phi'\psi = \phi$ and where α is a morphism $A(\psi)(a) \to a'$ in $A(m')$. The composite $(\psi'; \alpha')(\psi; \alpha)$ is $(\psi'\psi; \beta)$, where β is the composite

$$A(\psi'\psi)(a) \xrightarrow{\sigma(\psi',\psi)(a)} A(\psi')A(\psi)(a) \xrightarrow{A(\psi')(\alpha)} A(\psi')(a') \xrightarrow{\alpha'} a''.$$

The identity morphism of $(\phi; a)$ is $(1; \varrho(m)(a))$. For a morphism $\omega : n \to p$ in $\mathscr{G}$, $\tilde{A}(\omega) : \tilde{A}(n) \to \tilde{A}(p)$ is the functor specified on objects and morphisms by

$$\tilde{A}(\omega)(\phi; a) = (\omega\phi; a) \quad \text{and} \quad \tilde{A}(\omega)(\psi; \alpha) = (\psi; \alpha).$$

This completes the construction of the functor $\tilde{A} : \mathscr{G} \to \mathrm{Cat}$.

The functor $\varepsilon : \tilde{A}(n) \to A(n)$ is specified by

$$\varepsilon(\phi; a) = A(\phi)(a) \quad \text{and} \quad \varepsilon(\psi; \alpha) = A(\phi')(\alpha) \circ \sigma(\phi', \psi)(a).$$

The functor $\eta : A(n) \to \tilde{A}(n)$ is specified by

$$\eta(b) = (1; b) \quad \text{and} \quad \eta(\beta) = (1; \beta \circ \varrho(n)(b))$$

for $b \in A(n)$ and $\beta : b \to b'$ in $A(n)$. The counit $\varepsilon\eta \to \mathrm{id}$ and unit $\mathrm{id} \to \eta\varepsilon$ of the adjunction are specified by the morphisms

$$\varrho(n)(b) : A(1)(b) \to b \quad \text{and} \quad (\phi; 1) : (\phi; a) \to (1; A(\phi)(a)),$$

and the latter morphisms also specify the natural transformations

$$\eta(\phi) : \tilde{A}(\phi)\eta(m) \to \eta(n)A(\phi)$$

required for η to give a lax natural transformation $A \to \tilde{A}$.

For a lax natural transformation $d : A \to B$, the genuine natural transformation $\tilde{d} : \tilde{A} \to \tilde{B}$ is given by the functors $\tilde{d}(n) : \tilde{A}(n) \to \tilde{B}(n)$ specified on objects and morphisms by

$$\tilde{d}(n)(\phi; a) = (\phi; d(m)(a)) \quad \text{and} \quad \tilde{d}(n)(\psi; \alpha) = (\psi; d(m')(\alpha) \circ d(\psi)(a)).$$

For a natural homotopy $\delta : d \to d'$, the natural transformation $\tilde{\delta}(n) : \tilde{d}(n) \to \tilde{d}'(n)$ is specified by the morphisms

$$(1; \delta(m)(a) \circ \varrho(m) d(m)(a)) : (\phi; d(m)(a)) \to (\phi; d'(m)(a)).$$

We have ignored the topology so far in this section. We assume for simplicity that $\mathscr{G}$ is discrete, since this holds in our applications. Via disjoint unions and products, the sets $\mathscr{O}\tilde{A}(n)$ and $\mathscr{M}\tilde{A}(n)$ inherit topologies from the spaces $\mathscr{O}A(m)$ and $\mathscr{M}A(m)$. Here points of $\mathscr{M}\tilde{A}(n)$ must be regarded as triples (source, morphism, target) in order to obtain continuous source and target maps. When, as holds in our applications, the $\varrho(n)$ are given by identity morphisms, the identity functions $\mathscr{O}\tilde{A}(n) \to \mathscr{M}\tilde{A}(n)$ are cofibrations because the identity functions $\mathscr{O}A(m) \to \mathscr{M}A(m)$ are cofibrations.

We need a few general observations about the constructions above. For typographical simplicity, we write $SA = \tilde{A}$ in the remainder of this section.

Lemma 3.5. *Lax functors $A : \mathscr{G} \to \mathrm{Cat}$ and $B : \mathscr{H} \to \mathrm{Cat}$ induce a product lax functor $A \times B : \mathscr{G} \times \mathscr{H} \to \mathrm{Cat}$, and $S(A \times B) = SA \times SB$. Similar assertions hold for lax natural transformations and natural homotopies thereof.*

Lemma 3.6. *If $F : \mathscr{H} \to \mathscr{G}$ is a functor and $A : \mathscr{G} \to \mathrm{Cat}$ is a lax functor, then $AF : \mathscr{H} \to \mathrm{Cat}$ is a lax functor and application of F to the first coordinate of objects and morphisms specifies a natural transformation $\zeta : S(AF) \to S(A)F$ which is natural with respect to lax natural transformations $d : A \to B$ and natural homotopies thereof.*

Clearly the following diagram commutes when defined:

$$\begin{tikzcd}
S(AFF') \arrow[dr, "\zeta"] \arrow[dd, "\zeta"'] & \\
& S(AF)F' \arrow[dl, "\zeta F'"] \\
S(A)FF' &
\end{tikzcd}$$

We shall be interested in lax functors $\mathscr{F} \to \mathrm{Cat}$, where we shall have $\zeta : S(A \circ \wedge) \to S(A) \circ \wedge$ for $\wedge : \mathscr{F} \times \mathscr{F} \to \mathscr{F}$. If $\iota_1 : \mathscr{F} \to \mathscr{F} \times \mathscr{F}$ is specified by $\iota_1(\mathbf{n}) = (\mathbf{n}, \mathbf{1})$ and $\iota_1(\phi) = (\phi, 1)$ on objects and morphisms, then $\wedge \circ \iota_1$ is the identity. The diagram just given shows that the composite

$$S(A \circ \wedge \circ \iota_1) \xrightarrow{\zeta} S(A \circ \wedge) \circ \iota_1 \xrightarrow{\zeta \iota_1} S(A)$$

is the identity, and similarly for $\iota_2 = t\iota_1$. The diagram also implies the commutativity of the following associativity diagram, the composites having common value ζ:

$$\begin{tikzcd}
S(A \circ \wedge \circ (\wedge \times 1)) \arrow[r, "\zeta"] \arrow[d, equal] & S(A \circ \wedge) \circ (\wedge \times 1) \arrow[r, "\zeta(\wedge \times 1)"] & S(A) \circ \wedge \circ (\wedge \times 1) \arrow[d, equal] \\
S(A \circ \wedge \circ (1 \times \wedge)) \arrow[r, "\zeta"] & S(A \circ \wedge) \circ (1 \times \wedge) \arrow[r, "\zeta(1 \times \wedge)"] & S(A) \circ \wedge \circ (1 \times \wedge)
\end{tikzcd}$$

For the study of commutativity, we shall need a much less obvious analog. Recall the natural commutativity isomorphism $\tau : A\circ\wedge \to A\circ\wedge\circ t$ from section one and define a lax natural transformation $\tau : A\circ\wedge\to A\circ\wedge\circ t$ by letting the (m, n)th functor $\tau(m, n) : A(mn)\to A(nm)$ be $A(\tau(m, n))$ and letting

$$\tau(\phi, \psi) : A(\psi\wedge\phi)\tau(m, n)\to\tau(p, q)A(\phi\wedge\psi)$$

be the natural transformation $\sigma(\tau(p, q), \phi\wedge\psi)\circ\sigma(\psi\wedge\phi, \tau(m, n))^{-1}$. We assume the $\sigma(\psi, \phi)$ are isomorphisms, as that will be obvious in our examples; we see that the definition makes sense by applying A to the diagram

$$\begin{tikzcd}
\mathbf{m}\wedge\mathbf{n} \arrow[r, "\phi\wedge\psi"] \arrow[d, "\tau(m,n)"'] & \mathbf{p}\wedge\mathbf{q} \arrow[d, "\tau(p,q)"] \\
\mathbf{n}\wedge\mathbf{m} \arrow[r, "\psi\wedge\phi"] & \mathbf{q}\wedge\mathbf{p}
\end{tikzcd}$$

Lemma 3.7. *The following diagram of functors commutes up to a natural homotopy which is itself natural in A:*

$$\begin{tikzcd}
S(A\circ\wedge) \arrow[r, "\zeta"] \arrow[d, "S(\tau)"'] & S(A)\circ\wedge \arrow[d, "S(A)\tau"] \\
S(A\circ\wedge\circ\tau) \arrow[r, "\zeta"] & S(A)\circ\wedge\circ\tau
\end{tikzcd}$$

Proof. A typical object of the category $S(A\circ\wedge)(\mathbf{m}, \mathbf{n})$ has the form $((\mu, \nu); a)$, where $\mu : \mathbf{i}\to\mathbf{m}$, $\nu : \mathbf{j}\to\mathbf{n}$, and $a\in A(ij)$. We have

$$(S(A)\tau)\zeta((\mu, \nu); a) = (S(A)\tau)(\mu\wedge\nu; a) = (\tau(m, n)(\mu\wedge\nu); a)$$

and

$$\zeta\circ S(\tau)((\mu, \nu); a) = \zeta((\mu, \nu); \tau(i, j)(a)) = (\nu\wedge\mu; \tau(i, j)(a)).$$

The required natural transformations

$$(\zeta\circ S(\tau))(\mathbf{m}, \mathbf{n})\to(S(A)\tau\circ\zeta)(\mathbf{m}, \mathbf{n})$$

are specified by the morphisms

$$(\tau(i, j); 1) : (\nu\wedge\mu; \tau(i, j)(a))\to(\tau(m, n)(\mu\wedge\nu); a)$$

in $S(A)(\mathbf{m}, \mathbf{n})$. The remaining verifications are tedious exercises.

4. The passage from permutative categories to $\mathscr{F}$-spaces

We first apply Theorem 3.4 to construct a functor from permutative categories to $\mathscr{F}$-spaces and then use this functor to prove Theorems 2.1 and 2.2. Symmetric monoidal categories would work just as well but would serve only to complicate the notation.

We associate a lax functor $A : \mathscr{F} \to \mathrm{Cat}$ to a permutative category $\mathscr{A}$ as follows. Set $A(n) = \mathscr{A}^n$. For a morphism $\phi : \mathbf{m} \to \mathbf{n}$ in $\mathscr{F}$, specify the function $A(\phi) : A(m) \to A(n)$ by

$$A(\phi)\left(\mathop{\times}_{i=1}^{m} a_i\right) = \mathop{\times}_{j=1}^{n} \left(\sum_{\phi(i)=j} a_i\right)$$

on objects and morphisms, where the empty sum is interpreted as the object 0 or its identity morphism. Note that $A(1)$ is the identity functor and let $\varrho(n)$ be the identity transformation. For $\psi : n \to p$,

$$A(\psi\phi)\left(\mathop{\times}_{i=1}^{m} a_i\right) = \mathop{\times}_{k=1}^{p} \sum_{(\psi\phi)(i)=k} a_i$$

while

$$A(\psi)A(\phi)\left(\mathop{\times}_{i=1}^{m} a_i\right) = \mathop{\times}_{k=1}^{p} \sum_{\psi(j)=k} \sum_{\phi(i)=j} a_i.$$

The sums are taken in different order, and the commutativity isomorphism in $\mathscr{A}$ determines a natural isomorphism $\sigma(\psi,\phi) : A(\psi\phi) \to A(\psi)A(\phi)$; its kth coordinate rearranges sums ordered by increasing i to sums ordered by increasing j and for fixed j increasing i. If ϕ or ψ is the identity, no rearrangement is necessary and $\sigma(\psi,\phi)$ is the identity. The second diagram in Definition 3.1 commutes by coherence.

For a morphism $F : \mathscr{A} \to \mathscr{B}$ of permutative categories, the functors $F^n : \mathscr{A}^n \to \mathscr{B}^n$ and natural transformations $B(\phi)F^m \to F^n A(\phi)$ with jth coordinate

$$\sum_{\phi(i)=j} F(a_i) \to F\left(\sum_{\phi(i)=j} a_i\right)$$

determined by $\Phi : F \oplus F \to F \circ \oplus$ (or by $\lambda : 0 \to F(0)$ if $j \notin \mathrm{Im}\ \phi$) specify a lax natural transformation $A \to B$ of lax functors. Again, the second diagram of Definition 3.2 commutes by coherence.

We have associated functors $\tilde{A} : \mathscr{F} \to \mathrm{Cat}$ and natural transformations $\tilde{F} : \tilde{A} \to \tilde{B}$. We shall also write $\tilde{A} = \tilde{\mathscr{A}}$ to emphasize that it is a collection of categories $\tilde{A}(n) = \tilde{\mathscr{A}}_n$ to which the classifying space functor B can be applied. We give $\tilde{\mathscr{A}}_n$ the base object $0 = (0 : \mathbf{0} \to \mathbf{n}; 0)$; this uses the convention that $\mathscr{A}^0$ is the trivial category with object 0. Then the lax functor A and induced functor $\tilde{A}$ both take values in based categories (because $\omega \circ 0 = 0$ for any morphism ω in $\mathscr{F}$). It is now clear from the general discussion in the previous section that $B\tilde{\mathscr{A}}$ is an $\mathscr{F}$-space and $B\tilde{F}$ is a morphism of $\mathscr{F}$-spaces.

Remarks 4.1. A different functor $\tilde{\mathscr{A}} : \mathscr{F} \to \mathrm{Cat}$ was introduced by Segal [22, Section 2] and made precise in [18]. That construction is smaller and perhaps more elegant;

$\tilde{\mathscr{A}}_1$ is precisely $\mathscr{A}$, and $\tilde{\mathscr{A}}_n$ is equivalent to $\mathscr{A}^n$ rather than just related to it by an adjunction. It is a basic insight of Thomason that $\tilde{\mathscr{A}}$ is usually the more powerful tool. In particular, Theorems 2.1 and 2.2 are direct consequences of coherence theory using $\tilde{\mathscr{A}}$, but cannot be proven using $\bar{\mathscr{A}}$. On the other hand, the uniqueness theorem in [18] for the passage from permutative categories to spectra depends on the fact that each $\bar{\mathscr{A}}_n$ is a permutative category. I do not see how to prove such a result for $\tilde{\mathscr{A}}$, hence a generalization of the uniqueness assertion to non-strict morphisms is not quite immediate (in contradiction to a claim in [25]). However, we shall give a proof of such a generalization, due to Thomason, at the end of the section. The appendix will give a comparison of $\bar{\mathscr{A}}$ to $\tilde{\mathscr{A}}$ and will show how to develop a theory of pairings based on use of $\bar{\mathscr{A}}$.

With these preliminaries, we show how the category theorist's work on coherence proves Theorems 2.1 and 2.2 for us.

Let $\otimes : \mathscr{A}\times\mathscr{B}\to\mathscr{C}$ be a pairing of permutative categories. We define a lax natural transformation $\otimes : A\times B\to C\circ\wedge$ of lax functors $\mathscr{F}\times\mathscr{F}\to\mathrm{Cat}$ as follows. The (m,n)th functor

$$\otimes(m,n) : \mathscr{A}^m\times\mathscr{B}^n\to\mathscr{C}^{mn}$$

has (i,j)th coordinate the given pairing applied to the ith coordinate of $\mathscr{A}^m$ and the jth coordinate of $\mathscr{B}^n$. For morphisms $\phi : \mathbf{m}\to\mathbf{p}$ and $\psi : \mathbf{n}\to\mathbf{q}$ in $\mathscr{F}$, the functors

$$C(\phi\wedge\psi)\circ\otimes(m,n) \quad \text{and} \quad \otimes(p,q)\circ(A(\phi)\times B(\psi))$$

from $\mathscr{A}^m\times\mathscr{B}^n$ to $\mathscr{C}^{pq}$ have respective (r,s)th coordinates given on objects and morphisms $(\times_{i=1}^m a_i, \times_{j=1}^n b_j)$ by

$$\sum_{\phi(i)=r,\,\psi(j)=s} a_i\otimes b_j \quad \text{and} \quad \Big(\sum_{\phi(i)=r} a_i\Big)\otimes\Big(\sum_{\psi(j)=s} b_j\Big).$$

The given natural distributivity isomorphism provides a natural isomorphism

$$\otimes(\phi,\psi) : C(\phi\wedge\psi)\circ\otimes(m,n)\to\otimes(p,q)\circ(A(\phi)\times B(\psi)).$$

If $r\notin\operatorname{Im}\phi$ or $s\notin\operatorname{Im}\psi$, we use the nullity of zero here, and we could use nullity of zero isomorphisms if the zeros of $\mathscr{A}$ and $\mathscr{B}$ were not strict. If ϕ and ψ are identities, then so is $\otimes(\phi,\psi)$. The second diagram of Definition 3.2 commutes by coherence.

Converting $\otimes$ to a natural transformation by Street's first construction and applying Lemmas 3.5 and 3.6 to the source and target, we obtain a natural transformation $\tilde{\otimes} : \tilde{A}\times\tilde{B}\to\tilde{C}\circ\wedge$ of functors $\mathscr{F}\times\mathscr{F}\to\mathrm{Cat}$. By inspection of definitions, the zeros of $\tilde{A}(m)$ and $\tilde{B}(n)$ act as strict zeroes for $\otimes$ (even if this only holds up to isomorphism for $\mathscr{A}$ and $\mathscr{B}$). Upon usage of the commutation of the classifying space functor with products, we see that the induced maps

$$B\otimes : B\tilde{\mathscr{A}}_m\times B\tilde{\mathscr{B}}_n\to B\tilde{\mathscr{C}}_{mn}$$

factor through smash products and specify a pairing of $\mathscr{F}$-spaces.

For functoriality, suppose given a morphism of pairings

$$\begin{tikzcd}
\mathscr{A}\times\mathscr{B} \arrow[r, "\otimes"] \arrow[d, "F\times G"'] & \mathscr{C} \arrow[d, "H"] \\
\mathscr{A}'\times\mathscr{B}' \arrow[r, "\otimes"] & \mathscr{C}'
\end{tikzcd}$$

Then $F\otimes G$ and $H\circ\otimes$ induce lax natural transformations from $A\times B$ to $C'\circ\Lambda$ with induced natural transformations $\tilde{F}\otimes\tilde{G}\rightarrow(\tilde{H}\Lambda)\circ\tilde{\otimes}$. The natural transformation $\Psi : F\otimes G\rightarrow H\circ\otimes$ given as part of the definition of a morphism of pairings induces a natural homotopy of lax natural transformations, by coherence again, and thus a natural homotopy $\tilde{\Psi} : \tilde{F}\tilde{\otimes}\tilde{G}\rightarrow(\tilde{H}\Lambda)\circ\tilde{\otimes}$. Verification that composition behaves properly up to homotopy is an exercise in the use of the first notion of composition specified in Definition 3.3.

Turning to Theorem 2.2, let $\mathscr{A}$ be a ring permutative category under the coherently unital and associative pairing $\otimes : \mathscr{A}\times\mathscr{A}\rightarrow\mathscr{A}$. With $e : *\rightarrow\mathscr{A}$ the unit injection, the diagrams

$$\begin{tikzcd}
*\times A \arrow[r, "e\times 1"] \arrow[dr, equal] & (\mathscr{A}\times A)\iota_2 \arrow[d, "\otimes"] & \text{and} & (A\times\mathscr{A})\iota_1 \arrow[d, "\otimes"'] & A\times * \arrow[l, "1\times e"'] \arrow[dl, equal] \\
& A\circ\Lambda\circ\iota_2 & & A\circ\Lambda\circ\iota_1 &
\end{tikzcd}$$

of lax functors $\mathscr{F}\rightarrow$Cat commute up to natural homotopies determined coordinatewise by the unit isomorphism of $\mathscr{A}$. Similarly, the diagram of lax functors $\mathscr{F}^3\rightarrow$Cat

$$\begin{tikzcd}
A\times A\times A \arrow[r, "\otimes\times 1"] \arrow[d, "1\times\otimes"'] & (A\circ\Lambda)\times A=(A\times A)\circ(\Lambda\times 1) \arrow[d, "\otimes\circ(\Lambda\times 1)"] \\
A\times(A\circ\Lambda)=(A\times A)\circ(1\times\Lambda) \arrow[r, "\otimes\circ(1\times\Lambda)"] & A\circ\Lambda\circ(1\times\Lambda)=A\circ\Lambda\circ(\Lambda\times 1)
\end{tikzcd}$$

commutes up to a natural homotopy determined coordinatewise by the associativity isomorphism of $\mathscr{A}$. The diagram of Definition 3.3 amounts to a typical coherence diagram relating distributivity to associativity; specifically, it says that the two visible ways of going from

$$\left[\left(\sum_{\phi(i)=r} a_i\right)\otimes\left(\sum_{\psi(j)=s} b_j\right)\right]\otimes\left(\sum_{\chi(k)=t} c_k\right)$$

to

$$\sum_{\phi(i)=r,\,\psi(j)=s,\,\chi(k)=t} a_i\otimes(b_j\otimes c_k),$$

by first distributing and then associating or first associating and then distributing, coincide. Again, if $\mathscr{A}$ is commutative, the diagram of lax functors $\mathscr{F}^2\rightarrow$Cat

$$\begin{tikzcd}
A \times A \arrow[r, mapsto, "\otimes"] \arrow[d, "\tau"'] & A \circ \Lambda \arrow[d, "\tau"] \\
(A \times A) \circ t \arrow[r, "\otimes t"] & A \circ \Lambda \circ t
\end{tikzcd}$$

commutes up to a natural homotopy determined coordinatewise by the commutativity isomorphism of $\mathscr{A}$. Here τ on the right is specified above Lemma 3.7 and τ on the left is given by the transposition isomorphism in Cat. Let $1=(1 : \mathbf{1}\to\mathbf{1};\ 1)\in\tilde{\mathscr{A}}_1$. Upon application of Street's first construction, quotation of the unit, associativity, and commutativity properties of ζ given in and above Lemma 4.7, and passage to classifying spaces, we conclude that the diagrams above imply the desired homotopy commutativity of the analogous diagrams of Definition 1.3. This proves Theorem 2.2, its functoriality assertion following from that of Theorem 2.1 and a trivial check of units.

I probably should point out where the mistake occurs in my erroneous earlier treatment.

Remark 4.2. In [17, IX Section 1], I introduced a particular pairing of operads $(\mathscr{L},\mathscr{L})\to\mathscr{L}$ and defined the notion of a pairing of $\mathscr{L}$-spaces. That notion was exactly our present notion of a pairing of $\hat{\mathscr{L}}$-spaces between Π-spaces arising as powers of actual spaces. I then asserted [17, IX, 1.4] that a certain diagram of categories determined by a pairing of permutative categories was commutative. That diagram would have given a pairing of $\mathscr{L}$-spaces on passage to classifying spaces, but in fact it only commutes strictly on generating objects and morphisms. It does commute up to natural isomorphism, but that is not good enough for the machinery of infinite loop space theory. The point of using Street's theory is that it so beautifully converts diagrams which commute up to natural isomorphism to diagrams which commute strictly.

Finally, we give the promised generalization of the uniqueness theorem in [18]. An infinite loop space machine E defined on permutative categories is a functor from permutative categories to connective spectra together with a natural group completion $\iota : B\mathscr{A}\to E_0\mathscr{A}$. This makes sense with either strict or lax morphisms of permutative categories. We know by [18] that, up to equivalence, there is a unique such machine when strict morphisms are understood. The following result, which specializes constructions due to Thomason [26, 27], immediately implies the corresponding uniqueness assertion when lax morphisms are understood. The point is that, up to equivalences (in the sense of strict morphisms which induce equivalences upon passage to classifying spaces), we can convert lax morphisms to strict morphisms, and such equivalences $\mathscr{A}\to\mathscr{A}'$ induce equivalences $E\mathscr{A}\to E\mathscr{A}'$ by [18, Lemma 5].

Proposition 4.3. *There is a functor, written* $\mathscr{A}\to\hat{\mathscr{A}}$ *on objects and* $F\to\hat{F}$ *on* *morphisms, from the category of permutative categories and lax morphisms to the category of permutative categories and strict morphisms. There is a natural adjoint pair of functors*

$$\eta : \mathscr{A} \to \hat{\mathscr{A}} \quad \text{and} \quad \varepsilon : \hat{\mathscr{A}} \to \mathscr{A}$$

such that ε *is a strict morphism of permutative categories (and* η *is a non-unital lax morphism of permutative categories).*

Proof. Let $\hat{\mathscr{A}}$ have objects $[k; a_1, \dots, a_k]$, where $k \geq 0$ and the a_r are objects of $\mathscr{A}$. In particular, there is an object $0 = [0;\ ]$. Let $\hat{\mathscr{A}}$ have morphisms

$$[\psi; \alpha_1, \dots, \alpha_j] : [k; a_1, \dots, a_k] \to [j; b_1, \dots, b_j],$$

where $\psi : \{1, \dots, k\} \to \{1, \dots, j\}$ is a surjective function and α_q is a morphism $\sum_{\psi(r)=q} a_r \to b_q$ in $\mathscr{A}$. The composite of such a morphism with another morphism

$$[\phi; \beta_1, \dots, \beta_i] : [j; b_1, \dots, b_j] \to [i; c_1, \dots, c_i]$$

is $[\phi\psi; \gamma_1, \dots, \gamma_i]$, where γ_p is the following composite:

$$\sum_{(\phi\psi)(r)=p} a_r \xrightarrow{\sigma(\phi,\psi)} \sum_{\phi(q)=p} \sum_{\psi(r)=q} a_r \xrightarrow{\sum_{\phi(q)=p} \alpha_q} \sum_{\phi(q)=p} b_q \xrightarrow{\beta_p} c_p.$$

Here $\sigma(\phi, \psi)$ is the evident permutation isomorphism. There are no non-identity morphisms with source or target 0. (That is, 0 is a disjoint basepoint for $\hat{\mathscr{A}}$.) The sum on $\hat{\mathscr{A}}$ is specified by

$$[j; a_1, \dots, a_j] \oplus [k; b_1, \dots, b_k] = [j+k; a_1, \dots, a_j, b_1, \dots, b_k]$$

on objects and similarly on morphisms. Certainly $\oplus$ is strictly associative with strict unit 0. The evident block shuffle permutations of $j+k$ letters (and identity morphisms in $\mathscr{A}$) give the required natural commutativity isomorphism $\oplus \to \oplus \circ t$. For a lax morphism $F : \mathscr{A} \to \mathscr{A}'$, define a strict morphism $\hat{F} : \hat{\mathscr{A}} \to \hat{\mathscr{A}}'$ by

$$\hat{F}[k; a_1, \dots, a_k] = [k; Fa_1, \dots, Fa_k]$$

on objects and by

$$\hat{F}[\psi; \alpha_1, \dots, \alpha_j] = [\psi; \alpha'_1, \dots, \alpha'_j]$$

on morphisms, where α'_q is the composite

$$\sum_{\psi(r)=q} Fa_r \xrightarrow{\Phi} F\Bigl(\sum_{\psi(r)=q} a_r\Bigr) \xrightarrow{F\alpha_q} Fb_q.$$

Here Φ is the natural transformation required of a lax morphism, composition is preserved by coherence, and $\hat{F}$ strictly preserves $\oplus$ by a glance at the definitions. The functor $\eta : \mathscr{A} \to \hat{\mathscr{A}}$ is specified on objects and morphisms by

$$\eta(a) = [1; a] \quad \text{and} \quad \eta(\alpha) = [1; \alpha].$$

A natural transformation $\oplus \circ (\eta \times \eta) \rightarrow \eta \circ \oplus$ is specified by the morphisms

$$[\phi_2; 1] : [2; a+b] \rightarrow [1; a+b],$$

where ϕ_k is the unique surjection $\{1, \dots, k\} \rightarrow \{1\}$; however, there is no morphism in $\hat{\mathscr{A}}$ between 0 and $\eta(0) = [1; 0]$. The functor $\varepsilon : \hat{\mathscr{A}} \rightarrow \mathscr{A}$ is specified on objects and morphisms by

$$\varepsilon[k; a_1, \dots, a_k] = \sum_{r=1}^{k} a_r \quad \text{and} \quad \varepsilon[\psi; \alpha_1, \dots, \alpha_j] = \beta,$$

where β is the composite

$$\sum_{r=1}^{k} a_r \rightarrow \sum_{q=1}^{j} \sum_{\psi(r)=q} a_r \xrightarrow{\sum_{q=1}^{j} \alpha_q} \sum_{q=1}^{j} b_q;$$

the first morphism here is the evident permutation isomorphism. We set $\varepsilon(0) = 0$. It is obvious that ε strictly preserves sums; it preserves commutativity isomorphisms by virtue of the role played by permutations in its definition on morphisms. The composite $\varepsilon\eta$ is the identity functor, and the unit $\mathrm{Id} \rightarrow \eta\varepsilon$ of the required adjunction is the natural transformation specified by the morphisms

$$[\phi_k; 1] : [k; a_1, \dots, a_k] \rightarrow \left[1; \sum_{r=1}^{k} a_r\right].$$

5. Pairings of $\mathscr{I}_*$-prespectra and of spectra

The proofs of Theorems 1.6 and 1.7 will proceed by passage from pairings of $\hat{\mathscr{C}}$-spaces to pairings of $\mathscr{I}_*$-prespectra to pairings of spectra. We give the second step first, and we precede it by a sequence of definitions closely analogous to those of section one.

Let $\mathscr{I}_*$ denote the category of finite-dimensional real inner product spaces and their linear isometric isomorphisms. Observe that $\mathscr{I}_*$ admits the coherently associative, unital (with unit $\{0\}$), and commutative operation $\oplus$. The natural commutativity isomorphism $\tau : \oplus \rightarrow \oplus \circ t$ is given by the transposition isometries $V \oplus W \rightarrow W \oplus V$. Let $S : \mathscr{I}_* \rightarrow \mathscr{T}$ denote the sphere-valued functor obtained by one-point compactification; in particular, $S\{0\} = S^0$. We abbreviate $S\tau = \tau$ and write ω for the evident natural isomorphism $S \wedge S \rightarrow S \circ \oplus$. Define

$$\Sigma^V X = X \wedge SV \quad \text{and} \quad \Omega^V X = F(SV, X),$$

$F(Y, X)$ being the function space of based maps $Y \rightarrow X$.

Definition 5.1. An $\mathscr{I}_*$-prespectrum is a continuous functor $T : \mathscr{I}_* \rightarrow \mathscr{T}$ together with a natural transformation $\sigma : T \wedge S \rightarrow T \circ \oplus$ such that the following conditions hold.

(i) $\sigma : TV = TV \wedge S\{0\} \rightarrow T(V \oplus \{0\}) = TV$ is the identity map.

(ii) The following associativity diagram commutes:

$$\begin{tikzcd}
TV\wedge SW\wedge SZ \arrow[r, "\sigma\wedge 1"] \arrow[d, "1\wedge\omega"'] & T(V\oplus W)\wedge SZ \arrow[d, "\sigma"] \\
TV\wedge S(W\oplus Z) \arrow[r, "\sigma"] & T(V\oplus W\oplus Z)
\end{tikzcd}$$

(iii) The adjoint $\tilde{\sigma} : TV \to \Omega^W T(V\oplus W)$ of σ is an inclusion.

A morphism $f : T\to T'$ of $\mathscr{J}_*$-prespectra is a natural transformation such that the following diagram of functors commutes:

$$\begin{tikzcd}
T\wedge S \arrow[r, "\sigma"] \arrow[d, "f\wedge 1"'] & T\circ\oplus \arrow[d, "f\circ\oplus"] \\
T'\wedge S \arrow[r, "\sigma'"] & T'\circ\oplus
\end{tikzcd}$$

Let $\mathscr{J}_*[\mathscr{T}]$ denote the category of $\mathscr{J}_*$-prespectra.

Remark 5.2. For a based space Y, there is an evident $\mathscr{J}_*$-prespectrum $F(Y)$ with Vth space $Y\wedge SV$. In particular, $S=F(S^0)$ is an $\mathscr{J}_*$-prespectrum. The functor $F : \mathscr{T}\to\mathscr{J}_*[\mathscr{T}]$ is left adjoint to the zero$^{\text{th}}$ space functor. That is, a map $Y\to T\{0\}$ extends uniquely to a morphism $F(Y)\to T$ of $\mathscr{J}_*$-prespectra.

Observe that the transposition $X\wedge Y\to Y\wedge X$ induces a natural isomorphism $\tau : T\wedge T'\to(T'\wedge T)\circ t$ of functors $\mathscr{J}_*\times\mathscr{J}_*\to\mathscr{T}$ for $\mathscr{J}_*$-prespectra T and T'. The first diagram to follow may be viewed as one of functors $\mathscr{J}_*^4\to\mathscr{T}$. A more conceptual formulation will be given in the appendix.

Definition 5.3. A pairing $\omega : (P,Q)\to T$ of $\mathscr{J}_*$-prespectra is a natural transformation $\omega : P\wedge Q\to T\circ\oplus$ of functors $\mathscr{J}_*\times\mathscr{J}_*\to\mathscr{T}$ such that the following diagrams commute:

$$\begin{tikzcd}
PV\wedge SW\wedge QV'\wedge SW' \arrow[r, "1\wedge\tau\wedge 1"] \arrow[d, "\sigma\wedge\sigma"'] & PV\wedge QV'\wedge SW\wedge SW' \arrow[r, "\omega\wedge\omega"] & T(V\oplus V')\wedge S(W\oplus W') \arrow[d, "\sigma"] \\
P(V\oplus W)\wedge Q(V'\oplus W') \arrow[r, "\omega"] & T(V\oplus W\oplus V'\oplus W') \arrow[r, "T(1\oplus\tau\oplus 1)"] & T(V\oplus V'\oplus W\oplus W')
\end{tikzcd}$$

A morphism $\omega\to\omega'$ of such pairings is a triple (b,c,f) of morphisms of $\mathscr{J}_*$-prespectra such that the following diagram of functors commutes up to homotopy:

$$\begin{tikzcd}
P\wedge Q \arrow[r, "\omega"] \arrow[d, "b\wedge c"'] & T\circ\oplus \arrow[d, "f\oplus"] \\
P'\wedge Q' \arrow[r, "\omega'"] & T'\circ\oplus
\end{tikzcd}$$

Definition 1.2 makes the last notion precise.

Definition 5.4. An $\mathscr{I}_*$-prespectrum T is said to be a ring $\mathscr{I}_*$-prespectrum if it has a unit map $e : S \to T$ of $\mathscr{I}_*$-prespectra and a pairing $\omega : (T, T) \to T$ such that the following diagrams commute up to homotopy:

$$\begin{tikzcd}
T\wedge S \arrow[r, "1\wedge e"] \arrow[dr, "\sigma"'] & T\wedge T \arrow[r, "\tau"] & (T\wedge T)\circ t \arrow[d, "\omega t"] \\
 & T\circ\oplus \arrow[r, "T\tau"] & T\circ\oplus\circ t
\end{tikzcd}
\qquad
\begin{tikzcd}
T\wedge T \arrow[d, "\omega"'] & T\wedge S \arrow[l, "1\wedge e"'] \arrow[dl, "\sigma"] \\
T\circ\oplus &
\end{tikzcd}$$

and

$$\begin{tikzcd}
T\wedge T\wedge T \arrow[r, "\omega\wedge 1"] \arrow[d, "1\wedge\omega"'] & (T\circ\oplus)\wedge T = (T\wedge T)\circ(\oplus\times 1) \arrow[d, "\omega(\oplus\times 1)"] \\
T\wedge(T\circ\oplus) = (T\wedge T)\circ(1\times\oplus) \arrow[r, "\omega(1\times\oplus)"] & T\circ\oplus\circ(1\times\oplus) \cong T\circ\oplus\circ(\oplus\times 1)
\end{tikzcd}$$

T is said to be commutative if the following diagram commutes up to homotopy:

$$\begin{tikzcd}
T\wedge T \arrow[r, "\omega"] \arrow[d, "\tau"'] & T\circ\oplus \arrow[d, "T\tau"] \\
(T\wedge T)\circ t \arrow[r, "\omega t"] & T\circ\oplus\circ t
\end{tikzcd}$$

A morphism $f : T \to T'$ of ring $\mathscr{I}_*$-prespectra is a map of $\mathscr{I}_*$-prespectra such that $e' \simeq fe : S \to T'$ and the triple (f, f, f) is a morphism of pairings.

Remark 5.5. S is a commutative ring $\mathscr{I}_*$-prespectrum with the identity functor as unit e and with $\omega = \sigma : S\wedge S \to S\circ\oplus$. For any ring $\mathscr{I}_*$-prespectrum T, $e : S \to T$ is a morphism of ring $\mathscr{I}_*$-prespectra by virtue of the diagram

$$\begin{tikzcd}
S\wedge S \arrow[r, dash, "e\wedge 1"] \arrow[d, "\omega=\sigma"'] & T\wedge S \arrow[r, dash, "1\wedge e"] \arrow[d, "\sigma"] & T\wedge T \arrow[dl, "\omega"] \\
S\circ\oplus \arrow[r, "e\circ\oplus"] & T\circ\oplus &
\end{tikzcd}$$

Remarks 5.6. In [17, p. 73], Quinn, Ray, and I introduced the notion of an $\mathscr{I}_*$-prefunctor. This is precisely a strictly unital, associative, and commutative ring $\mathscr{I}_*$-prespectrum. That is, all diagrams in Definition 5.4 commute, without homotopies. (We only prescribed T, e, and ω there since we could then set $\sigma = \omega(1\wedge e)$ and deduce the diagrams in which it appears.) We showed that the spectra associated to $\mathscr{I}_*$-prefunctors are E_∞ ring spectra and observed that Thom spectra are naturally occurring examples. We shall see in the sequel that the derived notion of an $\mathscr{I}$-spectrum is general enough for all of multiplicative infinite loop space theory. In particular, the spectra associated to bipermutative categories will be seen to be $\mathscr{I}$-

spectra, provided that the May machine is used for the construction. (It will be seen that the Segal machine inevitably leads to considerably more complicated output when fed the same multiplicative input.)

The reader will surely not find it hard to believe that suitable machinery converts the input data of section one to the output data prescribed above. On the other hand, this output data feeds naturally into stable homotopy theory.

Theorem 5.7. *There is a functor E from $\mathscr{I}_*$-prespectra to spectra under which pairings $(P, Q)\to T$ of $\mathscr{I}_*$-prespectra functorially determine pairings $EP\wedge EQ\to ET$ in the stable category.*

Theorem 5.8. *If T is a ring $\mathscr{I}_*$-prespectrum, then ET is functorially a ring spectrum. If T is commutative, then ET is commutative.*

To begin the proofs, we require a notion of spectrum compatible with the notion of an $\mathscr{I}_*$-prespectrum. We follow the approach to spectra and the stable category outlined in [17, II]. Details will appear in [19] and also in [5], where an equivariant generalization is given.

Let U be any countably infinite dimensional real inner product space. We are thinking of $U=(R^\infty)^j$ for any $j\geq 1$. A prespectrum D indexed on U consists of based spaces DV for all finite-dimensional $V\subset U$ (or all V in a large enough family of subspaces) and based maps $\sigma : DV\wedge SW\to D(V+W)$ whenever V is orthogonal to W in U. These spaces and maps are required to satisfy conditions (i)–(iii) of Definition 5.1 with external direct sums replaced by internal direct sums. Clearly an $\mathscr{I}_*$-prespectrum determines a prespectrum indexed on U for any U.

A prespectrum D indexed on U is said to be a spectrum if each $\tilde{\sigma} : DV\to \Omega^W D(V+W)$ is a homeomorphism. Thus, when $U=R^\infty$, DR^i is homeomorphic to ΩDR^{i+1}. A prespectrum D functorially determines a spectrum LD by an obvious passage to limits. That is,

$$(LD)(V)=\bigcup_{V\perp W}\Omega^W D(V+W),$$

where the union is taken over loops of inclusions $\tilde{\sigma}$ and the required homeomorphisms are evident.

Morphisms of prespectra (or of spectra) are collections of maps $DV\to D'V$ strictly compatible with the structural maps σ. Two morphisms are homotopic if their component maps $DV\to D'V$ are homotopic through homotopies which at each time t comprise a map of prespectra. Clearly passage from $\mathscr{I}_*$-prespectra to prespectra indexed on U is functorial and homotopy-preserving, where homotopies between morphisms of $\mathscr{I}_*$-prespectra are homotopies of natural transformations.

Let $\mathscr{P}U$ and $\mathscr{S}U$ denote the categories of prespectra and spectra indexed on U, and use a prefix h to denote homotopy categories. There are sphere spectra S^q in $h\mathscr{S}U$, hence there are homotopy groups, and there is a concomitant notion of weak homotopy equivalence. There is a category $H\mathscr{S}U$ obtained from $h\mathscr{S}U$ by formally inverting its weak equivalences. Passage from $h\mathscr{S}U$ to $H\mathscr{S}U$ is equivalent to the familiar process of replacing spectra by CW-approximations (with the right notion of CW-spectrum in $\mathscr{S}U$). We abbreviate $\mathscr{P}R^\infty = \mathscr{P}$ and $\mathscr{S}R^\infty = \mathscr{S}$. It is $H\mathscr{S}$ that we understand to be the stable category, and we have an evident composite functor

$$E : \mathscr{I}_*[\mathscr{T}] \to h\mathscr{I}_*[\mathscr{T}] \to h\mathscr{P} \to h\mathscr{S} \to H\mathscr{S}.$$

This stable category, like any other worthy of the name, is equivalent to that introduced by Boardman [3] and explained in elementary terms by Adams [1]. The present construction has various advantages, the trivial passage from $\mathscr{I}_*$-prespectra to the stable category just given being an illustrative example.

The reason for bothering with different "universes" U is that there is an obvious "external" smash product functor $\mathscr{P}U \times \mathscr{P}U' \to \mathscr{P}(U \oplus U')$ specified by

$$(D \wedge D')(V \oplus V') = DV \wedge D'V'.$$

(Here we exploit the fact that prespectra need not be defined on all finite dimensional subspaces W of $U \oplus U'$.) We extend this to a functor on spectra by

$$E \wedge E' = L(\nu E \wedge \nu E'),$$

where ν is the evident forgetful functor from spectra to prespectra. Technically, the inclusion condition in our definition of a prespectrum need not be satisfied by $\nu E \wedge \nu E'$, so the functor L must be extended to prespectra defined without this condition. The extension is due to Lewis [11; see also 19]. It follows formally that L commutes with $\wedge$,

$$L(D \wedge D') \cong L(\nu LD \wedge \nu LD') = LD \wedge LD'.$$

To exploit these smash products, we need change of universe functors. For a linear isometry $g : U \to U'$, there is an evident functor $g^* : \mathscr{P}U' \to \mathscr{P}U$ specified by $(g^*D')(V) = D'(gV)$ for $V \subset U$. Clearly g^* restricts to $g^* : \mathscr{S}U' \to \mathscr{S}U$, and $g^*L = Lg^*$. When g is an isomorphism, these functors g^* are isomorphisms of categories with inverses $g_* = (g^{-1})^*$. In general, g^* admits a left adjoint $g_* : \mathscr{P}U \to \mathscr{P}U'$. We define $g_* = Lg_*\nu : \mathscr{S}U' \to \mathscr{S}U'$ and have that g_* is left adjoint to g^* and satisfies $g_*L = Lg_*$. The construction of g_* is given in [5, VIII and 19] and it is shown there that g_* and g^* induce inverse equivalences of categories between $H\mathscr{S}U$ and $H\mathscr{S}U'$ and that, up to coherent natural equivalence, these stable category level functors are the same for different linear isometries g and g'. We shall write $\simeq$ for equivalences that only hold in stable categories and $\cong$ for spectrum level isomorphisms. By the explicit definitions, there are coherent natural isomorphisms

$$g_*E \wedge h_*F \cong (g \oplus h)_*(E \wedge F) \quad \text{and} \quad (g'g)_*(E) \cong g'_*g_*E.$$

Write $\mathscr{S}(R^\infty)^j = \mathscr{S}_j$ so that $\mathscr{S}_1 = \mathscr{S}$. The "internal" smash product on the stable category $H\mathscr{S}$ is the composite functor

$$H\mathscr{S} \times H\mathscr{S} = H\mathscr{S}_1 \times H\mathscr{S}_1 \xrightarrow{\wedge} H\mathscr{S}_2 \xrightarrow{g_*} H\mathscr{S}_1 = H\mathscr{S}$$

determined by any linear isometry $g : R^\infty \oplus R^\infty \to R^\infty$. Technically, to pass from the spectrum level to the stable category level, we must first replace given spectra by CW-approximations; this is the standard procedure for handling functors, such as $\wedge$, which need not preserve weak equivalence. The internal smash product is unital, associative, and commutative up to coherent natural isomorphism, and to prove Theorem 5.8 we need to know exactly what these isomorphisms are.

Define the stabilization functor $\Sigma^\infty : \mathscr{T} \to \mathscr{S}$ by $\Sigma^\infty Y = LF_1(Y)$, where $F_1(Y) = \{\Sigma^V Y\}$ denotes the suspension prespectrum of Y. Let $i : R^\infty \to R^\infty \oplus R^\infty$ be the inclusion onto the first summand. We have a smash product

$$E \wedge Y = L(vE \wedge Y),$$

where the prespectrum level version is specified by

$$(T\wedge Y)(V) = TV \wedge Y = (T \wedge F_1(Y))(iV) = i^*(T \wedge F_1(Y))(V).$$

That is, $T \wedge Y = i^*(T \wedge F_1(Y))$. By adjunction and application of L, we obtain

$$i_*(T\wedge Y) = T \wedge F_1(Y) \text{ in } \mathscr{P}_2 \quad \text{and} \quad i_*(E\wedge Y) = E \wedge \Sigma^\infty Y \text{ in } \mathscr{S}_2.$$

Define a natural isomorphism β in $H\mathscr{S}$ by the diagram

$$\begin{tikzcd}
E\wedge Y \arrow[r, phantom, "\simeq"] \arrow[d, "\beta"'] & (gi)_*(E\wedge Y) \arrow[d, phantom, sloped, "\cong"] \\
g_*(E\wedge \Sigma^\infty Y) \arrow[r, phantom, "="] & g_*i_*(E\wedge Y)
\end{tikzcd}$$

Here the top equivalence is that between the functors 1_* and $(gi)_*$, where $1 : R^\infty \to R^\infty$ is the identity linear isometry. When $Y = S^0$, $S = \Sigma^\infty S^0$ is the sphere spectrum and $\beta : E \to E \wedge S$ is the required unit isomorphism in $H\mathscr{S}$.

For associativity, we use the following composite α, where we exploit the evident associativity of the external smash product in obtaining the vertical isomorphisms.

$$\begin{tikzcd}
g(g\oplus 1)_*(D\wedge E\wedge F) \arrow[r, phantom, "\simeq"] \arrow[d, phantom, sloped, "\cong"] & g(1\oplus g)_*(D\wedge E\wedge F) \arrow[d, phantom, sloped, "\cong"] \\
g_*(g_*(D\wedge E)\wedge F)) \arrow[r, "\alpha"] & g_*(D\wedge g_*(E\wedge F))
\end{tikzcd}$$

For commutativity, we use the following composite τ, where $\tau : R^\infty \oplus R^\infty \to R^\infty \oplus R^\infty$ is the transposition isometry and $F\wedge E \cong \tau_*(E\wedge F)$ expresses the evident commutativity of the external smash product.

$$\begin{tikzcd}
g_*(E\wedge F) \arrow[r, phantom, "\simeq"] \arrow[d, "\tau"'] & (g\tau)_*(E\wedge F) \arrow[d, phantom, sloped, "\cong"] \\
g_*(F\wedge E) \arrow[r, phantom, "\cong"] & g_*\tau_*(E\wedge F)
\end{tikzcd}$$

We can now prove Theorems 5.7 and 5.8. We already have our functor $E : \mathscr{J}_*[\mathscr{T}] \to H\mathscr{S}$. For an $\mathscr{J}_*$-prespectrum T, write T_j for the induced prespectrum indexed on $(R^\infty)^j$. Thus $ET = LT_1$. It is immediate from the definitions that a pairing $\omega : (P, Q) \to T$ induces a map

$$\omega : P_i \wedge Q_j \to T_{i+j} \quad \text{in } \mathscr{P}_{i+j}$$

for each i and j. Application of L then gives

$$L\omega : LP_i \wedge LQ_j \cong L(P_i \wedge Q_j) \to LT_{i+j} \quad \text{in } \mathscr{S}_{i+j}.$$

Let $f : (R^\infty)^i \to (R^\infty)^j$ be any linear isometry. We have a map $T_i \to f^*T_j$ in $\mathscr{P}_i$ specified by the maps

$$T(f|V) : T_iV = TV \to Tf(V) = (f^*T_j)(V)$$

given for $V \subset (R^\infty)^i$ by the fact that T is a functor $\mathscr{I}_* \to \mathscr{T}$. Passing to adjoints and then to spectra, we obtain

$$\phi : f_*T_i \to T_j \text{ in } \mathscr{P}_j \quad \text{and} \quad L\phi : f_*LT_i = Lf_*T_i \to LT_j \text{ in } \mathscr{S}_j.$$

The functoriality of T and naturality of ω on the $\mathscr{I}_*$-prespectrum level translate directly to give functoriality and naturality properties of the prespectrum level maps ϕ and ω. If f is an isomorphism, then ϕ and $L\phi$ are isomorphisms by inspection. We need a technical lemma. Its proof requires use of the explicit definition of the f_* and is deferred to the end of the section.

Lemma 5.9. *Let f and f' be linear isometries $(R^\infty)^i \to (R^\infty)^j$. Then the natural equivalence $f_* \simeq f'_*$ fits into the following commutative diagram in $H\mathscr{S}_j$:*

$$\begin{tikzcd}
f_*LT_i \arrow[dr, "L\phi"'] & \simeq & f'_*LT_i \arrow[dl, "L\phi"] \\
& LT_j &
\end{tikzcd}$$

It follows that all of the $L\phi$ are natural equivalences. Now specialize. Choose linear isometries $f : R^\infty \to R^\infty \oplus R^\infty$ and $g : R^\infty \oplus R^\infty \to R^\infty$. Application of g_* to $L\omega$ and use of $L\phi$ gives the desired pairing $E\omega$ via the diagram

$$\begin{tikzcd}
EP \wedge EQ = g_*(LP_1 \wedge LQ_1) \arrow[r] \arrow[d, "E\omega"'] & g_*LT_2 \arrow[d, phantom, "\wr"] \\
ET \simeq (gf)_*(LT_1) \cong & g_*f_*LT_1
\end{tikzcd}$$

in the stable category $H\mathscr{S}$, the last equivalence being the natural one between $(gf)_*$ and 1_*, $1 : R^\infty \to R^\infty$. Here we could have used g^* rather than f_* and exploited $g_*g^* \simeq 1$, but use of f_* will simplify the proof of Theorem 5.8. Since a morphism of pairings of $\mathscr{I}_*$-prespectra clearly induces a homotopy commutative diagram in $\mathscr{P}_2$, the functoriality of this passage from pairings of $\mathscr{I}_*$-prespectra to pairings of spectra is obvious. This proves Theorem 5.7 and the functoriality claim in Theorem 5.8.

For the proof of Theorem 5.8, observe that $E\omega$ is actually independent of the choice of f. Indeed, Lemma 5.9 and the coherence of our change of isometries equivalences gives the commutative diagram

$$\begin{tikzcd}
& g_*f_*LT_1 \cong (gf)_*(LT_1) \arrow[dr, phantom, sloped, "\simeq"] & \\
g_*LT_2 \arrow[ur, phantom, sloped, "\simeq"] \arrow[dr, phantom, sloped, "\simeq"] & \wr \qquad\qquad \wr & LT_1 \\
& g_*f'_*LT_1 \cong (gf')_*(LT_1) \arrow[ur, phantom, sloped, "\simeq"] &
\end{tikzcd}$$

Let T be a ring $\mathscr{I}_*$-prespectrum. We must derive the diagrams displayed in Theorem 1.7 (with X replaced by T) from the diagrams displayed in Definition 5.4. For the unit diagrams, we need the observation that

$$\sigma : T_1 \wedge S_1 \to T_2 \quad \text{and} \quad \phi : i_* T_1 \to T_2$$

coincide under the identification of $T_1 \wedge S_1$ with i_*T_1. (Here $S_1 = F_1(S^0)$ is the suspension prespectrum of S^0.) The unit condition involving $1 \wedge e$ in Theorem 1.7 is then easily verified simply by choosing $f = i$ in the construction of $E\omega$, the point being that $E\omega$ and the coherence isomorphism β are then defined in terms of exactly the same equivalence $(gi)_* = 1_*$. For the unit condition involving $e \wedge 1$, one chooses f to be the inclusion τi of R^∞ onto the second summand of $R^\infty \oplus R^\infty$ in defining $E\omega$ and notes that the relevant coherence isomorphism is $\tau\beta$. For this diagram, and for commutativity, one must observe that

$$\tau : T \wedge T \to (T \wedge T) \circ t \quad \text{and} \quad T\tau : T \circ \oplus \to T \circ \oplus \circ t$$

on the $\mathscr{I}_*$-prespectrum level correspond to the external commutativity isomorphism $T \wedge T \cong \tau_*(T \wedge T)$ and to instances of ϕ^{-1} on the prespectrum level. For the commutativity diagram, $E\omega \circ \tau = E\omega$, it is convenient to use any given f to define one of the instances of $E\omega$ and to use τf to define the other. Since the remaining work is the purely formal exercise of writing down large diagrams and verifying that the information above guarantees their commutativity, we leave further details to the interested reader.

We have left one unfinished piece of business.

Proof of Lemma 5.9. We shall be sketchy since we don't wish to go into full detail on the relevant constructions. The space of linear isometries $(R^\infty)^i \to (R^\infty)^j$ is contractible (e.g. [17, p. 10]), hence we can choose a path of isometries h connecting f to f'. By [5, VIII], h induces a functor $h_* : \mathscr{P}_i \to \mathscr{P}_j$ and thus $h_* = Lh_*v : \mathscr{S}_i \to \mathscr{S}_j$. For $E \in \mathscr{S}_i$, the inclusions of $\{0\}$ and $\{1\}$ in I induce natural weak equivalences $f_*E \to h_*E \leftarrow f'_*E$, by [5, VIII]. This diagram gives the equivalence $f_*E \simeq f'_*E$ in $H\mathscr{S}_j$ exploited in the arguments above. We claim that there is a map $\phi : h_*T_i \to T_j$ such that the following diagram commutes:

$$\begin{tikzcd}
f_*T_i \arrow[r] \arrow[dr, "\phi"'] & h_*T_i \arrow[d, "\phi"] & f'_*T_i \arrow[l] \arrow[dl, "\phi"] \\
 & T_j &
\end{tikzcd}$$

Upon application of L, this will imply the lemma. For $W\subset(R^\infty)^j$, $(h_*T_i)(W)$ is obtained by choosing $V\subset(R^\infty)^i$ such that $h(I\times V)\subset W$ and letting $(h_*T_i)(W)$ be the smash product of T_iV with the Thom complex of the complementary bundle over I of the bundle map $I\times V\to I\times W$ determined by h. For $t\in I$, the fibre is the complement W_t of $h_t(V)$ in W, and the maps $\sigma : TV\wedge SW_t\to TW$ glue together to specify $\phi : (h_*T_i)(W)\to T_jW$. The maps ϕ on f_*T_i and f'_*T_i are restrictions. With these indications, the details are straightforward from the constructions in [5, VIII Section 4], which are largely concerned with setting up sufficient language to explain how to choose the V's consistently so as to obtain a prespectrum and to show that everything becomes independent of choice on passage to spectra.

6. The passage from $\mathscr{C}$-spaces to $\mathscr{J}_*$-prespectra

We shall begin by proving the following result.

Theorem 6.1. *Let $\mathscr{C}$ be any operad whatever. Then there is a functor T from $\mathscr{C}$-spaces to $\mathscr{J}_*$-prespectra.*

The point is that the spacewise contractibility of $\mathscr{C}$ assumed in section one serves only to identify the homotopy type of the zero-th space of the associated spectrum. It has nothing to do with the general constructions. We shall then prove the following results.

Theorem 6.2. *Let $\wedge : (\mathscr{C},\mathscr{D})\to\mathscr{E}$ be a pairing of operads. Then pairings $f : (X, Y)\to Z$ of a $\mathscr{C}$-space X and $\mathscr{D}$-space Y to an $\mathscr{E}$-space Z functorially determine pairings $Tf : (TX, TY)\to TZ$ of $\mathscr{J}_*$-prespectra.*

Theorem 6.3. *Let $\mathscr{C}$ be a permutative operad and let X be a ring $\mathscr{C}$-space. Then TX is functorially a ring $\mathscr{J}_*$-prespectrum. If X is commutative, then TX is commutative.*

With $EX=ETX$, Theorems 6.2 and 6.3 combine with Theorems 5.7 and 5.8 to prove Theorems 1.6 and 1.7, generalized to arbitrary operads. While our main interest is of course in the spacewise contractible case, there may well be useful applications of the full generality. We record one amusing trivial case. There is a trivial permutative operad $\mathscr{P}$ with $\mathscr{P}(0)=\{0\}$, $\mathscr{P}(1)=\{1\}$, and $\mathscr{P}(j)$ empty for $j>1$. The associated category of operators is precisely Π. It will be immediate from our constructions that there is a natural equivalence $F(X_1)\to TX$ for a Π-space X, where

$F(X_1)$ is the suspension $\mathscr{I}_*$-prespectrum introduced in Remark 5.2. Therefore $\Sigma^\infty X_1 \simeq EX$ in the stable category.

Corollary 6.4. *A pairing $(X, Y) \to Z$ of Π-spaces induces a pairing $\Sigma^\infty X_1 \wedge \Sigma^\infty Y_1 \to \Sigma^\infty Z_1$ of spectra. If X is a ring Π-space, then $\Sigma^\infty X_1$ is a ring spectrum, and $\Sigma^\infty X_1$ is commutative if X is commutative.*

In fact, this observation has a bit more than just amusement value. Our constructions are natural in $\mathscr{C}$, and $\mathscr{P}$ is an initial object in the category of operads. For a $\hat{\mathscr{C}}$-space X, there results a natural map $\tilde{\iota} : \Sigma^\infty X_1 \to EX$. By the definitions in section one, the structures in Theorems 6.2 and 6.3 have underlying Π-structures.

Corollary 6.5. *Under the hypotheses of Theorem 6.2, the following diagram commutes in the stable category.*

$$\begin{tikzcd}
\Sigma^\infty X_1 \wedge \Sigma^\infty Y_1 \arrow[r] \arrow[d, "\tilde{\iota} \wedge \tilde{\iota}"'] & \Sigma^\infty Z_1 \arrow[d, "\tilde{\iota}"] \\
EX \wedge EY \arrow[r] & EZ
\end{tikzcd}$$

Under the hypotheses of Theorem 6.3, $\tilde{\iota} : \Sigma^\infty X_1 \to EX$ is a map of ring spectra.

The map $\tilde{\iota}$ is adjoint to a natural map $\iota : X_1 \to E_0X$. The natural equivalence $\Sigma^\infty(X \wedge Y) \simeq \Sigma^\infty X \wedge \Sigma^\infty Y$ in the stable category for spaces X and Y and the adjunction between Σ^∞ and the zero-th space functor $\Omega^\infty E = E_0$ yield the following composite natural map ω of spaces for spectra E and F:

$$\begin{tikzcd}
\Omega^\infty\Sigma^\infty(E_0 \wedge F_0) \arrow[r, phantom, "\simeq"] & \Omega^\infty(\Sigma^\infty E_0 \wedge \Sigma^\infty F_0) \arrow[d, "\Omega^\infty(\varepsilon \wedge \varepsilon)"] \\
E_0 \wedge F_0 \arrow[u, "\eta"] \arrow[r, "\omega"] & \Omega^\infty(E \wedge F)
\end{tikzcd}$$

Here η and ε are the unit and counit of the adjunction. An elementary diagram chase shows that the diagram of the previous corollary implies the following commutative diagram:

$$\begin{tikzcd}
X_1 \wedge Y_1 \arrow[r, "f_{11}"] \arrow[d, "\iota \wedge \iota"'] & Z_1 \arrow[d, "\iota"] \\
E_0X \wedge E_0Y \arrow[r, "(E_0f)\omega"] & E_0Z
\end{tikzcd}$$

We recall the following result, which is the characteristic property of infinite loop space machines [20].

Theorem 6.6. *If* $\mathscr{C}$ *is spacewise contractible, then* $\iota : X_1 \to E_0X$ *is a group completion.*

The proofs follow the same lines as in [17], but the arguments there can be made very much cleaner by virtue of a lovely improvement of my theory contributed by Steiner [23]. In [17, VII Section 1], I tried to prove the following result by taking $\mathscr{K}_V(j)$ to be the space of suitable j-tuples of embeddings $V \to V$. This worked only "partially", and the failure led me to introduce the perfectly hideous notions of partial operads and partial monads. Steiner very cleverly observed that everything I hoped for could be proven by using suitable j-tuples of paths of embeddings. The little convex bodies partial operads of [17] should therefore be consigned to oblivion, along with the partial notions to which they gave rise, and supplanted henceforward by Steiner's operads. We recall his definitions in the following proof.

Theorem 6.7 (Steiner). *There is a continuous functor* $\mathscr{K}$ *from* $\mathscr{I}_*$ *to the category of operads, written* $\mathscr{K}_V$ *on objects* V *and* $g : \mathscr{K}_V \to \mathscr{K}_W$ *on morphisms* $g : V \to W$. *Here continuity means that the evaluation maps*

$$\mathscr{I}_*(V, W) \times \mathscr{K}_V(j) \to \mathscr{K}_W(j)$$

are continuous for all j. $\mathscr{K}_0$ *is the trivial operad and, for* $V \neq 0$, $\mathscr{K}_V(j)$ *is* Σ_j*-free and has the* Σ_j*-equivariant homotopy type of the configuration space* $F(V, j)$ *of* j*-tuples of distinct points of* V. *Further, there is a system of pairings*

$$\wedge : (\mathscr{K}_V, \mathscr{K}_W) \to \mathscr{K}_{V \oplus W}$$

which is natural with respect to morphisms in $\mathscr{I}_* \times \mathscr{I}_*$ *and satisfies the following properties.*

(i) Inclusion: *The map* $\sigma : \mathscr{K}_V(j) \to \mathscr{K}_{V \oplus W}(j)$ *specified by* $\sigma(c) = c \wedge 1$, *where* $1 \in \mathscr{K}_W(1)$ *is the unit, is a closed inclusion.*

(ii) Unity: *For* $c \in \mathscr{K}_V(j)$ *and* $1 \in \mathscr{K}_0(1)$,

$$1 \wedge c = c \quad \textit{and} \quad c \wedge 1 = c \quad \textit{in } \mathscr{K}_V(j).$$

(iii) Associativity: *For* $b \in \mathscr{K}_Z(i)$, $c \in \mathscr{K}_V(j)$, *and* $d \in \mathscr{K}_W(k)$

$$(b \wedge c) \wedge d = b \wedge (c \wedge d) \quad \textit{in } \mathscr{K}_{Z \oplus V \oplus W}(ijk).$$

(iv) Commutativity: *For* $c \in \mathscr{K}_V(j)$ *and* $d \in \mathscr{K}_W(k)$,

$$c \wedge d = \tau(W, V)((d \wedge c)\tau(j, k)) \quad \textit{in } \mathscr{K}_{V \oplus W}(jk),$$

where $\tau(W, V) : W \oplus V \to V \oplus W$ *is the transposition.*

Proof. Let $\mathscr{A}_V$ be the space of embeddings $V \to V$ and let $\mathscr{E}_V$ be the space of paths $h : I \to \mathscr{A}_V$ such that $h(1)$ is the identity map of V and each $h(t)$ is distance reducing. Let $\mathscr{K}_V(j)$ be the Σ_j-invariant subspace of $\mathscr{E}_V^j$ consisting of j-tuples $\langle h_1, \ldots, h_j \rangle$ such that the $h_i(0)$ have disjoint images. We have composition maps $\mathscr{E}_V \times \mathscr{E}_V \to \mathscr{E}_V$ and product maps $\mathscr{E}_V\times\mathscr{E}_W\to\mathscr{E}_{V\oplus W}$ specified by

$$(h'\circ h)(t)=h'(t)\circ h(t)\quad\text{and}\quad(h\times j)(t)=h(t)\times j(t),$$

and we let $1\in\mathscr{K}_V(1)$ be the constant path at the identity map. The structural maps γ of the $\mathscr{K}_V$ are given by blocks of composites and the pairings $\wedge$ are given by lexicographically ordered pairwise products, exactly as for the little cubes operads [15, pp. 30 and 72]. We have action maps $\mathscr{I}_*(V,W)\times\mathscr{E}_V\to\mathscr{E}_W$ specified by $(gh)(t)=gh(t)g^{-1}$, and these apply coordinatewise on j-tuples to give $g:\mathscr{K}_V(j)\to\mathscr{K}_W(j)$. The formal verifications are trivial, and the topological assertions are proven by Steiner [23].

The maps σ of (i) specify inclusions of operads $\mathscr{K}_V\to\mathscr{K}_{V\oplus W}$. Define $\mathscr{K}_\infty$ to be the union of the operads $\mathscr{K}_V$ for $V\subset R^\infty$. The $\mathscr{K}_\infty(j)$ are Σ_j-free and contractible, hence $\mathscr{K}_\infty$ is an E_∞ operad.

Steiner's point is that this use of paths controls the homotopy type of the $\mathscr{K}_V(j)$, while use of just the initial embeddings $h(0)$ gives natural actions of the $\mathscr{K}_V$ on V-fold loop spaces just like those of the little n-cubes operads on n-fold loop spaces [15, p. 40].

Proposition 6.8. *There is a natural action θ of $\mathscr{K}_V$ on Ω^VY. The action of $\mathscr{K}_V$ on $\Omega^V\Omega^WY=\Omega^{V\oplus W}Y$ coincides with the restriction to $\mathscr{K}_V$ of the action of $\mathscr{K}_{V\oplus W}$, and there results a natural action of $\mathscr{K}_\infty$ on the* 0th *spaces of spectra indexed on R^∞.*

Operads naturally determine monads in $\mathscr{T}$ [15, Section 2]. As Steiner points out, one has the following assertion just as in [15, p. 44] for the little cubes operads, the group completion property being due independently to Cohen and Segal. Recall that $\Omega^V\Sigma^V$ is a monad in $\mathscr{T}$ and that suspension gives a map of monads $\sigma:\Omega^V\Sigma^V\to\Omega^{V\oplus W}\Sigma^{V\oplus W}$ (e.g., [15, pp. 17 and 42]).

Theorem 6.9. *Define $\alpha_V:K_VY\to\Omega^V\Sigma^VY$ to be the composite*

$$K_VY\xrightarrow{K_V\eta}K_V\Omega^V\Sigma^VY\xrightarrow{\theta}\Omega^V\Sigma^VY.$$

Then α_V is a weak equivalence if Y is connected and is a group completion in general. The α_V specify a morphism of monads $K_V\to\Omega^V\Sigma^V$, and the following diagram of monads commutes.

$$\begin{tikzcd}
K_V \arrow[r, "\alpha_V"] \arrow[d, "\sigma"'] & \Omega^V\Sigma^V \arrow[d, "\sigma"] \\
K_{V\oplus W} \arrow[r, "\alpha_{V\oplus W}"] & \Omega^{V\oplus W}\Sigma^{V\oplus W}
\end{tikzcd}$$

The adjoints $\beta_V:\Sigma^VK_V\to\Sigma^V$ specify an action of the monad K_V on the functor Σ^V, as explained in [15, pp. 86–88].

We can now prove Theorem 6.1 by arguments like those of [20, Sections 5–6]. There we constructed a monad $\hat{C}$ in the category $\Pi[\mathscr{T}]$ of Π-spaces such that a $\mathscr{C}$-space is the same thing as a $\hat{C}$-space. There is an adjoint pair of functors

$$L : \Pi[\mathscr{T}] \to \mathscr{T} \quad \text{and} \quad R : \mathscr{T} \to \Pi[\mathscr{T}]$$

specified by $LX = X_1$ and $RY = \{Y^n\}$. The monad $\hat{C}$ in $\Pi[\mathscr{T}]$ extends the monad C in $\mathscr{T}$ in the sense that $\hat{C}RY = R\hat{C}Y$. By a formal argument [20, p. 219], if C acts on a functor F, then $\hat{C}$ acts on the functor FL.

For simplicity of notation, write $\mathscr{C}_V$ for the product operad $\mathscr{C} \times \mathscr{K}_V$. The projections induce morphisms $\pi : \hat{C}_V \to \hat{K}_V$ and $\psi : \hat{C}_V \to \hat{C}$ of monads in $\Pi[\mathscr{T}]$. By pullback along ψ, a $\hat{C}$-space X is a $\hat{C}_V$-space. By pullback along π, the $\hat{K}_V$-functor $\Sigma^V L$ is a $\hat{C}_V$-functor. By [15, Sections 9 and 11], there results a two-sided bar construction

$$(TX)(V) = B(\Sigma^V L, \hat{C}_V, X).$$

More explicitly, $(TX)(V)$ is the geometric realization of a simplicial space whose space of q-simplices is $\Sigma^V L \hat{C}^q_V X$. The action of linear isometries on paths of embeddings and on spheres passes through the construction to yield maps

$$(TX)(g) : (TX)(V) \to (TX)(W)$$

for $g : V \to W$. All functors in sight are continuous, and we have the continuous functor $TX : \mathscr{I}_* \to \mathscr{T}$ required for Theorem 6.2.

To construct $\sigma : TX \wedge S \to (TX) \circ \oplus$, we recall first that suspension commutes with realization [15, 12.1]. By the adjoint of the diagram in Theorem 6.9, the maps $\sigma : \hat{C}_V \to \hat{C}_{V \oplus W}$ induced from $1 \times \sigma : \mathscr{C} \times \mathscr{K}_V \to \mathscr{C} \times \mathscr{K}_{V \oplus W}$ determine the required maps

$$\begin{tikzcd}
(TX)(V) \wedge SW \arrow[r, phantom, "\cong"] \arrow[dd, "\sigma"'] & \Sigma^W B(\Sigma^V L, \hat{C}_V, X) \arrow[d, phantom, "\wr"] \\
 & B(\Sigma^{V \oplus W} L, \hat{C}_V, X) \arrow[d, "{B(1, \sigma, 1)}"] \\
(TX)(V \oplus W) \arrow[r, phantom, "\cong"] & B(\Sigma^{V \oplus W} L, \hat{C}_{V \oplus W}, X)
\end{tikzcd}$$

Properties (i)–(iii) in Theorem 6.7 imply properties (i)–(iii) in Definition 5.1. We have proven Theorem 6.1.

The use of $\mathscr{I}_*$-prespectra here is just a reinterpretation of my earlier constructions and, with the little cubes operad there replaced by the Steiner operads here, the proof of Theorem 6.6 is exactly the same as in [20, 6.4].

We are ready to return to our theme of pairings. We take more or less seriously the full categorical generality of the constructions of [15, Section 9]. Given a monad C in any category $\mathscr{V}$ whatever and given a C-object P in $\mathscr{V}$ and a C-functor $F : \mathscr{V} \to \mathscr{W}$ in any other category $\mathscr{W}$, we are entitled to a simplicial object $B_*(F, C, P)$ in $\mathscr{W}$.

Let Pair $\mathscr{T}$ be the category of pairings $X \wedge Y \to Z$ of based spaces; its morphisms are homotopy commutative diagrams. This category will play the role of $\mathscr{W}$, but our $B_*(F, C, X)$ will have faces and degeneracies given by strictly commutative diagrams and our morphisms will be homotopy commutative diagrams of simplicial spaces, in the sense that the homotopies on q-simplicies will be compatible with the faces and degeneracies. By the product and homotopy preserving properties of geometric realization [15, 11.5 and 11.9], we will arrive back in Pair $\mathscr{T}$ upon realization.

Let Pair $\Pi[\mathscr{T}]$ be the category of pairings $X \wedge Y \to Z$ of Π-spaces; its morphisms are homotopy commutative diagrams of functors. This category will play the role of $\mathscr{V}$. Applied to all variables, the functors $L : \Pi[\mathscr{T}] \to \mathscr{T}$ and $R : \mathscr{T} \to \Pi[\mathscr{T}]$ induce functors

$$L : \text{Pair } \Pi[\mathscr{T}] \to \text{Pair } \mathscr{T} \quad \text{and} \quad R : \text{Pair } \mathscr{T} \to \text{Pair } \Pi[\mathscr{T}].$$

The following lemma may be viewed as giving monads $(\hat{C}, \hat{D}) \to \hat{E}$ in Pair $\Pi[\mathscr{T}]$. We refer to strict morphisms of pairings when no homotopies are required.

Lemma 6.10. *Let* $\wedge : (\mathscr{C}, \mathscr{D}) \to \mathscr{E}$ *be a pairing of operads. Pairings* $f : X \wedge Y \to Z$ *of spaces functorially determine pairings* $\tilde{f} : CX \wedge DY \to EZ$ *of spaces such that* $\eta = (\eta, \eta, \eta)$ *and* $\mu = (\mu, \mu, \mu)$ *are strict morphisms of pairings. Pairings* $f : (X, Y) \to Z$ *of* Π*-spaces functorially determine pairings* $\tilde{f} : (\hat{C}X, \hat{D}Y) \to \hat{E}Z$ *of* Π*-spaces such that* $\hat{\eta} = (\hat{\eta}, \hat{\eta}, \hat{\eta})$ *and* $\hat{\mu} = (\hat{\mu}, \hat{\mu}, \hat{\mu})$ *are strict morphisms of pairings. Both functors preserve strict morphisms. If* f *is a pairing of spaces with associated pairing of* Π*-spaces* Rf*, then* $\widetilde{Rf} = R\tilde{f}$.

Proof. For the first statement, $\tilde{f}$ is induced by the maps

$$\mathscr{C}(j) \times X^j \times \mathscr{D}(k) \times Y^k \xrightarrow{(\wedge \times (Rf)_{jk})(1 \times \tau \times 1)} \mathscr{E}(jk) \times Z^{jk}.$$

For the second statement, $\tilde{f}_{pq} : \hat{C}_p X \wedge \hat{D}_q Y \to \hat{E}_{pq} Z$ is induced by passage to coends from the maps

$$\hat{\mathscr{C}}(\mathbf{m}, \mathbf{p}) \times X_m \times \hat{\mathscr{D}}(\mathbf{n}, \mathbf{q}) \times Y_n \xrightarrow{(\wedge \times f_{mn})(1 \times \tau \times 1)} \hat{\mathscr{E}}(\mathbf{mn}, \mathbf{pq}) \times Z_{mn}.$$

The verifications are easy, being combinatorial from Definition 1.4 in the first case and categorical in the second. In both cases, the functoriality is immediate from the continuity of the functors involved, which allows their application to homotopies.

When $\mathscr{C} = \mathscr{D} = \mathscr{E}$, we write $\tilde{f} = \hat{C}f$ and $\tilde{\xi} = \hat{C}\xi$ on pairings and their morphisms.

The following lemma may be viewed as giving actions of these monads on objects of Pair $\Pi[\mathscr{T}]$. The proof is an immediate verification from Definition 1.1.

Lemma 6.11. *A pairing* $f : (X, Y) \to Z$ *of a* $\hat{\mathscr{C}}$*-space* X *and a* $\hat{\mathscr{D}}$*-space* Y *to an* $\hat{\mathscr{E}}$*-space* Z *determines and is determined by a strict morphism of pairings* ξ *from* $\tilde{f} : (\hat{C}X, \hat{D}Y) \to \hat{E}Z$ *to* f *such that* $\xi\hat{\eta} = 1$ *and* $\xi\hat{\mu} = \xi\tilde{\xi}$.

Finally, the following observation will lead to actions of our monads on functors. The proof is exactly the same as for the little cubes operads [15, p. 72].

Lemma 6.12. *For spaces X and Y, the following diagram commutes:*

$$\begin{tikzcd}
K_V X \wedge K_W Y \arrow[r, "\wedge"] \arrow[d, "\alpha_V \wedge \alpha_W"'] & K_{V\oplus W}(X \wedge Y) \arrow[d, "\alpha_{V\oplus W}"] \\
\Omega^V \Sigma^V X \wedge \Omega^W \Sigma^W Y \arrow[r, "\wedge"] & \Omega^{V\oplus W} \Sigma^{V\oplus W}(X \wedge Y)
\end{tikzcd}$$

Passing to adjoints, we see that a pairing $f: X \wedge Y \to Z$ of spaces gives rise to a commutative diagram

$$\begin{tikzcd}
\Sigma^V K_V X \wedge \Sigma^W K_W Y \arrow[r] \arrow[d, "\beta_V \wedge \beta_W"'] & \Sigma^{V\oplus W} K_{V\oplus W} Z \arrow[d, "\beta_{V\oplus W}"] \\
\Sigma^V X \wedge \Sigma^W Y \arrow[r] & \Sigma^{V\oplus W} Z
\end{tikzcd}$$

This may be interpreted as saying that we have a functor $(\Sigma^V, \Sigma^W) \to \Sigma^{V\oplus W}$ from the category Pair $\mathscr{T}$ to itself and an action of the monad $(K_V, K_W) \to K_{V\oplus W}$ on this functor. The formal arguments of [20, p. 219] apply equally well to pairings and, by pullback along the projections π, we deduce an analogous diagram which may be interpreted as asserting that the monad $(\hat{C}_V, \hat{D}_W) \to \hat{E}_{V\oplus W}$ in Pair $\Pi[\mathscr{T}]$ acts on the functor $(\Sigma^V L, \Sigma^W L) \to \Sigma^{V\oplus W} L$ from Pair $\Pi[\mathscr{T}]$ to Pair $\mathscr{T}$.

The verbiage may seem a bit strained, but the ideas should be clear enough. The lemmas above combine to show that a pairing f of a $\mathscr{C}$-space X and a $\mathscr{D}$-space Y to an $\mathscr{E}$-space Z gives rise to maps

$$T_q f : \Sigma^V L \hat{C}^q_V X \wedge \Sigma^W L \hat{D}^q_W Y \to \Sigma^{V\oplus W} L \hat{E}^q_{V\oplus W} Z$$

which together specify a map of simplicial spaces. Upon passage to geometric realization, we obtain

$$Tf : (TX)(V) \wedge (TY)(W) \to (TZ)(V \oplus W).$$

The Tf clearly specify a natural transformation of functors $\mathscr{I}_* \times \mathscr{I}_* \to \mathscr{T}$. The defining diagram for a pairing of $\mathscr{I}_*$-prefunctors in Definition 5.3 commutes by virtue of the case $k=1$ of Theorem 6.7(iv), in which $\tau(j, 1)$ is the identity permutation. The point is that the $\mathscr{C}$, $\mathscr{D}$ and X, Y coordinates of our constructions are obviously mapped in the same way by the two composites in that diagram, and the sphere and Steiner operad coordinates are also mapped in the same way by inspection and use of the cited commutativity relation. Since we have kept track of functoriality throughout, we have now proven Theorem 6.2 and the functoriality claim in Theorem 6.3.

It remains to show that the diagrams given by Definition 1.3 for a ring $\mathscr{C}$-space X imply the diagrams in Definition 5.4 required for a ring $\mathscr{I}_*$-prespectrum.

For the unit diagrams, $e: S^0 \to X_1$ determines

$$e : S = F(S^0) \to F(X_1) \subset TX,$$

where the inclusion is given by the spaces of 0-simplices $\Sigma^V LX = \Sigma^V X_1$ contained in the $(TX)(V)$. Note that $\Sigma^V X_1$ maps to $\Sigma^V(\hat{\eta}^q X_1)$ under the unique iterated degeneracy operator from 0-simplices to q-simplices. Here $\hat{\eta}(x) = [1; x] \in L\hat{C}_V X$ for $x \in X_1$, where $1 = (1, 1) \in \mathscr{C}(1) \times \mathscr{K}_V(1)$. These degeneracies are relevant because of the role played by degeneracies in the commutation of realization with products. The composite $(Tf)(1 \wedge e) : T \wedge S \to T \circ \oplus$ is given on q-simplices at the simplicial space level by

$$\Sigma^V L\hat{C}^q_V X \wedge SW \xrightarrow{1 \wedge e} \Sigma^V L\hat{C}^q_V X \wedge \Sigma^W(\hat{\eta}^q X_1) \xrightarrow{T_q f} \Sigma^{V \oplus W} L\hat{C}_{V \oplus W} X.$$

This differs from the corresponding level of σ only in that here the X coordinates of $\Sigma^V L\hat{C}^q_V X$ are paired with $1 \in X_1$, whereas they are mapped identically under σ. The homotopy $f(1 \wedge e) \simeq 1$ given in Definition 1.3 therefore implies the homotopy $(Tf)(1 \wedge e) \simeq \sigma$ required in Definition 5.4. If we transpose in the middle before applying $T_q f$ in the composite above, then the result differs in the symmetric way from the composite of σ with $T_q \tau(V, W)$, hence $f(e \wedge 1) \simeq 1$ implies the other unit diagram in Definition 5.4.

Similarly, to check the associativity or commutativity of Tf, it suffices to consider the relevant diagram at the level of smash products of spaces of q-simplices. By Definition 1.3 and Theorem 6.7, we have precise associativity and commutativity for the operad coordinates. The given associativity or commutativity homotopy for X induces homotopies on the relevant spaces of q-simplices, and these homotopies as q varies specify a simplicial homotopy and so pass to realizations. This proves Theorem 6.3.

Remark 6.13. The arguments above simplify to show that a pairing of a $\hat{\mathscr{K}}_V$-space X and a $\hat{\mathscr{K}}_W$-space Y to a $\hat{\mathscr{K}}_{V \oplus W}$-space Z induces a pairing

$$\begin{tikzcd}
B(\Sigma^V L, \hat{K}_V, X) \wedge B(\Sigma^W L, \hat{K}_W, Y) \arrow[d] \\
B(\Sigma^{V \oplus W} L, \hat{K}_{V \oplus W}, Z).
\end{tikzcd}$$

Such pairings also enjoy unity, associativity, and commutativity properties. Here $\Omega^V B(\Sigma^V L, \hat{K}_V, X)$ is a group completion of X_1. That is, our theory applies directly to these V-fold delooping machines.

Appendix. Generalizations and variants of the theory

The following is a straightforward generalization of Segal's definition [22, Section 5] of a multiplication on an $\mathscr{F}$-space (or Γ-space) and of our Definition 1.1.

Definition A.1. Let $\wedge : \hat{\mathscr{C}} \times \hat{\mathscr{D}} \to \hat{\mathscr{E}}$ be a pairing of categories of operators. Let X, Y, and Z be a $\hat{\mathscr{C}}$-space, a $\hat{\mathscr{D}}$-space, and an $\hat{\mathscr{E}}$-space, respectively. A (generalized) pairing $(X, Y)\to Z$ is a functor $M : \hat{\mathscr{C}}\times\hat{\mathscr{D}}\to\mathscr{T}$ together with natural transformations

$$g : M\to X\times Y \quad\text{and}\quad f : M\to Z\circ\wedge$$

such that each $g_{m,n} : M_{m,n}\to X_m\times Y_n$ is an equivalence. We require $(\phi, \psi) : M_{m,n}\to M_{p,q}$ to be a $\Sigma_\phi\times\Sigma_\psi$-equivariant cofibration if $\phi : \mathbf{m}\to\mathbf{p}$ and $\psi : \mathbf{n}\to\mathbf{q}$ are injections in Π (compare [20, 1.2]). Diagrammatically, we are given

$$\begin{tikzcd}
\hat{\mathscr{C}}\times\hat{\mathscr{D}} \arrow[r, "\wedge"] \arrow[d, "(X,Y)"'] \arrow[dr, "M"] \arrow[dr, phantom, shift right=4ex, "\overset{g}{\Leftarrow}"] \arrow[dr, phantom, shift left=4ex, "\overset{f}{\Rightarrow}"] & \hat{\mathscr{E}} \arrow[d, "Z"] \\
\mathscr{T}\times\mathscr{T} \arrow[r, "\times"] & \mathscr{T}
\end{tikzcd}$$

Among other things, the cofibration condition ensures that $g_{m,n}$ induces an equivalence of pairs

$$(M_{m,n}, M_{m,0}\vee M_{0,n})\to(X_m\times Y_n, (X_m\times Y_0)\vee(X_0\times Y_n)),$$

and naturality implies that $f_{m,n}(M_{m,0}\vee M_{0,n})\subset Z_0$. At least if $X_0=Y_0=Z_0=\{*\}$, as could be arranged functorially by [20, App. B] and will be assumed tacitly below (in order to ensure the appearance of smash products), the case when $M=X\times Y$ and g is the identity is precisely the notion of pairing given in Definition 1.1. The case when $\hat{\mathscr{C}}=\hat{\mathscr{D}}=\hat{\mathscr{E}}=\mathscr{T}$ and $X=Y=Z$ is essentially Segal's notion of a multiplication on X.

Morphisms of pairings are quadruples $(\alpha, \beta, \gamma, \delta)$ of natural transformations such that the following diagram of functors commutes up to homotopy:

$$\begin{tikzcd}
X\times Y \arrow[d, "\alpha\times\beta"'] & M \arrow[l, "g"'] \arrow[r, "f"] \arrow[d, "\delta"] & Z\circ\wedge \arrow[d, "\gamma\wedge"] \\
X'\times Y' & M' \arrow[l, "g'"'] \arrow[r, "f'"] & Z'\circ\wedge
\end{tikzcd}$$

Unit, associativity, and commutativity conditions on a $\hat{\mathscr{C}}$-space X with a pairing $(X, X)\to X$ can be expressed in terms of homotopy commutative diagrams involving auxiliary functors $P, P' : \hat{\mathscr{C}}\to\mathscr{T}$ and $Q : \hat{\mathscr{C}}^3\to\mathscr{T}$:

$$\begin{tikzcd}
X_1\times X & M_{1,*} \arrow[l, "g"'] \arrow[r, "f"] & X & X\times X_1 & M_{*,1} \arrow[l, "g"'] \arrow[r, "f"] & X \\
S^0\times X \arrow[u, "e\times 1"] & P \arrow[l, "\cong"'] \arrow[r] \arrow[u] & X \arrow[u, equal] & X\times S^0 \arrow[u, "1\times e"] & P' \arrow[l, "\cong"'] \arrow[r] \arrow[u] & X \arrow[u, equal]
\end{tikzcd}$$

$$\begin{tikzcd}
X\times X\times X & & M\times X \arrow[ll, "g\times 1"'] \arrow[r, "f\times 1"] & (X\circ\wedge)\times X=(X\times X)\circ(\wedge\times 1) \\
X\times M \arrow[u, "1\times g"] \arrow[d, "1\times f"'] & & Q \arrow[ll, "\cong"'] \arrow[u, "\cong"'] \arrow[r] \arrow[d] & M\circ(\wedge\times 1) \arrow[u, "g(\wedge\times 1)"'] \arrow[d, "f(\wedge\times 1)"] \\
X\times(X\circ\wedge)=(X\times X)\circ(1\times\wedge) & & M\circ(1\times\wedge) \arrow[ll, "g(1\times\wedge)"'] \arrow[r, "f(1\times\wedge)"] & (X\circ\wedge)\circ(1\times\wedge)=(X\circ\wedge)\circ(\wedge\times 1)
\end{tikzcd}$$

$$\begin{tikzcd}
X\times X \arrow[d, "\tau"'] & M \arrow[l, "g"'] \arrow[r, "f"] \arrow[d] & X\circ\wedge \arrow[d, "X\tau"] \\
(X\times X)t & Mt \arrow[l, "gt"'] \arrow[r, "ft"] & X\circ\wedge\circ t
\end{tikzcd}$$

While this does give a generalized notion of (commutative) ring $\mathscr{C}$-space, it is clearly all much more cumbersome than Definition 1.3 and therefore to be avoided whenever possible.

If we use the functor $\bar{\mathscr{A}}$ of [18, Const. 10] to pass from permutative categories to $\mathscr{F}$-spaces, then we are forced to use this generalized notion of pairing rather than the simpler notion of the body of the paper. We show this in the following elaboration of the cited construction. A special case was sketched by Robinson [30]. We assume familiarity with [18, Const. 10] and we write $\pi_{s,t}=i^{-1}_{(s,t)}$ in it. We sometimes write $\bar{A}$ instead of $\bar{\mathscr{A}}$, in parallel with Section 4.

Construction A.2. Let $\otimes : \mathscr{A}\times\mathscr{B}\rightarrow\mathscr{C}$ be a pairing of permutative categories. We construct a diagram

$$\begin{tikzcd}
\mathscr{F}\times\mathscr{F} \arrow[r, "\wedge"] \arrow[d, "(\bar{A},\bar{B})"'] \arrow[dr, "D"] \arrow[d, phantom, shift left=4ex, "\overset{g}{\Leftarrow}"] & \mathscr{T} \arrow[d, "C"] \arrow[d, phantom, shift right=4ex, "\overset{\otimes}{\Rightarrow}"] \\
\mathrm{Cat}\times\mathrm{Cat} \arrow[r, "\times"] & \mathrm{Cat}
\end{tikzcd}$$

of functors and natural transformations which yields a pairing $(B\mathscr{A},B\mathscr{B})\rightarrow B\mathscr{C}$ on passage to classifying spaces.

Step 1. Construction of the categories $\mathscr{D}_{m,n}$: The objects of $\mathscr{D}_{m,n}$ are systems $(A,B,C,\Pi_{r,s})$, where

$$A=\langle A_r,\pi_{r,r'}\rangle,\quad B=\langle B_s,\pi_{s,s'}\rangle,\quad\text{and}\quad C=\langle C_t,\pi_{t,t'}\rangle$$

are objects of $\bar{\mathscr{A}}_m$, $\bar{\mathscr{B}}_n$, and $\bar{\mathscr{C}}_{mn}$ and where $\Pi_{r,s}$ is a nullary and distributive system of isomorphisms $A_r\otimes B_s\rightarrow C_{r\wedge s}$. Here $r\subset\mathbf{m}$ $s\subset\mathbf{n}$, and the precise requirements are that

$$\Pi_{0,s}=0 : A_0\otimes B_s=0\otimes B_s\rightarrow C_0,\quad \Pi_{r,0}=0 : A_r\otimes B_0=A_r\otimes 0\rightarrow C_0,$$

and that the following diagram commutes:

$$\begin{tikzcd}
(A_r\oplus A_{r'})\otimes(B_s\oplus B_{s'}) \arrow[r, phantom, "\cong"] \arrow[d, "\pi_{r,r'}\otimes\pi_{s,s'}"'] & (A_r\otimes B_s)\oplus(A_r\otimes B_{s'})\oplus(A_{r'}\otimes B_s)\oplus(A_{r'}\otimes B_{s'}) \arrow[d, "\Pi_{r,s}\oplus\Pi_{r,s'}\oplus\Pi_{r',s}\oplus\Pi_{r',s'}"] \\
A_{r\cup r'}\otimes B_{s\cup s'} \arrow[d, "\Pi_{r\cup r',s\cup s'}"'] & C_{r\wedge s}\oplus C_{r\wedge s'}\oplus C_{r'\wedge s}\oplus C_{r'\wedge s'} \arrow[d, "\pi_{r\wedge(s\cup s'),r'\wedge(s\cup s')}\circ(\pi_{r\wedge s,r\wedge s'}\oplus\pi_{r'\wedge s,r'\wedge s'})"] \\
C_{(r\cup r')\wedge(s\cup s')} \arrow[r, phantom, "="] & C_{(r\wedge s)\cup(r\wedge s')\cup(r'\wedge s)\cup(r'\wedge s')}
\end{tikzcd}$$

The morphisms of $\mathscr{D}_{m,n}$ are triples (α,β,γ) of morphisms $A\to A'$ in $\bar{\mathscr{A}}_m$, $B\to B'$ in $\bar{\mathscr{B}}_n$, and $C\to C'$ in $\bar{\mathscr{C}}_{mn}$ which are strictly compatible with the $\Pi_{r,s}$.

Step 2. Construction of functors $(\phi,\psi) : \mathscr{D}_{m,n}\to\mathscr{D}_{p,q}$: Let $\phi : \mathbf{m}\to\mathbf{p}$ and $\psi : \mathbf{n}\to\mathbf{q}$ be morphisms of $\mathscr{F}\times\mathscr{F}$. Define

$$(\phi,\psi)(A,B,C,\Pi_{r,s})=(\phi A,\psi B,(\phi\wedge\psi)(C),A_{u,v}),$$

where $A_{u,v}=\Pi_{\phi^{-1}(u),\psi^{-1}(v)}$ for $u\subset\mathbf{p}$ and $v\subset\mathbf{q}$. Here $\phi^{-1}(u)$ is to be interpreted as $\{0\}\cup\{i\,|\,\phi(i)\in u-\{0\}\}$ and similarly for $\psi^{-1}(v)$. Define

$$(\phi,\psi)(\alpha,\beta,\gamma)=(\phi\alpha,\psi\beta,(\phi\wedge\psi)(\gamma)).$$

The definition makes sense in view of Step 2 of [18, Const. 10], and it is easily seen that these data specify a well-defined functor $D : \mathscr{F}\times\mathscr{F}\to\mathrm{Cat}$.

Step 3. Construction of $g : D\to\bar{A}\times\bar{B}$ and $f : D\to\bar{C}\circ\wedge$: The requisite (m,n)th functors $\mathscr{D}_{m,n}\to\bar{\mathscr{A}}_m\times\bar{\mathscr{B}}_n$ and $\mathscr{D}_{m,n}\to\bar{\mathscr{C}}_{mn}$ are specified by

$$(A,B,C,\Pi_{r,s})\mapsto(A,B),\quad(\alpha,\beta,\gamma)\mapsto(\alpha,\beta)$$

and

$$(A,B,C,\Pi_{r,s})\mapsto C,\qquad(\alpha,\beta,\gamma)\mapsto\gamma.$$

Naturality with respect to morphisms of $\mathscr{F}\times\mathscr{F}$ is clear. Construct a functor $\nu(m,n) : \bar{\mathscr{A}}_m\times\bar{\mathscr{B}}_n\to\mathscr{D}_{m,n}$ by sending (A,B) to $(A,B,C,\Pi_{r,s})$ and (α,β) to (α,β,γ), where C and γ are obtained from (A,B) and (α,β) via the composite functor

$$\bar{\mathscr{A}}_m\times\bar{\mathscr{B}}_n\xrightarrow{\delta(m)\times\delta(n)}\mathscr{A}^m\times\mathscr{B}^n\xrightarrow{\otimes(m,n)}\mathscr{C}^{mn}\xrightarrow{\nu(mn)}\bar{\mathscr{C}}_{mn}.$$

Here $\delta(m) : \bar{\mathscr{A}}_m\to\mathscr{A}^m$ and $\nu(m) : \mathscr{A}^m\to\bar{\mathscr{A}}_m$ are the inverse equivalences defined in Step 2 of [18, Const. 10]; the specification of the $\Pi_{r,s}$ is dictated by the conditions of Step 1. Then $g(m,n)\nu(m,n)=\mathrm{Id}$, and the $\Pi_{r,s}$ of general objects determine a natural equivalence $\xi(m,n) : \mathrm{Id}\to\nu(m,n)g(m,n)$, just as in [18, Const. 10].

Passage from $\mathscr{A}$ to $\bar{A}$ is only functorial on strict morphisms of permutative categories, and passage from $\mathscr{A}\times\mathscr{B}\to\mathscr{C}$ to $(\bar{A},\bar{B})\to\bar{C}$ is only functorial on morphisms of pairings given by natural isomorphisms. Thus use of this construction results in considerable loss of information. One could go on to treat unit, associativity, and commutativity conditions in terms of $\bar{A}$, but one would only end up with more complicated proofs of weaker results than in the body of the paper.

Instead, we use the following elaboration of [18, Const. 10] to obtain a direct comparison between $\bar{A}$ and $\tilde{A}$. Recall the lax functor $A : \mathscr{F} \to \mathrm{Cat}$ of Section 4.

Proposition A.3. *Let $\mathscr{A}$ be a permutative category. There are lax natural transformations $\delta : \bar{A} \to A$ and $\nu : A \to \bar{A}$ such that $\delta\nu = \mathrm{Id} : A \to A$ and there is a natural homotopy $\xi : \mathrm{Id} \to \nu\delta$ of lax natural transformations such that each $\xi(n)$ is a natural isomorphism of functors $\bar{\mathscr{A}}_n \to \bar{\mathscr{A}}_n$.*

Proof. The component functors $\delta(n) : \bar{\mathscr{A}}_n \to \mathscr{A}^n$ and $\nu(n) : \mathscr{A}^n \to \bar{\mathscr{A}}_n$ and the natural isomorphisms $\xi(n)$ are specified in [18, Const. 10]. For $\phi : \mathbf{m} \to \mathbf{n}$ in $\mathscr{F}$, the requisite natural transformations

$$\delta(\phi) : \mathscr{A}(\phi)\delta(m) \to \delta(n)\bar{\mathscr{A}}(\phi) \quad \text{and} \quad \nu(\phi) : \bar{\mathscr{A}}(\phi)\nu(m) \to \nu(n)\mathscr{A}(\phi)$$

are specified on objects $\langle A_s, \pi_{s,t} \rangle$ of $\bar{\mathscr{A}}_m$ and $(A_1, \ldots, A_m)$ of $\mathscr{A}^m$ by the isomorphisms

$$\sum_{\phi(i)=j} A_i \to A_{\{0\} \cup \phi^{-1}(j)} \quad \text{and} \quad \sum_{\phi(i) \in t - \{0\}} A_i \to \sum_{j \in t - \{0\}} \sum_{\phi(i)=j} A_i$$

given by the system $\{\pi_{s,t}\}$ and by the commutativity isomorphism of $\mathscr{A}$, respectively. Verification that the relevant coherence diagrams commute is left as an exercise.

Corollary A.4. *There is a natural transformation $\tilde{A} \to \bar{A}$ of functors $\mathscr{F} \to \mathrm{Cat}$ such that each $\tilde{\mathscr{A}}_n \to \bar{\mathscr{A}}_n$ induces an equivalence on passage to classifying spaces.*

Proof. With $\bar{\bar{A}}$ as an intermediary, this is immediate from Theorem 3.4; $\varepsilon\bar{\nu}$ is the required transformation.

Of course, there results an equivalence $EB\tilde{\mathscr{A}} \to EB\bar{\mathscr{A}}$ on passage to spectra.
We have a similar comparison on the level of pairings.

Proposition A.5. *Let $\otimes : \mathscr{A} \times \mathscr{B} \to \mathscr{C}$ be a pairing of permutative categories. Then there is a lax natural transformation $\nu : \bar{A} \times \bar{B} \to D$ (between actual functors $\mathscr{F} \times \mathscr{F} \to \mathrm{Cat}$) such that the left square commutes and the right square commutes up to natural homotopy in the diagram*

$$\begin{tikzcd}
A \times B \arrow[r, equal] \arrow[dd, "\nu \times \nu"'] & A \times B \arrow[r, "\otimes"] \arrow[d, "\nu \times \nu"] & C \circ \wedge \arrow[dd, "\nu\wedge"] \\
 & \bar{A} \times \bar{B} \arrow[d, "\nu"] & \\
\bar{A} \times \bar{B} & D \arrow[l, "g"'] \arrow[r, "f"] & \bar{C} \circ \wedge
\end{tikzcd}$$

Proof. Step 3 of Construction A.2 gave functors $\nu(m, n)$. Natural transformations

$$\nu(\phi, \psi) : D(\phi, \psi)\nu(m, n) \to \nu(p, q)(\bar{A}(\phi) \times \bar{B}(\psi))$$

(of functors $\bar{\mathscr{A}}_m \times \bar{\mathscr{B}}_n \to \mathscr{D}_{p,q}$ for ϕ : $\mathbf{m} \to \mathbf{p}$ and ψ : $\mathbf{n} \to \mathbf{q}$) are easily obtained. Indeed, they are dictated by requiring

$$g \circ v = \mathrm{Id} \quad \text{and} \quad f \circ v = v\wedge \circ \otimes \circ (\delta \times \delta)$$

as lax natural transformations between functors $\mathscr{F} \times \mathscr{F} \to \mathrm{Cat}$. With this specification, the result follows from Proposition A.3.

Corollary A.6. *There is a natural diagram of functors*

$$\begin{tikzcd}
\tilde{A} \times \tilde{B} \arrow[r, equal] \arrow[d] & \tilde{A} \times \tilde{B} \arrow[r, no head, "\bar{\otimes}"] \arrow[d] & \bar{C} \circ \wedge \arrow[d] \\
\bar{A} \times \bar{B} & D \arrow[l, "g"'] \arrow[r, "f"] & \bar{C} \circ \wedge
\end{tikzcd}$$

such that the left square commutes, the right square commutes up to natural homotopy, and the vertical arrows induce equivalences on passage to classifying spaces.

Proof. This is immediate from Theorem 3.4 and Lemmas 3.5 and 3.6.

Naturality refers to strict morphisms of pairings of permutative categories. We conclude that our results proved about the spectra $EB\tilde{\mathscr{A}}$ carry over to their equivalents $EB\bar{\mathscr{A}}$. Thus the generalized definition of pairing in Definition A.1 serves no useful purpose in the usual categorical applications. However, it is necessary in applications to étale homotopy theory, and for this reason it is worth going on to a generalization of our recognition principle. Since the generalization presents only notational complications, we shall be rather sketchy. We begin by generalizing the material of Section 5.

Definition A.7. An $\mathscr{I}_* \times \mathscr{I}_*$-prespectrum is a continuous functor $R : \mathscr{I}_* \times \mathscr{I}_* \to \mathscr{T}$ together with a natural transformation

$$\sigma : R \wedge S \wedge S \to R \circ (\oplus \times \oplus) \circ (1 \times t \times 1)$$

of functors $\mathscr{I}_*^4 \to \mathscr{T}$ such that σ is appropriately unital and associative and the adjoints

$$\tilde{\sigma} : R(V, V') \to F(SW \wedge SW', R(V \oplus W, V' \oplus W'))$$

are inclusions; compare Definition 5.1.

We can define $\mathscr{I}_*^n$-prespectra similarly; the case $n=3$ is needed for the study of associativity of pairings. The smash product of $\mathscr{I}_*$-prespectra P and Q is an $\mathscr{I}_* \times \mathscr{I}_*$-prespectrum with respect to the structural maps

$$(\sigma \wedge \sigma)(1 \wedge \tau \wedge 1) : PV \wedge QV' \wedge SW \wedge SW' \to P(V \oplus W) \wedge Q(V' \oplus W').$$

If T is an $\mathscr{I}_*$-prespectrum, then $T\circ\oplus$ is an $\mathscr{I}_*\times\mathscr{I}_*$-prespectrum with respect to the structural maps

$$T(1\oplus\tau\oplus 1)\circ\sigma\circ(1\wedge\omega) : T(V\oplus V')\wedge SW\wedge SW'\to T(V\oplus W\oplus V'\oplus W').$$

This allows the following generalization and conceptualization of Definition 5.3.

Definition A.8. A (generalized) pairing $(P,Q)\to T$ of $\mathscr{I}_*$-prespectra is an $\mathscr{I}_*\times\mathscr{I}_*$-prespectrum R and maps

$$\psi : R\to P\wedge Q \quad \text{and} \quad \omega : R\to T\circ\oplus$$

of $\mathscr{I}_*\times\mathscr{I}_*$-prespectra such that each component map

$$\psi(V,W) : R(V,W)\to PV\wedge QW$$

is an equivalence. Morphisms of pairings are defined in the evident way in terms of homotopy commutative diagrams of functors.

There is an analogous generalization of Definition 5.4, involving auxiliary $\mathscr{I}_*\times\mathscr{I}_*$-prespectra for the unit diagrams and an auxiliary $\mathscr{I}_*^3$-prespectrum for the associativity diagram. We leave the details to the interested reader.

Theorem A.9. *Theorems* 5.7 *and* 5.8 *remain valid as stated with respect to the generalized notions of a pairing of* $\mathscr{I}_*$*-prespectra and of a ring* $\mathscr{I}_*$*-prespectrum.*

Proof. For any pair of universes U and U', an $\mathscr{I}_*\times\mathscr{I}_*$-prespectrum determines a prespectrum indexed on $U\oplus U'$ in an evident and natural way. In particular, a pairing as above gives a diagram

$$P\wedge Q \xleftarrow{\psi} R \xrightarrow{\omega} T\circ\oplus$$

of prespectra indexed on $R^\infty\oplus R^\infty$, with ψ a spacewise equivalence. We pass to spectra indexed on $R^\infty\oplus R^\infty$ and then to spectra indexed on R^∞ just as in Section 5. Here ψ is a weak equivalence, hence an isomorphism in the stable category, and there results the required pairing $EP\wedge EQ\to ET$ in $H\mathscr{S}$. Unit, associativity, and commutativity conditions are also handled just as in Section 5, the only complications being purely notational.

Theorem A.10. *Theorems* 6.2 *and* 6.3, *and therefore also Theorems* 1.6 *and* 1.7, *remain valid as stated with respect to the generalized notions of a pairing of a* $\hat{\mathscr{C}}$*-space and a* $\hat{\mathscr{D}}$*-space to an* $\hat{\mathscr{E}}$*-space and of a ring* $\hat{\mathscr{C}}$*-space.*

Proof. On general nonsense grounds, the category $\hat{\mathscr{C}}\times\hat{\mathscr{D}}$ determines a monad $\hat{C}\times\hat{D}$ in the category $(\Pi\times\Pi)[\mathscr{T}]$ of functors $M : \Pi\times\Pi\to\mathscr{T}$ such that an action of $\hat{C}\times\hat{D}$ on M is the same thing as an extension of M to a continuous functor $\hat{\mathscr{C}}\times\hat{\mathscr{D}}\to\mathscr{T}$. The (p,q)th space is a coend

$$(\hat{C}\times\hat{D})_{p,q}(M) = \coprod_{(m,n)} \hat{\mathscr{C}}(\mathbf{m},\mathbf{p})\times\hat{\mathscr{D}}(\mathbf{n},\mathbf{q})\times M_{m,n}/(\sim),$$

and the equivalence relation is such that

$$(\hat{C}\times\hat{D})_{p,q}(X\times Y)=\hat{C}_pX\times\hat{D}_qY$$

for functors $X, Y : \Pi\to\mathscr{T}$. Let L' and L'' be the functors which assign $M_{0,*} : \Pi\to\mathscr{T}$ and $M_{*,0} : \Pi\to\mathscr{T}$ to $M : \Pi\times\Pi\to\mathscr{T}$. Then the functor (L', L'') has the evident product construction as right adjoint. With notations as in Section 6, the functor

$$\wedge\circ(\Sigma^VL, \Sigma^WL)=\Sigma^VL\wedge\Sigma^WL : \Pi[\mathscr{T}]\times\Pi[\mathscr{T}]\to\mathscr{T}$$

has an evident right action by $(\hat{C}_V, \hat{D}_W)$, and it follows formally that the functor

$$\Sigma^VLL'\wedge\Sigma^WLL'' : (\Pi\times\Pi)[\mathscr{T}]\to\mathscr{T}$$

has a right action by $\hat{C}_V\times\hat{D}_W$. For a functor $M : \hat{\mathscr{C}}\times\hat{\mathscr{D}}\to\mathscr{T}$, there results an $\mathscr{I}_*\times\mathscr{I}_*$-prespectrum RM with (V, W)th space

$$(RM)(V, W)=B(\Sigma^VLL'\wedge\Sigma^WLL'', \hat{C}_V\times\hat{D}_W, M).$$

When $M=X\times Y$, there is a natural homeomorphism

$$R(X\times Y)(V, W)\cong(TX)(V)\wedge(TY)(W).$$

Now consider a pairing $(X, Y)\to Z$ as in Definition A.1. The transformations g and f induce maps

$$\begin{tikzcd}\Sigma^VL\hat{C}^q_VX\wedge\Sigma^WL\hat{D}^q_WY & (\Sigma^VLL'\wedge\Sigma^WLL'')(\hat{C}_V\times\hat{D}_W)^q(M) \arrow[l] \arrow[d] \\ & \Sigma^{V\oplus W}L\hat{E}^q_{V\oplus W}Z.\end{tikzcd}$$

These specify maps of simplicial spaces and so induce maps

$$(TX)(V)\wedge(TY)(W)\leftarrow(RM)(V, W)\to(TZ)(V\oplus W)$$

on passage to realization. These maps specify a pairing of $\mathscr{I}_*$-prespectra in the sense of Definition A.8 (the cofibration condition of Definition A.1 being needed to ensure that the first map is an equivalence). The remaining verifications are exactly parallel to those in Section 6.

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