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THE COHOMOLOGY STRUCTURE OF CERTAIN FIBRE SPACES—I

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§1. INTRODUCTION

LET $p : E \to B$ be a fibre bundle with fibre F and group G, and let $i : F \to E$ denote the inclusion map. This paper is a contribution to the following long outstanding problem: Express the structure of the cohomology of E in terms of that of B and F and invariants of the bundle structure.

Of course an answer or a partial answer to this question can sometimes be obtained by use of the spectral sequence. However, even in cases where the spectral sequence is trivial (i.e. the fibre is totally non-homologous to zero and $E_2 = E_\infty$) it may be difficult or impossible to determine the multiplicative structure or the action of the Steenrod algebra on $H^*(E, Z_q)$, although the additive structure is completely determined. It is this aspect of the problem with which we are mainly concerned.

The problem is further complicated by the fact that $H^*(E, Z_q)$ has at least three different algebraic structures which are important: It is an algebra over Z_q, it is a module over the Steenrod algebra $\mathscr{A}_q$, and a module over the algebra $H^*(B, Z_q)$ via the induced homomorphism p^*. (Actually, it is more interesting to consider it as a module over the quotient algebra $R = H^*(B)/\text{kernel } p^*$).

Moreover, these different structures are not independent of each other; they must satisfy various identities. We list explicitly all these conditions and call any algebraic object which satisfies all of them an *unstable* $\mathscr{A}_q$*–R-algebra*. We also define the closely related notion of an *unstable* $\mathscr{A}_q$*–R-module*. Once these definitions are fixed, it is clear how to define the *free* unstable $\mathscr{A}_q$–R-algebra generated by a given unstable $\mathscr{A}_q$–R-module. These ideas may also be looked on as rather natural generalizations of some basic concepts introduced by Steenrod and Epstein [8]: namely, given any *unstable* module X over the Steenrod algebra $\mathscr{A}_q$, there is defined the *free* algebra, $U(X)$, over $\mathscr{A}_q$ generated by X.

Our main theorem then asserts that under "appropriate" conditions, $H^*(E, Z_2)$ is the free unstable $\mathscr{A}_2$–R-algebra generated by a certain unstable $\mathscr{A}_2$–R-module $M \subset H^*(E, Z_2)$.

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Thus in these cases, questions involving the structure of $H^*(E, Z_2)$ as an algebra are reduced to questions about the $\mathscr{A}_2$–R-module M. The "appropriate" conditions just referred to are the following: (a) $H^*(F, Z_2)$ must be the free algebra, $U(X)$, generated by some $\mathscr{A}_2$-module $X \subset H^*(F, Z_2)$, with elements of X transgressive. (b) The ideal in $H^*(B, Z_2)$ generated by the image of X under transgression satisfies a certain condition which has been considered by algebraists for many years (since Kronecker, 1882). (c) In the universal bundle with fibre F and group G, the projection induces an epimorphism of the cohomology of the base space onto that of the total space (Z_2 coefficients). This last condition seems less natural than the first two. However, it is automatically satisfied in the case of principal bundles ($F = G$), and also holds in many other cases.

As Steenrod and Epstein point out, condition (a) holds if G is one of the classical groups, and F is an associated Stiefel manifold. It also holds in case $G = F$ is a product of Eilenberg–MacLane spaces, and in various other cases; it is of fairly wide occurrence. Condition (b) holds in case the fibre F is totally non-homologous to zero in E (mod 2) and in certain other important cases. Thus the hypotheses of our main theorems are satisfied in many important cases.

Another of our achievements is to subsume the theory of $\mathscr{A}_q$–R-modules under the usual theory of modules over a single algebra, denoted by the symbol $R \odot \mathscr{A}_q$. Thus no elaborate new theory is needed to study these modules. The algebra $R \odot \mathscr{A}_q$ is called the *semi-tensor product of the algebras* R and $\mathscr{A}_q$ (with respect to the given operations of $\mathscr{A}_q$ on R) and it bears the same relation to the ordinary tensor product of R and $\mathscr{A}_q$ that the semi-direct product of two groups bears to the ordinary direct product. This notion should be of interest in its own right. In the appendix we give examples of the occurrence of this concept in classical algebra.†

As is often the case, this study raises several new problems. The most obvious one is the following: In the cases where $H^*(E, Z_2) =$ the free unstable $R \odot \mathscr{A}_2$-algebra generated by a sub-module M, how does one determine the structure of M? In many cases, M is an extension of two known modules over a rather complicated ring (the semi-tensor product, $R \odot \mathscr{A}_2$). Thus one has the problem of finding suitable invariants of the bundle (E, p, B, F) to determine this unknown module extension. We hope to consider this problem in a future paper.

The authors are deeply indebted to Dr. Edward D. Davis for information on the topics in §7, and to the referee for suggestions for improving the presentation of this paper.

§2. THE SEMI-TENSOR PRODUCT OF ALGEBRAS

Throughout this paper (except in Appendix II) all modules and algebras will be graded by the non-negative integers and assumed to be "locally finite". In this and the next two sections, we work over the fixed ground field Z_p (p prime). Let A be a Hopf algebra over Z_p with *commutative, associative* diagonal map $\psi : A \to A \otimes A$ [cf. Steenrod, [7] or [8] for the

† The notion of semi-tensor product has been developed independently by J. P. Meyer.

precise definitions]. As Steenrod points out in [7], §7, this is precisely the additional structure needed to convert the tensor product (over Z_p) of left A-modules into an A-module such that the usual natural equivalences for the functors "*Hom*" and "$\otimes$" remain true. If M and N are left A-modules, then $M \otimes N$ (tensor product over Z_p) is also a left A-module via the diagonal map ψ, and if $f : M \to M'$ and $g : N \to N'$ are homomorphisms of left A-modules, then $f \otimes g : M \otimes N \to M' \otimes N'$ is also a homomorphism of left A-modules.

Following Steenrod (*loc. cit.*), if R is both an algebra over Z_p and a left A-module, we will say that R is an *algebra over the Hopf algebra A* if the multiplication $\mu : R \otimes R \to R$ and the "unit" $\eta : Z_p \to R$ are homomorphisms of left A-modules (where A operates on Z_p via the augmentation $\varepsilon : A \to Z_p$). A natural generalization of this notion is the following: Let R be an algebra over the Hopf algebra A, and let M be a left module over *both* of the rings R and A. We will say that M is *an $A - R$ module* if the map $R \otimes M \to M$ defining the R-module structure is a homomorphism of left A-modules. In terms of elements, this condition can be written as follows: For any $\alpha \in A$, $r \in R$, and $m \in M$,

$$\alpha(rm) = \sum_i (-1)^{(\deg r)(\deg \alpha_i'')} (\alpha_i' r)(\alpha_i'' m) \tag{2.1}$$

where $\psi(\alpha) = \sum_i \alpha_i' \otimes \alpha_i''$.

As we shall see later, A–R modules are fairly common objects, especially in algebraic topology. As a trivial example, if R is an algebra over the Hopf algebra A, then R, considered as a left R-module in the usual way, is also an A–R module. A less trivial example is the following: Let X be a topological space, Y a subspace, and $\mathscr{A}_p$ the Steenrod algebra (mod p). Then $H^*(X, Y, Z_p)$ is an $\mathscr{A}_p - H^*(X, Z_p)$ module, with respect to the usual definitions. The following lemma, which will be needed shortly, gives an easy way to multiply the number of examples:

LEMMA (2.2). *Let A and R be as above. If M is a left A–R module, and N a left A-module, then $M \otimes N$ is a left A–R module in a natural way.*

Proof. Let $\mu : R \otimes M \to M$ be the map defining the R-module structure on M and $1 : N \to N$ the identity map. Since μ and 1 are homomorphisms of left A-modules, so is

$$\mu \otimes 1 : R \otimes M \otimes N \longrightarrow M \otimes N.$$

This defines a structure of left R-module, and hence A–R module, on $M \otimes N$.

It would clearly be desirable to be able to subsume the theory of A–R modules under the usual theory of modules over a single algebra. We will now show how this may be done. Our development of the ideas involved will be heuristic, since this has some advantages in this case.

Let M be an A–R module and $\mathscr{E}(M) = Hom_{Z_p}(M, M)$ the graded algebra (over Z_p) of all homogeneous Z_p-endomorphisms of M. The R-module structure and A-module structure on M determine homomorphisms

$$\varphi : R \longrightarrow \mathscr{E}(M),$$
$$\varphi' : A \longrightarrow \mathscr{E}(M)$$

of Z_p-algebras with unit. Define a Z_p-linear map

$$\Phi : R \otimes A \longrightarrow \mathscr{E}(M)$$

by the formula

$$\Phi(r \otimes a) = (\varphi r)(\varphi' a).$$

The following question now presents itself: Is it possible to define a multiplication in $R \otimes A$ such that Φ is a homomorphism of algebras (for any A–R module M)? If the answer to this question is "Yes", then any A–R module is also a module over $R \otimes A$ (with this new multiplication).

It is readily seen that to define such a multiplication in $R \otimes A$, one should set

(2.3) $$(r \otimes a)(s \otimes b) = \sum_i (-1)^{(\deg s)(\deg a_i')} r(a_i's) \otimes (a_i''b)$$

for any $r, s \in R$ and $a, b \in A$; here $\psi(a) = \sum_i a_i' \otimes a_i''$, as before. In terms of diagrams, this multiplication is a linear map $(R \otimes A) \otimes (R \otimes A) \to R \otimes A$ which is the composition of four homomorphisms, as indicated by the following diagram:

(2.4) $$\begin{array}{c} R \otimes A \otimes R \otimes A \\ \Big\downarrow {\scriptstyle 1 \otimes \psi \otimes 1 \otimes 1} \\ R \otimes A \otimes A \otimes R \otimes A \\ \Big\downarrow {\scriptstyle 1 \otimes 1 \otimes T \otimes 1} \\ R \otimes A \otimes R \otimes A \otimes A \\ \Big\downarrow {\scriptstyle 1 \otimes \mu \otimes 1 \otimes 1} \\ R \otimes R \otimes A \otimes A \\ \Big\downarrow {\scriptstyle m_R \otimes m_A} \\ R \otimes A. \end{array}$$

In this diagram, the symbol "1" denotes an identity map, $T : A \otimes R \to R \otimes A$ is the "twist", $\mu : A \otimes R \to R$ defines the A-module structure on R, and m_R and m_A define multiplications in R and A respectively.

It is clear that the multiplication thus defined in $R \otimes A$ is distributive and has a 2-sided unit, $1 \otimes 1$. It is not obvious that the multiplication is associative; since Φ is a homomorphism of algebras, and $\mathscr{E}(M)$ is an associative algebra, if we can choose M such that Φ is a monomorphism, it will follow that the multiplication is associative. But such a choice is readily at hand: take $M = R \otimes A$ (that $R \otimes A$ is an R–A-module follows by setting $M = R$ and $N = A$ in Lemma (2.2)). Then one has

$$\left[\Phi\left(\sum_i r_i \otimes a_i\right)\right](1 \otimes 1) = \sum_i r_i[a_i(1 \otimes 1)]$$
$$= \sum_i r_i(1 \otimes a_i) = \sum_i r_i \otimes a_i.$$

Hence kernel $\Phi = \{0\}$, as required.

DEFINITION (2.5). *The semi-tensor product of R and A, denoted by $R \odot A$, is the Z_p-algebra which has $R \otimes A$ as its underlying Z_p-vector space and its multiplication defined by* (2.3) *or* (2.4) *above.*

The reason for calling this algebra "the semi-tensor product" is that it bears the same relation to the ordinary tensor product of algebras that the semi-direct product of groups bears to the usual direct product; see Appendix I, where this relation is further developed, and relations to the semi-direct product of Lie algebras are established.

Note that the map $R \to R \odot A$ defined by $r \to r \otimes 1$ is a monomorphism of Z_p-algebras, as is the map $A \to R \odot A$ defined by $a \to 1 \otimes a$. Let M be an A–R module. Then M is an $R \odot A$-module if we define $(r \otimes a)m = r(am)$. Conversely, if M is an $R \odot A$-module, then M is an A–R module using the above imbeddings. Furthermore, a map of A–R-modules is linear over both A and R if and only if it is linear over $R \odot A$. Thus we have achieved our goal of subsuming the theory of A–R modules under the theory of modules over a single algebra.

Next, suppose that A and R are as above, M is a left A and right R-module such that the structure map $\mu : M \otimes R \to M$ is a homomorphism of (left) A-modules, and N is a left A–R-module with structure map $\nu : R \otimes N \to N$. Then we assert that $M \otimes_R N$ *has a natural structure of left A-module*. This follows from the fact that $M \otimes_R N$ is (by definition) the cokernel of the map

$$\mu \otimes 1 - 1 \otimes \nu : M \otimes R \otimes N \longrightarrow M \otimes N$$

which is clearly a homomorphism of left A-modules. One can carry this sort of thing even further, as follows: Let S and T also be algebras over A. Assume that M is an S–R bimodule and N is an R–T bimodule (in symbols, ${}_SM_R$ and ${}_RN_T$) such that all four structure maps

$$S \otimes M \longrightarrow M, \qquad \mu : M \otimes R \longrightarrow M,$$
$$\nu : R \otimes N \longrightarrow N, \qquad N \otimes T \longrightarrow T,$$

are homomorphisms of left A-modules. Then it is well known that $M \otimes_R N$ is an S–T-bimodule (cf. MacLane, [5], p. 143); in this case, on account of the additional hypotheses, one sees immediately that both of the structure maps of this S–T bimodule are homomorphisms of left A-modules.

An important special case occurs if R is commutative; then the distinction between right and left R-modules disappears, and any R-module is also an R–R bimodule. Thus we have the following fact which we record for later use:

LEMMA (2.6). *Let R be a commutative, associate algebra over the Hopf algebra A. If M and N are left $R \odot A$-modules, then $M \otimes_R N$ has a natural structure of left $R \odot A$-module.*

§3. THE FREE ALGEBRA $U_R(M)$

For the rest of this paper, we take for A the mod p Steenrod algebra $\mathscr{A}_p$. The operations of $\mathscr{A}_p$ on $H^*(X)$ satisfy further properties than those embodied in the definition of an algebra over the Hopf algebra $\mathscr{A}_p$. Namely, for the case $p = 2$,

(3.1) $$Sq^n x = 0 \quad \text{if} \quad \deg x < n, \qquad \text{and}$$

(3.2) $$Sq^n x = x^2 \quad \text{if} \quad \deg x = n.$$

When p is an odd prime, we have

(3.3) $$\mathscr{P}^n x = 0 \quad \text{if} \quad \deg x < 2n,$$

(3.4) $$\delta^* \mathscr{P}^n x = 0 \quad \text{if} \quad \deg x = 2n, \qquad \text{and}$$

(3.5) $$\mathscr{P}^n x = x^p \quad \text{if} \quad \deg x = 2n.$$

The following definitions are modelled on Steenrod and Epstein, [8].

DEFINITION (3.6). *A module M over $\mathscr{A}_p$ is called an unstable module over $\mathscr{A}_p$ if* (3.1) *holds when $p = 2$ or if* (3.3) *and* (3.4) *hold when p is an odd prime. An algebra R over $\mathscr{A}_p$ is called an unstable algebra over $\mathscr{A}_p$ if it is an unstable module over $\mathscr{A}_p$ and if* (3.2) *holds when $p = 2$ or if* (3.5) *holds when p is an odd prime.*

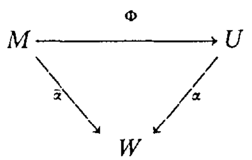
For the rest of this section, let R be a commutative, unstable algebra with unit over the Hopf algebra $\mathscr{A}_p$.

DEFINITION (3.7). *An algebra over $R \odot \mathscr{A}_p$ is an $R \odot \mathscr{A}_p$-module which is an algebra over R (in the classical sense) and over $\mathscr{A}_p$ (in the sense of Steenrod). An $R \odot \mathscr{A}_p$-module M is unstable if it is unstable as an $\mathscr{A}_p$-module. An algebra over $R \odot \mathscr{A}_p$ is unstable if it is unstable as an algebra over $\mathscr{A}_p$.*

DEFINITION (3.8). *Let M be an $R \odot \mathscr{A}_p$-module. A base point for M is an $R \odot \mathscr{A}_p$-homomorphism $\eta : R \to M$. If M is an $R \odot \mathscr{A}_p$-algebra, we also require that $\eta(1)$ be the unit for M.*

Steenrod and Epstein [8] have defined the free $\mathscr{A}_p$-algebra generated by an $\mathscr{A}_p$-module. We now generalize this definition to $R \odot \mathscr{A}_p$-modules.

DEFINITION (3.9). *Let M be an unstable $R \odot \mathscr{A}_p$-module with base point. A free $R \odot \mathscr{A}_p$ algebra generated by M is a pair (U, Φ), where U is a commutative unstable $R \odot \mathscr{A}_p$-algebra with base point and $\Phi : M \to U$ is an $R \odot \mathscr{A}_p$-homomorphism preserving the base point, such that the following "universal mapping condition" is satisfied: for any commutative unstable $R \odot \mathscr{A}_p$-algebra W with base point and any $R \odot \mathscr{A}_p$-homomorphism $\alpha : M \to W$ preserving the base point, there exists a unique $R \odot \mathscr{A}_p$-homomorphism of algebras, $\bar{\alpha} : U \to W$, preserving the base point such that*



is commutative.

It is easy to prove that (U, Φ) is unique up to isomorphism if it exists. In order to prove existence, one could appeal to the general existence theorem of P. Samuel [6]; however, we will give a more constructive proof. For $n > 0$ let $\otimes^n_R(M)$ denote the tensor product over R of n copies of M. By Lemma (2.6), $\otimes^n_R(M)$ is an $R \odot \mathscr{A}_p$-module. By [8; p. 27] it is an unstable module. Let $\otimes^*_R(M) = \sum_{n \geqq 0} \otimes^n_R(M)$, where $\otimes^0_R(M) = R$, an $R \odot \mathscr{A}_p$-algebra which is unstable as an $R \odot \mathscr{A}_p$-module. Let D be the ideal in $\otimes^*_R(M)$ generated by all elements of the form $xy - (-1)^{\deg x \cdot \deg y} yx$. Clearly D is an $R \odot \mathscr{A}_p$-submodule. Define $\mathscr{S}^*_R(M) = \otimes^*_R(M)/D$, and let $\mathscr{S}^n_R(M)$ denote the image of $\otimes^n_R(M)$ in $\mathscr{S}^*_R(M)$. $\mathscr{S}^*_R(M)$ is a commutative $R \odot \mathscr{A}_p$-algebra which is unstable as an $R \odot \mathscr{A}_p$-module. Clearly $\mathscr{S}^0_R(M) = R$ and $\mathscr{S}^1_R(M) = M$. Let $\eta : R \to M$ denote the base point in M. Let E be the ideal in $\mathscr{S}^*_R(M)$ generated by $1 - \eta(1)$. E is an $R \odot \mathscr{A}_p$-submodule, hence $\mathscr{S}^*_R(M)/E$ is also an $R \odot \mathscr{A}_p$-algebra. Define $\Phi_0 : M \to \mathscr{S}^*_R(M)/E$ by the natural map $M = \mathscr{S}^1_R(M) \subset \mathscr{S}^*_R(M) \to \mathscr{S}^*_R(M)/E$. Φ_0 is an $R \odot \mathscr{A}_p$-homomorphism preserving the base point. Define $\lambda : M \to M$ as follows. If $p = 2$, $\lambda | M^n = Sq^n$. If p is an odd prime, $\lambda | M^{2n+1} = 0$ and $\lambda | M^{2n} = \mathscr{P}^n$. Let F denote the ideal in $\mathscr{S}^*_R(M)/E$ generated by all elements of the form $\Phi_0(\lambda(m)) - (\Phi_0(m))^p$ for $m \in M$. The proof given in [8; p. 28] applies without change to prove that F is closed under $\mathscr{A}_2$. An analogous proof applies for p an odd prime. Define $U = \mathscr{S}^*_R(M)/E/F$ and $\Phi : M \to U$ as the composition of Φ_0 with the canonical map $\mathscr{S}^*_R(M)/E \to U$. It is readily seen that (U, Φ) has the required properties.

We will suppress the map Φ from our notation and denote by $U_R(M)$ the free $R \odot \mathscr{A}_p$-algebra generated by M. Clearly $U_R(\text{—})$ is a covariant functor. When $R = R^0 = Z_2$, $U_R(M)$ reduces to the notion defined by Steenrod and Epstein [8].

PROPOSITION (3.10). *Let M be an unstable $R \odot \mathscr{A}_p$-module with base point η. Let N_1 and N_2 be sub $R \odot \mathscr{A}_p$-modules such that $N_1 \cap N_2 = \operatorname{Im} \eta$ and $M = N_1 + N_2$. Then $U_R(M)$ is naturally isomorphic to $U_R(N_1) \otimes_R U_R(N_2)$.*

The proof of Proposition (3.10) is straightforward and left to the reader. Let M be an unstable $R \odot \mathscr{A}_p$-module with base point $\eta : R \to M$. Let M' denote $M \oplus M$ with the two base points identified. Then the diagonal $M \to M \oplus M$ induces an $R \odot \mathscr{A}_p$-homomorphism $M \to M'$. Since U_R is a covariant functor, we obtain the following corollary.

COROLLARY (3.11). *$U_R(M)$ has a diagonal map $\psi : U_R(M) \to U_R(M) \otimes_R U_R(M)$ which is an $R \odot \mathscr{A}_p$-homomorphism. In particular, $U_{Z_2}(M)$ is a Hopf algebra over Z_2.*

Of course $U_R(M)$ may have other diagonal maps. The map ψ is a canonical one, independent of any choices.

We need the following weak form of a structure theorem for the topological applications. The theorem is a straightforward consequence of the construction of $U_R(M)$. There is an analogous result for p an odd prime which we omit.

THEOREM (3.12). *Let $p = 2$ and let M be an unstable $R \odot \mathscr{A}_2$-module. Let $b_0 = \eta(1)$, $b_1, \ldots$ be a set of homogeneous generators for M as an R-module. Then the monomials $b_{i_1} \ldots b_{i_k}$, $0 < i_1 < i_2 < \ldots < i_k$ together with 1 generate $U_R(M)$ as an R-module.*

The following theorem is a more precise structure theorem for $U_R(M)$. Since we do not need this theorem for the topological applications, we postpone the proof until the appendix. In the topological applications, this result is obtained as a corollary free of charge.

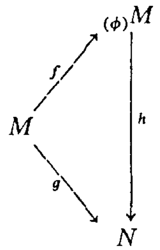
THEOREM (3.13). *Let $p = 2$ and let M be an unstable $R \odot \mathscr{A}_2$-module with base point. Let $b_0 = \eta(1)$, $b_1, \ldots, b_n$ be a homogeneous basis for M as an R-module. Then $b_1, \ldots, b_n$ is a simple system of generators* (in the sense of A. Borel [2]) *for $U_R(M)$ as an algebra over R.*

§4. COVARIANT ϕ-EXTENSIONS

The notion of the covariant ϕ-extension of a module is due to Cartan and Eilenberg [3; p. 28]. It provides the algebraic machinery for our discussion of naturality in §5. We first review the definition and then state various properties.

Let Λ and Γ be associative rings with unit and let $\phi : \Lambda \to \Gamma$ be a ring homomorphism. If M and N are Λ- and Γ-modules respectively, we say that $f : M \to N$ is semilinear with respect to ϕ if f is a Λ-homomorphism where Λ operates on N via ϕ.

DEFINITION (4.1). *Let M be a left Λ-module and let $\phi : \Lambda \to \Gamma$ be a ring homomorphism. A covariant ϕ-extension of M is a pair consisting of a left Γ-module, ${}_{(\phi)}M$, and a semi-linear map $f : M \to {}_{(\phi)}M$ such that the following "universal mapping property" holds: if $g : M \to N$ is a semi-linear map, then there exists a unique Γ-homomorphism $h : {}_{(\phi)}M \to N$ such that the diagram*



is commutative.

Clearly any two covariant ϕ-extensions are naturally isomorphic. To prove existence we take ${}_{(\phi)}M = \Gamma \otimes_\Lambda M$, and define $f : M \to {}_{(\phi)}M$ by $f(m) = 1 \otimes_\Lambda m$.

The proofs of the following propositions are easy and are omitted.

PROPOSITION (4.2). *If $f : M \to N$ is a Λ-homomorphism, there is defined an induced Γ-homomorphism ${}_{(\phi)}f : {}_{(\phi)}M \to {}_{(\phi)}N$.* This defines an additive, right-exact, covariant functor from the category of left Λ-modules to the category of left Γ-modules.

PROPOSITION (4.3). *If M is a free left Λ-module with basis $\{b_i\}$, then a semi-linear map $f : M \to {}_{(\phi)}M$ defines a covariant ϕ-extension if and only if $\{f(b_i)\}$ is a basis for ${}_{(\phi)}M$ as a left Γ-module.*

PROPOSITION (4.4). *Assume that Λ and Γ are graded, commutative rings. Let A be an algebra over Λ. Then ${}_{(\phi)}A$ is an algebra over Γ in a canonical way. If A is associative, has a unit, or is commutative, then so is ${}_{(\phi)}A$.*

PROPOSITION (4.5). *Assume that Λ and Γ are algebras over the Hopf algebra $\mathscr{A}_p$ and* *that* $\phi : \Lambda \to \Gamma$ *is an* $\mathscr{A}_p$*-homomorphism. If* M *is an* $\mathscr{A}_p$*-*Λ*-module, then* ${}_{(\phi)}M$ *is an* $\mathscr{A}_p$*-*Γ*-module in a canonical way. Furthermore, if* M *and* Γ *are unstable, so is* ${}_{(\phi)}M$.

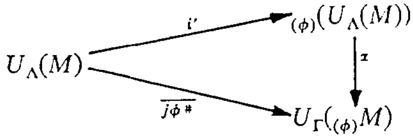
The proof uses the coassociativity of $\mathscr{A}_p$.

COROLLARY (4.6). *Let* $\bar{\phi} : \Lambda \odot \mathscr{A}_p \to \Gamma \odot \mathscr{A}_p$ *be the homomorphism induced by* ϕ. *Then* ${}_{(\phi)}M \approx {}_{(\bar{\phi})}M$; i.e. $(\Gamma \odot \mathscr{A}_p) \otimes_{\Lambda \odot \mathscr{A}_p} M \approx \Gamma \otimes_\Lambda M$.

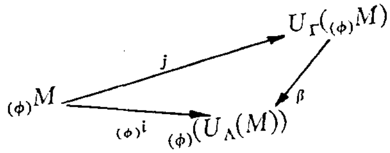
COROLLARY (4.7). *If* A *is an unstable algebra over* $\Lambda \odot \mathscr{A}_p$, *then* ${}_{(\phi)}A$ *is an unstable algebra over* $\Gamma \odot \mathscr{A}_p$.

THEOREM (4.8). *Let* Λ *and* Γ *be unstable algebras over the Hopf algebra* $\mathscr{A}_p$ *and let* $\phi : \Lambda \to \Gamma$ *be an* $\mathscr{A}_p$*-homomorphism. Let* M *be an unstable* $\Lambda \odot \mathscr{A}_p$*-module with base point. Then* ${}_{(\phi)}(U_\Lambda(M))$ *is naturally isomorphic to* $U_\Gamma({}_{(\phi)}M)$ *as unstable algebras over* $\Gamma \odot \mathscr{A}_p$.

Proof. Let $i : M \to U_\Lambda(M)$, $j : {}_{(\phi)}M \to U_\Gamma({}_{(\phi)}M)$, $\phi^\# : M \to {}_{(\phi)}M$. Then $j\phi^\# : M \to U_\Gamma({}_{(\phi)}M)$ is an $\Lambda \odot \mathscr{A}_p$-homomorphism. Let $\overline{j\phi^\#} : U_\Lambda(M) \to U_\Gamma({}_{(\phi)}M)$ be the unique extension; it is semi-linear with respect to ϕ. Let $\alpha : {}_{(\phi)}(U_\Lambda(M)) \to U_\Gamma({}_{(\phi)}M)$ be the unique Γ-homomorphism such that the diagram



s commutative. Furthermore, ${}_{(\phi)}(U_\Lambda(M))$ is an unstable $\Gamma \odot \mathscr{A}_p$-algebra by corollary (4.7) and ${}_{(\phi)}j : {}_{(\phi)}M \to {}_{(\phi)}(U_\Lambda(M))$ is a $\Gamma \odot \mathscr{A}_p$-homomorphism. Hence there is a unique $\Gamma \odot \mathscr{A}_p$-homomorphism $\beta : U_\Gamma({}_{(\phi)}M) \to {}_{(\phi)}(U_\Lambda(M))$ such that



is commutative. Using standard diagram chasing techniques, one easily proves that α and β are isomorphisms and inverses of one another.

§5. THE MAIN THEOREM

We now apply the algebraic notions developed so far to the study of the cohomology structure of the total space of certain fibre spaces. We will assume $p = 2$ throughout this and the next two sections.

Let $\xi = (E, p, B, F, G)$ be a fibre bundle with fibre F and group G. Let $\xi_0 = (E_0, p_0, B_0, F, G)$ denote the universal bundle with fibre F and group G. Following are the three basic assumptions on F and G, and we assume these throughout this section.

(i) There is an $\mathscr{A}_2$-module X such that $H^*(F) = U(X)$ and the elements of X are transgressive in the universal bundle.

(ii) The local coefficient system defined by $H^*(F)$ is trivial in the universal bundle.

(iii) $p_0^* : H^*(B_0) \to H^*(E_0)$ is an epimorphism and Ker p_0^* is the ideal generated by the image of X under transgression.

These conditions are fulfilled, for example, in the following five cases, the first of which will be treated more thoroughly in §6.

(1) $F = V_{n,r} = 0(n)/0(n-r)$ and $G = 0(n)$,
(2) $F = U(n)/U(n-r)$ and $G = U(n)$,
(3) $F = Sp(n)/Sp(n-r)$ and $G = Sp(n)$,
(4) $F = 0(2n)/U(n)$ and $G = 0(2n)$,
(5) $F = G =$ a product of Eilenberg–MacLane spaces.†

We need the following notions in order to state our assumptions on the particular bundle ξ. Let S be a graded algebra over Z_2. A sequence of homogeneous elements $s_1, \ldots, s_n, \ldots$ (finite or infinite) is called an S-sequence if s_{i+1} is not a zero divisor in the quotient algebra $S/(s_1, \ldots, s_i)$, where $(s_1, \ldots, s_i)$ denotes the ideal generated by $s_1, \ldots, s_i$. An ideal $I \subset S$ is called a *Borel ideal*‡ if there is an S-sequence $s_1, \ldots, s_n, \ldots$ such that $(s_1, \ldots) = I$ and dim $s_i > 0$.

Now given ξ, let $\xi_T = (E_T, p_T, B_T, T(F), G)$ denote the bundle associated to ξ with fibre $T(F)$, the cone on F. (E_T is the mapping cylinder of p.) Note that p_T^* is an isomorphism. Let $k : E \to E_T$ and $\bar{k} : E_T \to (E_T, E)$ be the inclusions. Let $l : H^*(E_T, E) \to H^*(B)$ be defined by $l = (p_T^*)^{-1}\bar{k}^*$ and note that $p^* = k^* p_T$. Hence, from the exact sequence of the pair (E_T, E) we obtain the exact sequence:

$$\ldots \longrightarrow H^q(E_T, E) \xrightarrow{\ l\ } H^q(B) \xrightarrow{\ p^*\ } H^q(E) \xrightarrow{\ \delta\ } H^{q+1}(E_T, E) \longrightarrow \ldots \tag{5.1}$$

Since p_T^* is an isomorphism, $H^*(E_T, E)$, $H^*(B)$, and $H^*(E)$ are modules over $H^*(B)$. By the results of §2, they are all $H^*(B) \odot \mathscr{A}_2$-modules. Furthermore, l, p^* and δ are $H^*(B) \odot \mathscr{A}_2$-homomorphisms. Let $f : B \to B_0$, $g : E \to E_0$, and $h : (E_T, E) \to (E_{0T}, E_0)$ be the classifying maps for ξ. We have the following commutative diagram.

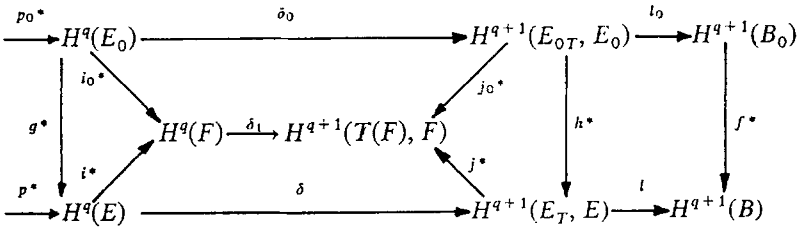


FIG. 1.

† Some applications of this case will be given in a future paper.

‡ These notions are quite old, at least in the case of polynomial rings, cf. the Cambridge Tract by F. S. Macauley entitled *The Algebraic Theory of Modular Systems* (1916) and Kronecker's paper in *Crelle's J.* **92** (1882), especially p. 80, where a Borel ideal is called an "ideal of the principal class". For a summary of recent work on local rings which uses these notions together with complete references, see MacLane, [5], Chapter VII. These ideas occur in §11 of Borel's thesis [2] in a context similar to ours. In Appendix 6 of Vol. II of *Commutative Algebra* by Zariski and Samuel, S-sequences are called "prime sequences". This seems to be a better name, but it has not been widely adopted.

Since it is not obvious at first sight, it should be pointed out that the homomorphisms in Fig. 1 determine the transgression. To be precise:

LEMMA (5.1). *The transgression in the bundle* (E, p, B, F) *is defined by* $-l(j^*)^{-1}\delta_1$.

Proof. Consider the diagram in Fig. 2. By definition, the transgression is determined

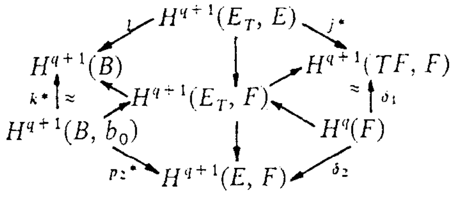


FIG. 2.

by $k^*(p_2^*)^{-1}\delta_2$. It follows from the hexagonal lemma that the two ways of going around this diagram are the negatives of each other; hence the lemma.

Let $M'(\xi)$ denote the $H^*(B)$-submodule of $H^*(E_T, E)$ generated by $h^*(H^*(E_{0T}, E_0))$. Since $H^*(E_{0T}, E_0)$ is an $\mathscr{A}_2$-module, $M'(\xi)$ is an $\mathscr{A}_2$-submodule and hence an $H^*(B)\odot\mathscr{A}_2$-submodule of $H^*(E_T, E)$.

Define $M(\xi) = M'(\xi) \cap \delta(H^*(E))$. $M(\xi)$ is an $H^*(B)\odot\mathscr{A}_2$-submodule. Ker p^* operates trivially on $\delta(H^*(E))$ and hence on $M(\xi)$. Thus $M(\xi)$ is a module over $R = H^*(B)/\text{Ker } p^*$. Clearly R is an algebra over the Hopf algebra $\mathscr{A}_2$ and hence $M(\xi)$ is an $R\odot\mathscr{A}_2$-module. Define $N(\xi) = \delta^{-1}(M(\xi))$. It is obvious that $0 \to R \xrightarrow{p^*} N(\xi) \xrightarrow{\delta} M(\xi) \to 0$ is an exact sequence of unstable $R\odot\mathscr{A}_2$-modules, and p^* is a base point for $N(\xi)$.

Let $\alpha : N(\xi) \to H^*(E)$ be the inclusion. Let $\bar{\alpha} : U_R(N(\xi)) \to H^*(E)$ be the induced homomorphism of algebras over $R\odot\mathscr{A}_2$ (see 3.9). Our main theorem is the following; it will be proved in §8.

THEOREM (5.2). *Let ξ be a fibre bundle such that F and G satisfy* (i), (ii), *and* (iii) *above. Also, assume that the ideal in $H^*(B)$ generated by the image of X under transgression is a Borel ideal. Then $\bar{\alpha} : U_R(N(\xi)) \to H^*(E)$ is an isomorphism.*

§6. SOME EXAMPLES OF APPLICATIONS OF THEOREM (5.2)

Before giving the proof of Theorem (5.2), we discuss some special cases in more detail for the sake of illustration. Let $F = V_{n,r}$, the Stiefel manifold of r-frames in R^n, and let $G = 0(n)$. Let $\{h_{i-1}\}$, $n - r < i \leqq n$, be the standard simple system of generators for $H^*(V_{n,r})$ (cf. [2], Proposition (10.3). Let X be the Z_2-module generated by $\{h_{i-1}\}$. Then $H^*(V_{n,r}) = U(X)$ by [8], chap. IV, §6. The universal bundle ξ_0 is $(B0(n-r), p_0, B0(n), V_{n,r}, 0(n)$. $H^*(B0(n))$ is a polynomial ring on the universal Stiefel–Whitney classes, W_i, $i = 1, \ldots, n$, and p_0^* is an epimorphism with Ker p_0^* the ideal generated by W_i, $i = n - r + 1, \ldots, n$. Since $\tau(h_{i-1}) = W_i$, $n - r < i \leqq n$, hypotheses (i), (ii), and (iii) are satisfied.

Since l is a monomorphism, there exist unique classes $U_i \in H^i(B0(n-r)_T, B0(n-r))$, $n - r < i \leqq n$ such that $l(U_i) = W_i$. Using the Wu formulae, we find that the action of $\mathscr{A}_2$ on U_i is given by

(6.1) $$Sq^t(U_i) = \sum_{s=0}^{t} \binom{i-t+s-1}{s} W_{t-s} \cdot U_{i+s}.$$

Given $\xi = (E, p, B, V_{n,r}, 0(n))$, define $U_i(\xi) = h^*(U_i) \in H^i(E_T, E)$, $n - r < i \leqq n$.

We now assume that ξ is a fibre bundle with fibre totally non-homologous to zero. This is equivalent to the condition that $f^*(W_i) = W_i(\xi) = 0$, $n - r < i \leqq n$ or to the condition that $p^*: H^*(B) \to H^*(E)$ is a monomorphism. In this case the image of X under transgression is $\{0\}$ and $\{0\}$ is a Borel ideal. Thus Theorem (5.2) applies. Here $R = H^*(B)$, $M(\xi) = M'(\xi)$ = the $H^*(B)$-submodule of $H^*(E_T, E)$ generated by $\{U_i(\xi)\}$, $n - r < i \leqq n$. Let $a_{i-1} \in N^{i-1}(\xi) \subset H^{i-1}(E)$ be such that $\delta(a_{i-1}) = U_i(\xi)$, $n - r < i \leqq n$. It follows from the proof of Theorem (5.2) that $\{1, a_{i-1}\}$ is an $H^*(B)$-basis for $N(\xi)$, that $\{U_i(\xi)\}$ is an $H^*(B)$-basis for $M(\xi)$, and that an $H^*(B)$-basis for $H^*(E)$ is given by 1 and monomials $a_{i_1} \ldots a_{i_k}$, $n - r \leqq i_1 < \ldots < i_k < n$. (6.1) gives the structure of $M(\xi)$ as an $H^*(B) \odot \mathscr{A}_2$-module. Hence to complete our knowledge of the structure of $H^*(E)$ as an $H^*(B) \odot \mathscr{A}_2$-algebra, we need only study the extension in the exact sequence

(6.2) $$0 \longrightarrow H^*(B) \xrightarrow{p^*} N(\xi) \xrightarrow{\delta} M(\xi) \longrightarrow 0.$$

This is an exact sequence of $H^*(B) \odot \mathscr{A}_2$-modules which splits over $H^*(B)$. In a subsequent paper, we hope to study such extensions in detail.

Using the techniques discussed in §4, we now discuss the naturality properties in this case.

Let $\xi = (E, p, B, V_{n,r}, 0(n))$ and $\xi' = (E', p', B', V_{n,r}, 0(n))$ and let $f: \xi \to \xi'$ be a bundle map. Assume ξ' (and hence ξ) has a totally non-homologous to zero fibre. Clearly $\hat{f}^*: M(\xi') \to M(\xi)$ and $f_E^*: N(\xi') \to N(\xi)$, and thus we have a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^*(B') & \xrightarrow{p'^*} & N(\xi') & \xrightarrow{\delta'} & M(\xi') & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_B^*} & & \downarrow{\scriptstyle f_E^*} & & \downarrow{\scriptstyle \hat{f}^*} & & \\ 0 & \longrightarrow & H^*(B) & \xrightarrow{p^*} & N(\xi) & \xrightarrow{\delta} & M(\xi) & \longrightarrow & 0. \end{array}$$

LEMMA (6.1). *f_E^* and $\hat{f}^*$ are covariant (f_B^*)-extensions.*

Proof. $\hat{f}^*$ is a covariant (f_B^*)-extension because $M(\xi')$ is a free $H^*(B')$-module on $U_i(\xi')$ and $M(\xi)$ is a free $H^*(B)$-module on $\hat{f}^*(U_i(\xi')) = U_i(\xi)$. f_E^* is a covariant (f_B^*)-extension because we can choose $a_{i-1} = f_E^*(a'_{i-1})$.

COROLLARY (6.2). *$f_E^*: H^*(E') \to H^*(E)$ is a covariant (f_B^*)-extension.*

Proof. This follows immediately from Theorem (5.2), Theorem (4.8), and Lemma (6.1).

COROLLARY (6.3). *$H^*(E) \approx H^*(B) \otimes_{H^*(B')} H^*(E')$ as an $H^*(B) \odot \mathscr{A}_2$-algebra.*

A similar discussion could be given in case the group $0(n)$ was replaced by the unitary group, $U(n)$, or the symplectic group, $Sp(n)$, with the corresponding Stiefel manifold as fibre. There is not as much interest in these cases, however.

Next, we consider a totally different kind of application of Theorem (5.2).

Let $f : Y \to X$ be a continuous mapping of topological spaces such that the kernel of the induced homomorphism $f^* : H^*(X, Z_2) \to H^*(Y, Z_2)$ is a Borel ideal in $H^*(X, Z_2)$. Let $p : E \to X$ be a principal fibre space over X with fibre a product of Eilenberg–MacLane spaces such that kernel $p^* =$ kernel f^*; it is well known that it is always possible to exactly "kill off" the kernel of f^* by such a fibre space construction. It then follows automatically from Theorem (5.2) that $H^*(E, Z_2)$ is a free unstable $R \odot \mathscr{A}_2$-algebra, $U_R(N)$, generated by some sub-module N (here $R = H^*(B)/\text{kernel } f^*$). This result is true independent of the choices of Eilenberg–MacLane spaces which compose the fibre.

§7. A THEOREM ON THE SPECTRAL SEQUENCE OF CERTAIN FIBRE SPACES

In this section we prove a theorem which shows that under certain hypotheses it is possible to determine the successive terms $E_2, E_3, \ldots$ of the spectral sequence of a fibre space from the knowledge of the transgression. While this theorem is more or less "known", it has never appeared in print in a form convenient for our purposes. It should be of independent interest.

Let S be a commutative algebra with unit over Z_2. Let $x_i \in S$, $i = 1, \ldots, n$. Let $\wedge(u_1, \ldots, u_n)$ denote the exterior algebra on $u_1, \ldots, u_n$ over Z_2. The *Koszul complex* (E, d) over S is defined to be $E = S \otimes \wedge(u_1, \ldots, u_n)$ with $d : E \to E$ defined by $d(u_i) = x_i$, $d/S = 0$. E is graded by setting edeg $S = 0$ and edeg $u_i = 1$, where "edeg" denotes "exterior degree". Clearly $d^2 = 0$ and it is easy to check that $H_0(E) = S/(x_1, \ldots, x_n)$ and $H_q(E) = 0$ for $q > n$.

PROPOSITION (7.1). *If $x_1, \ldots, x_n$ is an S-sequence, then $H_q(E) = 0$ for $q > 0$.*

In general, the converse of this proposition is not true. However, we do have the following partial converse:

PROPOSITION (7.2). *If S is a graded commutative algebra with all degrees $\geqq 0$, degree $x_i > 0$ for $1 \leqq i \leqq n$, and $H_q(E) = 0$ for $q > 0$, then $(x_1, \ldots, x_n)$ is an S-sequence.*

COROLLARY. *Under the hypotheses of Proposition (7.2), the property of $x_1, \ldots, x_n$ being an S-sequence is independent of the order of the x_i's.*

The proof of these two propositions is based on the following construction (cf. Exercise 3 on p. 218 of [5]). Let

$$E(k) = S \otimes \wedge(u_1, \ldots, u_k), \qquad 1 \leqq k \leqq n.$$

Then $E(k)$ is a sub-complex of $E(k+1)$, hence we have an exact sequence

$$(S_1) \ldots \xrightarrow{\partial_*} H_q(E(k-1)) \xrightarrow{i_*} H_q(E(k)) \xrightarrow{j_*} H_q(E(k)/E(k-1)) \xrightarrow{\partial_*} \ldots$$

Note that $E(k-1)$ is *not* an ideal in $E(k)$, only a sub-S-module.

Next, note that we have

$$E(k) = E(k-1) \oplus E(k-1) \cdot u_k$$

(direct sum of S-modules). Hence we have an isomorphism of S-modules,

$$E(k-1) \xrightarrow[\approx]{} E(k)/E(k-1)$$

defined by $x \to$ coset of $(x \cdot u_k)$ for any $x \in E(k-1)$. Moreover, this isomorphism is an isomorphism of complexes, although it shifts degrees by one unit. By making use of this isomorphism, the exact sequence (S_1) is transformed into the following exact sequence of S-modules:

$$(S_2) \ldots \xrightarrow{\mu} H_q(E(k-1)) \xrightarrow{i^*} H_q(E(k)) \longrightarrow H_{q-1}(E(k-1)) \xrightarrow{\mu} \ldots$$

and it is readily verified that the homomorphism $\mu : H_q(E(k-1)) \to H_q(E(k-1))$ is defined by $\mu(a) = a \cdot x_k$ for any $a \in H_q(E(k-1))$.

The exact sequence (S_2) suffices to prove Proposition (7.1) by induction on k.

To prove Proposition (7.2), we now give $E = S \otimes \wedge(u_1, \ldots, u_n)$ a different graded structure as follows. S is assumed to already have a non-trivial graded structure. Give the exterior algebra $\wedge(u_1, \ldots, u_n)$ a graded structure by the rule

$$\text{degree } u_i = \text{degree } x_i - 1, \qquad 1 \leqq i \leqq n.$$

Then give the tensor product E a graded structure according to the usual rule for tensor products. This new degree will be called the *total degree* and denoted by a superscript. Note that the differentiation d has total degree $+1$.

These two graded structures on E (defined by the exterior degree and the total degree) are compatible with one another in an obvious sense.

The exact sequences (S_1) and (S_2) are now exact sequences of graded S-modules, graded by the total degree. We have

$$\text{total degree } \mu = \text{degree } x_k > 0.$$

It follows that if $H_q(E(k-1)) \neq 0$, then $\mu : H_q(E(k-1)) \to H_q(E(k-1))$ is *not* an epimorphism, therefore $H_q(E(k)) \neq 0$. In other words, if $H_q(E(k)) = 0$, then $H_q(E(k-1)) = 0$.

Applying this argument in the case where $H_q(E(k)) = 0$ for all $q > 0$, we see that the exact sequence (S_2) reduces to the following:

$$\begin{array}{ccccccccc} 0 \longrightarrow & H_0(E(k-1)) & \xrightarrow{\mu} & H_0(E(k-1)) & \longrightarrow & H_0(E(k)) & \longrightarrow 0 \\ & \| & & \| & & & \\ & S/(x_1, \ldots, x_{k-1}) & & S/(x_1, \ldots, x_{k-1}) & & & \end{array}$$

Thus we see that μ is a monomorphism. We can now use this argument for $k = n, n-1, n-2, \ldots$ in succession to conclude that $(x_1, \ldots, x_n)$ is an S-sequence, as required.

We now generalize Propositions (7.1) and (7.2). Let A', A, and S be graded algebras over Z_2. Let $A' \subset A$ and assume that $y_1, \ldots, y_n$ is a simple system of generators for A as an algebra over A'. Let $x_i \in S$, $i = 1, \ldots, n$, with dim $x_i > 0$. Assume given $d : S \otimes A \to S \otimes A$ such that $d|A' = 0$, $d(y_i) = x_i$, and $d|S = 0$.

PROPOSITION (7.3). *If $x_1, \ldots, x_n$ is an S-sequence, then* $H(S \otimes A) \approx \dfrac{S}{(x_1, \ldots, x_n)} \otimes A'$ *as an algebra over Z_2 and as a module over S.*

Proof. Define $\phi : S \otimes A \to S \otimes \wedge(u_1, \ldots, u_n) \otimes A'$, a map of S-modules, by $\phi(s \otimes$

$a'y_{i_1} \ldots y_{i_k}) = s \otimes u_{i_1} \wedge \ldots \wedge u_{i_k} \otimes a'$. ϕ is an isomorphism and commutes with the differentiations, Hence $\phi_* : H(S \otimes A) \to H(S \otimes \wedge (u_1, \ldots, u_n) \otimes A')$ is an isomorphism. However, $H(S \otimes \wedge (u_1, \ldots, u_n) \otimes A') \approx H(S \otimes \wedge (u_1, \ldots, u_n)) \otimes A' \approx \dfrac{S}{(x_1, \ldots, x_n)} \otimes A'$, the latter isomorphism by Proposition (7.1). Let $\psi_1 : S \otimes A' \to S \otimes A$ and $\psi_2 : S \otimes A' \to S \otimes \wedge (u_1, \ldots, u_n) \otimes A'$ be the obvious maps. Clearly, $\psi_2 = \phi \psi_1$. Let $S \otimes A'$ have the trivial differentiation. Then $\psi_{2*} = \phi_* \psi_{1*}$ with ψ_{1*} and ψ_{2*} epimorphisms of algebras. Hence ϕ_* is an isomorphism of algebras.

We now prove the main theorem of this section.

THEOREM (7.4). *Let $p : E \to B$ be a fibre space with fibre F and let $i : F \to E$.* Assume

(a) *in the mod 2 spectral sequence,* $E_2^{p,q} \approx H^p(B) \otimes H^q(F)$,

(b) $H^*(F) = U(X)$ *for some $\mathscr{A}_2$-module X, and X is transgressive,*

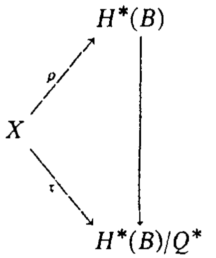
(c) *the ideal in $H^*(B)$ generated by the image of X under transgression is a Borel ideal.*

Then (α) $E_r^{p,q} = E_r^{p,0} \otimes E_r^{0,q} 2 \leqq r \leqq \infty$,

(β) Im $i^* = U(X')$, *where $X' \subset X$ is the kernel of the transgression (restricted to X). and* $X' = X \cap \text{Im } i^*$,

(γ) Ker p^* *is the ideal in* (c) *above.*

Proof. Let $\tau : X \to H^*(B)/Q^*$ denote the transgression restricted to X. Here Q^* is the subgroup by which $H^*(B)$ must be factored so that τ is well defined. Let $\rho : X \to H^*(B)$ be a Z_2-homomorphism such that



is commutative. Let $X' = \text{Ker } \tau$. We now prove the following three conditions by induction on $r \geqq 2$:

(i) $E_r^{p,q} = E_r^{p,0} \otimes E_r^{0,q}$,

(ii) $E_r^{0,*} = U\left(X' + \sum_{q=r-1}^{\infty} X^q\right)$,

(iii) $E_r^{*,0} = H^*(B)/(\rho(X^1 + \ldots + X^{r-2}))$.

The case $r = 2$ is true by hypothesis. Assume true for r. $E_{r+1}^{p,q} = H(E_r^{p,q})$ under d_r. By (i), d_r is determined by $d_r | E_r^{0,*}$. By (ii), d_r is determined by $\tau \Big| \left(X' + \sum_{q=r-1}^{\infty} X^q\right)$. By hypothesis (c), the ideal in $E_r^{*,0}$ generated by $\tau(X^{r-1})$ is a Borel ideal. We now apply Proposition (7.3) with $A = U(X' + \sum_{q=r-1}^{\infty} X^q)$, $A' = U(X' + \sum_{q=r}^{\infty} X^q)$, $S = E_r^{*,0}$, and $\{x_i\} = \{\tau(z_i)\}$, where z_i is a Z_2-basis for X^{r-1} mod X'. Hence, $E_{r+1}^{*,*} = \frac{E_r^{*,0}}{(\tau(X^{r-1}))} \otimes U\left(X' + \sum_{q=r}^{\infty} X^q\right)$. To complete the induction, we note that $E_r^{*,0}/(\tau(X^{r-1})) = H^*(B)/(\rho(X^i + \ldots + X^{r-1}))$. Condition ($\alpha$) is now proved by induction on r. Conditions (β) and (γ) follow easily by letting $r \to \infty$. Q.E.D.

§8. PROOF OF THEOREM (5.2)

First of all, we apply Theorem (7.4) to conclude that $E_\infty^{p,q} = E_\infty^{p,0} \otimes E_\infty^{0,q}$ and Im $i^* = U(X')$, where $X' = \text{Ker } \tau$. Hence $H^*(E)$ is a free R-module with basis obtained as follows: let $\{b_i\}$ be a Z_2-basis for Im i^*. Let $b_i' \in H^*(E)$ be such that $i^*(b_i') = b_i$. Then $\{b_i'\}$ is an R-basis for $H^*(E)$. This follows from the fact that $E_\infty^{p,q} = E_\infty^{p,0} \otimes E_\infty^{0,q}$. The rest of the proof will be devoted to showing the existence of a set of elements $\{z_i\} \subset H^*(E)$ such that

(I) $\{i^*(z_i)\}$ is a Z_2-basis for X' and

(II) $\{\delta(z_i)\}$ is a set of generators for $M(\xi)$ as an R-module. Theorem (5.2) follows from these two properties as follows: $\{z_i\} \cup \{1\}$ is an R-basis for $N(\xi)$ by (I) and (II) and hence the non-repeating monomials in $\{z_i\}$ and 1 generate $U_R(N(\xi))$ as an R-module by Theorem (3.12). Also, from (I) it follows that $\{i^*(z_i)\}$ is a simple system of generators for Im $i^* = U(X')$ over Z_2. Thus the non-repeating monomials† in $\{z_i\}$ and 1 give an R-basis for $H^*(E)$. Thus $\bar{\alpha} : U_R(N(\xi)) \to H^*(E)$ sends a system of generators over R onto an R-basis in a 1–1 manner, proving that $\bar{\alpha}$ is an isomorphism.

Choose generators $\{v_i'\}$ for Ker p^* as an ideal such that $v_1', v_2', \ldots$ is an $H^*(B)$-sequence. Let $v_i \in H^*(E_T, E)$ be such that $l(v_i) = v_i'$. Let W be the $H^*(B)$-submodule of $H^*(E_T, E)$ generated by $\{v_i\}$. We claim that $H^*(E_T, E) = \text{Im } \delta \oplus W$, a direct sum as $H^*(B)$-modules. Clearly Im δ and W generate $H^*(E_T, E)$, since $l(W) = \text{Ker } p^*$. We now show that Im $\delta \cap W = \{0\}$. Let $a = \sum_{i=1}^{k} b_i \cdot v_{q_i} \in W \cap \text{Im } \delta$, where $b_i \in H^*(B)$ and $b_i \neq 0$. We prove a contradiction by induction on k. If $k = 1$, then $l(a) = b_1 \cdot v_{q_1}' = 0$ with $b_1 \neq 0$ and $v_{q_1}' \neq 0$, which is a contradiction as v_{q_1}' is not a zero divisor. Assume that we have proved that $l(a) = 0$ implies $a = 0$ if $k < n$. Let $a = \sum_{i=1}^{n} b_i \cdot v_{q_i}$. Assume $l(a) = 0 = \sum_{i=1}^{n} b_i \cdot v_{q_i}'$. Since v_{q_n}' is not a zero divisor in

$H^*(B)/(v_{q_1}', \ldots, v_{q_{n-1}}')$, $b_n = \sum_{i=1}^{n-1} c_i \cdot v_{q_i}$. Hence $\sum_{i=1}^{n-1} (b_i + c_i v_{q_n}') \cdot v_{q_i}' = 0$. By induction,

$$\sum_{i=1}^{n-1} (b_i + c_i v_{q_n}') \cdot v_{q_i} = 0 = \sum_{i=1}^{n-1} b_i \cdot v_{q_i}, + \sum_{i=1}^{n-1} c_i v_{q_i}', \cdot v_{q_n} = \sum_{i=1}^{n} b_i \cdot v_{q_i} = a.$$

† Here we are also making use of the following fact which seems to be known, but is not mentioned in the literature: If X is an unstable module over $\mathscr{A}_2$, and $\{b_i\}$ is any Z_2-basis for X, then the b_i's are a simple system of generators for the free algebra $U(X)$. It is not too difficult to give a direct proof.

Since elements of X are transgressive in the universal bundle ξ_0, $\delta_1(X) \subset \operatorname{Im} j_0^*$ in Fig. 1 (cf. Lemma (5.1)). Hence we can choose a Z_2-homomorphism θ_0 of degree 1, $\theta_0 : X \to H^*(E_{0T}, E_0)$, such that $j_0^* \theta_0 = \delta_1 | X$. Since p_0^* is an epimorphism, $\delta_0 = 0$ and l_0 is a monomorphism onto $\operatorname{Ker} p_0^*$. By hypothesis (iii), $H^*(E_{0T}, E_0)$ is generated by $\theta_0(X)$ as an $H^*(B_0)$-module. Define $\theta : X \to H^*(E_T, E)$ by $\theta = h^* \theta_0$. Then $j^* \theta = \delta_1 | X$ and $l\theta(x)$ is a representative of $\tau(x)$ for any $x \in X$. Hence $M'(\xi)$ is the $H^*(B)$-submodule of $H^*(E_T, E)$ generated by $\theta(X)$.

Let $X = X' \oplus X''$ as a Z_2-module, where $X' = \operatorname{Ker} \tau$. Let $\{a_i\}$ be a homogeneous Z_2-basis for X' and $\{c_i\}$ be a homogeneous Z_2-basis for X'', arranged in order of increasing degree. Then $\{l\theta(c_i)\}$ generates $\operatorname{Ker} p^*$ and is an $H^*(B)$-sequence. Hence we can choose $v_i = \theta(c_i)$ and $\{v_i\}$ generates W as an $H^*(B)$-module.

Now let $x' \in X'$. Then $\theta(x') = e + w$, where $e \in \operatorname{Im} \delta$ and $w \in W$. We claim that $j^*(w) = 0$. To prove this claim, let $e = \delta(e')$ for some $e' \in H^*(E)$. Let $w = \Sigma b_i \cdot \theta(c_i)$, $b_i \in H^*(B)$. Then $\theta(x') = \delta(e') + \Sigma b_i \cdot \theta(c_i)$. Apply j^* and we obtain $\delta_1(x') = j^* \theta(x') = j^* \delta(e') + \Sigma \in(b_i) \cdot j^* \theta(c_i) = \delta_1 i^*(e') + \Sigma \in(b_i) \cdot \delta_1(c_i)$, where $\in : H^*(B) \to Z_2$ is the augmentation. Since δ_1 is an isomorphism, we see that $x' - i^*(e') = \Sigma \in(b_i) \cdot c_i$. Since $x' \in X'$ and $\Sigma \in(b_i) \cdot c_i \in X''$, $i^*(e') \in X \cap \operatorname{Im} i^* = X'$ by Theorem (7.3). Thus $x' - i^*(e') = 0 = \Sigma \in(b_i) \cdot c_i$. Therefore $\in(b_i) = 0$ for all i and $j^*(w) = 0$ as required.

For each basis element a_i of X', there exist unique $e_i \in \operatorname{Im} \delta$ and $w_i \in W$ such that $\theta(a_i) = e_i + w_i$ and $j^*(w_i) = 0$. Choose $z_i \in H^*(E)$ such that $e_i = \delta(z_i)$ for each i. Note that $\delta_1 i^*(z_i) = j^* \delta(z_i) = j^*(e_i) = j^*(\theta(a_i) - w_i) = \delta_1(a_i)$. Thus $i^*(z_i) = a_i$ and condition (I) is proven. Since $\theta(X)$ generates $M'(\xi)$ and $\{\theta(c_i)\}$ generates W, it is clear that $W \subset M'(\xi)$. Hence $M'(\xi) = M(\xi) \oplus W$. Therefore $\delta(z_i) = e_i$ generates $M(\xi)$ as an $H^*(B)$-module, and hence as an R-module, proving (II). This completes the proof of Theorem (5.2).

APPENDIX I

The semi-direct product of groups and the semi-direct product (or split extension) of Lie algebras are well established algebraic notions of long standing. It seems that the semi-tensor product of algebras as defined in §2 is the analogous notion in the theory of associative algebras, and it deserves to be considered as one of the basic constructions in the subject. In support of this thesis we present the following examples, all of which are of a classical nature. In all these examples, the grading will be trivial, i.e. every element has degree 0.

Let K be a field and π an abstract group. Let $K(\pi)$ denote the group algebra of π over K. It is easily seen that $K(\pi \times \pi) \approx K(\pi) \otimes_K K(\pi)$. Define $\psi : K(\pi) \to K(\pi) \otimes_K K(\pi)$ by $\psi(x) = x \otimes x$ for $x \in \pi$ and extend linearly. Thus $K(\pi)$ is a Hopf algebra over K; these facts are all well known. Now let π be the *semi-direct* product of π' and π'' with respect to a homomorphism $\alpha : \pi'' \to \operatorname{Aut}(\pi')$. $K(\pi'')$ is a Hopf algebra and the action of π'' on π' determines an action of $K(\pi'')$ on $K(\pi')$ so that $K(\pi')$ is an algebra over the Hopf algebra $K(\pi'')$ in the sense of Steenrod. One can now readily show that $K(\pi) \approx K(\pi') \odot K(\pi'')$.

Let R be a Galois extension of K of degree n with π as the Galois group. Then R is an algebra over the Hopf algebra $K(\pi)$ and $R \odot K(\pi)$ turns out to be a "crossed product with trivial factor set" ([1] Chap. VIII). In this case $R \odot K(\pi)$ is isomorphic to the algebra of all $n \times n$ matrices over K.

As another example, let L be a Lie algebra over K and let $U(L)$ denote the universal enveloping algebra of L. Then $U(L)$ has a natural augmentation [4; p. 268] and a natural diagonal map [4; p. 275] and is thus a Hopf algebra. If L is the semi-direct sum or split extension of L' and L'' [4; p. 17], the action of L'' on L' determines an action of the Hopf algebra $U(L'')$ on $U(L')$. Again, $U(L) \approx U(L') \odot U(L'')$.

Finally, let R be an associative algebra and L a Lie algebra of derivations on R. Then $U(L)$ acts on R and R is an algebra over the Hopf algebra $U(L)$. The algebra $R \odot U(L)$ was used by Jacobson [4; p. 175] to construct the standard complex for the cohomology of a Lie algebra.

APPENDIX II

We give here an outline of the proof of Theorem (3.13). We wish to acknowledge that this proof was suggested by N. Jacobson.

Since $p=2$, R is a commutative ring in the classical sense and we will ignore the grading. With respect to the given basis $b_0, b_1, \ldots, b_n$, we write

$$\text{(i)}\quad \lambda(b_i) = \sum_{j=0}^{n} \alpha_{ij} b_j, \qquad 1 \geqq i \geqq n, \quad \text{and} \quad \lambda(b_0) = b_0.$$

Note that

$$\text{(ii)}\quad \lambda\left(\sum_i r_i b_i\right) = \sum_i r_i^2 \lambda(b_i), \qquad \text{where } r_i \in R.$$

Referring to the construction of $U_R(M)$ in §3, we see that $\mathscr{S}_R^*(M)/E \approx R[b_1, \ldots, b_n]$, a polynomial ring over R and that F is the ideal generated by $\{\lambda(b_i) - b_i^2 | i = 1, \ldots, n\}$. It is clear, by induction, that the set of monomials
$\{b_1^{\varepsilon_1} \ldots b_n^{\varepsilon_n} | \varepsilon_i = 0 \text{ or } 1\}$ generates $R[b_1, \ldots, b_n]/F = U_R(M)$. Thus we must prove that these monomials are independent over R. We first prove it in the special case where R is the algebraic closure of $Z_2(\alpha_{10}, \ldots, \alpha_{nn})$, and the $\alpha_{ij} \in R$ are algebraically independent over Z_2. In this case $\lambda : M \to M$ is monomorphism. If λ were not a monomorphism, it would not be a monomorphism when we "specialized" the α_{ij}. Taking $\alpha_{ij} = \delta_{ij}$ we see that λ is a monomorphism. Now we apply Theorem (13) of [4; p. 192] to conclude that M has a basis $a_0 = b_0 = 1, a_1, \ldots, a_n$ such that $\lambda(a_i) = a_i$, $0 \leqq i \leqq n$. Hence $U_R(M) = R[a_1, \ldots, a_n]/F$, where F is generated by $\{a_i^2 - a_i\}$. The theorem for this case now follows easily by induction on n (cf. [4; chapt. II, §7]). Thus $U_R(M)$ has a basis $\{b_1^{\varepsilon_1} \ldots b_n^{\varepsilon_n} | \varepsilon_i = 0 \text{ or } 1\}$ over R. In terms of this basis one can compute explicit formulas for the constants of multiplication for this algebra (see [4; chapt. I]) by use of (i) and the fact that $b_i^2 = \lambda(b_i)$ in $U_R(M)$. These constants will be certain polynomials in $\{\alpha_{ij}\}$ over Z_2. The fact that $U_R(M)$ is commutative and associative is equivalent to the fact that these polynomials satisfy certain identities. These identities will still be satisfied if we replace the α_{ij}'s by arbitrary elements from a commutative associative algebra with unit over Z_2.

Now let R be arbitrary. We construct an associative, commutative algebra W with unit over R as follows. As an R-module, W is free on symbols $\{b_1^{\varepsilon_1} \dots b_n^{\varepsilon_n} \mid \varepsilon_i = 0 \text{ or } 1\}$. The constants of multiplication for W are the polynomials mentioned above with the indeterminants replaced by elements $\alpha_{ij} \in R$. Then W is a commutative, associative algebra. There is an obvious map of algebras $\beta : R[b_1, \dots, b_n] \to W$. β is an epimorphism and clearly Ker $\beta \supset F$. However, each monomial $b_1^{\varepsilon_1} \dots b_n^{\varepsilon_n}$, $\varepsilon_i = 0$ or 1 in $R[b_1, \dots, b_n]/F$ maps onto a basis element in W and hence $\{b_1^{\varepsilon_1} \dots b_n^{\varepsilon_n} | \varepsilon_i = 0 \text{ or } 1\}$ are independent in $U_R(M)$.

This proof would generalize to prove an analogous basis theorem for p an odd prime in case R had only elements of even degree and M were a free R-module on a finite basis. We conjecture that the theorem is true in the locally finite case with p arbitrary.

REFERENCES

1. E. ARTIN, C. J. NESBITT and R. M. THRALL: *Rings with Minimum Condition*, University of Michigan Press, 1946.
2. A. BOREL: Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lïe compacts. *Ann. Math., Princeton* **57** (1953), 115–207.
3. H. CARTAN and S. EILENBERG: *Homological Algebra*, Princeton University Press, 1956.
4. N. JACOBSON: *Lie Algebras*, Interscience, New York, 1962.
5. S. MACLANE: *Homology*, Springer-Verlag, Berlin, 1963.
6. P. SAMUEL: On universal mappings and free topological groups, *Bull. Amer. Math. Soc.*, **54** (1948), 591–598.
7. N. E. STEENROD: The cohomology algebra of a space, *Enseign. Math.* 7 (1961), 153–178.
8. N. E. STEENROD and D. B. A. EPSTEIN: Cohomology operations, *Ann. Math. Studies Number* 50, Princeton University Press, 1962.

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