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A BP analog of Lin's theorem and the realization of $A(n)^*$

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A BP analog of Lin's Theorem and the Realization of $A(n)^*$

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Curriculum Vitae

Binhua Mao was born in Hubei Province, China. He earned his B.S. in applied mathematics at Changsha Institute of Technology in July 1986 and M.A. in mathematics at Nankai University in July 1989, respectively. Since 1989, he has attended the Ph.D. program in mathematics at the University of Rochester. He received his M.A. in mathematics in May 1991. He worked on his Ph.D. in mathematics under the guidance of his advisor, Professor Douglas Ravenel. He was a teaching assistant and research assistant at the Mathematics Department of the University of Rochester from 1989 to 1994.

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I am greatly indebted to my advisor, Douglas Ravenel, for his advice and guidance. Without his continuous encouragement and valuable suggestions over the years, I could not have finished what I have achieved.

I am especially grateful to him for teaching me not only the mathematical methodology but also the philosophy of dealing with people, i.e. everyone has the right to preserve his or her own interest by all means even at expense of other people's interests. I did not realize that a student and an adviser do not necessarily have a common interest so that I did not successfully protect my own interests from be violated. As a student of his, I am afraid I will never be able to become a Doctor of THIS philosophy like him.

As Douglas points out, promotion is as important as research to the success of a mathematician. Unfortunately I am not brilliant enough to grasp his craftiness in promoting his own mathematics.

Douglas has been the best mentor a student can ask for. I admire him for MORE THAN his mathematical skill.

Abstract

This thesis consists of two parts. The first is about the BP analog of Lin's theorem and the second is about the realization of $A(n)$.

The BP analog of Lin's theorem is actually about the isomorphism between the Novikov E_2-term for S^{-1} and the inverse limit of Novikov E_2-term for the spectra P_{-n}. We construct spectral sequences which converge to the objects we are considering by filtering them I-adically. We can prove that the E_1-terms of the I-adic spectral sequences are isomorphic.

In the second part we find the spectrum whose cohomology is $A(n)^*$ as an A^*-module for any integer n and $p \geq n+3$. We also find a self-map of this spectrum which induces multiplication of v_{n+1} on its BP-cohomology for $p \geq n+4$. We modify Toda's technique to accomplish this goal. It involves BP-theory calcualtions, so we can regard this proof as obtaining ordinary cohomology information from cobordism.

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Chapter 1

Introduction and Main Results

We will consider two problems in this thesis. The first problem is about Lin's theorem, which is due to W. H. Lin. It is the first step to prove Segal's conjecture. The original proof by Lin is complicated and not published. We can find a much simpler proof in [LDMA80]. Here we state Lin's theorem in homology for convenience.

Theorem A ([LDMA80] for $p = 2$, [Gun81] [AGM] for $p > 2$) *There is an isomorphism*

$$\mathrm{Ext}^{s,t}_{\mathcal{A}_*}(\mathbf{Z}/(p), \Sigma^{-1}\mathbf{Z}/(p)) \cong \varprojlim \ \mathrm{Ext}^{s,t}_{\mathcal{A}_*}(\mathbf{Z}/(p), H_*(P_{-n}))$$

which is induced by $\mathcal{A}_*$-comodule homomorphism

$$r_{-n} \doteq (\psi_{-n})_* : \Sigma^{-1}\mathbf{Z}/(p) \longrightarrow H_*(P_{-n})$$

for $n \geq 0$, where $\mathcal{A}_*$ is the dual Steenrod algebra.

Throughout this paper, q denotes $2(p-1)$.

For each $n \geq 0$, there is a spectrum P_{-n} which is closely related to the Thom spectrum of $(-n)$-times of the tautological line bundle over $B\mathbf{Z}/(2) = RP^\infty$ for $p = 2$ and over $B\Sigma_p$ for $p > 2$. The P_{-n}, which we just referred, is denoted by the P_{-nq-1} in [Sad]. These spectra have the following properties:

a) P_{-n} has one cell in each dimension k when $k = k'q$ or $k'q-1$ for $k' \geq -n$;

b) There is a canonical "projection" map $p_{-n} : P_{-n} \to P_{-n+1}$ which collapses cells in dimension $-nq$ and $-nq-1$ to the base point;

c) There exist maps $\psi_{-n} : S^{-1} \to P_{-n}$ for all $n \geq 0$ compatible with the projection maps in b): $p_{-n} \circ \psi_{-n} = \psi_{-n+1}$. In particular ψ_0 is the inclusion of the bottom cell of P_0.

The cohomology of P_{-n} for $p = 2$ is

$$H^*(P_{-n}; \mathbf{Z}/(2)) = \mathbf{Z}/(2)\{x^i : i > -2(n+1);\ |x| = 1\}$$

with action of Steenrod squares given by

$$\mathrm{Sq}^i x^j = \binom{j}{i} x^{i+j} \tag{1.1.1}$$

for all $j \in \mathbf{Z}$, $i \geq 1$.

For $p > 2$, the cohomology of P_{-n} is

$$H^*(P_{-n}; \mathbf{Z}/(p)) = \mathbf{Z}/(p)\{x^i y^{\frac{kq}{2}-i} : k \geq -n;\ i = 0, 1, |x| = 1, |y| = 2\}$$

with the action of reduced powers $\mathcal{P}^i$ and Bockstein operation Q_0 given by

$$\mathcal{P}^i \alpha = \begin{cases} 0 & \alpha = x \\ \binom{n}{i} y^{n+i(p-1)} & \alpha = y^n \end{cases} \tag{1.1.2}$$

$$Q_0 \alpha = \begin{cases} y & \alpha = x \\ 0 & \alpha = y^n \end{cases} \tag{1.1.3}$$

for all $n \in \mathbf{Z}$ and $i \geq 1$.

We will always denote the homotopy group of the Brown-Peterson spectrum BP as

$$BP_* \doteq \mathbf{Z}_{(p)}[v_1, v_2, \cdots]$$

and $BP_*(BP)$ as

$$\Gamma \doteq BP_*[t_1, t_2, \cdots].$$

The BP-homology of P_{-n} is

$$BP_*(P_{-n}) = BP_*\langle b_{-n}, b_{-n+1}, \cdots \rangle / (\sum_{i=0}^{k} c_i b_{k-n-i} : k \geq 0)$$

which denotes the BP_*-module with the indicated generators modulo the indicated relations. Here c_i's are the coefficients in the p-series

$$[p](x) = x \sum_{i=0}^{\infty} c_i x^{i(p-1)} \tag{1.1.4}$$

and $|b_i| = iq - 1$. The Γ-coaction of $BP_*(P_{-n})$ is given by

$$\psi(b_k) = 1 \otimes b_k + \sum_{1 \leq i \leq k+n} w_i \otimes b_{k-i} \tag{1.1.5}$$

where the elements $w_i's$ can be determined as following: for $k = -m < 0$

$$(\sum^{F}(t_j \otimes x^{p^j-1}))^{\frac{mq}{2}} = w_i \otimes x^{\frac{iq}{2}} + \text{others} \tag{1.1.6}$$

and for $k = m > 0$

$$(\sum^{F}(t_j \otimes x^{p^j-1}))^{\frac{(m-i)q}{2}} = w_i \otimes x^{\frac{iq}{2}} + \text{others}. \tag{1.1.7}$$

Here F is the canonical formal group law over BP_* and x is a polynomial generator with dimension 2.

Interested readers may find these BP-theory calculations in detail in [Sad].

The main object of the first part of this thesis is the following BP analog of Lin's theorem.

Theorem B *There is an isomorphism*

$$\text{Ext}_\Gamma^{s,t}(BP_*, \Sigma^{-1} B\hat{P}_*) \cong \varprojlim \text{Ext}_\Gamma^{s,t}(BP_*, BP_*(P_{-n}))$$

which is induced by the extensions of Γ*-comodule homomorphisms* $(\psi_{-n})_*$

$$\begin{array}{ccc}
\Sigma^{-1}BP_* & \xrightarrow{(\psi_{-n})_*} & BP_*(P_{-n}) \\
\downarrow & \nearrow & \\
\Sigma^{-1}BP_*/(p^{n+1}) & &
\end{array}$$

for all $n \geq 0$. Here we use $\hat{BP}_*$ to denote the p-completion of BP_*, i.e. the inverse limit of $BP_*/(p^{n+1})$.

We can generalize Theorem B to the following form.

Theorem C *There is an isomorphism*

$$\mathrm{Ext}_\Gamma^{s,t}(BP_*, \hat{BP}_*(S^{-1} \wedge X)) \cong \varprojlim \mathrm{Ext}_\Gamma^{s,t}(BP_*, BP_*(P_{-n} \wedge X))$$

for any finite spectrum X.

Instead of proving Theorem B with BP-theory, we will find two natural spectral sequences converging to the indicated groups and then prove the corresponding E_2-terms are isomorphic. We obtain these spectral sequences by filtering the cobar complexes I-adically. I denotes the ideal in BP_* generated by p and all v_i with $i > 0$. The E_2-terms of these spectral sequences are E_2-terms of classical Adams spectral sequences and their inverse limit. The following diagram explains our idea.

$$\begin{array}{ccc}
\mathrm{Ext}_{P_*}(\mathrm{Ext}_{E_*}(\Sigma^{-1}\mathbf{Z}/(p))) & \Longrightarrow & \mathrm{Ext}_\Gamma(\Sigma^{-1}\hat{BP}_*) \\
\downarrow & & \downarrow \\
\varprojlim \mathrm{Ext}_{P_*}(\mathrm{Ext}_{E_*}(H_*(P_{-n}))) & \Longrightarrow & \varprojlim \mathrm{Ext}_\Gamma(BP_*, BP_*(P_{-n}))
\end{array} \qquad (1.1.8)$$

It is sufficient to prove that the first column is an isomorphism. The E_2-terms in the first column are only related to ordinary homology and classical Adams spectral sequence. This simplifies our job.

Before we state more results, we like to introduce some basic properties of dual Steenrod algebra $\mathcal{A}_*$ from [Rav86], which were originally proved by Milnor.

Lemma 1.1.9 ([Rav86], 3.1.1) *$\mathcal{A}_*$ is a graded commutative noncocommutative Hopf algebra.*

(1) *For $p = 2$, $\mathcal{A}_* = P[\xi_1, \xi_2, \cdots]$ as an algebra where $P[\ ]$ denotes a polynomial algebra over $\mathbf{Z}/(p)$ on the indicated generators, and $|\xi_n| = 2^n - 1$. The coproduct $\Delta : \mathcal{A}_* \to \mathcal{A}_* \otimes \mathcal{A}_*$ is given by $\Delta(\xi_n) = \sum_{0 \le i \le n} \xi_{n-i}^{2^i} \otimes \xi_i$, where $\xi_0 = 1$.*

(2) *For $p > 2$, $\mathcal{A}_* = P[\xi_1, \xi_2, \cdots] \otimes E(\tau_0, \tau_1, \cdots)$ as an algebra, where $E(\ )$ denotes the exterior algebra on the given generators, $|\xi_n| = 2(p^n - 1)$, and $|\tau_n| = 2p^n - 1$. The coproduct is given by $\Delta(\xi_n) = \sum_{0 \le i \le n} \xi_{n-i}^{p^i} \otimes \xi_i$, where $\xi_0 = 1$ and $\Delta(\tau_n) = \tau_n \otimes 1 + \sum_{0 \le i \le n} \xi_{n-i}^{p^i} \otimes \tau_i$.*

Let $P_* \subset \mathcal{A}_*$ be $P[\xi_1^2, \xi_2^2, \cdots]$ for $p = 2$ and $P[\xi_1, \xi_2, \cdots]$ for $p > 2$, and let $E_* = \mathcal{A}_* \otimes_{P_*} \mathbf{Z}/(p)$, i.e. $E_* = E(\xi_1, \xi_2, \cdots)$ for $p = 2$ and $E_* = E(\tau_0, \tau_1, \cdots)$ for $p > 2$. Then we have

Lemma 1.1.10 ([Rav86] 4.43) *With notations as above*

(1) $\mathrm{Ext}_{E_*}(\mathbf{Z}/(p), \mathbf{Z}/(p)) = P[u_0, u_1, \cdots]$ *with $u_i \in \mathrm{Ext}^{1, 2p^i - 1}$ represented in the cobar complex by $[\xi_i]$ for $p = 2$ and $[\tau_i]$ for $p > 2$.*

(2) *$P_* \to \mathcal{A}_* \to E_*$ is an extension of Hopf algebras.*

(3) *The P_*-coaction on $\mathrm{Ext}_{E_*}(\mathbf{Z}/(p), \mathbf{Z}/(p))$ is given by*

$$\psi(u_n) = \begin{cases} \sum_i \xi_{n-i}^{2^{i+1}} \otimes u_i & p = 2 \\ \sum_i \xi_{n-i}^{p^i} \otimes u_i & p > 2 \end{cases}$$

There are a series of compatible splittings ([LDMA80] Theorem 1.3) for $p = 2$:

Theorem D

$$\mathcal{A}_*\Box_{\mathcal{A}(r)_*}F_*^{lr} = \oplus_j\Sigma^{jp^rq}\mathcal{A}_*\Box_{\mathcal{A}(r-1)_*}\mathbf{Z}/(2)$$

where $\mathcal{A}(r)_*$ *is the quotient Hopf algebra* $\mathcal{A}_*/(\xi_1^{p^r},\cdots,\xi_r^p,\cdots;\tau_{r+1},\cdots)$ *and* F_*^{lr} *is an* $\mathcal{A}(r)_*$*-subcomodule of* $\varprojlim H_*(P_{-n})$ *and* $H_*(P_{-n})$ *for* $n \gg 0$.

It is crucial to the proof of Lin's theorem in the case of $p = 2$. Gunawardena ([Gun81]) proved the Lin's theorem in a different way for $p > 2$. However we can not take the advantage of his proof directly. We will give a proof of this theorem for $p > 2$ later which is similar to the proof in [LDMA80].

Applying the functor

$$\mathrm{Ext}_{P_*}(\mathbf{Z}/(p), \mathrm{Ext}_{E_*}(\mathbf{Z}/(p), -))$$

to the splittings in the above theorem and letting r and l pass to ∞, we obtain the isomorphism in (1.1.8).

The second problem is a realization problem.

Given a left module N over the Steenrod algebra $\mathcal{A}^*$, we can ask the following question: can we find a spectrum X whose cohomology is isomorphic to N as $\mathcal{A}^*$-modules? We call this spectrum X the realization of N if it exists. Of course the answer is not always yes; and in fact we can only answer this question for some special N's. In thesis we will consider the realization problem of the subalgebra $\mathcal{A}(n)^*$ of $\mathcal{A}^*$, which is generated by $Q_0, \mathcal{P}^1, \cdots, \mathcal{P}^{p^{n-1}}$ as an $\mathcal{A}^*$-module. $\mathcal{A}(n)^*$ may have more than one natural $\mathcal{A}^*$-module structure, i.e. there may exist more than one extension in the following diagram.

$$\begin{array}{ccc} \mathcal{A}(n)^* \otimes \mathcal{A}(n)^* & \xrightarrow{m} & \mathcal{A}(n)^* \\ \downarrow & \nearrow & \\ \mathcal{A}^* \otimes \mathcal{A}(n)^* & & \end{array}$$

Davis and Mahowald [DM81] found that for $\mathcal{A}(1)^*$ when $p = 2$ there are 4 different module structures. We also know that there are 1600 module structures for $\mathcal{A}(2)^*$ with $p = 2$ found by W. H. Lin. An important result in this direction is the existence of a self-dual $\mathcal{A}^*$-module structure of $\mathcal{A}(n)^*$ for any n and p shown by S. Mitchell in [Mit85]. This self-dual module structure has the property

$$Q_{n+i}\mathcal{A}(n)^* = 0$$

for all $i > 0$. In this thesis we will only consider module structures of $\mathcal{A}(n)^*$ with this property.

We have known the existence of realizations of $\mathcal{A}(1)^*$ [DM81], $\mathcal{A}(2)^*$ [Ino88] for $p = 2$ and direct sum of finite many copies of $\mathcal{A}(n)^*$ (the number is very large) [Mit85].

One of our main results is

Theorem E *For any possible $\mathcal{A}^*$-module structure with $Q_{n+i}\mathcal{A}(n)^* = 0$ for all $i > 0$, $\mathcal{A}(n)^*$ is realizable for all p, n when $p \geq n + 3$.*

Our tool is a modification of the algebraic sufficient condition of realization of a given $\mathcal{A}^*$-module, which was proved by H. Toda [Tod71] and was used by him to find the existence of $V(1)$, $V(2)$, $V(3)$. This condition requires the vanishing of $\mathrm{Ext}^{s,t}_{\mathcal{A}^*}(N, \mathbf{Z}/(p))$ for those (s, t) such that $s \geq 2$ and $t - s$ are dimensions of the cells of N. Unfortunately this condition is just satisfied for $\mathrm{Ext}_{\mathcal{A}^*}(\mathcal{A}(n)^*, \mathbf{Z}/(p))$ with $n \leq 2$, $p > 2$ and $n = 3$, $p > 3$. There is an obstruction when $n = 3$, $p = 3$. It is conceivable that there are more obstructions for $n \geq 4$ if we use this method.

We will generalize this method by considering the quotient algebra $B(n)^*$ of $\mathcal{A}^*$ whose dual is

$$\mathbf{Z}/(p)[\xi_i | i > 0] \otimes E(\tau_j | j \leq n)$$

for the realization problem of $\mathcal{A}(n)^*$. This is the homology of the spectrum called $P(n + 1)$ [JW75].

The new condition we need to realize $\mathcal{A}(n)^*$ is the vanishing of certain terms of $\mathrm{Ext}_{B(n)^*}(\mathcal{A}(n)^*, \mathbf{Z}/(p))$ similar to the old one. However this object is much simpler: it gets rid of all Q_i's for $i > n$.

Both Toda and Inoue used the May spectral sequence to calculate the Ext groups through certain range. However we do not need to do calculations in detail for our problem. What we do need is the sparseness, i.e the corresponding $\mathrm{Ext}^{s,t}$ are non-zero unless $t \equiv 0 \bmod q$.

Toda's idea to find geometric realization of N is, roughly speaking, building a Posnikov tower. He started from wedge of enough many copies of $H\mathbf{Z}/(p)$ which contain the cells we need to form N. Then he killed unnecessary cells and preserved the cells needed for N by aPostnikov tower. This procedure cannot always succeed. It fails when there is a nontrivial Steenrod operation connecting a cell we want to kill to a cell we want to preserve. That cell of N will be killed when we try to get rid of those unnecessary ones. Hence we need the vanishing of certain terms of Ext to assure that we can preserve all the cells we need. Our modification is based on this idea. The difference is that we begin from a wedge of copies of $P(n+1)$, which reduce the number of unnecessary cells from input and puts less restriction on Ext simultaneously. Then we kill unnecessary cells by constructing $P(n+1)$-Posnikov tower. sparseness and vanishing of certain terms of $\mathrm{Ext}_{B(n)^*}(\mathcal{A}(n)^*, \mathbf{Z}/(p))$ will secure the proceeding of the construction of $P(n+1)$-Posnikov tower.

$\mathcal{A}(n)$, which denotes the realization of $\mathcal{A}(n)^*$, is a $(n+1)$-type spectrum. So it has a self-map called v_{n+1}-map by the periodicity theorem of M. Hopkins and J. Smith [HS], namely

$$f : \Sigma^{d|v_{n+1}|}\mathcal{A}(n) \longrightarrow \mathcal{A}(n)$$

where f induces the multiplication of v_{n+1}^d on BP homology. But we do not have general knowledge about the lower bound of d. It is one of the main results in [DM81] that $\mathcal{A}(1)$ has a self-map such that $d = 4$ for $p = 2$. Another main theorem in this thesis is

Theorem F *For $p \geq n+4$, there is a self-map*

$$v_{n+1} : \Sigma^{|v_{n+1}|}\mathcal{A}(n) \longrightarrow \mathcal{A}(n)$$

which induces the multiplication of v_{n+1} on the Brown-Peterson homology.

We will work on the BP analog of Lin's Theorem in Chapter 2 and the realization of $\mathcal{A}(n)^*$ in Chapter 3.

In the first section of Chapter 1, we construct the I-adic spectral sequence and show its convergence. Then we prove Theorems B and D in Section 2.2. In the last section of this chapter, we generalize Theorem B to C.

We introduce some useful knowledge in Section 3.1. In the next section, we prove the sufficient conditions for the realization and existence of v_{n+1}^d self-maps for a special kind of $\mathcal{A}^*$-modules. Then we verify that these conditions are satisfied for $\mathcal{A}(n)^*$ in last section.

Chapter 2

A BP Analog of Lin's Theorem

2.1 I-adic spectral sequences

The I-adic spectral sequence is the object of this section. The idea of I-adic spectral sequence is simple. It has been applied to Adams-Novikov E_2-term $\mathrm{Ext}_\Gamma(BP_*, BP_*)$ by H. Miller; we can find these calculations in the end of chapter 4 in [Rav86]. It is called the algebraic Novikov spectral sequence there. The convergence problem of these spectral sequences in the case we are considering is nontrivial since it involves an inverse limit. On the other hand there is no general theorem about the I-adic spectral sequence of an arbitrary Γ-comodule, so we have to identify E_1-term as well.

Homology and inverse limits do not always commute, but we will show that they do in some relevant special cases.

Lemma 2.1.1 *Suppose $C(-n)$ is a chain complex which is finite in each dimension and $f_{-n} : C(-n) \to C(-n+1)$ is a chain map for each positive integer n. Then $\varprojlim H_*(C(-n)) = H_*(\varprojlim C(-n))$.*

Proof. It follows from the Mittag-Leffler condition (Thm 7.75 [Swi75]) that $\varprojlim{}^1 C(-n) = 0$. This is the only thing we have to worry in order to prove this lemma. ■

The first step to build our spectral sequence is to give I-adic filtrations on the cobar complexes $C^*_\Gamma(BP_*)$, $C^*_\Gamma(BP_*(P_{-n}))$ and the inverse limit of the latter. We will denote them as $C^*(BP_*)$, $C^*(BP_*(P_{-n}))$ and $\varprojlim C^*(BP_*(P_{-n}))$

for short. $BP_*(P_{-n})$, BP_* and $BP_*(BP)$ are finite in each dimension, as are $C^*(BP_*)$ and $C^*(BP_*(P_{-n}))$. By Lemma 2.1.1 we have

$$\varprojlim \mathrm{Ext}_\Gamma(BP_*, BP_*(P_{-n})) = H_*(\varprojlim C^*(BP_*(P_{-n}))).$$

The decreasing filtrations of $C^*(BP_*(P_{-n}))$ and $\varprojlim C^*(BP_*(P_{-n}))$, which we are interested in, are

$$C^*(BP_*(P_{-n})) = F^0C^*(BP_*(P_{-n})) \supseteq F^1C^*(BP_*(P_{-n})) \supseteq \cdots$$

where

$$F^iC^*(BP_*(P_{-n})) = \left\{\sum_{j=j_0}^{-n} \alpha_j b_j \colon \alpha_j \in I^iC^*(BP_*)\right\} / \left(\sum_{k=0}^{m+n} c_k b_{m-k} : m \geq -n\right)$$

and

$$\varprojlim C^*(BP_*(P_{-n})) = F^0\varprojlim C^*(BP_*(P_{-n})) \supseteq F^1\varprojlim C^*(BP_*(P_{-n})) \supseteq \cdots$$

where

$$F^i\varprojlim C^*(BP_*(P_{-n})) = \varprojlim F^iC^*(BP_*(P_{-n})).$$

They are compatible with the following decreasing filtrations of BP_* and Γ:

$$BP_* = F^0BP_* \supseteq F^1BP_* = I \supseteq F^2BP_* = I^2 \cdots,$$

$$\Gamma = F^0\Gamma \supseteq F^1\Gamma \supseteq \cdots$$

where

$$I = (p = v_0,\ v_1,\ v_2,\ \cdots),$$

$$F^i\Gamma = \{\textstyle\sum a_j x_j | a_j \in I^i,\ x_j \in \Gamma\}.$$

The associated graded objects are

$$E_0^*BP_* = \mathbf{Z}/(p)[v_0,\ v_1,\ \cdots,\ v_n,\ \cdots] \doteq P[v],$$

$$E_0^*\Gamma = \mathbf{Z}/(p)[v][t_1,\ t_2, \cdots, t_n, \cdots] \doteq P[v][t],$$

$$\begin{aligned} E_0^* BP_*(P_{-n}) &= \{\sum_{i=-n}^{i_0} \alpha_i b_i | \alpha_i \in P[v]\}/(\sum v_j b_{k-\frac{p^j-1}{p-1}} : k - \frac{p^j-1}{p-1} \geq -n) \\ &\doteq P[v]\langle b_* \rangle_{-n}/(\sim), \end{aligned}$$

$$E_0^* C^*(BP_*(P_{-n})) = C^*_{P[v][t]}(P[v], E_0^* BP_*(P_{-n})),$$

$$E_0^* \varprojlim C^*(BP_*(P_{-n})) = \varprojlim E_0^* C^*(BP_*(P_{-n})).$$

$P[v]\langle b_* \rangle_{-n}/(\sim)$ is a comodule over Hopf-algebroid $(P[v], P[v][t])$. Let us state their structures. They can be easily deduced from the structure of Hopf algebroid (BP_*, Γ) and (1.1.5), (1.1.6) and (1.1.7):

$$\eta_L(v_i) = v_i,$$

$$\eta_R(v_i) = \sum_{j=0}^{i} v_j t_{i-j}^{p^j} \text{ with } t_0 = 1,$$

$$\Delta(t_i) = \sum_{j=0}^{j=i} t_j^{p^{i-j}} \otimes t_{i-j},$$

$$\psi(b_k) = 1 \otimes b_k + \sum_{1 \leq i \leq k+n} \hat{w}_i \otimes b_{k-i}. \tag{2.1.2}$$

where the elements $\hat{w}_i's$ can be determined as follows: for $k = -m < 0$

$$(\sum(t_j \otimes x^{p^j-1}))^{\frac{mq}{2}} = \hat{w}_i \otimes x^{\frac{iq}{2}} + \text{others} \tag{2.1.3}$$

and for $k = m > 0$

$$(\sum(t_j \otimes x^{p^j-1}))^{\frac{(m-i)q}{2}} = \hat{w}_i \otimes x^{\frac{iq}{2}} + \text{others.} \tag{2.1.4}$$

We should be careful that (2.1.2), (2.1.3), (2.1.4) are over the field $\mathbf{Z}/(p)$ but (1.1.5), (1.1.6), (1.1.7) are over the p-local integers $\mathbf{Z}_{(p)}$.

We will draw a diagram to outline the proof in the rest of this section.

$$\begin{array}{ccc}
\mathrm{Ext}_{P_*}(\mathbf{Z}/(p), \mathrm{Ext}_{E_*}(\mathbf{Z}/(p), \Sigma^{-1}\mathbf{Z}/(p))) & \xrightarrow{(X)} & \varprojlim \mathrm{Ext}_{P_*}(\mathbf{Z}/(p), \mathrm{Ext}_{E_*}(\mathbf{Z}/(p), F_*(-n))) \\
\Big\downarrow{\scriptstyle (viii)} & & \Big\downarrow{\scriptstyle (\alpha)} \\
\mathrm{Ext}_{P[t]}(\mathbf{Z}/(p), \Sigma^{-1}P[v]) & \xrightarrow{(vii)} & \varprojlim \mathrm{Ext}_{P[t]}(\mathbf{Z}/(p), P[v]\langle b_*\rangle_{-n}/(\sim)) \\
\Big\downarrow{\scriptstyle (v)} & & \Big\downarrow{\scriptstyle (vi)} \\
\mathrm{Ext}_{P[v][t]}(P[v], \Sigma^{-1}P[v]) & \xrightarrow{(iv)} & \varprojlim \mathrm{Ext}_{P[v][t]}(P[v], P[v]\langle b_*\rangle_{-n}/(\sim)) \\
\Big\downarrow{\scriptstyle (ii)} & & \Big\downarrow{\scriptstyle (iii)} \\
\mathrm{Ext}_{\Gamma}(BP_*, \Sigma^{-1}B\hat{P}_*) & \xrightarrow{(i)} & \varprojlim \mathrm{Ext}_{\Gamma}(BP_*, BP_*(P_{-n}))
\end{array} \tag{2.1.5}$$

(ii) and (iii) are p-adic spectral sequences shown in Theorem 2.1.7. The homomorphism (i) is induced by

$$S^{-1} \longrightarrow P_{-n}.$$

The homomorphism (iv) is induced by

$$P[v] \longrightarrow P[v]\langle b_*\rangle_{-n}/(\sim) \tag{2.1.6}$$

which sends 1 to b_0. (v) is a change-of-rings isomorphism which is shown in Theorem 2.1.8. (vi) is also a change-of-rings isomorphism shown in Lemma

(2.1.11). The homomorphism (vii) is also induced by (2.1.6). (viii) is proved in Theorem 2.1.8 and (α) is proved in Lemma 2.1.9 and 2.1.11. From the proof, it is not hard to see the homomorphism is also induced by (2.1.6).

Theorem 2.1.7 *There are natural spectral sequences converging to*

$$\mathrm{Ext}_\Gamma(BP_*, \hat{BP_*})$$

and

$$\varprojlim \mathrm{Ext}_\Gamma(BP_*, BP_*(P_{-n}))$$

such that the E_1 terms are

$$E_1^{s,m,t} = \mathrm{Ext}_{P[v][t]}^{s,m,t}(P[v], P[v])$$

and

$$E_1^{s,m,t} = \varprojlim \mathrm{Ext}_{P[v][t]}^{s,m,t}(P[v], P[v]\langle b_*\rangle_{-n}/(\sim))$$

where the first degree is the homological degree, the second one comes from the filtration and

$$d_r : E_r^{s,m,t} \longrightarrow E_r^{s+1,m+r,t}.$$

Proof. This theorem is a consequence of [Rav86] A1.3.9. But we have to check that

$$\bigcap_{i\geq 0} F^i BP_* = 0$$

and

$$\bigcap_{i\geq 0} F^i \varprojlim C^*(BP_*(P_{-n})) = 0.$$

The first identity is trivial and the second one follows from

$$\bigcap_{i\geq 0} F^i C^*(BP_*(P_{-n})) = \varprojlim F^i C^*(BP_*(P_{-n})) = 0$$

and the fact that inverse limits commute. ∎

We denote the cohomology of P_{-n} by $F^*(-n)$ and the homology of P_{-n}, which is the dual of $F^*(-n)$, by $F_*(-n)$. Meanwhile we denote $\varinjlim F^*(-n)$ by F^* and $\varprojlim F_*(-n)$ by F_*.

We have a theorem about the relationship between the E_1-terms in Theorem 2.1.7 and $F_*(-n)$.

Theorem 2.1.8 *There are isomorphisms*

$$\mathrm{Ext}^{s,m,t}_{P[v][t]}(P[v], P[v]) = \mathrm{Ext}^{s,t}_{P_*}(\mathbf{Z}/(p), \mathrm{Ext}^{m}_{E_*}(\mathbf{Z}/(p), \mathbf{Z}/(p)))$$

and

$$\varprojlim \mathrm{Ext}^{s,m,t}_{P[v][t]}(P[v], P[v]\langle b_*\rangle_{-n}/(\sim)) = \varprojlim \mathrm{Ext}^{s,t}_{P_*}(\mathbf{Z}/(p), \mathrm{Ext}^{m}_{E_*}(\mathbf{Z}/(p), F_*(-n)))$$

where P_* and E_* were defined in Lemma 1.1.10.

The first isomorphism is Theorem 4.4.4 in [Rav86]. We have an object as intermediate between

$$\varprojlim \mathrm{Ext}_{P[v][t]}(P[v], P[v]\langle b_*\rangle_{-n}/(\sim))$$

and

$$\varprojlim \mathrm{Ext}_{P_*}(\mathbf{Z}/(p), \mathrm{Ext}_{E_*}(\mathbf{Z}/(p), F_*(-n))).$$

It is

$$\varprojlim \mathrm{Ext}_{P_*}(\mathbf{Z}/(p), P[v]\langle b_*\rangle_{-n}/(\sim)).$$

We will prove each of the first two isomorphic to the third. Before going to the proof of this theorem, we have to prove two lemmas.

Lemma 2.1.9

$$\mathrm{Ext}_{P_*}(\mathbf{Z}/(p), \mathrm{Ext}_{E_*}(\mathbf{Z}/(p), F_*(-n))) = \mathrm{Ext}_{P_*}(\mathbf{Z}/(p), P[u]\langle d_*\rangle_{-n}/(\sim))$$

where

$$P[u]\langle d_*\rangle_{-n}/(\sim) \doteq \left\{\sum_{i=m}^{-n} \alpha_i d_i | \alpha_i \in P[u_0, u_1, \cdots]\right\} / \left(\sum u_i d_{n-\frac{p^i-1}{p-1}}\right).$$

The coaction of $P[u]\langle d_*\rangle_{-n}/(\sim)$ as a $(\mathbf{Z}/(p), P_*)$-comodule is determined by

$$\mathcal{P}^{p^m} d_i^* = \binom{i(p-1)}{p^m} d_{i+p^n}^*$$

for any $d_i^* \in F^*$ which is the dual of d_i.

Remarks 2.1.10 *The dual of P_* is generated by $\mathcal{P}$, $\mathcal{P}^p$, $\cdots$. The coaction over P_* follows from the action of P on the dual of the given comodule.*

Proof. Considering the Adams spectral sequence for $\pi_*(P_{-n} \wedge BP)$, we will have the following result after the change of rings isomorphism

$$\begin{aligned} E_2 &= \mathrm{Ext}_{\mathcal{A}_*}(\mathbf{Z}/(p), F_*(-n) \otimes H_*(BP)) \\ &= \mathrm{Ext}_{E_*}(\mathbf{Z}/(p), F_*(-n)) \end{aligned}$$

It collapses for degree reasons: all elements in Ext have even topological degree. On the other hand we know $BP_*(P_{-n})$ and its associated graded object with respect to classic Adams spectral sequence, which is just

$$P[u]\langle d_*\rangle_{-n}/(\sim). \qquad \blacksquare$$

Lemma 2.1.11 *The following is an isomorphism*

$$\mathrm{Ext}_{P[v][t]}(P[v], P[v]\langle b_*\rangle_{-n}/(\sim)) \cong \mathrm{Ext}_{P[t]}(\mathbf{Z}/(p), P[v]\langle b_*\rangle_{-n}/(\sim))$$

Proof. We can make $(\mathbf{Z}/(p), P[t])$ a quotient Hopf-algebroid of $(P[v], P[v][t])$ by setting

$$f_1 : P[v] \longrightarrow \mathbf{Z}/(p) \quad \text{with} \quad f_1(v_i) = 0$$

$$f_2 : P[v][t] \longrightarrow P \quad \text{with} \quad f_2(v_i) = 0,\ f_2(t_i) = t_i$$

so that

$$P[v][t] \otimes_{P[v]} \mathbf{Z}/(p) \cong P[t]$$

By the change-of-rings isomorphism, we can obtain

$$\begin{aligned} E_1 &= \mathrm{Ext}_{P[v][t]}(P[v], P[v]\langle b_*\rangle_{-n}/(\sim)) \\ &= \mathrm{Ext}_{P[t]}(\mathbf{Z}/(p), (P[v][t] \otimes_{P[v]} \mathbf{Z}/(p)) \square_{P[t]} P[v]\langle b_*\rangle_{-n}/(\sim)) \\ &= \mathrm{Ext}_{P[t]}(\mathbf{Z}/(p), P[v]\langle b_*\rangle_{-n}/(\sim)). \qquad \blacksquare \end{aligned}$$

Proof of Theorem 2.1.8. It is elementary to verify that $(\mathbf{Z}/(p), P[t])$ is isomorphic to $(\mathbf{Z}/(p), P_*)$ as Hopf-algebroids. According to Lemma 2.1.9 and 2.1.11 what we have to do now is to find the isomorphism between $P[u]\langle d_*\rangle_{-n}/(\sim)$ and $P[v]\langle b_*\rangle_{-n}/(\sim)$ as comodules.

From [Rav86] 4.3.1 we know there is a formula in Γ relating η_R and η_L:

$$\sum_{i,j\geq 0}^{F} t_i \eta_R(v_j)^{p^i} = \sum_{i,j\geq 0}^{F} v_i t_j^{p^i}.$$

A new formula in $P[v][t]$ is obtained by reducing modulo I:

$$\sum_{j\geq 0} \eta_R(v_j) = \sum_{i,j\geq 0} v_i t_j^{p^i}.$$

Applying the conjugation and separating it by degree, we get

$$v_m = \sum_{i+j=m} \eta_R(v_i) t_j^{p^i}.$$

The coaction of v_m is

$$\begin{aligned} \psi(v_m) &= v_m \otimes 1 \\ &= \textstyle\sum_{i+j=m} \eta_R(v_i) t_j^{p^i} \otimes 1 \\ &= \textstyle\sum_{i+j=m} t_j^{p^i} \otimes v_i. \end{aligned}$$

It won't change when we pass it to $P[t] \otimes P[v]\langle b_*\rangle_{-n}/(\sim)$.

At last we have to show that the coactions of b_n and d_n coincide. Reducing (2.1.3) (2.1.4) modulo the ideal $(t_2, \cdots)$, we can find the coefficient of $t_1^{p^m} \otimes x^{\frac{p^m q}{2}}$ is $\binom{i(p-1)}{p^m}$. This fact is the same as the result of Theorem 2.1.9. $\blacksquare$

2.2 Proof of Theorem B

We will give the proof of Theorem B in this section. Our proof parallels that from [LDMA80]. The difference is that the proof in [LDMA80] only works for $p = 2$ but ours works for $p > 2$. It will be a long story. We will divide it into several lemmas. We write

$$\begin{aligned} F^* &= \varinjlim H^*(P_{-n}, \mathbf{Z}/(p)) \\ &= \mathbf{Z}/(p)\{h_i, h_i' : i \in \mathbf{Z}, h_i = xy^{i(p-1)-1} \text{ and } h_i' = y^{i(p-1)}\} \end{aligned}$$

and

$$\begin{aligned} F_* &= \varprojlim H_*(P_{-n}; \mathbf{Z}/(p)) \\ &= \mathbf{Z}/(p)\{d_i, d_i' : i \in \mathbf{Z}, d_i = h_i^* \text{ and } d_i' = (h_i')^*\}. \end{aligned}$$

Lemma 2.2.1 *As an $\mathcal{A}(r)^*$-module, F^* is generated by h_j with $j \equiv 0$ mod p^r.*

Proof. If $j \equiv 0 \bmod p^r$ and $0 \le i < p^r$, $k = 0, 1$, then

$$Q_0^k \mathcal{P}^i h_j = \begin{cases} a_{ij} h_{i+j} & k = 0 \\ a_{ij} h'_{i+j} & k = 1 \end{cases}$$

where

$$\begin{aligned} a_{ij} &= \binom{j(p-1)-1}{i} \\ &= \binom{kp^r(p-1)-1}{i} \\ &= \binom{p^r-1}{i} \\ &\neq 0. \qquad \blacksquare \end{aligned}$$

Let $F_{l,r}^*$ be the $\mathcal{A}(r)^*$-submodule of F^* generated by the h_j and h_j' with $j < l$. By Lemma 2.2.1 it is sufficient to consider those $F_{l,r}^*$'s with $l \equiv 0$ mod p^r.

Lemma 2.2.2 *There is an isomorphism of $\mathcal{A}^*$-modules*

$$\mathcal{A}^* \otimes_{\mathcal{A}(r)^*} (F^*/F^*_{lp^r,r}) \cong \bigoplus_{j\geq l} \Sigma^{jp^r q-1}(\mathcal{A}^* \otimes_{\mathcal{A}(r-1)^*} \mathbf{Z}/(p)).$$

Remark 2.2.3 *The above lemma is actually Theorem D in cohomology form.*

Remark 2.2.4 *We prove later in Lemma* 2.2.14 *that this splitting can be defined using explicit generators. Since it involves complicated calculations, we like to prove the splitting first and separately from naturality in order to give readers a clearer idea.*

We need four more lemmas to complete the proof of Lemma 2.2.2. We will state them and give proofs before proving Lemma 2.2.2.

The $\mathcal{A}(r)^*$-modules $F^*/F^*_{l,r}$ for different values of l become isomorphic after we regrade them; so it is sufficient to consider one value of l, say $l = 0$. And as we only have to consider one value of r at one time, there is no need to display r either, so for brevity let us write

$$F = F^*, \ F_i = F^*_{(i-1)p^r,r}.$$

Lemma 2.2.5 *In F we have $\mathcal{P}^{p^i} h_0 \in F_1$ if $i < r - 1$.*

Proof. It is sufficient to display the following identities

$$\begin{aligned}
\mathcal{P}^{p^{r-1}} h_{p^i - p^{r-1}} &= \binom{(p^i - p^{r-1})(p-1)-1}{p^{r-1}} h_{p^i} \\
&= \binom{p^{r-1}}{p^{r-1}} \binom{p^{i+1} - p^i - 1}{0} h_{p^i} \\
&= h_{p^i} \\
&= \binom{-1}{p^i} h_{p^i} \\
&= \mathcal{P}^{p^i} h_0. \qquad \blacksquare
\end{aligned}$$

Lemma 2.2.6 *We have the following short exact sequence of $\mathcal{A}(r)^*$-modules:*

$$0 \longrightarrow \Sigma^{-1}(\mathcal{A}(r)^* \otimes_{\mathcal{A}(r-1)^*} \mathbf{Z}/(p)) \longrightarrow F/F_1 \longrightarrow F/F_2 \longrightarrow 0.$$

Proof. It is clear that we have a short exact sequence

$$0 \longrightarrow F_2/F_1 \longrightarrow F/F_1 \longrightarrow F/F_2 \longrightarrow 0.$$

Meanwhile Lemma 2.2.5 shows that we can define a map

$$\alpha : \Sigma^{-1}(\mathcal{A}(r)^* \otimes_{\mathcal{A}(r-1)^*} \mathbf{Z}/(p)) \longrightarrow F_2/F_1$$

by sending $a \otimes 1$ to ah_0. This map is onto by Lemma 2.2.1. To show it is an isomorphism, it is sufficient to show that both sides have rank $2p^r$ over $\mathbf{Z}/(p)$. This is known for $\Sigma^{-1}(\mathcal{A}(r)^* \otimes_{\mathcal{A}(r-1)^*} \mathbf{Z}/(p))$, and we have to prove it for F_2/F_1.

Consider

$$j : F_2 \longrightarrow \Sigma^{p^r q} F_1$$

which sends h_n to h_{n-p^r} and h'_n to h'_{n-p^r}. It is an $\mathcal{A}(r)^*$-module isomorphism, so we have a monomorphism

$$F_1 \overset{i}{\hookrightarrow} F_2 \overset{j}{\longrightarrow} \Sigma^{p^r q} F_1.$$

The rank of Coker$(j \circ i)$ is the same as F_2/F_1. It is clear that the element h_n or (h'_n) is not in the image $j \circ i$ if and only if h_{n+p^r} (or h'_{n+p^r}) is not in F_1. So the rank of Coker$(j \circ i)$ as $\mathbf{Z}/(p)$ vector space equals to $2p^r$. ■

Lemma 2.2.7 *We have the following short exact sequence of $\mathcal{A}^*$-modules:*

$$0 \longrightarrow \Sigma^{-1}(\mathcal{A}^* \otimes_{\mathcal{A}(r-1)^*} \mathbf{Z}/(p)) \overset{\alpha}{\longrightarrow} \mathcal{A}^* \otimes_{\mathcal{A}(r)^*} F/F_1 \longrightarrow \mathcal{A}^* \otimes_{\mathcal{A}(r)^*} F/F_2 \longrightarrow 0.$$

Proof. This follows by applying the functor $\mathcal{A}^* \otimes_{\mathcal{A}(r)^*} -$ to the short exact sequence in Lemma 2.2.6, which preserves exactness since $\mathcal{A}^*$ is free as a right module over $\mathcal{A}(r)^*$. ■

We now introduce a quotient of $\mathcal{A}_*$, namely

$$B_* = \mathcal{A}_*/(\xi_2^{p^{r-1}}, \cdots, \xi_r^p, \xi_{r+1}, \cdots, \tau_{r+1}, \cdots).$$

It is easy to verify that B_* is a left-comodule with respect to $\mathcal{A}(r)_*$ and a right comodule with respect to $\mathcal{A}(r-1)_*$. Let B^* denote the dual of B_*; it is a sub-vector-space of $\mathcal{A}^*$, a left module over $\mathcal{A}(r)^*$ and a right module over $\mathcal{A}(r-1)^*$.

Lemma 2.2.8 *There is an isomorphism of $\mathcal{A}(r)^*$-modules*

$$\beta : \Sigma^{-1}B^* \otimes_{\mathcal{A}(r-1)^*} \mathbf{Z}/(p) \longrightarrow F/F_1$$

which sends $b \otimes 1$ to bh_0.

Proof. The prescription $\beta(b \otimes 1) = bh_0$ gives a well-defined map from $\Sigma^{-1}(B^* \otimes_{\mathcal{A}(r-1)^*} \mathbf{Z}/(p))$ by Lemma 2.2.5; and it is a $\mathcal{A}(r)^*$-map. It is onto because for $Q_0^j\mathcal{P}^i \in B^*(i \geq 0, j = 1, 2)$,

$$Q_0^j\mathcal{P}^i h_0 = \begin{cases} h_i & j = 0 \\ h_i' & j = 1 \end{cases}$$

and h_i, h_i' span F/F_1. In order to prove that β is an isomorphism, it is sufficient to note that $\Sigma^{-1}(B^* \otimes_{\mathcal{A}(r-1)^*} \mathbf{Z}/(p))$ and F/F_1 have the same Poincaré series. In fact, since we know the structure of B and B is free as a right-module over $\mathcal{A}(r-1)^*$ we can find that the Poincaré series for $B^* \otimes_{\mathcal{A}(r-1)^*} \mathbf{Z}/(p)$ is

$$\frac{1 + t^{2p^r - 1}}{1 - t^{2p^{r-1}(p-1)}} \prod_{i=2}^{r} \frac{1 - t^{2p^{r+1-i}(p^i - 1)}}{1 - t^{2p^{r-i}(p^i - 1)}}.$$

On the other hand, using Lemma 2.2.6 we can filter F/F_1 so as to obtain a subquotient $\mathcal{A}(r)^* \otimes_{\mathcal{A}(r-1)^*} \mathbf{Z}/(p)$ every $2p^r(p-1)$ dimension, then we can find the Poincaré series for F/F_1 is the same as above.

This fact proves Lemma 2.2.8. ∎

Proof of Lemma 2.2.2. Consider the following diagram.

$$\begin{array}{ccc}
\Sigma^{-1}(\mathcal{A}^* \otimes_{\mathcal{A}(r-1)^*} \mathbf{Z}/(p)) & \xrightarrow{\alpha} & \mathcal{A}^* \otimes_{\mathcal{A}(r-1)^*} F/F_1 \\
\Big\| \cong & & \Big\uparrow 1\otimes\beta \\
\Sigma^{-1}(\mathcal{A}^* \otimes_{\mathcal{A}(r-1)^*} \mathbf{Z}/(p)) & \xleftarrow[\mu\otimes 1]{} & \Sigma^{-1}(\mathcal{A}^* \otimes_{\mathcal{A}(r)^*} B^* \otimes_{\mathcal{A}(r-1)^*} \mathbf{Z}/(p))
\end{array} \tag{2.2.9}$$

Here α and β are as in Lemma 2.2.7 and Lemma 2.2.8, while μ is given by the product map for $\mathcal{A}$, that is, $\mu(a\otimes b) = ab$. We claim that

$$(\mu \otimes 1)(1 \otimes \beta)^{-1}(\alpha) = \text{id}.$$

It can be easily verified by the fact that

$$(1 \otimes \beta)(a\otimes 1\otimes h_0) = a\otimes h_0$$

and $1 \otimes \beta$ is an isomorphism.

Thus the short exact sequence in Lemma 2.2.7 splits and gives

$$\mathcal{A}^* \otimes_{\mathcal{A}(r)} F/F_1 \cong \Sigma^{-1}(\mathcal{A}^* \otimes_{\mathcal{A}(r-1)^*} \mathbf{Z}/(p)) \oplus (\mathcal{A}^* \otimes_{\mathcal{A}(r)^*} F/F_2). \tag{2.2.10}$$

But the same conclusion applies to F/F_2, so that

$$\mathcal{A}^* \otimes_{\mathcal{A}(r)^*} F/F_1$$

is isomorphic to

$$\Sigma^{-1}(\mathcal{A}^* \otimes_{\mathcal{A}(r-1)^*} \mathbf{Z}/(p)) \oplus \Sigma^{p^r q-1}(\mathcal{A}^* \otimes_{\mathcal{A}(r-1)^*} \mathbf{Z}/(p)) \oplus (\mathcal{A}^* \otimes_{\mathcal{A}(r)^*} F/F_2).$$

Continuing by induction, we obtain Lemma 2.2.2. ■

Our next target is to prove that the splitting above is natural. Consider the following diagrams.

$$\begin{array}{ccc}
\mathcal{A}^* \otimes_{\mathcal{A}(r)_*} F^*/F^*_{lp^r,r} & \overset{\cong}{\longleftrightarrow} & \bigoplus_{j\geq l} \Sigma^{jp^r q-1}(\mathcal{A}^* \otimes_{\mathcal{A}(r-1)_*} \mathbf{Z}/(p)) \\
\Big\downarrow & & \Big\downarrow \theta \\
\mathcal{A}^* \otimes_{\mathcal{A}(r)_*} F^*/F^*_{mp^r,r} & \underset{\cong}{\longleftrightarrow} & \bigoplus_{k\geq m} \Sigma^{kp^r q-1}(\mathcal{A}^* \otimes_{\mathcal{A}(r-1)_*} \mathbf{Z}/(p))
\end{array} \tag{2.2.11}$$

Here the left-hand vertical arrow is the obvious quotient map, which exists when $l \leq m$. The map θ has the obvious components, namely the zero map of $\Sigma^{jp^r q-1}(\mathcal{A}^* \otimes_{\mathcal{A}(r-1)_*} \mathbf{Z}/(p))$ if $j < m$, and the identity map of $\Sigma^{jp^r q-1}(\mathcal{A}^* \otimes_{\mathcal{A}(r-1)_*} \mathbf{Z}/(p))$ if $j \geq m$.

$$\begin{array}{ccc}
\mathcal{A}^* \otimes_{\mathcal{A}(r)_*} F^*/F^*_{lp^{r+1},r} & \overset{\cong}{\longleftrightarrow} & \bigoplus_{j\geq lp} \Sigma^{jp^r q-1}(\mathcal{A}^* \otimes_{\mathcal{A}(r-1)_*} \mathbf{Z}/(p)) \\
\Big\downarrow & & \psi \Big\downarrow \\
\mathcal{A}^* \otimes_{\mathcal{A}(r+1)_*} F^*/F^*_{lp^{r+1},r+1} & \overset{\cong}{\longleftrightarrow} & \bigoplus_{k\geq m} \Sigma^{kp^{r+1}-1}(\mathcal{A}^* \otimes_{\mathcal{A}(r)_*} \mathbf{Z}/(p))
\end{array} \tag{2.2.12}$$

Here the left-hand vertical arrow is the obvious quotient map. The map ψ has the obvious components: if

$$j = kp$$

we take the obvious quotient map

$$\Sigma^{jp^r q-1}(\mathcal{A}^* \otimes_{\mathcal{A}(r-1)_*} \mathbf{Z}/(p)) \longrightarrow \Sigma^{kp^{r+1} q-1}(\mathcal{A}^* \otimes_{\mathcal{A}(r)_*} \mathbf{Z}/(p)),$$

and if $j \neq kp$ then we take the zero map of $\Sigma^{jp^r q-1}(\mathcal{A}^* \otimes_{\mathcal{A}(r-1)_*} \mathbf{Z}/(p))$.

The horizontal maps in the above diagrams can be constructed after we find the explict generators for splittings in Lemma 2.2.14.

Lemma 2.2.13 *The isomorphism in Lemma 2.2.2 can be chosen so that the diagram (2.2.11) and (2.2.12) commute, and for $l \leq 0$ the composite*

$$\Sigma^{-1}(\mathcal{A}^* \otimes_{\mathcal{A}(r-1)^*} \mathbf{Z}/(p)) \longrightarrow \mathcal{A}^* \otimes_{\mathcal{A}(r)^*} F^*/F^*_{lp^r,r} \xrightarrow{1\otimes\gamma} \mathcal{A}^* \otimes_{\mathcal{A}(r)^*} \Sigma^{-1}\mathbf{Z}/(p)$$

is the obvious quotient map.

We first introduce the element

$$y_k = \sum_{i+j=k} \chi(\mathcal{P}^i) \otimes h_j \in \mathcal{A}^* \otimes_{\mathcal{A}(r)^*} F^*/F^*_{l,r}.$$

Here χ is the canonical anti-automorphism of $\mathcal{A}^*$; and the sum is finite since we only have to consider the range $i \geq 0$, $j \geq lp^r$. Then we have the following more precise form of Lemma 2.2.2.

Lemma 2.2.14 *The $\mathcal{A}^*$-module $\mathcal{A}^* \otimes_{\mathcal{A}(r)^*} F^*/F^*_{lp^r,r}$ is a direct sum of cyclic summands $\Sigma^{kp^r q-1}(\mathcal{A}^* \otimes_{\mathcal{A}(r-1)^*} \mathbf{Z}/(p))$ over k such that $k \geq l$ with generators y_{kp^r}.*

Remark 2.2.15 *We can define homomorphism*

$$\mathcal{A}^* \otimes_{\mathcal{A}(r)^*} F^*/F^*_{lp^r,r} \longrightarrow \oplus_{k\geq l}\Sigma^{kp^r q-1}(\mathcal{A}^* \otimes_{\mathcal{A}(r-1)^*} \mathbf{Z}/(p))$$

by sending a to $\oplus_{k\geq l} a_k$ for any

$$a = \Sigma_{k\geq l} a_k y_k \in \mathcal{A}^* \otimes_{\mathcal{A}(r)^*} F^*/F^*_{lp^r,r}$$

.

Proof. Consider the explicit splitting used in proving Lemma 2.2.2. It displays $F/F_{0,r}$ as the direct sum of the cyclic submodule $\Sigma^{-1}\mathcal{A}^* \otimes_{\mathcal{A}(r-1)^*} \mathbf{Z}/(p)$ on the generator $h_0 = y_0$ and the complementary summand, namely the kernel of the splitting map

$$(\mu\otimes 1)(1\otimes\beta)^{-1}.$$

We claim that this kernel contains the remaining elements y_{kp^r}, i.e. those with $k > 0$. In fact we have

$$\beta(\mathcal{P}^i \otimes 1) = h_i$$

so we have calculation

$$\begin{aligned} & (\mu \otimes 1)(1 \otimes \beta)^{-1}(\textstyle\sum_{i+j=kp^r} \chi(\mathcal{P}^i) \otimes h_j) \\ = & \textstyle\sum_{i+j=kp^r} \chi(\mathcal{P}^i)\mathcal{P}^j \\ = & 0 \end{aligned}$$

if $k > 0$. This means that in the splitting (2.2.10) the first direct summand is generated by y_0 and all y_{kp^r} with $k > 0$ are in the second summand. Suppose we have proved that in the following splitting

$$\mathcal{A}^* \otimes_{\mathcal{A}(r)^*} F/F_{0,r} \cong \oplus_{0 \le i < k} \Sigma^{ip^r q-1}(\mathcal{A}^* \otimes_{\mathcal{A}(r)^*} \mathbf{Z}/(p)) \bigoplus \mathcal{A}^* \otimes_{\mathcal{A}(r)^*} F/F^*_{kp^r,r},$$

the first k summands are generated by y_{ip^r} with $0 \le i < k$ and all y_{jp^r} are in the last summand if $j \ge k$. Let us consider following composite

$$\mathcal{A}^* \otimes_{\mathcal{A}(r)^*} F^*/F^*_{0,r} \longrightarrow \mathcal{A}^* \otimes_{\mathcal{A}(r)^*} F^*/F^*_{kp^r q,r} \xrightarrow{\cong} \Sigma^{kp^r q-1}\mathcal{A}^* \otimes_{\mathcal{A}(r)^*} F^*/F^*_{0,r}$$

which send y_{kp^r} to y_0. This is equivalent to saying y_{kp^r} is the generator of next copy of $\mathcal{A}^* \otimes_{\mathcal{A}(r-1)^*} \mathbf{Z}/(p)$. We have finished the induction. ■

It is now clear that diagram (2.2.11) commutes since we constructed the splitting by induction. Furthermore the composite

$$\Sigma^{-1}(\mathcal{A}^* \otimes_{\mathcal{A}(r-1)^*} \mathbf{Z}/(p)) \longrightarrow \mathcal{A}^* \otimes_{\mathcal{A}(r)^*} F^*/F^*_{l,r} \xrightarrow{1\otimes\gamma} \mathcal{A}^* \otimes_{\mathcal{A}(r)^*} \Sigma^{-1}\mathbf{Z}/(p)$$

carries the generators 1,via y_0, to 1. To complete to proof of Lemma 2.2.13, we have to show the commutativity of diagram (2.2.12). We need one more lemma.

Lemma 2.2.16 *The element $y_k \in \mathcal{A}^* \otimes_{\mathcal{A}(r)^*} F^*/F^*_{lp^r,r}$ is zero unless $k \equiv 0$ mod p^r. It is equal to the sum*

$$\sum_{i+j=k} \chi(\mathcal{P}^i) \otimes h_j$$

where i and j are restricted to

$$i, j \equiv 0 \bmod (p^r).$$

Proof of Lemma 2.2.13. It follows from Lemma 2.2.16. ∎

In order to prove Lemma 2.2.16 , we have to use some identities in $\mathcal{A}^*$ and F^*.

Lemma 2.2.17 *There exist a finite number of elements $a_{i,l} = a_{i,l}(k) \in \mathcal{A}(r)^*$ of degree $(ip^r + lp^{r-1})q$ for $i \leq 0$, such that*

(*i*) $\mathcal{P}^{(kp^r+lp^{r-1})} = \sum_{i+j=k} a_{i,l}\mathcal{P}^{jp^r}$,

(*ii*) $\displaystyle\sum_{i+k=m, 0<j<p} \chi(a_{i,j})h_{kp^r+(p-j)p^{r-1}} = 0,$

(*iii*) $\displaystyle\sum_{(i+k)p+j+l=mp+n, 0\leq j,l<p} \chi(a_{i,j})h_{kp^r+lp^{r-1}} = 0.$

Proof. B^* is a free as left module over $\mathcal{A}(r)^*$ and we can take $\mathcal{P}^{p^r}$ as an $\mathcal{A}(r)^*$-basis. Therefore, we have for each k a unique formula

$$\mathcal{P}^{kp^r+lp^{r-1}} = \sum_{i+j=k} a_{i,l}(k)\mathcal{P}^{jp^r}$$

with coefficients $a_{i,l}(k) \in \mathcal{A}(r)^*$. Since $\mathcal{A}(r)^*$ is a finite algebra and $a_{i,l}(k)$ is of degree $(ip^r + lp^{r-1})q$, the sum can be taken over a finite number of i which does not depend on k. In the dual, the multiplication by $\xi_1^{p^r}$ gives a homomorphism of $\mathcal{A}(r)_*$–comodule. The $\mathcal{A}(r)^*$-module homomorphism induced by the multiplication of $\xi_1^{p^r}$ will send $\mathcal{P}^{kp^r+lp^{r-1}}$ to $\mathcal{P}^{(k-1)p^r+lp^{r-1}}$ and $a_{i,l}(k)\mathcal{P}^{jp^r}$ to $a_{i,l}(k)\mathcal{P}^{(j-1)p^r}$ because $a_{i,l}(k)$ is in $\mathcal{A}(r)^*$. We can deduce that $a_{i,l}(k)$ is independent to k, i.e. (i).

Before giving the proof of the left two formulas, we need to build some techniques. For any $a \in \mathcal{A}^*$, $h = h_i$ or h_i', ah is a scalar multiple of certain h_j or h_j'. We will denote the scalar as $c(ah)$. Now we can define

$$\langle a, h \rangle = c(ah)$$

where $a \in \mathcal{A}^*$, $h = h_i$ or h_i' and $c(ah)$ is the coefficient of ah. We claim that we have the following duality relation for $\langle , \rangle$:

$$\langle a, h_i \rangle = \langle \chi(a), h'_{-i-m} \rangle.$$

where $|a| = mq$. We will prove this in two steps. First we note that

$$\begin{aligned} \langle ab, h_i \rangle &= c(abh_i) \\ &= c(ac(bh_i)h_{i+m'}) \\ &= c(bh_i)c(ah_{i+m'}) \\ &= \langle b, h_i \rangle \langle a, h_{i+m'} \rangle \end{aligned}$$

where we suppose that $|b| = mq$. This identity implies that we only need to prove the the duality formula for the generators $\mathcal{P}^{p^r}$. We start from the following formula about the anti-automorphism

$$\sum_{g+f=p^r} \mathcal{P}^g \chi(\mathcal{P}^f) = 0.$$

Apply both sides to h_i and assume the duality formula for $f < p^r$ then we have

$$\begin{aligned}
-\chi(\mathcal{P}^{p^r})h_i &= \textstyle\sum_{g>0} \mathcal{P}^g \chi(\mathcal{P}^f) h_i \\
&= \textstyle\sum_{g>0} \mathcal{P}^g \langle \chi(\mathcal{P}^f), h_i \rangle h_{i+f} \\
&= \textstyle\sum_{g>0} \binom{(i+f)(p-1)-1}{g} \langle \mathcal{P}^f, h'_{-i-f} \rangle h_{i+p^r} \\
&= \textstyle\sum_{g>0} \binom{(i+f)(p-1)-1}{g} \binom{-(i+f)(p-1)}{f} h_{i+p^r} \\
&= \textstyle\sum_{g>0} (-1)^f \binom{(i+f)(p-1)-1}{g} \binom{(i+f)(p-1)+f-1}{f} h_{i+p^r} \\
&= \textstyle\sum_{g>0} (-1)^f \binom{(i+f)(p-1)+f-1}{p^r} \binom{p^r}{f} h_{i+p^r} \\
&= \binom{i(p-1)-1}{p^r} h_{i+p^r}.
\end{aligned}$$

On the other hand we have another formula

$$\begin{aligned}
\mathcal{P}^{p^r} h'_{-i-p^r} &= \binom{-(i+p^r)(p-1)}{p^r} h'_{-i} \\
&= (-1)^{p^r} \binom{(i+p^r)(p-1)+p^r-1}{p^r} \\
&= -\binom{i(p-1)-1}{p^r}.
\end{aligned}$$

These two imply

$$\langle \chi(\mathcal{P}^{p^r}), h_i \rangle = \langle \mathcal{P}^{p^r}, h'_{-i-p^r} \rangle.$$

There is another property for this bioperation:

$$\langle a, h_{p^r+i} \rangle = \langle a, h_i \rangle$$

$$\langle a, h'_{p^r+i} \rangle = \langle a, h'_i \rangle$$

where $a \in \mathcal{A}(r)^*$. We can begin to prove the part (ii) and (iii).

The formula (ii) is equivalent to the following formula

$$\sum_{0<j<p}\left(\sum_{i+k=m}\chi(a_{i,j})h_{kp^r+(p-j)p^{r-1}}\right)$$
$$= \sum_{0<j<p} x_{m,p,j}h_{(m+1)p^r}$$
$$= 0.$$

(iii) is equivalent to

$$\Sigma_{0\leq j\leq n}(\sum_{i+k=m}\chi(a_{i,j})h_{kp^r+(n-j)p^{r-1}}) + \sum_{n<j<p}(\sum_{i+k=m-1}\chi(a_{i,j})h_{kp^r+(p+n-j)p^{r-1}})$$
$$= \sum_{0\leq j\leq n} x_{m,n,j}h_{mp^r+np^{r-1}} + \sum_{n<j<p} x_{m-1,p+n,j}h_{mp^r+np^{r-1}}$$
$$= 0.$$

We can calculate these x's explicitly by the technique we prepared before. By the duality formula we have the following identity

$$\begin{aligned} x_{m,n,j} &= \sum_{i+k=m}\langle\chi(a_{i,j}), h_{kp^r+(n-j)p^{r-1}}\rangle \\ &= \sum_{i+k=m}\langle a_{i,j}, h'_{-kp^r-np^{r-1}}\rangle \\ &= \sum_{i+k=m}\langle a_{i,j}, h'_{-np^{r-1}}\rangle. \end{aligned}$$

The last identity comes from the periodicity property of operation $\langle,\rangle$. From the other direction we can consider the formula defining the $a_{i,j}$: applying it to the cohomology class $h'_{-\frac{(n-1)(p^v-1)}{p-1}p^r-np^{r-1}}$ where $v \gg 0$. Then the left side is

$$\langle \mathcal{P}^{mp^r+jp^{r-1}}, h'_{-\frac{(n-1)(p^v-1)}{p-1}p^r-np^{r-1}} \rangle$$

$$= \binom{-(n-1)(p^v-1)p^r-np^{r-1})(p-1)}{mp^r+jp^{r-1}}$$

$$= (-1)^{m+j}\binom{(n-1)(p^v-1)p^r+np^{r-1})(p-1)+mp^r+jp^{r-1}-1}{mp^r+jp^{r-1}}$$

$$= (-1)^{m+j}\binom{(m+1)p+j-n-1}{mp+j}$$

$$= (-1)^{m+j}\binom{p+j-n-1}{j}.$$

The right side can be simplified to

$$\sum_{i+k=m} \langle a_{i,j}\mathcal{P}^{kp^r}, h'_{-\frac{(n-1)(p^v-1)}{p-1}p^r-np^{r-1}} \rangle$$

$$= \sum_{i+k=m} \binom{(-\frac{(n-1)(p^v-1)}{p-1}p^r - np^{r-1})(p-1)}{kp^r} \langle a_{i,j}, h'_{kp^r-\frac{(n-1)(p^v-1)}{p-1}p^r-np^{r-1}} \rangle$$

$$= \sum_{i+k=m} (-1)^{kp^r} \binom{(\frac{(n-1)(p^v-1)}{p-1}p^r + np^{r-1})(p-1) + kp^r - 1}{kp^r} \langle a_{i,j}, h'_{kp^r-\frac{(n-1)(p^v-1)}{p-1}p^r-np^{r-1}} \rangle$$

$$= \sum_{i+k=m} (-1)^k \binom{(n-1)p^v + k}{k} \binom{p^r - np^{r-1} - 1}{0} \langle a_{i,j}, h'_{kp^r-\frac{(n-1)(p^v-1)}{p-1}p^r-np^{r-1}} \rangle$$

$$= \sum_{i+k=m} \langle (-1)^{m-i} a_{i,j}, h'_{kp^r-\frac{(n-1)(p^v-1)}{p-1}p^r-np^{r-1}} \rangle.$$

Let us denote the above constant by $x^*_{m,n,j}$. It is easy to verify that

$$\langle a_{m,j}, h'_{-np^{r-1}} \rangle = x^*_{m,n,j} + x^*_{m-1,n,j}$$

which is 0. This finishes the proof of (ii) and (iii). ■

Proof of Lemma 2.2.16. We proceed by induction over r. The result is obviously true for $r = -1$ if we deal with $\mathcal{A}(-1)^*$ as $\mathbf{Z}/(p)$. So we can assume this lemma is true for $r - 1$. Then y_k is zero in $\mathcal{A}^* \otimes_{\mathcal{A}(r)^*} F^*/F^*_{l,r}$ unless $k \equiv 0$ *mod* p^r. If $k = mp^r$ then the inductive hypothesis gives

$$y_{mp^r} = \sum_{i+j=m} \chi(\mathcal{P}^{ip^r}) \otimes h_{jp^r} + \sum_{i+j=m-1, 0<l<p} \chi(\mathcal{P}^{ip^r+lp^{r-1}}) \otimes h_{ip^r+(p-l)p^{r-1}}.$$

We can rewrite the second sum using Lemma 2.2.17 (i) and obtain

$$\begin{aligned} & \textstyle\sum_{i+j=m-1,0<l<p} \chi(\mathcal{P}^{ip^r+lp^{r-1}}) \otimes h_{ip^r+(p-l)p^{r-1}} \\ = & \textstyle\sum_{e+h+j=m-1,0<l<p} \chi(\mathcal{P}^{ep^r})\chi(a_{h,l}) \otimes h_{jp^r+(p-l)p^{r-1}} \\ = & \textstyle\sum_{e+h+j=m-1,0<l<p} \chi(\mathcal{P}^{ep^r}) \otimes \chi(a_{h,l})h_{jp^r+(p-l)p^{r-1}}. \end{aligned}$$

But this gives zero by Lemma 2.2.17 (ii) .

If $k = mp^r + lp^{r-1}$ then the inductive hypothesis gives

$$y_{mp^r+lp^{r-1}} = \sum_{ip+jp+l_1+l_2=mp+l} \chi(\mathcal{P}^{ip^r+l_1p^{r-1}}) \otimes h_{jp^r+l_2p^{r-1}}$$

where $0 \le l_1, l_2 \le p$. We can rewrite the second sum using Lemma 2.2.17 (i) and obtain

$$\begin{aligned} y_{mp^r+lp^{r-1}} &= \textstyle\sum_{ip+jp+l_1+l_2=mp+l} \chi(\mathcal{P}^{ip^r+l_1p^{r-1}}) \otimes h_{jp^r+l_2p^{r-1}} \\ &= \textstyle\sum_{ep+hp+jp+l_1+l_2=mp+l} \chi(\mathcal{P}^{ep^r})\chi(a_{h,l_1}) \otimes h_{jp^r+l_2p^{r-1}} \\ &= \textstyle\sum_{ep+hp+jp+l_1+l_2=mp+l} \chi(\mathcal{P}^{ep^r}) \otimes \chi(a_{h,l_1})h_{jp^r+l_2p^{r-1}}. \end{aligned}$$

So we see that $y_k = 0$ in this case from Lemma 2.2.17 (iii) . ■

Lemma 2.2.2 is stated in cohomology for convenience in its proof. We can rewrite it in homology as

$$\mathcal{A}_*\square_{\mathcal{A}(r)_*}(F_*^{lr}) \cong \bigoplus_j \Sigma^{jp^rq-1}(\mathcal{A}_*\square_{\mathcal{A}(r-1)_*}\mathbf{Z}/(p)).$$

Here j runs over all $j \equiv 0 \bmod p^r$, $j \ge l$ and F_*^{lr} is the dual of F^*/F^*_{lr} , i.e. a bounded below subcomodule of F_* over $\mathcal{A}(r)_*$. We claim that we can deduce the following theorem from above.

Theorem 2.2.18 *There is an isomorphism between*

$$\varprojlim \operatorname{Ext}_{P_*}(\mathbf{Z}/(p), \operatorname{Ext}_{E_*}(\mathbf{Z}/(p), H_*(P_{-n})))$$

and

$$\operatorname{Ext}_{P_*}(\mathbf{Z}/(p), \operatorname{Ext}_{E_*}(\mathbf{Z}/(p), \Sigma^{-1}\mathbf{Z}/(p))).$$

induced by

$$\Sigma^{-1}\mathbf{Z}/(p) \longrightarrow H_*(P_{-n})$$

which sends 1 *to* b_0.

Proof.

$$\begin{aligned}
& \varprojlim_n \operatorname{Ext}^{s,u}_{P_*}(\mathbf{Z}/(p), \operatorname{Ext}^{l}_{E_*}(\mathbf{Z}/(p), H_*(P_{-n}))) \\
\cong\ & \varprojlim_n \varprojlim_r \operatorname{Ext}^{s,u}_{P(r)_*}(\mathbf{Z}/(p), \operatorname{Ext}^{l}_{E(r)_*}(\mathbf{Z}/(p), H_*(P_{-n}))) && (1) \\
\cong\ & \varprojlim_r \varprojlim_n \operatorname{Ext}^{s,u}_{P(r)_*}(\mathbf{Z}/(p), \operatorname{Ext}^{l}_{E(r)_*}(\mathbf{Z}/(p), H_*(P_{-n}))) && (2) \\
\cong\ & \varprojlim_r \operatorname{Ext}^{s,u}_{P(r)_*}(\mathbf{Z}/(p), \operatorname{Ext}^{l}_{E(r)_*}(\mathbf{Z}/(p), H_*(P_{-n_r}))) && (3) \\
\cong\ & \varprojlim_r \operatorname{Ext}^{s,u}_{P(r)_*}(\mathbf{Z}/(p), \operatorname{Ext}^{l}_{E(r)_*}(\mathbf{Z}/(p), F^{l_r}_*)) && (4) \\
\cong\ & \varprojlim_r \operatorname{Ext}^{s,u}_{P_*}(\mathbf{Z}/(p), \operatorname{Ext}^{l}_{E_*}(\mathbf{Z}/(p), \mathcal{A}_*\square_{\mathcal{A}(r)_*} F^{l_r}_*)) && (5) \\
\cong\ & \varprojlim_r \oplus_{j\in\mathbf{Z}} \operatorname{Ext}^{s,u}_{P_*}(\mathbf{Z}/(p), \operatorname{Ext}^{l}_{E_*}(\mathbf{Z}/(p), \Sigma^{jp^r q-1}\mathcal{A}_*\square_{\mathcal{A}(r-1)_*}\mathbf{Z}/(p))) \\
\cong\ & \varprojlim_r \oplus_{j\in\mathbf{Z}} \operatorname{Ext}^{s,u}_{P(r-1)_*}(\mathbf{Z}/(p), \operatorname{Ext}^{l}_{E(r-1)_*}(\mathbf{Z}/(p), \Sigma^{jp^r q-1}\mathbf{Z}/(p))) \\
\cong\ & \operatorname{Ext}^{s+l,u}_{P_*}(\mathbf{Z}/(p), \operatorname{Ext}_{E^*}(\mathbf{Z}/(p), \Sigma^{-1}\mathbf{Z}/(p))). && (6)
\end{aligned}$$

For a fixed n, $H_*(P_{-n})$ is bounded below and $P(r)_*$ and $E(r)_*$ are isomorphic to P_* and E_* up to some degree increasing to infinite with r. (1) follows naturally. (2) comes from the commutativity of the inverse limits. Both (3) and (4) come from the degree reason since we can choose n_r and $-l$ big enough with respect to r for fixed (s,u,t). With the help of the well known change of rings theorem, we have the following isomorphisms

$$\begin{aligned}
& \mathrm{Ext}^{s,u}_{P(r)_*}(\mathbf{Z}/(p), \mathrm{Ext}^{t}_{E(r)_*}(\mathbf{Z}/(p), H_*(P_{-n}))) \\
\cong\ & \mathrm{Ext}^{s,u}_{P_*}(\mathbf{Z}/(p), P_*\Box_{\mathrm{P(r)}_*}\mathrm{Ext}^{t}_{E(r)_*}(\mathbf{Z}/(p), H_*(P_{-n}))) \\
\cong\ & \mathrm{Ext}^{s,u}_{P_*}(\mathbf{Z}/(p), P_*\Box_{\mathrm{P(r)}_*}\mathrm{Cotor}^{t}_{E(r)_*}(\mathbf{Z}/(p), H_*(P_{-n}))) \\
\cong\ & \mathrm{Ext}^{s,u}_{P_*}(\mathbf{Z}/(p), P_*\Box_{\mathrm{P(r)}_*}\mathrm{Cotor}^{t}_{\mathcal{A}(\mathrm{r})_*}(\mathbf{Z}/(p)\Box_{E(r)_*}\mathcal{A}(r)_*, H_*(P_{-n}))) \\
\cong\ & \mathrm{Ext}^{s,u}_{P_*}(\mathbf{Z}/(p), P_*\Box_{\mathrm{P(r)}_*}\mathrm{Cotor}^{t}_{\mathcal{A}(\mathrm{r})_*}(P(r)_*, H_*(P_{-n}))) \\
\cong\ & \mathrm{Ext}^{s,u}_{P_*}(\mathbf{Z}/(p), \mathrm{Cotor}^{t}_{\mathcal{A}(\mathrm{r})_*}(P_*\Box_{\mathrm{P(r)}_*}P(r)_*, H_*(P_{-n}))) \qquad (7) \\
\cong\ & \mathrm{Ext}^{s,u}_{P_*}(\mathbf{Z}/(p), \mathrm{Cotor}^{t}_{\mathcal{A}_*}(P_*, \mathcal{A}_*\Box_{\mathcal{A}(\mathrm{r})_*}H_*(P_{-n}))) \\
\cong\ & \mathrm{Ext}^{s,u}_{P_*}(\mathbf{Z}/(p), \mathrm{Cotor}^{t}_{E_*}(\mathbf{Z}/(p), \mathcal{A}_*\Box_{\mathcal{A}(\mathrm{r})_*}H_*(P_{-n}))) \\
\cong\ & \mathrm{Ext}^{s,u}_{P_*}(\mathbf{Z}/(p), \mathrm{Ext}^{t}_{E_*}(\mathbf{Z}/(p), \mathcal{A}_*\Box_{\mathcal{A}(\mathrm{r})_*}H_*(P_{-n})))
\end{aligned}$$

where (7) comes from the fact that the $P(r)_*$-comodule structure of

$$\mathrm{Cotor}^{t}_{\mathcal{A}(\mathrm{r})_*}(P(r)_*, H_*(P_{-n}))$$

inherits from the $P(r)_*$-comodule structure of $P(r)_*$. (5) is proved. It is easy to obtain (6) after we can get the following commutative diagram from (2.2.12).

$$\begin{array}{ccc}
\mathrm{Ext}_r(F_*^{l_r}) & \overset{\cong}{\longleftrightarrow} & \bigoplus_j \mathrm{Ext}_{r-1}(\Sigma^{jp^r q-1}\mathbf{Z}/(p)) \\
\uparrow & & \uparrow \psi_* \\
\mathrm{Ext}_{r+1}(F_*^{l_{r+1}}) & \overset{\cong}{\longleftrightarrow} & \bigoplus_j \mathrm{Ext}_r(\Sigma^{jp^{r+1} q-1}\mathbf{Z}/(p))
\end{array}$$

Here $\mathrm{Ext}_r(-)$ means $\mathrm{Ext}_{P(r)_*}(\mathrm{Ext}_{E(r)_*}(-))$. The details of maps are exactly as in Lemma 2.2.13 . Moreover the composite

$$\begin{aligned}
\mathrm{Ext}_{P_*}(\mathrm{Ext}_{E_*}(\Sigma^{-1}\mathbf{Z}/(p))) &\longrightarrow \mathrm{Ext}_{P_*}(\mathrm{Ext}_{E_*}(F_*)) \\
&\longrightarrow \mathrm{Ext}_{P(r)_*}(\mathrm{Ext}_{E(r)_*}(F_*^{kr})) \\
&\longrightarrow \bigoplus_j \mathrm{Ext}_{P(r-1)_*}(\mathrm{Ext}_{E(r-1)_*}(\Sigma^{jp^r q-1}\mathbf{Z}/(p))) \\
&\longrightarrow \mathrm{Ext}_{P(r-1)_*}(\mathrm{Ext}_{E(r-1)_*}(\Sigma^{-1}\mathbf{Z}/(p)))
\end{aligned}$$

is the obvious projection map. ∎

Proof of Theorem B. By diagram (2.1.5), we can see that Theorem 2.2.18 implys Theorem B. ∎

2.3 Generalization of Theorem B

Our task in this section is to prove Theorem C. Before that we need a lemma on which our generalization depends heavily.

Lemma 2.3.1 *If*

$$0 \longrightarrow M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \longrightarrow 0$$

is a short exact sequence of finitely presented (BP_*, Γ)*-comodules and* $M_3 = \Sigma^s BP_*/I_k$*, then the system of short exact sequence*

$$0 \longrightarrow Im(1 \otimes f) \longrightarrow BP_*(P_{-n}) \otimes M_2 \longrightarrow BP_*(P_{-n}) \otimes M_3 \longrightarrow 0$$

gives rise to a long exact sequence

$$\cdots \longrightarrow \varprojlim \mathrm{Ext}^s_\Gamma(BP_*(P_{-n}) \otimes M_1) \longrightarrow \varprojlim \mathrm{Ext}^s_\Gamma(BP_*(P_{-n}) \otimes M_2)$$
$$\longrightarrow \varprojlim \mathrm{Ext}^s_\Gamma(BP_*(P_{-n}) \otimes M_3) \longrightarrow$$

Proof. By the Key lemma in [Lan82] we can deduce that

$$\begin{aligned} &\mathrm{Ker}\{BP_*(P_{-n}) \otimes M_1 \longrightarrow \mathrm{Im}(1 \otimes f)\} \\ = \ &\mathrm{Im}\{\mathrm{Tor}(BP_*(P_{-n}), M_3) \longrightarrow BP_*(P_{-n}) \otimes M_1\}. \end{aligned}$$

is finitely presented if we can show that $\mathrm{Tor}(BP_*(P_{-n}), M_3)$ is finitely presented and $BP_*(P_{-n}) \otimes M_1$ has finite homological dimension. The first is actually Lemma 5.1.4 in [Sad] and the second comes from the fact that M_1 is finitely presented. Since Ker is independent of n except for degree, the fact that it is finitely generated implies that

$$\varprojlim \mathrm{Ker} = 0.$$

Thus the systems of $\{BP_*(P_{-n}) \otimes M_1\}$ and $\{\mathrm{Im}(1 \otimes f)\}$ are pro-isomorphic, so

$$\varprojlim \mathrm{Ext}_{\mathrm{BP_*BP}}(BP_*(P_{-n}) \otimes M_1) \cong \varprojlim \mathrm{Ext}(\mathrm{Im}(1 \otimes \mathrm{f})).$$

This completes the proof. ■

Corollary 2.3.2 *There is an isomorphism*

$$\mathrm{Ext}_{\Gamma}^{s,t}(BP_*, \Sigma^{-1}(BP_*/I_k)) \cong \varprojlim \mathrm{Ext}_{\Gamma}^{s,t}(BP_*, BP_*(P_{-n})/I_k)).$$

Proof. We can apply Lemma 2.3.1 to the following series

$$0 \longrightarrow BP_*/I_i \longrightarrow BP_*/I_i \longrightarrow BP_*/I_{i+1} \longrightarrow 0$$

for $i = 1, \cdots, k-1$. The isomorphism is proved by induction with the help of five-lemma. ∎

Proof of Theorem C: It is actually another corollary of Lemma 2.3.1. For any finite complex X, we have a series of short exact sequences of (BP_*, BP_*BP)-comodules

$$0 \longrightarrow L_i \longrightarrow L_{i+1} \longrightarrow \Sigma^{s_{i+1}} BP_*/I_{j_i} \longrightarrow 0$$

for $i = 0, \cdots, m$ with $L_0 = \Sigma^{s_0} BP_*/I_{j_0}$ and $L_m = BP_*(X)$ by the Landweber filtration theorem in [Lan76]. We can apply Lemma 2.3.1 to the short exact sequences above to obtain that

$$\mathrm{Ext}_{\Gamma}(\Sigma^{-1} BP_*(X)) \longrightarrow \varprojlim \mathrm{Ext}_{\Gamma}(BP_*(P_{-n}) \otimes BP_*(X))$$

is an isomorphism. On the other hand by Corollary 5.1.7 in [Sad] we have

$$\varprojlim \mathrm{Ext}_{BP_*BP}(BP_*(P_{-n}) \otimes BP_*(X)) \cong \varprojlim \mathrm{Ext}_{BP_*BP}(BP_*(P_{-n} \wedge X)).$$

This completes the proof. ∎

Chapter 3

The Realization of $\mathcal{A}(n)^*$

3.1 Notations and Preliminaries

It is easy to see from the coproduct formulas in Lemma 1.1.9 that $E_* = E(\tau_0, \cdots)$ is a quotient-Hopf-algebra of $\mathcal{A}_*$ and $P_* = P(\xi_1, \cdots)$ is a sub-Hopf-algebra of $\mathcal{A}_*$. We denote $E(\tau_0, \tau_1, \cdots, \tau_n)$ as $E(n)_*$. It is a sub-Hopf-algebra of E_*. We will denote $B(n)_*$ as $P_* \otimes E(n)_*$. It is a sub-Hopf-algebra of $\mathcal{A}_*$. The quotient-Hopf-algebra $E(\tau_{n+1}, \cdots) = Q(n)_*$, of $\mathcal{A}_*$ appears in the following Hopf-algebra extension

$$B(n)_* \longrightarrow P_* \otimes E_* \cong \mathcal{A}_* \longrightarrow Q(n)_*.$$

We will use P^*, E^*, $E(n)^*$, $B(n)^*$ and $Q(n)^*$ to denote the dual of P_*, E_*, $E(n)_*$, $B(n)_*$ and $Q(n)_*$ respectively. Out of them, P^* and $B(n)^*$ are quotient-Hopf-algebras of $\mathcal{A}^*$ and E^* and $Q(n)^*$ are sub-Hopf-algebras of $\mathcal{A}^*$. $E(n)^*$ is a quotient-Hopf-algebra of E^*. As usual we denote the dual of τ_i as Q_i. So we have

$$E^* = E(Q_0, Q_1, \cdots),$$
$$E(n)^* = E(Q_0, \cdots, Q_n),$$
$$Q(n)^* = E(Q_{n+1}, \cdots).$$

Now we like to introduce some basic knowledge of BP-theory.

Theorem 3.1.1 [BP66] , [Qui69] For each prime p there is an associative commutative ring spectrum BP such that

(i) $BP_* = \pi_*(BP) = Z_{(p)}[v_1, \cdots]$ with $v_i \in \pi_{2(p^n-1)}(BP)$.

(ii) $H_*(BP, \mathbf{Z}/(p)) = \mathbf{Z}/(p)[t_1, \cdots] \cong P_*$ as comodule of $\mathcal{A}_*$.

(iii) $(BP_*, BP_*(BP) = BP_*[t_1, \cdots])$ is a Hopf-algebroid(See [Rav86] Appendix A1).

(iv) The map

$$BP \wedge BP \xrightarrow{T \wedge 1_{BP}} H\mathbf{Z}/(p) \wedge BP \xrightarrow{1_{H\mathbf{Z}/(p)} \wedge T} H\mathbf{Z}/(p) \wedge H\mathbf{Z}/(p)$$

induces a homomorphism

$$BP_*(BP) \longrightarrow H_*(BP) \longrightarrow A_*$$

as Hopf-algebroids. Here $[T] = 1 \in H^0(BP)$. $(T \wedge 1_{BP})_*$ is a projection from BP_* to $\mathbf{Z}/(p)$ and identity on t_i's. $(1_{H\mathbf{Z}/(p)} \wedge T)_*$ sends t_i to $c(\xi_i)$.

Lemma 3.1.2 [Qui69] Let $R = \{r_E : E = (e_1, e_2, \cdots)\}$. Then $BP^*BP \cong BP^*\hat{\otimes}R$. r_E is a lifting in the following diagram.

$$\begin{array}{ccc} BP & \xrightarrow{r_E} & \Sigma^{|r_E|}BP \\ \Big\downarrow T & & \Big\downarrow T \\ H\mathbf{Z}/(p) & \xrightarrow{c(\mathcal{P}^E)} & \Sigma^{|r^E|}H\mathbf{Z}/(p) \end{array}$$

Here $c(\ )$ is the conjugacy in Steenrod algebra and $\mathcal{P}^E$ is the reduced power correspondent to E.

$\mathcal{A}^*$ is the ordinary $\mathbf{Z}/(\mathrm{p})$-cohomology of spectrum $H\mathbf{Z}/(p)$. $H\mathbf{Z}/(p)$ plays an important role in Toda's construction in [Tod71]. For our consideration of $B(n)^*$-modules, we have a similar spectrum called $P(n+1)$. It has cohomology

of $B(n)^*$-modules, we have a similar spectrum called $P(n+1)$. It has cohomology

$$H^*(P(n+1)) \cong B(n)^*$$

Theorem 3.1.3 ([JW75] 2.9, 2.12, 2.14, 2.15 , [Wur77] 2.13) For each prime p and integer $n \geq 0$, there is a BP-module spectrum $P(n)$ with the following properties.

(i) $P(n)$ is a ring spectrum. For $p > 2$, the multiplication is unique and commutative;

(ii) It is defined inductively by the following fibre sequences

$$\Sigma^{|v_i|}P(i) \xrightarrow{v_i} P(i) \longrightarrow P(i+1)$$

for $i \geq 0$. Here v_0 denotes p and v_i is defined by

$$\Sigma^{|v_i|}P(i) \xrightarrow{\simeq} S^{|v_i|} \wedge P(i) \xrightarrow{v_i \wedge 1_{P(i)}} P(i) \wedge P(i) \xrightarrow{m} P(i).$$

The homotopy group of $P(n)$ is $\pi_*(P(n)) = BP_*/I_n$.

(iii) We can associate each BP operation r_E to a $P(n)$ operation $\Phi_n(r_E)$. Inparticularly, when $n = \infty$ we recover $c(\mathcal{P}^E)$. We have the following homotopy commutative diagram.

$$\begin{array}{ccc}
BP & \xrightarrow{r_E} & \Sigma^{|r_E|}BP \\
\downarrow & & \downarrow \\
P(n) & \xrightarrow{\Phi_n(r_E)} & \Sigma^{|r_E|}P(n) \\
\downarrow & & \downarrow \\
H\mathbf{Z}/(p) & \xrightarrow{c(\mathcal{P}^E)} & \Sigma^{|r_E|}H\mathbf{Z}/(p)
\end{array}$$

Furthermore when $2p - 2 > n$, the choice of $\Phi_n(r^E)$ is unique.

(iv)

$$\Phi_n : BP^*/I_n \hat{\otimes} R \otimes E[Q_0, Q_1, \cdots, Q_{n-1}] \longrightarrow P(n)^*P(n)$$

is an isomorphism of left BP^*/I_n-modules. Furthermore when $2p - 2 > n$, $P(n)^{iq}(P(n)) = \Phi_n((P(n)^* \hat{\otimes} R)^{iq}$. $\Phi_n((P(n)^* \hat{\otimes} R))$ inherits its $BP^*(BP)$ comodule structure from BP^*BP via the projection

$$BP^* \hat{\otimes} R \longrightarrow P(n)^* \hat{\otimes} R.$$

(v)

$$P(n)_*(P(n)) = BP_*/I_n[t_1, \cdots] \otimes E(a_0, \cdots, a_{n-1})$$

Here a_i is the dual of Q_i. The coproduct of t_i inherits from coproduct in $BP_*(BP)$ and

$$\Delta(a_s) = \Sigma_{i+j=s} a_i \otimes t_j^{p^i} + 1 \otimes a_s.$$

(vi) The Thom map

$$T : P(n) \longrightarrow H\mathbf{Z}/(p)$$

induces a homomorphism

$$H^*(H\mathbf{Z}/(p)) \cong A^* \longrightarrow H^*(P(n)) \cong B(n-1)^*.$$

This homomorphism is the projection onto $B(n-1)^*$ and sends $c(\mathcal{P}^i)$ to $\mathcal{P}^i$.

The next lemma is useful in the proof of Lemma 3.2.1.

Lemma 3.1.4 *Let*

$$h : \Sigma^k B(n)^* \longrightarrow H^*(X)$$

be a $B(n)^*$-homomorphism and

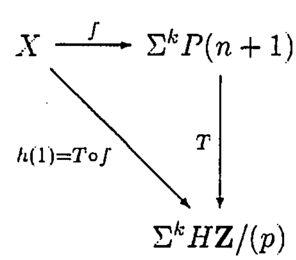
$$P(n+1)^k(X) \longrightarrow H^k(X)$$

be surjective. Then there is a map

$$f : X \longrightarrow \Sigma^k P(n+1)$$

such that $h = H^*(f)$.

Proof. Take $h(1) \in H^k(X)$. It should be hit by an element from $P(n+1)^*(X)$. In other words, we have the following homotopy commutative diagram.



Since $1 \in H^0(\mathit{\Gamma}(n+1))$ is represented by T, we can deduce from the above that $H^*(f)(1) = h(1)$. For any $a \in B(n)^* = H^*(\mathit{\Gamma}(n+1))$, we can find $\hat{a} \in \mathcal{A}^*$ which projects to a. It follows that

$$\begin{aligned} H^*(f)(a) &= H^*(f)(\hat{a}1) \\ &= \hat{a}H^*(f)(1) \\ &= \hat{a}h(1) \\ &= ah(1) \\ &= h(a) \end{aligned}$$

because $H^*(\mathit{\Gamma}(n+1))$ is actually a $B(n)^*$-module. Thus

$$h = H^*(f).$$

■

Remark 3.1.5 *Take $X = \mathit{\Gamma}(n+1)$ and $k \equiv 0$ modulo q. Then $t_\alpha \in B^k(n)$ is hit by $r_\alpha \in \mathit{\Gamma}(n+1)^k\mathit{\Gamma}(n+1)$. Hence we can take f such that*

$$[f] = r_\alpha \in R \subset \mathit{\Gamma}(n+1)^*\mathit{\Gamma}(n+1).$$

3.2 New sufficient conditions

The goal of this section is to prove two key lemmas which are actually the sufficient conditions for realizations of certain $\mathcal{A}^*$-modules and existence of v_{n+1}-maps. The first lemma is about the realization, the analog of Lemma 3.1 in [Tod71].

Lemma 3.2.1 *An $\mathcal{A}^*$-module N over $\mathbf{Z}/(p)$ is realizable if it satisfies the following conditions.*

(i) *N is an $E(n)^*$-free module and has trivial Q_{n+i} actions for all $i > 0$.*

(ii) *M, which denotes $N \otimes_{E(n)^*} \mathbf{Z}/(p)$, is concentrated in the degrees congruent to 0 modulo q.*

(iii) *For all $t - s \leq w$, there exists a positive integer $m < q - n - 1$ such that*

$$\mathrm{Ext}^{s,t}_{B(n)^*}(N, \mathbf{Z}/(p)) \doteq \{a_{si} \in Ext^{s,qr_{si}} | i \in I_s\} = 0$$

when all $s > m$. Here w is not less than the largest dimension of N and congruent to $n+1$ respect to q. Without loss of generality, we will assume that $I_0 = \{1\}$.

(iv) *For any $a_{1\alpha}$ with $\alpha \in I_1$ which can be represented by $t_\alpha \in B(n)^*$, we can find correspondent $r_\alpha \in P(n+1)^* P(n+1)$ as in Remark 3.1.5. The cokernel of $P(n+1)^*(\oplus_{\alpha \in I_1} r_\alpha)$ is isomorphic to $N \otimes BP^*/I_{n+1}$ and the cokernel of $BP^*(\oplus_{\alpha\in I_1} r_\alpha)$ is isomorphic to $M \otimes BP^*/I_{n+1}$ respectively.*

Remark 3.2.2 *(ii) implies that*

$$\mathrm{Ext}^{s,t}_{B(n)^*}(N, \mathbf{Z}/(p)) = 0$$

if $t \not\equiv 0$ mod q because of the following change-of-rings isomorphism

$$\mathrm{Ext}_{B(n)^*}(N, \mathbf{Z}/(p)) = Ext_{P^*}(M, \mathbf{Z}/(p))$$

and the fact that all elements in P^* have degrees congruent to 0 modulo q.

Proof. By condition (i) we know that N is actually a module over $B(n)^*$. We consider the following minimal resolution of N as module over $B(n)^*$

$$0 \longleftarrow N \xleftarrow{\iota} C^0 \xleftarrow{d_1} C^1 \longleftarrow \cdots \tag{3.2.3}$$

where

$$C^s = B(n)^* \otimes \mathrm{Ext}^s(N, \mathbf{Z}/(p)).$$

By the change-of-rings isomorphism in Remark 3.2.2, the minimal resolution above is the same as the tensor product of $E(n)^*$ and the minimal resolution of M as P^*-module. Hence all kernels and cokernels in the above resolution are $E(n)^*$-free.

We claim that there exist a series of spectra X_s for $s = 1, \cdots, m$ such that they satisfy the following inductive conditions.

(a) There is a fibre sequence

$$X_{s+1} \xrightarrow{i_s} X_s \xrightarrow{\pi_s} B_s$$

such that

$$H^{t-s+1}(B_s) = C^{s,t}$$

for each $m \geq s \geq 1$. In fact we will choose B_s to be $\bigvee \Sigma^{qr_{si}-s+1} P(n+1)$.

(b) Applying the $H^*(-)$ to the long exact sequence in (a), we can obtain the following short exact sequence

$$0 \longrightarrow N \longrightarrow H^*(X_{s+1}) \longrightarrow \mathrm{Ker}(d_s) \longrightarrow 0.$$

Meanwhile $H^*(X_{s+1})$ is split into the direct sum of N^* and $\mathrm{Ker}(d_s)$ as A^*-module.

(c) The homomorphism induced by the Thom map

$$P(n+1)^k(X_{s+1}) \longrightarrow H^k(X_{s+1})$$

is surjective for $k \equiv -s \bmod q$.

As the first step of the inductive proof, we have to prove (a), (b) and (c) for $s = 1$. Let

$$X_1 = \Sigma^{qr_{0,1}} P(n+1).$$

Then

$$H^*(X_1) = B(n)^* \otimes \mathrm{Ext}^0_{B(n)^*}(N, \mathbf{Z}/(p)).$$

We want to find a map

$$\pi_1 : X_1 \longrightarrow B_1$$

such that the homomorphism

$$H^*(\pi_1) : H^*(B_1) = C^1 \longrightarrow H^*(X_1) = C^0$$

is identical to d_1. Since both X_1 and B_1 are wedges of $P(n)$, it suffices to establish similar result on the following canonical example.

Given

$$h : H^*(\Sigma^{qk} P(n+1)) \longrightarrow H^*(P(n+1))$$

to be a homomorphism of $\mathcal{A}^*$-modules, we can find a map

$$f : P(n+1) \longrightarrow \Sigma^{qk} P(n+1)$$

such that $H^*(f) = h$ according to Remark 3.1.5. (a) for $s = 1$ is proved.

Because N is isomorphic to $\mathrm{Coker}(d_1)$, we can easily identify the short exact sequence in (b) from the long exact sequence obtained by applying $H^*(-)$ to fibre sequence in (a). $H^*(X_2)$ can be split into a direct sum of N^* and $\mathrm{Ker}(d_1)$ as $\mathbf{Z}/(p)$-module. We will abuse the notations N^* and $\mathrm{Ker}(d_1)$ to denote the corresponding summands which are $\mathbf{Z}/(p)$ sub-modules of $H^*(X_2)$. The splitting of $H^*(X_2)$ as an $\mathcal{A}^*$-module is equivalent to

$$\mathcal{A}^*\mathrm{Ker}(d_1) = \mathrm{Ker}(d_1).$$

Assuming

$$E^*\mathrm{Ker}(d_1) = \mathrm{Ker}(d_1),$$

we can show

$$P^*\mathrm{Ker}(d_1) = \mathrm{Ker}(d_1).$$

Since $\mathrm{Ker}(d_1)$ is a free $E(n)^*$-module with generators at dimensions congruent to -1 modulo$q$ and

$$N^i = 0 \qquad i \equiv n+2, \cdots, q-1 \qquad \bmod \quad q,$$

we can deduce that

$$P^*(\mathbf{Z}/(p) \otimes_{E(n)^*} \mathrm{Ker}(d_1)) = (\mathbf{Z}/(p) \otimes_{E(n)^*} \mathrm{Ker}(d_1)).$$

Hence

$$\begin{aligned} P^*\mathrm{Ker}(d_1) &= P^*E(n)^*(\mathbf{Z}/(p) \otimes_{E(n)^*} \mathrm{Ker}(d_1)) \\ &\subset P^*E^*(\mathbf{Z}/(p) \otimes_{E(n)^*} \mathrm{Ker}(d_1)) \\ &= E^*P^*(\mathbf{Z}/(p) \otimes_{E(n)^*} \mathrm{Ker}(d_1)) \\ &= E^*(\mathbf{Z}/(p) \otimes_{E(n)^*} \mathrm{Ker}(d_1)) \\ &\subset E^*\mathrm{Ker}(d_1) \\ &= \mathrm{Ker}(d_1). \end{aligned}$$

We still have to prove

$$E^*\mathrm{Ker}(d_1) = \mathrm{Ker}(d_1).$$

Because $\mathrm{Ker}(d_1)$ is $E(n)^*$-free, we just need to prove there is no nontrivial Q_{n+i} action with $i > 0$ on $\mathbf{Z}/(p) \otimes_{E(n)^*} \mathrm{Ker}(d_1)$.

Applying Adams spectral sequence to $[X_2, BP]$, the E_2 term is

$$\begin{aligned} \mathrm{Ext}_{\mathcal{A}^*}(H_*(X_2), P_*) &\cong \mathrm{Ext}_{\mathcal{A}^*}(H_*(X_2), \mathcal{A}^*\square_{E_*}\mathbf{Z}/(p)) \\ &\cong \mathrm{Ext}_{E_*}(\mathbf{Z}/(p)\square_{Q(n)^*}(\mathbf{Z}/(p)\square_{E(n)_*}H_*(X_2)), \mathbf{Z}/(p)) \\ &\cong \mathrm{Ext}_{Q(n)_*}(\mathbf{Z}/(p)\square_{E(n)_*}H_*(X_2), \mathbf{Z}/(p)). \end{aligned}$$

We can say that the E_2 term is a sub-quotient of $(\mathbf{Z}/(p) \otimes_{E(n)^*} H^*(X_2))\hat{\otimes} BP^*/I_{n+1}$. If there is a nontrivial Q_{n+1} action from $\mathbf{Z}/(p) \otimes_{E(n)^*} \mathrm{Ker}(d_1)$ to N, at least one element of

$$M\hat{\otimes}\mathbf{Z}/(p)[v_{n+1}, \cdots] \subset (\mathbf{Z}/(p) \otimes_{E(n)^*} H^*(X_2))\hat{\otimes} BP^*/I_{n+1}$$

will not appear in Adams spectral sequence E_2. This contradicts to the fact that $BP^*/I_{n+1}\hat{\otimes}M$ is mapped into $BP^*(X_2)$ in condition (iv). This proves (b) for $s = 1$.

Similarly we can show that the E_2-term of Adams spectral sequence for $[X_2, P(n+1)]$ is

$$H^*(X_2)\hat{\otimes}BP^*/I_{n+1} = (N \oplus \mathrm{Ker}(d_1))\hat{\otimes}BP^*/I_{n+1}.$$

Here $N\hat{\otimes}BP^*/I_{n+1}$ comes from the E_2term of Adams spectral sequence for $[\Sigma^{r_{0}1q}P(n+1), P(n+1)]$ and $\mathrm{Ker}(d_1)\hat{\otimes}BP^*/I_{n+1}$ is mapped into the E_2-term of Adams spectral sequence for $\bigvee \Sigma^{r_{1i}q-1}[P(n+1), P(n+1)]$. On the other hand the Adams spectral sequence for $[P(n+1), P(n+1)]$ collapses from E_2 term. Thus the only possible nontrivial Adams differential is between two summands. But if this happens, $P(n)^*(X_2)$ will not have a summand of $N\hat{\otimes}BP^*/I_{n+1}$. We have a contradiction to the condition (iv). So

$$P(n+1)^*(X_2) = H^*(X_2)\hat{\otimes}BP^*/I_{n+1}.$$

This will imply the condition (c) for $s = 1$.

Suppose we can show that (a), (b) and (c) are correct for 1, 2, $\cdots$, s. We have to prove (a) and (b) and (c) for $s+1$ in order to finish the proof of this lemma.

The following homomorphism of $\mathcal{A}^*$ modules

$$d_{s+1} : C^{s+2} \longrightarrow \mathrm{Ker}(d_s) \subset C^{s+1}$$

is actually a homomorphism of $B(n)^*$ modules. By (b) for s we can extend the above homomorphism to

$$C^{s+2} \longrightarrow H^*(X_{s+1}).$$

This is still a homomorphism of $\mathcal{A}^*$ and $B(n)^*$ modules. The image of $B(n)^*$ generators is in $H^k(X_{s+1})$ with $k \equiv -s$ modulo q. On the other hand by (c) for s, we know all elements in cohomology in these dimensions are hit by Thom homomorphism from $P(n+1)$ cohomology. According to Remark 3.1.5, we can obtain that the homomorphism can be realized by

$$\pi_{s+1} : X_{s+2} \longrightarrow B_{s+2}.$$

(a) for $s+1$ is proved.

We can easily obtain the short exact sequence and just need to show

$$Q(n)^*\mathrm{Ker}(d_{s+1}) \subset \mathrm{Ker}(d_{s+1})$$

in order to show the splitting as $\mathcal{A}^*$-module. It suffices to show that

$$Q_{n+i}(\mathbf{Z}/(p) \otimes_{E(n)^*} \mathrm{Ker}(d_{s+1})) \cap N = 0$$

as we do for $s = 0$. It is obvious this time because

$$N^k = 0$$

for $k \equiv -s$ modulo q for $s \geq 1$ and

$$(\mathbf{Z}/(p) \otimes_{E(n)^*} \mathrm{Ker}(d_{s+1}))^k =$$

except $k \equiv -s-1$ modulo q. (b) is proved for $s+1$.

Apply Adams spectral sequence to

$$[X_{s+2}, P(n)].$$

Since the cohomology of X_{s+2} is a direct sum of N and $\mathrm{Ker}(d_{s+1})$, we can obtain the E_2 term as following

$$E_2 = (N \oplus \mathrm{Ker}(d_{s+1}))\hat{\otimes} BP^*/I_{n+1}.$$

Considering homology degree, we can see that

$$(N\hat{\otimes} BP^*/I_{n+1})^k = 0$$

except when

$$k \equiv 0, -1, \cdots, -n-1 \qquad \mathrm{mod} \quad q$$

and

$$(\mathrm{Ker}(d_{s+1})\hat{\otimes} BP^*/I_{n+1})^k = 0$$

except when

$$k \equiv s+1, s, \cdots, s-n \qquad \mathrm{mod} \quad q.$$

We know that the differentials in Adams spectral sequence lower degree by 1. Thus

$$(\mathbf{Z}/(p) \otimes_{E(n)^*} \mathrm{Ker}(d_{s+1}))\hat{\otimes} BP^*/I_{n+1}$$

which only has elements at dimensions congruent to $s+1$ modulo q will survive in Adams spectral sequence when

$$s+1 < q-n-1.$$

According to the condition (ii) , we know $s+1 \leq m$ and $m < q-n-1$. Even though we do not prove the convergence of Adams spectral sequence for $[X_{s+1}, P(n+1)]$ here, we do know that it converges to $\varprojlim P(n+1)^*(X_{s+1}^{(i)})$ where $X_{s+1}^{(i)}$ is the i-the skeleton of X_{s+1}. The survival of

$$(\mathbf{Z}/(p) \otimes_{E(n)^*} H^*(X_{s+2}))\hat{\otimes} BP^*/I_{n+1}$$

in the Adams spectral sequence will guarantee the sujectivity we need for (c). We just need to notice that $\varprojlim H^*(X_{s+1}^{(i)}) = H^*(X_{s+1})$. Hence (c) is proved for $s+1$ and we finish the induction.

Now we have a spectrum X_m which satisfies

$$H^*(X_{m+1}) = N \oplus \mathrm{Ker}(d_m).$$

On the other hand condition (iii) will imply that

$$(\mathrm{Ker}(d_m))^k = 0.$$

for $k \leq w$.

$$(\mathrm{Ker}(d_m))^k = 0$$

except when $k \equiv -m, \cdots, n+1-m$ modulo q and $w \equiv n+1$ modulo q. On the other hand, $n+2 < q-m$. Hence we conclude

$$H^{w+1}(X_{m+1}) = 0.$$

We can take the w-skeleton of X_{m+1} which has cohomology N.

Lemma 3.2.3 *Consider an $\mathcal{A}^*$ module N which satisfies the conditions (i), (ii) and (iv) and the condition (iii) for w, m and $w + d|v_{n+1}|$, $m' \geq m$ respectively. The conditions (i), (ii), (iii) and (iv) are the same as in Lemma 3.2.1. Then there exists v_{n+1}-map*

$$v_{n+1}^d : \Sigma^{d|v_{n+1}|} N \longrightarrow N.$$

Here we use the notation N to denote the realization of $\mathcal{A}^*$-module N.

Proof. According to the proof of Lemma 3.2.1, we realize N by constructing a series of fibre sequences. The realization is the w-skeleton of X_{m+1}. Similarly we can replace w by $w + d|w_{n+1}|$ and m by m' and construct more fibre sequences. N can also be obtained by taking the $w + d|v_{n+1}|$ skeleton of $X_{m'+1}$. We claim we can construct the following commutative diagrams

$$\begin{array}{ccccc}
\Sigma^{d|v_{n+1}|} X_{s+1} & \xrightarrow{i_s} & \Sigma^{d|v_{n+1}|} X_s & \xrightarrow{\Sigma^{d|v_{n+1}|}\pi_s} & \Sigma^{d|v_{n+1}|} B_s \\
\downarrow v_{n+1}^d & & \downarrow v_{n+1}^d & & \downarrow v_{n+1}^d \\
X_{s+1} & \xrightarrow{i_s} & X_s & \xrightarrow{\pi_s} & B_s
\end{array}$$

for $s = 1, \cdots, m$. Here v_{n+1}^d will induced v_{n+1}^d multiplication in BP homology. We will prove it by induction.

X_1 and B_s for $s = 1, \cdots, m$ are wedges of $P(n+1)$. As we can see in Theorem 3.2.1, v_{n+1}^d is an element of $P(n+1)^* P(n+1)$. Hence we can always find v_{n+1} self-maps for B_s and X_1. These maps induce v_{n+1}^d multiplications in BP homology. The commutativity for the second square in the above diagram is equivalent to

$$P(n+1)^*(v_{n+1}^d)([\pi_s]) = v_{n+1}^d[\pi_s]. \tag{3.2.4}$$

because we can regard $[\pi_s]$ as an element in $P(n+1)^*(X_s)$. We claim that we can show that (3.2.4) is true by showing that v_{n+1}^d induces v_{n+1}^d homomorphism on BP homology and $P(n)^k(X_{s+1})$ with $k \equiv -s \bmod q$ for $s = 1, \cdots, m$.

For $s = 1$, (3.2.4) can be obtained from Theorem 3.1.3 since both B_1 and X_1 are wedges of $P(n+1)$ and we have

$$\eta_L(v_{n+1}) = v_{n+1}$$

in BP^*/I_{n+1} and $P(n+1)^*P(n+1)$. Thus we can find self-map v_{n+1}^d for X_2. Applying Adams spectral sequenceto $BP_*(X_2)$, we have

$$\begin{aligned} E_2 &\cong \mathrm{Ext}_{\mathcal{A}_*}(\mathbf{Z}/(p), P_* \otimes H_*(X_2)) \\ &\cong \mathrm{Ext}_{\mathcal{A}_*}(\mathbf{Z}/(p), \mathcal{A}_* \square_{Q(n)^*} (M_* \oplus (\mathbf{Z}/(p) \square_{E(n)_*} \mathrm{Ker}(d_1)_*)) \\ &\cong \mathrm{Ext}_{Q(n)_*}(\mathbf{Z}/(p), M_* \oplus (\mathbf{Z}/(p) \square_{E(n)_*} \mathrm{Ker}(d_1)_*) \\ &\cong BP_*/I_{n+1} \otimes (M_* \oplus \mathbf{Z}/(p) \square_{E(n)_*} \mathrm{Ker}(d_1)_*). \end{aligned}$$

By condition (iv), we can deduce that the E_2-term collapses as we do for the Adams spectral sequence for $P(n+1)$ cohomology in the proof of Lemma 3.2.1.We can conclude that the BP homology of X_2 consists of two direct summands

$$BP_*/I_{n+1} \otimes M_*$$

and

$$BP_*/I_{n+1} \otimes (\mathbf{Z}/(p) \square_{E(n)_*} \mathrm{Ker}(d_1)_*).$$

as BP_*-modules. We also know from (b) in the proof of Lemma 3.2.1 that the first summand goes to $BP_*(X_1)$ and the second summand comes from $BP_*(\Sigma^{-1}B_1)$. On the other hand M_* and $\mathbf{Z}/(p)\square_{E(n)_*}\mathrm{Ker}(d_1)_*$ are concentrated in dimensions congruent to 0 and -1 modulo q respectively. Hence

$$BP_*(v_{n+1}^d)(BP_*/I_{n+1} \otimes M_*) \cap (BP_*/I_{n+1} \otimes (\mathbf{Z}/(p) \square_{E(n)_*} \mathrm{Ker}(d_1)_*) = 0.$$

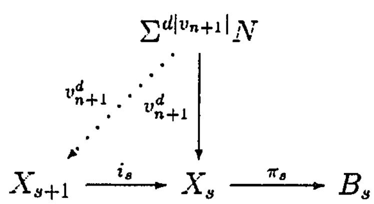
Noting that v_{n+1}^d induces v_{n+1}^d-multiplication on the BP homology of X_1 and B_1, we can see that so does the self-map v_{n+1}^d on the BP homology of X_2. Similarly we can show that all elements of $P(n+1)^k(X_2)$ go to $P(n+1)^k(B_1)$ for $k \equiv -1$ modulo q. Hence we can conclude that self-map v_{n+1}^d of X_2 has the required property on BP homology and $P(n+1)$ cohomology.

Suppose we can construct the diagram with required properties for s. We have to show that we can also do it for $s+1$. The formula (3.2.4) and commutativity of the second square follow from the fact that π is a wedges of maps which represent the elements in $P(n+1)^k(X_{s+1})$ with $k \equiv -s$ modulo q. Hence we can construct the self-map for X_{s+2}. Applying Adams spectral sequence to $BP \wedge X_{s+2}$, we obtain

$$E_2 = BP_*/I_{n+1} \otimes (M_* \oplus (\mathbf{Z}/(p)\square_{E(n)_*}\mathrm{Ker}(d_{s+1})_*)).$$

from our knowledge about $H^*(X_{s+2})$. The above Adams spectral sequence collapses since there is no differential between two summands by degree reason. Hence $BP_*(X_{s+2})$ has two summands as BP_*-modules which are from $BP_*(X_{s+1})$ and $BP_*(B_s)$. It follows that $BP_*(v_{n+1}^d)$ is v_{n+1}^d-multiplication. We can also prove the $P(n+1)^*(v_{n+1}^d)$ has the required property on the $P(n+1)^*(X_{s+2})$ as we do for X_2.

So far we find the self-map of X_m and it induces v_{n+1}^d-multiplication on BP homology. We claim we can construct the following diagrams



for $s = m, \cdots, m'$ such that $BP_*(v_{n+1}^d)$ is v_{n+1}^d multiplication. Here $BP_*(N)$ can be seen as a summand of $BP_*(X_s)$. Since $\Sigma^{d|v_{n+1}|}N$ is the skeleton of $\Sigma^{d|v_{n+1}|}X_m$, we have the map v_{n+1}^d for $\Sigma^{d|v_{n+1}|}N$ to X_m. The composition of v_{n+1}^d and π_s represents wedges of elements in $P(n+1)^k(\Sigma^{d|v_{n+1}|}N)$ with $k \equiv -s$ modulo q. Fortunately this group is 0. Thus we can life v_{n+1}^d to X_{m+1}. We have all necessary elements for another induction on $s = m, \cdots, m'$.

Finally we have the following diagram.

$$\begin{array}{ccc} & & \Sigma^{d|v_{n+1}|}N \\ & \overset{v_{n+1}^{d}}{\swarrow} & \Big\downarrow v_{n+1}^{d} \\ X_{m'}^{(w+d|v_{n+1}|)} = N & \xrightarrow{\text{inclusion}} & X_{m'+1} \end{array}$$

The lifting exists since both $\Sigma^{d|v_{n+1}|}N$ and $(w + d|w_{n+1}|)$-skeleton of $X_{m'+1}$, which is also N, have the same highest dimensions. It is obvious that the lifting induces v_{n+1}^{d}-multiplication in BP homology. ■

3.3 Realization and Existence of Self-map

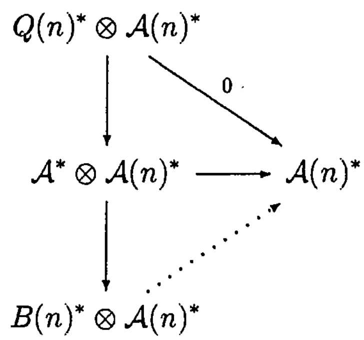
As we said in Chapter 1, we will only consider the $\mathcal{A}^*$-module structure of $\mathcal{A}(n)^*$ with the following property:

$$Q_{n+i}\mathcal{A}(n)^* = 0.$$

for each $i \geq 1$. In other words

$$Q(n)^*\mathcal{A}(n)^* = 0.$$

So we can have the following commutative diagram for this kind of module structure.



Lemma 3.3.1 *We consider $\mathcal{A}(n)^*$ as $B(n)^*$-module for any n over coefficient ring $\mathbf{Z}/(p)$. When $p > n+1$, we can show that*

$$\mathrm{Ext}^{s,t}_{B(n)^*}(\mathcal{A}(n)^*, \mathbf{Z}/(p)) = 0$$

if (s,t) satisfies $t - s \leq w_n$ and $s > n+1$. Here w_n is the top dimension of $\mathcal{A}(n)^*$:

$$\begin{aligned} w_n &= \textstyle\sum_{1\leq i\leq n} 2(p^i-1)(p^{n+1-i}-1) + \sum_{0\leq i\leq n}(2p^i-1) \\ &= 2np^{n+1} - 2(p+\cdots+p^n) - n + 3. \end{aligned}$$

Proof. Applying the May spectral sequence, we have

$$E_1 = E(h_{i,j} : i+j \geq n+1) \otimes P[b_{i,j} : i+j \geq n+1] \Longrightarrow \mathrm{Ext}_{B(n)^*}(\mathcal{A}(n)^*, \mathbf{Z}/(p)).$$

Here $h_{ij} \in \mathrm{Ext}^{1,2p^j(p^i-1)}$ represented by $\xi_i^{p^j}$ in cobar complex and $b_{i,j} \in \mathrm{Ext}^{2,2p^{j+1}(p^i-1)}$. When $p > n+1$,

$$\begin{aligned} |b_{i,j}| - w_n &\geq 2pp^{n+1} - 2p^{n+1} - 2np^{n+1} \\ &= (2p - 2 - 2n)p^{n+1} \\ &> 0 \end{aligned}$$

for all $i + j \geq n+1$ and

$$|h_{i,j}| - w_n > 0$$

for all $i+j \geq n+2$. We only need to consider those $h_{i,j}$'s with $i+j = n+1$. So we can simplify our E_1 term in the range we are considering to

$$E(h_{i,j} : i+j = n+1).$$

It is easy to see that there is no element in $\mathrm{Ext}^{s,t}_{B(n)^*}(\mathcal{A}(n)^*, \mathbf{Z}/(p))$ in the range we are considering. ■

Remark 3.3.2 *When $p > n+2$, we can prove similarity that $\mathrm{Ext}^{s,t} = 0$ if*

$$t - s \leq w_n + |v_{n+1}| = w_n + 2p^{n+1} - 1$$

and $s > n+1$.

Lemma 3.3.3 *In the following diagram*

$$\begin{array}{ccc} BP_*(BP) & \xrightarrow{BP_*(r_E)} & BP_*(\Sigma^{|r_E|}BP) \cong \Sigma^{-|r_E|}BP_*BP \\ \downarrow & & \downarrow \\ BP_*(P(n)) \cong BP_*BP/I_n & \xrightarrow{BP_*(\Phi_n(r_E))} & BP_*(\Sigma^{|r_E|}P(n)) \cong \Sigma^{-|r_E|}BP_*BP/I_n \end{array}$$

the vertical maps are just projection modulo I_n.

Proof. From [Rav86, Corollary 4.3.21], we obtain

$$\eta_R(v_i) \equiv v_i \mod I_i.$$

In the construction of $P(n)$ in Theorem 3.1.3(i), v_i-multiplication can be seen as both left and right multiplication by the above formula. We can prove this lemma inductively. ∎

Lemma 3.3.4 *Let N be $\mathcal{A}(n)^*$. The condition (iv) in Lemma 3.2.1 is satisfied.*

Proof. It is well known that

$$BP^*(P(n+1)) = Hom_{BP^*}(BP_*(P(n+1)), BP^*)$$

and

$$P(n+1)^*P(n+1) = Hom_{P(n+1)^*}(P(n+1)_*(P(n+1)), P(n+1)^*).$$

Since the Adams spectral sequence for them collapse, we can easily prove there is no lim^1 problem. The above facts can also be proved accordingly.

It suffices to prove that

$$\mathrm{Ker}(\oplus_{\alpha\in I_1} BP_*(r_\alpha)) = (\mathbf{Z}/(p)\Box_{E_*}\mathcal{A}(n)_*) \otimes BP_*/I_{n+1}$$

and

$$\mathrm{Ker}(\oplus_{\alpha\in I_1} P(n+1)_*(r_\alpha)) = \mathcal{A}(n)_* \otimes BP_*/I_{n+1}.$$

We will deal with BP homology first. Lemma 3.3.3 actually tells us that we can work on BP_*BP and then take the quotient modulo I_{n+1}. By the duality we know that

$$BP_*(r_E)(t^F) = \Sigma v^{F_1} t^{F_2}$$

where $v^{F_1} t^E \otimes t^{F_2}$ appears in the coproduct of t^F. On the other hand we can obtain the following formula from [Rav86, 4.3.13].

$$\Delta(t_i) \equiv \Sigma_{j+k=i} t_j \otimes t_k^{p^j} \mod I_{n+1}$$

We can see that this formula is the same as the formula in homology. Since

$$\mathrm{Ker}(\oplus H_*(r_\alpha)) = A(n)_*$$

and $\mathbf{Z}/(p)\square_{E_*}A(n)_*$ is hit by $BP_*(P(n+1))$, it follows

$$(\mathbf{Z}/(p)\square_{E_*}A(n)_*) \otimes BP_*/I_{n+1} \subset \mathrm{Ker}(\oplus BP_*(r_\alpha)).$$

For any BP_*/I_{n+1} generators in $BP_*(P(n+1))$ other than those in $\mathbf{Z}/(p)\square_{E_*}A(n)_*$, they are not in kernel of $\oplus H_*(r_\alpha)$. This implies that they are also not in the kernel of $\oplus BP_*(r_\alpha)$. We can conclude the kernel of $\oplus BP_*(r_\alpha)$ is as we expected because it is a homomorphism of BP_* modules.

From Theorem 3.1.3 (v), we know that the coproduct of a_i is the same as the coproduct of Q_i in $H_*(P(n+1)) = B(n)_*$ for $0 \le i \le n$. This time $B(n)_*$ is hit by $P(n+1)_*P(n+1) = B(n)_* \otimes BP_*/I_{n+1}$. So we can prove similarly that the kernel of $\oplus P(n+1)_*(r_\alpha)$ is also what we expected. ∎

Proof of Theorem E and F. We just need to verify the conditons (i), (ii), (iv) and (iv) for $m = m' = n+1$, $d = 1$, $w = w_n$ and $w_n + |v_{n+1}|$ respectively. (i) and (ii) are obvious. (iv) comes from Lemma 3.3.4 and (iii) comes from Lemma 3.3.1 and Remark 3.3.2.

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