THIS IS THE (CO)END, MY ONLY (CO)FRIEND

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ABSTRACT. The present note is a recollection of the most striking and useful applications of *co/end calculus*. We put a considerable effort in making arguments and constructions rather explicit: after having given a series of preliminary definitions, we characterize co/ends as particular co/limits; then we derive a number of results directly from this characterization. The last sections discuss the most interesting examples where co/end calculus serves as a powerful abstract way to do explicit computations in diverse fields like Algebra, Algebraic Topology and Category Theory as well as some generalizations to higher dimensional category theory.

The appendices serve to sketch a number of results in theories heavily relying on co/end calculus; the reader who dares to arrive at this point, being completely introduced to the mysteries of *co/end fu* (端楔術, duānxiē shù; literally "the art [of handling] terminal wedges"), can regard basically every statement as a guided exercise.

CONTENTS

Introduction.

The purpose of the present survey is to familiarize its readers with what in category theory is called *co/end calculus*, gathering a series of examples of its application; the author would like to stress clearly, from the very beginning, that the material presented here makes no claim of originality: we put a special care in acknowledging carefully, where possible, each of the countless authors whose work was a source of inspiration in compiling this note. Among these, every erroneous or missing attribution must be ascribed to the mere ignorance of the author, which invites everyone he didn't acknowledge to contact him.

The introductory material appearing in section **1** is the most classical and comes almost *verbatim* from the classical reference [**ML98**]; the nerve-realization formalism is a *patchwork* of various results, scattered in the (algebraic) topology literature; these results are often presented as a *deus ex machina*, mysterious machineries leaving the reader without any clue about *why* they work so well. The reformulation of operads using coends comes *verbatim* from [**Kel05**]; the chapter discussing the theory of *relators* (or *profunctors*, or *distributors*; in French it is common to call them *distributeurs*, following an idea of J. Bénabou; we deviate from this unfortunate use) comes from the work of J. Bénabou, and I immensely profited from some notes taken by T. Streicher, [**Béner**]; during May 2016 I had the pleasant opportunity to meet Thomas, and I fell in love with his mathematical style, near to the craftsmanship of certain watchmakers, but also vital and passionate. I hope this text is better, in view of those pleasant days of friendly mathematics.

Chapter **4** on weighted co/limits is taken almost verbatim from [**Rie14**, **II.7**]: only a couple of implicit conceptual dependencies on co/end calculus have been made explicit, and some examples left as exercises there are discussed in full detail; the fact that we follow so faithfully the exposition of [**Rie14**] must be interpreted not as an act of plagiarism, but instead ad an implicit invitation to get acquainted with such a wonderful book. I count Emily as one of the most fierce supporters of Category Theory among the "new generation", and I'm proud to be considered a good mathematician by her.

A subsequent section discusses co/end calculus in higher category theory, introducing lax co/ends in 2-categories [**Boz77**, **Boz75**, **Boz80**], *homotopy ends* [**DF78**, **Isa09**] and $(\infty, 1)$-*co/ends* in various models: first of all Joyal-Lurie's ∞-categories following [**GHN15**], and then Bergner's simplicially enriched categories mainly following [**CP97**] and the general theory for enriched co/ends in [**Dub70**, **Gra**].

The relation between co/end calculus and homotopy co/limits is discussed, and a(n already) classical result [**Gam10**] is presented to unify the two

constructions classically given for the homotopy co/limit of a diagram; the compatibility of the co/end operation as a Quillen functor $\int \colon \mathbf{Cat}(\mathbf{C}^{\mathrm{op}} \times \mathbf{C}, \mathbf{D}) \to \mathbf{D}$ is discussed when $\mathbf{D}$ is a sufficiently nice model category (our main reference to prove that $\int$ is a Quillen functor is [**Lur09**, **A.2.9.28**]).

The last chapter about promonoidal categories comes from [**Day74, Str12**]; the subsequent appendices propose the reader to familiarize with the theory of promonoidal categories, and to show the initial results of [**Day11**, §**1-3**], a delightful and deep paper whose importance is far more than a source of unusual exercises.

Because of all these remarks, the reader has to keep in mind that the value of this work –if there is any– lies, rather than in the originality of the discussion (which is affected by a series of unforgivable sins of omission), in its strong will to be simple and clear remaining an exhaustive account, collecting as much material as possible without leaving obscure or unreferenced passages.

Our aim is to serve the inexperienced reader (either the beginner in the study of category theory, or the experienced student who exhausted the "primary" topics of her education) as a guide to familiarize with this extraordinarily valuable tool-set producing a large number of *abstract-nonsense* proofs, most of which are "formally formal" strings of natural isomorphisms. It is our firm opinion that this document could serve as a pedagogical tool even for some experienced mathematicians, maybe working nearby category theory, that are never been exposed to this beautiful machinery. It is our hope that they could find their way to exploit such a powerful language.

After the first examples, the keen reader will certainly prefer to re-write most of the proofs in the silence of her room, and we warmly invite her to do so; *imitation* of the basic techniques is for sure an unavoidable step in getting acquainted with the machinery of co/end calculus, and more generally with any machinery whatsoever. This is especially true for mathematical language, in the same sense we all learned integration rules by imitation and training reading precalculus books.

Maybe it's not a coincidence that the one you are about to see is *another* integral calculus to be learned by means of examples and exercises. This analogy *could* be pushed further; we refrain to do it, lest you think we are arrogantly claiming to be able to reduce the subtle art of integration to category theory. Nevertheless, we can't help but mention several insightful formal and informal analogies between mathematical analysis and co/end calculus: these are scattered throughout all the discussion, and we denote them with the special symbol ◉◉.[1] Whenever it appears, we advise the reader

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[1]I learned this funny notation during my freshman year, when I was handed [**DM96**] for the first time; the "small-eyes" notation accompanied me throughout all my mathematical life until today. Various (facial) expressions advise different ways the reader is supposed to

feeling uncomfortable with a certain dose of hand-waving and categorical juggling to raise her eyes and skip the paragraph.

It has been said that "Universal history is, perhaps, the history of a few metaphors" [**Bor64**]: differential and integral calculus is undoubtedly one of such recurring themes. In some sense, the fortune of co/end calculus is based on the analogy which represents these universal objects by means of an integral symbol [**Yon60**, **DK69**, **Str12**]; this analogy is motivated by the Fubini theorem on the interchange of "iterated integrals": unfortunately, despite its immense expressive and unifying power the language of coends seems to be woefully underestimated. No elementary book in category theory (apart a single chapter in the aforementioned [**ML98**]) seems to contain something more than a bare introduction of the basic elements of the language.

What's missing, in the humble opinion of the author, is an exhaustive and unitary source of examples, exercises and computations, showing their readers how co/end calculus can literally *disintegrate* involved computations and reduce them to a bunch of canonical isomorphisms.

Trying to fill this gap in the literature has been the main motivation for the text you're about to read. It's up to you to decide it it is a clumsy attempt, or a partial success.

A final note, January 3, 2017. It is impossible, at this point, to keep track of all the influences I received since when this project started. The present document doubled its length since its first version appeared on the arXiv. Probably someday a "version 3" will appear. Thanks everybody, especially to the people I acknowledge below.

Acknowledgements. In some sense, I am not the only, and for sure I'm the youngest and less experienced, author of this note. I would like to thank T. Trimble, E. Rivas and A. Mazel-Gee for having read carefully the preliminary version of this document, suggesting improvements and corrections, having spotted a disgraceful number of errors, misprints and incoherent choices of notation. Their attentive proofreading has certainly increased the value –again, if there is any– of the document you're about to read.

This humongous amount of errors did not prevent the first version of *coend-cofriend* to circulate, and being read, far more than we expected. And this is true to the point that at the moment of writing this is the paper with the highest number of (formal or moral) citations. I warmly thank the

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behave when she meets them. There are four such expressions: ◐◑, abstract material; ◉◉, standard exercises; ◔◕, material that you are supposed to meditate a lot; ◉◉, shattering exercises.

people who judged this clumsy list of examples a document worth to be read, improved and even studied.

Foundations, notation and conventions. The main foundational convention we adopt throughout the paper is the assumption ([**GV72**]) that every set lies in a suitable Grothendieck universe. We implicitly fix such an universe $\mho$, whose elements are termed *sets*; categories are always considered to be small with respect to *some* universe: in particular we choose to adopt, whenever necessary, the so-called *two-universe convention*, where we postulate the existence of a universe $\mho^+ \supset \mho$ in which all the non-$\mho$-small categories live. This rather common choice has nevertheless subtle consequences: as it is recorded in [**Wat75**, **Low13**] the existence and good behaviour of some co/limits and Kan extensions critically depends from the particular choice of a universe.

At least in some situations, it is still possible to keep this problem under control, appealing some "boundedness" conditions keeping track of the cardinality of the involved constructions. There are few places where this caveat could become a real problem (like for example our **3.9**).

Several kinds of categorical structures (categories, and often also 2-categories and bicategories, as well as instances of higher categories) will be denoted as boldface letters $\mathbf{C}, \mathbf{D}, \dots$; the context always clarifies which structure is considered each time. Functors between categories are denoted as capital Latin letters like F, G, H, K and suchlike (there can be little deviations to this rule, like for example in §**5**); the category of functors $\mathbf{C} \to \mathbf{D}$ between two categories is denoted as $\mathsf{Fun}(\mathbf{C}, \mathbf{D})$, $\mathbf{D}^{\mathbf{C}}$, $[\mathbf{C}, \mathbf{D}]$ and suchlike; following a long-standing tradition, $\widehat{\mathbf{C}}$ is a shorthand for the category

[$\mathbf{C}^{\mathrm{op}}, \mathbf{Sets}$] of presheaves on $\mathbf{C}$; the canonical hom-bifunctor of a category $\mathbf{C}$ sending (c, c') to the set of all arrows $\hom(c, c') \subseteq \hom(\mathbf{C})$ is almost always denoted as $\mathbf{C}(\_, =)\colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{Sets}$, and the symbols $\_$, $=$ are used as placeholders for the "generic argument" of a functor or bifunctor; morphisms in the category $\mathsf{Fun}(\mathbf{C}, \mathbf{D})$ (i.e. natural transformations between functors) are often written in Greek, or Latin lowercase alphabet, and collected in the set $\mathsf{Nat}(F, G) = \mathbf{D}^{\mathbf{C}}(F, G)$. The simplex category $\boldsymbol{\Delta}$ is the *topologist's delta* (opposed to the *algebraist's delta* $\boldsymbol{\Delta}_+$ which has an additional initial object $[-1] \triangleq \varnothing$), having objects *nonempty* finite ordinals $[n] \triangleq \{0 < 1 \cdots < n\}$; we denote $\Delta[n]$ the representable presheaf on $[n] \in \boldsymbol{\Delta}$, i.e. the image of $[n]$ under the Yoneda embedding of $\boldsymbol{\Delta}$ in the category $\mathbf{sSet} = \widehat{\boldsymbol{\Delta}}$ of simplicial sets. More generally, we indicate the Yoneda embedding of a category $\mathbf{C}$ into its presheaf category with $よ_{\mathbf{C}}$ –or simply $よ$–, i.e. with the hiragana symbol for "yo"; this choice comes from [**LB15**], and I share a similar aesthetics for peculiar notation. Whenever there is an adjunction $F \dashv G$ between functors, the arrow $Fa \to b$ in the codomain of F and the corresponding arrow $a \to Gb$ in its domain are called *mates* or *adjuncts*; so, the notation "the mate/adjunct of $f\colon Fa \to b$" means "the unique arrow $g\colon a \to Gb$ determined by f".

> Los idealistas arguyen que las salas hexagonales son una forma necesaria del espacio absoluto o, por lo menos, de nuestra intuición del espacio.
>
> J.L. Borges, *La biblioteca de Babel.*

1. Dinaturality, extranaturality, co/wedges.

Let's start with a simple example. Let $\mathbf{Sets}$ be the category of sets and functions, considered with its natural cartesian closed structure: this means we have a bijection of sets

$$\mathbf{Sets}(A \times B, C) \cong \mathbf{Sets}(A, C^B) \tag{1}$$

where C^B is the set of all functions $B \to C$. The adjunction $\_ \times B \dashv (\_)^B$ has a counit, which is a natural transformation

$$\epsilon_{X,(B)}\colon X^B \times B \to X \tag{2}$$

where the codomain can be considered "mutely depending on the variable B". The collection of functions $\{\epsilon_X\colon X^B \times B \to X\}$ is natural in the classical sense in the variable X; as for the variable B, the most we can say is the commutativity above: it doesn't remind naturality so much. See also Exercise 2 at the end of [**ML98**, **IV.7**] and our Exercise **1**.**E1**: the naturality in B of the adjunction $\mathbf{Sets}(A \times B, C) \cong \mathbf{Sets}(A, C^B)$ implies a "more complicated" dependence on the variable B.

Fortunately a suitable generalization of naturality (a "super-naturality" condition), encoding the result of Exercise **1.E1**, is available to describe this and other similar phenomena in the same common framework. A notion of super-naturality adapted to describe co/ends as suitable universal objects comes in two flavours: one of them is *di*naturality, which we now introduce.

Definition 1.1 [DINATURAL TRANSFORMATION]**:** Given two functors $P, Q \colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{D}$ a *dinatural transformation*, depicted as an arrow $\alpha \colon P \overset{\cdot\cdot}{\to} Q$, consists of a family of arrows $\{\alpha_c \colon P(c,c) \to Q(c,c)\}_{c \in \mathbf{C}}$ such that for any $f \colon c \to c'$ the following hexagonal diagram commutes

$$
\begin{array}{ccccc}
P(c',c) & \xrightarrow{P(f,c)} & P(c,c) & \xrightarrow{\alpha_c} & Q(c,c) \\
{\scriptstyle P(c',f)}\big\downarrow & & & & \big\downarrow{\scriptstyle Q(c,f)} \\
P(c',c') & \xrightarrow[\alpha_{c'}]{} & Q(c',c') & \xrightarrow[Q(f,c')]{} & Q(c,c')
\end{array}
\tag{3}
$$

Definition 1.2 [WEDGE FOR A FUNCTOR]**:** Let $P \colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{D}$; a *wedge* for P is a dinatural transformation $\Delta_d \overset{\cdot\cdot}{\to} P$ from the constant functor on the object $d \in \mathbf{D}$ (often denoted simply by $d \colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{D}$), defined sending $(c,c') \mapsto d$, $(f,f') \mapsto \mathrm{id}_d$.

Definition 1.3 [END OF A FUNCTOR]**:** The *end* of a functor $F \colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{D}$ consists of a universal wedge $\underline{\mathrm{end}}(F) \overset{\cdot\cdot}{\to} F$; the constant $\underline{\mathrm{end}}(F) \in \mathbf{D}$ is itself termed, by abuse, the end of the functor.

Spelled out explicitly, the universality requirement means that for any other wedge $\beta \colon d \overset{\cdot\cdot}{\to} F$ the diagram

$$\tag{4}$$

commutes for a unique arrow $h \colon d \to \underline{\mathrm{end}}(F)$, for any arrow $f \colon c \to c'$.

Remark 1.4 : Uniqueness requirements imply functoriality: given a natural transformation $\eta \colon F \Rightarrow F'$ there is an induced arrow $\underline{\mathrm{end}}(\eta) \colon \underline{\mathrm{end}}(F) \to$

$\underline{\text{end}}(F')$ between their ends, as depicted in the diagram

$$
\begin{array}{c}
\text{(diagram)}
\end{array}
\tag{5}
$$

This implies that taking the end of a functor is a (covariant) functor $\mathbf{D}^{\mathbf{C}^{\text{op}} \times \mathbf{C}} \to \mathbf{D}$. The case of a coend is dually analogous: filling the details is an easy dualization exercise.

A slightly less general, but better behaved notion of super-naturality (we say "better behaved" since it admits a *graphical calculus* translating commutativity theorems into controlling that certain string diagrams can be deformed one into the other), allowing again to define co/wedges, is available: this is called *extra-naturality* and it was introduced in [**EK66**].

Definition 1.5 [EXTRANATURAL TRANSFORMATION]**:** Let P, Q be functors

$$
P \colon \mathbf{A} \times \mathbf{B}^{\text{op}} \times \mathbf{B} \to \mathbf{D}
$$
$$
Q \colon \mathbf{A} \times \mathbf{C}^{\text{op}} \times \mathbf{C} \to \mathbf{D}.
$$

An *extranatural transformation* $\alpha \colon P \overset{..}{\Rightarrow} Q$ consist of a collection of arrows

$$
\big\{ \alpha_{abc} \colon P(a, b, b) \longrightarrow Q(a, c, c) \big\}
\tag{6}
$$

indexed by triples of object in $\mathbf{A} \times \mathbf{B} \times \mathbf{C}$ such that the following hexagonal diagram commutes for every $f \colon a \to a'$, $g \colon b \to b'$, $h \colon c \to c'$, all taken in their suitable domains:

$$
\begin{array}{ccc}
P(a, b', b) \xrightarrow{P(f,b',g)} P(a', b', b') \xrightarrow{\alpha_{a'b'c}} Q(a', c, c) \\
{\scriptstyle P(a,g,b)} \downarrow \qquad\qquad\qquad\qquad\qquad\qquad \downarrow {\scriptstyle Q(a',c,h)} \\
P(a, b, b) \xrightarrow[\alpha_{abc'}]{} Q(a, c', c') \xrightarrow[Q(f,h,c')]{} Q(a', c, c');
\end{array}
\tag{7}
$$

This commutative hexagon can be equivalently described as the juxtaposition of three distinguished commutative squares, depicted in [**EK66**]: the three can be obtained letting respectively f and h, f and g, or g and h be

identities in the former diagram, which collapses to

$$P(a,b,b) \xrightarrow{P(f,b,b)} P(a',b,b) \qquad P(a,b',b) \xrightarrow{P(a,b',g)} P(a,b',b') \qquad P(a,b,b) \xrightarrow{\alpha_{abc}} Q(a,c,c)$$

$$\downarrow{\alpha_{abc}} \qquad \qquad \downarrow{\alpha_{a'bc}} \qquad P(a,g,b)\downarrow \qquad \qquad \downarrow{\alpha_{ab'c}} \qquad \alpha_{abc'}\downarrow \qquad \qquad \downarrow{Q(a,c,h)}$$

$$Q(a,c,c) \xrightarrow{Q(f,c,c)} Q(a',c,c) \qquad P(a,b,b) \xrightarrow{\alpha_{abc}} Q(a,c,c) \qquad Q(a,c',c') \xrightarrow{Q(a,h,c')} Q(a,c,c')$$

$$(8)$$

Remark 1.6 : We can again define co/wedges in this setting: if $\mathbf{B} = \mathbf{C}$ and in $F(a,b,b) \to G(a,c,c)$ the functor F is constant in $d \in \mathbf{C}$, $G(a,c,c) = \bar{G}(c,c)$ is mute in a, we get a wedge condition for $d \xrightarrow{\cdot\cdot} G$; dually we obtain a cowedge condition for $F(b,b) \to G(a,b,b) \equiv d'$ for all a,b,c.

Both notions give rise to the same notion of co/end as a universal co/wedge for a bifunctor $F \colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{D}$. The main reason we should prefer extranaturality, as was pointed out to the author in a (semi)private conversation with T. Trimble, is that

> For the purposes of describing end/coend calculus, I wouldn't emphasize dinatural transformations so much as I would extranatural transformations. Most dinatural transformations that arise in the wild can be analyzed in terms of extranatural (extraordinary natural, in the old lingo) transformations. [...]
>
> [Co/wedges can be regarded as particular examples of two slightly different, but related (see Prop **1.8**) constructions:] first, they are special examples of dinatural transformations. Second, they are special cases of an extranatural (*extraordinary natural*) transformation, which generally is a family of maps $F(a,a,b) \to G(b,c,c)$ which combines naturality in the argument b with a cowedge condition on a and a wedge condition on c.

We now briefly describe the promised graphical calculus for extranatural transformations: it depicts the components α_{abc}, and arrows $f \colon a \to a'$, $g \colon b \to b'$, $h \colon c \to c'$, as planar diagrams like
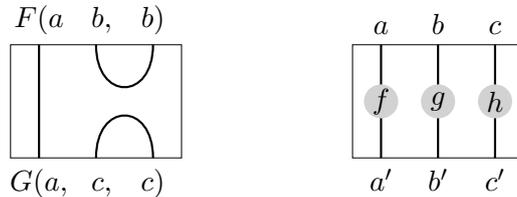
where wires are labeled by objects and must be thought oriented from top to bottom. The commutative squares of (**8**) become, in this representation, the following three string diagrams, whose equivalence is graphically obvious (the labels f, g, h can "slide" along the wire they live in):
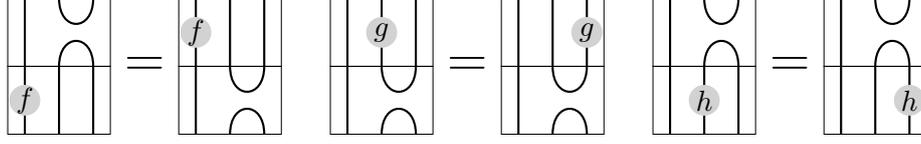
Remark 1.7 : The notion of extranatural transformation can be specialized to encompass various other constructions: simple old naturality arises when F, G are both constant in their co/wedge components, like in
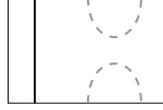
whereas wedge and cowedge conditions arise when either F, G are constant:

All the others mixed situations (a wedge-cowedge condition, naturality and a wedge, etc. which lack a specified name) admit a graphical representation of the same sort, and follow similar graphical rules of juxtaposition.

Proposition 1.8 : Extranatural are particular kinds of dinatural transformations.

Proof (due to T. Trimble). Suppose you have functors $F\colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \times \mathbf{C} \to \mathbf{D}$ and $G\colon \mathbf{C} \times \mathbf{C} \times \mathbf{C}^{\mathrm{op}} \to \mathbf{D}$. Now put $\mathbf{A} = \mathbf{C} \times \mathbf{C}^{\mathrm{op}} \times \mathbf{C}^{\mathrm{op}}$, and form two new functors $F', G' : \mathbf{A}^{\mathrm{op}} \times \mathbf{A} \to \mathbf{D}$ by taking the composites

$$F' = (\mathbf{C}^{\mathrm{op}} \times \mathbf{C} \times \mathbf{C}) \times (\mathbf{C} \times \mathbf{C}^{\mathrm{op}} \times \mathbf{C}^{\mathrm{op}}) \xrightarrow{\mathrm{proj}} \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \times \mathbf{C} \xrightarrow{F} \mathbf{D}$$

$$(x', y', z'; x, y, z) \longmapsto (x', x, y') \xmapsto{F} F(x', x, y')$$

$$G' = (\mathbf{C}^{\mathrm{op}} \times \mathbf{C} \times \mathbf{C}) \times (\mathbf{C} \times \mathbf{C}^{\mathrm{op}} \times \mathbf{C}^{\mathrm{op}}) \xrightarrow{\mathrm{proj}'} \mathbf{C} \times \mathbf{C} \times \mathbf{C}^{\mathrm{op}} \xrightarrow{G} \mathbf{D}$$

$$(x', y', z'; x, y, z) \mapsto (y', z', z) \xmapsto{G} G(y', z', z)$$

Now let's put $a' = (x', y', z')$ and $a = (x, y, z)$, considered as objects in $\mathbf{A}$. An arrow $\varphi\colon a' \to a$ in $\mathbf{A}$ thus amounts to a triple of arrows $f\colon x' \to x$, $g\colon y \to y'$, $h\colon z \to z'$ all in $\mathbf{C}$.

Following the instructions above, we have $F'(a', a) = F(x', x, y')$ and $G(a', a) = G(y', z', z)$.

Now if we write down a dinaturality hexagon for $\alpha : F' \overset{..}{\to} G'$, we get a diagram of shape

$$
\begin{array}{ccccc}
F'(a,a') & \xrightarrow{F'(1,\varphi)} & F'(a,a) & \xrightarrow{\alpha_a} & G'(a,a) \\
{\scriptstyle F(\varphi,1)}\downarrow & & & & \downarrow {\scriptstyle G'(\varphi,1)} \\
F'(a',a') & \xrightarrow[\alpha_{a'}]{} & G'(a',a') & \xrightarrow[G'(1,\varphi)]{} & G(a',a)
\end{array}
\qquad (9)
$$

which translates to a hexagon of shape

$$
\begin{array}{ccccc}
F(x,x',y) & \xrightarrow{F(1,f,1)} & F(x,x,y) & \longrightarrow & G(y,z,z) \\
{\scriptstyle F(f,1,g)}\downarrow & & & & \downarrow {\scriptstyle G(g,h,1)} \\
F(x',x',y') & \longrightarrow & G(y',z',z') & \xrightarrow[G(1,h,1)]{} & G(y',z',z)
\end{array}
\qquad (10)
$$

where the unlabeled arrows refer to the extranatural transformation. $\square$

1.1. **The integral notation for co/ends.** A suggestive and useful notation alternative to the anonymous one "$\underline{\mathrm{co}}/\underline{\mathrm{end}}(F)$" is due to N. Yoneda, which in [**Yon60**] introduces most of the notions we are dealing with, specialized to **Ab**-enriched functors $\mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{Ab}$: the *integral notation* denotes the end of a functor $F \in \mathbf{D}^{\mathbf{C}^{\mathrm{op}} \times \mathbf{C}}$ as a "subscripted-integral" $\int_c F(c,c)$, and the coend $\underline{\mathrm{coend}}(F)$ as the "superscripted-integral" $\int^c F(c,c)$.

From now on we will systematically adopt this notation to denote the universal co/wedge $\underline{\mathrm{co}}/\underline{\mathrm{end}}(F)$ or, following a well-established abuse of notation, the object itself; we also accept slightly more pedantic variants of this, as

$$
\int_{c \in \mathbf{C}}^{/c \in \mathbf{C}} F(c,c), \qquad \int_{c:\mathbf{C}}^{/c:\mathbf{C}} F(c,c).
\qquad (11)
$$

Remark 1.9 : One should be aware that Yoneda's notation in [**Yon60**] is "reversed" as he calls *integration* our coends, which he denotes as $\int_{c \in \mathbf{C}} F(c,c)$, and *cointegrations* our ends, which he denotes $\int_{c \in \mathbf{C}}^{*} F(c,c)$.

Remark 1.10 : Properties of co/ends acquire a particularly suggestive flavour when written in this notation:

1) Functoriality (the *B-rule* for integrals or the *freshman's dream*): the unique arrow $\underline{\mathrm{end}}(\eta)$ induced by a natural transformation $\eta \colon F \Rightarrow G$ between $F, G \in \mathbf{D}^{\mathbf{C}^{\mathrm{op}} \times \mathbf{C}}$ can be written as $\int \eta \colon \int F \to \int G$, and uniqueness of this induced arrow entails that $\int(\eta \circ \sigma) = \int \eta \circ \int \sigma$.

2) The "Fubini theorem" for ends, which first appeared as equation (**4.0.1**) of [**Yon60**]: given a functor $F \colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \times \mathbf{E}^{\mathrm{op}} \times \mathbf{E} \to \mathbf{D}$, we can form the end $\int_c F(c, c, \_, =)$ obtaining a functor $\mathbf{E}^{\mathrm{op}} \times \mathbf{E} \to$

D whose end is $\int_e \int_c F(c, c, e, e) \in \mathbf{D}$; we can also form the ends $\int_c \int_e F(c, c, e, e) \in \mathbf{D}$ and $\int_{(c,e)} F(c, c, e, e)$ identifying $\mathbf{C}^{\mathrm{op}} \times \mathbf{C} \times \mathbf{E}^{\mathrm{op}} \times \mathbf{E}$ with $(\mathbf{C} \times \mathbf{E})^{\mathrm{op}} \times (\mathbf{C} \times \mathbf{E})$. *Fubini's theorem* for ends states that there is a canonical isomorphism between the three:

$$\int_{(c,e)} F(c, c, e, e) \cong \int_e \int_c F(c, c, e, e) \cong \int_c \int_e F(c, c, e, e) \qquad (12)$$

Remark 1.11 : In some sense, the Fubini rule for coends seems a rather weak analogy between integrals and coends; indeed there is no doubt that the following passage ([**ML98**, **IX.5**])

> [...] the "variable of integration" c [in $\int_c F$] appears twice under the integral sign (once contravariant, once covariant) and is "bound" by the integral sign, in that the result no longer depends on c and so is unchanged if c is replaced by any other letter standing for an object of the category **C**

motivates the integral notation, and nevertheless, this analogy seems quite elusive to justify in a precise way. In the eye of the author, it seems worthwhile to remember that in view of the characterization for co/ends in terms of co/equalizers given in **1.15**, $\int_c \colon \mathbf{D}^{\mathbf{C}^{\mathrm{op}} \times \mathbf{C}} \to \mathbf{D}$ can be thought as an *averaging* operation on a functor, giving the "fixed points" of the "action" induced by $F(\varphi, c'), F(c, \varphi)$ as $\varphi \colon c \to c'$ runs over $\hom(\mathbf{C})$.

Although the author prefers to abstain from any further investigation, having no chance to give a valid (or rather, formal) explanation of the integral notation, it is nevertheless impossible to underestimate the power of this convenient shorthand.

1.2. **Co/ends as co/limits.** A general tenet of elementary category theory is that you can always characterize a universal construction as an element of the triad

$$\text{limit - adjoint - representation of functors.}$$

The formalism of co/ends makes no exception: the scope of the following subsection is to characterize, whenever it exists, the co/end of a functor $F \colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{D}$ as a co/limit over a suitable diagram, and finally as the co/equalizer of a single pair of arrows.

First of all notice that given $F\colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{D}$ and a wedge $\tau\colon d \overset{..}{\to} F$, we can build the following commutative diagram

$$
\begin{array}{c}
\text{(13)}
\end{array}
$$

where $c \xrightarrow{f} c' \xrightarrow{g} c''$ are two arbitrarily chosen, but fixed, arrows in $\mathbf{C}$. From this commutativity we deduce the following relations:

$$
\begin{aligned}
\tau_{gf} &= F(gf, c'') \circ \tau_{c''} = F(c, gf) \circ \tau_c \\
&= F(f, c'') \circ F(g, c'') \circ \tau_{c''} = F(f, c'') \circ \tau_g \\
&= F(c, g) \circ F(c, f) \circ \tau_c = F(c, g) \circ \tau_f.
\end{aligned}
$$

where τ_f, τ_g are the common values $F(f, c')\tau_{c'} = F(c, f)\tau_c$ and $F(c', g)\tau_{c'} = F(g, c'')\tau_{c''}$ respectively, τ_{gf} is the common value $F(c, g)\tau_f = F(f, c'')\tau_g$. These relations imply that there is a link between co/wedges and co/cones, encoded in the following definition.[2]

Definition 1.12 [THE TWISTED ARROW CATEGORY OF $\mathbf{C}$]**:** For every category $\mathbf{C}$ we define $\mathrm{TW}(\mathbf{C})$, the category of *twisted arrows* in $\mathbf{C}$ as follows:

- $\mathrm{Ob}(\mathrm{TW}(\mathbf{C})) = \hom(\mathbf{C})$;
- Given $f\colon c \to c'$, $g\colon d \to d'$ a morphism $f \to g$ is given by a pair of arrows $(h\colon d \to c, k\colon c' \to d')$, such that the obvious square commutes (asking that the arrow between domains is in reversed order is *not* a mistake!).

Endowed with the obvious rules for composition and identity, $\mathrm{TW}(\mathbf{C})$ is easily seen to be a category, and now we can find a functor

$$
\mathrm{Fun}(\mathbf{C}^{\mathrm{op}} \times \mathbf{C}, \mathbf{D}) \longrightarrow \mathrm{Fun}(\mathrm{TW}(\mathbf{C}), \mathbf{D}) \qquad (14)
$$

defined sending F to the functor $\overline{F}\colon \mathrm{TW}(\mathbf{C}) \to \mathbf{D}\colon f \mapsto F(\mathrm{src}(f), \mathrm{trg}(f))$; it is extremely easy to check that bifunctoriality for F exactly corresponds to functoriality for $\overline{F}$, but there is more.

---

[2]I am indebted to Giorgio Mossà (Università degli studi di Pisa, e chissà per quanto) for having revealed me this argument when I was still unable to manipulate co/ends.

Remark 1.13 : The family $\{\tau_f\}_{f\in\mathrm{hom}(\mathbf{C})}$ constructed before is a cone for the functor $\overline{F}$, and conversely any such cone determines a wedge for F, given by $\{\tau_c \triangleq \tau_{\mathrm{id}_c}\}_{c\in\mathbf{C}}$. Again, a morphism of cones goes to a morphism between the corresponding wedges, and conversely any morphism between wedges induces a morphism between the corresponding cones; these operations are mutually inverse and form an equivalence between the category $\mathrm{CN}(\overline{F})$ of cones for $\overline{F}$ and the category $\mathrm{WD}(F)$ of wedges for F (see Exercise **1**.**E5**).

Equivalences of categories obviously respect initial/terminal objects, and since co/limits are initial/terminal objects in the category of co/cones, and co/ends are initial/terminal objects in the category of co/wedges, we obtain that[3]

$$\int_c F \cong \varprojlim_{\mathrm{TW}(\mathbf{C})} \overline{F}; \qquad \int^c F \cong \varinjlim_{\mathrm{TW}(\mathbf{C}^{\mathrm{op}})^{\mathrm{op}}} \overline{F} \tag{15}$$

Remark 1.14 : There is another (slightly *ad hoc* and cumbersome, in the humble opinion of the author) characterization of co/ends as co/limits, given by [**ML98**, Prop. **IX.5.1**], which relies upon the *subdivision (associative) plot* (see [**Trib**, Def. **2**, **3**]) $(\wp\mathbf{C})^{\S}$ of $\mathbf{C}$, whose

- objects are the set $\mathrm{Ob}(\mathbf{C}) \sqcup \mathrm{hom}(\mathbf{C})$, in such a way that there exists a "marked" object $c^{\S}$ for each $c \in \mathbf{C}$, and another marked object $f^{\S}$ for each $f \in \mathrm{hom}(\mathbf{C})$. The reader must have clear in mind that $c^{\S}$ and $\mathrm{id}_c^{\S}$ are *different* objects of $\mathbf{C}^{\S}$;
- arrows are the set of all symbols $\mathrm{src}(f)^{\S} \to f^{\S}$, or $\mathrm{trg}(f)^{\S} \to f^{\S}$, as f runs over $\mathrm{hom}(\mathbf{C})$;
- composition law is the empty function.

The *subdivision category* $\mathbf{C}^{\S}$ is obtained from $(\wp\mathbf{C})^{\S}$ formally adding identities and giving to the resulting category the trivial composition law (composition is defined only if one of the arrows is the identity). The discussion before [**ML98**, Prop. **IX.5.1**] now sketches the proof that every functor $F\colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{D}$ induces a functor $\overline{F}\colon \mathbf{C}^{\S} \to \mathbf{D}$, whose limit (provided it exists) is isomorphic to the end of F.

Remark 1.15 : Co/limits in a category exist whenever the category has co/products and equalizers. So we would expect a characterization of co/ends in terms of these simpler pieces; such a characterization exists, and it turns out to be extremely useful in explicit computations (see for example our Remark **5.11** and the argument therein), characterizing co/ends

---

[3]Notice that the colimit is taken over the category $\mathrm{TW}^{\mathrm{op}}(\mathbf{C})$, the *opposite* of $\mathrm{TW}(\mathbf{C}^{\mathrm{op}})$: an *object* of $\mathrm{TW}^{op}(\mathbf{C})$ is an arrow $f\colon c' \to c$ in $\mathbf{C}^{\mathrm{op}}$, and a *morphism* from $f\colon c \to c'$ to $g\colon d \to d'$ is a commutative square such that $vgu = f$.

as co/equalizers of pairs of maps. In fact it is rather easy to see that

$$\int_c F(c,c) \cong \mathrm{eq}\left(\prod_{c\in\mathbf{C}} F(c,c) \xrightarrow[F_*]{F^*} \prod_{\varphi\colon c\to c'} F(c,c')\right) \qquad (16)$$

where the product over $\varphi\colon c \to c'$ can be expressed as a double product (over the objects $c, c' \in \mathbf{C}$, and over the arrows φ between these two fixed objects), and the arrows F^*, F_* are easily obtained from the arrows whose (φ, c, c')-component is (respectively) $F(\varphi, c')$ and $F(c, \varphi)$.

It is useful to stress that this characterization is compatible with the description of a co/limit as a co/equalizer, when F is mute in one variable. From this we deduce a different argument showing that the co/end of a mute functor coincides with its co/limit.

Definition 1.16 : There is an obvious definition of *preservation* of co/ends from their description as co/limits, which reduces to the preservation of the particular kind of co/limit involved in the definition of $\underline{\mathrm{end}}(F)$ and $\underline{\mathrm{coend}}(F)$.

This remark entails easily that

Theorem 1.17 : Every co/continuous functor $F\colon \mathbf{D} \to \mathbf{E}$ preserves every co/end that exists in $\mathbf{D}$, namely if $T\colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{D}$ has a co/end $\int_c^{/c} T(c,c)$, then

$$F\left(\int_c^{/c} T(c,c)\right) \cong \int_c^{/c} FT(c,c) \qquad (17)$$

in the obvious meaning that the two objects are canonically isomorphic having the same universal property.

As a particular example of this, we have

Corollary 1.18 [THE hom FUNCTOR COMMUTES WITH INTEGRALS]**:** Continuity of the hom bifunctor $\mathbf{C}(\_,=)\colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{Sets}$ gives its co/end preservation properties: for every $c \in \mathbf{C}$ we have the canonical isomorphisms

$$\mathbf{C}\left(\int^x F(x,x), c\right) \cong \int_x \mathbf{C}(F(x,x), c)$$
$$\mathbf{C}\left(c, \int_x F(x,x)\right) \cong \int_x \mathbf{C}(c, F(x,x))$$

The power of this remark can't be overestimated: co/continuity of the hom functor is a fundamental *kata* of coend-fu. Basically *every* example in the rest of the paper involves a computation carried on using this co/end preservation property, plus the fully faithfulness of the Yoneda embedding.

1.3. **Natural transformations as ends.**   A basic example exploiting
the whole machinery introduced so far is the proof that the set of natural
transformations between two functors $F, G \colon \mathbf{C} \to \mathbf{D}$ can be characterized as
an end:

Theorem 1.19 : Given functors $F, G \colon \mathbf{C} \to \mathbf{D}$ between small categories we
have the canonical isomorphism of sets

$$\mathsf{Nat}(F, G) \cong \int_c \mathbf{D}(Fc, Gc). \tag{18}$$

Proof. Giving a wedge $\tau_c \colon Y \to \mathbf{D}(Fc, Gc)$ consists in giving a function
$y \mapsto \tau_{c,y} \colon Fc \to Gc$, which is natural in $c \in \mathbf{C}$ (this is simply a rephrasing of
the wedge condition):

$$G(f) \circ \tau_{c,y} = \tau_{c',y} \circ F(f) \tag{19}$$

for any $f \colon c \to c'$; this means that there exists a unique way to close the
diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{\ \tau_c\ } & \mathbf{D}(Fc, Gc) \\
 & {\raise2pt\hbox{h}}\ \diagup & \big\uparrow \\
 & & \mathsf{Nat}(F, G)
\end{array}
\tag{20}
$$

with a function sending $y \mapsto \tau_{-,y} \in \prod_{c \in \mathbf{C}} \mathbf{D}(Fc, Gc)$, and where $\mathsf{Nat}(F, G) \to$
$\mathbf{D}(Fc, Gc)$ is the wedge sending a natural transformation to its c-component;
the diagram commutes for a single $h \colon Y \to \mathsf{Nat}(F, G)$, and this is precisely
the desired universal property for $\mathsf{Nat}(F, G)$ to be $\int_c \mathbf{D}(Fc, Gc)$.   □

Remark 1.20 : A suggestive way to express naturality as a "closure" condition
is given in [**Yon60**, **4.1.1**], where for an **Ab**-enriched functor $F \colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{Ab}$
between suitably complete **Ab**-categories, one can prove that $\mathsf{Nat}(F, G) =$
$\ker \delta$, for a suitable map δ defined among $\mathbf{Ab}(Fx, Gx) \oplus \mathbf{Ab}(Fy, Gy)$ and
$\mathbf{Ab}(Fy, Gy)$.

    After this, we can embark in more sophisticated and pervasive examples.
In particular, the following section is the gist of the paper, demonstrating
the power of co/end calculus to prove highly technical and involved results
by means of abstract-nonsense only.

Exercises for §1

E1 Show that for each $f \in \mathbf{Sets}(B, B')$ the following square is commutative:

$$
\begin{array}{ccc}
X^{B'} \times B & \xrightarrow{X^f \times B} & X^B \times B \\
{\scriptstyle X^{B'} \times f} \downarrow & & \downarrow {\scriptstyle \epsilon} \\
X^{B'} \times B' & \xrightarrow{\quad \epsilon \quad} & X
\end{array} \tag{21}
$$

E2 A dinatural transformation serves to define natural transformations between functors having the same co/domain but different variance; try to do this.

E3 Show with an example that dinatural transformations $\alpha \colon P \xrightarrow{\cdot\cdot} Q, \beta \colon Q \xrightarrow{\cdot\cdot} R$ cannot be composed in general. Nevertheless, there exists a "composition" of a dinatural $\alpha \colon P \xrightarrow{\cdot\cdot} Q$ with a natural $\eta \colon P' \to P$ which is again dinatural $P' \xrightarrow{\cdot\cdot} Q$, as well as a composition $P \xrightarrow{\cdot\cdot} Q \to Q'$ (hint: the appropriate diagram results as the pasting of a dinaturality hexagon and two naturality squares).

E4 Coends are obtained by a suitable dualization. State the definition of a *cowedge* for a functor $F \colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{D}$; a *coend* for F consists of a universal cowedge $\underline{\mathrm{coend}}(F)$ for P. Prove functoriality for coends. Show that coends are ends in the dual category: the coend of F is the end of $F^{\mathrm{op}} \colon \mathbf{C} \times \mathbf{C}^{\mathrm{op}} \cong \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{D}^{\mathrm{op}}$.

E5 Define a category $\mathrm{WD}(F)$ having objects the wedges for $F \colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{D}$ and show that the end of F is the terminal object of $\mathrm{WD}(F)$; dualize to coends (initial objects of a category $\mathrm{CWD}(F)$ of cowedges).

E6 Show that extranatural transformations compose according to these rules:

- (stalactites) Let F, G be functors of the form $\mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{D}$. If $\alpha_{x,y} \colon F(x, y) \to G(x, y)$ is natural in x, y and $\beta_x \colon G(x, x) \to H$ is extranatural in x (for some object H of D), then

$$
\beta_x \circ \alpha_{x,x} \colon F(x, x) \to H \tag{22}
$$

  is extranatural in x.

- (stalagmites) Let G, H be functors of the form $\mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{D}$. If $\alpha_x \colon F \to G(x, x)$ is extranatural in x (for some object F of D) and $\beta_{x,y} \colon G(x, y) \to H(x, y)$ is natural in x, y, then

$$
\beta_{x,x} \circ \alpha_x \colon F \to H(x, x) \tag{23}
$$

  is extranatural in x.

- (yanking) Let F, H be functors of the form $\mathbf{C} \to \mathbf{D}$, and let $G \colon \mathbf{C} \times \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{D}$ be a functor. If $\alpha_{x,y} \colon F(y) \to G(x, x, y)$ is natural in y and extranatural in x, and if $\beta_{x,y} \colon G(x, y, y) \to H(x)$ is natural in x and extranatural in y, then

$$
\beta_{x,x} \circ \alpha_{x,x} \colon F(x) \to H(x) \tag{24}
$$

  is natural in x.

Express these laws as equalities between suitable string diagrams (explaining also the genesis of the names "stalactite" and "stalagmite").

E7 Prove the Fubini theorem for ends embarking in a long exercise in universality; prove that if $F \colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{D}$ is *mute* in one of the two variables (i.e. $F(c', c) = \bar{F}(c)$ or $\hat{F}(c')$ for each $c, c' \in \mathbf{C}$ and suitable functors $\bar{F} \colon \mathbf{C} \to \mathbf{D}$ or $\hat{F} \colon \mathbf{C}^{\mathrm{op}} \to \mathbf{D}$), then the co/end of F is canonically isomorphic to its co/limit.

This gives an alternative proof of a similar *Fubini rule for co/limits*: given a functor $F: I \times J \to \mathbf{D}$ we have

$$\varinjlim_I \varinjlim_J F \cong \varinjlim_J \varinjlim_I F \cong \varinjlim_{I \times J} F \tag{25}$$

(and similarly for limits).

E8 Introduced to vector analysis in basic calculus courses, students learn that if (X, Ω, μ) is a measure space, the integral of a vector function $\vec{F}: X \to \mathbb{R}^n$ such that each $F_i = \pi_i \circ F: X \to \mathbb{R}$ is measurable and has finite integral, is the vector whose entries are $\left(\int_X F_1 d\mu, \cdots, \int_X F_n d\mu \right)$.

Prove that category theory possesses a similar formula, i.e. that if $F: \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{A}_1 \times \cdots \times \mathbf{A}_n$ is a functor towards a product of categories, such that for $1 \leq i \leq n$

- each $\mathbf{A}_i$ has both an initial and a terminal object, respectively denoted $\varnothing$ and 1;
- each co/end $\int_c^{/c} \pi_i \circ F$ exists

then the "vector" of all these co/ends, as an object $\left(\int_c^{/c} F_1, \ldots, \int_c^{/c} F_n \right) \in \mathbf{A}_1 \times \cdots \times \mathbf{A}_n$, is the (base of a universal co/wedge forming the) co/end of F.

E9 The categories $\mathrm{TW}(\mathbf{C})$ and $\mathbf{C}^{\S}$ are linked by a *final* (see [**Bor94a**, **2.11.1**]) functor $K: \mathbf{C}^{\S} \to \mathrm{TW}(\mathbf{C})$; this motivates the fact that the colimit is the same when indexed by one of the two. Define K and show that is is final, i.e. that for every object in $\mathrm{TW}(\mathbf{C})$ the comma category $(\varphi \downarrow K)$ is nonempty and connected (see [**Rie14**, Remark **7.2.10**, Example **8.3.9**]).

E10 Show that the end of a functor $T: \Delta[1]^{\mathrm{op}} \times \Delta[1] \to \mathbf{Sets}$ is the pullback of the morphisms $T(0,0) \xrightarrow{T(0,d_0)} T(0,1) \xleftarrow{T(d_0,1)} T(1,1)$, i.e.

$$\begin{array}{ccc}
\int_{i \in \Delta[1]} T(i,i) & \longrightarrow & T(0,0) \\
\downarrow & & \downarrow \\
T(1,1) & \longrightarrow & T(0,1)
\end{array} \tag{26}$$

is a pullback in $\mathbf{Sets}$ (of course there's nothing special about sets here!).

E11 Let G be a topological group, and $\mathrm{Sub}(G)$ the partially ordered (with respect to inclusion) set of its subgroups; let X be a G-space, i.e. a topological space with a continuous action $G \times X \to X$.

We can define two functors $\mathrm{Sub}(G) \to \mathbf{Top}$, sending $(H \leq G) \mapsto G/H$ (this is a covariant functor, and G/H has the induced quotient topology) and $(H \leq G) \mapsto X^H$ (the subset of H-fixed points for the action; this is a contravariant functor).

(1) Compute the coend

$$\int^{H \leq G} X^G \times G/H \tag{27}$$

in $\mathbf{Top}$ if $G = \mathbb{Z}/2$ has the discrete topology;

(2) Give a general rule for $\int^{H \leq G} X^G \times G/H$ when H is cyclic with n elements;

(3) Let instead $\mathrm{Orb}(G)$ be the *orbit category* of subgroups of G, whose objects are subgroups but $\hom(H, K)$ contains G-*equivariant* maps $G/H \to G/K$. Let again X^- and $G/\_$ define the same functors, now with different

action on arrows. Prove that $\int^{H \in \mathrm{Orb}(G)} X^H \times G/H \cong X$ (*Elmendorf reconstruction*, [**Elm83**]).

(4) Let $E|F$ be a field extension, and $\{H \subseteq \mathrm{Gal}(E|F)\}$ the partially ordered set of subgroups of the Galois group of the extension. Compute (in the category of *rings*)

$$\int^H E^H \times \mathrm{Gal}(H|F) \tag{28}$$

E12 Dualize the above construction, to obtain a similar characterization for the coend $\int^c F(c,c)$, characterized as the coequalizer of a similar pair (F^*, F_*).

E13 Show that the hom functors $\mathbf{C}(\_, y)$ *jointly preserve* ends (see [**ML98**] for a precise definition of joint preservation); the rough idea is that whenever $\mathrm{f}\mathbf{C}(W, y)$ is the end of $\mathbf{C}(F(c,c), y)$ for each y, naturally in $y \in \mathbf{C}$, then W is canonically isomorphic to $\int^c F(c,c)$.

E14 Define the map δ above, and show that $\ker \delta \cong \int_y \mathbf{Ab}(Fy, Gy)$ in the above notation; dualize to express a coend as a suitable coequalizer.

E15 Prove again Theorem **1.19**, using the characterization of $\int_c \mathbf{D}(Fc, Gc)$ as an equalizer: the subset of $\prod_{c \in \mathbf{C}} \mathbf{D}(Fc, Gc)$ you will have to consider is precisely the subset of natural transformations $\{\tau_c : Fc \to Gc \mid Gf \circ \tau_c = \tau_{c'} \circ Ff, \ \forall f : c \to c'\}$. This yields the evocative formula

$$\mathsf{Nat}(1_\mathbf{C}, 1_\mathbf{C}) = \mathsf{End}(1_\mathbf{C}) \cong \int_c \mathbf{C}(c,c). \tag{29}$$

Is it possible to give an explicit meaning to the dual construction $\int^c \mathbf{C}(c,c)$ (start with simple examples: $\mathbf{C}$ discrete, $\mathbf{C}$ a finite group, $\mathbf{C}$ a finite groupoid...)? Compare also Example **2.11**.

E16 What is the co/end of the identity functor $1_{\mathbf{C}^\mathrm{op} \times \mathbf{C}} : \mathbf{C}^\mathrm{op} \times \mathbf{C} \to \mathbf{C}^\mathrm{op} \times \mathbf{C}$? Use the bare definition; use the characterization of co/ends as co/limits; feel free to invoke Exercise **E8**.

E17 A preliminary definition for this exercise is the following: a set of objects $S \subset \mathbf{C}$ *finitely generates* a category $\mathbf{C}$ if for each object $X \in \mathbf{C}$ and each arrow $f : s \to c$ from $s \in S$ there is a factorization

$$s \xrightarrow{g} \coprod_{i=1}^n s_i \xrightarrow{h_c} c \tag{30}$$

where h_c is an epimorphism and $\{s_1, \ldots, s_n\} \subset S$ (n depends on c and f). Suppose $T : \mathbf{C}^\mathrm{op} \times \mathbf{C} \to \mathbf{Sets}$ is a functor, finitely continuous in both variables, and $\mathbf{C}$ is finitely generated by S. Then, considering S as a full subcategory $\mathbf{S} \subseteq \mathbf{C}$, and calling $T' := T|_\mathbf{S} : \mathbf{S}^\mathrm{op} \times \mathbf{S} \to \mathbf{Sets}$, we have an isomorphism

$$\int^{c \in \mathbf{S}} T'(c,c) \cong \int^{c \in \mathbf{C}} T(c,c) \tag{31}$$

induced by a canonical arrow $\int^{c \in \mathbf{S}} T'(c,c) \to \int^{c \in \mathbf{C}} T(c,c)$.

E18 Let $F \dashv U : \mathbf{C} \leftrightarrows \mathbf{D}$ be an adjunction, and $G : \mathbf{D}^\mathrm{op} \times \mathbf{C} \to \mathscr{E}$ a functor; then there is an isomorphism

$$\int^c G(Fc, c) \cong \int^d G(d, Ud). \tag{32}$$

Show that a converse of this result is true: if the above isomorphism is true for any G and natural therein, then there is an adjunction $F \dashv U$.

2. Yoneda reduction, Kan extensions

One of the most famous results about the category $\mathbf{Cat}(\mathbf{C}^{\mathrm{op}}, \mathbf{Sets})$ of presheaves on a category $\mathbf{C}$ is that every object in it can be canonically presented as a colimit of representable functors; see [**ML98**, Theorem **III.7.1**] for a description of this classical result (and its dual holding in $\mathbf{Cat}(\mathbf{C}, \mathbf{Sets})$).

Now, co/end calculus allows us to rephrase this result in an extremely compact way, called *Yoneda reduction*; in a few words, it says that *every co/presheaf can be expressed as a co/end*[4].

Proposition 2.1 [NINJA YONEDA LEMMA]**:** For every functor $K \colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Sets}$ and $H \colon \mathbf{C} \to \mathbf{Sets}$, we have the following isomorphisms (natural equivalences of functors):

$$(i)\ K \cong \int^c Kc \times \mathbf{C}(\_, c) \qquad (ii)\ K \cong \int_c Kc^{\mathbf{C}(c,-)}$$

$$(iii)\ H \cong \int^c Hc \times \mathbf{C}(c, \_) \qquad (iv)\ H \cong \int_c Hc^{\mathbf{C}(-,c)}$$

Remark 2.2 : The name *ninja Yoneda lemma* is a pun coming from a `Math-Overflow` comment by T. Leinster, whose content is basically the proof of the above statement:

> Th[e above one is] often called the *Density Formula*, [. . .] or (by Australian ninja category theorists) simply the Yoneda Lemma. (but Australian ninja category theorists call *everything* the Yoneda Lemma. . .).

Undoubtedly, there is a link between the above result and the Yoneda Lemma we all know: in fact, the proof heavily relies on the Yoneda isomorphism, and in enriched setting (see [**Dub70**, §**I.5**]) the ninja Yoneda lemma, interpreted as a theorem about Kan extensions, is *equivalent* to the one from the Northern hemisphere.

We must admit to feel somewhat unqualified to properly discuss the topic, as we live in the wrong hemisphere of the planet to claim any authority on it, and to be acquainted with the rather unique taste of Australian practicioners in choosing evocative (or obscure) terminology. Nevertheless, along the whole note, we keep the name "ninja Yoneda Lemma" as a (somewhat witty) nickname for the above isomorphisms, without pretending any authoritativeness whatsoever.

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[4] The reader looking for a nifty explanation of this result should wait for a more thorough discussion, which can be deduced from the material in §**4**, thanks to the machinery of *weighted co/limits*.

Proof. We prove case (i) only, all the others being totally analogous. We put a certain emphasis on the style of this proof, as it is paradigmatic of most of the subsequent ones. Consider the chain of isomorphisms

$$
\begin{aligned}
\mathbf{Sets}\Big(\textstyle\int^{c\in\mathbf{C}} Kc \times \mathbf{C}(x,c), y \Big) &\cong \int_{c\in\mathbf{C}} \mathbf{Sets}(Kc \times \mathbf{C}(x,c), y) \\
&\cong \int_{c\in\mathbf{C}} \mathbf{Sets}(\mathbf{C}(x,c), \mathbf{Sets}(Kc, y)) \\
&\cong \mathsf{Nat}(\mathbf{C}(x, \_\,), \mathbf{Sets}(K-, y)) \\
&\cong \mathbf{Sets}(Kx, y)
\end{aligned}
$$

where the first step is motivated by the coend-preservation property of the hom functor, the second follows from the fact that **Sets** is a cartesian closed category, where

$$\mathbf{Sets}(X \times Y, Z) \cong \mathbf{Sets}(X, \mathbf{Sets}(Y, Z)) \tag{33}$$

for all three sets X, Y, Z (naturally in all arguments), and the final step exploits Theorem **1.19** plus the classical Yoneda Lemma.

Every step of this chain of isomorphisms is natural in y; now we have only to notice that the natural isomorphism of functors

$$\mathbf{Sets}\Big(\textstyle\int^c Kc \times \mathbf{C}(x,c), y \Big) \cong \mathbf{Sets}(Kx, y) \tag{34}$$

ensures that there exists a (natural) isomorphism $\int^c Kc \times \mathbf{C}(x,c) \cong Kx$. This concludes the proof and it is, if you want, a way to motivate and partially solve Exercise **1.E13**. $\qquad\square$

From now one we will make frequent use of the notion of (**Sets**-)*tensor* and (**Sets**-)*cotensor* in a category; these standard definitions are in the chapter of any book about enriched category theory (see for example [**Bor94b**, Ch. **6**], its references, and in particular its Definition **6.5.1**, which we report for the ease of the reader:

Definition 2.3 [Tensor and cotensor in a $\mathscr{V}$-category]**:** In any $\mathscr{V}$-enriched category **C** (see [**Bor94b**, Def. **6.2.1**]), the *tensor* $\cdot: \mathscr{V} \times \mathbf{C} \to \mathbf{C}$ is a functor $(V, c) \mapsto V \cdot c$ such that there is the isomorphism

$$\mathbf{C}(V \cdot c, c') \cong \mathscr{V}(V, \mathbf{C}(c, c')), \tag{35}$$

natural in all components; dually, the *cotensor* in an enriched category **C** is a functor $(V, c) \mapsto c^V$ (contravariant in V) such that there is the isomorphism

$$\mathbf{C}(c', c^V) \cong \mathscr{V}(V, \mathbf{C}(c', c)), \tag{36}$$

natural in all components.

Example 2.4 : Every co/complete, locally small category **C** is naturally **Sets**-co/tensored by choosing $c^V \cong \prod_{v\in V} c$ and $V \cdot c \cong \coprod_{v\in V} c$.

Remark 2.5 : The tensor, hom and cotensor functors are the prototype of a THC *situation* (see our Remark **3.13** and [**Gra80**, §**1.1**] for a definition); given the hom-objects of a $\mathcal{V}$-category **C**, the tensor $\cdot \colon \mathcal{V} \times \mathbf{C} \to \mathbf{C}$ and the cotensor $(_=)^{(-)} \colon \mathcal{V}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{C}$ can be characterized as adjoint functors: usual co/continuity properties of the co/tensor functors are implicitly derived from this characterization.

Remark 2.6 [☺☺, THE YONEDA EMBEDDING IS A DIRAC DELTA]**:** In functional analysis, the Dirac delta appears in the following convenient abuse of notation:

$$\int_{-\infty}^{\infty} f(x)\delta(x-y)dx = f(y) \tag{37}$$

(the integral sign, here, is not a co/end). Here $\delta_y(x) \coloneqq \delta(x-y)$ is the *y-centered delta-distribution*, and $f \colon \mathbb{R} \to \mathbb{R}$ is a continuous, compactly supported function on $\mathbb{R}$.

It is really tempting to draw a parallel between this relation and the ninja Yoneda lemma, conveying the intuition that representable functors on an object $c \in \mathbf{C}$ play the rôle of c-centered delta-distributions. If the relation above is written as $\langle f, \delta_y \rangle = f(y)$, interpreting integration as an inner product between functions, then the ninja Yoneda lemma says formally the same thing: for each presheaf $F \colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Sets}$, the "inner product" $\langle \, \natural_c, F \rangle = \int^x \natural_c(x) \times Fc$ equals Fc (obviously, there's nothing special about sets here).

2.1. **Kan extensions as co/ends.**

Definition 2.7 : Given a functor $F \colon \mathbf{C} \to \mathbf{D}$, its *left* and *right Kan extensions* are defined[5] to be, respectively, the left and right adjoint to the "precomposition" functor

$$F^* \colon \mathsf{Fun}(\mathbf{D}, \mathbf{E}) \to \mathsf{Fun}(\mathbf{C}, \mathbf{E}) \tag{38}$$

given by $H \mapsto F^*(H) = H \circ F$, in such a way that there are two isomorphisms

$$\mathsf{Nat}(\mathrm{Lan}_F\, G, H) \cong \mathsf{Nat}(G, H \circ F)$$
$$\mathsf{Nat}(H \circ F, G) \cong \mathsf{Nat}(F, \mathrm{Ran}_F\, G).$$

Now, we want to show that in "nice" situations it is possible to describe Kan extensions via co/ends: whenever the co/tensors (see Def. **2.3**) involved in the definition of the following co/ends exist in **D** for any choice of functors F, G (a blatant example is when **D** is co/complete, since in that case as said

---

[5]This is not true, strictly speaking. Nevertheless we prefer to cheat the reader with this useful insight instead of obscuring the general idea keeping track of all possible pathologies. Furthermore, we are interested only in the cases when Kan extensions can be written as co/ends, so we will not consider any pathology whatsoever and we can safely assume that this is a proper definition.

before it is always **Sets**-co/tensored by suitable iterated co/products, see the remark above **2.5**), then the left/right Kan extensions of $G\colon \mathbf{C} \to \mathbf{E}$ along $F\colon \mathbf{C} \to \mathbf{D}$ exists and there are isomorphisms

$$\mathrm{Lan}_F\, G \cong \int^c \mathbf{D}(Fc, \_) \cdot Gc \qquad\qquad \mathrm{Ran}_F\, G \cong \int_c Gc^{\mathbf{D}(\_, Fc)}. \qquad (39)$$

Proof. The proof consists of a string of canonical isomorphisms, exploiting simple remarks in elementary Category Theory and the results established so far: the same argument is offered in [**ML98**, Thm. **X.4.1, 2**].

$$
\begin{aligned}
\mathsf{Nat}\Big(\int^c \mathbf{D}(Fc, \_) \cdot Gc, H \Big) &\overset{\mathbf{1.19}}{\cong} \int_x \mathbf{D}\Big(\int^c \mathbf{D}(Fc, x) \cdot Gc, Hx \Big) \\
&\overset{\mathbf{1.18}}{\cong} \int_{cx} \mathbf{D}(\mathbf{D}(Fc, x) \cdot Gc, Hx) \\
&\overset{(35)}{\cong} \int_{cx} \mathbf{Sets}(\mathbf{D}(Fc, x), \mathbf{E}(Gc, Hx)) \\
&\overset{\mathbf{1.19}}{\cong} \int_c \mathsf{Nat}(\mathbf{D}(Fc, \_), \mathbf{E}(Gc, H-)) \\
&\overset{\mathrm{Yon}}{\cong} \int_c \mathbf{E}(Gc, HFc) \cong \mathsf{Nat}(G, HF).
\end{aligned}
$$

The case of $\mathrm{Ran}_F\, G$ is dually analogous. $\qquad\qquad\qquad\qquad\qquad\square$

Remark 2.8 : This is the pattern of every "proof by coend-juggling" we will meet in the rest of the paper; from now on we feel free to abandon a certain pedantry in justifying every single deduction in the chains of isomorphisms leading to conclude a proof.

Proposition 2.9 : Left/right adjoint functors commute with left/right Kan extensions, whenever they can be expressed as the coends above.

Proof. An immediate corollary of Theorem **1.17**, once it has been proved that a left adjoint commutes with tensors, i.e. $F(X \cdot a) \cong X \cdot Fa$ for any $(X, a) \in \mathbf{Sets} \times \mathbf{C}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Example 2.10 : Let $T\colon \mathbf{C} \to \mathbf{C}$ be a *monad* on the category $\mathbf{C}$; the *Kleisli category* $\mathbf{Kl}(T)$ of T is defined having the same objects of $\mathbf{C}$ and morphisms $\mathbf{Kl}(T)(a, b) := \mathbf{C}(a, Tb)$.

Given any functor $F\colon \mathbf{A} \to \mathbf{C}$, the right Kan extension $T_F = \mathrm{Ran}_F\, F$ is a monad on $\mathbf{C}$, the *codensity monad* of F; hom-sets in the Kleisli category $\mathbf{Kl}(T_F)$ can be characterized as

$$\mathbf{Kl}(T_F)(c, c') \cong \int_a \mathbf{Sets}(\mathbf{C}(c', Fa), \mathbf{C}(c, Fa)). \qquad (40)$$

The proof is an exercise in coend-juggling, recalling that $T_F(\_) \overset{(39)}{\cong} \int_c Fc^{\mathbf{C}(\_, Fc)}$.

Example 2.11 : Let V be a finite dimensional vector space over the field K; let $V^\vee$ denote the dual vector space of linear maps $V \to K$. Then there is a canonical isomorphism

$$\int^V V^\vee \otimes V \cong K. \tag{41}$$

The fastest way to see this is to notice that

$$\int^V \hom(V, \_) \otimes V \cong \mathrm{Lan}_{\mathrm{id}}(\mathrm{id}) \cong \mathrm{id}_{\mathbf{Vect}} \tag{42}$$

(compare this argument with any proof trying to explicitly evaluate the coend!)

Remark 2.12 : The universal cowedge $\hom(V, V) \xrightarrow{\alpha_V} K$ sends an endomorphism $f \colon V \to V$ to its *trace* $\tau(f) \in K$ (which in this way acquires a universal property).

The above argument holds in fact in fair generality, adapting to the case where V is an object of a compact closed monoidal category $\mathscr{C}$.

Exercises for §2

E1 Show that presheaf categories are cartesian closed, via coends: if $[\mathbf{C}^{\mathrm{op}}, \mathbf{Sets}]$ is the category of presheaves on a small $\mathbf{C}$, then there exists an adjunction

$$\mathsf{Nat}(P \times Q, R) \cong \mathsf{Nat}(P, R^Q) \tag{43}$$

by showing that $R^Q(c) = \mathsf{Nat}(\mathord{\text{\maltese}}_c \times Q, R)$ does the job (use the ninja Yoneda lemma, as well as Thm. **1.19**).

E2 Use equations (**39**) and the ninja Yoneda lemma that $\mathrm{Lan}_{\mathrm{id}}$ and $\mathrm{Ran}_{\mathrm{id}}$ are the identity functors, as expected. Use again (**39**) and the ninja Yoneda lemma to complete the proof that $F \mapsto \mathrm{Lan}_F$ is a pseudofunctor, by showing that for $\mathbf{A} \xrightarrow{F} \mathbf{B}$, $\mathbf{A} \xrightarrow{G} \mathbf{C} \xrightarrow{H} \mathbf{D}$ there is a uniquely determined laxity cell for composition

$$\mathrm{Lan}_H(\mathrm{Lan}_G(F)) \cong \mathrm{Lan}_{HG}(F) \tag{44}$$

(hint: coend-juggle with $\mathrm{Lan}_H(\mathrm{Lan}_G(F))d$ until you get $\int^{xy} (\mathbf{D}(Hx, d) \times \mathbf{C}(Gy, x)) \cdot Fy$; now use the ninja Yoneda lemma plus co-continuity of the tensor, as suggested in Remark **2.5**).

E3 Prove in a similar way isomorphisms $(ii), (iii), (iv)$ in Thm. **2.1** (hint: for (ii) and (iv) start from $\mathbf{Sets}\left(y, \int_c Hc^{\mathbf{C}(x,c)}\right)$ and use again the end preservation property, cartesian closure of $\mathbf{Sets}$, and Thm. **1.19**).

E4 Let $\mathbf{C}$ be a *compact* closed monoidal category [**Day77**]; Show that the functor $y \mapsto \int^x x^\vee \otimes y \otimes x$ carries the structure of a monad on $\mathbf{C}$.

---

3. The nerve and realization paradigm.

3.1. **The classical nerve and realization.** The most fruitful application of the machinery of Kan extensions is the "Kan construction" for the

realization of simplicial sets. It is impossible to underestimate the value of this construction as a unification tool in algebra and algebraic topology; here, we briefly sketch this construction.

Consider the category $\mathbf{\Delta}$ of finite ordinals and monotone maps, as defined in [**GJ09**], and the Yoneda embedding よ$_\Delta \colon \mathbf{\Delta} \to \mathbf{sSet}$; we can define two functors $\rho \colon \mathbf{\Delta} \to \mathbf{Top}$ and $i \colon \mathbf{\Delta} \to \mathbf{Cat}$ which "represent" every object $[n] \in \mathbf{\Delta}$ either as a topological space or as a small category:

- The category $i[n]$ is $\{0 \to 1 \to \cdots \to n\}$ (there is a similar functor $P \colon \mathbf{Pos} \to \mathbf{Cat}$ regarding any poset $(P, \leq)$ as the category $\mathbf{P}$ where the composition function is induced by the partial order relation $\leq$: i here is the restriction $P|_\Delta \colon \mathbf{\Delta} \subset \mathbf{Pos}$);
- The topological space $\rho[n]$ is defined as the *standard n-simplex* Δ^n embedded in $\mathbb{R}^{n+1}$,

$$\rho[n] = \left\{ (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid 0 \leq x_i \leq 1, \sum_{i=0}^n x_i = 1 \right\}. \qquad (45)$$

In a few words, we are in the situation depicted by the following diagrams:

$$
\begin{array}{ccc}
\mathbf{\Delta} \xrightarrow{\ i\ } \mathbf{Cat} & \qquad \mathbf{\Delta} \xrightarrow{\ \rho\ } \mathbf{Top} & \qquad (46) \\
\text{よ} \downarrow \phantom{\mathbf{Cat}} & \text{よ} \downarrow \phantom{\mathbf{Top}} & \\
\mathbf{sSet} & \qquad \mathbf{sSet} &
\end{array}
$$

The two functors i, ρ can be (left) Kan-extended along the Yoneda embedding よ$_\Delta \colon \mathbf{\Delta} \to \mathbf{sSet}$, and these extensions happen to be left adjoints (this can be proved directly, but we will present in a while a completely general statement).

We denote these adjunctions

$$\mathrm{Lan}_{\text{よ}}\, i \dashv N_i \ \text{ and } \ \mathrm{Lan}_{\text{よ}}\, \rho \dashv N_\rho; \qquad (47)$$

these two right adjoint functors are called the *nerves* associated to i and ρ respectively, and are defined, respectively, sending a category $\mathbf{C}$ to the simplicial set $N_i(\mathbf{C}) \colon [n] \mapsto \mathsf{Fun}(i[n], \mathbf{C})$ (the *classical nerve* of a category), and to the simplicial set $N_\rho(X) \colon [n] \mapsto \mathbf{Top}(\rho[n], X) = \mathbf{Top}(\Delta^n, X)$ (the *singular complex* of a space X).[6]

The left adjoints to N_ρ and N_i must be thought as "realizations" of a simplicial set as an object of $\mathbf{Top}$ or $\mathbf{Cat}$:

---

[6]The name is motivated by the fact that if we consider the free-abelian group on $N_\rho(X)_n$, the various $C_n = \mathbb{Z} \cdot N_\rho(X)_n = \coprod_{N_\rho(X)_n} \mathbb{Z}$ organize as a chain complex, whose homology is precisely the singular homology of X.

- The left Kan extension $\mathrm{Lan}_{\mathcal{L}}\, \rho$ is the "geometric" realization $|X_\bullet|$ of a simplicial set $X_\bullet$, resulting as the coend

$$\int^{n\in\mathbf{\Delta}} \Delta^n \times X_n \tag{48}$$

which is equal to a suitable coequalizer in **Top**. The shape of this object is fairly easy to motivate, keeping open any book in algebraic topology. The topological space $|X_\bullet|$ is obtained choosing a n-dimensional disk Δ^n (we are, of course, reasoning up to isomorphism!) for each n-simplex $x \in X_n$ and gluing these disks along their boundaries $\delta_i(\Delta^n)$ according to the degeneracy maps of $X_\bullet$. The resulting space is (almost by definition) a CW-complex (this means that $|X_\bullet|$ has the topology induced by a sequential colimit of pushouts of spaces $X_{(0)} \to X_{(1)} \to \dots$).

- The left Kan extension $\mathrm{Lan}_{\mathcal{L}}\, i$ is the "categorical realization" $\tau_1(X_\bullet)$ of a simplicial set $X_\bullet$, resulting as the coend

$$\int^{n\in\mathbf{\Delta}} i[n] \times X_n. \tag{49}$$

This is the category whose objects are 0-simplices of $X_\bullet$, arrows are 1-simplices, and the map $X_1 \times_{X_0} X_1 \to X_1$ obtained pulling back $d_0, d_1 \colon X_1 \to X_0$ works as a composition law. The category $i[n]$ can be regarded as the "universal composable string of n arrows", and the action of degeneracies (given by composition of contiguous arrows) induces a composition law for n-tuples of arrows under the quotient that defines the coend.

We leave the reader think about why we only mention degeneracies here, leaving out faces (they are necessary, aren't they?), and we depict the geometric realization of $X_\bullet$ in figure **3.1**.

It should now be evident that there is a pattern (we will call it the "nerve-realization paradigm") acting behind the scenes, and yielding the classical/singular nerve as particular cases of a general construction interpreted from time to time in different settings; unraveling this machinery with the power of co/end calculus is the scope of the following section.

3.2. **Some famous realizations and their associated nerves.** Algebraic topology, representation theory, and more generally every setting where a "well-behaved" categorical structure is involved constitute natural factories for examples of the nerve-realization paradigm. We now want to lay down organic foundations for a general theory and a general terminology allowing us to collect several examples (leaving outside many interesting others!) of
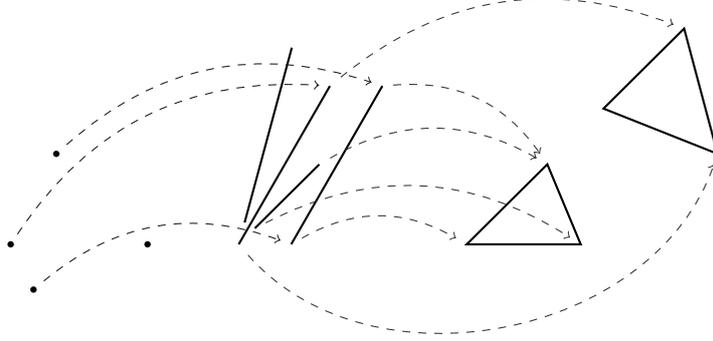
FIGURE 1. The construction of the geometric realization of
$X \in \mathbf{sSet}$ as a sequential colimit of pushouts of disks.

nerve-realizations pairs, obtained varying the domain category of "geometric shapes" or the category where this fundamental shapes are "represented".

Definition 3.1 : Any functor $\varphi \colon \mathbf{C} \to \mathbf{D}$ from a small category $\mathbf{C}$ to a (locally small) *cocomplete* category $\mathbf{D}$ is called a *nerve-realization context* (a NR-*context* for short).

Given a nerve-realization context φ, we can prove the following result:

Proposition 3.2 [NERVE-REALIZATION PARADIGM]**:** The left Kan extension of φ along the Yoneda embedding $\yen \colon \mathbf{C} \to [\mathbf{C}^{\mathrm{op}}, \mathbf{Sets}]$, i.e. the functor $R_\varphi = \mathrm{Lan}_{\yen}\, \varphi \colon [\mathbf{C}^{\mathrm{op}}, \mathbf{Sets}] \to \mathbf{D}$ is a left adjoint, $R_\varphi \dashv N_\varphi$. R_φ is called the $\mathbf{D}$-*realization functor* or the *Yoneda extension* of φ, and its right adjoint the $\mathbf{D}$-*coherent nerve.*

Proof. The cocomplete category $\mathbf{D}$ is $\mathbf{Sets}$-tensored, and hence $\mathrm{Lan}_{\yen}\, \varphi$ can be written as the coend in equation (**39**); so the claim follows from the chain of isomorphisms

$$\mathbf{D}\big(\mathrm{Lan}_{\yen}\, \varphi(P), d \big) \cong \mathbf{D}\Big(\int^c [\mathbf{C}^{\mathrm{op}}, \mathbf{Sets}](\yen_c, P) \cdot \varphi c, d \Big)$$

$$\cong \int_c \mathbf{D}\big([\mathbf{C}^{\mathrm{op}}, \mathbf{Sets}](\yen_c, P) \cdot \varphi c, d \big)$$

$$\cong \int_c \mathbf{Sets}\big([\mathbf{C}^{\mathrm{op}}, \mathbf{Sets}](\yen_c, P), \mathbf{D}(\varphi c, d) \big)$$

$$\cong \int_c \mathbf{Sets}\big(Pc, \mathbf{D}(\varphi c, d) \big).$$

If we define $N_\varphi(d)$ to be $c \mapsto \mathbf{D}(\varphi c, d)$, this last set becomes canonically isomorphic to $[\mathbf{C}^{\mathrm{op}}, \mathbf{Sets}](P, N_\varphi(d))$. $\square$

Remark 3.3 : The nerve-realization paradigm can be rewritten in the following equivalent form: there is an equivalence of categories, induced by the universal property of the Yoneda embedding,

$$\mathsf{Fun}(\mathbf{C}, \mathbf{D}) \cong \mathrm{RAdj}(\widehat{\mathbf{C}}, \mathbf{D}) \tag{50}$$

whenever $\mathbf{D}$ is a cocomplete locally small category (in such a way that "the category of nerve-realization contexts" is a high-sounding name for the category of functors $\mathsf{Fun}(\mathbf{C}, \mathbf{D})$).

A famous result in Algebraic Topology (see for example [**GZ67**, **GJ09**]) says that the geometric realization functor $R\colon \mathbf{sSet} \to \mathbf{Top}$ commutes with finite products: coend calculus gives a massive simplification of this result.

Proof. The main point of the proof is showing that the geometric realization commutes with products *of representables*: the rest of the proof relies on a suitable application of coend-fu. We could appeal conceptual ways to show this preliminary result ([**ABLR02**, §**2**]):

Proposition 3.4 : The following properties for a functor $F\colon \mathbf{C} \to \mathbf{Sets}$ are equivalent:

- F commutes with finite limits;
- $\mathrm{Lan}_{\mkern1mu\rotatebox{90}{\tiny $\curlywedge$}}\, F$ commutes with finite limits;
- F is a filtered colimit of representable functors;
- The *category of elements* $\mathbf{C} \int F$ of F (see Def. **4.1** and Prop. **4.14**) is cofiltered.

Nevertheless, this result isn't powerful enough to show that the geometric realization ρ commutes with $\mathbf{Top}$-products, since it gives only a *bijection* $|\Delta[n] \times \Delta[m]| \cong \Delta^n \times \Delta^m$; a certain amount of dirty work is necessary to show that this bijection is a homeomorphism. However, if we take the commutativity of R with finite products of representables for granted, the proof is a *kata* in coend-fu, recalling that

- The geometric realization is a left adjoint, hence it commutes with colimits and tensors;
- Every simplicial set is a colimit of representables.

Starting the usual machinery, we have that

$$\begin{aligned}
R(X \times Y) &\cong R\left[\left(\textstyle\int^m X_m \cdot \Delta[m]\right) \times \left(\textstyle\int^n Y_n \cdot \Delta[n]\right)\right] \\
&\cong R\left[\textstyle\int^{mn} (X_m \cdot \Delta[m]) \times (Y_n \cdot \Delta[n])\right] \\
&\cong R\left[\textstyle\int^{mn} (X_m \times Y_n) \cdot (\Delta[m] \times \Delta[n])\right] \\
&\cong \textstyle\int^{mn} (X_m \times Y_n) \cdot R(\Delta[m] \times \Delta[n])
\end{aligned}$$

$$\cong \int^{mn} (X_m \times Y_n) \cdot \Delta^m \times \Delta^n$$
$$\cong \int^{mn} (X_m \cdot \Delta^m) \times (Y_n \cdot \Delta^n)$$
$$\cong \left(\int^m X_m \cdot \Delta^m \right) \times \left(\int^n Y_n \cdot \Delta^n \right)$$
$$\cong R(X) \times R(Y)$$

where we applied, respectively, the ninja Yoneda lemma, the colimit preservation property of R, its commutation with tensors, and its commutativity with finite products of representables. $\qquad\square$

3.3. **Examples of nerves and realizations.** A natural factory of nerve-realization contexts is homotopical algebra, as such functors are often used to build Quillen equivalences between model categories. This is somewhat related to the fact that "transfer theorem" for model structures often apply to the well-behaved nerve functor.

But Quillen adjunctions between model categories are certainly not the only examples of NR-paradigms!

The following list attempts to gather important examples of NR-contexts: for the sake of completeness, we repeat the description of the two above-mentioned examples of the topological and categorical realizations.

Example 3.5 [CATEGORICAL NERVE AND REALIZATION]**:** In the case of $\varphi = i \colon \mathbf{\Delta} \to \mathbf{Cat}$, we obtain the *classical nerve* $N_{\mathbf{Cat}}$ of a (small) category $\mathbf{C}$, whose left adjoint is the *categorical realization* (the *fundamental category* $\tau_1 X$ of X described in [**Joy02b**]). The nerve-realization adjunction

$$\tau_1 \colon \mathbf{sSet} \leftrightarrows \mathbf{Cat} \colon N_{\mathbf{Cat}} \tag{51}$$

gives a Quillen adjunction between the Joyal model structure on **sSet** (see [**Joy02b**]) and the folk model structure on **Cat**.

Example 3.6 [GEOMETRIC NERVE AND REALIZATION]**:** If $\varphi = \rho \colon \mathbf{\Delta} \to \mathbf{Top}$ is the realization of a representable $[n]$ in the standard topological simplex, we obtain the adjunction between the *geometric realization* $|X|$ of a simplicial set X and the *singular complex* of a topological space Y, i.e. the simplicial set having as set of n-simplices the continuous functions $\Delta^n \to Y$.

Example 3.7 [**sSet**-COHERENT NERVE AND REALIZATION]**:** If $\varphi \colon \mathbf{\Delta} \to \mathbf{Cat}_{\mathbf{\Delta}}$ is the functor which realizes every representable $[n]$ as a simplicial category having objects the same set $[n] = \{0, 1, \ldots, n\}$ and as $\hom(i, j)$ the simplicial set obtained as the nerve of the poset $P(i, j)$ of subsets of the interval $[i, j]$ which contain both i and j,[7] we obtain the *(Cordier) simplicially coherent nerve and realization*, which sends **C** into a simplicial set constructed "coherently remembering" that **C** is a simplicial category. This adjunction establishes a

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[7]In particular if $i > j$ then $P(i, j)$ is empty and hence so is its nerve.

Quillen adjunction $\mathbf{sSet} \leftrightarrows \mathbf{Cat}_{\Delta}$ which restricts to an equivalence between quasicategories (fibrant objects in the Joyal model structure on $\mathbf{sSet}$) and fibrant simplicial categories (with respect to the Bergner model structure on $\mathbf{Cat}_{\Delta}$).

Example 3.8 [Moerdijk generalized intervals]**:** The construction giving the topological realization of $\Delta[n]$ extends to the case of any "interval" in the sense of [**Moe95**, §III.1], i.e. any ordered topological space J having "endpoints" $0, 1$; indeed every such space J defines a "generalized" (in the sense of [**Moe95**, §III.1]) topological n-simplex $\Delta^n(J)$, i.e. a nerve-realization context $\varphi_J \colon \Delta \to \mathbf{Top}$.

Example 3.9 [Toposophic nerve and realization]**:** The correspondence $\delta \colon [n] \mapsto \mathrm{Sh}(\Delta^n)$ defines a *cosimplicial topos*, i.e. a cosimplicial object in the category of toposes, which serves as a NR-context. Some geometric properties of this nerve/realization are studied in [**Moe95**, §III]: we outline an instance of a problem where this adjunction naturally arises: let $\mathbf{X}, \mathbf{Y}$ be the categories of sheaves over topological spaces X, Y. Let $\mathbf{X} \star \mathbf{Y}$ be the join of the two toposes seen as categories: this blatantly fails to be a topos, but there is a rather canonical "replacement" procedure

$$\mathbf{Cat} \times \mathbf{Cat} \xrightarrow{\ \star\ } \mathbf{Cat} \xrightarrow{\ N\ } \mathbf{sSet} \xrightarrow{\mathrm{Lan}_{\sharp}(\delta)} \mathbf{Topos}$$
$$(\mathbf{X}, \mathbf{Y}) \longmapsto \mathbf{X} \star \mathbf{Y} \longmapsto \mathbf{Cat}(\Delta^{\bullet}, \mathbf{X} \star \mathbf{Y}) \longmapsto \mathbf{X} \circledast \mathbf{Y}$$

Example 3.10 [The Dold-Kan correspondence]**:** The well-known Dold-Kan correspondence, which asserts that there is an equivalence of categories between simplicial abelian groups $[\Delta^{\mathrm{op}}, \mathbf{Ab}]$ and chain complexes $\mathrm{Ch}^+(\mathbf{Ab})$ with no negative homology, and it can be seen as an instance of the nerve-realization paradigm.

In this case, the functor $\Delta \to \mathrm{Ch}^+(\mathbf{Ab})$ sending $[n]$ to $\mathbb{Z}^{\Delta[n]}$ (the free abelian group on $\Delta[n]$) and then to the *Moore complex* $M(\mathbb{Z}^{\Delta[n]})$ determined by any simplicial group $A \in [\Delta^{\mathrm{op}}, \mathbf{Ab}]$ as in [**GJ09**] is the nerve-realization context.

Example 3.11 [Étale spaces as Kan extensions]**:** Let X be a topological space, and $\mathbf{Opn}(X)$ its poset of open subsets. There exists a natural functor

$$A \colon \mathbf{Opn}(X) \to \mathbf{Top}_{/X} \tag{52}$$

sending $U \subseteq X$ to the same morphism $U \to X$; this works as a NR-context giving the pair of adjoint functors

$$\mathrm{Lan}_{\sharp} A \dashv N_A \tag{53}$$

where N_A is defined precisely taking the (pre)sheaf of sections of $p \in \mathbf{Top}_{/X}$. The resulting left adjoint is precisely the functor sending a presheaf $F \in$

$[\mathbf{Opn}(X), \mathbf{Sets}]$ to the space whose carrier is the disjoint union of stalks $\tilde{F} = \coprod_{x \in X} F_x$, endowed with the final topology turning all maps of the form $\tilde{s} \colon U \to \tilde{F}$ sending y to the equivalence class $[s]_y \in F_y$.

This adjunction restricts to an equivalence of categories between the subcategory $\mathbf{Sh}(X)$ of sheaves on X and the subcategory $\mathbf{\acute{E}t}(X)$ of *étale spaces* over X, giving a formal method to prove [**MLM92**, Thm. **II.6.2**]. A complete proof can be found at [**Car**], lectures 3 and 4.

Example 3.12 [THE TENSOR PRODUCT AS A COEND]**:** Any ring R can be regarded as an $\mathbf{Ab}$-category with a single object, whose set of endomorphisms is the ring R itself; once noticed this, we obtain natural identifications for the categories of modules over R:

$$\mathbf{Mod}_R \cong \mathrm{Fun}(R^{\mathrm{op}}, \mathbf{Ab})$$

$$_R\mathbf{Mod} \cong \mathrm{Fun}(R, \mathbf{Ab}).$$

Given $A \in \mathbf{Mod}_R, B \in {}_R\mathbf{Mod}$, we can define a functor $T_{AB} \colon R^{\mathrm{op}} \times R \to \mathbf{Ab}$ which sends the unique object to the tensor product $A \otimes_{\mathbb{Z}} B$ of abelian groups. The coend of this functor can be computed as the coequalizer

$$\mathrm{coker}\Big(\coprod_{r \in R} A \otimes_{\mathbb{Z}} B \xrightarrow[1 \otimes r]{r \otimes 1} A \otimes_{\mathbb{Z}} B \Big), \qquad (54)$$

or in other words, $\int^{* \in R} T_{AB} \cong A \otimes_R B$. This point of view on tensor products can be extremely generalized (see [**ML98**, §**IX.6**], but more on this has been written in [**Yon60**, §**4**]): given functors $F, G \colon \mathbf{C}^{\mathrm{op}}, \mathbf{C} \to \mathscr{V}$ having values in a cocomplete monoidal category, we can define the *tensor product* of F, G as the coend

$$F \boxtimes_{\mathbf{C}} G := \int^c Fc \otimes_{\mathscr{V}} Gc. \qquad (55)$$

Remark 3.13 : This can be regarded as part of a general theory which defines a THC *situation* (see [**Gra80**, §**1.1**]; these are also called *adjunctions of two variables* in newer references) as a triple $\mathfrak{t} = (\otimes, \wedge, [\,_-, \,_=])$ of (bi)functors between three categories $\mathbf{S}, \mathbf{A}, \mathbf{B}$, defined via the adjunctions

$$\mathbf{B}(S \otimes A, B) \cong \mathbf{S}(S, [A, B]) \cong \mathbf{A}(A, S \wedge B). \qquad (56)$$

Such an isomorphism uniquely determines the domains ant the variance of the three functors involved, in each variable; to be more clear, however, we notice that $\otimes \colon \mathbf{S} \times \mathbf{A} \to \mathbf{B}$, and then $\wedge \colon \mathbf{S}^{\mathrm{op}} \times \mathbf{B} \to \mathbf{A}$, and $[\,_-, \,_=] \colon \mathbf{A}^{\mathrm{op}} \times \mathbf{B} \to \mathbf{S}$.

Example 3.14 [GIRAUD THEOREM USING COENDS]**:** The gist of *Giraud theorem* is the following statement: left exact localizations of presheaf categories

$[\mathbf{C}^{\mathrm{op}}, \mathbf{Sets}]$ classify Grothendieck toposes (i.e. categories of *sheaves* $\mathbf{Sh}(\mathbf{C}, J)$ with respect to a Grothendieck topology J).

A proof of this classical "representation" theorem, intertwined with the theory of locally presentable categories (see for example [**Vit06**]), is contained at the end of [**MLM92**].

We now try to outline an argument giving the localization between a presheaf category and a category $\mathscr{E}$ satisfying the axioms of Giraud, hence "realizing" $\mathscr{E}$ as a full subcategory of $[\mathbf{C}^{\mathrm{op}}, \mathbf{Sets}]$. presheaves on $\mathbf{C} = \mathscr{E}^{\mathrm{c}} \subset \mathscr{E}$, the subcategory of *compact* object of $\mathscr{E}$.

The trick in the proof is to choose $\mathbf{C}$ wisely: to do this we use the fact that there is a small full subcategory $\mathbf{C} \subseteq \mathscr{E}$ of compact objects, $\mathscr{E}_{<\omega}$, and a full embedding $\iota \colon \mathbf{C} \subset \mathscr{E}$; this is a nerve-realization context (Def. **3.1**), that activates coend calculus to prove that

(1) The ι-nerve N_ι is full and faithful and coincides with the inclusion of sheaves into presheaves $[\mathbf{C}^{\mathrm{op}}, \mathbf{Sets}]$;

(2) $\mathrm{Lan}_{\yon} \iota$ is the left exact reflection.

Let then

$$\mathrm{Lan}_{\yon}(\iota) \colon [\mathbf{C}^{\mathrm{op}}, \mathbf{Sets}] \underset{\longleftarrow}{\overset{\longrightarrow}{}} \mathscr{E} \colon N_\iota \qquad (57)$$

be the nerve-realization adjunction.

Since the functor $\yon_{\mathbf{C}}$ is dense, the associated nerve N_ι is fully faithful, and this gives the first point: it remains then only to prove that the functor $\mathrm{Lan}_{\yon}(\iota)$ behaves like sheafification. This, in view of our characterization of the unit and counit of the nerve-realization adjunction (Remark **E4**) means that we have to manipulate the following chain of (iso)morphisms:

$$\begin{aligned}
\mathrm{Lan}_{\yon}(\iota)(P) &\cong \mathscr{E}(\iota C, \mathrm{Lan}_{\yon} \iota(P)) \\
&\cong \mathscr{E}\left(\iota C, \int^A PA \times \iota A\right) \\
&\leftarrow \int^A \mathscr{E}(\iota C, PA \times \iota A) \\
&\cong \int^A PA \times \mathscr{E}(\iota C, \iota A) \\
&\cong \int^A PA \times \mathscr{E}(C, A) \\
&\cong PC
\end{aligned}$$

It only remains to prove that this functor is left exact. To do this we invoke Prop. **3.4**. It also remains to characterize *sheaves* as those P such that η_P is invertible (this to a certain amount seems to be implied by taking only those P that preserve finite limits, and yet...).

Example 3.15 [Simplicial subdivision functor]**:** Let $\mathbf{\Delta}$ be the standard "topologist's" simplex category. The *Kan Ex$^\infty$ functor* is an endofunctor of

$\mathbf{sSet} = [\boldsymbol{\Delta}^{\mathrm{op}}, \mathbf{Sets}]$ turning every simplicial set X into a *Kan complex*[8]. This construction is of fundamental importance in simplicial homotopy theory, and we now want to organize the classical construction in the modern terms of a nerve-realization paradigm on $\boldsymbol{\Delta}$.

First of all, recall ([**GJ09**]) that the nondegenerate m-simplices of $\Delta[n]$ are in bijective correspondence with the subsets of $\{0, \ldots, n\}$ of cardinality $m+1$; this entails that the set of nondegenerate simplices of $\Delta[n]$ becomes a poset $\mathsf{s}[n]$ ordered by inclusion when this partial order is tacitly transported. We can then consider the nerve $N_\rho(\mathsf{s}[n]) \in \mathbf{sSet}$ (see Example **3.6**). This organizes into a functor $\mathsf{sd} \colon \boldsymbol{\Delta} \to \mathbf{sSet}$, which forms a nerve-realization paradigm: using Prop. **3.2** we obtain the adjunction

$$\mathbf{sSet} \xleftarrow{\quad \mathrm{Ex} \quad} \perp \xrightarrow{\quad \mathsf{Sd} \quad} \mathbf{sSet}$$

where Ex is the nerve N_{sd} associated to the NR-paradigm sd: the set of m-simplices $\mathrm{Ex}(X)_n$ is $\mathbf{sSet}(\mathsf{sd}(\Delta[n]), X)$.

There is a canonical map $\mathsf{sd}(\Delta[n]) \to \Delta[n]$ which in turn, by the Yoneda lemma, induces a map $X \to \mathrm{Ex}(X)$, natural in $X \in \mathbf{sSet}$. This gives to $\mathrm{Ex}(\_)$ the structure of a pointed functor, and in fact a *well-pointed* functor in the sense of [**Kel80**]; this, finally, means that we can define

$$\mathrm{Ex}^\infty(X) \cong \varinjlim \left(X \to \mathrm{Ex}(X) \to \mathrm{Ex}^2(X) \to \cdots \right) \tag{58}$$

as an endofunctor on $\mathbf{sSet}$. The functor Ex^∞ enjoys a great deal of formal properties useful in the study of simplicial homotopy theory (the most important of which is that $\mathrm{Ex}^\infty(X)$ is a Kan complex for each $X \in \mathbf{sSet}$, see [**GJ09**]). A more intrinsic characterization of this construction is contained in [**EP08**], and defines not only $\mathsf{Sd} = \mathrm{Lan}_{\,\mathsf{k}}\, \mathsf{sd}$ is a Left Kan extension, but also sd: they consider the diagram of 2-cells

where $\oplus \colon \boldsymbol{\Delta} \times \boldsymbol{\Delta} \to \boldsymbol{\Delta}$ is the *ordinal sum* defined by $[m] \oplus [n] = [m+n+1]$.

---

[8]A Kan complex is a simplicial set Y such that the functor $\hom(\_, X)$ turns each *horn inclusion* $\Lambda^k[n] \to \Delta[n]$ into an epimorphism.

Example 3.16 [ISBELL DUALITY]**:** Isbell duality consists of the following statement: let $\mathscr{V}$ be a Bénabou cosmos, and $\mathbf{C} \in \mathscr{V}$-**Cat**; if we denote, as always, $[\mathbf{C}, \mathscr{V}]$ and $[\mathbf{C}^{\mathrm{op}}, \mathscr{V}]$ the categories of covariant and contravariant functors $\mathbf{C} \to \mathscr{V}$, then we have an adjunction

$$[\mathbf{C}, \mathscr{V}]^{\mathrm{op}} \underset{\mathsf{Spec}}{\overset{\mathfrak{O}}{\underset{\longrightarrow}{\longleftarrow}}} [\mathbf{C}^{\mathrm{op}}, \mathscr{V}].$$

This means that we find a bijection of hom-sets

$$[\mathbf{C}, \mathscr{V}]^{\mathrm{op}}(\mathfrak{O}(X), Y) = [\mathbf{C}, \mathscr{V}](Y, \mathfrak{O}(X)) \cong [\mathbf{C}^{\mathrm{op}}, \mathscr{V}](X, \mathsf{Spec}(Y)) \qquad (59)$$

induced by the functors

$$\mathfrak{O} \colon X \mapsto \Big(c \mapsto [\mathbf{C}, \mathscr{V}](X, \text{よ}_{\mathbf{C}^{\mathrm{op}}}(c))\Big)$$

$$\mathsf{Spec} \colon Y \mapsto \Big(c \mapsto [\mathbf{C}, \mathscr{V}]^{\mathrm{op}}(\text{よ}_{\mathbf{C}^{\mathrm{op}}}(c), Y)\Big)$$

Executed by an expert in coend-fu, this statement is almost a tautology thanks to Thm. **1.19**:

$$[\mathbf{C}, \mathscr{V}](Y, \mathfrak{O}(X)) \cong \int_d \mathscr{V}\Big(Yd, \int_a \mathscr{V}(Xa, \mathbf{C}(a,d))\Big)$$

$$\cong \int_{da} \mathscr{V}(Yd, \mathscr{V}(Xa, \mathbf{C}(a,d)))$$

$$\cong \int_a \mathscr{V}\Big(Xa, \int_d \mathscr{V}(Yd, \mathbf{C}(a,d))\Big)$$

$$\cong [\mathbf{C}^{\mathrm{op}}, \mathscr{V}](X, \mathsf{Spec}(Y)).$$

Exercises for §3

E1 Use coend-fu to show that starting from a given THC-situation $\mathfrak{t} = (\otimes, \wedge, [\_, =])$, we can induce a new one $\mathfrak{t}' = (\boxtimes, \curlywedge, \langle \_, = \rangle)$, on the categories $\mathbf{S}^{\mathbf{I}^{\mathrm{op}} \times \mathbf{J}}, \mathbf{A}^{\mathbf{I}}, \mathbf{B}^{\mathbf{J}}$, for any $\mathbf{I}, \mathbf{J} \in \mathbf{Cat}$; start defining $F \boxtimes G \in \mathbf{B}^{\mathbf{J}}$ out of $F \in \mathbf{S}^{\mathbf{I}^{\mathrm{op}} \times \mathbf{J}}, G \in \mathbf{A}^{\mathbf{I}}$, as the coend

$$\int^i F(i, \_) \otimes Gi \qquad (60)$$

and show that there is an adjunction

$$\mathbf{B}^{\mathbf{J}}(F \boxtimes G, H) \cong \mathbf{S}^{\mathbf{I}^{\mathrm{op}} \times \mathbf{J}}(F, \langle G, H \rangle) \cong \mathbf{A}^{\mathbf{I}}(G, F \curlywedge H) \qquad (61)$$

developing $\mathbf{B}^{\mathbf{J}}(F \boxtimes G, H) = \dots$ in two ways.

E2 In this exercise **Top** is a nice category for algebraic topology. Define the category Γ having objects the power-sets of finite sets, and morphisms the functions $f \colon 2^n \to 2^m$ preserving unions and set-theoretical differences.

  (1) Show that there is a functor $\mathbf{\Delta} \hookrightarrow \Gamma$, sending the chain $\{0 < 1 < \cdots < n\}$ in $\mathbf{\Delta}$ to $\{\varnothing \subset \{0\} \subset \cdots \subset \{0, \dots, n\}\}$ in Γ.

(2) The category of presheaves of spaces $\Gamma^{\mathrm{op}} \to \mathbf{Top}$ is called the category of Γ-*spaces*; a Γ-space is *Segal* if it turns pushout in Γ (describe them) into homotopy pullback in $\mathbf{Top}$.

More explicitly, let $A\colon \Gamma^{\mathrm{op}} \to \mathbf{Top}$ be a Γ-space, it is Segal if (a) $A(0)$ is contractible; (b) the canonical map $A(n) \to \prod_{i=1}^{n} A(1)$ is a homotopy equivalence in $\mathbf{Top}$.

(3) Let $X \in \mathbf{Top}$ and $A\colon \Gamma^{\mathrm{op}} \to \mathbf{Top}$; define $X \otimes A$ to be the coend (in $\mathbf{Top}$)

$$\int^{n\in\Gamma} X^n \times A(n) \tag{62}$$

Show that $S^1 \otimes_\Gamma A$ is homeomorphic to the geometric realization of the simplicial space $\mathbf{\Delta}^{\mathrm{op}} \to \Gamma^{\mathrm{op}} \xrightarrow{A} \mathbf{Top}$. If A is Segal, $S^1 \otimes_\Gamma A \cong BA(1)$, where $B(-)$ is the *classifying space* functor.

(4) Let $\mathbf{C}\colon \Gamma^{\mathrm{op}} \to \dots$; let $X \otimes_\Gamma \mathbf{C}$ be the coend (in the category of topological categories)

$$\int^{n\in\Gamma} X^n \times \mathbf{C}(n). \tag{63}$$

Show that $X \otimes_\Gamma (\_)\colon \mathbf{Top\text{-}Cat} \to \mathbf{Cat}$ commutes with finite products, namely if $\mathbf{C}, \mathbf{D}$ are topological categories, then

$$X \otimes_\Gamma (\mathbf{C} \times \mathbf{D}) \cong (X \otimes_\Gamma \mathbf{C}) \times (X \otimes_\Gamma \mathbf{D}). \tag{64}$$

E3 Compute the J-realization (see Example **3.8**) of $X \in \mathbf{sSet}$ in the case J is the Sierpiński space $\{0 < 1\}$ with topology $\{\varnothing, J, \{1\}\}$.

E4 Write explicitly the unit and counit of the nerve and realization adjunction $\mathrm{Lan}_{\text{よ}}\varphi \dashv N_\varphi$.

E5 Show that the nerve functor N_φ is canonically isomorphic to $\mathrm{Lan}_\varphi \text{よ}$, so that there is an adjunction

$$\mathrm{Lan}_{\text{よ}}\varphi \dashv \mathrm{Lan}_\varphi \text{よ}. \tag{65}$$

E6 [◉◉] Example **3.9** can be expanded and studied more deeply:

- Is $\odot$ a monoidal structure on $\mathbf{Topos}$?
- Under which conditions on X, Y is $\mathbf{X} \odot \mathbf{Y}$ equivalent to a topos of sheaves on a topological space $X \odot Y$?
- What are the properties of the bifunctor $(X, Y) \mapsto X \odot Y$? Does this operation resemble or extend the topological join?

E7 Generalize the nerve-realization paradigm to the setting of *separately cocontinuous*, or *multilinear* functors. Given $\varphi\colon \mathbf{C}_1 \times \cdots \times \mathbf{C}_n \to \mathbf{D}$, where each $\mathbf{C}_i$ is small and $\mathbf{D}$ is cocomplete, show that there exists an equivalence of categories

$$\mathbf{Cat}(\mathbf{C}_1 \times \cdots \times \mathbf{C}_n, \mathbf{D}) \cong \mathsf{Mult}(\widehat{\mathbf{C}_1} \times \cdots \times \widehat{\mathbf{C}_n}, \mathbf{D}) \tag{66}$$

where $\mathsf{Mult}(\_, =)$ is the category of all functors that are cocontinuous in each variable once all the others have been fixed (hint: show it 'by induction' composing multiple Kan extensions). Given $\theta \in \mathbf{Cat}(\mathbf{C}_1 \times \cdots \times \mathbf{C}_n, \mathbf{D})$, describe the right adjoint of each $\theta(c_1, \dots, {}_{-i}, \dots, c_n)\colon \widehat{\mathbf{C}_i} \to \mathbf{D}$. All these functors assemble to a 'vector-nerve' $N\colon \mathbf{D} \to \widehat{\mathbf{C}_1} \times \cdots \times \widehat{\mathbf{C}_n}$.

4. Weighted limits.

The theory of weighted co/limits constitutes an extremely interesting argument even *per se* and it constitutes one of the pillars of enriched category

theory. It can be easily formulated and understood in terms of co/end calculus.

The whole discussion in this chapter comes from [**Rie14**, **II.7**]; even most of the notation is basically the same. A classical reference for the subject is [**Kel82**, **KS74**]: we do not make any attempt to recall the fundamental definitions of enriched category theory, heavily relying on these classical references.

It's easy to motivate that the calculus of co/limits is a cornerstone of basic category theory. Nevertheless, this notion becomes too strict when one deals with enriched categories; the "conical" shape of a classical co/limit is not general enough to encompass the fairly rich variety of shapes in which co/limits in enriched categories arise; naïvely speaking, one of these shapes gives rise to the notion of a cone, having as limit the well-known initial/terminal object. In full generality, one can compute a co/limit of a diagram $F \colon \mathbf{J} \to \mathbf{C}$ *weighing* it with a presheaf with the same domain of F: conical colimits arise weighing with the terminal presheaf on **Sets**.

We cannot touch but the surface of this intricate topic: the interested reader can consult [**Kel89**], a presentation of unmatched lucidity filled with enlightening examples. We choose to follow a more modern approach, for we are interested in translating the fundamentals about weighted co/limits into a *kata* of co/end-fu. To do this, we follow another really useful and well-written presentation of the theory, *c'est à dire* [**Rie14**].[9]

4.1. **A brief prelude: the category of elements of a presheaf.**

Definition 4.1 : Let $W \colon \mathbf{C} \to \mathbf{Sets}$ be a functor; the *category of elements* $\mathbf{C}\!\int W$ of W is the category having objects the pairs $(c \in \mathbf{C}, u \in Wc)$, and morphisms $(c, u) \to (c', v)$ those $f \in \mathbf{C}(c, c')$ such that $W(f)(u) = v$.

Notation 4.2 : The exotic notation "$\mathbf{C}\!\int W$" for the category of elements of W comes from the wondrous paper [**Gra**].

Proposition 4.3 : The category $\mathbf{C}\!\int W$ can be equivalently characterized as

- The category which results from the pullback

$$
\begin{array}{ccc}
\mathbf{C}\!\int W & \longrightarrow & \mathbf{Sets}_* \\
\downarrow & \lrcorner & \downarrow{\scriptstyle U} \\
\mathbf{C} & \underset{W}{\longrightarrow} & \mathbf{Sets}
\end{array}
\tag{67}
$$

  where $U \colon \mathbf{Sets}_* \to \mathbf{Sets}$ is the forgetful functor which sends a pointed set to its underlying set;

---

[9]The reader is warned, now, that the present section results in a rough imitation of this source. Every error is obviously due to the author's misunderstanding.

- The comma category of the cospan $\{*\} \to \mathbf{Sets} \xleftarrow{W} \mathbf{C}$, where $\{*\} \to \mathbf{Sets}$ chooses the terminal object of $\mathbf{Sets}$;
- The opposite of the comma category $(\rotatebox{-45}{\text{よ}} \downarrow \lceil W \rceil)$, where $\lceil W \rceil \colon \{*\} \to [\mathbf{C}, \mathbf{Sets}]$ is the *name* of the functor W, i.e. the unique functor choosing the presheaf $W \in [\mathbf{C}, \mathbf{Sets}]$:

$$
\begin{array}{ccc}
(\mathbf{C}\!\int\! W)^{\mathrm{op}} & \longrightarrow & * \\
\downarrow & \nearrow\!\!\!\nearrow & \downarrow \\
\mathbf{C}^{\mathrm{op}} & \longrightarrow & [\mathbf{C}, \mathbf{Sets}]
\end{array}
$$

Proof. It is an exercise in Yoneda lemma and universal properties. $\qquad\square$

Remark 4.4 : There is a fourth characterization for the category of elements of a presheaf which couches it as a particular weighted colimit which, using the formalism exposed in this section, can be written as a coend. Example **4.22** below is a guided exercise in proving and clarifying this statement.

Proposition 4.5 : The category of elements $\mathbf{C}\!\int\! W$ of a functor $W \colon \mathbf{C} \to \mathbf{Sets}$ comes equipped with a canonical "Grothendieck fibration" to the domain of W, which we denote $\Sigma \colon \mathbf{C}\!\int\! W \to \mathbf{C}$, defined forgetting the distinguished element $u \in Wc$.

Proof. We only have to prove that $\Sigma \colon \mathbf{C}\!\int\! W \to \mathbf{C}$ is an isofibration; given an isomorphism $\varphi \colon x \to c = \Sigma(c, u)$, we can define $v \in Wx$ to be $W(\varphi^{-1})(u)$. $\qquad\square$

Notation 4.6 : All along the present section, we assume that $\mathcal{V}$ denotes a *Bénabou cosmos*, i.e. a symmetric monoidal closed, complete and cocomplete category which is the "base" for our enriched category theory.

Remark 4.7 : Let $F \colon \mathbf{C} \to \mathbf{A}$ be a functor between small ordinary categories. The *limit* $\varprojlim F$ of F can be characterized as the representing object of a suitable presheaf: indeed, we have the natural isomorphism

$$
\mathbf{A}(a, \varprojlim F) \cong \mathbf{Sets}^{\mathbf{C}}(*, \mathbf{A}(a, F(\_))) \tag{68}
$$

where $*$ is a shorthand to denote the terminal functor $\mathbf{C} \to \mathbf{Sets} \colon X \mapsto *$ sending every object to the terminal set, and $\mathbf{A}(a, F(\_)) = \mathbf{A}(a, F)$ is the functor $\mathbf{C} \to \mathbf{Sets}$ sending c to $\mathbf{A}(a, Fc)$ (so we represent the functor $a \mapsto \mathbf{A}(a, F)$).

Dually, the colimit $\varinjlim F$ can be characterized, in the same notation, as the representing object in the natural isomorphism

$$
\mathbf{A}(\varinjlim F, a) \cong \mathbf{Sets}^{\mathbf{C}}(*, \mathbf{A}(F(\_), a)). \tag{69}
$$

$\mathbf{Sets^C}(*, \mathbf{A}(F, a))$ is a set of natural transformations and $a \mapsto \mathbf{Sets^C}(*, \mathbf{A}(F, a))$ is a functor: the leading idea behind the definition of weighted co/limit is to generalize this construction to admit shapes other than the terminal presheaf for the domain functor. We can package this intuition in the following definition:

Definition 4.8 [WEIGHTED LIMIT AND COLIMIT]**:** Given functors $F \colon \mathbf{C} \to \mathbf{A}$ and $W \colon \mathbf{C} \to \mathbf{Sets}$, we define the *weighted limit* of F by W as a representative for the functor sending $a \in \mathbf{A}$ to $\mathbf{Sets^C}(W, \mathbf{A}(a, F(\_)))$, in other words the weighted limit of F by W is an object $\varprojlim^W F \in \mathbf{A}$ such that

$$\mathbf{A}\big(a, \varprojlim^W F\big) \cong \mathbf{Sets^C}(W, \mathbf{A}(a, F(\_))) \tag{70}$$

naturally in $a \in \mathbf{A}$. Dually we define the *colimit* of $F \colon \mathbf{C} \to \mathbf{A}$ weighted by $W \colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Sets}$ to be an object $\varinjlim^W F \in \mathbf{A}$ such that

$$\mathbf{A}\big(\varinjlim^W F, a\big) \cong \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}(W, \mathbf{A}(F(\_), a)). \tag{71}$$

Example 4.9 : Let $f \colon \Delta^1 \to \mathbf{A}$ the functor choosing an arrow $f \colon x \to y$ in $\mathbf{A}$, and $W \colon \Delta^1 \to \mathbf{Sets}$ the functor sending $\{0 < 1\}$ to the single arrow $\{0, 1\} \to \{0\}$; then a natural transformation $W \Rightarrow \mathbf{A}(a, f)$ consists of arrows $W0 \to \mathbf{A}(a, x), W1 \to \mathbf{A}(a', y)$, namely on the choice of two arrows $h, k \colon a \to x$ such that $fh = fk$: the universal property for $\varprojlim^W f$ implies that this is the *kernel pair* of the arrow f, namely that h, k fill in the pullback

$$\tag{72}$$

Proposition 4.10 [WEIGHTED CO/LIMITS AS CO/ENDS]**:** When the indicated universal objects (the end below and the **Sets**-cotensor $(X, a) \mapsto a^X$ used to define it) exist, we can express the weighted limit $\varprojlim^W F$ as an end:

$$\mathbf{Sets^C}(W, \mathbf{A}(m, F)) \cong \int_{c \in \mathbf{C}} \mathbf{Sets}(Wc, \mathbf{A}(m, Fc))$$

$$\cong \int_{c \in \mathbf{C}} \mathbf{A}\big(m, Fc^{Wc}\big)$$

$$\cong \mathbf{A}\Big(m, \int_{c \in C} Fc^{Wc}\Big)$$

This string of natural isomorphisms implies that there is a canonical isomorphism

$$\varprojlim{}^W F \cong \int_{c \in \mathbf{C}} Fc^{Wc}. \tag{73}$$

Example 4.11 : Consider the particular case of two parallel functors $W, F \colon \mathbf{C} \to \mathbf{Sets}$; then we can easily see that $\varprojlim{}^W F$ coincides with the set of natural transformations $W \Rightarrow F$, since the cotensor Fc^{Wc} amounts to the set $\mathbf{Sets}(Wc, Fc)$.

Example 4.12 : The ninja Yoneda lemma, rewritten in this notation, says that $\varprojlim{}^{\mathbf{C}(c,-)} F \cong Fc$ (or, in case F is contravariant, $\varprojlim{}^{\mathbf{C}(-,c)} F \cong Fc$). This can be memorized as "representably-weighted co/limits are evaluation" (see Remark **2.6** on the Dirac delta) and suggests that *Kan extensions* can be expressed as suitable weighted co/limits, and more precisely that they can be characterized as those weighted co/limits where the weight is a representable functor:

$$\mathrm{Ran}_K F(\_) \cong \int_{c \in \mathbf{C}} Fc^{\mathbf{D}(-,Kc)} \cong \varprojlim{}^{\mathbf{D}(-,K=)} F. \tag{74}$$

The following Remark and Proposition constitute a central observation.

Remark 4.13 [❂❂, THE GROTHENDIECK CONSTRUCTION TRIVIALIZES WEIGHTS]: Definition **4.8** can be extended in the case $F \colon \mathbf{C} \to \mathbf{A}$ is a $\mathscr{V}$-enriched functor between $\mathscr{V}$-categories, and $W \colon \mathbf{C} \to \mathscr{V}$ is a $\mathscr{V}$-co/presheaf; this is the setting where the notion of a weighted co/limit acquires a supremacy over the "classical" one (where the weight is the terminal presheaf). When $\mathscr{V} = \mathbf{Sets}$, indeed, the *Grothendieck construction* sending a (co)presheaf into its category of elements turns out to trivialize almost completely the theory of $\mathbf{Sets}$-weighted limits: as the following discussion shows, in such a situation every weighted limit can be expressed as a classical (we call them *conical*, due to the shape of the weight) limit.

Proposition 4.14 [$\mathbf{Sets}$-WEIGHTED LIMITS ARE LIMITS]: As shown in Prop. **4.5**, the category $\mathbf{C} \int W$ comes equipped with a fibration $\Sigma \colon \mathbf{C} \int W \to \mathbf{C}$, such that for any functor $F \colon \mathbf{C} \to \mathbf{A}$ one has

$$\varprojlim{}^W F \cong \varprojlim_{(c,x) \in \mathbf{C} \int W} F \circ \Sigma. \tag{75}$$

Proof. The proof goes by inspection, using the characterization of the end $\int_{c \in \mathbf{C}} Fc^{Wc}$ as an equalizer (see Proposition **1.15**), and the characterization of $\mathbf{Sets}$-cotensors as iterated products, showing that

$$\int_{c \in \mathbf{C}} Fc^{Wc} \cong \mathrm{eq}\Big(\prod_{c \in \mathbf{C}} Fc^{Wc} \rightrightarrows \prod_{c \to c'} Fc'^{Wc} \Big)$$

$$\cong \mathrm{eq}\Big(\prod_{c\in\mathbf{C}} \prod_{x\in Wc} Fc \rightrightarrows \prod_{c\to c'} \prod_{x\in Wc} Fc' \Big)$$

$$(\star) \cong \mathrm{eq}\Big(\prod_{(c,x)\in\mathbf{C}\int W} Fc \rightrightarrows \prod_{(c,x)\to(c',x')\in\mathbf{C}\int W} Fc' \Big)$$

$$\cong \varprojlim_{(c,x)\in\mathbf{C}\int W} F\circ\Sigma$$

(equation $(\star)$ is motivated by the fact that every arrow $\varphi\colon \Sigma(c,x)\to c'$ has a unique lift $(c,x)\to(c',x')$ since $W(\varphi)(x)=x'm$). $\qquad\square$

Remark 4.15 : When we consider Kan extensions as weighted co/limits, this result agrees with the classical theory: if the weight has the form $W=\mathbf{D}(d,K-)$ for an object $d\in\mathbf{D}$, and a functor $K\colon\mathbf{C}\to\mathbf{D}$, then the category of elements $\mathbf{C}\int W$ is precisely the *comma category* $(d\downarrow K)$: the right Kan extension of F along K can be computed as the conical limit of the functor FU, where $U\colon(d\downarrow K)\to\mathbf{C}$ is the obvious forgetful functor.

Obviously, when *every* weighted limit exists in $\mathbf{A}$, we can prove that the correspondence $(W,F)\mapsto\varprojlim^W F$ is a bifunctor:

$$\varprojlim^{(-)}(=)\colon (\mathbf{Sets}^{\mathbf{C}})^{\mathrm{op}}\times\mathbf{A}^{\mathbf{C}}\longrightarrow\mathbf{A}. \tag{76}$$

A number of useful corollaries of this fact:
- The unique, terminal natural transformation $W\to *$ induces a *comparison arrow* between the weighted limit of any $F\colon\mathbf{C}\to\mathbf{A}$ and the classical (conical) limit: $\varprojlim F\to\varprojlim^W F$. For example, the classical limit of the functor $f\colon\Delta[1]\to\mathbf{A}$ described in Example **4.9** consists of the object $a=\mathrm{src}(f)$; hence the comparison arrow consists of the unique factorization of two copies of id_a along the kernel pair of f.
- The functor $\varprojlim^{(-)}F$ is continuous, namely we can prove the suggestive isomorphism

$$\varprojlim^{\big(\varinjlim_J W_j\big)} F \cong \varprojlim_J\big(\varprojlim^{W_j} F\big), \tag{77}$$

valid for any small diagram of weights $\mathbf{J}\to[\mathbf{C},\mathbf{Sets}]\colon j\mapsto W_j$.

Example 4.16 : Ends can be computed as weighted limits: given $H\colon\mathbf{C}^{\mathrm{op}}\times\mathbf{C}\to\mathbf{D}$ we can take the hom functor $\mathbf{C}(\_,=)\colon\mathbf{C}^{\mathrm{op}}\times\mathbf{C}\to\mathbf{Sets}$ as a weight, and if the weighted limit exists, we have the chain of isomorphisms

$$\varprojlim^{\mathbf{C}(-,=)} H \cong \int_{(c,c')\in\mathbf{C}^{\mathrm{op}}\times\mathbf{C}} H(c,c')^{\mathbf{C}(c,c')} \cong \int_c\Big(\int_{c'} H(c,c')^{\mathbf{C}(c,c')}\Big) \overset{\mathbf{2.1}}{\cong} \int_c H(c,c). \tag{78}$$

Remark 4.17 : This characterization will turn out to be useful during our discussion of *simplicially coherent* co/ends. See **7.38**.

Remark 4.18 : Weighted *colimits* are discussed in Exercise **E3** and stem from a straightforward dualization process; we refer freely to both concepts from now on.

Remark 4.19 : Aside from showing that "weighted limits are the true enriched-categorical limits", the above examples and the last characterization of ends as weighted limits are fundamental steps towards a sensible definition of *enriched* ends: given a Bénabou cosmos $\mathcal{V}$ (see Notation **4.6**) and a $\mathcal{V}$-functor $H\colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathcal{V}$, we *define* $\int_c H(c,c)$ to be the limit of H weighted by $\mathbf{C}(\_,=)\colon \mathbf{C}^{\mathrm{op}} \boxtimes \mathbf{C} \to \mathcal{V}$ (see Definition **4.25** for the notation $\mathbf{C} \boxtimes \mathbf{D}$).

Remark 4.20 : Let $F\colon \mathbf{C} \to \mathbf{A}, W\colon \mathbf{C} \to \mathcal{V}, U\colon \mathbf{C}^{\mathrm{op}} \to \mathcal{V}$ be $\mathcal{V}$-functors and let $\mathbf{A}$ be $\mathcal{V}$-tensored. There are canonical isomorphisms

$$\mathbf{A}(\varinjlim^W F, x) \cong \varprojlim^W \mathbf{A}(F, x) \tag{79}$$

$$\mathbf{A}(y, \varprojlim^U F) \cong \varprojlim^W \mathbf{A}(y, F). \tag{80}$$

This can be recorded in the motto "the hom functor commutes with weighted limits" (prove it as an exercise).

Example 4.21 [THE CONE CONSTRUCTION AS A WEIGHTED COLIMIT]**:** Let K be a ring, and $\mathcal{V} = \mathbf{Ch}(K)$ the category of chain complexes of K-modules. Considering $\mathcal{V}$ as a $\mathcal{V}$-category in the obvious way, we aim to prove that the *mapping cone* $\mathrm{C}(f) = X_*[1] \oplus Y_*$ of a chain map $f\colon X_* \to Y_*$ [**Wei94**, **1.5.1**] in $\mathcal{V}$ can be characterized as the weighted colimit $\varinjlim^W f$, where $f\colon \Delta[1] \to \mathcal{V}$ is the arrow f, and $W\colon \Delta[1]^{\mathrm{op}} \to \mathcal{V}$ is the functor which chooses the map $S^1(K)_* \to D^2(K)_*$, where $S^n(K)_* = K[n]_*$ is the chain complex with the only term K concentrated in degree $-n$, and $D^n(K)_*$ is the complex

$$\cdots \longrightarrow 0 \longrightarrow K = \!\!= K \longrightarrow 0 \longrightarrow \cdots ,$$

where the first nonzero term is in degree $-n$. There is an obvious inclusion $S^n_* \hookrightarrow D^{n+1}_*$:

$$
\begin{array}{ccccccccc}
\cdots \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & K & \longrightarrow & 0 & \longrightarrow \cdots \\
& & & \downarrow & & \| & & & \\
\cdots \longrightarrow & 0 & \longrightarrow & K & = \!\!= & K & \longrightarrow & 0 & \longrightarrow \cdots .
\end{array}
$$

We now aim to prove that

$$\mathrm{C}(f) \cong \int^{i \in \Delta^1} W(i) \otimes f(i). \tag{81}$$

In view of (the dual of) Exercise **1.E10**, it is enough to show that there is a pushout square

$$
\begin{array}{ccc}
W(1) \otimes f(0) & \longrightarrow & W(1) \otimes f(1) \\
\downarrow & \ulcorner & \vdots \\
W(0) \otimes f(0) & \cdots\cdots\cdots\cdots\!\!\dashrightarrow & \mathrm{C}(f)
\end{array}
$$

This is a rather simple exercise in universality, given the maps

$$
B \xrightarrow{\;\binom{0}{1}\;} \mathrm{C}(f) \xleftarrow{\;\left(\begin{smallmatrix} 0 & 1 \\ f & 0 \end{smallmatrix}\right)\;} A \oplus A[1]. \tag{82}
$$

Example 4.22 [THE CATEGORY OF ELEMENTS OF A PRESHEAF]**:** The scope of this example is to prove that the category of elements of a functor $F\colon \mathbf{C} \to \mathbf{Sets}$ introduced in Def. **4.1** can be characterized as a **Cat**-weighted colimit: it is, in particular, the category isomorphic to the colimit

$$
\mathbf{C}\textstyle\int W \cong \int^{c\in\mathbf{C}} c/\mathbf{C} \times Wc \tag{83}
$$

where Wc is the set, regarded as a discrete category; it is, in other words, isomorphic to the weighted colimit $\varinjlim^{J} W$, where $J\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Cat}$ is the functor $c \mapsto c/\mathbf{C}$ (the "coslice" category of arrows $c \to x$).

To prove this statement, we verify that $\mathbf{C}\int W$ has the universal property of the coequalizer of the pair

$$
\coprod_{f\colon a\to b} b/\mathbf{C} \times Wa \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \coprod_{c\in\mathbf{C}} c/\mathbf{C} \times Wc \tag{84}
$$

where α has components $\alpha_f\colon b/\mathbf{C}\times Wa \xrightarrow{1\times Ff} b/\mathbf{C}\times Wb$ sending $\left(\begin{bmatrix} b \\ \downarrow \\ x \end{bmatrix}, u\right) \mapsto \left(\begin{bmatrix} b \\ \downarrow \\ x \end{bmatrix}, F(f)u\right)$ and β has components $\beta_f\colon b/\mathbf{C} \times Wa \xrightarrow{f^*\times Wa} a/\mathbf{C} \times Wa$ sending $\left(\begin{bmatrix} b \\ \downarrow \\ x \end{bmatrix}, u\right) \mapsto \left(\begin{bmatrix} a \xrightarrow{f} b \\ \downarrow \\ x \end{bmatrix}, u\right)$.

It's rather easy to define a functor

$$
\theta\colon \coprod_{a\in\mathbf{C}} a/\mathbf{C} \times Wa \longrightarrow \mathbf{C}\textstyle\int W \tag{85}
$$

having components $\theta_a\colon a/\mathbf{C} \times Wa \to \mathbf{C}\int W$ sending $\left(\begin{bmatrix} a \\ f\downarrow \\ b \end{bmatrix}, u \in Fa\right) \mapsto (b, F(f)(u) \in Fb)$, which coequalizes α and β. This functor θ has the universal property of the coequalizer: given any other $\zeta\colon \coprod_{a\in\mathbf{C}} a/\mathbf{C} \times Wa \to \mathbf{K}$ we can define a functor $\overline{\zeta}\colon \mathbf{C}\int W \to \mathbf{K}$ such that

$$
\overline{\zeta}(a, u \in Fa) = \zeta(\mathrm{id}_a, u). \tag{86}
$$

Now notice that every map ζ' that coequalizes (α, β) has the property that

$$\zeta'\left(\left[\begin{smallmatrix} b \\ g\downarrow \\ x \end{smallmatrix}\right], F(f)u\right) = \zeta'\left(\left[\begin{smallmatrix} a \xrightarrow{f} b \\ \downarrow \\ x \end{smallmatrix}\right], u\right) \tag{87}$$

It is now a routine verification to see that $\overline{\zeta} \circ \theta_a = \zeta_a$, and every other functor with this property must coincide with our $\overline{\zeta}$. This concludes the proof.

Remark 4.23 : The careful reader may have noticed that all the above discussion gives a fifth presentation for the category of elements $\mathbf{C}\int W$, as the image of W under the Kan extension $\mathrm{Lan}_{ \downarrow} J$: in the language of §**3**, $J \colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Cat}$ is the NR-context of the paradigm

$$\mathbf{C}\int {}_{\_} \colon [\mathbf{C}, \mathbf{Sets}] \xrightleftharpoons{\hspace{2cm}} \mathbf{Cat} \colon N_J \tag{88}$$

where $N_J \colon \mathbf{Cat} \to [\mathbf{C}, \mathbf{Sets}]$ is the "nerve" functor sending $\mathbf{A}$ to $c \mapsto \mathbf{Cat}(c/\mathbf{C}, \mathbf{A})$.

Remark 4.24 : An alternative approach to characterize $\mathbf{C}\int W$ is the following: the category $\mathbf{C}\int W$ is precisely the lax limit of W regarded as a $\mathbf{Cat}$-valued presheaf [**Kel89**, §4], [**Gra, Str76**].

A fundamental step to write the theory of weighted limits relies upon the above-mentioned isomorphism

$$\mathbf{A}\big(m, {\varprojlim}^W F\big) \cong \mathscr{V}^{\mathbf{C}}(W, \mathbf{A}(m, F({}_{\_}))) \tag{89}$$

valid in a $\mathscr{V}$-category $\mathbf{A}$ naturally in any object $m \in \mathbf{A}$; this isomorphism has to be interpreted in the base-cosmos $\mathscr{V}$, and this means that we have to find a way to interpret the category $\mathscr{V}^{\mathbf{C}}$ as an object $[\mathbf{C}, \mathscr{V}]$ of $\mathscr{V}$: to do this, we must endow $\mathscr{V}\,{}_{\_}\mathbf{Cat}$ with a closed symmetric monoidal structure, such that

$$\mathscr{V}\text{-}\mathsf{Fun}(\mathbf{C} \boxtimes \mathbf{E}, \mathbf{D}) \cong \mathscr{V}\text{-}\mathsf{Fun}(\mathbf{E}, [\mathbf{C}, \mathbf{D}]). \tag{90}$$

Definition 4.25 : Given two $\mathscr{V}$-categories $\mathbf{C}, \mathbf{D}$ we define the $\mathscr{V}$-category $\mathbf{C} \boxtimes \mathbf{D}$ having

- as objects the set $\mathbf{C} \times \mathbf{D}$, and
- as $\mathscr{V}$-object of arrows $(c, d) \to (c', d')$ the object

$$\mathbf{C}(c, c') \otimes \mathbf{D}(d, d') \in \mathscr{V}. \tag{91}$$

The free $\mathscr{V}$-category $\mathbf{I}$ associated to the terminal category is the unit object for this monoidal structure.

Proposition 4.26 : $(\mathscr{V}\text{-}\mathbf{Cat}, \boxtimes)$ is a closed monoidal structure, with internal hom denoted $[{}_{\_}, {}_{=}] \colon \mathscr{V}\text{-}\mathbf{Cat}^{\mathrm{op}} \times \mathscr{V}\text{-}\mathbf{Cat} \to \mathscr{V}\text{-}\mathbf{Cat}$.

Proof. Given $\mathbf{C}, \mathbf{D} \in \mathscr{V}\text{-}\mathbf{Cat}$ we define a $\mathscr{V}$-category whose objects are $\mathscr{V}$-functors $F, G \colon \mathbf{C} \to \mathbf{D}$ and where ('abstracting' Theorem **1.19** to the enriched

setting) the $\mathscr{V}$-object of natural transformations $F \Rightarrow G$ is defined via the end

$$[\mathbf{C}, \mathbf{D}](F, G) := \int_{c \in \mathbf{C}} \mathbf{D}(Fc, Gc). \tag{92}$$

Recall that in the unenriched case, the end was better understood as the equalizer of a pair of arrows:

$$\int_{c \in \mathbf{C}} \mathbf{D}(Fc, Gc) \cong \mathrm{eq}\Big(\prod_{c \in \mathbf{C}} \mathbf{D}(Fc, Gc) \rightrightarrows \prod_{c,c'} \prod_{c \to c'} \mathbf{D}(Fc, Gc') \Big) \tag{93}$$

In the enriched case, we can consider the same symbol, and re-interpret the product $\prod_{\mathbf{C}(c,c')}$ as a suitable *power* in $\mathscr{V}$:

$$\int_{c \in \mathbf{C}} \mathbf{D}(Fc, Gc) \cong \mathrm{eq}\Big(\prod_{c \in \mathbf{C}} \mathbf{D}(Fc, Gc) \rightrightarrows \prod_{c,c'} \mathbf{D}(Fc, Gc')^{\mathbf{C}(c,c')} \Big) \tag{94}$$

(see also [Gra80, §2.3], [Dub70] for a more detailed discussion about co/ends in enriched setting.)

It remains to prove, now, that the isomorphism (**90**) holds: this is rather easy, since in the above notations, any functor $F \colon \mathbf{C} \boxtimes \mathbf{E} \to \mathbf{D}$ defines a unique functor $\hat{F} \colon \mathbf{E} \to [\mathbf{C}, \mathbf{D}]$.[10] $\qquad\square$

The given definition for the enriched end allows us to state an elegant form of the $\mathscr{V}$-enriched Yoneda lemma:

Remark 4.27 [$\mathscr{V}$-YONEDA LEMMA]**:** Let $\mathbf{D}$ be a small $\mathscr{V}$-category, $d \in \mathbf{D}$ an object, and $F \colon \mathbf{D} \to \mathscr{V}$ a $\mathscr{V}$-functor. Then the canonical map

$$Fd \longrightarrow [\mathbf{D}, \mathscr{V}](\mathbf{D}(d, \_), F) \tag{95}$$

induced by the universal property of the involved end[11] is a $\mathscr{V}$-isomorphism.

Homotopy co/limits as weighted co/limits. We can express the Bousfield-Kan construction for the homotopy co/limit functor using co/end calculus (see **7.2.1** for a crash course on what's an homotopy co/limit). We condense Bousfield-Kan construction in the following series of examples.

Theorem 4.28 [THE BOUSFIELD-KAN FORMULA FOR HOMOTOPY CO/LIMITS]**:** Let $F \colon \mathbf{J} \to \mathscr{M}$ be a diagram in a model category $(\mathscr{M}, \mathrm{WK}, \mathrm{COF}, \mathrm{FIB})$ which is tensored and cotensored over the category $\mathbf{sSet}$ of simplicial sets by functors

---

[10]Notice that for any two objects $e, e' \in \mathbf{E}$, the collection of arrows $\hom(e, e') \to \hom(F(c, e), F(c, e')]$ is a wedge in $c \in \mathbf{C}$.

[11]Notice that this is an alternative point of view on the proof of the ninja Yoneda lemma **2.1**; this arrow is induced by a wedge $\{Fd \to \mathscr{V}(\mathbf{D}(d, a), Fa)\}_{a \in \mathbf{C}}$, whose members are the mates of the various $\mathbf{D}(d, a) \to \mathscr{V}(Fd, Fa)$ giving the action of F on arrows.

$\pitchfork\colon \mathbf{sSet}^{\mathrm{op}} \times \mathscr{M} \to \mathscr{M}$ and $\cdot\colon \mathbf{sSet} \times \mathscr{M} \to \mathscr{M}$. Then the homotopy limit $\underleftarrow{\operatorname{holim}} F$ of F can be computed as the end

$$\int_j N(J/j) \pitchfork F(j), \tag{96}$$

and the homotopy colimit $\underrightarrow{\operatorname{holim}} F$ of F can be computed as the coend

$$\int^j N(j/J) \cdot F(j). \tag{97}$$

Remark 4.29 : These two universal objects are weighted co/limits in an evident way: it is possible to rewrite them as $\underleftarrow{\lim}^{N(J/-)} F$ and $\underrightarrow{\lim}^{N(-/J)} F$.

The idea behind this characterization is that the co/limit functor results as the weighted colimit over the terminal weight. When we want to pass to the homotopy invariant version of the $\underrightarrow{\lim}^{(-)}(_=)$ bifunctor we can "derive" the diagram part as well as the weight part. Bousfield-Kan formula arises precisely when we derive the weight: $N(j/J)$ and $N(J/j)$ are contractible categories, and they are linked to $N(*)$ by an homotopy equivalence induced by the terminal functor.

Then, these categories must be thought as proper *replacements* for the homotopy co/limit functor.

Example 4.30 : The *mapping cylinder* of $f\colon A \to B$ is the topological space obtained from $(A \times [0,1]) \coprod B$ from the smallest quotient that identifies $(f(a), (a,0))$ for all $a \in A$ (compare this example with Example **4.21** and Exercise **E4**).

Exercises for §4

E1 Prove Equation (**77**) using the characterization of $\underleftarrow{\lim}^W F \cong \int_c Fc^{Wc}$, plus its universal property.

E2 What is the category of elements of the hom functor? Compare with Definition **1.12**.

E3 Every definition we gave until now can be dualized to obtain a theory of weighted *co*limits: fill in the details.

  (1) (weighted colimits as coends) If **A** is cocomplete, we can express the weighted colimit $\underrightarrow{\lim}^W F$ as a coend: more precisely

$$\underrightarrow{\lim}{}^W F \cong \int^{c \in \mathbf{C}} Wc \cdot Fc \tag{98}$$

  where we used, like everywhere else, the **Sets**-tensoring of **A**.

  (2) (left Kan extensions as weighted colimits) Let $F\colon \mathbf{C} \to \mathbf{A}$ and $K\colon \mathbf{C} \to \mathbf{D}$ be functors; then

$$\operatorname{Lan}_K F(\_) \cong \int^{c \in \mathbf{C}} \mathbf{D}(Kc, \_) \cdot Fc \cong \underrightarrow{\lim}{}^{\mathbf{D}(K=,-)} F \tag{99}$$

(3) (coends as hom-weighted colimits) The coend of $H \colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{D}$ can be written as $\varinjlim^{\mathbf{C}(-,=)} H$.

(4) If the weight W is **Sets**-valued, the colimit of F weighted by W can be written as a conical colimit over $\mathbf{C}^{\mathrm{op}} \int W$:

$$\varinjlim{}^{W} F \cong \varinjlim_{(c,x) \in \mathbf{C}^{\mathrm{op}} \int W} F\Sigma \tag{100}$$

(5) (functoriality) If the W-colimit of $F \colon \mathbf{C} \to \mathbf{A}$ always exists, then the correspondence $(W, F) \mapsto \varinjlim^{W} F$ is a functor, cocontinuous in its first variable:

$$\varinjlim{}^{(-)}(=) \colon \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}} \times \mathbf{A}^{\mathbf{C}} \longrightarrow \mathbf{A},$$

$$\varinjlim{}^{\left(\varinjlim_{J} W_{j} \right)} F \cong \varinjlim_{J} \left(\varinjlim{}^{W_{j}} F \right) \tag{101}$$

(6) (comparison) There is a canonical natural transformation $W \to *$, inducing a canonical *comparison arrow* from the W-colimit of any $F \colon \mathbf{C} \to \mathbf{A}$ to the conical colimit.

E4 Let $w \colon S^{0} \hookrightarrow D^{1}$ be the canonical inclusion of $\{0, 1\}$ into $[0, 1] \subset \mathbb{R}$, with the usual topology; prove that the mapping cone of a continuous map $f \colon X \to Y$ is precisely the weighted colimit $\varinjlim^{w} f$.

E5 Fill in the details of the above proof; for those who need a hint, show that $\mathbf{E}(e, e') \to \mathbf{D}(F(x, e), F(x, e'))$ is a wedge in $x \in \mathbf{C}$.

E6 Show that there are canonical isomorphisms $\varprojlim^{W} FJ \cong \varprojlim^{\mathrm{Lan}_{J} W} F$, and dually $\varinjlim^{WJ} F \cong \varinjlim^{W} \mathrm{Lan}_{J} F$.

5. The theory of relators.

The lucid presentation in the notes [**Béner**] and in [**CP08**, §**4**], [**Bor94b**] are standard references to follow this section. Our only merit here is having expressed several proofs using coend calculus. The arguments are essentially unchanged, and yet the employment of coends is implicit in [**Béner**], and only partially spelled out in [**Bor94b**].The lucid presentation in the notes [**Béner**] and in the book [**CP08**, §**4**] are standard references to follow this section. Our only merit here is that we have restated several proofs using coends; these arguments are somewhat implicit in [**Béner**] and only partially spelled out in [**Bor94b**].

Definition 5.1 [THE BICATEGORY OF RELATORS]**:** There exists a bicategory **Relt** having

- 0-cells (objects) those of **Cat** (small categories $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \dots$);
- 1-cells $\mathrm{P}, \mathrm{Q} \dots$, depicted as arrows $\mathbf{A} \rightsquigarrow \mathbf{B}$, the functors $\mathbf{A}^{\mathrm{op}} \times \mathbf{B} \to$ **Sets**;
- 2-cells $\alpha \colon \mathrm{P} \Rightarrow \mathrm{Q}$ the natural transformations between these functors.

Given two contiguous 1-cells $\mathbf{A} \overset{\mathrm{P}}{\rightsquigarrow} \mathbf{B} \overset{\mathrm{Q}}{\rightsquigarrow} \mathbf{C}$ we define their *composition* $\mathrm{Q} \diamond \mathrm{P}$ as the coend

$$\mathrm{Q} \diamond \mathrm{P}(a, c) := \int^{x \in \mathbf{B}} \mathrm{P}(a, x) \times \mathrm{Q}(x, c) \tag{102}$$

Definition 5.2 : This definition works well also with **Sets** replaced by an arbitrary Bénabou cosmos $\mathscr{V}$, i.e. in any symmetric monoidal closed and bicomplete category: in this case we speak of $\mathscr{V}$-relators in the bicategory $\mathbf{Relt}(\mathscr{V})$.

Remark 5.3 [Naming a category]**:** The 1-cells of **Relt** are more often called *profunctors* or *distributors* ([**Béner**] follows the equation funct*ions* : funct*ors* = distribut*ions* : distribut*ors*), *correspondences* (consider the case when $\mathscr{V} = \{0, 1\}$ i.e. where $\mathbf{A}, \mathbf{B}$ are sets regarded as discrete categories), or *bimodules* (consider the case where $\mathscr{V} = \mathbf{Ab}$ and $\mathbf{A}, \mathbf{B}$ are rings; see [**Gen15**] and several examples below).

As we will see during the present section, the bicategory **Relt** carries a fairly rich structure: it is then quite difficult to choose a name for its 1-cells able to convey this richness or the main features thereof.

There are several reasons why most of the above choices are unsatisfying: naming the 1-cells of **Relt** "profunctors" may generate confusion as the prefix *pro-C* denotes the pro-completion of a category $\mathbf{C}$, i.e. the collection of all "formal" cofiltered limits of objects of $\mathbf{C}$; then a pro-functor, etymologically, should be an object of the pro-completion of some functor category, which is not true; justifying the name "distributor" would request a more strict analogy between $\mathrm{P} \colon \mathbf{A} \to \mathbf{B}$ and distributions in mathematical analysis; *correspondence* is a name so inflated that it doesn't conveys any intuition at all;...

While taking note of this situation, we decide to add a new term to this vast zoology, and we call the 1-cells of **Relt** "relators", motivated by the analogy in the following example.

Example 5.4 [Relators as generalized relations]**:** A relator between $\{0,1\}$-categories is a function between sets $A^{\mathrm{op}} \times B \to \{0, 1\}$, namely a function $A \times B \to \{0, 1\}$, or in other words a relation regarded as a subset $R \subseteq A \times B$.

From this point of view, relators $\mathbf{A} \rightsquigarrow \mathbf{B}$ can be thought as *generalized relations*, taking values in more complicated, or structured, enriching cosmoi. This point of view is what Lawvere [**Law73**, §**4**,**5**] calls *generalized logic*, and it regards the coend in Def. **5.1**, and the product $\mathrm{P}(a, x) \times \mathrm{Q}(x, b)$ therein, as a generalized existential quantification and a generalized conjunction respectively, giving a composition rule for generalized relations: the coend stands as

$$(x, z) \in R \circ S \iff \exists y \in Y : \big((x, y) \in R\big) \wedge \big((y, z) \in S\big) \tag{103}$$

for two relations $R \subseteq X \times Y, S \subseteq Y \times Z$: we can depict this analogy as in Figure **5**: this is valid for suitable relations $R\colon X \rightsquigarrow Y$ and $S\colon Y \rightsquigarrow Z$ or suitable relators $\mathrm{R}\colon \mathbf{X} \rightsquigarrow \mathbf{Y}$ and $\mathrm{S}\colon \mathbf{Y} \to \mathbf{Z}$.

$$
\boxed{(x,z) \in S \circ R} \quad \boxed{\iff} \quad \boxed{\exists y \in Y} \; \boxed{((x,y) \in R)} \; \boxed{\wedge} \; \boxed{((y,z) \in S)}
$$

$$
\boxed{(\mathrm{S} \diamond \mathrm{R})(x,z)} \quad \boxed{=} \quad \boxed{\int^{y \in \mathbf{Y}}} \; \boxed{\mathrm{R}(x,y)} \; \boxed{\times} \; \boxed{\mathrm{S}(y,z)}
$$

FIGURE 2. The analogy between the composition of relators between categories and the composition of relations between sets gives rise to what lawvere calls *generalized logic*.

Example 5.5 : Let A, B be sets, considered as categories having only identity arrows via the embedding **Sets** $\subset$ **Cat**. A relator $A \rightsquigarrow B$ is then simply a collection of sets P_{ab}, one for each $a \in A, b \in B$. Their composition then results in a "categorified" matrix multiplication, in that the coend simplifies to be a mere coproduct: given $\mathrm{P}\colon A \to B$, $\mathrm{Q}\colon B \to C$ we have

$$
(\mathrm{P} \diamond \mathrm{Q})_{ac} = \coprod_{b \in B} P_{ab} \times Q_{bc} \tag{104}
$$

if $P_{ab} = \mathrm{P}(a,b)$ and $Q_{bc} = \mathrm{Q}(b,c)$.

Remark 5.6 : There is an alternative, but equivalent definition for $\mathrm{Q} \diamond \mathrm{P}$ which exploits the universal property of $\widehat{\mathbf{C}}$ as a free cocompletion: any relator $\mathrm{P}\colon \mathbf{A} \rightsquigarrow \mathbf{B}$ can be identified with its *mate* under the adjunction giving the cartesian closed structure of **Cat**,

$$
\mathsf{Fun}(\mathbf{A}^{\mathrm{op}} \times \mathbf{B}, \mathbf{Sets}) \cong \mathsf{Fun}(\mathbf{B}, [\mathbf{A}^{\mathrm{op}}, \mathbf{Sets}]) \tag{105}
$$

i.e. with a functor $\widehat{\mathrm{P}}\colon \mathbf{B} \to \widehat{\mathbf{A}}$ obtained as $b \mapsto \mathrm{P}(\_, b)$. Hence we can define the composition $\mathbf{A} \overset{\mathrm{P}}{\rightsquigarrow} \mathbf{B} \overset{\mathrm{Q}}{\rightsquigarrow} \mathbf{C}$ to be $\mathrm{Lan}_{\sharp}\, \widehat{\mathrm{P}} \circ \widehat{\mathrm{Q}}$:

$$
\begin{array}{ccc}
\mathbf{B} & \xrightarrow{\widehat{\mathrm{P}}} & \widehat{\mathbf{A}} \\
{\scriptstyle \sharp}\Big\downarrow & \nearrow \scriptstyle{\mathrm{Lan}_{\sharp}\, \widehat{\mathrm{P}}} & \\
\mathbf{C} \xrightarrow{\widehat{\mathrm{Q}}} & \widehat{\mathbf{B}} &
\end{array} \tag{106}
$$

This is equivalent to the previous definition, in view of the characterization of a left Kan extension as a coend in $\widehat{\mathbf{A}}$, given in Equation **39**:

$$\mathrm{Lan}_{\text{よ}}\,\widehat{\mathrm{P}} \cong \int^b \widehat{\mathbf{B}}(\text{よ}_b,\,{-}) \cdot \widehat{\mathrm{P}}(b). \tag{107}$$

Since in **Sets** copower coincides with product (i.e. $X \cdot Y \cong X \times Y$, since $\mathbf{Sets}(X \cdot Y, B) \cong \mathbf{Sets}(X, \mathbf{Sets}(Y, B)) \cong \mathbf{Sets}(X \times Y, B)$, naturally in B), we have

$$\mathrm{Lan}_{\text{よ}}\,\widehat{\mathrm{P}}(\widehat{\mathrm{Q}}(c)) \cong \int^b \widehat{\mathbf{B}}(\text{よ}_b, \widehat{\mathrm{Q}}(c)) \cdot \widehat{\mathrm{P}}(b)$$

$$\cong \int^b \widehat{\mathrm{Q}}(c)(b) \cdot \widehat{\mathrm{P}}(b)$$

$$\cong \int^b \mathrm{P}({-}, b) \times \mathrm{Q}(b, c).$$

Remark 5.7 : The properties of (strong) associativity and unitality for the composition of relators follow directly from the associativity of cartesian product, its cocontinuity as a functor of a fixed variable, and from the ninja Yoneda lemma **2.1**, as shown by the following computation:

- Composition of relators is associative (up to isomorphism), giving the *associator* of a bicategory structure:

$$\mathrm{P} \diamond (\mathrm{Q} \diamond \mathrm{H}) = \int^x \mathrm{P}(b, x) \times (\mathrm{Q} \diamond \mathrm{H})(x, a)$$

$$= \int^x \mathrm{P}(b, x) \times \left(\int^y \mathrm{Q}(x, y) \times \mathrm{H}(y, a) \right)$$

$$\cong \int^{xy} \mathrm{P}(b, x) \times \left(\mathrm{Q}(x, y) \times \mathrm{H}(y, a) \right)$$

$$(\mathrm{P} \diamond \mathrm{Q}) \diamond \mathrm{H} = \int^x (\mathrm{P} \diamond \mathrm{Q})(b, x) \times \mathrm{H}(x, a)$$

$$\cong \int^{xy} \left(\mathrm{P}(b, y) \times \mathrm{Q}(y, x) \right) \times \mathrm{H}(x, a)$$

  and these results are clearly isomorphic, and naturally so, once we changed name to "integration" variables (which are obviously "mute"). See Remark **5.8** below for a discussion on the coherence laws of this associator.
- Any object $\mathbf{A}$ has an identity arrow, given by the "diagonal" relator $\mathbf{A}({-}, {=}) = \mathrm{hom}_{\mathbf{A}} \colon \mathbf{A}^{\mathrm{op}} \times \mathbf{A} \to \mathbf{Sets}$: the fact that $\mathrm{P} \diamond \mathrm{hom} \cong \mathrm{P}$, $\mathrm{hom} \diamond \mathrm{Q} \cong \mathrm{Q}$ simply rewrites the ninja Yoneda lemma.

Remark 5.8 : The isomorphism above is part of the data turning **Relt** into a bicategory; the *associator* realizes the identification between different

parenthesizations of 1-cells, and the *unitor* realizes the identification between $\mathrm{P} \diamond \hom \cong \mathrm{P}$.

To ensure that "every" diagram which can be constructed from these data commutes some coherence conditions have to be imposed. One of these is the *pentagon* identity [**ML98**], encoded in the following diagram:

$$
\begin{array}{ccc}
& (wx)(yz) & \\
((wx)y)z & & w(x(yz)) \\
(w(xy))z & \longrightarrow & w((xy)z)
\end{array}
$$

This equality is natural in each argument (as a consequence of being a composition of natural transformations).

It's immediate to observe that the validity of the pentagon identity in the case of the cartesian monoidal structure of **Sets**, and the naturality thereof, ensure that the associator (whose components are) $(\mathrm{P} \diamond \mathrm{Q}) \diamond \mathrm{H} \Rightarrow \mathrm{P} \diamond (\mathrm{Q} \diamond \mathrm{H})$ satisfies the pentagon identity; a similar argument shows that the unitor satisfies similar (left and right) triangular identities, as a consequence of the naturality of the ninja Yoneda lemma **2.1**.

Definition 5.9 [EINSTEIN NOTATION]**:** There is a useful notation which can be implied to shorten involved computations with coends, and which is particularly evocative when dealing with relators; we choose to call it *Einstein convention* for evident graphical reasons.[12]

Let $\mathrm{P} \colon \mathbf{A} \rightsquigarrow \mathbf{B}$, $\mathrm{Q} \colon \mathbf{B} \rightsquigarrow \mathbf{C}$ be two composable relators. If we adopt the notation $\mathrm{P}_b^a, \mathrm{Q}_c^b$ to denote the images $\mathrm{P}(a,b), \mathrm{Q}(b,c) \in \mathbf{Sets}$ (keeping track that superscripts are contravariant and subscripts are covariant components), then composition of relators acquires again the form of a "matrix product":

$$
\mathrm{P} \diamond \mathrm{Q}(a,c) = \int^b \mathrm{P}_b^a \times \mathrm{Q}_c^b = \int^b \mathrm{P}_b^a \mathrm{Q}_c^b. \tag{108}
$$

From now on, we feel free to adopt the Einstein summation convention during long calculations.

---

[12]This notation has been adopted also in the beautiful [**RV14**], a valuable reading for several reasons, last but not least the fact that its authors adopt some coend-fu to simplify the discussion about "Reedy calculus".

Remark 5.10 [(Co)presheaves are relators]**:** Presheaves on $\mathbf{C}$ obviously correspond to relators $\mathbf{C} \rightsquigarrow \mathbf{1}$; copresheaves, i.e. functors $\mathbf{C} \to \mathbf{Sets}$, correspond to relators $\mathbf{1} \rightsquigarrow \mathbf{C}$.

5.1. **Embeddings and adjoints.** There are two identity-on-objects embeddings $\mathbf{Cat} \to \mathbf{Relt}$ (respectively the *covariant* and the *contravariant* one, looking at the behaviour on 2-cells), and send a diagram in $\mathbf{Cat}$ respectively to

$$(109)$$

This clearly defines a (pseudo)functor, since it's easy to see that

- $\mathrm{P}_{FG} \cong \mathrm{P}_F \diamond \mathrm{P}_G$, and $\mathrm{P}^{FG} \cong \mathrm{P}^G \diamond \mathrm{P}^F$;
- $\mathrm{P}_{\mathrm{id}_{\mathbf{A}}} = \mathrm{P}^{\mathrm{id}_{\mathbf{A}}} = \mathbf{A}(\,{}_-\,,\,{}_=\,)$.

Natural transformations $\alpha \colon F \Rightarrow G$ are obviously sent to 2-cells in $\mathbf{Relt}$, and the covariancy of this assignment is uniquely determined as in the diagram above.

Remark 5.11 : The 1-cells P_F, P^F are not independent: they are *adjoint* 1-cells in the bicategory $\mathbf{Relt}$. Indeed, for every $F \in \mathbf{Cat}(\mathbf{A}, \mathbf{B})$ we can define 2-cells

$$\epsilon = \epsilon_F \colon \mathrm{P}_F \diamond \mathrm{P}^F \Longrightarrow \mathbf{B}(\,{}_-\,,\,{}_=\,) \tag{110}$$

$$\eta = \eta_F \colon \mathbf{A}(\,{}_-\,,\,{}_=\,) \Longrightarrow \mathrm{P}^F \diamond \mathrm{P}_F \tag{111}$$

(*counit* and *unit* of the adjunction): here we unravel the coends involved in these definitions.

- For what concerns the counit, we write the coend $\mathrm{P}_F \diamond \mathrm{P}^F$ as the quotient set

$$\int^x \mathbf{B}(a, Fx) \times \mathbf{B}(Fx, b) = \Big(\coprod_{x \in \mathbf{A}} \mathbf{B}(a, Fx) \times \mathbf{B}(Fx, b) \Big) / \simeq \tag{112}$$

  where $\simeq$ is the equivalence relation generated by $\big(a \xrightarrow{u} Fx, Fx \xrightarrow{v} b\big) \simeq \big(a \xrightarrow{u'} Fy, Fy \xrightarrow{v'} b\big)$ if there is $t \colon x \to y$ such that $v' = Ft \circ v$ e $Ft \circ u = u'$. This can be visualized as the commutativity of the

square

$$
\begin{array}{c}
(113)
\end{array}
$$

$$
\xymatrix{ & Fx & \\ a \ar[ur]^{u} \ar[dr]_{u'} & \vdots{Ft} & b \\ & Fy & }
$$

Now it's easily seen that sending $(a \xrightarrow{u} Fx, Fx \xrightarrow{v} b)$ in the composition $v \circ u$ descend to the quotient with respect to $\simeq$, hence $\epsilon \colon \mathrm{P}_F \diamond \mathrm{P}^F \to \mathbf{B}(\_,\, =)$ is well defined. All boils down to notice that the composition

$$
c \colon \mathbf{B}(a, Fx) \times \mathbf{B}(Fx, b) \to \mathbf{B}(a, b) \tag{114}
$$

defines a cowedge in the variable x.

- The unit of the adjunction is the 2-cell

$$
\eta \colon \mathbf{A}(\_,\, =) \Longrightarrow \mathrm{P}^F \diamond \mathrm{P}_F \tag{115}
$$

obtained when we noticed that $\mathrm{P}^F \diamond \mathrm{P}_F(a, b) = \int^X \mathbf{B}(Fa, x) \times \mathbf{B}(x, Fb) \cong \mathbf{B}(Fa, Fb)$ (as a consequence of the ninja Yoneda lemma), is simply determined by the action of F on arrows, $\mathbf{A}(a, b) \to \mathbf{B}(Fa, Fb)$.

We now have to verify that the zig-zag identities (see [**Bor94a**, Thm.**3.1.5.(2)**]) hold:

$$
(\mathrm{P}^F \diamond \epsilon) \circ (\eta \diamond \mathrm{P}^F) = \mathrm{id}_{\mathrm{P}^F}
$$
$$
(\epsilon \diamond \mathrm{P}_F) \circ (\mathrm{P}_F \diamond \eta) = \mathrm{id}_{\mathrm{P}_F}
$$

As for the first, we must verify that the diagram

$$
(116)
$$

$$
\xymatrix{
\mathrm{P}^F \ar[r]^-{\sim} \ar@{=}[d] & \mathbf{B}(\_,\,=) \diamond \mathrm{P}^F \ar[r]^-{\eta \diamond \mathrm{P}^F} & (\mathrm{P}^F \diamond \mathrm{P}_F) \diamond \mathrm{P}^F \ar[d]^{\cong} \\
\mathrm{P}^F & \mathrm{P}^F \diamond \mathbf{B}(\_,\,=) \ar[l]^-{\sim} & \mathrm{P}^F \diamond (\mathrm{P}_F \diamond \mathrm{P}^F) \ar[l]^-{\mathrm{P}^F \diamond \epsilon}
}
$$

commutes. One has to send $h \in \mathrm{P}^F(u, v) = \mathbf{B}(Fu, v)$ in the class $[(\mathrm{id}_u, h)] \in \int^x \mathbf{B}(u, x) \times \mathbf{B}(Fx, v)$, which must go under $\eta \diamond \mathrm{P}^F$ in the class $[(F(\mathrm{id}_u), h)] \in \int^{xy} \mathbf{B}(Fa, x) \times \mathbf{B}(x, Fy) \times \mathbf{B}(Fy, b)$, canonically identified with $\int^y \mathbf{B}(Fa, Fy) \times \mathbf{B}(Fy, b)$. Now $\mathrm{P}^F \diamond \epsilon$ acts composing the two arrows, and one obtains $F(\mathrm{id}_A) \circ h = h$ back.

Similarly, to prove the second identity, the diagram

$$\begin{array}{ccc}
\mathrm{P}_F & \xrightarrow{\ \sim\ } \mathrm{P}_F \diamond \mathbf{B}(\,_-,\,_=) \xrightarrow{\ \mathrm{P}_F \diamond \eta\ } \mathrm{P}_F \diamond \big(\mathrm{P}^F \diamond \mathrm{P}_F\big) & \quad (117) \\
\Big\| & & \Big\downarrow{\cong} \\
\mathrm{P}_F & \xleftarrow{\ \sim\ } \mathbf{B}(\,_-,\,_=) \diamond \mathrm{P}_F \xleftarrow{\ \epsilon \diamond \mathrm{P}_F\ } \big(\mathrm{P}_F \diamond \mathrm{P}^F\big) \diamond \mathrm{P}_F &
\end{array}$$

must commute (all the unlabeled isomorphisms are the canonical ones). This translates into

$$\Big(a \xrightarrow{u} Fb\Big) \longmapsto (u, \mathrm{id}_b)_\sim \longmapsto (u, F(\mathrm{id}_b)) \longmapsto u \circ F(\mathrm{id}_b) = u, \quad (118)$$

which is what we want; hence $\mathrm{P}_F \dashv \mathrm{P}^F$. $\qquad\qquad\qquad\square$

Remark 5.12 : Two functors $F\colon \mathbf{A} \leftrightarrows \mathbf{B}\colon G$ are adjoints if and only if $\mathrm{P}_F \cong \mathrm{P}^G$ (and therewith $G \dashv F$) or $\mathrm{P}_G \cong \mathrm{P}^F$ (and therewith $F \dashv G$).

Remark 5.13 : It is a well-known fact (see [**Bor94a**, dual of Prop. **3.4.1**]) that if $F \dashv G$, then F is fully faithful if and only if the unit of the adjunction $\eta\colon 1 \to GF$ is an isomorphism.

This criterion can be extended also to functors which do not admit a "real" right adjoint, once noticed that F is fully faithful if and only if $\mathbf{A}(a, b) \cong \mathbf{B}(Fa, Fb)$ for any two $a, b \in \mathbf{A}$, i.e. if and only if the unit $\eta\colon \hom_{\mathbf{A}} \Rightarrow \mathrm{P}^F \diamond \mathrm{P}_F$ is an isomorphism.

Example 5.14 : Given a relator $\mathrm{P}\colon \mathbf{A} \rightsquigarrow \mathbf{B}$ and a functor $F\colon \mathbf{B} \to \mathbf{D}$ we can define $\mathrm{P} \otimes F$ to be the functor $\mathbf{A} \to \mathbf{D}$ given by $\mathrm{Lan}_y F \circ \widehat{\mathrm{P}}$ (provided this colimit exists), where $\widehat{\mathrm{P}}\colon \mathbf{B} \to \widehat{\mathbf{A}}$ is the adjunct of P.

More explicitly,

$$\mathrm{P} \otimes F(a) = \int^b \mathsf{Nat}(y_b, \mathrm{P}(\,_-, a)) \cdot Fb \cong \int^b \mathrm{P}_a^b \cdot F_b \qquad (119)$$

Exploiting this definition, several things can be proved via coend-fu:

- $\hom_{\mathbf{B}} \otimes F \cong F$ as a consequence of the ninja Yoneda lemma;
- If $\mathbf{C} \xrightarrow{\mathrm{Q}} \mathbf{A} \xrightarrow{\mathrm{P}} \mathbf{B} \xrightarrow{F} \mathbf{X}$, then $(\mathrm{P} \diamond \mathrm{Q}) \otimes F \cong \mathrm{Q} \otimes (\mathrm{P} \otimes F)$: indeed

$$[(\mathrm{P} \diamond \mathrm{Q}) \otimes F]a = \int^b (\mathrm{P} \diamond \mathrm{Q})_a^b \times F_b$$

$$\cong \int^{bx} \mathrm{P}_x^b \times \mathrm{Q}_a^x \times F_b$$

$$\cong \int^x \mathrm{Q}_a^x \times \Big(\int^b \mathrm{P}_x^b \times F_b \Big)$$

$$\cong \int^x \mathrm{Q}_a^x \times (F \otimes \mathrm{P})_x = [\mathrm{Q} \otimes (\mathrm{P} \otimes F)]a$$

Example 5.15 [KAN EXTENSIONS IN **Relt**]**:** Any relator P has a right Kan extension Ran_P in the sense that the notion has in any bicategory, where composition of functors or natural transformations is replaced by composition of 1- or 2-cells.

One has the following chain of isomorphisms in **Relt** (see Definition **5.9** for the Einstein convention):

$$
\begin{aligned}
\mathsf{Nat}(\mathrm{G} \diamond \mathrm{P}, \mathrm{H}) &\cong \int_{ab} \mathbf{Sets}\big((\mathrm{G} \diamond \mathrm{P})_b^a, \mathrm{H}_b^a\big) \\
&\cong \int_{ab} \mathbf{Sets}\Big(\int^x \mathrm{G}_x^a \times \mathrm{P}_b^x, \mathrm{H}_b^a \Big) \\
&\cong \int_{abx} \mathbf{Sets}\big(\mathrm{G}_x^a, \mathbf{Sets}(\mathrm{P}_b^x, \mathrm{H}_b^a)\big) \\
&\cong \int_{ax} \mathbf{Sets}\Big(\mathrm{G}_x^a, \int_b \mathbf{Sets}(\mathrm{P}_b^x, \mathrm{H}_b^a) \Big) \\
&\cong \int_{ax} \mathbf{Sets}\Big(\mathrm{G}_x^a, \mathrm{Ran}_\mathrm{P}\, \mathrm{H}_x^a \Big) \\
&\cong \mathsf{Nat}(\mathrm{G}, \mathrm{Ran}_\mathrm{P}\, \mathrm{H})
\end{aligned}
$$

when we define $\mathrm{Ran}_\mathrm{P}\,\mathrm{H}(a, x)$ to be $\mathsf{Nat}\big(\mathrm{P}(x, \_), \mathrm{H}(a, \_)\big)$.

Remark 5.16 [THE MULTIBICATEGORY OF RELATORS]**:** The bicategory of relators can be promoted to a *multibicategory* in the sense of [**CKS03**, **1.4**]; this means that we exploit the (partial) monoidal structure on each **Relt**(**C**, **D**) to specify a class of multimorphisms $\eta\colon \mathrm{P}_1, \dots, \mathrm{P}_n \rightsquigarrow \mathrm{Q}$, depicted as diagrams

$$X_0 \xrightarrow{\ \mathrm{P}_1\ } X_1 \xrightarrow{\ \mathrm{P}_2\ } \dots \xrightarrow{\ \mathrm{P}_n\ } X_n \qquad (120)$$

$$\eta \Big\Downarrow$$

$$\mathrm{Q}$$

and composition, associativity and unitality thereof, follow at once from pasting laws for 2-cells in 2-categories [**Kel82**] (try to outline them as a straightforward exercise).

Exercises for §5

E1 Describe relators between monoids, regarded as one-object categories; describe relators between posets regarded as thin categories.

E2 Given relators $\mathrm{K}\colon \mathbf{C} \rightsquigarrow \mathbf{D}$ and $\mathrm{L}\colon \mathbf{C} \rightsquigarrow \mathbf{E}$ define

$$\mathrm{K} \rhd \mathrm{L} = \int_c [\mathrm{K}(c, \_), \mathrm{L}(c, =)]$$

Show that this operation is a Kan lifting (of $\textsc{l}$ along $\textsc{k}$); dually, given $\textsc{h}\colon \mathbf{D} \rightsquigarrow \mathbf{A}$ and $\textsc{l}\colon \mathbf{E} \rightsquigarrow \mathbf{A}$ we can define

$$\textsc{l} \lhd \textsc{h} = \int_a [\textsc{h}(=,a), \textsc{l}(\_,a)]. \tag{121}$$

Show that this second operation is a Kan extension (some vagueness is intended to be fixed as part of the exercise), and that these two operations "behave like an action" on $(\mathbf{Relt}, \diamond, \mathrm{hom})$ on the bicategory $\mathbf{Relt}$:

   i) $(\textsc{k} \diamond \textsc{h}) \rhd \textsc{l} \cong \textsc{k} \rhd (\textsc{h} \rhd \textsc{l})$;

   ii) $\textsc{l} \lhd (\textsc{k} \diamond \textsc{h}) \cong (\textsc{l} \lhd \textsc{k}) \lhd \textsc{h}$;

   iii) $\mathrm{hom} \rhd \textsc{l} \cong \textsc{l} \cong \textsc{l} \lhd \mathrm{hom}$.

This is a naïve way to see that the structure on $\mathbf{Relt}$ given by $\diamond$ is *biclosed* (i.e., $\diamond$ is a bifunctor $\mathbf{Relt}(\mathbf{A},\mathbf{B}) \times \mathbf{Relt}(\mathbf{B},\mathbf{C}) \to \mathbf{Relt}(\mathbf{A},\mathbf{C})$ and each $\textsc{p} \diamond \_$, as well as each $\_ \diamond \textsc{q}$ have right adjoints).

E3 The *collage* of two categories $\mathbf{A}, \mathbf{B}$ along a relator $\textsc{p}\colon \mathbf{A} \rightsquigarrow \mathbf{B}$ is defined to be the category $\mathbf{A} \uplus_{\textsc{p}} \mathbf{B}$ with the same objects as $\mathbf{A} \amalg \mathbf{B}$ and morphisms given by the rule

$$\mathbf{A} \uplus_{\textsc{p}} \mathbf{B}(x,y) = \begin{cases} \mathbf{A}(x,y) & \text{if } x,y \in \mathbf{A} \\ \mathbf{B}(x,y) & \text{if } x,y \in \mathbf{B} \\ \textsc{p}(x,y) & \text{if } x \in \mathbf{A}, y \in \mathbf{B} \end{cases} \tag{122}$$

and empty in every other case. Show that $\mathbf{A} \uplus_{\textsc{p}} \mathbf{B}$ has the universal property of the category of elements of $\textsc{p}$, regarded as a presheaf.

E4 Show that the composition laws $\textsc{p}(A,B) \times \mathbf{B}(B,B') \to \textsc{p}(A,B')$, $\mathbf{A}(A,A') \times \textsc{p}(A',B) \to \textsc{p}(A,B)$ of arrows in $\mathbf{A} \uplus_{\textsc{p}} \mathbf{B}$ are governed by the universal property of a coend.

E5 The *cocomma object* (F/G) of two functors $\mathbf{X} \xleftarrow{F} \mathbf{A} \xrightarrow{G} \mathbf{Y}$ is defined to be the pushout of

$$\begin{array}{ccc} \mathbf{A} \amalg \mathbf{A} & \longrightarrow & \mathbf{A} \times \Delta[1] \\ \big\downarrow & & \\ \mathbf{X} \amalg \mathbf{Y} & & \end{array} \tag{123}$$

in $\mathbf{Cat}$, where the horizontal arrow is the "cylinder" embedding. Show that (F/G) is the collage of $\mathbf{X}$ and $\mathbf{Y}$ along the relator $\textsc{p}^G \diamond \textsc{p}_F\colon \mathbf{X} \rightsquigarrow \mathbf{Y}$.

E6 Given relators $\mathbf{A} \stackrel{\textsc{p}}{\rightsquigarrow} \mathbf{B} \stackrel{\textsc{q}}{\rightsquigarrow} \mathbf{C}$ consider the categories $\mathbf{A} \uplus_{\varphi} \mathbf{B}$ and $\mathbf{B} \uplus_{\psi} \mathbf{C}$. Describe the pushout

$$\begin{array}{ccc} \mathbf{B} & \longrightarrow & \mathbf{A} \uplus_{\varphi} \mathbf{B} \\ \big\downarrow & & \big\downarrow \\ \mathbf{B} \uplus_{\psi} \mathbf{C} & \longrightarrow & \mathbf{H} \end{array} \tag{124}$$

in $\mathbf{Cat}$. Is there a relation between $\mathbf{H}$ and the collage $\mathbf{A} \uplus \mathbf{C}$ along $\textsc{q} \diamond \textsc{p}$?

6. Operads using coends.

Since they were introduced by P. May in his [**May72**] to solve a problem in algebraic topology,[13] it has been clear that operads are monoid-like objects

---

[13]There are several reasons why algebraic topologists are interested in spaces Y which are homotopy equivalent to ΩX; they are much more interested in spaces $Y \simeq \Omega^n X$, and

in some category of functors; making this analogy a precise statement, using the power of coend-fu, is the content of Kelly's [**Kel05**], which we follow here almost *verbatim.*

A certain acquaintance with the machinery of operads is a fundamental prerequisite to follow the discussion; unfortunately, given the plethora of different interpretation of the theory, and different areas of mathematics where the notion of operad arises, the beginners (the author of the present note is undoubtedly among them) may feel rather disoriented when approaching any book on the subject, so it's extremely difficult to advise a single, comprehensive reference.

Among classical textbooks, we can't help but mention the founder [**May72**], as well as more recent monographies like [**LV12, MSS02**] written respectively from the algebraist's and topologist's point of view. Among less classical and yet extremely valid points of view, the author profited a lot from a lucid, and unfortunately still unfinished, online draft [**Tria**] written by T. Trimble.

Local conventions. Along the whole section we will adopt the following notation and conventions:

- **P** is the *groupoid of natural numbers*, i.e. the category having objects the nonempty sets $\{1, \ldots, n\}$ (denoted as n for short, assuming that $0 = \varnothing$) where $\mathbf{P}(m, n) = \varnothing$ if $n \neq m$ and S_n (the group of bijections of n-element sets) if $n = m$. It is evident that **P** is the disjoint union of groups $\coprod_{n \geq 0} S_n$ in the category **Gpd** of groupoids.
- $\mathscr{V}$ is a fixed *Bénabou cosmos* (i.e. a bicomplete closed symmetric monoidal category, "a good setting to do enriched category theory", see Notation **4.6**).

Notice that **P** has a symmetric monoidal structure, with tensor the *sum* of natural numbers; the action on arrows is given by $(\sigma, \tau) \mapsto \sigma + \tau$ defined acting as σ on the set $\{1, \ldots, m\}$ and as τ on the set $\{m + 1, \ldots, m + n\}$ (these permutations are called *shuffles*).

6.1. **Convolution product.** We begin our discussion presenting a general theorem on monoidal categories, first outlined by B. Day: it will be utterly generalized in our Appendix **A**.

The rough idea is the following: in the same way the set of regular functions $f \colon G \to \mathbb{C}$ on a topological group G acquires a *convolution* product given by $(f, g)(x) = \int_G f(xy^{-1})g(y)dy$ (the integral sign here is not an end!), we can endow the *category of co/presheaves* $F \colon \mathbf{C} \to \mathscr{V}$ with a monoidal structure induced by a monoidal structure on **C**, which is different from the pointwise one, induced by the monoidality of the codomain. This is called

---

in spaces such that "$Y \simeq \Omega^\infty X$": these are called *infinite loop spaces.* [**May72**] offers a way to recognize infinite loop spaces among all spaces. See [**Ada78**] for more informations.

the *convolution product* of functors; appendix **A** will give a generalization of this point of view in form of exercises (see in particular Exercises **A.4**, **A.5**).

Definition 6.1 [DAY CONVOLUTION]**:** If **C** is a symmetric monoidal category, then the functor category $[\mathbf{C}, \mathscr{V}]$ is itself a Bénabou cosmos with respect to the monoidal structure given by *Day convolution product*: given $F, G \in [\mathbf{C}, \mathscr{V}]$ we define

$$F * G := \int^{cd} \mathbf{C}(c \otimes d, {}_{-}) \cdot Fc \otimes Gd \tag{125}$$

where we recall that $X \cdot V$ for $X \in \mathbf{Sets}, V \in \mathscr{V}$ is the *copower* (or *tensor*) $X \cdot V$ such that

$$\mathscr{V}(X \cdot V, W) \cong \mathbf{Sets}(X, \mathscr{V}(V, W)). \tag{126}$$

Notation 6.2 : In the following sections we will make use of the "Einstein notation" for co/ends defined in **5.9**; this will compactify a lot the exposition. A fundamental rule to avoid getting lost is the following, really akin to the Einstein convention for tensor operations: variables of integration are paired as subscript-superscript, and whenever they are paired a co/end operation is implicit.

Without smart ideas to specify the difference we are forced to maintain all integral signs, to discern ends from coends. In Einstein notation we write the convolution as

$$F * G = \int^{cd} \mathbf{C}_{(-)}^{c \otimes d} F_c G_d \tag{127}$$

Proof. We have to show that this really defines a monoidal structure:

- Associativity follows from the associativity of the tensor product on **C** and the ninja Yoneda lemma (see the remark above for the Einstein convention; it is also harmless to suppress the distinction between monoidal products in $\mathscr{V}$ and **Sets**-tensors, since the distinction can be easily devised with a "dimensionality check"):

$$
\begin{aligned}
[F * (G * H)]_x &= \int^{ab} \mathbf{C}_x^{a \otimes b} F_a (G * H)_b \\
&\cong \int^{ab} \mathbf{C}_x^{a \otimes b} \int^{cd} \mathbf{C}_b^{c \otimes d} F_a G_c H_d \\
&\cong \int^{abcd} \mathbf{C}_x^{a \otimes b} \mathbf{C}_b^{c \otimes d} F_a G_c H_d \\
&\cong \int^{acd} \mathbf{C}_x^{a \otimes (c \otimes d)} F_a G_c H_d \\
[(F * G) * H]_x &\cong \int^{acd} \mathbf{C}_x^{(a \otimes c) \otimes d} F_a G_c H_d.
\end{aligned}
$$

- (Right) unitality : choose $J = \mathord{\natural}_I = \mathbf{C}(I, \_)$ and notice that the ninja Yoneda lemma implies that

$$
\begin{aligned}
[F * J]_x &\cong \int^{cd} \mathbf{C}_x^{c \otimes d} F_c J_d \\
&\cong \int^{cd} \mathbf{C}_x^{c \otimes d} \mathbf{C}_d^I F_c \\
&\cong \int^c \mathbf{C}_x^{c \otimes I} F_c \cong F_x.
\end{aligned}
$$

- Left unitality is totally analogous.

Example 6.3 [Subdivision and joins as convolutions]**:** Compare Example **3.15** and the definition of *join* of *augmented*[14] simplicial sets given in [**Joy08**]: given $X, Y \in \mathbf{sSet}_+$ we define

$$
X \star Y = \int^{p,q} X_p \times Y_q \times \mathbf{\Delta}(\_, p \oplus q) \tag{128}
$$

where $\oplus$ is the *ordinal sum* operation (see again [**Joy08**] or rather our **3.15**).

The category $[\mathbf{C}, \mathcal{V}]$ is left and right closed with respect to this monoidal structure: the exponential G/H (or rather the functor $G/\_$ which is right adjoint to $\_ * G$) is given by

$$
G/H := \int_c [Gc, H(c \otimes \_)] \tag{129}
$$

where $[\_, =]$ is the internal hom in $\mathcal{V}$, and often denoted $[\![G, H]\!]$. We can compute directly that

$$
\begin{aligned}
\mathsf{Nat}(F * G, H) &\cong \int_c \mathcal{V}((F * G)c, Hc) \\
&\cong \int_c \mathcal{V}\left(\int^{ab} \mathbf{C}_c^{a \otimes b} F_a G_b,\ H_c\right) \\
&\cong \int_{abc} \mathcal{V}(\mathbf{C}_c^{a \otimes b} F_a G_b,\ H_c) \\
&\cong \int_{abc} \mathcal{V}(F_a, [\mathbf{C}_c^{a \otimes b} G_b, H_c])
\end{aligned}
$$

---

[14]The category $\mathbf{\Delta}$ lacks an initial object $[-1] = \varnothing$; if we add this colimit we get a category $\mathbf{\Delta}_+$, and an *augmented* simplicial set is a presheaf on $\mathbf{\Delta}_+$; the category of augmented simplicial sets is denoted $\mathbf{sSet}_+$. There is a triple of adjoints induced by the inclusion $i \colon \mathbf{\Delta} \subset \mathbf{\Delta}_+$ and linking the categories of simplicial and augmented simplicial sets. The join operation is easily seen to restrict to simplicial sets (identified with augmented simplicial sets with empty set of (-1)-simplices) giving a monoidal structure on $\mathbf{sSet}$.

$$\cong \int_{abc} \mathscr{V}\left(F_a, [G_b, [\mathbf{C}_c^{a\otimes b}, H_c]]\right)$$

$$\cong \int_{ab} \mathscr{V}\left(F_a, [G_b, \int_c [\mathbf{C}_c^{a\otimes b}, H_c]]\right)$$

$$\cong \int_{ab} \mathscr{V}\left(F_a, [G_b, H_{a\otimes b}]\right)$$

$$\cong \int_a \mathscr{V}\left(F_a, \int_b [G_b, H_{a\otimes b}]\right)$$

$$\cong \int_a \mathscr{V}\left(F_a, [\![G, H]\!]_a\right)$$

$$\cong \mathsf{Nat}(F, [\![G, H]\!]). \qquad \square$$

Remark 6.4 : In the particular case $\mathbf{C} = \mathbf{P}$ this means that $[\mathbf{P}, \mathscr{V}]$ is monoidal closed if we define

$$(F * G)_k := \int^{mn} \mathbf{P}(m+n, k) \cdot F_m \otimes G_n$$

$$[\![F, G]\!]_k := \int_n [F_n, G_{n+k}]$$

In particular, we have the formula

$$F_1 * \cdots * F_n = \int^{k_1, \ldots, k_n} \mathbf{P}\left(\sum k_i, \_\right) \cdot F_1 k_1 \otimes \cdots \otimes F_n k_n. \qquad (130)$$

for the iterated convolution of $F_1, \ldots, F_n \in [\mathbf{C}, \mathscr{V}]$, which will become useful in a while.

The gist of the definition of a $\mathscr{V}$-operad lies in an additional monoidal structure on $[\mathbf{P}, \mathscr{V}]$, defined by means of the Day convolution:

Definition 6.5 [DIAMOND PRODUCT ON $[\mathbf{P}, \mathscr{V}]$]**:** Let $F, G \in [\mathbf{P}, \mathscr{V}]$. Define

$$F \diamond G := \int^m Fm \otimes G^{*m}, \qquad (131)$$

where $G^{*m} := G * \cdots * G$.

Associativity exploits the following

Lemma 6.6 : There exists a natural equivalence $(F \diamond G)^{*m} \cong F^{*m} \diamond G$.

Proof. It's a formal manipulation:

$$(F \diamond G)^{*m} = \int^{\vec{n}_i} \mathbf{P}\left(\sum n_i, \_\right) \cdot (F \diamond G)n_1 \otimes \cdots \otimes (F \diamond G)n_m$$

$$\cong \int^{\vec{n}_i, k_i} \mathbf{P}\left(\sum n_i, \_\right) \cdot Fk_1 \otimes G^{*k_1}n_1 \otimes \cdots \otimes Fk_m \otimes G^{*k_m}n_m$$

$$\cong \int^{\vec{n}_i, \vec{k}_i} Fk_1 \otimes \cdots \otimes Fk_m \otimes \mathbf{P}\Big(\sum n_i, \_\Big) \cdot G^{*k_1} n_1 \otimes \ldots G^{*k_m} n_m$$

$$\cong \int^{\vec{k}_i} Fk_1 \otimes \cdots \otimes Fk_m \otimes \big(G^{*k_1} * \cdots * G^{*k_m}\big)$$

$$\cong \int^{\vec{k}_i} Fk_1 \otimes \cdots \otimes Fk_m \otimes G^{* \sum k_i}$$

$$\text{NINJA} \cong \int^{\vec{k}_i, r} \mathbf{P}\Big(\sum k_i, r\Big) \otimes Fk_1 \otimes \cdots \otimes Fk_m \otimes G^{*r}$$

$$\cong \int^{r} F^{*m} r * Gr = F^{*m} \diamond G$$

(we used a compact notation for $\int^{\vec{n}_i} = \int^{n_1, \ldots, n_m}$; the ninja Yoneda Lemma is used in the form $G^{*n} \cong \int^r \mathbf{P}(n, t) \cdot Gt = \mathbf{P}(n, \_) \diamond G$, because $(n, G) \mapsto G^{*n}$ is a bifunctor $\mathbf{P} \times [\mathbf{P}, \mathscr{V}] \to [\mathbf{P}, \mathscr{V}]$). $\qquad\qquad\qquad \square$

Associativity of the diamond product now follows at once: we have

$$(F \diamond (G \diamond H))(k) = \int^m Fm \otimes (G \diamond H)^{*m} k$$

$$\cong \int^m Fm \otimes (G^{*m} \diamond H) k$$

$$\cong \int^{m, l} Fm \otimes G^{*m} l \otimes H^{*l} k$$

$$\cong \int^l (F \diamond G) l \otimes H^{*l} k$$

$$= ((F \diamond G) \diamond H)(k).$$

A unit object for the $\diamond$-product is $J = \mathbf{P}(1, \_) \cdot I$; indeed $J(1) = I$, $J(n) = \varnothing_{\mathscr{V}}$ for any $n \neq 1$ and the ninja Yoneda lemma applies on both sides to show unitality rules:

- On the left one has

$$J \diamond F = \int^m Jm \otimes F^{*m} = \int^m \mathbf{P}(1, m) \cdot F^{*m} \cong F^{*1} = F. \qquad (132)$$

- On the right, $G \diamond J \cong G$ once noticed that $J^{*m} \cong \mathbf{P}(m, \_) \cdot I$ since

$$J^{*m} = \int^{\vec{n}_i} \mathbf{P}\Big(\sum n_i, \_\Big) \cdot \mathbf{P}(1, n_1) \cdot \cdots \cdot \mathbf{P}(1, n_m) \cdot I$$

$$\text{NINJA} \cong \mathbf{P}(1 + \cdots + 1, \_) \cdot I = \mathbf{P}(m, \_) \cdot I$$

because

$$\int^{\vec{n}_i} \mathbf{P}(n_1+\cdots+n_m,\,{}_-)\cdot\mathbf{P}(1,n_i) \cong \mathbf{P}(n_1+\cdots+n_{i-1}+1+n_{i+1}+\cdots+n_m,\,{}_-),$$
(133)

for any $1 \leq i \leq m$ (it is again an instance of the ninja Yoneda Lemma). One has

$$G \diamond J = \int^m Gm \otimes J^{*m} \cong \int^m Gm \otimes \mathbf{P}(m,\,{}_-) \cdot I \cong G. \qquad (134)$$

Theorem 6.7 : The $\diamond$-monoidal structure is left closed, but not right closed.

Proof. It is a formal manipulation:

$$\mathsf{Nat}(F \diamond G, H) \cong \mathsf{Nat}\Big(\int^m Fm \otimes G^{*m}, H\Big)$$

$$\cong \int_k \mathscr{V}\Big(\int^m Fm \otimes G^{*m}, H\Big)$$

$$\cong \int_{km} \mathscr{V}(Fm, [G^{*m}k, Hk])$$

$$\cong \int_m \mathscr{V}(Fm, \int_k [G^{*m}k, Hk])$$

which is equal to $\mathsf{Nat}(F, \{G, H\})$ if we define $\{G,H\}m = \int_k [G^{*m}k, Hk]$. Hence the functor $({}_-) \diamond G$ has a right adjoint for any G.

The functor $F \diamond ({}_-)$ can't have such an adjoint (Incidentally, this shows also that the diamond product can't come from a convolution product with respect to a *promonoidal structure* in the sense of Proposition **A.3**. Re-read this result after having gone through Appendix **A**!). We leave the reader think about the reason. $\square$

Definition 6.8 : An *operad* in $\mathscr{V}$ consists of a monoid object in $[\mathbf{P}, \mathscr{V}]$ endowed with the $(\diamond, \{\,{}_-, {}_=\})$ left-closed monoidal structure.

More explicitly, an operad is a functor $T \in [\mathbf{P}, \mathscr{V}]$ endowed with a natural transformation called *multiplication*, $\mu\colon T \diamond T \to T$ and a *unit* $\eta\colon J \to T$ such that

$$
\begin{array}{ccc}
T \diamond T \diamond T & \xrightarrow{\;T \diamond \mu\;} & T \diamond T \\
{\scriptstyle \mu \diamond T}\downarrow & & \downarrow{\scriptstyle \mu} \\
T \diamond T & \xrightarrow{\;\mu\;} & T
\end{array}
\qquad
\begin{array}{ccccc}
J \diamond T & \xrightarrow{\;\eta \diamond T\;} & T \diamond T & \xleftarrow{\;T \diamond \eta\;} & T \diamond J \\
& {\scriptstyle \cong}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle \cong} & \\
& & T & &
\end{array}
$$

are commutative diagrams.

Definition 6.9 [ENDOMORPHISM OPERAD]: For any $F \in [\mathbf{P}, \mathcal{V}]$ the object $\{F, F\}$ is an operad whose multiplication is the adjunct of the arrow

$$\{F, F\} \diamond \{F, F\} \diamond F \xrightarrow{\ 1\diamond\text{ev}\ } \{F, F\} \diamond F \xrightarrow{\quad\text{ev}\quad} F$$

and whose unit is the adjunct of the isomorphism $J \diamond F \cong F$.

Unraveling the previous definition, we can notice that an operad in $\mathcal{V}$ consists of

- Giving a natural transformation $\eta \colon J \to T$ amounts to a map $\eta_1 \colon I \to T(1)$, since $J(1) = I, J(n) = \varnothing$ for $n \neq 1$;
- Giving a natural transformation $\mu \colon T \diamond T \to T$, in view of the universal property of the two coends involved, amounts to give a cowedge

$$Tm \otimes \mathbf{P}(n_1 + \cdots + n_m, k) \cdot Tn_1 \otimes \cdots \otimes Tn_m \xrightarrow{\quad \tau \quad} Tk$$

for any $m, n_1, \ldots, n_m, k \in \mathbb{N}$, natural in k and the n_i and such that the following diagram commutes:

$$
\begin{array}{ccc}
Tm \otimes \mathbf{P}(\vec{n}, k) \cdot T\vec{n} & \xrightarrow{\ \sigma^*\ } & Tm \otimes \mathbf{P}(\vec{n}, k) \cdot T\vec{n} \\
\downarrow{\scriptstyle \sigma_*} & & \downarrow \\
Tm \otimes \mathbf{P}(\vec{n}, k) \cdot T\vec{n} & \xrightarrow{\hspace{3cm}} & Tk
\end{array}
$$

(the notation is self-evident) for every morphism $\sigma \in \mathbf{P}$. This is equivalent to a transformation

$$Tm \otimes Tn_1 \otimes \cdots \otimes Tn_m \xrightarrow{\quad \widehat{\tau} \quad} [\mathbf{P}(n_1 + \cdots + n_m, {}_-), T({}_-)]$$

(considering the n_i fixed and the first functor constant in k) i.e., by the Yoneda Lemma a natural transformation

$$Tm \otimes Tn_1 \otimes \cdots \otimes Tn_m \xrightarrow{\quad \widehat{\tau} \quad} T(n_1 + \cdots + n_m).$$

This concludes the discussion, as it is precisely the definition of operad given in [May97]. It only remains to verify that all the axioms given there are satisfied. This is a tedious but necessary exercise.

Exercises for §6

E1 Show that the convolution product on $[\mathbf{C}, \mathscr{V}]$ results as the following left Kan extension:

$$\mathbf{C} \times \mathbf{C} \xrightarrow{F \times G} \mathscr{V} \times \mathscr{V} \xrightarrow{\otimes_{\mathscr{V}}} \mathscr{V} \qquad (135)$$

with $\odot$ going down to $\mathbf{C}$.

E2 Define two functors

$$\Phi \colon [\mathbf{P}, \mathscr{V}] \to \mathscr{V} \qquad \text{(evaluation at 0)}$$

$$\Psi \colon \mathscr{V} \to [\mathbf{P}, \mathscr{V}] \qquad \text{(the left adjoint to } \Phi)$$

Prove that $\Psi(a \otimes b) \cong \Psi a \otimes \Psi b$ and $\Phi \circ \Psi \cong 1$; finally, if $a \in \mathscr{V}$ is identified with the constant functor in a, then $[\mathbf{P}, \mathscr{V}](a * F, G) \cong \mathscr{V}(a, [F, G])$, where

$$[F, G] := \int_n \mathscr{V}(Fn, Gn). \qquad (136)$$

7. Higher dimensional coend calculus.

> Entonces desaparecerán del planeta el inglés y el francés y el mero español. El mundo será Tlön. Yo no hago caso, yo sigo revisando en los quietos días del hotel de Adrogué una indecisa traducción quevediana (que no pienso dar a la imprenta) del *Urn Burial* de Browne.
>
> J.L. Borges, *Tlön, Uqbar, Orbis Tertius*

Category theory was born in 1945 when Mac Lane and Eilenberg [**EM45**] isolated the correct definition of natural transformation between functors.

By doing this, they introduced the paradigmatic example of a 2-category: in this precise sense then *higher category theory* is a field as old as category theory itself. And yet, despite its age, it remains an area where even basic questions, burdened by an intrinsic computational difficulty, are still intricate, subtle and very challenging.

Even though for many years higher-dimensional category theory remained confined to well-defined geographical areas, the last 15 years witnessed a super-exponential growth of interest across several areas of mathematics: higher categorical structures have been recognized to lie at the heart of modern approaches to geometry [**TV05**], [**Lur09**], [**BZFN10**], logic [**Uni13**], topology and mathematical physics [**Sch13**]; the slow but constant diffusion of a dialect which is powerful enough to encompass all these developments, and yet sufficiently simple to be studied led to the present situation and led to a fairly general "reinterpretation" of known theories in a new language, inspired by homotopy theory. Higher category theory and the fundamental

constructions therein are hence interpreted in a homotopy-invariant way.[15] The process of passing from a 1-categorical (also called "classical" in the following) setting to an higher-categorical one can be seen as a process of "heightening".

Co/end calculus, as a part of the categorical toolbox, makes no exception and can be heightened (in fact, in several ways).

The scope of the present chapter is to give a compact but lucid presentation of this "higher co/end-fu". The struggle here is on two separate and opposing fields: on one side, "ancient" higher category theory [**GPS95, Hof11**] with its baroque equational approach to coherence conditions is a true nightmare, both for the listener and the exposer. On the other side the "new" approach to higher category theory based on homotopy theory (mainly that of simplicial sets) reinterprets those very coherence conditions allowing a precious bookkeeping device, which is nevertheless often too far from the taste of some practitioners of categorical algebra.

This simple observation does not distance the author from the current fashion and the current faith: the "homotopical" approach to higher categories has proved itself a valid tool to actually *do* beautiful mathematics, and speaks a subtle and intricate language, forbidden to the inhabitants of the 1-dimensional world. Nevertheless, we feel this is the right place to clarify our position in the battlefield.

At the moment of writing this note we (as a community) witness several attempts to acquire a deeper understanding of the landscape of current mathematics at the level of its fundamental architecture; category theory, with its indisputable unification power, can be encoded in homotopy theory, and this is part of a philosophical turnaround which takes the notion of *homotopy* as a primitive idea interacting with the similarly primitive notion of structure and operation on a space (seen from this perspective, sets are discrete groupoids and hence particular cases of topological spaces): there is a growing feeling that any attempt to rewrite a piece of "old" mathematics turning it into an "homotopically meaningful" statement is, in a suitable sense, a piece of higher category theory.

This is a peculiar, transitional moment in pure mathematics; several generations used to think in terms of set theory resist the revolution, thinking that the homotopy groups of spheres are too complicated an object to "lie at the foundation" of current mathematics. This point of view in some sense

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[15]One of the main tenets of higher category theory is that, whatever they are, these objects live in the world of (abstract) homotopy theory. There are several ways to justify this apparently strange remark, but this margin is too narrow to contain any of them: the starting point for most of them is that the nerve functor N_i (as described in **§3**) provides a fully faithful embedding **Cat** $\subseteq$ **sSet**; the wild variety of model structures on the category of simplicial sets becomes then a tool to better understand category theory.

indisputable, and yet the author feels that it underlies a subtle epistemological position (what is "simple mathematics"? What has, and what has not, the right to be considered a primitive entity? What is, in the end, the "right" primitive entity to choose for a foundation of mathematics?) which has no hope to be solved ever (so a fortiori, not in the space of this paragraph).

At the very least, this suggests that the extremely positive attitude towards the outreach of category theory among the barbarians[16] must be taken with a grain of salt. Categorical jargon gets more and more acknowledged as more and more applications for it are found, and yet it still suffers from a certain ostracism by some parts of the mathematical community. This is not, or at least not completely, their fault, and results in a generalized discomfort.

Together with the development of the language, nobody cared about a complementary development of self-contained presentations, as elementary as possible, forgetting how this is essential for a rapid and " healthy " spread of ideas. The canonical sources in which by now higher category theory has been distilled are difficult and time-consuming readings; this leaves many people behind. The technical subtleties, the intrinsic difficulty of simplicial homotopy theory and the lack of a meta-language shared by its users makes the big picture visible only to a handful of people, often the same creators of the theory; this is a deterrent for others to come in.

In this age where categorical language is expanding, it is more necessary than ever to write, to write well, and write simply, addressing to everyone. It is a deep author's conviction that by doing this the overall health of mathematical practice significantly improves, as well as the prompt and faithful diffusion of profound ideas that would otherwise be exclusively dedicated to an élite. Rather, it is the consequence of a precise historical moment and ideology. This kind of commitment, carried out in the past, allowed generations of totally unprepared students to appreciate the depth of rich and complex definitions (now for real, undoubtedly, a root of the mathematical language essential to every practitioner).

As in all "critical" moments –in which you carefully consider the risks and benefits of a language-shift– we should be oriented towards ideological openness and a multilingual attitude: this simple pedagogical principle is unfortunately often forgotten by innovators of all ages.

It is difficult to foresee where the current landscape will lead us; eventually, the world will become Tlön. All we can do then is wait, and have fun learning something beautiful.

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[16]The word here only means "the people living outside the *polis* of category theory", without any pejorative meaning.

7.1. **2-coends.** One of the most immediate generalization of co/end-fu lives in the world of 2-categories, where "1-cells" are allowed to be transformed by "2-cells".

In view of Prop. **1.2**, to define 2-co/ends we must exploit some flavour of limit and colimit in 2-category theory: this calculus has a rather natural interpretation in terms of enriched category theory ([**KS74**] and [**Kel89**], for example, offer an invaluably complete survey on this topic, but the reader should be aware that 2-category theory is *not* the theory of **Cat**-enriched categories). Here the reader is also warned that the following discussion has little or no hope to be a self-contained exposition, and instead heavily relies on classical sources as [**Kel82, Dub70**].

Notation 7.1 : As always when dealing with higher dimensional cells and their compositions, there are several "flavours" in which one can weaken strict commutativity: besides this strictness (where every diagram commutes with an implicit identity 2-cell filling it), there is a notion of *strong* commutativity and universality, where filling 2-cells are requested to be invertible, and *lax* commutativity/universality, where 2-cells are possibly non-invertible.[17]

The definition of *lax end* of a 2-functor S (given in terms of a universal *lax wedge* $\omega\colon b \xdashrightarrow{} S$) is the most general and less symmetric one can give (it can still be dualized to a colax coend, though!); definitions and theorems in this subsection are designed and proved in such a way to reduce to the strong and strict cases as particular examples of lax co/ends, where filling 2-cells are invertible (resp., identities).

Local notation. For the rest of the section we adopt some local conventions: first of all, we denote $\mathbf{A} \dashrightarrow \mathbf{B}$ a lax functor between two 2-categories $\mathbf{A}, \mathbf{B}$; as sketched above, this means that we have a correspondence $F_0\colon \mathrm{Ob}(\mathbf{A}) \to \mathrm{Ob}(\mathbf{B})$ and a correspondence $F_1\colon \hom_1(\mathbf{A}) \to \hom_1(\mathbf{B})$ at the level of 1-cells, such that there exist "laxity cells" $F(g)F(f) \Rightarrow F(gf)$ and $\mathrm{id}_{F_0a} \to F_1(\mathrm{id}_a)$, satisfying "obvious" coherence conditions.

This said, 2-category theory exists in many dialects: we mainly follow a natural and auto-explicative notation based on the canonical reference [**Kel89**], but we feel free to diverge from it from time to time: the operations of *whiskering* of an higher cell with a lower cell is denoted with the symbol $*$, so that $F * \alpha\colon FH \Rightarrow FK$ and $\alpha * G\colon HG \Rightarrow KG$ for natural transformation $\alpha\colon H \Rightarrow K$ and suitable functors F, G.

Definition 7.2 : Let $S\colon \mathbf{A}^{\mathrm{op}} \times \mathbf{A} \to \mathbf{B}$ be a strict 2-functor between strict 2-categories. A *lax wedge* based at S consists of a triple $\{b, \underline{\omega}_{\mathrm{Ob}}, \underline{\omega}_{\hom}\}$, where

---

[17]Obviously there are two possible choices for a direction in which a non-invertible "laxity cell" can go: a map endowed with a canonical 2-cell $F(g)F(f) \Rightarrow F(gf)$; the same correspondence, with 2-cells $F(gf) \Rightarrow F(g)F(f)$ is called *op*lax functor. The reader will find a personal mnemonic trick to remember how this nomenclature is chosen.

$b \in \mathrm{Ob}(\mathbf{B})$ (the *tip* of the wedge) and collections of 1-cells $\{\omega_a \colon b \to S(a,a)\}$, one for each $a \in \mathrm{Ob}(\mathbf{A})$, and 2-cells $\{\omega_f \colon S(a,f) \circ \omega_a \Rightarrow S(f,a') \circ \omega_{a'}\}$, in a diagram
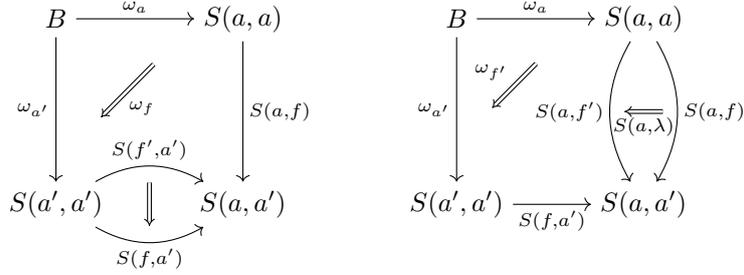
These data must fit together in such a way that the following coherence axioms, expressed by the commutation and pasting of the following diagrams of 2-cells, are satisfied:

(1) The diagram of 2-cells
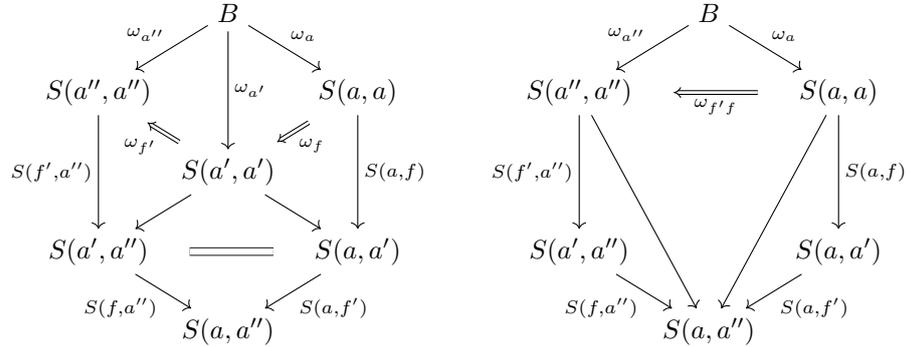
is commutative for any $\lambda \colon f \Rightarrow f'$, i.e. the equation

$$\omega_{f'} \circ (S(a,\lambda) * \omega_a) = (S(\lambda,a') * \omega_{a'}) \circ \omega_f \tag{137}$$

holds.

(2) For each pair $a \xrightarrow{f} a' \xrightarrow{f'} a''$ of composable arrows in $\mathbf{A}$, the diagram of 2-cells

is commutative, i.e. the equation

$$(S(f, a'') * \omega_{f'}) \circ (S(a, f') * \omega_f) = \omega_{f'f} \tag{138}$$

holds.

(3) For each $a \in \mathbf{A}$, $\omega_{\mathrm{id}_a} = \mathrm{id}_{\omega_a}$.

Notation 7.3 : A lax wedge will be often denoted $\underline{\omega}\colon b \xrightarrow{\cdot\cdot} S$ for short; this is evidently reminiscent of our Def. **1.5** and [**ML98**].

Definition 7.4 [MODIFICATION]**:** A *modification* $\Theta\colon \omega \Rrightarrow \sigma$ between two lax wedges $\underline{\omega}, \underline{\sigma}\colon b \xrightarrow{\cdot\cdot} S$ for $S\colon \mathbf{A}^{\mathrm{op}} \times \mathbf{A} \to \mathbf{B}$ consists of a collection of 2-cells $\{\Theta_a\colon \omega_a \Rightarrow \sigma_a\}_{a\in\mathrm{Ob}(\mathbf{A})}$ such that the diagram of 2-cells

is commutative, i.e.

$$(S(a, f) * \Theta_a) \circ \omega_f = \sigma_f \circ (S(f, a') * \omega_{a'}) \tag{139}$$

The definition of a modification is modeled on the definition of modification between (lax, if necessary) natural transformations; modifications form the 3-cells of the 3-category 2-**Cat**, whose objects are 2-categories, 1-cells are (lax, if necessary) functors, 2-cells are (lax, if necessary) natural transformations.

Modifications follow rules for "whiskering" which are similar to those for natural transformations and 2-cells, only in higher dimension (and in a much complicated web of relations between all the possible compositions: there are now *three* possible directions in which 3-cells can be composed!).

Remark 7.5 : There is another more general definition for a modification $\Theta\colon \omega \Rightarrow \sigma$ between lax wedges having different domains, say $\{b, \underline{\omega}\}$ and $\{b', \underline{\sigma}\}$: it consists of a morphism $\varphi\colon b \to b'$ and a 2-cell $\lambda_a\colon \sigma_a \circ \varphi \Rightarrow \omega_a$ such that

$$(\sigma_f * \varphi) \circ (S(a, f) * m_a) = (S(f, a') * m_{a'}) \circ \omega_f \qquad (140)$$

(draw the corresponding diagram of 2-cells!). Nevertheless, we are not interested in this alternative definition; Def. **7.4** entails that the set $\mathsf{LFun}(b, S)$ of lax wedges $b \xdashrightarrow{} S$ is a category having morphisms precisely the modifications $\Theta\colon \underline{\omega} \Rightarrow \underline{\sigma}$, and the correspondence $\beta_S\colon b \mapsto \mathsf{LFun}(b, S)$ is functorial. The definition of *lax end* for S relies on the representability of this 2-functor.

Definition 7.6 [Lax end of S]**:** Let $S\colon \mathbf{A}^{\mathrm{op}} \times \mathbf{A} \to \mathbf{B}$ be a 2-functor; a lax wedge $\underline{\omega}\colon b \xdashrightarrow{} S$ is called the *lax end* of S, and denoted $\oiint_a S(a, a) \xdashrightarrow{} S$ if for any other lax wedge $\underline{\sigma}\colon b' \to S$ there exists a single 1-cell $x\colon b' \to b$ between the tips of the wedges such that

$$\omega_a \circ x = \sigma_a, \qquad\qquad \omega_f * x = \sigma_f$$

i.e. the diagram of 2-cells

commutes, and if every modification $\Theta\colon \underline{\sigma} \Rightarrow \underline{\sigma}'$ induces a unique 2-cell $\lambda\colon x \Rightarrow x'$ (x' is the arrow induced by $\underline{\sigma}'$) in such a way that $\lambda * \omega_a = \Theta_a$. This realizes the isomorphism of categories between lax wedges $b \xdashrightarrow{} S$ and $\mathbf{B}(b, \oiint_a S(a, a))$.

We denote, with an evident and harmless abuse of notation,

$$B_S = \oiint_a S(a, a). \qquad (141)$$

Remark 7.7 : The notation chosen for $\oiint_a S$ has a meaning: ideally, the n-co/end operation is depicted by an integral symbol (accordingly super- or subscripted) overlapped by an 2^n-agon; in this way, a 2-end has the right to be denoted as a "square-integral" $\oiint$, and a ∞-end (see Def. **7.31**) should be denoted as a $\oint$ symbol, the circle being a polygon with an infinite number of sides.

7.1.1. **Lax co/end calculus.** Several *kata* of coend-fu proved in our §**1** and §**2** remain true after a proper "laxification", justifying the intuition of lax co/ends as the right 2-categorical generalization of strict co/ends. We

collect the most notable examples of this phenomenon in the rest of the section; the content of **7.1.2** surely deserves a special mention, as well as other remarks chosen to convey a sense of continuity and analogy. In **7.1.2** we prove that the lax counterpart of the ninja Yoneda lemma **2.1** provides a reflection (using coends) and a reflection (using ends) of the category of strong presheaves into the category of lax presheaves.

Example 7.8 : The *comma objects* (f/g) [**Gra74**] of a 2-category **B** can be identified with the lax end of functors $\mathbf{2}^{\mathrm{op}} \times \mathbf{2} \to \mathbf{B}$ choosing the two 1-cells f, g; this is a perfect analogy of Exercise **1**.**E10**, in view of the characterization of the comma object (f/g) as a lax pullback in **Cat**.

Example 7.9 : If $F, G\colon \mathbf{A} \to \mathbf{B}$ are 2-functors, then the lax end of the functor

$$\mathbf{B}(F, G)\colon \mathbf{A}^{\mathrm{op}} \times \mathbf{A} \to \mathbf{Cat} \tag{142}$$

is given by the formula

$$\oint_a \mathbf{B}(Fa, Ga) \cong \mathsf{LFun}(\mathbf{A}, \mathbf{B})(F, G) \tag{143}$$

where $\mathsf{LFun}(\mathbf{A}, \mathbf{B})(F, G)$ is the set of lax natural transformations between lax functors $F, G\colon \mathbf{A} \to \mathbf{B}$ defined in [**Gra74**].

Proof. A lax wedge for the 2-functor $(a, a') \mapsto \hom(Fa, Ga')$ amounts to a square

$$
\begin{array}{ccc}
a & \xrightarrow{\ h\ } & a' \\
 & \searrow_{\alpha} \nearrow & \\
u \searrow & & \swarrow u' \\
 & c. &
\end{array}
$$

filled by a 2-cell $\tau_f\colon G(f)_* \circ \tau_a \Rightarrow F(f)^* \circ \tau_{a'}$. Each of the functors $\tau_a\colon \mathbf{E} \to \hom(Fa, Ga)$ sends $e \in \mathbf{E}$ into an object $\tau_a(e)$ such that

$$G(f) \circ \tau_a(e) \overset{\tau_f}{\Longrightarrow} \tau_{a'}(e) \circ F(f) \tag{144}$$

which is precisely what is needed to show that the correspondence $e \mapsto \{\tau_a(e)\}_{a \in \mathbf{A}}$ factors through $\mathsf{LFun}(\mathbf{A}, \mathbf{B})(F, G) \subseteq \prod_{a \in \mathbf{A}} \hom(Fa, Ga)$. $\qquad\square$

Lax natural transformations $\eta\colon F \dashrightarrow G$, described as the lax end above, can also be characterized as *lax limits* in the enriched sense: this motivates the search for a description of lax co/ends which is analogue to our **1.2**, where instead of strict co/equalizers we use [**Kel89**]'s notion of *co/inserter*.

For the ease of the discussion, we recall now how these lax limits are defined (see [**Kel89**, §**4**]):

Definition 7.10 : Let $f, g\colon x \to y$ be two parallel 1-cells in the 2-category **C**; the *inserter* $\mathsf{Ist}(f, g)$ is defined as a pair (p, λ) where $p\colon \mathsf{Ist}(f, g) \to x$ is a 1-cell in **C** and $\lambda\colon fp \Rightarrow gp$ is a 2-cell, universal with respect to the property of connecting fp, gp: this means that whenever we are given a diagram
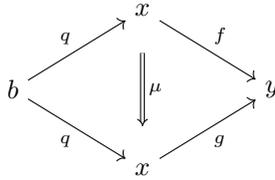
this can be split as the whiskering
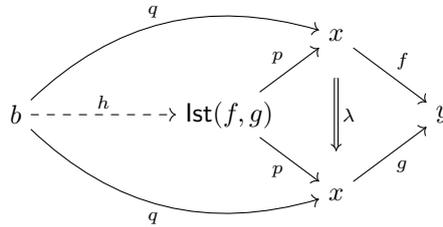
for a unique 1-cell $h\colon b \to \mathsf{Ist}(f, g)$ in **C**: this means, again, that $ph = q$ and $\lambda * h = \mu$.

This remark motivates the search for a description of lax co/ends as lax co/limits, on the same lines of our Remark **1.13**; this is the content of Prop. **7.19** in this section.

Moreover, as an analogue of the claim for 1-dimensional co/ends, we will prove that the lax co/end of a functor which is mute in a variable coincides with the lax co/limit of the same functor restricted to the unmuted component:

Remark 7.11 [COMMUTATION OF LAX LIMITS]**:** Strict limits are obviously particular cases of lax limits; since the classical argument, slightly modified to encompass the presence of non trivial laxity cells, applies to show that lax limits commutes with lax limits (in an obvious sense which we invite the keen reader to make precise), we obtain that lax co/limits (and lax co/ends) commute with strict co/limits (and lax co/ends).

This simple remark will be used all along the present section, and in a similar way we can deduce the "lax Fubini rule" for iterated lax co/ends: here is an ordered exposition for these results.

Proposition 7.12 [CO/ENDS OF MUTE FUNCTORS]**:** Suppose that the 2-functor $S\colon \mathbf{A}^{\mathrm{op}} \times \mathbf{A} \to \mathbf{B}$ is mute in the contravariant variable, i.e. that there is a

factorization $S = S' \circ p \colon \mathbf{A}^{\mathrm{op}} \times \mathbf{A} \xrightarrow{p} \mathbf{A} \xrightarrow{S'} \mathbf{B}$

$$\oint_a S(a,a) \cong q\text{-}\varprojlim S' \tag{145}$$

hence every lax co/limit can be computed as a lax co/end.

Example 7.13 : As a particular example of this, if $\mathbf{A}$ is locally discrete (i.e. identified with a locally small 1-category) and if the functor $S' \colon \mathbf{A} \to \mathbf{B}$ is constant, i.e. $S'(a) \equiv b$ for each $a \in \mathbf{A}$, then $\oint_a S$ is called *cotensor* of b by $\mathbf{A}$ and is denoted $b \pitchfork \mathbf{A}$.

Theorem 7.14 [PARAMETRIC LAX ENDS]**:** Whenever a functor $F \colon \mathbf{A}^{\mathrm{op}} \times \mathbf{A} \times \mathbf{B} \to \mathbf{C}$ is defined, and the lax end $\oint_a F(b,a,a)$ exists for every $B \in \mathbf{B}$, then $b \mapsto \oint_a F(b,a,a)$ extends to a 2-functor $\mathbf{B} \to \mathbf{C}$ which has the universal property of the lax end of its mate $\hat{F} \colon \mathbf{A}^{\mathrm{op}} \times \mathbf{A} \to \mathbf{C}^{\mathbf{B}}$ under the obvious adjunction.

Theorem 7.15 [FUBINI RULE FOR LAX CO/ENDS]**:** If one among the following lax ends exists, then so does the others, and the three are canonically isomorphic:

$$\oint_{b,c} T(b,c,b,c) \qquad \oint_b \left(\oint_c T(b,c,b,c) \right) \qquad \oint_c \left(\oint_b T(b,c,b,c) \right) \tag{146}$$

Corollary 7.16 [FUBINI RULE FOR LAX CO/LIMITS]**:** Lax limits commute: if $T \colon \mathbf{B} \times \mathbf{C} \to \mathbf{D}$ is a 2-functor, we have

$$q\text{-}\varprojlim_{b \in \mathbf{B}} \; q\text{-}\varprojlim_{c \in \mathbf{C}} \; T(b,c) \cong q\text{-}\varprojlim_{c \in \mathbf{C}} \; q\text{-}\varprojlim_{b \in \mathbf{B}} \; T(b,c). \tag{147}$$

7.1.2. The lax ninja Yoneda lemma. The lax analogue of Prop. **2.1** acquires an extremely particular flavour in this context, since it is the gist of an argument which shows the co/reflectivity of the category of *strict* presheaves $\mathbf{C}^{\mathrm{op}} \to \mathbf{Cat}$ in the category of lax presheaves $\mathbf{C}^{\mathrm{op}} \dashrightarrow \mathbf{Cat}$: there is a diagram of adjoint 2-functors

$$\mathsf{Fun}(\mathbf{C}^{\mathrm{op}}, \mathbf{Cat})$$

$$(-)^{\sharp} \Big\uparrow \dashv \Big\downarrow \dashv \Big\uparrow (-)^{\flat}$$

$$\mathsf{LFun}(\mathbf{C}^{\mathrm{op}}, \mathbf{Cat})$$

This means that for each strict 2-functor $H \in \mathsf{Fun}(\mathbf{C}^{\mathrm{op}}, \mathbf{Cat})$ there are two natural isomorphisms

$$\mathsf{Fun}(\mathbf{C}^{\mathrm{op}}, \mathbf{Cat})(H, F^{\flat}) \cong \mathsf{LFun}(\mathbf{C}^{\mathrm{op}}, \mathbf{Cat})(H, F),$$

$$\mathsf{Fun}(\mathbf{C}^{\mathrm{op}}, \mathbf{Cat})(F^{\sharp}, H) \cong \mathsf{LFun}(\mathbf{C}^{\mathrm{op}}, \mathbf{Cat})(F, H) \tag{148}$$

where the functors $F^\flat$ and $F^\sharp$ are defined by the lax coends

$$F^\sharp \cong \oint^a \mathbf{C}(\_,a) \times Fa \qquad\qquad F^\flat \cong \oint_a Fa^{\mathbf{C}(a,-)}. \tag{149}$$

Proof. The proof exploits Example **7.9** as well as the commutation of co/ends and lax co/ends, the preservation of lax co/ends by the hom functor, and the strict ninja Yoneda lemma:

$$
\begin{aligned}
\mathsf{Fun}(\mathbf{C}^{\mathrm{op}}, \mathbf{Cat})(H, F^\flat) &= \int_c \mathbf{Cat}(Hc, F^\flat c) \\
&= \int_c \mathbf{Cat}\left(Hc, \oint_a \mathbf{Cat}(\mathbf{C}(a,c), Fa)\right) \\
&\cong \int_c \oint_a \mathbf{Cat}(Hc, \mathbf{Cat}(\mathbf{C}(a,c), Fa)) \\
&\cong \oint_a \int_c \mathbf{Cat}(Hc \times \mathbf{C}(a,c), Fa) \\
&\cong \oint_a \mathbf{Cat}\left(\int^c Hc \times \mathbf{C}(a,c), Fa\right) \\
&\cong \oint_a \mathbf{Cat}(Ha, Fa) \overset{(\mathbf{7.9})}{=} \mathsf{LFun}(\mathbf{C}^{\mathrm{op}}, \mathbf{Cat})(H, F).
\end{aligned}
$$

The proof that $\mathsf{Fun}(\mathbf{C}^{\mathrm{op}}, \mathbf{Cat})(F^\sharp, H) \cong \mathsf{LFun}(\mathbf{C}^{\mathrm{op}}, \mathbf{Cat})(F, H)$ is done in a similar fashion. $\qquad\square$

Example 7.17 : Let $\mathbf{1}\colon \mathbf{C}^{\mathrm{op}} \dashrightarrow \mathbf{Cat}$ be the functor sending $c \in \mathbf{C}$ into the terminal category, regarded as a lax functor. Then the strict functor $\mathbf{1}^\sharp \colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Cat}$ is the lax coend

$$\mathbf{1}^\sharp(c) \cong \oint^a \mathbf{1}(a) \times \mathbf{C}(a,c) \cong \oint^a \mathbf{C}(a,c). \tag{150}$$

Hence the category $\oint^a \mathbf{C}(a,c)$ coincides with the lax colimit of the strict presheaf $\mathbf{C}^{\mathrm{op}} \to \mathbf{Cat}$, $a \mapsto \mathbf{C}(a,c)$, which is [**Str76**, p. **171**] the *lax slice category* $\mathbf{C}/\!\!/c$ of commutative diagrams of 2-cells

$$
\begin{array}{ccc}
B & \xrightarrow{\ \omega_a\ } & S(a,a) \\
{\scriptstyle \omega_{a'}}\downarrow & \overset{\displaystyle /\!\!/_{\omega_f}}{} & \downarrow{\scriptstyle S(a,f)} \\
S(a',a') & \xrightarrow[S(f,a')]{} & S(a,a')
\end{array}
$$

A long and straightforward unwinding of universal properties shows that the category $\mathbf{C}/\!\!/c$ enjoys the universal property of the lax colimit $q\text{-}\varinjlim \mathbf{C}(\_,c)$.

Remark 7.18 [THE TWISTED ARROW CATEGORY AS A LAX COLIMIT]**:** This is an interesting remark. It is possible to characterize the twisted arrow category of Def. **1.12** as the lax colimit of the diagram $a \mapsto \mathbf{A}/a$, i.e. as the lax coend

$$\oint^{a} \mathbf{A}/a. \tag{151}$$

This is another long and straightforward exercise in unwinding universal properties, that we leave to the reader.

Proposition 7.19 : [Boz80, §2] There is a canonical isomorphism between the lax end of a 2-functor $T\colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{B}$ and the limit of T weighted by the bifunctor $\mathbf{C}((\_)^{\sharp}, =)$, i.e.

$$\varprojlim{}^{\mathbf{C}((-)^{\sharp},=)}T \cong \oint_{a} T(a,a) \tag{152}$$

where $\mathbf{C}((\_)^{\sharp}, =)\colon (c,c') \mapsto \mathbf{C}(\_,c')^{\sharp}(c) = \oint^{a} \mathbf{C}(c,a) \times \mathbf{C}(a,c')$ is the *lax composition* of relators. A dual statement holds for lax coends.

Proof. The classical argument which exploits the conservativity of the Yoneda embedding applies: we can compute

$$\mathbf{B}\left(b, \varprojlim{}^{\mathbf{C}((-)^{\sharp},=)}T\right) \overset{(\mathbf{4.20})}{=} \varprojlim{}^{\mathbf{C}((-)^{\sharp},=)}\mathbf{B}(b,T)$$

$$(\mathbf{4.10}) \cong \int_{c,d} \mathbf{B}(b,T(c,d))^{\mathbf{C}(c^{\sharp},d)}$$

$$= \int_{c,d} \mathbf{B}(b,T(c,d))^{\oint^{a} \mathbf{C}(c,a) \times \mathbf{C}(a,d)}$$

$$\cong \oint_{a} \int_{c,d} \left(\mathbf{B}(b,T(c,d))^{\mathbf{C}(a,d)}\right)^{\mathbf{C}(c,a)}$$

$$(\mathbf{7.15}) \cong \oint_{a} \int_{c} \mathbf{Cat}\left(\mathbf{C}(c,a), \int_{d} \mathbf{B}(b,T(c,d))^{\mathbf{C}(a,d)}\right)$$

$$\cong \oint_{a} \int_{c} \mathbf{Cat}\left(\mathbf{C}(c,a), \mathbf{B}(b,T(c,a))\right)$$

$$\cong \oint_{a} \mathbf{B}(b,T(a,a)) = \mathbf{B}\left(b, \oint_{a} T(a,a)\right). \qquad \square$$

We can define the *tensor* of a category $\mathbf{T}$ with an object $a \in \mathbf{A}$, denoted $\mathbf{T} \cdot a$ by means of a lax coend; it is characterized by the natural isomorphism (in x)

$$\mathbf{Cat}(\mathbf{T}, \mathbf{A}(a,x)) \cong \mathbf{A}(\mathbf{T} \cdot a, x). \tag{153}$$

Now, let $S\colon \mathbf{A}^{\mathrm{op}} \to \mathbf{Cat}$, $T\colon \mathbf{A} \to \mathbf{B}$ be two functors and suppose $\mathbf{B}$ has $\mathbf{Cat}$-tensors; then the lax coend of the 2-functor $\mathbf{A}^{\mathrm{op}} \times \mathbf{A} \xrightarrow{S \times T} \mathbf{Cat} \times \mathbf{B} \overset{\cdot}{\to} \mathbf{B}$

is called *q-tensor product* of S and T, denoted

$$S \otimes T =: \oint^a Sa \cdot Ta. \tag{154}$$

More fundamental and subtle results, like the reduction of co/ends to co/limits, and the preservation of co/ends by the hom functors, remain valid for lax co/ends:

Theorem 7.20 : In a 2-category $\mathbf{B}$, lax co/ends exist provided that $\mathbf{B}$ has co/comma objects and $\mathbf{Cat}$-co/limits.

Theorem 7.21 : The lax end of the functor $T \colon \mathbf{A}^{\mathrm{op}} \times \mathbf{A} \to \mathbf{B}$, if it exists, is uniquely determined by the natural isomorphism

$$\mathbf{B}\left(x, \oint_a T(a,a)\right) \cong \oint_a \mathbf{B}(x, T(a,a)) \tag{155}$$

for every object $x \in \mathbf{A}$. A dual statement holds for lax coends.

7.1.3. **Applications: 2-distributors and lax Kan extensions.** A *2-distributor* $\varphi \colon \mathbf{A} \rightsquigarrow \mathbf{B}$ is a 2-functor $\varphi \colon \mathbf{B}^{\mathrm{op}} \times \mathbf{A} \to \mathbf{Cat}$. lax coends gives a method to compose 2-distributors, as in the 1-dimensional case: more precisely, let

$$\mathbf{A} \overset{\varphi}{\rightsquigarrow} \mathbf{B} \overset{\psi}{\rightsquigarrow} \mathbf{C} \tag{156}$$

be a couple of composable 2-distributors, namely two 2-functors $\varphi \colon \mathbf{B}^{\mathrm{op}} \times \mathbf{A} \to \mathbf{Cat}$ and $\psi \colon \mathbf{C}^{\mathrm{op}} \times \mathbf{B} \to \mathbf{Cat}$; then the composition $\psi \diamond \varphi$ is defined by the coend

$$\psi \diamond \varphi(c, a) = \oint^b \varphi(b, a) \times \psi(c, b) \tag{157}$$

The compatibility between lax colimits and products ensures that the expected associativity holds up to a canonical identification:

$$(\omega \diamond \psi) \diamond \varphi \cong \omega \diamond (\psi \diamond \varphi) \tag{158}$$

for any three $\mathbf{A} \overset{\varphi}{\rightsquigarrow} \mathbf{B} \overset{\psi}{\rightsquigarrow} \mathbf{C} \overset{\omega}{\rightsquigarrow} \mathbf{D}$.

Let $T \colon \mathbf{A} \to \mathbf{B}$ and $\varphi \colon \mathbf{A} \to \mathbf{C}$ be two 2-functors.

Definition 7.22 : We call *(left) quasi-Kan extension* of φ along T a 2-functor $\mathrm{Lan}_T \varphi \colon \mathbf{B} \to \mathbf{C}$ endowed with a quasi-natural transformation $a \colon \varphi \Rightarrow \mathrm{Lan}_T \varphi \circ T$ (a *unit*) such that, for each 2-functor $S \colon \mathbf{B} \to \mathbf{C}$ endowed with $\lambda \colon \varphi \Rightarrow S \circ T$ there exists a unique $\mathbf{Cat}$-natural transformation $\zeta \colon \mathrm{Lan}_T \varphi \Rightarrow S$ such that

$$(\zeta * T) \circ \alpha = \lambda \tag{159}$$

and moreover, if $\Sigma \colon \lambda \Rightarrow \lambda'$ is a modification between quasi-natural transformations, there is a unique modification $\Omega \colon \zeta \Rightarrow \zeta'$ (where ζ is induced by λ,

and ζ' by λ') between **Cat**-natural transformations such that $(\Omega * T) \circ \alpha = \Sigma$. This can be expressed with the isomorphism

$$\mathsf{Fun}(\mathbf{A}, \mathbf{C})\Big[\varphi, S \circ T\Big] \cong \mathbf{C}^{\mathbf{B}}\Big[\operatorname{Lan}_T \varphi, S\Big] \tag{160}$$

which is natural in S.

Example 7.23 : The quasi-Kan extension of a 2-functor $\varphi \colon \mathbf{A} \to \mathbf{C}$ along the trivial 2-functor $\mathbf{A} \to \mathbf{1}$ is the lax colimit of φ.

Remark 7.24 : We can obtain different notions of quasi-Kan extension by reversing the directions of α, λ, ζ etc.

The example above, as well as the following theorem, shows that the choice of **Cat**-natural transformations instead of quasi-natural transformations is the right choice (see also [**Boz80**] for a dual statement):

Theorem 7.25 : Notations as above. If $\mathbf{C}$ has at least tensors $\mathbf{B}(Ta, b) \cdot \varphi a'$ for each $a, a' \in \mathbf{A}$, and the lax coends

$$\oint^a \mathbf{B}(Ta, b) \cdot \varphi a \tag{161}$$

then the quasi-Kan extension of φ along T exists, and it is given by the formula above.

We can mimick also Exercise **2**.**E1** to obtain a lax analogue of it:

Proposition 7.26 : Let $\mathsf{LNat}(U, V)$ denote the category of lax natural transformations between two 2-functors U, V. Then

$$\mathsf{LNat}(F \times G, H) \cong \mathsf{LNat}(F, H^G), \tag{162}$$

where $H^G(x) = \mathsf{LNat}(\text{よ}(A) \times G, H) = \oint_y \mathbf{Sets}(\hom(y, x) \times Gy, Hy)$

Proof. It is a computation in coend-fu, and every step can be motivated by results in the present section: calculemus.

$$\begin{aligned}
\mathsf{LNat}(F, H^G) &= \oint_x \mathbf{Sets}(Fx, \mathsf{LNat}(\text{よ}(A) \times G, H)) \\
&\cong \oint_x \oint_y \mathbf{Sets}(Fx, \mathbf{Sets}(\hom(y, x) \times Gy, Hy)) \\
&\cong \oint_y \mathbf{Sets}\Big(\Big(\oint^x Fx \times \hom(y, x)\Big) \times Gy, Hy\Big) \\
&\cong \oint_y \mathbf{Sets}(Fy \times Gy, Hy) \\
&= \mathsf{LNat}(F \times G, H). \qquad \qquad \square
\end{aligned}$$

7.2. **Homotopy coends and ∞-coends.** Higher category theory is now living a Renaissance, thanks to a massive collaboration of several people drawing from various fields of research, and cooperating to re-analyze every feature of category theory in the topos of simplicial sets.

Several reasons, and the urge to keep this chapter finite-dimensional force us to take for granted a certain acquaintance with the language of ∞-categories *à la* Joyal-Lurie, but we also try to offer at least an intuition for what's going on and why things are done that way. The reader seeking a deeper understanding of this topic is advised to quit their job and move to Mojave desert with [**Lur09, Joy08, Joy02a, JT07**] and a bag full of good weed.

Here we present the theory of *homotopy co/ends* in model category theory, and then we move to the theory of *simplicially coherent* and *quasicategorical* co/end calculus. A final paragraph explores the definition of a co/end in a *derivator*, and this concludes the discussion of co/end calculus in each of the most common models for higher categories (model categories, enriched categories, simplicial sets, derivators). We leave aside a rather important question, that is *model dependence*: for example, does an homotopy co/end correspond to an ∞-co/end if we pass from model categories to quasicategories?

In our discussion we follow the unique references available: [**Isa09**] for co/ends in model categories, [**CP97**] for **sSet**-enriched co/ends, and [**GHN15**] for quasicategorical ones. The definition of a coend in a derivator comes from the extensive treatment of category theory in Grothendieck's derivators started by M. Groth [**Gro13**].

7.2.1. **Co/ends in model categories.** One of the most important parts of model category theory (the study of those structures that set homotopy theory in a purely formal framework) is the study of homotopy co/limits.

It is a fact, inherent to the theory, that colimit functors $\varinjlim_{\mathbf{J}} \colon \mathbf{C}^{\mathbf{J}} \to \mathbf{C}$ are often quite ill-behaved with respect to a homotopical structure: such a thing is determined by the specification of a distinguished class of arrows $\mathscr{W} \subseteq \hom(\mathbf{C})$ (these are called *weak equivalences*) which is the class of isomorphisms in a "localization" of $\mathbf{C}$. If $\mathbf{C}$ has such a structure, then every category $\mathbf{C}^{\mathbf{J}}$ acquires an analogous structure $\mathscr{W}^{\mathbf{J}}$ where $\eta \colon F \Rightarrow G$ is in $\mathscr{W}^{\mathbf{J}}$ if and only if each component $\eta_j \colon Fj \to Gj$ is in $\mathscr{W}$.

It is a fact that the image of such a natural transformation $\eta \colon F \Rightarrow G$ under the colimit functor, $\varinjlim \eta \colon \varinjlim F \to \varinjlim G$ is not always a weak equivalence.[18]

---

[18]A minimal instructive example goes as follows: take $\mathbf{J}$ to be the generic span $1 \leftarrow 0 \to 2$ and the functor sending it to $* \leftarrow S^{n-1} \to *$; the colimit of F is the terminal space $*$. We can replace F with the diagram $D^2 \leftarrow S^{n-1} \to D^2$, and since disks are contractible there

This is an unavoidable feature of the colimit functor $\varinjlim_{\mathbf{J}} \colon \mathbf{C}^{\mathbf{J}} \to \mathbf{C}$. One of the main tenets of homotopy theory is, nevertheless, that it doesn't matter if we replace an object with another, as soon as the two yield equivalent results. There is hope, then, that the *category* of functors $\mathbf{C}^{\mathbf{J}}$ contains a better-behaved representative for the functor $\varinjlim$, and that the two are linked by some sort of weak- or homotopy equivalence.

That's what a homotopy colimit is: a deformation $\underrightarrow{\mathrm{holim}}$ of $\varinjlim$ that preserves pointwise weak equivalences. And this is a general procedure in homotopy theory, where most objects X are not "compatible" with the homotopical structures we superimpose on our category of spaces, and yet one is often able to find better-behaved representatives $\tilde{X}$ in the same homotopy class.

This said, there are mainly two ways to link co/end calculus and homotopy theory:

- In nice situations, homotopy co/limits can be computed as co/ends: the first attempt to clarify this construction was given in [**BK72**]; nice explanatory surveys about this theory (touching also homological algebra) are [**Hör14**] and [**Gam10**].
- The co/end functor $\int \colon \mathbf{Cat}(\mathbf{C}^{\mathrm{op}} \times \mathbf{C}, \mathbf{D}) \to \mathbf{D}$ (as a particular colimit) can be "derived" yielding an *homotopy co/end* functor $\int \colon \mathbf{Cat}(\mathbf{C}^{\mathrm{op}} \times \mathbf{C}, \mathbf{D}) \to \mathbf{D}$ that preserves weak equivalences. This perspective, which is not independent from [**CP97**]'s point of view, is expanded in [**DF78**] and [**Isa09**].

These are respectively a co/end calculus *applied to* model categories and *interpreted in* model categories. The interplay and mutual completion of these two perspectives is evident.

Remark 7.27 : Let $\boxtimes \colon \mathbf{A} \times \mathbf{B} \to \mathbf{C}$ be the T part of a THC situation (see **3.13**), which is moreover left Quillen, and let $\mathscr{I}$ be a Reedy category ([**Hov99**, Def. **5.2.1**]). Then the coend functor

$$\int \colon \mathbf{Cat}(\mathscr{I}^{\mathrm{op}}, \mathbf{A}) \times \mathbf{Cat}(\mathscr{I}, \mathbf{B}) \to \mathbf{C} \tag{163}$$

is a left Quillen bifunctor if we regard the functor categories $\mathbf{Cat}(\mathscr{I}^{\mathrm{op}}, \mathbf{A})$ and $\mathbf{Cat}(\mathscr{I}, \mathbf{B})$ endowed with the Reedy model structure.

7.2.2. Co/ends in quasicategories.

Remark 7.28 : As a rule of thumb, the translation procedure from category to ∞-category theory is based on the following meta-principle: first you rephrase the old definition in a "simplicially meaningful" way, so that the

---

is a homotopy equivalence $\tilde{F} \Rightarrow F$; unfortunately, the induced arrow $\varinjlim \tilde{F} = S^2 \to *$ is not a weak equivalence.

∞-categorical definition specializes to the old one for quasicategories $N(\mathbf{C})$ which arise as nerves of categories. Then you forget about the original gadget and keep the simplicial one; this turns out to be the right definition.

The first victim of this procedure is the twisted arrow category **1.12** of an ∞-category.

Definition 7.29 [Twisted arrow ∞-category]**:** Let $\varepsilon\colon \mathbf{\Delta} \to \mathbf{\Delta}$ be the functor $[n] \mapsto [n] \star [n]^{\mathrm{op}}$, where $\star$ is the join of simplicial sets [**Joy08, EP08**]. Let $\mathbf{C}$ be a ∞-category; the twisted arrow category $\mathrm{TW}(\mathbf{C})$ is defined to be the simplicial set $\varepsilon^*\mathbf{C}$, where $\varepsilon^*\colon \mathbf{sSet} \to \mathbf{sSet}$ is the induced functor. More explicitly, and consequently, the n-simplices of $\mathrm{TW}(\mathbf{C})$ are characterized by the relation

$$\mathrm{TW}(\mathbf{C})_n \cong \mathbf{sSet}(\Delta[n], \mathrm{TW}(\mathbf{C})) \cong \mathbf{sSet}(\Delta[n] \star \Delta[n]^{\mathrm{op}}, \mathbf{C}). \tag{164}$$

The most important feature of the twisted arrow category is that it admits a fibration over $\mathbf{C}^{\mathrm{op}} \times \mathbf{C}$ (part of its essential properties can be deduced from this); the machinery of left and right fibrations exposed in [**Lur09**, Def. **2.0.0.3**] gives that

- There is a canonical simplicial map $\Sigma\colon \mathrm{TW}(\mathbf{C}) \to \mathbf{C}^{\mathrm{op}} \times \mathbf{C}$ (induced by the two join inclusions $\Delta[\_], \Delta[\_]^{\mathrm{op}} \to \Delta[\_] \star \Delta[\_]^{\mathrm{op}}$);
- This ∞-functor is a right fibration in the sense of [**Lur09**, Def. **2.0.0.3**].

Remark 7.30 : It is rather easy to see that a 0-simplex in $\mathrm{TW}(\mathbf{C})$ is an edge $f\colon \Delta[1] \to \mathbf{C}$, and a 1-simplex of $\mathrm{TW}(\mathbf{C})$ is a 3-simplex thereof, that we can depict as a pair of edges (u, v), such that the square having twisted edges

$$
\begin{array}{ccc}
 & \xleftarrow{\ u\ } & \\
f \downarrow & & \downarrow f' \\
 & \xrightarrow[\ v\ ]{} &
\end{array}
\tag{165}
$$

commutes. This suggest (as it must be) that the definition of $\mathrm{TW}(\mathbf{C})$ for a ∞-category specializes to the 1-dimensional one and adds higher-dimensional informations to it.

Definition 7.31 : Let $\mathbf{C}, \mathbf{D}$ be two ∞-categories; the *co/end* of a simplicial map $F\colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{D}$ is the co/limit of the composition

$$\mathrm{TW}(\mathbf{C}) \xrightarrow{\Sigma} \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \xrightarrow{F} \mathbf{D} \tag{166}$$

The main interest of the authors in [**GHN15**] is to formulate an analogue of **4.5**, which characterizes the Grothendieck construction of a **Cat**-valued functor as a particular weighted colimit (see **4.22**).

It is rather easy to formulate such an analogue definition: this appears as [**GHN15**, Def. **2.8**].

Definition 7.32 [OP/LAX COLIMIT OF F]**:** Let $F\colon \mathbf{C} \to \mathbf{Cat}_\infty$ be a functor between ∞-categories. We define

- the *lax colimit* of F to be the coend

$$\oint^c \mathbf{C}_{c/} \times Fc \tag{167}$$

- the *oplax colimit* of F to be the coend

$$\oint^c \mathbf{C}_{/c} \times Fc \tag{168}$$

The Grothendieck construction associated to F, discussed in [**Lur09**] with the formalism of un/straightening functors results precisely as the oplax colimit of F. This is coherent with our **4.22** and **4.23**.

7.3. **Simplicially coherent co/ends.** All the material in the following subsection comes from [**CP97**]. Since we are forced to divert from [**CP97**]'s notation by our personal choices and a slight pedantry, we begin the exposition establishing a convenient notation and a series of useful short-hands. We decided to keep this introduction equally self-contained and simple, but we can't help but admit that

- there is a sheer amount of (unavoidable, and yet annoying) sins of omissions in this survey, basically due to the ignorance of the author; moreover, the price to pay to obtain a self-contained exposition is to deliberately ignore several subtleties, exposing the theory to several simplifications.
- there are newer and more systematic approaches to this topic, owing a great debt to [**CP97**] but capable to generalize sensibly their constructions; among many, the reader should consult the exceptionally clear [**Rie14, Shu06**]. All these references reduce the construction of a simplicially coherent co/end to the "unreasonably effective co/bar construction" [**Rie14**, Ch. **4**].

It is our sincere hope that this does not affect the outreach of this elegant but neglected flavour in which to do higher category theory, and the clumsy attempt to popularize an account of [**CP97**] has to be seen as an attempt to communicate how beautiful we find this writing, as it is (one of) the beginner(s) of *categorical homotopy theory.*

Local notation. All categories $\mathbf{A}, \mathbf{B}, \dots$ appearing in this subsection are enriched over $\mathbf{sSet} = [\boldsymbol{\Delta}^{\mathrm{op}}, \mathbf{Sets}]$. All of them possess the co/tensors (see Def. **2.3**) needed to state definitions and perform computations. These functors assemble into a THC situation (see Remark **3.13**) $\mathbf{t} = (\cdot, \hom, \pitchfork)$ where $\cdot\colon \mathbf{sSet} \times \mathbf{A} \to \mathbf{B}$ determines the variance of the other two functors. A useful shorthand to denote the functor $\pitchfork (K, A) = K \pitchfork A$ (especially when

it is necessary to save space or invoke the "exponential" behaviour of this operation) is A^K. We switch rather freely among these two notations.

Definition 7.33 [Totalization]**:** Let $Y^\bullet \colon \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta} \to \mathbf{B}$ be a simplicial-cosimplicial object; we define the *totalization* $\mathsf{tot}(Y_\bullet^\bullet)$ of $Y^\bullet$ to be the end

$$\Big(\int_{n \in \mathbf{\Delta}} \Delta[n] \pitchfork Y^n \Big)^\bullet \tag{169}$$

(it is a cosimplicial object). The totalization of $Y^\bullet$ is also denoted with the shorthand $\mathbf{\Delta}^\bullet \pitchfork Y^\bullet$ or similar.

Definition 7.34 [Diagonalization]**:** Let $X_\bullet \colon \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{B}$ be a bisimplicial object; we define the *diagonalization* $\mathsf{diag}(X_\bullet)$ of $X_\bullet$ to be the coend

$$\Big(\int^{n \in \mathbf{\Delta}} \Delta[n] \cdot X_n \Big)_\bullet \tag{170}$$

(it is a simplicial object) The diagonalization of $X_\bullet$ is also denoted with the shorthand $\mathbf{\Delta}^\bullet \cdot X_\bullet$ or similar.

Notation 7.35 [Chain co/product]**:** Let $\mathbf{A} \in \mathbf{sSet\text{-}Cat}$, and $\vec{x}_n = (x_0, \dots, x_n)$ the "generic n-tuple of objects" in $\mathbf{A}$; given additional objects $a, b \in \mathbf{A}$ define a bisimplicial set

$$\amalg \mathbf{A}[a|\vec{x}_n|b]_\bullet := \coprod_{x_0, \dots, x_n \in \mathbf{A}} \mathbf{A}(a, x_0) \times \mathbf{A}(x_0, x_1) \times \cdots \times \mathbf{A}(x_n, b).^{[19]} \tag{171}$$

Faces and degeneracies are induced, respectively, by composition and identity-insertion (see Exercise **7.E4**).

Finally we define the simplicial set $\delta\mathbf{A}(a, b)$ to be $\mathsf{diag}(\amalg\mathbf{A}[a|\vec{x}_\bullet|b])$. Couched as a coend, $\delta\mathbf{A}(a, b)$ is written

$$\delta\mathbf{A}(a, b) \cong \int^{n \in \mathbf{\Delta}} \Delta[n] \times \amalg\mathbf{A}[a|\vec{x}_n|b]$$

$$= \int^{n \in \mathbf{\Delta}} \Delta[n] \times \coprod_{x_0, \dots, x_n \in \mathbf{A}} \mathbf{A}(a, x_0) \times \mathbf{A}(x_0, x_1) \times \cdots \times \mathbf{A}(x_n, b)$$

$$\cong \int^{n \in \mathbf{\Delta}} \coprod_{x_0, \dots, x_n \in \mathbf{A}} \Delta[n] \times \mathbf{A}(a, x_0) \times \mathbf{A}(x_0, x_1) \times \cdots \times \mathbf{A}(x_n, b)$$

---

[19]It is useful to extend this notation in a straightforward way: $\mathbf{A}[a|\vec{x}|b]$ denotes the product $\mathbf{A}(a, x_0) \times \mathbf{A}(x_0, x_1) \times \cdots \times \mathbf{A}(x_n, b)$, and $\Pi\mathbf{A}[a|\vec{x}|b]$, $\mathbf{A}[\vec{x}]$, $\Pi\mathbf{A}[\vec{x}]$, $\amalg\mathbf{A}[\vec{x}]$ are defined similarly. Note that $\amalg\mathbf{A}[a|\vec{x}_n|b]_\bullet$ does not depend on $\vec{x}_n$ since the coproduct is quantified over all such $\vec{x}_n$'s.

Definition 7.36 [THE FUNCTORS $\mathfrak{Y}$ AND $\mathfrak{W}$]: Let $T\colon \mathbf{A}^{\mathrm{op}} \times \mathbf{A} \to \mathbf{B}$ be a functor; we define $\mathfrak{Y}(T)^\bullet$ to be the cosimplicial object (in $\mathbf{B}$)

$$\mathfrak{Y}(T)^n := \prod_{\vec{x}=(x_0,\dots,x_n)} \mathbf{A}[\vec{x}_n] \pitchfork T(x_0, x_n)) \tag{172}$$

where $\mathbf{A}[\vec{x}_n] = \mathbf{A}(x, x_1) \times \cdots \times \mathbf{A}(x_{n-1}, x_n)$. Dually, given the same T, we define $\mathfrak{W}(T)_n$ to be the simplicial object (in $\mathbf{B}$)

$$\mathfrak{W}(T)_n := \coprod_{\vec{x}=(x_0,\dots,x_n)} \mathbf{A}[\vec{x}_n] \cdot T(x_0, x_n). \tag{173}$$

Example 7.37 : If we consider $\mathbf{A}$ to be trivially enriched (or as someone says, a *discrete* simplicial category), then the object $\mathbf{A}[a|\vec{x}_n|b]\bullet$ coincides with the nerve of the category $(a \downarrow \mathbf{A} \downarrow b)$ of arrows "under a and above b".

Definition 7.38 [SIMPLICIALLY COHERENT CO/END]: Let $T\colon \mathbf{A}^{\mathrm{op}} \times \mathbf{A} \to \mathbf{B}$ be a functor. We define

$$\oint_a T(a, a) := \int_{a', a''} T(a', a'')^{\delta\mathbf{A}(a', a'')} \quad \oint^a T(a, a) := \int^{a', a''} \delta\mathbf{A}(a', a'')\cdot T(a', a'')$$

$$\tag{174}$$

to be the *simplicially coherent co/end* of T.

In a few words, the definition of a **sSet**-coherent co/end involves the classical construction but adds to the scene "fattened up" co/tensors $\mathbf{A}[a|\vec{x}|b]$ and suchlike, organized as a functor $\mathbf{A}^{\mathrm{op}} \times \mathbf{A} \times \mathbf{A}^{\mathrm{op}} \times \mathbf{A} \to \mathbf{B}$ in such a way that the co/end is on two variables $a', a'' \in \mathbf{A}$.

Remark 7.39 [◉◉, COHERENT CO/ENDS AS DEFORMATIONS]: Example **4.16** gives that co/end are weighted co/limits, and precisely weighted co/limits with the hom weight. This perspective is useful here, as in some sense we are writing that $\oint_a T$ is the end $\int_{(a', a'') \in \mathbf{A}^{\mathrm{op}} \times \mathbf{A}} T(a', a'')^{\mathrm{hom}(a', a'')}$ where we applied a suitable "deformation" (or "resolution", or "replacement") functor δ to the hom functor $\mathbf{A}(\_, =)$, seen as the identity relator (Remark **5.7**). To some extent this point of view is explored in [**Gen15**] in the particular case where $\mathscr{V}\text{-}\mathbf{Cat} = \text{DG-}\mathbf{Cat}$; we have taken something from, as well as given something to, that document.

This perspective is of great importance to encompass coherent co/ends into a general theory "compatible" with some model structure on $\mathscr{V}\text{-}\mathbf{Cat}$, for some monoidal model $\mathscr{V}$ and the Bousfield-Kan model structure on $\mathscr{V}\text{-}\mathbf{Cat}$. Expanding this point of view, will be, hopefully, the subject of [**GL**].

Homotopy coherent calculus. Classical co/end calculus (in the triptych Fubini - Yoneda - Kan) is an invaluable tool (in fact, the only and most natural one) to prove several results even in the simplicial setting: as it is customary in the salons of higher category theory, we will always reduce a

computation involving a coherent, and yet indirectly defined object, to a computation involving a "concrete", and yet complicated object which takes into account the coherence introduced in the deformation.

[**CP97**] succeed in the indeed quite ambitious task to to rewrite *all* the most important pieces of classical category theory in this "higher" model (the paper contains a calculus for co/limits, mapping spaces, Yoneda, and Kan extensions). The aim of the rest of this subsection is to sketch some of these original definitions, hopefully helping an alternative formulation of $(\infty, 1)$-category theory (the authors of whom we owe a great debt, were it only for having proved –in 1997!– that the "∞-categorical dogma" can be avoided[20]) to escape oblivion: we do not claim to give a complete account of this theory here, and instead address the interested reader to their beautiful original paper.

Proposition 7.40 : Let $T\colon \mathbf{A}^{\mathrm{op}} \times \mathbf{A} \to \mathbf{B}$ be a **sSet**-functor. Then there is a canonical isomorphism

$$\oint_a T(a,a) \cong \mathsf{tot}(\mathfrak{Y}(T)^\bullet) \tag{175}$$

Proof. We use heavily the ninja Yoneda lemma **2.1** in its enriched form, where

$$\int_X \mathbf{A}(X,B) \pitchfork F(X) \cong F(B) \tag{176}$$

and the fact that $K \pitchfork (H \pitchfork A) \cong (K \otimes H) \pitchfork A$, naturally in all arguments.

With this remark in hand we can move to the real proof: $\vec{x} = (x_0, \ldots, x_n)$ is a generic tuples of objects of $\mathbf{A}$, and to save some space we switch to the notation A^K to denote $K \pitchfork A$.

$$\begin{aligned}
\oint_a T(a,a) &:= \int_{a',a''} T(a',a'')^{\delta \mathbf{A}(a',a'')} \\
&\cong \int_{a',a''} T(a',a'')^{\int_n \Delta[n] \times \mathrm{II}\mathbf{A}[a'|\vec{x}|a'']} \\
&\cong \int_{a',a'',n} T(a',a'')^{\Delta[n] \times \mathrm{II}\mathbf{A}[a'|\vec{x}|a'']} \\
&\cong \int_{a',a'',n} \left(T(a',a'')^{\mathbf{A}(x_n,a'')} \right)^{\Delta[n] \times \mathrm{II}\mathbf{A}[a'|\vec{x}]} \\
&\cong \int_{a',a'',n} \prod_{x_0,\ldots,x_n} \left(T(a',a'')^{\mathbf{A}(x_n,a'')} \right)^{\Delta[n] \times \mathbf{A}[a'|\vec{x}]}
\end{aligned}$$

---

[20]The subtle monophysism called "∞-categorical dogma" asserts that 'quasicategory' and '∞-category' are synonyms.

$$\cong \int_{a',n} \prod_{x_0,\dots,x_n} \left(\int_{a''} T(a',a'')^{\mathbf{A}(x_n,a'')} \right)^{\Delta[n] \times \mathbf{A}[a'|\vec{x}]}$$

$$\cong \int_{a',n} \prod_{x_0,\dots,x_n} T(a',x_n)^{\Delta[n] \times \mathbf{A}[a'|\vec{x}]}$$

$$\cong \int_{n} \prod_{x_0,\dots,x_n} \left(\int_{a'} T(a',x_n)^{\mathbf{A}(a',x_0)} \right)^{\Delta[n] \times \mathbf{A}[\vec{x}]}$$

$$\cong \int_{n} \left(\prod_{x_0,\dots,x_n} T(x_0,x_n)^{\mathbf{A}[\vec{x}_n]} \right)^{\Delta[n]} \cong \mathsf{tot}(\mathfrak{Y}(T)). \qquad \square$$

For the sake of completeness, we notice that the universal wedge testifying that $\oint_a T(a,a) \cong \mathsf{tot}(\mathfrak{Y}(T))$ is induced by the morphisms

$$\oint_a T(a,a) = \int_{a',a''} \delta\mathbf{A}(a',a'') \pitchfork T(a',a'')$$

$$\downarrow \wr$$

$$\int_{a',a'',n} \left(\Delta[n] \times \mathrm{II}\mathbf{A}[a'|\vec{x}|a''] \right) \pitchfork T(a',a'')$$

$$\downarrow$$

$$\Delta[n] \pitchfork \mathfrak{Y}(T)^n.$$

Prove the dual statement as an exercise (to finish the proof it is of vital importance to exploit a good notation):

Proposition 7.41 : Let $T \colon \mathbf{A}^{\mathrm{op}} \times \mathbf{A} \to \mathbf{B}$ be a **sSet**-functor. Then there is a canonical isomorphism

$$\oint^a T(a,a) \cong \mathsf{diag}(\mathfrak{W}(T)_\bullet) \tag{177}$$

7.3.1. **Simplicially coherent natural transformations.**

Remark 7.42 : The homotopy coherent co/ends admit "comparison" maps to the classical co/ends; this is part of a general tenet of higher category theory, where homotopically correct objects result as a *deformation* of classical ones, and this deformations maps into/out of the classical object.

The comparison map $\oint T(a,a) \to \int T(a,a)$ arises, here, as an homotopy equivalence between the simplicial set $\mathbf{A}(a,b)$ (seen as bisimplicial, and constant in one direction) and the bisimplicial set $\delta\mathbf{A}(a,b) = \mathsf{diag}\mathbf{A}[a| \bullet |b]_\bullet$: this is [**CP97**, p. 15].

The map

$$d_0 \colon \coprod_{x_0} \mathbf{A}(a,x_0) \times \mathbf{A}(x_0,b) \to \mathbf{A}(a,b) \tag{178}$$

given by composition has an homotopy inverse given by

$$s_{-1} \colon \mathbf{A}(a,b) \to \mathbf{A}(a,a) \times \mathbf{A}(a,b) \colon g \mapsto (\mathrm{id}_a, g). \tag{179}$$

Indeed, the composition $d_0 s_{-1}$ is the identity on $\mathbf{A}(a,b)$, whereas the composition $s_{-1} d_0$ admits is homotopic to the identity on $\delta\mathbf{A}(a,b)$ (we use the same name for the maps d_0, s_{-1} and the induced maps $\bar{d}_0 \colon \delta\mathbf{A}(a,b) \to \mathbf{A}(a,b)$, induced by the universal property, and $\bar{s}_{-1} \colon \mathbf{A}(a,b) \to \delta\mathbf{A}(a,b)$).

There is an important difference between these two maps, though: whilst d_0 is natural in both arguments, s_1 is natural in B but not in A. This has an immediate drawback: whilst d_0 can be obtained canonically, as the universal arrow associated to a certain natural isomorphism (see (**183**) below), s_{-1} can't (the best we can do is to characterize the natural argument of s_{-1} via [**CP97**, Example **2**, p. 16]).

As we have seen in **1.19**, the set of natural transformations between two functors $F, G \colon \mathbf{C} \to \mathbf{D}$ coincides with the end $\int_x \mathbf{D}(Fx, Gx)$, and (see **7.9**) the category of lax natural transformations between two 2-functors coincides with the lax end $\oint_x \mathbf{D}(Fx, Gx)$. It comes as no surprise, then, that the following characterization of *homotopy coherent* natural transformations between two simplicial functors hold:

Definition 7.43 [COHERENT NATURAL TRANSFORMATIONS]**:** Let $F, G \colon \mathbf{C} \to \mathbf{D}$ be two simplicial functors; then the simplicial set of *coherent transformations* between F and G is defined to be

$$\mathsf{Coh}(F,G) := \oint_a \mathbf{D}(Fa, Ga). \tag{180}$$

Definition 7.44 [MEAN TENSOR AND COTENSOR]**:** Let $F \colon \mathbf{A} \to \mathbf{B}$, $G \colon \mathbf{A} \to \mathbf{sSet}$, $H \colon \mathbf{A}^{\mathrm{op}} \to \mathbf{sSet}$. We define $G \,\overline{\pitchfork}\, F$, $H \underline{\otimes} F$ respectively as

$$G \,\overline{\pitchfork}\, F := \oint_a Ga \pitchfork Fa \qquad\qquad H \underline{\otimes} F := \oint^a Ha \otimes Fa. \tag{181}$$

Definition 7.45 [STANDARD RESOLUTIONS]**:** Let $F \colon \mathbf{A} \to \mathbf{B}$ be a simplicial functor; we define

$$\overline{F}a := \mathbf{A}(a, \_) \,\overline{\pitchfork}\, F = \oint_x \mathbf{A}(a,x) \pitchfork Fx$$

$$\underline{F}a := \mathbf{A}(\_, a) \underline{\otimes} F = \oint^x \mathbf{A}(x,a) \otimes Fx.$$

Example 7.46 : We specialize the above definition to compute the functors $\overline{\hom}(a, \_)$ and $\underline{\hom}(a, \_)$: in particular we concentrate on the second case, since the first is completely dual.

$$\underline{\hom}(a, b) = \oint^a \mathbf{A}(a,x) \times \mathbf{A}(x,b)$$

$$\cong \int^{xy} \delta\mathbf{A}(x,y) \times \mathbf{A}(a,x) \times \mathbf{A}(y,b)$$

$$\cong \int^{xy} \mathbf{A}(a,x) \times \delta\mathbf{A}(x,y) \times \mathbf{A}(y,b)$$

$$\cong \int^{xyn} \coprod_{x_0,\ldots,x_n} \mathbf{A}(a,x) \times \mathbf{A}(x,x_0) \times \cdots \times \mathbf{A}(x_n,y) \times \mathbf{A}(y,b) \times \Delta[n]$$

$$\cong \int^{xyn} \mathbf{A}[a|\tilde{x}_n|b] \times \Delta[n] \cong \delta\mathbf{A}(a,b).$$

We leave as an easy exercise in co/end-fu the proof of the following result (see Exercise **E5**), which shows that the standard resolutions $\underline{(\,\_\,)}, \overline{(\,\_\,)}$ 'absorb the coherence informations':

Proposition 7.47 : Let $F, G \colon \mathbf{C} \to \mathbf{D}$ be two simplicial functors; then there are canonical isomorphisms

$$\mathsf{Nat}(F, \overline{G}) \cong \mathsf{Coh}(F, G) \cong \mathsf{Nat}(\underline{F}, G). \tag{182}$$

This result has a number of pleasant consequences: the simplicially coherent setting is powerful enough to retrieve several classical constructions.

- Example **7.46** above shows that $\underline{\hom}(a, \_)(b) \cong \delta\mathbf{A}(a,b)$; this entails that there is an isomorphism

$$\mathsf{Nat}(\delta\mathbf{A}(a, \_), \mathbf{A}(a, =)) \cong \mathsf{Coh}(\mathbf{A}(a, \_), \mathbf{A}(a, =)) \tag{183}$$

  and it is a matter of verifying some additional nonsense to see that the **sSet**-natural transformation corresponding to the identity coherent transformation is precisely d_0.

- The map d_0 defines additional universal maps η_F, η^F which "resolve" a functor $F \colon \mathbf{A} \to \mathbf{B}$ whenever $\underline{F}, \overline{F}$ exist (it is sufficient that $\mathbf{B}$ admits all the relevant co/limits to perform the construction of $\underline{F}, \overline{F}$). From the chain of isomorphisms

$$\eta^F : \overline{F}b = \oint_a \mathbf{A}(b,a) \pitchfork Fa$$

$$\cong \int_{a',a''} \delta\mathbf{A}(a',a'') \pitchfork \mathbf{A}(b,a'') \pitchfork Fa'$$

$$\leftarrow \int_{a',a''} \mathbf{A}(a',a'') \pitchfork \mathbf{A}(b,a'') \pitchfork Fa'$$

$$(\mathbf{2.1}) \cong Fb;$$

$$\eta_F : \underline{F}b = \oint^a Fa \otimes \mathbf{A}(a,b)$$

$$\cong \int^{a',a''} Fa' \otimes \mathbf{A}(a'',b)\delta\mathbf{A}(a',a'')$$

$$\to \int^{a',a''} Fa' \otimes \mathbf{A}(a'',b)\mathbf{A}(a',a'')$$
$$\cong Fb;$$

we obtain natural transformations corresponding to suitable coherent identities under the isomorphism of Prop. **7.47**.

- The maps η_F, η^F behave like resolutions: [**CP97**, Prop. **3.4**] shows that they are level-wise homotopy equivalences (meaning that $\eta_F \colon Fa \to \overline{F}a$ induces homotopy equivalences of simplicial sets $\mathbf{B}(b, Fa) \xrightarrow{(\eta_F)_*} \mathbf{B}(b, \overline{F}a)$ for each b, naturally in b).[21]

7.3.2. **Simplicially coherent Kan extensions.** The universal property characterizing a Kan extension is inherently 2-dimensional: uniqueness is stated at the level of 2-cells, and any sensible generalization of it to the higher world involves a "space" of 2-cells between 1-cells. This entails that any reasonable definition of a (left or right) Kan extension ultimately relies on a nice definition for a space of coherent natural transformations between functors, which has been the subject of the previous subsection. There are, nevertheless, several subtleties as there are many choices available for a definition: in the words of [**CP97**],

> Clearly one can replace Nat by Coh [in the definition of a Kan extension], but should isomorphism be replaced by homotopy equivalence, should this be natural, in which direction should this go...?

As it turns out from [**CP97**], the right way to preserve a reasonably vast calculus for Kan extensions is to ask that the isomorphisms

$$\mathsf{Nat}(H, \operatorname{Ran}_G K) \cong \mathsf{Coh}(HG, K)$$
$$\mathsf{Nat}(\operatorname{Lan}_G H, K) \cong \mathsf{Coh}(H, KG)$$

hold. This can be achieved defining the left and right Kan extensions as follows:

Definition 7.48 [Coherent Kan extensions]**:** Let $F \colon \mathbf{A} \to \mathbf{C}$ and $G \colon \mathbf{A} \to \mathbf{B}$ be a span of simplicial functors; we define

$$\operatorname{Ran}_G F(\_) = \oint_a \mathbf{B}(\_, Ga) \pitchfork Fa$$
$$\operatorname{Lan}_G F(\_) = \oint^a \mathbf{B}(\_, Ga) \otimes Fa$$

---

[21]We decide to skip the proof of this proposition, as it is quite long, technical, and even though it relies on co/end-fu it doesn't add much to the present discussion.

Remark 7.49 : This can be seen as a simplicially coherent analogue of our **7.25**; it is not a coincidence that lax and simplicially coherent co/end calculi mimick each other: 2-co/ends correspond to suitable "truncated" simplicially coherent co/ends (and this correspondence can be made functorial).

In the same spirit of [**Boz80**], a co/endy view on categorical homotopy theory sheds a light on several geometric constructions (see [**CP97**] for more informations and links with [**CP90**, **Seg74**]).

We would like to prove, now, that the isomorphisms defining a Kan extension hold with the definitions above. This is a computation in co/end-fu, which at this point can be left as an exercise for the reader.

7.3.3. **Co/ends in a derivator.**   The theory of derivators serves as a purely 2-categorical model for higher category theory, where all the coherence informations are encoded in coherence conditions for suitable diagrams of 2-cells. Here we only sketch some of the basic definitions needed to pave the way to Def. **7.51** below.

Definition 7.50 [THE 2-CATEGORY PDER]**:** A *prederivator* is a strict 2-functor $\mathbf{D} \colon \mathbf{Cat}^{\mathrm{op}} \to \mathbf{CAT}$ (where $\mathbf{CAT}$ is the category of $\mho^+$-categories, see the two-universe convention in the introduction); a *morphism of prederivators* is a pseudonatural transformation between pseudofunctors, $\eta \colon \mathbf{D} \Rightarrow \mathbf{D}'$; a *2-cell* between morphisms of prederivators is a modification (see Def. **7.4**) $\Theta \colon \eta \Rrightarrow \eta'$ between pseudonatural transformations.

These data form the *2-category of prederivators.*

The notion of a *derivator* arises as a refinement of this; apart from some minor (milder, but not less important) assumptions, a derivator is a prederivator $\mathbf{D}$ such that every $\mathbf{D}(u) \colon \mathbf{D}(J) \to \mathbf{D}(I)$, induced by $u \colon I \to J$ has both a left and a right adjoint, fitting into a triple

$$u_! \dashv u^* \dashv u_* \colon \qquad \mathbf{D}(J) \mathrel{\substack{\xleftarrow{\quad u_! \quad} \\ \xrightarrow{\quad u^* \quad} \\ \xleftarrow{\quad u_* \quad}}} \mathbf{D}(I)$$

(see [**Gro13**, Def. **1.10**]). These functors are called respectively the *homotopy left and right Kan extensions along* $u \colon I \to J$. Axiom (Der4) in [**Gro13**, Def. **1.10**] states that these Kan extensions can always be computed with a *pointwise* formula; this can be interpreted as a rephrasing of the theory exposed in our **2.1**, in view of the equivalence between all the following characterizations of $\mathrm{Lan}_G F(b)/\mathrm{Ran}_G F(b)$:

- the (right or left) Kan extension of $F \colon \mathbf{A} \to \mathbf{C}$ along $G \colon \mathbf{A} \to \mathbf{B}$ computed in b;
- the weighted co/limit of F with respect to the representable $\hom(G, b)$;
- the conical co/limit of F over the category of elements of $\hom(G, b)$;

- the conical colimit of the diagram $(G \downarrow b) \to \mathbf{A} \xrightarrow{F} \mathbf{B}$.

Let $\mathrm{Tw}(K)$ be the category of elements (Def. **4.1**) of $\hom_K$, for a small category K; then there exists a functor $\Sigma_K = (t,s)\colon \mathrm{Tw}(K) \to K^{\mathrm{op}} \times K$ (Prop. **4.5**).

Definition 7.51 [Homotopy coend in a derivator]**:** Let $\mathbf{D}$ be a derivator, and $K \in \mathbf{Cat}$ a category. The *homotopy coend* $\int^K \colon \mathbf{D}(J \times K^{\mathrm{op}} \times K) \to \mathbf{D}(J)$ is defined as the composition

$$\int^K \colon \mathbf{D}(J \times K^{\mathrm{op}} \times K) \xrightarrow{\Sigma_K^*} \mathbf{D}(J \times \mathrm{Tw}(K)) \xrightarrow{\mathrm{p}_!} \mathbf{D}(J) \qquad (184)$$

Remark 7.52 : Let $\mathbf{D}(J|_-) \colon \mathbf{Cat}^{\mathrm{op}} \to \mathbf{CAT}$ be the *shifted derivator* of $\mathbf{D}$, i.e. the functor $I \xmapsto{-\times J} I \times J \xmapsto{\mathbf{D}(-)} \mathbf{D}(I \times J)$. Then the coend $\int^K$ is a morphism between the shifted derivators $\mathbf{D}(K^{\mathrm{op}} \times K|_-) \to \mathbf{D}(_-)$.

Remark 7.53 : There is obviously a similar notion of homotopy *end* in $\mathbf{D}$: one only has to replace $\mathrm{p}_!$ with the right adjoint p_* in the definition above (taking the *limit* over the twisted arrow category, instead of the colimit):

$$\int_K \colon \mathbf{D}(J \times K^{\mathrm{op}} \times K) \xrightarrow{\Sigma_K^*} \mathbf{D}(J \times \mathrm{Tw}(K)) \xrightarrow{\mathrm{p}_*} \mathbf{D}(J) \qquad (185)$$

Lemma 7.54 : If $F \colon \mathbf{D} \to \mathbf{D}'$ is a morphism of derivators, there is a canonical "comparison" morphism

$$\int^K \circ F \to F \circ \int^K \qquad (186)$$

obtained as the composition

where the second morphism results as the BC pasting

$$\mathrm{p}_! F_J \xRightarrow{\mathrm{p}_! F_J \eta_{(\mathrm{p}_! \dashv \mathrm{p}^*)}} \mathrm{p}_! F_J \mathrm{p}^* \mathrm{p}_! \xRightarrow{\mathrm{p}_! \varphi_J \mathrm{p}_!} \mathrm{p}_! \mathrm{p}^* F_e \mathrm{p}_! \xRightarrow{\epsilon_{(\mathrm{p}_! \dashv \mathrm{p}^*)} F_e \mathrm{p}_!} F e \mathrm{p}_! \qquad (187)$$

represented in the following diagram of 2-cells:

$$\begin{array}{ccccccc}
\mathbf{D}'(e) & \xleftarrow{\ p_!\ } & \mathbf{D}'(J) & \xleftarrow{\ F_J\ } & \mathbf{D}(J) & \xrightarrow{\quad\mathrm{id}\quad} & \\
& \epsilon_{(\mathrm{pr}_! \dashv \mathrm{pr}^*)} \Big\uparrow p^* & & & p^* \Big\uparrow & \eta_{(\mathrm{pr}_! \dashv \mathrm{pr}^*)} & \\
\mathrm{id} & & \mathbf{D}'(e) & \xleftarrow{\ F_e\ } & \mathbf{D}(e) & \xleftarrow{\ p_!\ } & \mathbf{D}(J)
\end{array}$$

It is almost a triviality that a derivator morphism F *preserves homotopy coends* (i.e. the above 2-cell is invertible) if it preserves colimits, or more generally Kan extensions.

Exercises for §7

E1 A *lax colimit* for a diagram $F\colon \mathcal{J} \to \mathbf{K}$ in a 2-category $\mathbf{K}$ is an object L with a *lax cocone* $\{Fj \to L\}$ satisfying a suitable universal property (state it, mimicking –the dual of– Def. **7.2**). Show that the opposite of the twisted arrow category $\mathrm{TW}(\mathbf{C})$ of Def. **1.12** is the lax colimit of the diagram $\mathbf{C} \to \mathbf{Cat}\colon c \mapsto \mathbf{C}/c$ sending every object to its slice category.

E2 State the definition of lax *cowedge* $S \overset{..}{\to} d$ for a 2-functor $S\colon \mathbf{A}^{\mathrm{op}} \times \mathbf{A} \to \mathbf{B}$; state the definition of lax coend for S as an *initial* cowedge, the representing object of the functor $d \mapsto \mathsf{LCwd}(S, d)$.

E3 Show that $\mathsf{lst}(f, g)$ share the universal property of the $\mathbf{Cat}$-limit of the diagram $\{0 \rightrightarrows 1\} \to \mathbf{C}$ choosing f, g weighted by the $\mathbf{Cat}$-presheaf $\{0 \rightrightarrows 1\} \to \mathbf{Cat}$ choosing the categories $\{0\} \overset{d_0}{\underset{d_1}{\rightrightarrows}} \{0 < 1\}$.

E4 Define co/faces and co/degeneracies for the objects $\mathfrak{Y}(T)$ and $\mathfrak{W}(T)$ (hint: there is an isomorphism $\tau\colon \Pi\mathbf{A}[\vec{x}] \pitchfork T(x_0, x_n) \cong \mathbf{A}[x_1|\vec{y}|x_n]_{n-1}\Big(\mathbf{A}(x_0, x_1) \pitchfork T(x_0, x_n)\Big)$, and you want to assemble a map $\mathfrak{Y}(T)^{n-1} \to \mathfrak{Y}(T)^n$ from its components $\Pi\mathbf{A}[\vec{x}] \pitchfork T(x_1, x_n) \to \mathfrak{Y}(T)^n$; this defines d^0. The map d^n is defined via an isomorphism σ and a similar argument).

E5 Prove that $\mathsf{Nat}(F, \overline{G}) \cong \mathsf{Coh}(F, G) \cong \mathsf{Nat}(\underline{F}, G)$, using Def. **7.38** and a formal argument.

E6 Prove that the isomorphisms

$$\mathsf{Nat}(H, \mathrm{Ran}_G K) \cong \mathsf{Coh}(HG, K)$$
$$\mathsf{Nat}(\mathrm{Lan}_G H, K) \cong \mathsf{Coh}(H, KG)$$

hold defining coherent Kan extensions as in **7.48**.

E7 Prove that $\int^K\colon \mathbf{D}(K^{\mathrm{op}} \times K|-) \to \mathbf{D}$ defines a morphism of derivators (you can either prove that a functor $u\colon K \to L$ induces a morphism between the shifted derivators $\mathbf{D}(L|-) \to \mathbf{D}(J|-)$, or prefer an explicit argument –both ways are considerably long).

E8 Prove that "coends in a derivator are pointwise", i.e. that given an arrow $j\colon e \to J$ there is a canonical isomorphism $j^*\big(\int^K X\big) \cong \int^K j^*X$ for each $X \in \mathbf{D}(J \times K^{\mathrm{op}} \times K)$.

E9 State and prove the Fubini theorem for homotopy coends in $\mathbf{D}$: the diagram

commutes for canonically determined 2-cells α and β.

E10 State and prove an existence theorem for *weighted colimits* in a derivator: given a bimorphism $\boxplus\colon (\mathbf{D}^{\mathbf{Sets}}, \mathbf{D}) \to \mathbf{D}(I|-)$, we define the colimit of $X \in \mathbf{D}(J)$, weighted by $W \in \mathbf{D}^{\mathbf{Sets}}$ as the coend (in $\mathbf{D}$) $\int^J W \boxplus X$, i.e. as the image of the pair (W, X) under the composition

$$\mathbf{D}^{\mathbf{Sets}}(J^{\mathrm{op}}) \times \mathbf{D}(J) \mathbin{\dot{\to}} \mathbf{D}(I|J^{\mathrm{op}} \times J) \xrightarrow{\int^J} \mathbf{D}(I). \tag{188}$$

Appendix A. Promonoidal categories

A *promonoidal category* is what we obtain taking the definition of a monoidal category and we replace every occurrence of the word *functor* with the word *profunctor* (here we refrain to adopt the name "relator" since *rel-monoidal* category is a terrible name).

More precisely, we define

Definition A.1 [Promonoidal structure on a category]**:** A *promonoidal category* consists of a category $\mathbf{C}$ which is a monoid object in $\mathbf{Relt}$, the category of profunctors defined in **5.1**.

A monoid object in $\mathbf{Relt}$ is a category $\mathbf{C}$, endowed with a bi-profunctor $P\colon \mathbf{C} \times \mathbf{C} \rightsquigarrow \mathbf{C}$ (the monoidal multiplication) and a profunctor $J\colon 1 \rightsquigarrow \mathbf{C}$ (the monoidal unit), such that the following two diagrams are filled by the indicated 2-cells (respectively, the *associator* and *left/right unitor*) in $\mathbf{Relt}$:

$$\tag{189}$$

These data form what is called a *promonoidal structure* on the category $\mathbf{C}$, denoted $\mathfrak{P} = (P, J, \alpha, \rho, \lambda)$.

Remark A.2 : Coend calculus allows us to turn the conditions

$$P \diamond (P \times \mathrm{hom}) \cong P \diamond (\mathrm{hom} \times P) \tag{190}$$

$$P \diamond (J \times \text{hom}) \cong \text{hom} \cong P \diamond (\text{hom} \times J) \tag{191}$$

giving the associativity and unit of the promonoidal structure into explicit relations involving the functors $P \colon \mathbf{C}^{\text{op}} \times \mathbf{C}^{\text{op}} \times \mathbf{C} \to \mathbf{Sets}$ and $J \colon \mathbf{C} \to \mathbf{Sets}$;[22] we have the following rules, written in Einstein notation:

- The associativity condition for $P \colon \mathbf{C} \times \mathbf{C} \rightsquigarrow \mathbf{C}$ amounts to saying that the following boxed sets, obtained as coends, are naturally isomorphic (via a natural transformation $\alpha_{abc;d}$ having four components, three contravariant and one covariant).

$$\begin{aligned}
(P \diamond (\text{hom} \times P))_{abc;d} &= \int^{xy} P_d^{xy} H_a^x P_y^{bc} \\
&\cong \int^y z \big(\int^x P_d^{xy} H_a^x \big) P_y^{bc} \\
&\cong \boxed{\int^z P_d^{ay} P_y^{bc}} \\
(P \diamond (P \times \text{hom}))_{abc;d} &\cong \int^{xy} P_d^{xy} H_y^c P_x^{ab} \\
&\cong \boxed{\int^z P_x^{ab} P_d^{xc}}.
\end{aligned}$$

- The left unit axiom is equivalent to the isomoprhism

$$(a,b) \mapsto \int^{yz} J_z H_y^a P_b^{yz} \int^z J_z \big(\int^y H_y^a P_b^{yz} \big) \cong \int^z J_z P_b^{az} \cong \text{hom}(a,b). \tag{192}$$

The most interesting feature of promonoidal structure in categories is that they correspond bijectively with monoidal structures on the category of functors $[\mathbf{C}, \mathbf{Sets}]$, heavily generalizing the Day construction of Definition **6.1**.

Proposition A.3 : Let $\mathfrak{P} = (P, J, \alpha, \rho, \lambda)$ be a promonoidal structure on the category $\mathbf{C}$; then we can define a $\mathfrak{P}$-*convolution* on the category $[\mathbf{C}, \mathbf{Sets}]$ (or more generally, on the category $[\mathbf{C}, \mathscr{V}]$), via

$$[F *_{\mathfrak{P}} G]_c = \int^{ab} P(a,b;c) \times Fa \times Gb \tag{193}$$

$$J_{\mathfrak{P}} = J \tag{194}$$

and this turns out to be a monoidal structure on $[\mathbf{C}, \mathbf{Sets}]$. We denote the monoidal structure $([\mathbf{C}, \mathbf{Sets}], *_{\mathfrak{P}}, J_{\mathfrak{P}})$ shortly as $[\mathbf{C}, \mathbf{Sets}]_{\mathfrak{P}}$.

Exercise A.4 : Prove the above statement using associativity and unitality for $\mathfrak{P}$.

Exercise A.5 [Day and Cauchy convolutions]**:** Outline the promonoidal structure $\mathfrak{P}$ giving the Day convolution described in Definition **6.1**. If $\mathbf{C}$ is any small category, we define $P(a,b;c) = \mathbf{C}(a,c) \times \mathbf{C}(b,c)$ and J to be the

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[22]Obvoiusly, there is nothing special about sets here. The whole discussion performed in the setting of $\mathscr{V}$-profunctors leads to the definition of a $\mathscr{V}$-promonoidal structure.

terminal functor $\mathbf{C} \to \mathbf{Sets}$. Outline the convolution product on $[\mathbf{C}, \mathbf{Sets}]$, called the *Cauchy convolution*, obtained from this promonoidal structure.

Definition A.6 : A functor $\Phi \colon [\mathbf{A}, \mathbf{Sets}]_{\mathfrak{P}} \to [\mathbf{B}, \mathbf{Sets}]_{\mathfrak{Q}}$ is said to *preserve the convolution product* if the obvious isomorphisms hold in $[\mathbf{B}, \mathbf{Sets}]_{\mathfrak{Q}}$:

- $\Phi(F *_{\mathfrak{P}} G) \cong \Phi(F) *_{\mathfrak{Q}} \Phi(G)$;
- $\Phi(J_{\mathfrak{P}}) = J_{\mathfrak{Q}}$.

Remark A.7 : It is observed in [**IK86**] that for a monoidal $\mathbf{A}$ the category of presheaves $[\mathbf{A}^{\mathrm{op}}, \mathscr{V}]$ endowed with the convolution monoidal structure is the *free monoidal cocompletion* of $\mathbf{A}$, having in $\mathbf{Mon}$ (monoidal categories, monoidal functors and monoidal natural transformations) the same universal property that $\widehat{\mathbf{A}}$ has in $\mathbf{Cat}$.

Appendix B. Fourier transforms via coends.

Definition B.1 : Let $\mathbf{A}, \mathbf{C}$ be two promonoidal categories with pormonoidal structures $\mathfrak{P}$ and $\mathfrak{Q}$ respectively; a *multiplicative kernel* from $\mathbf{A}$ to $\mathbf{C}$ consists of a profunctor $K \colon \mathbf{A} \rightsquigarrow \mathbf{C}$ such that there are the two natural isomorphisms

$k_1)$ $\int^{yz} K_y^a K_z^b P_x^{yz} \cong \int^c K_x^c P_c^{ab}$;

$k_2)$ $\int^c K_x^c J_c \cong J_x$.

These isomorphisms say that K "behaves like hom" even if it is not an endofunctor (in fact, hom is a particular example of a kernel $\mathbf{A} \rightsquigarrow \mathbf{A}$, since the isomorphisms k_1, k_2 follow from the ninja Yoneda lemma **2.1**). A *multiplicative* natural transformation $\alpha \colon K \to H$ is a 2-cell between profunctors commuting with the structural isomorphisms given in $(k_1), (k_2)$ in the obvious sense.

Exercise B.2 : Define the category of multiplicative kernels $\ker(\mathbf{A}, \mathbf{C}) \subset \mathbf{Relt}(\mathbf{A}, \mathbf{C})$ in such a way that the composition of two kernels is again a kernel.

Exercise B.3 : Show that a profunctor $K \colon \mathbf{A} \rightsquigarrow \mathbf{C}$ is a multiplicative kernel if and only if the cocontinuous functor $\mathrm{Lan}_{\, \sharp} K = \hat{K} \colon [\mathbf{A}, \mathbf{Sets}] \to [\mathbf{C}, \mathbf{Sets}]$ corresponding to $\bar{K} \colon \mathbf{A} \to \widehat{\mathbf{C}}$ under the construction in **5.6** preserves the convolution monoidal structure on both categories $[\mathbf{A}, \mathbf{Sets}]_{\mathfrak{P}}$ and $[\mathbf{C}, \mathbf{Sets}]_{\mathfrak{Q}}$.

Describe the isomorphisms k_1, k_2 when $\mathfrak{P}$ is Day convolution.

Exercise B.4 : Show that a functor $F \colon (\mathbf{A}, \otimes_{\mathbf{A}}, i) \to (\mathbf{C}, \otimes_{\mathbf{C}}, j)$ between monoidal categories is strong monoidal, i.e.

- $F(a \otimes b) \cong Fa \otimes Fb$;
- $Fi \cong j$

naturally in a, b if and only if $\mathrm{P}^F = \hom(F, 1)$ is a multiplicative kernel.

Dually, show that for $\mathbf{A}, \mathbf{C}$ promonoidal, $F\colon \mathbf{C} \to \mathbf{A}$ preserves convolution on $[\mathbf{A}, \mathbf{Sets}]_{\mathfrak{P}}, [\mathbf{C}, \mathbf{Sets}]_{\mathfrak{Q}}$ precisely if $\mathrm{P}_F = \hom(1, F)$ is a multiplicative kernel.

Definition B.5 : Let $K\colon \mathbf{A} \rightsquigarrow \mathbf{C}$ be a multiplicative kernel between promonoidal categories; define the *K-Fourier transform $f \mapsto \hat{K}(f)\colon \mathbf{C} \to \mathbf{Sets}$*, obtained as the image of $f\colon \mathbf{A} \to \mathbf{Sets}$ under the left Kan extension $\mathrm{Lan}_{\, \natural}\, K\colon [\mathbf{A}, \mathbf{Sets}] \to [\mathbf{C}, \mathbf{Sets}]$.

Exercise B.6 : Show the following properties of the K-Fourier transform:

- There is the canonical isomorphism

$$\hat{K}(f) \cong \int^a K(a, \_) \times f(a) \tag{195}$$

- $\hat{K}$ preserves the convolution monoidal structure (this is the *Parseval identity* for the Fourier transform);
- $\hat{K}$ has a right adjoint defined by

$$\check{K}(g) \cong \int_x [K(\_, x), g(x)]. \tag{196}$$

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