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Topology and its Applications 155 (2008) 459–496

Topology and its Applications

www.elsevier.com/locate/topol

$Ext_A^{4,*}(\mathbb{Z}/2, \mathbb{Z}/2)$ and $Ext_A^{5,*}(\mathbb{Z}/2, \mathbb{Z}/2)$

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Received 3 September 2007; received in revised form 13 November 2007; accepted 13 November 2007

Abstract

Let A denote the mod 2 Steenrod algebra. In this paper we make calculations to completely determine the *Ext* groups $Ext_A^{4,*}(\mathbb{Z}/2, \mathbb{Z}/2)$ and also to determine the structure of $\mathbb{Z}/2$-submodule of decomposable elements in $Ext_A^{5,*}(\mathbb{Z}/2, \mathbb{Z}/2)$.

MSC: 55S10; 55T15

Keywords: Steenrod algebra; Adams spectral sequence

1. Introduction

Let A denote the mod 2 Steenrod algebra. The *Ext* groups $Ext_A^{s,t}(\mathbb{Z}/2, \mathbb{Z}/2)$ are well known to form the E_2-term of the Adams spectral sequence for computing the 2-primary stable homotopy groups of spheres [1]. Here s is the homological degree and t is associated with the degree in the Steenrod algebra A. We will simply write $Ext_A^{s,t}$ to denote $Ext_A^{s,t}(\mathbb{Z}/2, \mathbb{Z}/2)$ and let $Ext_A^{s,*}$ denote $\bigoplus_t Ext_A^{s,t}$.

The structure of $Ext_A^{s,*}$ for $s \leqslant 3$ is known and this will be recalled in a moment. The purpose of this paper is to make calculations to determine completely $Ext_A^{4,*}$ and also to determine the structure of $\mathbb{Z}/2$-submodule of decomposable elements in $Ext_A^{5,*}(\mathbb{Z}/2, \mathbb{Z}/2)$.

We shall describe these results in terms of the mod 2 lambda algebra Λ [3]. Recall that Λ is a bigraded differential algebra over $\mathbb{Z}/2$ generated by $\lambda_j \in \Lambda^{1,j}$, $j \geqslant 0$, with relations

$$\text{(a)}\quad \lambda_j \lambda_{2j+1+m} = \sum_{\nu \geqslant 0} \binom{m-\nu-1}{\nu} \lambda_{j+m-\nu} \lambda_{2j+1+\nu}$$

for $m \geqslant 0$ and the differential

$$\text{(b)}\quad \delta(\lambda_k) = \sum_{\nu \geqslant 0} \binom{k-\nu-1}{\nu+1} \lambda_{k-\nu-1} \lambda_\nu \text{ on the generators } \lambda_k$$

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0166-8641/$ – see front matter
doi:10.1016/j.topol.2007.11.003

and that $H^{s,t}(\Lambda) = H^{s,t}(\Lambda,\delta) = Ext_A^{s,t+s}$. From (a) we see the set $\{\lambda_{j_1}\cdots\lambda_{j_s} \mid j_i \geqslant 2j_{i+1}\}$ is a $\mathbb{Z}/2$-base for Λ. Such monomials in the $\lambda_j{}'^s$ are said to be admissible. There is an operation $Sq^0 : \Lambda \to \Lambda$ given by

(c) $Sq^0(\lambda_{j_1}\cdots\lambda_{j_s}) = \lambda_{2j_1+1}\cdots\lambda_{2j_s+1}$,

where $\lambda_{j_1}\cdots\lambda_{j_s}$ is not necessarily admissible. This operation respects the relations in (a) and commutes with the differential in (b). So it induces a map

$$Sq^0 : H^{s,t-s}(\Lambda) = Ext_A^{s,t} \to H^{s,2t-s}(\Lambda) = Ext_A^{s,2t}$$

which is precisely the first Steenrod operation $Sq^0 : Ext_A^{s,t} \to Ext_A^{s,2t}$ in [7]. In what follows, $(Sq^0)^i : \Lambda \to \Lambda$ (or $(Sq^0)^i : H^{*,*}(\Lambda) \to H^{*,*}(\Lambda)$) denotes the composite $\underbrace{Sq^0\cdots Sq^0}_{i}$ if $i > 1$, is Sq^0 if $i = 1$ and is the identity map if $i = 0$.

In (1.1) below we list some classes in $Ext_A^{*,*}$ where each chain in Λ as given is easily seen to be a cycle (by direct computations from (a) and (b)) representing the corresponding class as named.

(1.1) (1) $h_i = \{\lambda_{2^i-1} = (Sq^0)^i(\lambda_0)\} \in Ext_A^{1,2^i}$,
that corresponds to the generator $Sq^{2^i} \in A$, $i \geqslant 0$.

(2) $c_i = \{(Sq^0)^i(\lambda_2\lambda_3^2)\} \in Ext_A^{3,2^{i+3}+2^{i+1}+2^i}$, $\quad i \geqslant 0$.

(3) $d_i = \{(Sq^0)^i(\lambda_6\lambda_2\lambda_3^2 + \lambda_4^2\lambda_3^2 + \lambda_2\lambda_4\lambda_5\lambda_3 + \lambda_1\lambda_5\lambda_1\lambda_7)\} \in Ext_A^{4,2^{i+4}+2^{i+1}}$, $\quad i \geqslant 0$.

(4) $e_i = \{(Sq^0)^i(\lambda_8\lambda_3^3 + \lambda_4(\lambda_5^2\lambda_3 + \lambda_7\lambda_3^2) + \lambda_2(\lambda_3\lambda_5\lambda_7 + \lambda_9\lambda_3^2))\} \in Ext_A^{4,2^{i+4}+2^{i+2}+2^i}$, $\quad i \geqslant 0$.

(5) $f_i = \{(Sq^0)^i(\lambda_4\lambda_0\lambda_7^2 + \lambda_3(\lambda_9\lambda_3^2 + \lambda_3\lambda_5\lambda_7) + \lambda_2\lambda_4\lambda_5\lambda_7)\} \in Ext_A^{4,2^{i+4}+2^{i+2}+2^{i+1}}$, $\quad i \geqslant 0$.

(6) $g_{i+1} = \{(Sq^0)^i(\lambda_6\lambda_0\lambda_7^2 + \lambda_5(\lambda_9\lambda_3^2 + \lambda_3\lambda_5\lambda_7) + \lambda_3(\lambda_5\lambda_9\lambda_3 + \lambda_{11}\lambda_3^2))\} \in Ext_A^{4,2^{i+4}+2^{i+3}}$, $\quad i \geqslant 0$.

(7) $p_i = \{(Sq^0)^i(\lambda_{14}\lambda_5\lambda_7^2 + \lambda_{10}\lambda_9\lambda_7^2 + \lambda_6\lambda_9\lambda_{11}\lambda_7)\} \in Ext_A^{4,2^{i+5}+2^{i+2}+2^i}$, $\quad i \geqslant 0$.

(8) $D_3(i) = \{(Sq^0)^i(\lambda_{22}\lambda_1\lambda_7\lambda_{31} + \lambda_{16}\lambda_{15}^3 + \lambda_{14}\lambda_9\lambda_7\lambda_{31})\} \in Ext_A^{4,2^{i+6}+2^i}$, $\quad i \geqslant 0$.

(9) $p_i' = \left\{(Sq^0)^i\begin{pmatrix}\lambda_{38}\lambda_1\lambda_{15}^2 + \lambda_{30}\lambda_9\lambda_{15}^2 + \lambda_{28}\lambda_{11}\lambda_{15}^2 + \lambda_{22}\lambda_{17}\lambda_{15}^2\\ + \lambda_{20}\lambda_{19}\lambda_{15}^2 + \lambda_{14}\lambda_1\lambda_{23}\lambda_{31} + \lambda_{12}\lambda_{19}\lambda_{23}\lambda_{15}\end{pmatrix}\right\} \in Ext_A^{4,2^{i+6}+2^{i+3}+2^i}$, $\quad i \geqslant 0$.

Theorem 1.2 below recalls the already known result on $Ext_A^{s,*}$ for $s \leqslant 3$.

Theorem 1.2. *(See [2,9].) The algebra $Ext_A^{s,*}$ for $s \leqslant 3$ is generated by $h_i \neq 0$ and $c_i \neq 0$ for $i \geqslant 0$, where h_i, c_i are as in* (1.1)(1) *and* (1.1)(2)*, and subject only to the relations $h_ih_{i+1} = 0$, $h_ih_{i+2}^2 = 0$ and $h_i^3 = h_{i-1}^2h_{i+1}$. In particular, $\{c_i \mid i \geqslant 0\}$ is a $\mathbb{Z}/2$-base for the indecomposable elements in $Ext_A^{3,*}$.*

Now we state our main results of the paper as follows. Theorem 1.3 is the result on $Ext_A^{4,*}$ and Theorem 1.4 is the result on $Ext_A^{5,*}$.

Theorem 1.3.

(1) *The subalgebra E of the algebra $Ext_A^{s,*}$ for $s \leqslant 4$ generated by h_i and c_i for $i \geqslant 0$ is subject only to the relations in Theorem* 1.2 *together with the relations: $h_i^2h_{i+3}^2 = 0$, $h_jc_i = 0$ for $j = i-1$, i, $i+2$ and $i+3$.*
(2) *The set S of the classes d_i, e_i, f_i, g_{i+1}, p_i, $D_3(i)$ and p_i' for $i \geqslant 0$ in* (1.1)(3) *through* (1.1)(9) *is a $\mathbb{Z}/2$-base for the indecomposable elements in $Ext_A^{4,*}$.*

Here in (1.3)(2), "S is a $\mathbb{Z}/2$-base for the indecomposable elements in $Ext_A^{4,*}$" means that the projection of S to $Ext_A^{4,*}/E$ is a $\mathbb{Z}/2$-base where E is as in (1.3)(1).

Theorem 1.4. *The subalgebra $\bar{E}$ of the algebra $Ext_A^{s,*}$ for $s \leqslant 5$ generated by h_i, c_i, d_i, e_i, f_i, g_{i+1}, p_i, $D_3(i)$ and p'_i for $i \geqslant 0$ is subject only to the relations in Theorem* 1.3 (*note that these include those in Theorem* 1.2) *together with the following relations* (1) *through* (39) *where classes, if not specified to be zero, are all non-zero, and where $j \geqslant 0$ except* (34) *in which $j \geqslant 1$.*

(1) $h_{j+4}^2 c_j = 0$, (2) $h_{j+3}h_j c_{j+2} = 0$, (3) $h_{j+1}^2 c_j = 0$,
(4) $h_j d_{j+1} = 0$, (5) $h_{j+3} d_j = 0$, (6) $h_{j+4} d_j = 0$, (7) $h_j e_{j+1} = 0$,
(8) $h_{j+4} e_j = 0$, (9) $h_{j+1} f_j = 0$, (10) $h_{j+3} f_j = 0$, (11) $h_{j+4} f_j = 0$,
(12) $h_{j+3} g_{j+1} = 0$, (13) $h_j p_{j+1} = 0$, (14) $h_{j+1} p_j = 0$, (15) $h_{j+2} p_j = 0$,
(16) $h_{j+4} p_j = 0$, (17) $h_{j+5} p_j = 0$, (18) $h_j D_3(j+1) = 0$, (19) $h_j D_3(j) = 0$,
(20) $h_{j+5} D_3(j) = 0$, (21) $h_{j+6} D_3(j) = 0$, (22) $h_j p'_{j+1} = 0$, (23) $h_{j+2} p'_j = 0$,
(24) $h_{j+3} p'_j = 0$, (25) $h_{j+6} p'_j = 0$, (26) $h_{j+4} h_{j+1} c_j = h_{j+3} e_j$,
(27) $h_{j+4} h_j c_{j+3} = h_{i+5} p'_i$, (28) $h_{j+5}^2 c_j = h_{j+1} p'_j$,
(29) $h_j d_{j+2} = h_{j+3} D_3(j)$, (30) $h_{j+1} d_{j+1} = h_j p_j$,
(31) $h_{j+2} d_{j+1} = h_{j+4} g_{j+1}$, (32) $h_{j+2} d_j = h_j e_j$, (33) $h_{j+1} e_j = h_j f_j$,
(34) $h_{j+1} e_j = h_j f_j = h_{j-1}^2 c_{j+1}$, (35) $h_{j+2} e_j = h_j g_{j+1}$,
(36) $h_j f_{j+2} = h_{j+4} p'_j$, (37) $h_j f_{j+1} = h_{j+3} p_j$,
(38) $h_{j+2} f_j = h_{j+1} g_{j+1}$, (39) $h_{j+3} g_{j+2} = h_{j+5} g_{j+1}$.

The result (1.3) is announced in [6]. We apologize for the delay of its proof given here. Some of the relations at $Ext_A^{5,*}$ in (1.4) are known [8]. Here we will give complete proofs of all of these relations.

Theorems 1.3 and 1.4 will be proved by making calculations for the *Ext* groups

$$Ext_A^{s,*}(P) = Ext_A^{s,*}\big(\widetilde{H}^*(P), \mathbb{Z}/2\big)$$

over the Steenrod algebra A where P denotes the infinite real projective space $\mathbb{R}P^\infty$. More precisely, we are going to make calculations in a spectral sequence $\{E_r^{i,s,t}\}$ for $Ext_A^{s,*}(P)$ with $s \leqslant 4$ from which to deduce (1.3) and (1.4). This spectral sequence is considered in [4] where the differentials

$$(*) \quad E_r^{i,s,t} \xrightarrow{d_r} E_r^{i-r,s+1,t-1} \quad \text{for } s \leqslant 2$$

are determined. Our main work here is to determine completely the differentials

$$(**) \quad E_r^{i,3,t} \xrightarrow{d_r} E_r^{i-r,4,t-1}$$

in the spectral sequence. To get $(**)$ we need to recall $(*)$. All of these will be given in the next section. In Section 3 we recall a connection from $Ext_A^{*,*}(P)$ to $Ext_A^{*,*}$ and also a connection from $Ext_A^{*,*}$ back to $Ext_A^{*,*}(P)$ and use the differentials $(*)$ and $(**)$ in Section 2 plus some extensive calculations to complete the proofs of Theorems 1.3 and 1.4.

2. Some calculations in a spectral sequence for $Ext_A^{*,*}(P)$

Given a locally finite graded left module N over the mod 2 Steenrod algebra A. The lambda algebra Λ in Section 1 can also be used to compute the *Ext* groups $Ext_A^{s,t}(N) = Ext_A^{s,t}(N, \mathbb{Z}/2)$ and this is described as follows. Let N_* be the $\mathbb{Z}/2$-dual of N which is a right A-module by transposing the left A-module structure on N. Consider $N_* \otimes \Lambda$ and bigrade it by

$$(N_* \otimes \Lambda)^{s,t} = \sum_k N_k \otimes \Lambda^{s,t-k}.$$

For any sequence $I = (i_1, \ldots, i_s)$ of non-negative integers we write λ_I to denote $\lambda_{i_1} \cdots \lambda_{i_s} \in \Lambda$. For $m_* \in N_*$ write $m_*\lambda_I$ to denote $m_* \otimes \lambda_I \in N_* \otimes \Lambda$ and let $m_* = m_*1$. $N_* \otimes \Lambda$ is a bigraded differential right Λ-module with differential δ given by

$$(1) \quad \delta(m_*\lambda_I) = m_*\delta(\lambda_I) + \sum_{j \geqslant 0} m_* Sq^{j+1}\lambda_j\lambda_I.$$

Then $Ext_A^{s,t+s}(N) = H^{s,t}(N_* \otimes \Lambda, \delta)$, and the differential in (1) induces a right action of $Ext_A^{*,*}$ on $Ext_A^{*,*}(N)$ making the latter a right $Ext_A^{*,*}$-module.

We will be interested in $N = \widetilde{H}^*(P)$, the reduced mod 2 cohomology of the infinite real projective space P. To simplify, let $Ext_A^{s,t}(P) = Ext_A^{s,t}(\widetilde{H}^*(P))$. We recall that $N_* = \widetilde{H}_*(P)$, the reduced mod 2 homology of P, has

$$\widetilde{H}_k(P) = \begin{cases} \mathbb{Z}/2 & \text{for } k \geqslant 1, \\ 0 & \text{otherwise} \end{cases}$$

and that if e_k is the generator of $\widetilde{H}_k(P) = \mathbb{Z}/2$ for $k \geqslant 1$ then the Steenrod algebra A acts on $\widetilde{H}_*(P)$ from the right by

$$(2) \quad e_k Sq^l = \binom{k-l}{l} e_{k-l}.$$

From (1) and (2) we see the differential δ on $\widetilde{H}_*(P) \otimes \Lambda$ is given by

$$(3) \quad \delta(e_k \lambda_I) = e_k \delta(\lambda_I) + \sum_{j \geqslant 0} \binom{k-j-1}{j+1} e_{k-j-1} \lambda_j \lambda_I.$$

Thus $Ext_A^{s,t+s}(P) = H^{s,t}(\widetilde{H}_*(P) \otimes \Lambda, \delta)$ with δ given as in (3).

Define a filtration $\{F(i)\}_{i \geqslant 1}$ of the differential Λ-module $\widetilde{H}_*(P) \otimes \Lambda$ by $(F(i)) = \sum_{1 \leqslant k \leqslant i} \widetilde{H}_k(P) \otimes \Lambda$. Clearly, $F(i)/F(i-1) \cong \Sigma^i \Lambda (F(0) = 0)$; so $H^{s,t}(F(i)/F(i-1)) = \Sigma^i Ext_A^{s,t+s-i}$. This filtration gives rise to a spectral sequence $\{E_r^{i,s,t}\}_{r \geqslant 1}$ with

$$(4) \quad E_1^{i,s,t} = H^{s,t}\big(F(i)/F(i-1)\big) = \Sigma^i Ext_A^{s,t+s-i}$$

and $\bigoplus_{i \geqslant 1} E_\infty^{i,s,t} \cong Ext_A^{s,t+s}(P)$ as $\mathbb{Z}/2$-modules. For each $r \geqslant 1$ the differential d_r of the spectral sequence goes from $E_r^{i,s,t}$ to $E_r^{i-r,s+1,t-1}$. We will simply write $E_r^{*,s,*} \xrightarrow{d_r} E_r^{*,s+1,*}$ to indicate that we are considering these differentials for a fixed s and for all i, t and r.

From (4) we see that, for a given $s > 0$, if $Ext_A^{s',*}$ are known for all $s' \leqslant s$ (and all $*$) then one can compute the differentials

$$E_r^{*,\bar{s},*} \xrightarrow{d_r} E_r^{*,\bar{s}+1,*} \quad \text{for } \bar{s} \leqslant s-1.$$

In particular, one can compute the differentials

$$(*) \quad E_r^{*,\bar{s},*} \xrightarrow{d_r} E_r^{*,\bar{s}+1,*} \quad \text{for } 0 \leqslant \bar{s} \leqslant 2$$

since $Ext_A^{s',*}$ for $s' \leqslant 3$ are known by Theorem 1.2. This has been completely done in [4]. In order to compute the next stage differentials $E_r^{*,3,*} \xrightarrow{d_r} E_r^{*,4,*}$, which will be the main work here, we need to recall from [4] the results on the differentials $(*)$. These will be stated in (2.1), (2.2) and (2.3) below.

We will use some conventions in stating these results. Note that, by (4), if α is a basis element in $Ext_A^{s,*}$ then $e_i\alpha = e_i \otimes \alpha$ is a basis element in $E_1^{i,s,*}$. We will write $e_i\alpha \to e_{i-r}\beta$, where $e_{i-r}\beta$ is a basis element in $E_1^{i-r,s+1,*}$, to mean that both $e_i\alpha$ and $e_{i-r}\beta$ survive to $E_r^{*,*,*}$ and $d_r(e_i\alpha) = e_{i-r}\beta$ in the spectral sequence. Such a differential is a non-trivial one. If $d_r(e_i\alpha) = 0$ for all $r > 0$ so that $e_i\alpha$ is an infinite cycle then we write $e_i\alpha \to 0$. We will only consider those $e_i\alpha$ with $e_i\alpha \to 0$ which are not boundaries so that they survive to $E_\infty^{*,*,*}$ representing non-trivial elements in $Ext_A^{*,*}(P)$.

From Theorem 1.2 we have the following.

(5) (i) $\{e_i = e_i 1 \mid i \geqslant 1\}$ is a $\mathbb{Z}/2$-base for $E_1^{*,0,*}$,

(ii) $\{e_i h_j \mid i \geqslant 1, j \geqslant 0\}$ is a $\mathbb{Z}/2$-base for $E_1^{*,1,*}$,

(iii) $\{e_i h_j h_k \mid i \geqslant 1, 0 \leqslant j = k \text{ or } 0 \leqslant j < k-1\}$ is a $\mathbb{Z}/2$-base for $E_1^{*,2,*}$,

(iv) $\left\{e_i h_j h_k h_l \mid \begin{pmatrix} i \geqslant 1,\ 0 \leqslant j < k-1 < l-2 \text{ or} \\ 0 \leqslant j = k < l-1 \text{ or } 0 \leqslant j < k-2 = l-2 \end{pmatrix}\right\} \cup \{e_i c_j \mid i \geqslant 1,\ j \geqslant 0\}$ is a $\mathbb{Z}/2$-base for $E_1^{*,3,*}$.

The results on the differentials $(*)$ are recalled as follows. (2.1), (2.2) and (2.3) record all the non-trivial differentials for $(*)$. From these all the non-trivial infinite cycles in $E_\infty^{*,s,*}$ for $0 \leqslant s \leqslant 2$ will be extracted and listed in $\overline{(2.1)}$, $\overline{(2.2)}$ and $\overline{(2.3)}$ that follow.

(2.1.1) $e_{2^l(2m+1)-1} \to e_{2^{l+1}m-1}h_l$ for $m \geqslant 1,\ l \geqslant 0$.

(2.2.1) $e_{2^l(2m+1)-1}h_j \to e_{2^{l+1}m-1}h_lh_j$
for $m \geqslant 1,\ 0 \leqslant l < j-1$ or $0 \leqslant l = j$.

(2.2.2) $e_{2^{l+n}m+2^l-1}h_{l+1} \to e_{2^{l+n}m-2^l-1}h_{l+1}^2$ for $m \geqslant 1,\ l \geqslant 0$ and $n \geqslant 2$.

(2.2.3) $e_{2^{l+n+1}m+2^{l+n}-2^l-1}h_{l+1}$

$$\to \begin{cases} e_{2^{l+n+1}m-2^l-1}h_{l+1}h_{l+n} & \text{for } m \geqslant 1,\ l \geqslant 0,\ n \geqslant 3, \\ e_{2^{l+3}m-2^{l+1}-2^l-1}h_{l+2}^2 & \text{for } m \geqslant 1,\ l \geqslant 0,\ n = 2. \end{cases}$$

(2.3.1) $e_{2m}h_0^2 \to e_{2m-1}h_0^3$ for $m \geqslant 1$.

(2.3.2) $e_{2^{j+2}m+2^j-1}h_j^2 \to e_{2^{j+2}m-2^{j+1}-2^{j-1}-1}c_{j-1}$ for $m \geqslant 1,\ j \geqslant 1$.

(2.3.3) $e_{2^{j+3}m+2^{j+1}+2^j-1}h_j^2 \to e_{2^{j+3}m-2^{j+1}-2^j-2^{j-1}-1}h_{j-1}h_{j+2}^2$
for $m \geqslant 1,\ j \geqslant 1$.

(2.3.4) $e_{2^{j+n+1}m+2^{j+n}-2^j-1}h_j^2 \to e_{2^{j+n+1}m-2^j-1}h_j^2h_{j+n}$
for $m \geqslant 1,\ j \geqslant 1$ and $n \geqslant 3$.

(2.3.5) $e_{2^{j+1}m+2^{j-1}-1}h_j^2 \to e_{2^{j+1}m-2^{j-1}-1}h_j^3$ for $m \geqslant 1,\ j \geqslant 1$.

(2.3.6) $e_{2^jm+2^{j-1}+2^{j-2}-1}h_j^2 \to e_{2^jm-1}c_{j-2}$ for $m \geqslant 1,\ j \geqslant 2$.

(2.3.7) $e_{2^{j+1}m+2^{j-2}-1}h_j^2 \to e_{2^{j+1}m-2^j+2^{j-2}-1}h_j^3$ for $m \geqslant 1,\ j \geqslant 2$.

(2.3.8) $e_{2^l(2m+1)-1}h_j^2 \to e_{2^{j+1}m-1}h_lh_j^2$ for $m \geqslant 1,\ 0 \leqslant l \leqslant j-3$.

(2.3.9) $e_{2^j(2m+1)-1}h_jh_k \to e_{2^{j+1}m-1}h_j^2h_k$ for $m \geqslant 1,\ 0 \leqslant j \leqslant k-2$.

(2.3.10) $e_{2^{j+n}m+2^{j-1}-1}h_jh_k \to e_{2^{j+1}m-2^{j-1}-1}h_j^2h_k$
for $m \geqslant 1,\ 1 \leqslant j \leqslant k-2$ and $n \geqslant 1$.

(2.3.11) $e_{2^{l+n}m+2^n-2^{j-1}-1}h_jh_k$

$$\to \begin{cases} e_{2^{l+n}m-2^{j-1}-1}h_jh_nh_k & \text{for } m \geqslant 1,\ l \geqslant 1,\ 1 \leqslant j \leqslant n-2 \\ & \text{and } n < k-1 \text{ or } k = n \geqslant j+3, \\ e_{2^{l+j+1}m-2^{j+1}+2^{j-1}-1}h_{j+1}^2h_k & \text{for } m \geqslant 1,\ l \geqslant 1 \\ & \text{and } 1 \leqslant j = n-1 \leqslant k-3. \end{cases}$$

(2.3.12) $e_{2^{l+k}m+2^{k-1}-2^{j-1}-1}h_jh_k$

$$\to \begin{cases} e_{2^{l+k}m-2^{k-1}-2^{j-1}-1}h_jh_k^2 & \text{for } m \geqslant 1,\ l \geqslant 1,\ 1 \leqslant j \leqslant k-3, \\ e_{2^{l+j+2}m-2^{j+2}-2^{j-1}-1}c_j & \text{for } m \geqslant 1,\ l \geqslant 1,\ 1 \leqslant j = k-2. \end{cases}$$

(2.3.13) $e_{2^{l+k+1}m+2^{l+k}-2^{k-1}-2^{j-1}-1}h_jh_k \to e_{2^{l+k+1}m-2^{k-1}-2^{j-1}-1}h_jh_kh_{l+k}$
for $m \geqslant 1,\ l \geqslant 2,\ 1 \leqslant j \leqslant k-2$.

(2.3.14) $e_{2^{k+2}m+2^k+2^{k-1}-2^{j-1}-1}h_jh_k \to e_{2^{k+2}m-2^k-2^{k-1}-2^{j-1}-1}h_jh_{k+1}^2$
for $m \geqslant 1,\ 1 \leqslant j \leqslant k-2$.

(2.3.15) $e_{2^{l+j+2}m+2^{j+2}-2^{j-1}-1}h_jh_{j+2} \to e_{2^{l+j+2}m-2^{j+1}-2^j-2^{j-1}-1}h_{j+2}^3$
for $m \geqslant 1,\ l \geqslant 1,\ j \geqslant 1$.

(2.3.16) $e_{2^l(2m+1)-1}h_jh_k \to e_{2^{l+1}m-1}h_lh_jh_k$ for $0 \leqslant l < j-1 < k-2$.

Note that the differentials above from (2.1.1) through (2.3.16) are all of the form $e_{f(m)}\alpha \to e_{g(m)}\beta$ for a common integral variable m which is $\geqslant 1$ so that $g(m) > 0$. If we put $m = 0$ in these differential formulas then $g(0) < 0$. Thus $e_{f(0)}\alpha \to 0$, that is, $e_{f(0)}\alpha$ is an infinite cycle provided $f(0) > 0$ which is satisfied for (2.1.1) through (2.3.16) except (2.3.1). These infinite cycles $e_{f(0)}\alpha$ are listed in $\overline{(2.1.1)}$ through $\overline{(2.3.16)}$ below (there is no $\overline{(2.3.1)}$).

$\overline{(2.1.1)}$ $e_{2^l-1},\ l \geqslant 1.$ $\overline{(2.2.1)}$ $e_{2^l-1}h_j,\ 1 \leqslant l < j-1$ or $1 \leqslant l = j.$
$\overline{(2.2.2)}$ $e_{2^l-1}h_{l+1},\ l \geqslant 1.$ $\overline{(2.2.3)}$ $e_{2^{l+n}-2^l-1}h_{l+1},\ l \geqslant 0,\ n \geqslant 2.$
$\overline{(2.3.2)}$ $e_{2^j-1}h_j^2,\ j \geqslant 1.$ $\overline{(2.3.3)}$ $e_{2^{j+1}+2^j-1}h_j^2,\ j \geqslant 1.$
$\overline{(2.3.4)}$ $e_{2^{j+n}-2^j-1}h_j^2,\ j \geqslant 1,\ n \geqslant 3.$ $\overline{(2.3.5)}$ $e_{2^{j-1}-1}h_j^2,\ j \geqslant 2.$
$\overline{(2.3.6)}$ $e_{2^{j-1}+2^{j-2}-1}h_j^2,\ j \geqslant 2.$ $\overline{(2.3.7)}$ $e_{2^{j-2}-1}h_j^2,\ j \geqslant 3.$
$\overline{(2.3.8)}$ $e_{2^l-1}h_j^2,\ 1 \leqslant l \leqslant j-3.$ $\overline{(2.3.9)}$ $e_{2^j-1}h_jh_k,\ 1 \leqslant j < k-1.$
$\overline{(2.3.10)}$ $e_{2^{j-1}-1}h_jh_k,\ 2 \leqslant j < k-1.$
$\overline{(2.3.11)}$ $e_{2^n-2^{j-1}-1}h_jh_k,\ 1 \leqslant j = n-1 < k-2$ or $1 \leqslant j < n-1 < k-2$
or $1 \leqslant j < k-2 = n-2.$
$\overline{(2.3.12)}$ $e_{2^{k-1}-2^{j-1}-1}h_jh_k,\ 1 \leqslant j < k-1.$
$\overline{(2.3.13)}$ $e_{2^{l+k}-2^{k-1}-2^{j-1}-1}h_jh_k,\ 1 \leqslant j < k-1$ and $l \geqslant 2.$
$\overline{(2.3.14)}$ $e_{2^k+2^{k-1}-2^{j-1}-1}h_jh_k,\ 1 \leqslant j < k-1.$
$\overline{(2.3.15)}$ $e_{2^{j+2}-2^{j-1}-1}h_jh_{j+2},\ j \geqslant 1.$
$\overline{(2.3.16)}$ $e_{2^l-1}h_jh_k,\ 1 \leqslant l < j-1 < k-2.$

We recall that the differentials in (2.1.1) through (2.3.16) are of the form $e_{f(m)}\alpha \to e_{g(m)}\beta$. Call $e_{f(m)}\alpha$ a source element and $e_{g(m)}\beta$ a boundary in the spectral sequence. We also recall that $(2.k)$, for $1 \leqslant k \leqslant 3$, consists of the $(2.k.j)'^s$, that is, $(2.k) = \bigcup_j (2.k.j)$. Similarly, $\overline{(2.k)} = \bigcup_j \overline{(2.k.j)}$. For each k with $0 \leqslant k \leqslant 2$ let $S(k)$ (resp., $B(k+1)$) be the set of all the source elements (resp., all the boundaries) in $(2.k+1)$ and let $I(k)$ be the set of all the infinite cycles in $\overline{(2.k+1)}$. It is not difficult to check the following.

(6) (i) $S(0) \cup I(0)$ and $S(k) \cup B(k) \cup I(k)$, for $k = 1, 2$, are disjoint unions.
(ii) $S(0) \cup I(0)$ is a $\mathbb{Z}/2$-base for $E_1^{*,0,*}$.
(iii) $S(k) \cup B(k) \cup I(k)$ is a $\mathbb{Z}/2$-base for $E_1^{*,k,*}$, $k = 1, 2$.

In particular, this implies $I(k)$ is a $\mathbb{Z}/2$-base for $E_\infty^{*,k,*}$ for $k = 0, 1, 2$. From this the $Ext_A^{*,*}$-module structure of $Ext_A^{s,*}(P)$ for $0 \leqslant s \leqslant 2$ is determined in [4]. The result on this is recalled as Theorem 2.4 below. To state the result we note that e_{2^i-1} for $i \geqslant 1$ and $e_{2^{j+1}+2^j-1}\lambda^2_{2^{j+2}-1}$ for $j \geqslant 0$ are easily seen to be cycles in $\widetilde{H}_*(P) \otimes \Lambda$. Define certain classes in $Ext_A^{*,*}(P)$ as follows.

(7) (i) $\widehat{h}_i = \{e_{2^i-1}\} \in Ext_A^{0,2^i-1}(P),\ i \geqslant 1.$
(ii) $\widehat{c}_j = \{e_{2^{j+1}+2^j-1}\lambda^2_{2^{j+2}-1}\} \in Ext_A^{2,2^{j+3}+2^{j+1}+2^j-1}(P),\ j \geqslant 0.$

In the following statement the result in Theorem 1.2 on the algebra structure of $Ext_A^{s,*}$ for $s \leqslant 2$ is implicitly used.

Theorem 2.4. *The $Ext_A^{*,*}$-module $Ext_A^{s,*}(P)$, for $s \leqslant 2$, is generated by $\widehat{h}_i$ for $i \geqslant 1$ and $\widehat{c}_j$ for $j \geqslant 0$, described in* (7)(i), (ii)*, subject only to the relations*: $\widehat{h}_ih_{i-1} = 0,\ i \geqslant 1$, $\widehat{h}_{i+2}h_i^2 = \widehat{h}_{i+1}h_{i+1}^2$, $i \geqslant 0$ *and* $\widehat{h}_{i+2}h_{i+2}h_i = 0$, $i \geqslant 0$.

We proceed to describe the differentials

$$(**) \quad E_r^{*,3,*} \xrightarrow{d_r} E_r^{*,4,*}$$

which is the main work in this section. The source elements in $E_r^{*,3,*}$ are known by (5)(iv). By (4), $E_1^{i,4,*} = \Sigma^i Ext_A^{4,*}$. Although the *Ext* groups $Ext_A^{4,*}$ are yet to be determined, which essentially is Theorem 1.3, we may still make calculations to do the differentials $(**)$ by induction on the internal degree t in $Ext_A^{4,t}$ and this is explained as follows.

First of all, we note from [5] that there is a map

$$Ext_A^{s,t}(P) \xrightarrow{t_*} Ext_A^{s+1,t+1}$$

which is onto for $t-s>0$. This is known as the algebraic Kahn–Priddy theorem. The map t_* will be explicitly described in Section 3. For a given $k>0$ if $Ext_A^{s,t}(P)$ can be computed for $t-s \leqslant k$ then from the explicit formula for t_* one can compute $Ext_A^{s+1,t+1}$ for $t-s \leqslant k$.

Next we recall again that any non-trivial differential in $(**)$ is of the form

$(**)'$ $e_j\alpha \to e_k\beta$ where $j>k$, α is some basis element in $Ext_A^{3,t}$ for a certain t and β is some non-zero class in $Ext_A^{4,t'}$ with $t'=t+j-k>t$ (since $j>k$).

Given an integer $\bar{t}>0$ and suppose, as an inductive hypothesis, that $Ext_A^{4,t'}$ are known for t' up to $\bar{t}$ and that the differentials $e_j\alpha \to e_k\beta$ can be determined for any possible $e_j\alpha$ and $e_k\beta$ with $\beta \in Ext_A^{4,t'}$, $t' \leqslant \bar{t}$.

From this and from the relations of the internal degrees in $(**)'$ one can then compute $\bigcup_i E_\infty^{i,3,\widetilde{t}-3} \cong Ext_A^{3,\widetilde{t}}(P)$ for $\widetilde{t}$ up to $\bar{t}+1$ (since $k \geqslant 1$) and therefore $Ext_A^{4,\widetilde{t}+1}$ for $\widetilde{t}+1$ up to $\bar{t}+2$ by the algebraic Kahn–Priddy theorem.

In this way one may thus compute the differentials

$$(**) \quad E_r^{*,3,*} \xrightarrow{d_r} E_r^{*,4,*}$$

by simply assuming the results on $Ext_A^{4,*}$ in Theorem 1.3. We have made calculations to determine all the differentials for $(**)$ and the results are to be given in (2.5) below from (2.5.1) to (2.5.74). They are obtained by making calculations in the lambda algebra Λ using the same method in [4] by which the differentials (2.1), (2.2) and (2.3) recalled earlier were obtained. Details of these calculations will not be given here.

Now we list these differentials from (2.5.1) to (2.5.74) as follows. Note that the source elements in these differentials are taken into consideration from excluding the boundaries in (2.3). Also, a possible confusion on notations should be cautioned. Recall that e_k denotes the generator of $\widetilde{H}_k(P)=\mathbb{Z}/2$ for $k \geqslant 1$. Now the same notation e_k is also used to denote the cohomology class in $Ext_A^{4,l}$ described in (1.1)(4) of Section 1 where $l=2^{k+4}+2^{k+2}+2^k$. Thus if $e_i\alpha \to e_je_k$, say, is a differential in this list then e_j is the homology class for P and e_k is the Ext group class in $Ext_A^{4,*}$.

(2.5.1) $e_{2m}h_0^3 \to e_{2m-1}h_0^4$ for $m \geqslant 1$.

(2.5.2) $e_{2^{j+n}m+2^{j-1}-1}h_j^3 \to e_{2^{j+n}m-2^{j+1}-2^j-1}h_jc_{j-1}$
for $m \geqslant 1$, $j \geqslant 1$ and $n \geqslant 2$.

(2.5.3) $e_{2^{j+n}(2m+1)-2^j-2^{j-1}-1}h_j^3 \to e_{2^{j+n+1}m-2^j-2^{j-1}-1}h_j^3h_{j+n}$
for $m \geqslant 1$, $j \geqslant 1$ and $n \geqslant 3$.

(2.5.4) $e_{2^{j+3}m+2^{j+1}+2^{j-1}-1}h_j^3 \to e_{2^{j+3}m-2^{j+2}-2^{j-1}-1}e_{j-1}$
for $m \geqslant 1$, $j \geqslant 1$.

(2.5.5) $e_{2^{j+n}m+2^{j-2}-1}h_j^3 \to e_{2^{j+n}m-2^{j+1}-1}e_{j-2}$
for $m \geqslant 1$, $j \geqslant 2$ and $n \geqslant 2$.

(2.5.6) $e_{2^{j+2}m+2^{j+1}+2^{j-2}-1}h_j^3 \to e_{2^{j+2}m-1}e_{j-2}$
for $m \geqslant 1$, $j \geqslant 2$.

(2.5.7) $e_{2^{j+n}m+2^{j-3}-1}h_j^3 \to e_{2^{j+n}m-2^j-2^{j-1}-1}p_{j-3}$
for $m \geqslant 1$, $j \geqslant 3$ and $n \geqslant 1$.

(2.5.8) $e_{2^{j+n-1}m+2^{j-2}+2^{j-3}-1}h_j^3 \to e_{2^{j+n-1}m-1}c_{j-3}h_{j+1}$
for $m \geqslant 1$, $j \geqslant 3$ and $n \geqslant 0$.

(2.5.9) $e_{2^l(2m+1)-1}h_j^3 \to e_{2^{l+1}m-1}h_lh_j^3$ for $m \geqslant 1$, $0 \leqslant l < j-3$.

(2.5.10) $e_{2m}h_0^2h_k \to e_{2m-1}h_0^3h_k$ for $m \geqslant 1$, $k \geqslant 3$.

(2.5.11) $e_{2^{j+2}m+2^j-1}h_j^2h_k \to e_{2^{j+2}m-2^{j+1}-2^{j-1}-1}c_{j-1}h_k$
for $m \geqslant 1$, $1 \leqslant j < k-2$.

(2.5.12) $e_{2^{j+3}m+2^{j+1}+2^j-1}h_j^2h_k \to e_{2^{j+3}m-2^{j+1}-2^j-2^{j-1}-1}h_{j-1}h_{j+2}^2h_k$
for $m \geqslant 1,\ 1 \leqslant j < k-3$.

(2.5.13) $e_{2^{j+4}m+2^{j+1}+2^j-1}h_j^2h_{j+3} \to e_{2^{j+4}m-2^{j+3}-2^j-1}f_j$
for $m \geqslant 1,\ j \geqslant 1$.

(2.5.14) $e_{2^{j+n+1}m+2^{j+n}-2^{j+2}-2^j-1}h_j^2h_{j+3}$
$\to e_{2^{j+n+1}m-2^{j+2}-2^j-1}h_j^2h_{j+3}h_{j+n}$ for $m \geqslant 1,\ j \geqslant 1,\ n \geqslant 5$.

(2.5.15) $e_{2^{j+5}m+2^{j+4}-2^{j+2}-2^j-1}h_j^2h_{j+3}$
$\to e_{2^{j+5}m-2^{j+3}-2^{j+2}-2^j-1}h_j^2h_{j+4}^2$ for $m \geqslant 1,\ j \geqslant 1$.

(2.5.16) $e_{2^{n+1}m+2^n-2^j-1}h_j^2h_k \to e_{2^{n+1}m-2^j-1}h_j^2h_nh_k$
for $m \geqslant 1$ and $1 \leqslant j < n-2 < k-3$ or $1 \leqslant j < n-3 = k-3$.

(2.5.17) $e_{2^{k+1}m+2^{k-1}-2^j-1}h_j^2h_k \to e_{2^{k+1}m-2^{k-1}-2^j-1}h_j^2h_k^2$
for $m \geqslant 1,\ 1 \leqslant j < k-3$.

(2.5.18) $e_{2^{k+n+1}m+2^{k+n}-2^{k-1}-2^j-1}h_j^2h_k \to e_{2^{k+n+1}m-2^{k-1}-2^j-1}h_j^2h_kh_{k+n}$
for $m \geqslant 1,\ 1 \leqslant j < k-3$ and $n \geqslant 2$.

(2.5.19) $e_{2^{k+2}m+2^{k+1}-2^{k-1}-2^j-1}h_j^2h_k \to e_{2^{k+2}m-2^k-2^{k-1}-2^j-1}h_j^2h_{k+1}^2$
for $m \geqslant 1,\ 1 \leqslant j < k-3$.

(2.5.20) $e_{2^{j+4}m+2^{j+3}-2^j-1}h_j^2h_{j+3} \to e_{2^{j+4}m-2^{j+3}+2^{j-1}-1}h_{j-1}h_{j+3}^3$
for $m \geqslant 1,\ j \geqslant 1$.

(2.5.21) $e_{2^{j+1}m+2^{j-1}-1}h_j^2h_k \to e_{2^{j+1}m-2^{j-1}-1}h_j^3h_k$
for $m \geqslant 1,\ 1 \leqslant j < k-2$.

(2.5.22) $e_{2^{j+n}m+2^{j-2}-1}h_j^2h_k \to e_{2^{j+n}m-2^j+2^{j-2}-1}h_j^3h_k$
for $m \geqslant 1,\ 2 \leqslant j < k-2$ and $n \geqslant 1$.

(2.5.23) $e_{2^jm+2^{j-1}+2^{j-2}-1}h_j^2h_k \to e_{2^jm-1}c_{j-2}h_k$
for $m \geqslant 1,\ 2 \leqslant j < k-2$.

(2.5.24) $e_{2^l(2m+1)-1}h_j^2h_k \to e_{2^{l+1}m-1}h_lh_j^2h_k$
for $m \geqslant 1,\ 0 \leqslant l < j-2 < k-4$.

(2.5.25) $e_{2^j(2m+1)-1}h_jh_k^2 \to e_{2^{j+1}m-1}h_j^2h_k^2$ for $m \geqslant 1,\ 0 \leqslant j < k-3$.

(2.5.26) $e_{2^{j+4}m+2^j-1}h_jh_{j+3}^2 \to e_{2^{j+4}m-2^{j+3}-2^j-1}c_jh_{j+4}$
for $m \geqslant 1,\ j \geqslant 0$.

(2.5.27) $e_{2^{j+3}m+2^{j+2}+2^j-1}h_jh_{j+3}^2 \to e_{2^{j+3}m-1}f_j$ for $m \geqslant 1,\ j \geqslant 0$.

(2.5.28) $e_{2^{j+3}m+2^{j+1}+2^j-1}h_jh_{j+3}^2 \to e_{2^{j+3}m-2^{j+1}-1}f_j$ for $m \geqslant 1,\ j \geqslant 0$.

(2.5.29) $e_{2^{j+3}m+2^{j+2}+2^{j+1}+2^j-1}h_jh_{j+3}^2 \to e_{2^{j+3}m-1}g_{j+1}$
for $m \geqslant 1,\ j \geqslant 0$.

(2.5.30) $e_{2^{n+1}m+2^n-2^{j-1}-1}h_jh_k^2 \to e_{2^{n+1}m-2^{j-1}-1}h_jh_nh_k^2$
for $m \geqslant 1,\ 1 \leqslant j < n-1 < k-3$.

(2.5.31) $e_{2^{j+1}m+2^{j-1}-1}h_jh_k^2 \to e_{2^{j+1}m-2^{j-1}-1}h_j^2h_k^2$
for $m \geqslant 1,\ 1 \leqslant j < k-3$.

(2.5.32) $e_{2^{j+1}m+2^{j-1}-1}h_jh_{j+3}^2 \to e_{2^{j+1}m-2^j-1}p_{j-1}$
for $m \geqslant 1,\ j \geqslant 1$.

(2.5.33) $e_{2^{j+2}m+2^j+2^{j-1}-1}h_jh_k^2 \to e_{2^{j+2}m-2^j-2^{j-1}-1}h_{j+1}^2h_k^2$
for $m \geqslant 1,\ 1 \leqslant j < k-4$.

(2.5.34) $e_{2^{j+2}m+2^j+2^{j-1}-1}h_jh_{j+4}^2 \to e_{2^{j+2}m-2^{j+1}-1}p'_{j-1}$
for $m \geqslant 1,\ j \geqslant 1$.

(2.5.35) $e_{2^{j+3}m+2^{j+2}+2^j+2^{j-1}-1}h_jh_{j+3}^2 \to e_{2^{j+3}m-2^{j+2}+2^{j-1}-1}c_{j+1}h_{j+2}$
for $m \geqslant 1,\ j \geqslant 1$.

(2.5.36) $e_{2^{j+4}m+2^j+2^{j-1}-1}h_jh_{j+3}^2 \to e_{2^{j+4}m-2^{j+3}-2^{j-1}-1}c_jh_{j+4}$
for $m \geqslant 1,\ j \geqslant 1$.

(2.5.37) $e_{2^{k+1}m+2^{k-1}-2^{j-1}-1}h_jh_k^2 \to e_{2^{k+1}m-2^{k-1}-2^{j-1}-1}h_jh_k^3$
for $m \geqslant 1,\ 1 \leqslant j < k-3$.

(2.5.38) $e_{2^{j+4}m+2^{j+4}+2^{j+1}+2^j+2^{j-1}-1}h_jh_{j+3}^2 \to e_{2^{j+4}m-1}p'_{j-1}$
for $m \geqslant 1,\ j \geqslant 1$.

(2.5.39) $e_{2^km+2^{k-1}+2^{k-2}-2^{j-1}-1}h_jh_k^2 \to e_{2^km-2^{j-1}-1}h_jc_{k-2}$
for $m \geqslant 1,\ 1 \leqslant j < k-3$.

(2.5.40) $e_{2^{k+1}m+2^{k-2}-2^{j-1}-1}h_jh_k^2 \to e_{2^{k+1}m-2^{k-1}-2^{k-2}-2^{j-1}-1}h_jh_k^3$
for $m \geqslant 1,\ 1 \leqslant j < k-3$.

(2.5.41) $e_{2^{k+2}m+2^k-2^{j-1}-1}h_jh_k^2 \to e_{2^{k+2}m-2^{k+1}-2^{k-1}-2^{j-1}-1}h_jc_{k-1}$
for $m \geqslant 1,\ 1 \leqslant j < k-2$.

(2.5.42) $e_{2^{k+3}m+2^{k+2}-2^k-2^{j-1}-1}h_jh_k^2$
$\to e_{2^{k+3}m-2^{k+2}+2^{k-1}-2^{j-1}-1}h_jh_{k-1}h_{k+2}^2$
for $m \geqslant 1,\ 1 \leqslant j < k-2$.

(2.5.43) $e_{2^{k+n+1}m+2^{k+n}-2^k-2^{j-1}-1}h_jh_k^2 \to e_{2^{k+n+1}m-2^k-2^{j-1}-1}h_jh_k^2h_{k+n}$
for $m \geqslant 1,\ 1 \leqslant j < k-2$ and $n \geqslant 3$.

(2.5.44) $e_{2^l(2m+1)-1}h_jh_k^2 \to e_{2^{l+1}m-1}h_lh_jh_k^2$ for $m \geqslant 1,\ 0 \leqslant l < j-1 < k-3$.

(2.5.45) $e_{2^j(2m+1)-1}h_jh_kh_i \to e_{2^{j+1}m-1}h_j^2h_kh_i$ for $m \geqslant 1,\ 0 \leqslant j < k-1 < i-2$.

(2.5.46) $e_{2^{n+1}m+2^n-2^{j-1}-1}h_jh_kh_i \to e_{2^{n+1}m-2^{j-1}-1}h_jh_nh_kh_i$
for $m \geqslant 1$ and either $1 \leqslant j < n-1 < k-2 < i-3$
for $m \geqslant 1$ or $1 \leqslant j < n-2 = k-2 < i-3$.

(2.5.47) $e_{2^{j+1}m+2^{j-1}-1}h_jh_kh_i \to e_{2^{j+1}m-2^{j-1}-1}h_j^2h_kh_i$
for $m \geqslant 1,\ 1 \leqslant j < k-1 < i-2$.

(2.5.48) $e_{2^{k+1}m+2^{k-1}-2^{j-1}-1}h_jh_kh_i \to e_{2^{k+1}m-2^{k-1}-2^{j-1}-1}h_jh_k^2h_i$
for $m \geqslant 1,\ 1 \leqslant j < k-2 < i-3$.

(2.5.49) $e_{2^{j+3}m+2^{j+1}-2^{j-1}-1}h_jh_{j+2}h_i \to e_{2^{j+3}m-2^{j+2}-2^{j-1}-1}c_jh_i$
for $m \geqslant 1,\ 1 \leqslant j < i-3$.

(2.5.50) $e_{2^{k+n}m+2^k+2^{k-1}-2^{j-1}-1}h_jh_kh_i$
$\to e_{2^{k+n}m-2^{k+1}+2^{k-1}-2^{j-1}-1}h_jh_{k+1}^2h_i$
for $m \geqslant 1,\ 1 \leqslant j < k-1 < i-3$ and $n \geqslant 2$.

(2.5.51) $e_{2^{l+n}m+2^l-2^{k-1}-2^{j-1}-1}h_jh_kh_i$
$\rightarrow e_{2^{l+n}m-2^{k-1}-2^{j-1}-1}h_jh_kh_lh_i$
for $m \geqslant 1$, $1 \leqslant j < k-1 < l-2 < i-3$ and $n \geqslant 1$.

(2.5.52) $e_{2^{i+n}m+2^i-2^{k-1}-2^{j-1}-1}h_jh_kh_i$
$\rightarrow e_{2^{i+n}m-2^{k-1}-2^{j-1}-1}h_jh_kh_i^2$
for $m \geqslant 1$, $1 \leqslant j < k-1 < i-3$ and $n \geqslant 1$.

(2.5.53) $e_{2^{i+n}m+2^{i-1}-2^{k-1}-2^{j-1}-1}h_jh_kh_i$
$\rightarrow e_{2^{i+n}m-2^{i-1}-2^{k-1}-2^{j-1}-1}h_jh_kh_i^2$
for $m \geqslant 1$, $1 \leqslant j < k-1 < i-3$ and $n \geqslant 1$.

(2.5.54) $e_{2^{k+n}m+2^{k+1}-2^{k-1}-2^{j-1}-1}h_jh_kh_{k+2}$
$\rightarrow e_{2^{k+n}m-2^{k+2}-2^{k-1}-2^{j-1}-1}h_jc_k$
for $m \geqslant 1$, $1 \leqslant j < k-1$ and $n \geqslant 3$.

(2.5.55) $e_{2^{i+n+1}m+2^{i+n}-2^{i-1}-2^{k-1}-2^{j-1}-1}h_jh_kh_i$
$\rightarrow e_{2^{i+n+1}m-2^{i-1}-2^{k-1}-2^{j-1}-1}h_jh_kh_ih_{i+n}$
for $m \geqslant 1$, $1 \leqslant j < k-1 < i-2$ and $n \geqslant 2$.

(2.5.56) $e_{2^{i+2}m+2^i+2^{i-1}-2^{k-1}-2^{j-1}-1}h_jh_kh_i$
$\rightarrow e_{2^{i+2}m-2^i-2^{i-1}-2^{k-1}-2^{j-1}-1}h_jh_kh_{i+1}^2$
for $m \geqslant 1$, $1 \leqslant j < k-1 < i-2$.

(2.5.57) $e_{2^{k+n}m+2^{k+2}-2^{k-1}-2^{j-1}-1}h_jh_kh_{k+2}$
$\rightarrow e_{2^{k+n}m-2^{k+1}-2^k-2^{k-1}-2^{j-1}-1}h_jh_{k+2}^3$
for $m \geqslant 1$, $1 \leqslant j < k-1$ and $n \geqslant 3$.

(2.5.58) $e_{2^{j+2}m+2^j+2^{j-1}-1}h_jh_kh_i \rightarrow e_{2^{j+2}m-2^{j+1}+2^{j-1}-1}h_{j+1}^2h_kh_i$
for $m \geqslant 1$, $1 \leqslant j < k-2 < i-3$.

(2.5.59) $e_{2^{j+3}m+2^{j+2}-2^{j-1}-1}h_jh_{j+2}h_i \rightarrow e_{2^{j+3}m-2^{j+2}+2^{j-1}-1}h_{j+2}^3h_i$
for $m \geqslant 1$, $1 \leqslant j < i-4$.

(2.5.60) $e_{2^{j+3}m+2^{j+3}+2^{j+2}-2^{j-1}-1}h_jh_{j+2}h_{j+4} \rightarrow e_{2^{j+3}m-1}D_3(j-1)$
for $m \geqslant 1$, $j \geqslant 1$.

(2.5.61) $e_{2^l(2m+1)-1}h_jh_kh_i \rightarrow e_{2^{l+1}m-1}h_lh_jh_kh_i$
for $m \geqslant 1$, $0 \leqslant l < j-1 < k-2 < i-3$.

(2.5.62) $e_{2^{j+1}(2m+1)-1}c_j \rightarrow e_{2^{j+2}m-1}h_{j+1}c_j$ for $m \geqslant 1$, $j \geqslant 0$.

(2.5.63) $e_{2^{j+2}m+2^j-1}c_j \rightarrow e_{2^{j+2}m-2^j-1}c_jh_{j+1}$ for $m \geqslant 1$, $j \geqslant 0$.

(2.5.64) $e_{2^{j+4}m+2^{j+3}-2^j-1}c_j \rightarrow e_{2^{j+4}m-1}d_j$ for $m \geqslant 1$, $j \geqslant 0$.

(2.5.65) $e_{2^{j+m+3}-2^j-1}c_j \rightarrow e_{2^{j+m+3}-2^{j+3}-1}d_j$ for $m \geqslant 1$, $j \geqslant 0$.

(2.5.66) $e_{2^{j+n+1}m+2^{j+n}-2^{j+3}+2^{j+1}+2^{j-1}-1}c_j \rightarrow e_{2^{j+n+1}m-2^{j+3}+2^{j+1}+2^{j-1}-1}h_{j+n}c_j$
for $m \geqslant 1$, $j \geqslant 1$ and $n \geqslant 4$.

(2.5.67) $e_{2^{j+4}m+2^{j+1}+2^{j-1}-1}c_j \rightarrow e_{2^{j+4}m-2^{j+3}-2^{j+2}-2^{j-1}-1}c_{j+1}h_{j+2}$
for $m \geqslant 1$, $j \geqslant 1$.

(2.5.68) $e_{2^{j+2}m+2^{j-1}-1}c_j \rightarrow e_{2^{j+2}m-2^{j+1}+2^{j-1}-1}h_{j+1}c_j$
for $m \geqslant 1$, $j \geqslant 1$.

(2.5.69) $e_{2^{j+4}m+2^{j+2}+2^{j+1}+2^{j-1}-1}c_j \to e_{2^{j+4}m-2^{j-1}-1}d_j$ for $m \geqslant 1$, $j \geqslant 1$.

(2.5.70) $e_{2^{j+m+3}-2^{j+1}+2^{j-1}-1}c_j \to e_{2^{j+m+3}-2^{j+3}-2^{j-1}-1}d_j$ for $m \geqslant 1$, $j \geqslant 1$.

(2.5.71) $e_{2^{j+2}m+2^j+2^{j-1}-1}c_j \to e_{2^{j+2}m-2^{j-1}-1}h_{j+1}c_j$ for $m \geqslant 1$, $j \geqslant 1$.

(2.5.72) $e_{2^{j+4}m+2^{j+3}-2^{j-1}-1}c_j \to e_{2^{j+4}m-1}p_{j-1}$ for $m \geqslant 1$, $j \geqslant 1$.

(2.5.73) $e_{2^{j+3+m}-2^{j-1}-1}c_j \to e_{2^{j+3+m}-2^{j+3}-1}p_{j-1}$ for $m \geqslant 1$, $j \geqslant 1$.

(2.5.74) $e_{2^l(2m+1)-1}c_j \to e_{2^{l+1}m-1}h_lc_j$ for $m \geqslant 1$, $0 \leqslant l < j-1$.

This concludes the statements of all the non-trivial differentials $E_r^{*,3,*} \xrightarrow{d_r} E_r^{*,4,*}$.

We are going to obtain from the differentials (2.5.l) above the corresponding infinite cycles $\overline{(2.5.l)}$ analogous to the process that we get the infinite cycles $\overline{(2.k.l)}$ from the differentials (2.k.l) for $1 \leqslant k \leqslant 3$ discussed earlier. The process from $(2.5.l)'^s$ to $\overline{(2.5.l)}'^s$ will be described in a moment. The resulting infinite cycles $\overline{(2.5.l)}$ will then be listed together with the cohomology classes they represent. For this purpose we need to describe certain classes in $Ext_A^{3,*}(P)$ which, as will be seen, turn out to be indecomposable elements in the sense that they are not classes of the form $\widehat{h}_ih_jh_kh_l$ or $\widehat{c}_jh_k$ or $\widehat{h}_lc_i$. These classes will be given in (2.6) that follows. In order to describe these classes we note that there is an operation

$$(8)\quad Sq^0 : \widetilde{H}_*(P) \otimes \Lambda \to \widetilde{H}_*(P) \otimes \Lambda \quad \text{given by}$$
$$Sq^0(e_k\lambda_{i_1}\cdots\lambda_{i_s}) = e_{2k+1}\lambda_{2i_1+1}\cdots\lambda_{2i_s+1}$$

analogous to the operation $Sq^0 : \Lambda \to \Lambda$ in (c) of Section 1. Here again $\lambda_{i_1}\cdots\lambda_{i_s}$ is not necessarily admissible. The operation Sq^0 in (8) also commutes with the differential δ of $\widetilde{H}_*(P) \otimes \Lambda$ given in (3), and so induces an operation

$$Sq^0 : Ext_A^{s,t}(P) \to Ext_A^{s,2t+1}(P).$$

For example, $Sq^0(\widehat{h}_i = \{e_{2^i-1}\}) = \widehat{h}_{i+1} = \{e_{2^{i+1}-1}\}$ for $i \geqslant 1$ and $Sq^0(\widehat{c}_j = \{e_{2^{j+1}+2^j-1}\lambda^2_{2^{j+2}-1}\}) = \widehat{c}_{j+1} = \{e_{2^{j+2}+2^{j+1}-1}\lambda^2_{2^{j+3}-1}\}$ for $j \geqslant 0$.

Now we describe in (2.6) below the classes in $Ext_A^{3,*}(P)$ we want, where $(Sq^0)^i$ again denotes the composite $\underbrace{Sq^0\cdots Sq^0}_{i}$ if $i > 1$, is Sq^0 if $i = 1$ and is the identity map if $i = 0$.

(2.6) (1) $\widehat{d}_i = \{(Sq^0)^i(e_6\lambda_2\lambda_3^2 + e_4\lambda_4\lambda_3^2 + e_2\lambda_4\lambda_5\lambda_3 + e_1\lambda_5\lambda_1\lambda_7)\} \in Ext_A^{3,2^{i+4}+2^{i+1}-1}(P)$, $i \geqslant 0$.

(2) $\widehat{e}_i = \{(Sq^0)^i(e_8\lambda_3^3 + e_4(\lambda_5^2\lambda_3 + \lambda_7\lambda_3^2) + e_2(\lambda_3\lambda_5\lambda_7 + \lambda_9\lambda_3^2))\} \in Ext_A^{3,2^{i+4}+2^{i+2}+2^i-1}(P)$, $i \geqslant 0$.

(3) $\widehat{f}_i = \{(Sq^0)^i(e_4\lambda_0\lambda_7^2 + e_3(\lambda_9\lambda_3^2 + \lambda_3\lambda_5\lambda_7) + e_2\lambda_4\lambda_5\lambda_7)\} \in Ext_A^{3,2^{i+4}+2^{i+2}+2^{i+1}-1}(P)$, $i \geqslant 0$.

(4) $\widehat{g}_{i+1} = \{(Sq^0)^i(e_6\lambda_0\lambda_7^2 + e_5(\lambda_9\lambda_3^2 + \lambda_3\lambda_5\lambda_7) + e_3(\lambda_5\lambda_9\lambda_3 + \lambda_{11}\lambda_3^2))\} \in Ext_A^{3,2^{i+4}+2^{i+3}-1}(P)$, $i \geqslant 0$.

(5) $\widehat{p}_i = \{(Sq^0)^i(e_{14}\lambda_5\lambda_7^2 + e_{10}\lambda_9\lambda_7^2 + e_6\lambda_9\lambda_{11}\lambda_7)\} \in Ext_A^{3,2^{i+5}+2^{i+2}+2^i-1}(P)$, $i \geqslant 0$.

(6) $\widehat{D}_3(i) = \{(Sq^0)^i(e_{22}\lambda_1\lambda_7\lambda_{31} + e_{16}\lambda_{15}^3 + e_{14}\lambda_9\lambda_7\lambda_{31})\} \in Ext_A^{3,2^{i+6}+2^i-1}(P)$, $i \geqslant 0$.

(7) $\widehat{p}'_i = \left\{(Sq^0)^i\begin{pmatrix} e_{38}\lambda_1\lambda_{15}^2 + e_{30}\lambda_9\lambda_{51}^2 + e_{28}\lambda_{11}\lambda_{15}^2 + e_{22}\lambda_{17}\lambda_{15}^2 \\ + e_{20}\lambda_{19}\lambda_{15}^2 + e_{14}\lambda_1\lambda_{23}\lambda_{31} + e_{12}\lambda_{19}\lambda_{23}\lambda_{15} \end{pmatrix}\right\} \in Ext_A^{3,2^{i+6}+2^{i+3}+2^i-1}(P)$, $i \geqslant 0$.

(8) $\alpha_{16}(i) = \{(Sq^0)^i(e_2\lambda_0\lambda_7^2 + e_1(\lambda_9\lambda_3^2 + \lambda_3\lambda_5\lambda_7))\} \in Ext_A^{3,2^{i+4}+2^{i+2}-1}(P)$, $i \geqslant 0$.

(9) $\alpha_{21}(i) = \{(Sq^0)^i(e_2\lambda_5\lambda_7^2)\} \in Ext_A^{3,2^{i+4}+2^{i+3}+2^i-1}(P)$, $i \geqslant 0$.

(10) $\xi_{31}(i) = \left\{(Sq^0)^i\begin{pmatrix} e_{12}\lambda_5\lambda_7^2 + e_{10}\lambda_7^3 + e_6\lambda_1^2\lambda_{23} \\ + e_6\lambda_7\lambda_3\lambda_{15} + e_3\lambda_0\lambda_5\lambda_{23} \end{pmatrix}\right\} \in Ext_A^{3,2^{i+5}+2^{i+1}+2^i-1}(P)$, $i \geqslant 0$.

These classes in $Ext_A^{3,*}(P)$ are well defined in the following sense. For each k with $1 \leqslant k \leqslant 10$ let $f(k)$ denote the chain of $\widetilde{H}_*(P) \otimes \Lambda$ given in (2.6)(k). For example,

$$f(1) = e_6\lambda_2\lambda_3^2 + e_4\lambda_4\lambda_3^2 + e_2\lambda_4\lambda_5\lambda_3 + e_1\lambda_5\lambda_1\lambda_7.$$

It is not difficult to check, by direct computations, that $f(k)$ and therefore $(Sq^0)^i(f(k))$ are cycles for all k and i.

If one compares the classes $\widehat{d}_i, \widehat{e}_i, \ldots, \widehat{p}_i'$ in (2.6) above to the classes $d_i, e_i, \ldots, p_i'$ described in (1.1) of Section 1, one will notice a similarity between the formulations of these two sets of classes. For example, if each e_j in the cycle $f(1)$ above, that represents $\widehat{d}_0$, is replaced by the corresponding λ_j, the resulting

$$\lambda_6\lambda_2\lambda_3^2 + \lambda_4\lambda_4\lambda_3^2 + \lambda_2\lambda_4\lambda_5\lambda_3 + \lambda_1\lambda_5\lambda_1\lambda_7$$

is a cycle in Λ representing the class $d_0 \in Ext_A^{4,*}$ described in (1.1)(3). In this way we get correspondences

$$\widehat{d}_i \longleftrightarrow d_i, \quad \widehat{e}_i \longleftrightarrow e_i, \quad \ldots, \quad \widehat{p}_i' \longleftrightarrow p_i'.$$

These correspondences are relevant to the proofs of Theorems 1.3 and 1.4 and will be described more precisely in the next section.

We now proceed to describe the process "$(2.5.l) \to \overline{(2.5.l)}$" which is to obtain from the differentials $(2.5.l)$ the corresponding infinite cycles $\overline{(2.5.l)}$ and then to explain how we are going to list these infinite cycles together with the cohomology classes in $Ext_A^{3,*}(P)$ they represent.

We will consider the differentials $(2.5.l)$ for $1 \leqslant l \leqslant 74$ with $l \neq 1, 10$. For each such l let $\overline{(2.5.l)}$ denote the family of the elements $e_{f(0)}\alpha$ obtained by letting $m = 0$ in the source elements $e_{f(m)}\alpha$ of $(2.5.l)$ ($l = 1,\ 10$ excluded so that $f(0) > 0$). Then $e_{f(0)}\alpha \to 0$, that is, they are infinite cycles in the spectral sequence. In letting $m = 0$ in $(2.5.l)$ to get the corresponding $\overline{(2.5.l)}$ we have to adjust some of the restrictions on the integral variable j so that the resulting infinite cycles are in the "existing range". For example, in (2.5.2), the restriction on j is $j \geqslant 1$. The corresponding family in $\overline{(2.5.2)}$ is $e_{2^{j-1}-1}h_j^3$ for which the condition on j must be adjusted to $j \geqslant 2$ in order to have $2^{j-1} - 1 > 0$.

In addition to the resulting families $\overline{(2.5.l)}$ of infinite cycles thus obtained we will consider two families of infinite cycles that can not be obtained from any of the $(2.5.l)'^s$ by this process. These two additional families are $e_{2^{j+2}-2^{j-1}-1}h_jh_{j+3}^2$ for $j \geqslant 1$ and $e_{2^{j+2}-2^{j-1}-1}h_jh_{j+2}h_{j+4}$ for $j \geqslant 1$ and will be numbered respectively as $\overline{(2.5.75)}$ and $\overline{(2.5.76)}$ although there are no (2.5.75) and (2.5.76). As will be shown later, the set of the infinite cycles in $\overline{(2.5.l)}$ for $2 \leqslant l \leqslant 76$ with $l \neq 10$ will form a $\mathbb{Z}/2$-base for $E_\infty^{*,3,*}$.

We are going to list below these $\overline{(2.5.l)}$. Each will have the form $C \leftrightarrow D$. For example, $\overline{(2.5.3)}$ will be

$$\overline{(2.5.3)}\ \ e_{2^{j+n}-2^j-2^{j-1}-1}h_j^3 \longleftrightarrow \widehat{h}_{j+n}h_{j-1}^3, \quad n \geqslant 3,\ j \geqslant 1.$$

Here $C = e_{2^{j+n}-2^j-2^{j-1}-1}h_j^3$, for each $n \geqslant 3$ and $j \geqslant 1$, is the infinite cycle obtained by letting $m = 0$ in the source element $e_{2^{j+n}(2m+1)-2^j-2^{j-1}-1}h_j^3$ of the differential (2.5.3) and that $D = \widehat{h}_{j+n}h_{j-1}^3$ is the class in $Ext_A^{3,*}(P)$ represented by this infinite cycle C. Just how and why the class D is represented by the infinite cycle C, in this example and also in other $\overline{(2.5.l)}$, will be explained and proved later.

In order to make the list shorter, we will list these $\overline{(2.5.l)}$ in groups. Various $\overline{(2.5.l)}$ are put together in the same group if the infinite cycles in these $\overline{(2.5.l)}$ are of the "same type". For example, the infinite cycles in $\overline{(2.5.2)}$, $\overline{(2.5.5)}$, $\overline{(2.5.7)}$ and $\overline{(2.5.9)}$ are $e_{2^{j-1}-1}h_j^3$, $e_{2^{j-2}-1}h_j^3$, $e_{2^{j-3}-1}h_j^3$ and $e_{2^l-1}h_j^3$ with $l < j - 3$, respectively, and all of these are of the type "$e_{2^l-1}h_j^3$" with $1 \leqslant l \leqslant j - 1$. These four are put together in one group which is the first group $\overline{(2.5.l_1)}$. Most groups have at least two of the $\overline{(2.5.l)}'^s$. There are exactly 10 groups, each of which consisting of only one $\overline{(2.5.l)}$. These are precisely the infinite cycles representing the classes in (2.6)(1) through (2.6)(10). Finally we note that $\overline{(2.5.l)}$ is equal to $\overline{(2.5.l-1)}$ for $l = 65$, 70 and 73.

The grouped $\overline{(2.5.l)}'^s$ are listed as follows.

$\overline{(2.5.l_1)}$ $e_{2^l-1}h_j^3 \longleftrightarrow \widehat{h}_lh_j^3,\ 1 \leqslant l \leqslant j-1$ for $l_1 = 2,\ 5,\ 7,\ 9$.

$\overline{(2.5.l_2)}$ $e_{2^{j+n}-2^j-2^{j-1}-1}h_j^3 \longleftrightarrow \widehat{h}_{j+n}h_{j-1}^3,\ n \geqslant 2,\ j \geqslant 1$ for $l_2 = 3,\ 4$.

$\overline{(2.5.l_3)}$ $e_{2^{j+1}+2^{j-2}-1}h_j^3 \longleftrightarrow \widehat{e}_{j-2},\ j \geqslant 2$ for $l_3 = 6$.

$\overline{(2.5.l_4)}$ $e_{2^{j-1}+2^{j-2}-1}h_j^2h_k \longleftrightarrow \widehat{c}_{j-2}h_k,\ 2 \leqslant j < k-1$ for $l_4 = 8,\ 23$.

$\overline{(2.5.l_5)}$ $e_{2^l-1}h_j^2h_k \longleftrightarrow \widehat{h}_l h_j^2 h_k,\ 1 \leqslant l \leqslant j < k-2$
for $l_5 = 11,\ 21,\ 22,\ 24$.

$\overline{(2.5.l_6)}$ $e_{2^n-2^j-1}h_j^2h_k \longleftrightarrow \widehat{h}_n h_{j-1}^2 h_k,\ 3 \leqslant j+2 \leqslant n \leqslant k,\ j < k-2$
for $l_6 = 12, 13, 16, 17, 20$.

$\overline{(2.5.l_7)}$ $e_{2^{k+n}-2^{k-1}-2^j-1}h_j^2h_k \longleftrightarrow \widehat{h}_{k+n} h_{j-1}^2 h_{k-1},\ n \geqslant 1,\ 1 \leqslant j < k-2$
for $l_7 = 14, 15, 18, 19$.

$\overline{(2.5.l_8)}$ $e_{2^l-1}h_j h_k^2 \longleftrightarrow \widehat{h}_l h_j h_k^2,\ 1 \leqslant l \leqslant j < k-2$
for $l_8 = 25, 26, 31, 32, 44$.

$\overline{(2.5.l_9)}$ $e_{2^{j+2}+2^j-1}h_j h_{j+3}^2 \longleftrightarrow \widehat{f}_j,\ j \geqslant 0$ for $l_9 = 27$.

$\overline{(2.5.l_{10})}$ $e_{2^{j+1}+2^j-1}h_j h_{j+3}^2 \longleftrightarrow \alpha_{16}(j),\ j \geqslant 0$ for $l_{10} = 28$.

$\overline{(2.5.l_{11})}$ $e_{2^{j+2}+2^{j+1}+2^j-1}h_j h_{j+3}^2 \longleftrightarrow \widehat{g}_{j+1},\ j \geqslant 0$ for $l_{11} = 29$.

$\overline{(2.5.l_{12})}$ $e_{2^n-2^{j-1}-1}h_j h_k^2 \longleftrightarrow \widehat{h}_n h_{j-1} h_k^2,\ 2 \leqslant j+1 \leqslant n \leqslant k,\ j < k-2$
for $l_{12} = 30, 33, 34, 36, 37, 40, 41$ and 75.

$\overline{(2.5.l_{13})}$ $e_{2^{k-1}+2^{k-2}-2^{j-1}-1}h_j h_k^2 \longleftrightarrow \widehat{c}_{k-2} h_{j-1},\ 1 \leqslant j < k-2$
for $l_{13} = 35, 39$.

$\overline{(2.5.l_{14})}$ $e_{2^{j+4}+2^{j+1}+2^j+2^{j-1}-1}h_j h_{j+3}^2 \longleftrightarrow \widehat{p}'_{j-1},\ j \geqslant 1$ for $l_{14} = 38$.

$\overline{(2.5.l_{15})}$ $e_{2^{k+n}-2^k-2^{j-1}-1}h_j h_k^2 \longleftrightarrow \widehat{h}_{k+n} h_{j-1} h_{k-1}^2,\ n \geqslant 2,\ 1 \leqslant j < k-2$
for $l_{15} = 42, 43$.

$\overline{(2.5.l_{16})}$ $e_{2^l-1}h_j h_k h_i \longleftrightarrow \widehat{h}_l h_j h_k h_i,\ 1 \leqslant l \leqslant j < k-1 < i-2$
for $l_{16} = 45, 47, 61$.

$\overline{(2.5.l_{17})}$ $e_{2^n-2^{j-1}-1}h_j h_k h_i \longleftrightarrow \widehat{h}_n h_{j-1} h_k h_i,\ 2 \leqslant j+1 \leqslant n \leqslant k < i-1$ and $j < k-1$
for $l_{17} = 46, 48, 49, 58, 59, 76$.

$\overline{(2.5.l_{18})}$ $e_{2^l-2^{k-1}-2^{j-1}-1}h_j h_k h_i \longleftrightarrow \widehat{h}_l h_{j-1} h_{k-1} h_i,\ 4 \leqslant j+3 \leqslant k+1 \leqslant l \leqslant i$ and $k < i-1$
for $l_{18} = 50, 51, 52, 53, 54, 57$.

$\overline{(2.5.l_{19})}$ $e_{2^{i+n}-2^{i-1}-2^{k-1}-2^{j-1}-1}h_j h_k h_i \longleftrightarrow \widehat{h}_{i+n} h_{j-1} h_{k-1} h_{i-1},\ 1 \leqslant j < k-1 < i-2$ and $n \geqslant 1$
for $l_{19} = 55, 56$.

$\overline{(2.5.l_{20})}$ $e_{2^{j+3}+2^{j+2}-2^{j-1}-1}h_j h_{j+2} h_{j+4} \longleftrightarrow \widehat{D}_3(j-1),\ j \geqslant 1$
for $l_{20} = 60$.

$\overline{(2.5.l_{21})}$ $e_{2^l-1}c_j \longleftrightarrow \widehat{h}_l c_j,\ 1 \leqslant l \leqslant j+1$ for $l_{21} = 62, 63, 68, 74$.

$\overline{(2.5.l_{22})}$ $e_{2^{j+3}-2^j-1}c_j \longleftrightarrow \widehat{d}_j,\ j \geqslant 0$ for $l_{22} = 64, 65$
(note that $\overline{(2.5.64)} = \overline{(2.5.65)}$).

$\overline{(2.5.l_{23})}$ $e_{2^{j+n}-2^{j+3}+2^{j+1}+2^{j-1}-1}c_j \longleftrightarrow \widehat{h}_{j+n} c_{j-1},\ j \geqslant 1,\ n \geqslant 3$
for $l_{23} = 66, 67$.

$\overline{(2.5.l_{24})}$ $e_{2^{j+2}+2^{j+1}+2^{j-1}-1}c_j \longleftrightarrow \xi_{31}(j-1),\ j \geqslant 1$ for $l_{24} = 69, 70$
(note that $\overline{(2.5.69)} = \overline{(2.5.70)}$).

$\overline{(2.5.l_{25})}$ $e_{2^j+2^{j-1}-1}c_j \longleftrightarrow \alpha_{21}(j-1),\ j \geqslant 1$ for $l_{25} = 71$.

$\overline{(2.5.l_{26})}$ $e_{2^{j+3}-2^{j-1}-1}c_j \longleftrightarrow \widehat{p}_{j-1},\ j \geqslant 1$ for $l_{26} = 72, 73$
(note that $\overline{(2.5.72)} = \overline{(2.5.73)}$).

Each group $\overline{(2.5.l_j)}$ above has the form $C \leftrightarrow D$ where C are infinite cycles and D are the cohomology classes in $Ext_A^{3,*}(P)$ represented by these infinite cycles. We proceed to explain how these representations are obtained.

Given a chain $x = \sum_{j=1}^{n} e_{i_j}\lambda(j)$ in $\widetilde{H}_*(P) \otimes \Lambda^{s,*}$ where $i_1 < \cdots < i_n$ and each $\lambda(j)$ is a non-zero chain in $\Lambda^{s,*}$. We write $x \equiv e_{i_n}\lambda(n) \bmod F(i_n - 1)$. Here we recall that $F(i) = \sum_{k=1}^{i} \widetilde{H}_k(P) \otimes \Lambda$. From this equivalence equation we call i_n the filtration degree of x which is denoted by $m(x)$. For a non-zero class α in $Ext_A^{s,*}(P)$ we can always find an integer $m(\alpha) > 0$ and a cycle $x = \sum_{j=1}^{n} e_{i_j}\lambda(j) \in \widetilde{H}_*(P) \otimes \Lambda^{s,*}$ with $i_1 < \cdots < i_n$ such that

(i) $\alpha = \{x\}$, that is, α is represented by the cycle x.
(ii) $m(x) = i_n = m(\alpha)$.
(iii) α cannot be represented by any cycle y with $m(y) < m(\alpha)$.
(iv) $\lambda(n) \in \Lambda^{s,*}$ is a cycle representing a non-zero class $\{\lambda(n)\}$ in $Ext_A^{s,*}$.

In fact, property (iv) is a consequence of properties (i), (ii) and (iii). From the theory of the spectral sequence $\{E_r^{*,*,*}\}$ defined by the filtration $\{F(i) \mid i \geqslant 1\}$ for $\widetilde{H}_*(P) \otimes \Lambda$, we see these properties imply that $e_{i_n}\{\lambda(n)\}$ is a non-trivial cycle in $E_\infty^{*,s,*}$ representing the class α. And conversely, if $e_k\beta$ is a non-trivial infinite cycle in $E_\infty^{*,s,*}$ then there is a cycle $x = \sum_{j=1}^{n} e_{i_j}\lambda(j)$ in $\widetilde{H}_*(P) \otimes \Lambda$ with $m(x) = i_n = k$ such that $\lambda(n)$ is a cycle in $\Lambda^{s,*}$ with $\{\lambda(n)\} = \beta$ and such that $\gamma = \{x\}$ is a non-zero class in $Ext_A^{s,*}(P)$ which is represented by the infinite cycle $e_k\beta$.

We recall again that the expression $C \leftrightarrow D$ in each $\overline{(2.5.l_j)}$ above claims that the exhibited C are infinite cycles representing the cohomology classes D exhibited.

From the above theory of representations cohomology classes by infinite cycles we see immediately that the representations $C \leftrightarrow D$ as claimed in some of the $\overline{(2.5.l_j)}'^s$ are true. These include $\overline{(2.5.l_k)}$ for

$$k = 1, 3, 4, 5, 8, 9, 10, 11, 14, 16, 20, 21, 22, 24, 25 \text{ and } 26$$

noting the explicit cycle representations of $\widehat{h}_i$, $\widehat{c}_j$ in (7) and those for $\widehat{d}_i$, $\widehat{e}_i$, $\widehat{f}_i$, $\widehat{g}_{i+1}$, $\widehat{p}_i$, $\widehat{D}_3(i)$, $\widehat{p}'_i$, $\alpha_{16}(i)$, $\alpha_{21}(i)$ and $\xi_{31}(i)$ in (2.6) (and also the cycle representations for h_i, c_j in (1.1) of Section 1). For the claims $C \leftrightarrow D$ in the remaining $\overline{(2.5.l_j)}$ we have to make calculations to prove them. We will illustrate such calculations for two of these $\overline{(2.5.l_j)}$ and leave the proofs of the rest to the reader.

The first of these two is

$$\overline{(2.5.l_2)}\ e_{2^{j+n}-2^j-2^{j-1}-1}h_j^3 \longleftrightarrow \widehat{h}_{i+n}h_{j-1}^3, \quad n \geqslant 2,\ j \geqslant 1.$$

Now $\widehat{h}_{j+n}h_{j-1}^3$ is represented by the cycle $e_{2^{j+n}-1}\lambda_{2^{j-1}-1}^3$. We have to show that $e_{2^{j+n}-1}\lambda_{2^{j-1}-1}^3 \sim x$ for some cycle x such that

$$x \equiv e_{2^{j+n}-2^j-2^{j-1}-1}\lambda_{2^j-1}^3 \bmod F(2^{j+n} - 2^j - 2^{j-1} - 2).$$

Here "$\sim$" means "homologous". It suffices to do this for $j = 1$ and for any $j + n = n + 1 \geqslant 3$ since one can apply appropriate $(Sq^0)^i$. So we need to show

$\overline{(2.5.l_2)}^*$ $e_{2^i-1}\lambda_0^3 \sim x$ for some cycle x with

$$x \equiv e_{2^i-4}\lambda_1^3 \bmod F(2^i - 5) \quad \text{for any } i \geqslant 3.$$

By direct computations we find that

$$\delta(e_{2^i}\lambda_0^2 + e_{2^i-2}\lambda_2\lambda_0 + e_{2^i-3}\lambda_1\lambda_2) \equiv e_{2^i-1}\lambda_0^3 + e_{2^i-4}\lambda_3\lambda_0^2 \bmod F(2^i - 5).$$

$\overline{(2.5.l_2)}^*$ follows from this since $\lambda_3\lambda_0^2 \sim \lambda_1^3$. This proves $\overline{(2.5.l_2)}$.

The other one is

$$\overline{(2.5.l_{23})}\ e_{2^{j+n}-2^{j+3}+2^{j+1}+2^{j-1}-1}c_j \longleftrightarrow \widehat{h}_{j+n}c_{j-1}, \quad j \geqslant 1,\ n \geqslant 3.$$

Recall that $\lambda_{2^{k+1}+2^k-1}\lambda_{2^{k+2}-1}^2$ is a cycle representing c_k for $k \geqslant 0$. Again to prove $\overline{(2.5.l_{23})}$ it suffices to show it for $j = 1$ which is equivalent to proving

$\overline{(2.5.l_{23})}^*$ $e_{2^i-1}\lambda_2\lambda_3^2 \sim x$ for some cycle x such that

$$x \equiv e_{2^i-12}\lambda_5\lambda_7^2 \bmod F(2^i - 13) \quad \text{for any } i \geqslant 4.$$

We find that $\delta(z) \equiv e_{2^i-1}\lambda_2\lambda_3^2 + e_{2^i-12}\lambda_5\lambda_7^2 \bmod F(2^i - 13)$ where

$$z = e_{2^i+2}\lambda_3^2 + e_{2^i}\lambda_1\lambda_7 + e_{2^i-3}\lambda_0\lambda_{11} + e_{2^i-4}\lambda_5\lambda_7 + e_{2^i-5}\lambda_2\lambda_{11} + e_{2^i-6}(\lambda_{11}\lambda_3 + \lambda_7^2) + e_{2^i-8}\lambda_9\lambda_7 + e_{2^i-9}\lambda_6\lambda_{11} + e_{2^i-10}\lambda_7\lambda_{11}.$$

This proves $\overline{(2.5.l_{23})}^*$ and therefore $\overline{(2.5.l_{23})}$.

Recall that $B(3)$ denotes the set of all the boundaries in the differentials (2.3.1) through (2.3.16). Let $S(3)$ be the set of all the source elements in the differentials (2.5.1) through (2.5.74). Let $I(3)$ be the set of all the infinite cycles C in $\overline{(2.5.l_1)}$ through $\overline{(2.5.l_{26})}$ (recall each $\overline{(2.5.l_j)}$ is of the form $C \leftrightarrow D$). It is not difficult to check the following.

(9) $B(3) \cup S(3) \cup I(3)$ is a disjoint union and is a $\mathbb{Z}/2$-base for $E_1^{*,3,*}(P)$.

This implies $I(3)$ is a $\mathbb{Z}/2$-base $E_\infty^{*,3,*}$. From this we have the following conclusion. We recall again that each $\overline{(2.5.l_j)}$ is of the form $C \leftrightarrow D$.

Theorem 2.7. *The set of the cohomology classes D exhibited in $\overline{(2.5.l_1)}$ through $\overline{(2.5.l_{26})}$ is a $\mathbb{Z}/2$-base for $Ext_A^{3,*}(P)$. Thus the set of the classes $\widehat{d}_i$, $\widehat{e}_i$, $\widehat{f}_i$, $\widehat{g}_{i+1}$, $\widehat{p}_i$, $\widehat{D}_3(i)$, $\widehat{p}'_i$, $\alpha_{16}(i)$, $\alpha_{21}(i)$ and $\xi_{31}(i)$ for $i \geqslant 0$ is a $\mathbb{Z}/2$-base for the indecomposable elements in $Ext_A^{3,*}(P)$. Other classes in $Ext_A^{3,*}(P)$ are either of the form $\widehat{h_l}h_jh_kh_i$ or of the form $\widehat{c_i}h_j$ or of the form $\widehat{h_l}c_i$.*

Here "the set of the classes $\widehat{d}_i, \widehat{e}_i, \ldots, \xi_{31}(i)$ for $i \geqslant 0$ is a $\mathbb{Z}/2$-base for the indecomposable elements in $Ext_A^{3,*}(P)$" means the following. Let this set be $\widehat{S}$. Let $\widehat{D}$ be the $\mathbb{Z}/2$-submodule of $Ext_A^{3,*}(P)$ generated by $\widehat{h_l}h_jh_kh_i$, $\widehat{c_i}h_j$ and $\widehat{h_l}c_i$. Then the projection of $\widehat{S}$ to $Ext_A^{3,*}(P)/\widehat{D}$ is a $\mathbb{Z}/2$-base.

By some slight extra work, one can actually determine from $\overline{(2.5.l_j)}'^s$ the complete structure of $Ext_A^{3,*}(P)$ pertaining the decomposables $\widehat{h_l}h_jh_kh_i$, $\widehat{c_i}h_j$ and $\widehat{h_l}c_i$. This, however, will not be described here. In Section 3 we are going to use Theorem 2.7 to prove Theorem 1.3 and what has been stated in (2.7) suffices for this purpose.

3. Proofs of Theorems 1.3 and 1.4

We begin with the proof of Theorem 1.3 that occupies approximately two fifths of the section.

Recall from Theorem 1.2 that the algebra $Ext_A^{s,*}$ for $s \leqslant 3$ is generated by the generators $h_i \in Ext_A^{1,2^i}$, $c_i \in Ext_A^{3,2^{i+3}+2^{i+1}+2^i}$ for $i \geqslant 0$ as described in (1.1)(1), (2) of Section 1 subject only to the relations

(a) $h_ih_{i+1} = 0,\ h_ih_{i+2}^2 = 0$ and $h_i^3 = h_{i-1}^2h_{i+1}$.

We want to show for Theorem 1.3 the following.

$\overline{(1.3)}$ (1) The subalgebra of the algebra $Ext_A^{s,*}$ for $s \leqslant 4$ generated by h_i and c_i for $i \geqslant 0$ is subject only to the relations in (a) above together with the relations $h_i^2h_{i+3}^2 = 0$ for $i \geqslant 0$ and $h_jc_i = 0$ for $j = i-1, i, i+2, i+3$.

(2) The set of the classes d_i, e_i, f_i, g_{i+1}, p_i, $D_3(i)$, p'_i for $i \geqslant 0$ as described in (1.1)(3) through (1.1)(9) of Section 1 is a $\mathbb{Z}/2$-base for the indecomposable elements in $Ext_A^{4,*}$.

To prove these we first recall from [5] the following result.

Theorem 3.1. *(See [5].)*

(1) *The map $\widetilde{H}_*(P) \otimes \Lambda \xrightarrow{t} \Lambda$ given by $t(e_k\lambda_I) = \lambda_k\lambda_I$ is a chain map and commutes with the operations $Sq^0 : \widetilde{H}_*(P) \otimes \Lambda \to \widetilde{H}_*(P) \otimes \Lambda$ and $Sq^0 : \Lambda \to \Lambda$ described in* (8) *of Section* 2 *and* (c) *of Section* 1, *respectively.*

(2) *The induced map $Ext_A^{s,t}(P) \xrightarrow{t_*} Ext_A^{s+1,t+1}$ is onto for $t - s > 0$.*

This is known as the algebraic Kahn–Priddy theorem. (3.1)(1) is easy to see. We will also recall later how the "onto" result in (3.1)(2) is proved.

Let B_1 be the set of the classes

$$\widehat{h}_i,\ \widehat{c}_i,\ \widehat{d}_i,\ \widehat{e}_i,\ \widehat{f}_i,\ \widehat{g}_{i+1},\ \widehat{p}_i,\ \widehat{D}_3(i),\ \widehat{p}'_i,\ \alpha_{16}(i),\ \alpha_{21}(i) \text{ and } \xi_{31}(i) \quad \text{for } i \geqslant 0$$

in $Ext_A^{*,*}(P)$ and let B_2 be the set of the classes

$$h_i,\ c_i,\ d_i,\ e_i,\ f_i,\ g_{i+1},\ p_i,\ D_3(i) \text{ and } p'_i \quad \text{for } i \geqslant 0$$

in $Ext_A^{*,*}$. From the cycle representations for these classes described in (7) of Section 2, (2.6) and (1.1), and also from the formula for the chain map $\widetilde{H}_*(P) \otimes \Lambda \to \Lambda$ in (3.1)(1) we see the following.

(3.2) $t_*(\widehat{h}_i) = h_i$ for $i \geqslant 1$ and $t_*(\widehat{c}_i) = c_i,\ t_*(\widehat{d}_i) = d_i,\ t_*(\widehat{e}_i) = e_i,$
$t_*(\widehat{f}_i) = f_i,\ t_*(\widehat{g}_{i+1}) = g_{i+1},\ t_*(\widehat{p}_i) = p_i,\ t_*(\widehat{D}_3(i)) = D_3(i)$ and
$t_*(\widehat{p}'_i) = p'_i$ for $i \geqslant 0$.

We will prove in a moment the following.

(3.3) $t_*(\alpha_{16}(i)) = 0,\ t_*(\alpha_{21}(i)) = 0$ and $t_*(\xi_{31}(i)) = 0$ in $Ext_A^{4,*}$ for the remaining families $\alpha_{16}(i),\ \alpha_{21}(i)$ and $\xi_{31}(i)$ in the set

$$B_1 = \{\widehat{h}_i, \widehat{c}_i, \widehat{d}_i, \ldots, \widehat{p}'_i, \alpha_{16}(i), \alpha_{21}(i), \xi_{31}(i) \mid i \geqslant 0\}.$$

From Theorems 2.7, 3.1(2), (3.2) and (3.3) we deduce the following.

(3.4) The algebra $Ext_A^{s,*}$ for $s \leqslant 4$ is generated by

$$B_2 = \{h_i, c_i, d_i, e_i, f_i, g_{i+1}, p_i, D_3(i), p'_i \mid i \geqslant 0\}.$$

To prove $\overline{(1.3)}$ is then equivalent to proving the following.

$\overline{(1.3)}'$ (i) $h_i^2 h_{i+3}^2 = 0$ for $i \geqslant 0$ and $h_j c_i = 0$ for $j = i-1, i, i+2, i+3$ in $Ext_A^{4,*}$.

(ii) The set of the following classes in (1) through (18) is a $\mathbb{Z}/2$-base for $Ext_A^{4,*}$ where $i \geqslant 0$ in (1) through (8).

(1) d_i, (2) e_i, (3) f_i, (4) g_{i+1}, (5) p_i, (6) $D_3(i)$, (7) p'_i, (8) $c_i h_{i+1}$,
(9) $c_i h_j,\ 0 \leqslant j < i-1$, (10) $c_i h_j,\ 0 \leqslant i < j-3$, (11) $h_i^3 h_l,\ 0 \leqslant l < i-3$,
(12) $h_i^2 h_k^2,\ 0 \leqslant k < i-3$, (13) $h_i^2 h_k h_l,\ 0 \leqslant l < k-1 < i-3$,
(14) $h_i h_j^2 h_l,\ 0 \leqslant l < j-2 < i-4$, (15) $h_i h_j h_k^2,\ 0 \leqslant k < j-2 < i-3$,
(16) $h_i h_j^3,\ 0 \leqslant j < i-2$, (17) $h_i h_j h_k h_l,\ 0 \leqslant l < k-2 < j-2 < i-3$,
(18) h_0^4.

We note that in $\overline{(1.3)}'$(ii), the class h_0^4 lies in $Ext_A^{4,4}$ and is intentionally put at the end of the statement. All other classes lie in $Ext_A^{4,t}$ with $t - 4 > 0$.

We proceed to prove (3.3) and $\overline{(1.3)}'$ and we begin with the proofs of the triviality results (3.3) and $\overline{(1.3)}'$(i) which are easier.

Since $Sq^0(h_i^2 h_{i+3}^2) = h_{i+1}^2 h_{i+4}^2$ and $Sq^0(h_j c_i) = h_{j+1} c_{i+1}$, to prove $\overline{(1.3)}'$(i) it suffices to show

(b) $h_0^2 h_3^2 = 0,\ h_0 c_1 = 0,\ h_0 c_0 = 0,\ h_2 c_0 = 0$ and $h_3 c_0 = c_0 h_3 = 0$ in $Ext_A^{4,*}$.

Since t_* commutes with the operations $(Sq^0)^i$ (see (3.1)(1)), to prove (3.3) is to prove

(c) $t_*(\alpha_{16}(0)) = 0,\ t_*(\alpha_{21}(0)) = 0,\ t_*(\xi_{31}(0)) = 0$ in $Ext_A^{4,*}$.

The classes in (b) are represented respectively by the cycles

$$\lambda_0^2 \lambda_7^2, \quad \lambda_0 \lambda_5 \lambda_7^2 = \lambda_4 \lambda_1 \lambda_7^2 = 0, \quad \lambda_0 \lambda_2 \lambda_3^2 = \lambda_1^2 \lambda_3^2 = 0, \quad \lambda_3 \lambda_2 \lambda_3^2 \quad \text{and} \quad \lambda_2 \lambda_3^2 \lambda_7 = 0$$

in Λ. From (2.6) and the formula for t in (3.1)(1) we see the classes in (c) are represented respectively by the cycles

$$\lambda_2\lambda_0\lambda_7^2+\lambda_1(\lambda_3\lambda_5\lambda_7+\lambda_9\lambda_3^2)=\lambda_3\lambda_5^2\lambda_3+\lambda_5^2\lambda_3^2,\ \lambda_2\lambda_5\lambda_3^2=0 \quad \text{and}$$
$$\lambda_{12}\lambda_5\lambda_7^2+\lambda_{10}\lambda_7^3+\lambda_6(\lambda_1^2\lambda_{23}+\lambda_7\lambda_3\lambda_{15})+\lambda_3\lambda_0\lambda_5\lambda_{23}.$$

In Λ, we have $\delta(\lambda_9\lambda_3^2+\lambda_3\lambda_5\lambda_7)=\lambda_0^2\lambda_7^2$, $\delta(\lambda_6\lambda_3^2)=\lambda_3\lambda_2\lambda_3^2$, $\delta(\lambda_5\lambda_1\lambda_{11})=\lambda_5^2\lambda_3^2+\lambda_3\lambda_5^2\lambda_3$ and

$$\begin{aligned}&\delta\big[\lambda_{12}(\lambda_{17}\lambda_3+\lambda_9\lambda_{11})+\lambda_8(\lambda_{13}\lambda_{11}+\lambda_9\lambda_{15})\big]\\ &\quad=\lambda_{12}\lambda_5\lambda_7^2+\lambda_{10}\lambda_7^3+\lambda_6(\lambda_1^2\lambda_{23}+\lambda_7\lambda_3\lambda_{15})+\lambda_3\lambda_0\lambda_5\lambda_{23}.\end{aligned}$$

This proves (b) and (c) and therefore (3.3) and $\overline{(1.3)}'$(i).

To prove $\overline{(1.3)}'$(ii) we need to do some preparatory work. We begin by recalling from [5] a map $Ext_A^{s+1,t+1} \xrightarrow{\phi_*} Ext_A^{s,t}(P)$ which "essentially" is the right inverse of the map t_* in (3.1)(2). Precise meaning of this "essentially" is not important here and so will not be explained. The map ϕ_* is induced by a chain map $\Lambda \xrightarrow{\phi} \widetilde{H}_*(P)\otimes\Lambda$. In (3.5) below we recall the construction of this chain map which not only will be crucial to the proof of $\overline{(1.3)}'$(ii) but also will be crucial to the proof of Theorem 1.4 later.

(3.5) Define a map $\Lambda \xrightarrow{\phi} \widetilde{H}_*(P)\otimes\Lambda$ on any admissible monomial $\lambda_I=\lambda_{i_1}\cdots\lambda_{i_s}$ as follows. $\phi(\lambda_{i_1})=e_{i_1}$ for $i_1\geqslant 1$ if $s=1$. If $s\geqslant 2$ then $\phi(\lambda_I)=0$ for $i_1=0$ (which implies $i_k=0$ for $k\geqslant 2$) and, for $i_1\geqslant 1$, $\phi(\lambda_I)$ is defined to be

$$(*)\quad \phi(\lambda_I=\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_s})=e_{i_1}\lambda_{I'}+\sum_\nu e_{j_1(\nu)}\lambda_{J'(\nu)},$$

where $I'=(i_2,\dots,i_s)$ and the second sum is described as follows. First we require each $J'(\nu)=(j_2(\nu),\dots,j_s(\nu))$ be admissible and $J(\nu)=(j_1(\nu),j_2(\nu),\dots,j_s(\nu))$ be inadmissible. Secondly, choose any large integer m (compared to i_j and s) and let

$$J(\nu,m)=\big(2^m+j_1(\nu),j_2(\nu),\dots,j_s(\nu)\big)$$

which is admissible. Then $e_{j_1(\nu)}\lambda_{J'(\nu)}$ appears in the second sum of $(*)$ if and only if, for some $q\geqslant 2$, $\lambda_{J(\nu,m)}$ appears in the admissible expansion of $\lambda_{i_1}\cdots\lambda_{i_{q-1}}\lambda_{i_q+2^m}\lambda_{i_{q+1}}\cdots\lambda_{i_s}$.

The following result (3.6) on some properties of the map ϕ above is proved in [4,5] ((3.6)(1), (2) are proved in [5] and (3.6)(3), (4) are proved in [4]). To state the result we recall that the filtration $\{F(i)\mid i\geqslant 1\}$ of $\widetilde{H}_*(P)\otimes\Lambda$ is given by $F(i)=\sum_{k=1}^i \widetilde{H}_k(P)\otimes\Lambda$. Extend this filtration to $\{F(i)\mid i\geqslant 0\}$ by letting $F(0)=0$. Define an increasing filtration $\{\Lambda(i)\mid i\geqslant 0\}$ of Λ as follows. For each $i\geqslant 0$ let $\Lambda(i)$ be the $\mathbb{Z}/2$-submodule of Λ generated by the admissible monomials $\lambda_{i_1}\cdots\lambda_{i_s}$ with $i_1\leqslant i$. It is not difficult to show that each $\Lambda(i)$ is indeed a subcomplex of Λ.

Theorem 3.6. *(See [4,5].)*

(1) *The map* $\Lambda \xrightarrow{\phi} \widetilde{H}_*(P)\otimes\Lambda$ *constructed in* (3.5) *is well defined, is a chain map and commutes with the operations* $\Lambda \xrightarrow{Sq^0} \Lambda$ *and* $\widetilde{H}_*(P)\otimes\Lambda \xrightarrow{Sq^0} \widetilde{H}_*(P)\otimes\Lambda$.
(2) *Let* ψ *be the composite* $\Lambda \xrightarrow{\phi} \widetilde{H}_*(P)\otimes\Lambda \xrightarrow{t} \Lambda$ *where* t *is as in* (3.1)(1). *Then for each admissible* $\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_s}$ *in* Λ *with* $i_1\geqslant 1$ *there is the relation*

$$\psi(\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_s})\equiv\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_s} \bmod \Lambda(i_1-1).$$

(3) $\phi(\Lambda(i))\subseteq F(i)$ *for all* $i\geqslant 0$.
(4) $\phi(\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_s})\equiv e_{i_1}\lambda_{i_2}\cdots\lambda_{i_s} \bmod F(i_1-1)$ *for any admissible* $\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_s}$ *with* $i\geqslant 1$.

We should remark that the last conclusion in (3.6)(1) is actually not proved in [5]. But from the construction of ϕ in (3.5) it is not difficult to see that this result is true.

It is easy to see that the result (3.1)(2) (the algebraic Kahn–Priddy theorem) follows from (3.6)(2). It is only the properties (1), (3) and (4) in (3.6) that we will need for proving $\overline{(1.3)}'$(ii) (and also Theorem 1.4 later).

Given a class α in $Ext_A^{s,t}$ with $t-s>0$ and $s \geqslant 1$. Let x be a cycle in $\Lambda^{s,t-s}$ representing the class α. Suppose $x \neq 0$. Since $t-s>0$ we should have the following.

(d) $x \equiv \lambda_j \lambda(j) \bmod \Lambda(j-1)$ for some $j \geqslant 1$ where $\lambda(j)$ is a non-zero chain in $\Lambda^{s-1,t-s-j}$ and $\lambda_j \lambda_{j_1} \cdots \lambda_{j_{s-1}}$ is admissible for any admissible $\lambda_{j_1} \cdots \lambda_{j_{s-1}}$ appearing in the admissible expansion of $\lambda(j)$.

Call $\lambda_j \lambda(j)$ in (d) the leading term of the cycle x and we write $x = \overline{\lambda_j \lambda(j)}$ to denote this relation. Thus $\alpha = \{x\} = \{\overline{\lambda_j \lambda(j)}\}$. Since each $\Lambda(i)$ is a subcomplex of Λ, the chain $\lambda(j)$ in (d) is actually a cycle.

Now apply the chain map ϕ in (3.6)(1) to x in (d) above. From (3.6) we see the induced map $Ext_A^{s,t} \xrightarrow{\phi_*} Ext_A^{s-1,t-1}(P)$ of ϕ carries α to $\phi_*(\alpha)$ which is represented by the cycle $\phi(x) \in \widetilde{H}_*(P) \otimes \Lambda$ with

(e) $\phi(x) \equiv e_j \lambda(j) \bmod F(j-1)$.

Since $\lambda(j)$ is a cycle we can consider $e_j\{\lambda(j)\}$ which is an element in the E_1-term $E_1^{*,s-1,*}$ of the spectral sequence $\{E_r^{*,*,*}\}_{r \geqslant 1}$ considered in Section 2. The process from α to $e_j\{\lambda(j)\}$ in $E_1^{*,s-1,*}$ via (d), (e) depends on the representing cycle x for α. Since $\phi(x)$ is a cycle, from (e), we see $e_j\{\lambda(j)\}$ is actually an infinite cycle in the spectral sequence which may or may not be a non-zero one. We will write $e_j\{\lambda(j)\} \neq 0$ if $e_j\{\lambda(j)\}$ is a non-zero infinite cycle and write $e_j\{\lambda(j)\} = 0$ to mean that it is a boundary in the spectral sequence.

In case $e_j\{\lambda(j)\} \neq 0$ we will let $\bar{\phi}(\alpha = \{x\})$, or simply, $\bar{\phi}(\alpha)$, to denote $e_j\{\lambda(j)\}$ and use the correspondence

(f) $\alpha = \{x\} = \{\overline{\lambda_j \lambda(j)}\} \to e_j\{\lambda(j)\} = \bar{\phi}(\alpha = \{x\}) = \bar{\phi}(\alpha)$

to denote the connection from α to the non-trivial infinite cycle $e_j\{\lambda(j)\}$ via (d) and (e).

Suppose $e_j\{\lambda(j)\} = 0$. This does not necessarily imply that $\phi_*(\alpha) = \{\phi(x)\}$ is zero in $Ext_A^{s-1,*}(P)$. It only implies that the cycle $\phi(x)$ is homologous to some cycle $z \in \widetilde{H}_*(P) \otimes \Lambda$ with $z \in F(j-1)$ where j is as in (d), (e). Suppose one can find such a cycle z having the properties in (g) below. To state (g) we fix a notation. Given two cycles u and v in $\widetilde{H}_*(P) \otimes \Lambda$. We write $u \sim v \equiv e_l \lambda(l) \bmod F(l-1)$ to mean that u is homologous to v with $v \equiv e_l \lambda(l) \bmod F(l-1)$.

(g) (i) $\phi(x) \sim z \equiv e_k \lambda(k) \bmod F(k-1)$ for some k with $1 \leqslant k < j$ and some $\lambda(k) \in \Lambda^{s-1,*}$which is necessarily a cycle, where j is as in (d), (e).

(ii) $e_k\{\lambda(k)\}$ is a non-trivial infinite cycle in $E_\infty^{*,s-1,*}$.

In this case we will write $\tilde{\phi}(\alpha)$ to denote the non-trivial cycle $e_k\{\lambda(k)\}$ and use the correspondence

(h) $\alpha = \{x\} = \{\overline{\lambda_j \lambda(j)}\} \to e_k\{\lambda(k)\} = \tilde{\phi}(\alpha)$

to denote the connection from α to the non-trivial infinite cycle $e_k\{\lambda(k)\}$ via (d) and (g).

The reason to consider the notion (g) that leads to the correspondence (h) is the following. Given a non-zero class $\alpha \in Ext_A^{s,*}$. It may happen that no matter what cycle $x = \overline{\lambda_j \lambda(j)} \in \Lambda^{s,*}$ one chooses to represent α, the class $\{\lambda(j)\} \in Ext_A^{s-1,*}$ is always zero; so α has no correspondence of type (f). For such an α, since $\alpha \neq 0$, from Theorem 3.6 we see α always has property (g) and therefore has a correspondence of type (h). A typical example for such an α is the class f_i for any $i \geqslant 0$. This will be seen later when we come to prove (3.8.3) in which we have to consider the correspondence (h) for f_i. Many other examples will arise when we come to prove Theorem 1.4 later.

If $\alpha \in Ext_A^{s,*}$ is a class such that either there is a correspondence as that in (f) for α or there is a correspondence as that in (h) for α then we say α has either (f) or (h).

Proposition 3.7.

(1) *If $\alpha \in Ext_A^{s,*}$ is a class having either* (f) *or* (h) *then α is non-zero.*
(2) *If S is a set of classes in $Ext_A^{s,*}$ such that each $\alpha \in S$ has either* (f) *or* (h)*, so that either $\bar{\phi}(\alpha)$ or $\tilde{\phi}(\alpha)$ is defined, and such that the set $\{\bar{\phi}(\alpha)$ or $\tilde{\phi}(\alpha) \mid \alpha \in S\}$ is a linearly independent subset of non-trivial infinite cycles in $E_\infty^{*,s-1,*}$ then S is a linearly independent subset of $Ext_A^{s,*}$.*

Proof. If $\alpha \in Ext_A^{s,*}$ has either (f) or (h) then $\phi_*(\alpha) \in Ext_A^{s-1,*}$ is represented either by the non-trivial infinite cycle $e_j\{\lambda(j)\} = \overline{\phi}(\alpha)$ or by the non-trivial infinite cycle $e_k\{\lambda(k)\} = \widetilde{\phi}(\alpha)$ in the spectral sequence. So $\phi_*(\alpha) \neq 0$ and this implies $\alpha \neq 0$. This proves (3.7)(1). (3.7)(2) is clear. This proves Proposition 3.7. □

We are going to use Proposition 3.7 to prove $\overline{(1.3)}'$(ii). Recall that we want to prove that the set of the classes in $\overline{(1.3)}'$(ii) is a $\mathbb{Z}/2$-base for $Ext_A^{4,*}$. Call this set B. By (3.4) and $\overline{(1.3)}'$(i) we see this is equivalent to proving that B is linearly independent in $Ext_A^{4,*}$. We shall prove this by showing the following result (3.8).

In order to state this result we fix some notations. Recall that, for each $i \geqslant 0$, the class $h_i \in Ext_A^{1,*}$ is represented by the cycle $\lambda_{2^i-1} \in \Lambda^{1,*}$ and the class $c_i \in Ext_A^{3,*}$ is represented by the cycle $\lambda_{2^{i+1}+2^i-1}\lambda^2_{2^{i+2}-1} \in \Lambda^{3,*}$. To simplify, we will let h_i^* denote the cycle λ_{2^i-1} and let c_i^* denote the cycle $\lambda_{2^{i+1}+2^i-1}\lambda^2_{2^{i+2}-1}$. Note that $\lambda_n h_i^* h_j^* h_k^* = \lambda_n\lambda_{2^i-1}\lambda_{2^j-1}\lambda_{2^k-1}$ is admissible if and only if $2n \geqslant 2^i-1$ and $i \geqslant j \geqslant k$, and $\lambda_m c_i^* = \lambda_m\lambda_{2^{i+1}+2^i-1}\lambda^2_{2^{i+2}-1}$ is admissible if and only if $2m \geqslant 2^{i+1}+2^i-1$.

The result (3.8) below consists of the statements (3.8.1) through (3.8.17). For each n with $1 \leqslant n \leqslant 17$ and $n \neq 3$, (3.8.n) describes, for each class α in $\overline{(1.3)}'(\mathrm{ii})(n)$, a correspondence of type (f) for α which is of the form $\alpha = \{\overline{\lambda_j\lambda(j)}\} \to e_j\{\lambda(j)\} = \overline{\phi}(\alpha)$. And (3.8.3) describes, for each class α in $\overline{(1.3)}'$(ii)(3), a correspondence of type (h) for α which is of the form $\alpha = \{\overline{\lambda_j\lambda(j)}\} \to e_k\{\lambda(k)\} = \widetilde{\phi}(\alpha)$. These $\overline{\phi}(\alpha)$ or $\widetilde{\phi}(\alpha)$ are to be non-trivial infinite cycles in $E_\infty^{*,3,*}$. Recall that $\overline{(2.5.l_1)}$ through $\overline{(2.5.l_{26})}$ in Section 2 is the list of a basis for $E_\infty^{*,3,*}$. Right after $\overline{\phi}(\alpha)$ or $\widetilde{\phi}(\alpha)$ in these $(3.8.n)'^s$ we attach an appropriate $\overline{(2.5.l_j)}$ to indicate that $\overline{\phi}(\alpha)$ or $\widetilde{\phi}(\alpha)$ belongs to the family $\overline{(2.5.l_j)}$. For example, "$e_{2^{i+3}-2^i-1}c_i = \overline{\phi}(d_i)$, $\overline{(2.5.l_{22})}$" at the end of (3.8.1) indicates that the non-trivial infinite cycle $e_{2^{i+3}-2^i-1}c_i$ belongs to the family $\overline{(2.5.l_{22})}$. Finally, the restriction on i in (3.8.1) through (3.8.8) is $i \geqslant 0$.

(3.8.1) $d_i = \{\overline{\lambda_{2^{i+3}-2^i-1}c_i^*}\} \to e_{2^{i+3}-2^i-1}c_i = \overline{\phi}(d_i)$, $\overline{(2.5.l_{22})}$.

(3.8.2) $e_i = \{\overline{\lambda_{2^{i+3}+2^i-1}(h^*_{i+2})^3}\} \to e_{2^{i+3}+2^i-1}h^3_{i+2} = \overline{\phi}(e_i)$, $\overline{(2.5.l_3)}$.

(3.8.3) $f_i = \{\overline{\lambda_{2^{i+2}+2^{i+1}-1}h^*_{i+3}(h^*_{i+2})^2}\}$
$\to e_{2^{i+2}+2^i-1}h^2_{i+3}h_i = \widetilde{\phi}(f_i)$, $\overline{(2.5.l_9)}$.

(3.8.4) $g_{i+1} = \{\overline{\lambda_{2^{i+2}+2^{i+1}+2^i-1}(h^*_{i+3})^2h^*_i}\}$
$\to e_{2^{i+2}+2^{i+1}+2^i-1}h^2_{i+3}h_i = \overline{\phi}(g_{i+1})$, $\overline{(2.5.l_{11})}$.

(3.8.5) $p_i = \{\overline{\lambda_{2^{i+4}-2^i-1}c^*_{i+1}}\} \to e_{2^{i+4}-2^i-1}c_{i+1} = \overline{\phi}(p_i)$, $\overline{(2.5.l_{26})}$.

(3.8.6) $D_3(i) = \{\overline{\lambda_{2^{i+4}+2^{i+3}-2^i-1}h^*_{i+5}h^*_{i+3}h^*_{i+1}}\}$
$\to e_{2^{i+4}+2^{i+3}-2^i-1}h_{i+5}h_{i+3}h_{i+1} = \overline{\phi}(D_3(i))$, $\overline{(2.5.l_{20})}$.

(3.8.7) $p'_i = \{\overline{\lambda_{2^{i+5}+2^{i+2}+2^{i+1}+2^i-1}(h^*_{i+4})^2h^*_{i+1}}\}$
$\to e_{2^{i+5}+2^{i+2}+2^{i+1}+2^i-1}h^2_{i+4}h_{i+1} = \overline{\phi}(p'_i)$, $\overline{(2.5.l_{14})}$.

(3.8.8) $h_{i+1}c_i = \{\overline{\lambda_{2^{i+i}-1}c_i^*}\} \to e_{2^{i+1}-1}c_i = \overline{\phi}(h_{i+1}c_i)$, $\overline{(2.5.l_{21})}$.

(3.8.9) $c_ih_j = \{\overline{\lambda_{2^{i+1}+2^i-2^j-1}(h^*_{i+2})^2h^*_{j+1}}\}$
$\to e_{2^{i+1}+2^i-2^j-1}h^2_{i+2}h_{j+1} = \overline{\phi}(c_ih_j)$
for $0 \leqslant j < i-1$, $\overline{(2.5.l_{13})}$.

(3.8.10) $c_ih_j = h_jc_i = \{\overline{\lambda_{2^j-2^{i+4}+2^{i+2}+2^i-1}c^*_{i+1}}\}$
$\to e_{2^j-2^{i+4}+2^{i+2}+2^i-1}c_{i+1} = \overline{\phi}(c_ih_j)$
for $0 \leqslant i < j-3$, $\overline{(2.5.l_{23})}$.

(3.8.11) $h_i^3h_l = \{\overline{\lambda_{2^i-2^l-1}(h_i^*)^2h^*_{l+1}}\}$
$\to e_{2^i-2^l-1}h_i^2h_{l+1} = \overline{\phi}(h_i^3h_l)$, for $0 \leqslant l < i-3$, $\overline{(2.5.l_{12})}$.

(3.8.12) $h_i^2 h_k^2 = \{\overline{\lambda_{2^i-2^{k+1}-1} h_i^* (h_{k+1}^*)^2}\}$
$\quad \to e_{2^i-2^{k+1}-1} h_i h_{k+1}^2 = \overline{\phi}(h_i^2 h_k^2)$, for $0 \leqslant k < i-3$, $\overline{(2.5.l_6)}$.

(3.8.13) $h_i^2 h_k h_l = \{\overline{\lambda_{2^i-2^k-2^l-1} h_i^* h_{k+1}^* h_{l+1}^*}\}$
$\quad \to e_{2^i-2^k-2^l-1} h_i h_{k+1} h_{l+1} = \overline{\phi}(h_i^2 h_k h_l)$
$\quad$ for $0 \leqslant l < k-1 < i-3$, $\overline{(2.5.l_{18})}$.

(3.8.14) $h_i h_j^2 h_l = \{\overline{\lambda_{2^i-2^{j+1}-2^l-1} (h_{j+1}^*)^2 h_{l+1}^*}\}$
$\quad \to e_{2^i-2^{j+1}-2^l-1} h_{j+1}^2 h_{l+1} = \overline{\phi}(h_i h_j^2 h_l)$
$\quad$ for $0 \leqslant l < j-2 < i-4$, $\overline{(2.5.l_{15})}$.

(3.8.15) $h_i h_j h_k^2 = \{\overline{\lambda_{2^i-2^j-2^{k+1}-1} h_{j+1}^* (h_{k+1}^*)^2}\}$
$\quad \to e_{2^i-2^j-2^{k+1}-1} h_{j+1} h_{k+1}^2 = \overline{\phi}(h_i h_j h_k^2)$
$\quad$ for $0 \leqslant k < j-2 < i-3$, $\overline{(2.5.l_7)}$.

(3.8.16) $h_i h_j^3 = \{\overline{\lambda_{2^i-2^{j+1}-2^j-1} (h_{j+3}^*)^3}\}$
$\quad \to e_{2^i-2^{j+1}-2^j-1} h_{j+1}^3 = \overline{\phi}(h_i h_j^3)$, for $0 \leqslant j < i-2$, $\overline{(2.5.l_2)}$.

(3.8.17) $h_i h_j h_k h_l = \{\overline{\lambda_{2^i-2^j-2^k-2^l-1} h_{j+1}^* h_{k+1}^* h_{l+1}^*}\}$
$\quad \to e_{2^i-2^j-2^k-2^l-1} h_{j+1} h_{k+1} h_{l+1} = \overline{\phi}(h_i h_k h_k h_l)$
$\quad$ for $0 \leqslant l < k-1 < j-2 < i-3$, $\overline{(2.5.l_{19})}$.

We recall again that the left-sided classes in $\overline{(2.5.l_1)}$ through $\overline{(2.5.l_{26})}$ in Section 2 is a $\mathbb{Z}/2$-base for $E_\infty^{*,3,*}$. Let B' be the set of the classes in $Ext_A^{4,*}$ exhibited in $(3.8) = \bigcup_{n=1}^{17}(3.8.n)$ above. For each $\alpha \in B'$ the corresponding non-trivial infinite cycle $\overline{\phi}(\alpha)$ or $\widetilde{\phi}(\alpha)$ belongs to some $\overline{(2.5.l_j)}$ as depicted in (3.8). Since different $\overline{(2.5.l_j)}$ are attached to different (3.8.k), and this is easy to check, it follows that the set $\{\overline{\phi}(\alpha)$ or $\widetilde{\phi}(\alpha) \mid \alpha \in B'\}$ is a linearly independent subset of $E_\infty^{*,3,*}$. This implies B' is a linearly independent subset of $Ext_A^{4,*}$ by Proposition 3.7. Now B' is precisely the set of the classes listed in $\overline{(1.3)}'$(ii)(1) through $\overline{(1.3)}'$(ii)(17). Together with the class h_0^4 in $\overline{(1.3)}'$(ii)(18) the set $B = B' \cup \{h_0^4\}$ is therefore also linearly independent in $Ext_A^{*,4}$ since $h_0^4 \neq 0$ lies in $Ext_A^{4,4}$ while each $\alpha \in B'$ lies in $Ext_A^{4,t}$ for some t with $t > 4$. This proves $\overline{(1.3)}'$(ii) modulo the proof of (3.8).

We proceed to prove (3.8). First we explain what are to be proved. To prove (3.8.1), for example, is to show that, for each $i \geqslant 0$, the class d_i can be represented by a cycle $x \in \Lambda^{4,*}$ whose leading term is the admissible monomial

$$\lambda_{2^{i+3}-2^i-1} c_i^* = \lambda_{2^{i+3}-2^i-1} \lambda_{2^{i+1}+2^i-1} \lambda_{2^{i+2}-1}^2.$$

Once this done, that the corresponding $\overline{\phi}(d_i) = e_{2^{i+3}-2^i-1} c_i$ is a non-trivial infinite cycle belonging to the family $\overline{(2.5.l_{22})}$ is clear. All other (3.8.n) with $n \neq 3$ are to be proved this way. The correspondence (3.8.3) is of type (h) and will be given a special treatment of its proof.

To prove (3.8.1) through (3.8.7) we have to recall from (1.1) some specific cycle representations in $\Lambda^{5,*}$ for the classes in $B_3 = \{d_i, e_i, f_i, g_{i+1}, p_i, D_3(i), p_i' \mid i \geqslant 0\}$. Actually, for the purpose of making calculations for proving Theorem 1.4 later, we will give, for each class α in B_3, either

(i) a specific cycle representation $\overline{\alpha}(1)$, or
(ii) two specific cycle representations $\overline{\alpha}(1)$ and $\overline{\alpha}(2)$, or
(iii) three specific cycle representations $\overline{\alpha}(1)$, $\overline{\alpha}(2)$ and $\overline{\alpha}(3)$.

Only the cycle representations $\overline{\alpha}(1)$, for $\alpha \in B_3$, which are in admissible forms, will be relevant to the proofs of (3.8.1) through (3.8.7). If $\alpha \in B_3$ is a class in the case (ii) then the cycle representations $\overline{\alpha}(1)$ and $\overline{\alpha}(2)$, either

(ii)$'$ are different cycles but homologous, or
(ii)$''$ are equal but $\overline{\alpha}(2)$ is in inadmissible form.

In case (ii)′ we will give a specific chain $y \in \Lambda^{3,*}$ with $\delta(y) = \bar{\alpha}(1) + \bar{\alpha}(2)$ showing that $\bar{\alpha}(1) \sim \bar{\alpha}(2)$. In case (ii)″ we will write $\bar{\alpha}(1) = \bar{\alpha}(2)$. If $\alpha \in B_3$ is in the case (iii) then $\bar{\alpha}(1)$ and $\bar{\alpha}(2)$ will be in the situation (ii)′, and $\bar{\alpha}(1)$ and $\bar{\alpha}(3)$ will be in the situation (ii)″. These cycle representations are described in (3.9) below where $i \geqslant 0$ in each (3.9.k).

(3.9.1) $\bar{d}_i(1) = (Sq^0)^i(\lambda_6\lambda_2\lambda_3^2 + \lambda_4^2\lambda_3^2 + \lambda_2\lambda_4\lambda_5\lambda_3)$,
$\bar{d}_i(2) = (Sq^0)^i(\lambda_6\lambda_2\lambda_3^2 + \lambda_3\lambda_0^2\lambda_{11})$,
$\delta[(Sq^0)^i(\lambda_4\lambda_0\lambda_{11} + \lambda_2^2\lambda_{11} + \lambda_1\lambda_{11}\lambda_3 + \lambda_3\lambda_5\lambda_7)] = \bar{d}_i(1) + \bar{d}_i(2)$.

(3.9.2) $\bar{e}_i(1) = (Sq^0)^i(\lambda_8\lambda_3^3 + \lambda_4\lambda_5^2\lambda_3 + \lambda_4\lambda_7\lambda_3^2 + \lambda_2\lambda_3\lambda_5\lambda_7)$,
$\bar{e}_i(2) = (Sq^0)^i(\lambda_0\lambda_{11}\lambda_3^2 + \lambda_0\lambda_5\lambda_9\lambda_3 + \lambda_1\lambda_2\lambda_7^2)$,
$\delta[(Sq^0)^i(\lambda_8\lambda_7\lambda_3 + \lambda_7\lambda_0\lambda_{11})] = \bar{e}_i(1) + \bar{e}_i(2)$.

(3.9.3) $\bar{f}_i(1) = (Sq^0)^i(\lambda_5\lambda_7\lambda_3^2 + \lambda_4\lambda_6\lambda_5\lambda_3 + \lambda_3^2\lambda_5\lambda_7 + \lambda_2\lambda_4\lambda_5\lambda_7)$.

(3.9.4) $\bar{g}_{i+1}(1) = (Sq^0)^i(\lambda_6^2\lambda_5\lambda_3 + \lambda_5\lambda_9\lambda_3^2 + \lambda_5\lambda_3\lambda_5\lambda_7 + \lambda_3\lambda_5\lambda_9\lambda_3)$.

(3.9.5) $\bar{p}_i(1) = (Sq^0)^i(\lambda_{14}\lambda_5\lambda_7^2 + \lambda_{10}\lambda_9\lambda_7^2 + \lambda_6\lambda_9\lambda_{11}\lambda_7)$,
$\bar{p}_i(2) = (Sq^0)^i(\lambda_{14}\lambda_5\lambda_7^2 + \lambda_7\lambda_0\lambda_{19}\lambda_7)$,
$\bar{p}_i(3) = (Sq^0)^i[\lambda_0(\lambda_{19}\lambda_7^2 + \lambda_7\lambda_{11}\lambda_{15})] = \bar{p}_i(1)$,
$\delta[(Sq^0)^i(\lambda_{10}\lambda_1\lambda_{23} + \lambda_6\lambda_5\lambda_{23})] = \bar{p}_i(1) + \bar{p}_i(2)$.

(3.9.6) $\overline{D}_3(i)(1) = (Sq^0)^i[\lambda_{22}\lambda_{21}\lambda_{11}\lambda_7 + \lambda_{22}\lambda_{13}\lambda_{11}\lambda_{15} + \lambda_{16}\lambda_{15}^3 + \lambda_{14}\lambda_{13}\lambda_{19}\lambda_{15}]$,
$\overline{D}_3(i)(2) = (Sq^0)^i(\lambda_{23}\lambda_0\lambda_7\lambda_{31} + \lambda_{15}\lambda_8\lambda_7\lambda_{31} + \lambda_{16}\lambda_{15}^3)$,
$\overline{D}_3(i)(3) = (Sq^0)^i(\lambda_0\lambda_{23}\lambda_7\lambda_{31}) = \overline{D}_3(i)(1)$,
$\delta[(Sq^0)^i(\lambda_{24}\lambda_7\lambda_{31})] = \overline{D}_3(i)(1) + \overline{D}_3(i)(2)$.

(3.9.7) $$\bar{p}_i'(1) = (Sq^0)^i\begin{bmatrix}\lambda_{38}\lambda_{13}\lambda_{11}\lambda_7 + \lambda_{30}\lambda_9\lambda_5^2 + \lambda_{28}\lambda_{11}\lambda_{15}^2 + \lambda_{22}\lambda_{17}\lambda_{15}^2\\ + \lambda_{20}\lambda_{19}\lambda_{15}^2 + \lambda_{14}\lambda_{21}\lambda_{27}\lambda_7 + \lambda_{14}\lambda_{21}\lambda_{19}\lambda_{15}\\ + \lambda_{14}\lambda_{17}\lambda_{23}\lambda_{15} + \lambda_{14}\lambda_{13}\lambda_{19}\lambda_{23} + \lambda_{12}\lambda_{19}\lambda_{23}\lambda_{15}\end{bmatrix},$$
$\bar{p}_i'(2) = (Sq^0)^i[\lambda_0(\lambda_{39}\lambda_{15}^2 + \lambda_{15}\lambda_{23}\lambda_{31})] = \bar{p}_i'(1)$.

Let $\widetilde{d}_i$, $\widetilde{e}_i$, $\widetilde{f}_i$, $\widetilde{g}_{i+1}$, $\widetilde{p}_i$, $\widetilde{D}_3(i)$ and $\widetilde{p}_i'$ be the cycle representations for the classes d_i, e_i, f_i, g_{i+1}, p_i, $D_3(i)$ and p_i' respectively described in (1.1)(3) through (1.1)(9) of Section 1. The admissible cycle representations $\bar{d}_i(1), \ldots, \bar{p}_i'(1)$ in (3.9) are derived from $\widetilde{d}_i, \ldots, \widetilde{p}_i'$ with the following relations.

(i) (1) $\bar{f}_i(1) = \widetilde{f}_i$, $\bar{p}_i(1) = \widetilde{p}_i$, $\overline{D}_3(i)(1) = \widetilde{D}_3(i)$ and $\bar{p}_i'(1) = \widetilde{p}_i'$ in $\Lambda^{4,*}$.
(2) $\bar{d}_i(1)$ differs from $\widetilde{d}_i$ by one term $(Sq^0)^i(\lambda_1\lambda_5\lambda_1\lambda_7)$ which is ~ 0 as $\lambda_1\lambda_5\lambda_1\lambda_7 = \lambda_3^2\lambda_1\lambda_7 = \delta(\lambda_3\lambda_5\lambda_7)$.
(3) $\bar{e}_i(1)$ differs from $\widetilde{e}_i$ by one term $(Sq^0)^i(\lambda_2\lambda_9\lambda_3^2)$ which is ~ 0 as $\lambda_2\lambda_9\lambda_3^2 = \lambda_6\lambda_5\lambda_3^2 = \lambda_6\lambda_1\lambda_7\lambda_3 = \lambda_0\lambda_7^2\lambda_3 = \delta(\lambda_0\lambda_7\lambda_{11})$.
(4) $\bar{g}_{i+1}(1)$ differs from $\widetilde{g}_{i+1}$ by one term $(Sq^0)^i(\lambda_3\lambda_{11}\lambda_3^2)$ which is ~ 0 $\lambda_3\lambda_{11}\lambda_3^2 = \lambda_7^2\lambda_3^2 = \delta(\lambda_7\lambda_{11}\lambda_3)$.

This implies that $\bar{d}_i(1), \bar{e}_i(1), \ldots, \bar{p}_i'(1)$ are indeed cycle representations for $d_i, e_i, \ldots, p_i'$, respectively. From their admissible forms as given in (3.9) we see the following.

(3.8.1)* $d_i = \{\overline{(Sq^0)^i(\lambda_6\lambda_2\lambda_3^2 = \lambda_6 c_0^*)} = \overline{\lambda_{2^{i+3}-2^i-1} c_i^*}\}$.

(3.8.2)* $e_i = \{\overline{(Sq^0)^i(\lambda_8\lambda_3^3)} = \overline{\lambda_{2^{i+3}+2^i-1}(h_{i+2}^*)^2}\}$.

(3.8.4)* $g_{i+1} = \{\overline{(Sq^0)^i(\lambda_6^2\lambda_5\lambda_3 = \lambda_6\lambda_0\lambda_7^2)}\}$
$= \{\overline{(Sq^0)^i(\lambda_6\lambda_7^2\lambda_0)} = \overline{\lambda_{2^{i+2}+2^{i+1}+2^i-1}(h_{i+3}^*)^2 h_i^*}\}$, since
$\delta(\lambda_6\lambda_8\lambda_7 + \lambda_6\lambda_7\lambda_8) \equiv \lambda_6\lambda_0\lambda_7^2 + \lambda_6\lambda_7^2\lambda_0 \bmod \Lambda(5)$.

(3.8.5)* $p_i = \{\overline{(Sq^0)^i(\lambda_{14}\lambda_5\lambda_7^2 = \lambda_{14}c_1^*)} = \overline{\lambda_{2^{i+4}-2^i-1}c_{i+1}^*}\}$.

(3.8.6)* $D_3(i) = \{\overline{(Sq^0)^i[\lambda_{22}(\lambda_{21}\lambda_{11}\lambda_7 + \lambda_{13}\lambda_{11}\lambda_{15}) = \lambda_{22}\lambda_1\lambda_7\lambda_{31}]}\}$

$= \{\overline{(Sq^0)^i(\lambda_{22}\lambda_{31}\lambda_7\lambda_1)} = \overline{\lambda_{2^{i+4}+2^{i+3}-2^i-1}h_{i+5}^*h_{i+3}^*h_{i+1}^*}\}$, since

$\delta[\lambda_{22}(\lambda_9\lambda_{31} + \lambda_7\lambda_{33} + \lambda_{39}\lambda_1)] \equiv \lambda_{22}\lambda_1\lambda_7\lambda_{31} + \lambda_{22}\lambda_{31}\lambda_7\lambda_1 \bmod \Lambda(21)$.

(3.8.7)* $p_i' = \{\overline{(Sq^0)^i(\lambda_{38}\lambda_{13}\lambda_{11}\lambda_7 = \lambda_{38}\lambda_1\lambda_{15}^2)}\}$

$= \{\overline{(Sq^0)^i(\lambda_{38}\lambda_{15}^2\lambda_1)} = \overline{\lambda_{2^{i+5}+2^{i+2}+2^{i+1}+2^i-1}(h_{i+4}^*)^2h_{i+1}^*}\}$ since

$\delta[\lambda_{38}(\lambda_{17}\lambda_{15} + \lambda_{15}\lambda_{17})] \equiv \lambda_{38}\lambda_1\lambda_{15}^2 + \lambda_{38}\lambda_{15}\lambda_1^2 \bmod \Lambda(37)$.

This proves (3.8.k) for $k = 1, 2, 4, 5, 6$ and 7.

Next we prove (3.8.3). Apply the chain map $\Lambda \xrightarrow{\phi} \widetilde{H}_*(P)\otimes\Lambda$ in (3.5) to the cycle $\overline{f}_i(1)$ in (3.9.3) to get

(i) $\phi(\overline{f}_i(1)) = (Sq^0)^i[\phi(\lambda_5\lambda_7\lambda_3^2) + \phi(\lambda_4\lambda_6\lambda_5\lambda_3) + \phi(\lambda_3^2\lambda_5\lambda_7) + \phi(\lambda_2\lambda_4\lambda_5\lambda_7)]$.

Note that $\lambda_5\lambda_7\lambda_3^2$, $\lambda_4\lambda_6\lambda_5\lambda_3$, $\lambda_3^2\lambda_5\lambda_7$ and $\lambda_2\lambda_4\lambda_5\lambda_7$ are all admissibles. From (3.6)(4) to get

(ii) $\phi(\lambda_4\lambda_6\lambda_5\lambda_3) + \phi(\lambda_3^2\lambda_5\lambda_7) + \phi(\lambda_2\lambda_4\lambda_5\lambda_7) \equiv e_4(\lambda_6\lambda_5\lambda_3 = \lambda_0\lambda_7^2) \bmod F(3)$.

From the construction of the map ϕ described in (3.5) it is easy to see the following.

(iii) $\phi(\lambda_5\lambda_7\lambda_3^2) = e_5\lambda_7\lambda_3^2 + e_1\lambda_{11}\lambda_3^2 \equiv e_5\lambda_7\lambda_3^2 \bmod F(3)$.

From (i), (ii) and (iii) we conclude

(iv) $\phi(\overline{f}_i(1)) \equiv (Sq^0)^i(e_5\lambda_7\lambda_3^2 + e_4\lambda_0\lambda_7^2) \bmod F(2^{i+2}-1)$.

In $\widetilde{H}_*(P)\otimes\Lambda$ we have $\delta(e_5\lambda_{11}\lambda_3) \equiv e_5\lambda_7\lambda_3^2 \bmod F(3)$ and $\delta[e_4(\lambda_8\lambda_7 + \lambda_7\lambda_8)] \equiv e_4\lambda_0\lambda_7^2 + e_4\lambda_7^2\lambda_0 \bmod F(3)$. These imply

(v) $\delta[(Sq^0)^i(e_5\lambda_{11}\lambda_3)] \equiv (Sq^0)^i(e_5\lambda_7\lambda_3^2) \bmod F(2^{i+2}-1)$,
$\delta[(Sq^0)^i(e_4(\lambda_8\lambda_7 + \lambda_7\lambda_8))] \equiv (Sq^0)^i(e_4\lambda_0\lambda_7^2 + e_4\lambda_7^2\lambda_0) \bmod F(2^{i+2}-1)$.

From (iv) and (v) we deduce that

$$\phi(\overline{f}_i(1)) \sim y \equiv (Sq^0)^i(e_4\lambda_7^2\lambda_0) \equiv e_{2^{i+2}+2^i-1}(h_{i+3}^*)^2h_i^* \bmod F(2^{i+2}-1).$$

This implies $\widetilde{\phi}(f_i) = e_{2^{i+2}+2^i-1}h_{i+3}^2h_i$ by the definition of $\widetilde{\phi}(f_i)$ in (h). This proves (3.8.3).

Some calculations are needed in order to prove the remaining (3.8.8) through (3.8.17). We are not going to do all these calculations. Rather, we will just illustrate our method of calculations for two of these which are (3.8.9) and (3.8.11), as the proofs of the rest are similar which we leave to the reader.

We want to show for (3.8.9) that

$$c_ih_j = \{\overline{\lambda_{2^{i+1}+2^i-2^j-1}(h_{i+2}^*)^2h_{j+1}^*}\} \quad \text{for } 0 \leqslant j < i-1.$$

It is clear that it suffices to show this for $j = 0$, that is, to prove

$$(*) \quad c_ih_0 = \{\overline{\lambda_{2^{i+1}+2^i-2}(h_{i+2}^*)^2h_1^*}\} \quad \text{for } i \geqslant 2.$$

We already know that $\lambda_{2^{i+1}+2^i-1}\lambda_{2^{i+2}-1}^2\lambda_0$ is a cycle representing c_ih_0. Let $\lambda(1) = \lambda_{2^{i+2}-1}\lambda_{2^{i+2}} + \lambda_{2^{i+2}}\lambda_{2^{i+2}-1}$, $\lambda(2) = \lambda_{2^{i+2}-1}^2$ and $\lambda(3) = \lambda_{2^{i+2}+1}\lambda_{2^{i+2}-1} + \lambda_{2^{i+2}-1}\lambda_{2^{i+2}+1}$. We have

$$\delta\big(\lambda_{2^{i+1}+2^i-1}\lambda(1)+\lambda_{2^{i+1}+2^i}\lambda(2)+\lambda_{2^{i+1}+2^i-2}\lambda(3)\big)$$
$$\equiv \lambda_{2^{i+1}+2^i-1}\lambda^2_{2^{i+2}-1}\lambda_0+\lambda_{2^{i+1}+2^i-2}\lambda^2_{2^{i+2}-1}\lambda_1 \bmod \Lambda(2^{i+1}+2^i-3).$$

This proves $(*)$ and therefore (3.8.9).

We want to show for (3.8.11) that

$$h_i^3h_l=\big\{\overline{\lambda_{2^i-2^l-1}(h_i^*)^2h^*_{l+1}}\big\} \quad \text{for } 0\leqslant l<i-3.$$

Again it suffices to show

$$(**)\quad h_i^3h_0=\big\{\overline{\lambda_{2^i-2}(h_i^*)^2h_1^*}\big\} \quad \text{for } i\geqslant 4.$$

The cycle $\lambda^3_{2^i-1}\lambda_0$ represents $h_i^3h_0$. We have

$$\delta\big[\lambda_{2^i-1}(\lambda_{2^i-1}\lambda_{2^i}+\lambda_{2^i}\lambda_{2^i-1})+\lambda_{2^i}\lambda^2_{2^i-1}+\lambda_{2^i-2}(\lambda_{2^i+1}\lambda_{2^i-1}+\lambda_{2^i-1}\lambda_{2^i+1})\big]$$
$$\equiv \lambda^3_{2^i-1}\lambda_0+\lambda_{2^i-2}\lambda^2_{2^i-1}\lambda_1 \bmod F(2^i-3).$$

This proves $(**)$ and therefore (3.8.11).

This completes the proof of Theorem 1.3 and therefore also the "inductive proof" of the differentials $E_r^{*,3,*}\xrightarrow{d_r} E_r^{*,4,*}$ given in (2.5).

The remainder of this section is devoted to proving Theorem 1.4.

Let $\bar{E}$ be the subalgebra of the algebra $Ext_A^{s,*}$ for $s\leqslant 5$ generated by the classes in $B_2=\{h_i, c_i, d_i, e_i, f_i, g_{i+1}, p_i, D_3(i), p_i' \mid i\geqslant 0\}$. We already know from Theorem 1.3 the following relations in $\bar{E}$:

(1.3)* $h_ih_{i+1}=0,\ h_ih^2_{i+2}=0,\ h_i^3=h^2_{i-1}h_{i+1},\ h_i^2h^2_{i+3}=0$ and $h_jc_i=0$
for $j=i-1, i, i+2$ and $i+3$.

We want to prove for Theorem 1.4 that in $\bar{E}$ the only relations among the generators in $B_2=\{h_i, c_i, d_i, \ldots, p_i' \mid i\geqslant 0\}$ are those in (1.3)* together with the set of the relations (1) through (39) in the statement (1.4) in Section 1. Let $R(1.4)$ denote this set. Then $R(1.4)$ is the set of the relations obtained by applying $(Sq^0)^i$ for all $i\geqslant 0$ to the following relations (1.4)*(1) through (1.4)*(39) where $D_3=D_3(0)$.

(1.4)* (1) $h_4^2c_0=0$, (2) $h_3h_0c_2=0$, (3) $h_1^2c_0=0$, (4) $h_0d_1=0$,
(5) $h_3d_0=0$, (6) $h_4d_0=0$, (7) $h_0e_1=0$, (8) $h_4e_0=0$,
(9) $h_1f_0=0$, (10) $h_3f_0=0$, (11) $h_4f_0=0$, (12) $h_3g_1=0$,
(13) $h_0p_1=0$, (14) $h_1p_0=0$, (15) $h_2p_0=0$, (16) $h_4p_0=0$,
(17) $h_5p_0=0$, (18) $h_0D_3(1)=0$, (19) $h_0D_3=0$, (20) $h_5D_3=0$,
(21) $h_6D_3=0$, (22) $h_0p_1'=0$, (23) $h_2p_0'=0$, (24) $h_3p_0'=0$,
(25) $h_6p_0'=0$, (26) $h_4h_1c_0=h_3e_0$, (27) $h_4h_0c_3=h_5p_0'$,
(28) $h_5^2c_0=h_1p_0'$, (29) $h_0d_2=h_3D_3$, (30) $h_1d_1=h_0p_0$,
(31) $h_2d_1=h_4g_1$, (32) $h_2d_0=h_0e_0$, (33) $h_1e_0=h_0f_0$,
(34) $h_2e_1=h_1f_1=h_0^2c_2$, (35) $h_2e_0=h_0g_1$, (36) $h_0f_2=h_4p_0'$,
(37) $h_0f_1=h_3p_0$, (38) $h_2f_0=h_1g_1$, (39) $h_3g_2=h_5g_1$.

To complete the proof of Theorem 1.4 it suffices to show the following (3.10) and (3.11).

(3.10) In $\bar{E}$ there are the relations (1.4)*(1) through (1.4)*(39), which imply that the relations in $R(1.4)$ also hold in $\bar{E}$.

(3.11) The set of the monomials $h_ih_jh_kh_lh_m$, $c_ih_jh_k$ and $h_jd_i, h_je_i, h_jf_i, h_jg_{i+1}, h_jp_i, h_jD_3(i), h_jp_i'$ in $\bar{E}\cap Ext_A^{5,*}$ which are obtained by avoiding the relations in (1.3)* and also the relations in $R(1.4)$ is linearly independent.

Precise description of the set in (3.11) will be given later when we come to prove it.

To prove (3.10), and also (3.11) later on, we shall need the specific cycle representations $\bar{d}_i(1), \bar{d}_i(2), \dots, \bar{p}'_i(1)$, $\bar{p}'_i(2)$ in (3.9) for the classes $d_i, \dots, p'_i$, respectively. Recall that $\lambda^2_{2^j-1}$ and $\lambda_{2^{j+1}+2^j-1}\lambda^2_{2^{j+2}-1}$ are the standard cycle representations for the classes h_j and c_j, respectively. We shall use the cycle representations $\lambda_{2^j-1}\bar{d}_i(1)$ (or $\bar{d}_i(1)\lambda_{2^j-1}$), $\lambda_{2^j-1}\bar{d}_i(2), \dots, \lambda_{2^j-1}\bar{p}'_i(1), \lambda_{2^j-1}\bar{p}'_i(2)$ for the classes $h_jd_i = d_ih_j, \dots, h_jp'_i = p'_ih_j$ in $\widehat{B}_3 = \{h_jd_i, h_je_i, \dots, h_jD_3(i), h_jp'_i \mid i \geqslant 0,\ j \geqslant 0\}$. For example, $\lambda_7\bar{d}_0(2) = \lambda_7(\lambda_6\lambda_2\lambda_3^2 + \lambda_3\lambda_0^2\lambda_{11})$ represents the class h_3d_0 and $\bar{e}_0(1)\lambda_7 = [\lambda_8\lambda_3^3 + \lambda_4(\lambda_5^2\lambda_3 + \lambda_7\lambda_3^2) + \lambda_2\lambda_3\lambda_5\lambda_7]\lambda_7 = \lambda_2\lambda_3\lambda_5\lambda_7^2$ (since $\lambda_3\lambda_7 = 0$ in Λ) represents the class $h_3e_0 = e_0h_3$. Besides these cycle representations we shall also need some "exotic" cycle representations for some of the classes in $\widehat{B}_3$. These "exotic" cycle representations will be obtained from the following result which is new.

Proposition 3.12. *Let α be a class in $Ext_A^{s-1,*}(P)$ represented by a cycle $x = \sum_{k=1}^n e_{i_k}\lambda(k) \in \widetilde{H}_*(P) \otimes \Lambda^{s-1,*}$ where $s-1>0$. Consider the class $t_*(\alpha)$ in $Ext_A^{s,*}$ where t_* is as in (3.1)(2), so that $t_*(\alpha)$ is represented by the cycle $t(x) = \sum_{k=1}^n \lambda_{i_k}\lambda(k)$ in $\Lambda^{s,*}$. Then for any $j \geqslant 0$ the class $h_jt_*(\alpha) \in Ext_A^{s+1,*}$ is represented by*

$$\sum_{k=1}^n\Big[\sum_{l\geqslant 0}\binom{i_k-l}{l}\lambda_{i_k-l}\lambda_{2^j+l-1}\Big]\lambda(k) \in \Lambda^{s+1,*}.$$

Proposition 3.12 will be proved at the end of this section after we finish the proof of Theorem 1.4.

The classes in $\widehat{B}_3 = \{h_jd_i, \dots, h_jp'_i\}$ to be given exotic cycle representations via (3.12), which are relevant to the proof of (3.10), are the classes h_1d_1, h_0p_0, h_0e_0, h_1g_1, h_0p_1 and h_1p_0. These exotic cycle representations are to be described in (3.13) below. Recall that we use $\widetilde{d}_i, \widetilde{e}_i, \dots, \widetilde{p}'_i$ to denote the cycle representations for the classes $d_i, e_i, \dots, p_i$ respectively described in (1.1) of Section 1. For example, we have

(j) $\widetilde{g}_1 = \lambda_6\lambda_0\lambda_7^2 + \lambda_5(\lambda_9\lambda_3^2 + \lambda_3\lambda_5\lambda_7) + \lambda_3(\lambda_5\lambda_9\lambda_3 + \lambda_{11}\lambda_3^2)$,
$\widetilde{e}_0 = \lambda_8\lambda_3^3 + \lambda_4\lambda_5^2\lambda_3 + (\lambda_4\lambda_7\lambda_3^2 = \lambda_3\lambda_8\lambda_3^2) + \lambda_2(\lambda_3\lambda_5\lambda_7 + \lambda_9\lambda_3^2)$.

Recall also from (3.9) that the following specific cycles $\bar{d}_1(2)$, $\bar{p}_0(2)$ and $\bar{p}_1(2)$ represent d_1, p_0 and p_1 respectively.

(k) $\bar{d}_1(2) = Sq^0(\lambda_6\lambda_2\lambda_3^2 + \lambda_3\lambda_0^2\lambda_{11}) = \lambda_{13}\lambda_5\lambda_7^2 + \lambda_7\lambda_1^2\lambda_{23}$,
$\bar{p}_0(2) = \lambda_{14}\lambda_5\lambda_7^2 + \lambda_7\lambda_0\lambda_{19}\lambda_7$,
$\bar{p}_1(2) = \lambda_{29}\lambda_{11}\lambda_{15}^2 + \lambda_{15}\lambda_1\lambda_{39}\lambda_{15} = Sq^0(\bar{p}_0(2))$.

The following are cycles in $\widetilde{H}_*(P) \otimes \Lambda^{3,*}$ which is easy to check.

(l) $\widetilde{g}_1^* = e_6\lambda_0\lambda_7^2 + e_5(\lambda_9\lambda_3^2 + \lambda_3\lambda_5\lambda_7) + e_3(\lambda_5\lambda_9\lambda_3 + \lambda_{11}\lambda_3^2)$,
$\widetilde{e}_0^* = e_8\lambda_3^3 + e_4\lambda_5^2\lambda_3 + e_3\lambda_8\lambda_3^2 + e_2(\lambda_3\lambda_5\lambda_7 + \lambda_9\lambda_3^2)$,
$\bar{d}_1^*(2) = e_{13}\lambda_5\lambda_7^2 + e_7\lambda_1^2\lambda_{23}$, $\qquad \bar{p}_0^*(2) = e_{14}\lambda_5\lambda_7^2 + e_7\lambda_0\lambda_{19}\lambda_7$,
$\bar{p}_1^*(2) = e_{29}\lambda_{11}\lambda_{15}^2 + e_{15}\lambda_1\lambda_{39}\lambda_{15}$.

From (j), (k), (l) we see $t(\widetilde{g}_1^*) = \widetilde{g}_1$, $t(\widetilde{e}_0^*) = \widetilde{e}_0$, $t(\bar{d}_1^*(2)) = \bar{d}_1(2)$ and $t(\bar{p}_i^*(2)) = \bar{p}_i(2)$ for $i = 0, 1$. From (j), (k) and Propositions 3.12 we obtain the following exotic cycle representations for the classes h_1d_1, h_0p_0, h_0e_0, h_1g_1, h_0p_1 and h_1p_0, respectively.

(3.13) (1) $\overline{h_1d_1} = (\lambda_{13}\lambda_1\lambda_5\lambda_7^2 = 0) + \lambda_{11}\lambda_3\lambda_5\lambda_7^2 + \lambda_7^2\lambda_5\lambda_7^2$
$\qquad + (\lambda_7\lambda_1^3\lambda_{23} = \lambda_7\lambda_0^2\lambda_{19}\lambda_7)$.

(2) $\overline{h_0p_0} = (\lambda_{14}\lambda_0\lambda_5\lambda_7^2 = 0) + (\lambda_{13}\lambda_1\lambda_5\lambda_7^2 = 0) + \lambda_{11}\lambda_3\lambda_5\lambda_7^2$
$\qquad + \lambda_7^2\lambda_5\lambda_7^2 + \lambda_7\lambda_0^2\lambda_{19}\lambda_7$.

(3) $\overline{h_0e_0} = (\lambda_8\lambda_0\lambda_3^3 = 0) + (\lambda_7\lambda_1\lambda_3^3 = 0) + \lambda_6\lambda_2\lambda_3^3 + \lambda_4^2\lambda_3^3$
$\qquad + (\lambda_4\lambda_0\lambda_5^2\lambda_3 = \lambda_4^2\lambda_3^3) + \lambda_3\lambda_1\lambda_5^2\lambda_3 + (\lambda_2^2\lambda_5^2\lambda_3 = 0)$

$$+ \lambda_3\lambda_0\lambda_8\lambda_3^2 + (\lambda_2\lambda_0\lambda_3\lambda_5\lambda_7 = 0) + (\lambda_1^2\lambda_3\lambda_5\lambda_7 = 0)$$
$$+ (\lambda_2\lambda_0\lambda_9\lambda_3^2 = 0) + (\lambda_1^2\lambda_9\lambda_3^2 = \lambda_0^2\lambda_7^2\lambda_3).$$

(4) $\overline{h_1g_1} = \lambda_6\lambda_1\lambda_0\lambda_7^2 + \lambda_5\lambda_2\lambda_0\lambda_7^2 + \lambda_3\lambda_4\lambda_0\lambda_7^2 + \lambda_5\lambda_1\lambda_9\lambda_3^2$
$$+ \lambda_3^2\lambda_9\lambda_3^2 + (\lambda_5\lambda_1\lambda_3\lambda_5\lambda_7 = 0) + \lambda_3^3\lambda_5\lambda_7 + \lambda_3\lambda_1\lambda_5\lambda_9\lambda_3$$
$$+ \lambda_3\lambda_1\lambda_{11}\lambda_3^2.$$

(5) $\overline{h_0p_1} = \lambda_{29}\lambda_0\lambda_{11}\lambda_{15}^2 + \lambda_{27}\lambda_2\lambda_{11}\lambda_{15}^2 + \lambda_{23}\lambda_6\lambda_{11}\lambda_{15}^2$
$$+ \lambda_{15}\lambda_{14}\lambda_{11}\lambda_{15}^2 + (\lambda_{15}\lambda_0\lambda_1\lambda_{39}\lambda_{15} = 0).$$

(6) $\overline{h_1p_0} = (\lambda_{14}\lambda_1\lambda_5\lambda_7^2 = 0) + (\lambda_{13}\lambda_2\lambda_5\lambda_7^2 = 0) + \lambda_{11}\lambda_4\lambda_5\lambda_7^2$
$$+ \lambda_7\lambda_8\lambda_5\lambda_7^2 + \lambda_7\lambda_1\lambda_0\lambda_{19}\lambda_7.$$

Now we prove (3.10). Recall we want to prove for (3.10) that the relations (1.4)*(1) through (1.4)*(39) hold in $Ext_A^{5,*}$. The following Eqs. (3.10.k)* for $1 \leqslant k \leqslant 39$ in the lambda algebra Λ constitute a proof for these relations where, for each k, Eq. (3.10.k)* corresponds to the relation (1.4)*(k). In these equations we use the specific cycles $\bar{d}_i(1), \bar{d}_i(2), \ldots, \overline{p}_i'(1), \overline{p}_i'(2)$ in (3.9) to represent the classes $d_i, \ldots, p_i'$ respectively and, in Eqs. (3.10.k)* for $k =$ 13, 14, 30, 32 and 38, we use the exotic cycle representations in (3.13) for the classes $h_1d_1, h_0p_0, h_0e_0, h_1g_1, h_0p_1$ and h_1p_0. And we recall again that h_j and c_j are represented by the standard cycles λ_{2^j-1} and $\lambda_{2^{j+1}+2^j-1}\lambda_{2^{j+2}-1}^2 = c_j^*$, respectively. We also note that if α and β are classes in $Ext_A^{*,*}$ represented respectively by cycles x and y in Λ then $\{xy\} = \{yx\} = \alpha\beta = \beta\alpha$ since the algebra $Ext_A^{*,*}$ is commutative.

(3.10.1)* $h_4^2c_0 = 0$ by $\lambda_2\lambda_3^2\lambda_{15}^2 = 0$.

(3.10.2)* $h_3h_0c_2 = 0$ by $\lambda_7\lambda_0c_2^* = \lambda_7\lambda_0\lambda_{11}\lambda_{15}^2 = \delta(\lambda_8\lambda_{11}\lambda_{15}^2)$.

(3.10.3)* $h_1^2c_0 = 0$ by $\lambda_1^2c_0^* = \lambda_1^2\lambda_2\lambda_3^2 = \delta(\lambda_0^3\lambda_{11})$.

(3.10.4)* $h_0d_1 = 0$ by $\lambda_0\bar{d}_1(2) = \lambda_0(\lambda_{13}\lambda_5\lambda_7^2 + \lambda_7\lambda_1^2\lambda_{23}) = \delta(\lambda_8\lambda_1^2\lambda_{23})$.

(3.10.5)* $h_3d_0 = 0$ by $\lambda_7\bar{d}_0(2) = \lambda_7(\lambda_6\lambda_2\lambda_3^2 + \lambda_3\lambda_0^2\lambda_{11})$
$$= \delta(\lambda_{14}\lambda_2\lambda_3^2 + \lambda_{11}\lambda_0^2\lambda_{11}).$$

(3.10.6)* $h_4d_0 = 0$ by $\bar{d}_0(1)\lambda_{15} = (\lambda_6\lambda_2\lambda_3^2 + \lambda_4^2\lambda_3^2 + \lambda_2\lambda_4\lambda_5\lambda_3)\lambda_{15}$
$$= \delta(\lambda_0\lambda_{16}\lambda_7^2 + \lambda_0\lambda_{15}\lambda_8\lambda_7 + \lambda_0\lambda_{23}\lambda_0\lambda_7).$$

(3.10.7)* $h_0e_1 = 0$ by $\lambda_0\bar{e}_1(1) = \lambda_0(\lambda_{17}\lambda_7^3 + \lambda_9\lambda_{11}^2\lambda_7 + \lambda_9\lambda_{15}\lambda_7^2 + \lambda_5\lambda_7\lambda_{11}\lambda_{15}) = 0$.

(3.10.8)* $h_4e_0 = 0$ by $\bar{e}_0(1)\lambda_{15} = (\lambda_8\lambda_3^3 + \lambda_4\lambda_5^2\lambda_3 + \lambda_4\lambda_7\lambda_3^2 + \lambda_2\lambda_3\lambda_5\lambda_7)\lambda_{15}$
$$= \delta(\lambda_4\lambda_3^2\lambda_{23}).$$

(3.10.9)* $h_1f_0 = 0$ by $\lambda_1\overline{f}_0(1) = \lambda_1\big[\lambda_4\lambda_0\lambda_7^2 + \lambda_3(\lambda_9\lambda_3^2 + \lambda_3\lambda_5\lambda_7) + \lambda_2\lambda_4\lambda_5\lambda_7\big]$
$$= \delta(\lambda_2\lambda_4\lambda_7^2).$$

(3.10.10)* $h_3f_0 = 0$ by $\overline{f}_0(1)\lambda_7 = \big[\lambda_4\lambda_0\lambda_7^2 + \lambda_3(\lambda_9\lambda_3^2 + \lambda_3\lambda_5\lambda_7) + \lambda_2\lambda_4\lambda_5\lambda_7\big]\lambda_7$
$$= \lambda_3^2\lambda_5\lambda_7^2 + \lambda_2\lambda_4\lambda_5\lambda_7^2 = \delta(\lambda_1^3\lambda_{23} + \lambda_0^2\lambda_{11}\lambda_{15}).$$

(3.10.11)* $h_4f_0 = 0$ by $\overline{f}_0(1)\lambda_{15} = \big[\lambda_4\lambda_0\lambda_7^2 + \lambda_3(\lambda_9\lambda_3^2 + \lambda_3\lambda_5\lambda_7) + \lambda_2\lambda_4\lambda_5\lambda_7\big]\lambda_{15}$
$$= \lambda_3\lambda_9\lambda_3^2\lambda_{15} = \lambda_3\lambda_9\lambda_7^3 = \delta(\lambda_3\lambda_{17}\lambda_7^2).$$

(3.10.12)* $h_3g_1 = 0$ by $\bar{g}_1(1)\lambda_7 = \big[\lambda_6\lambda_0\lambda_7^2 + \lambda_5(\lambda_9\lambda_3^2 + \lambda_3\lambda_5\lambda_7) + \lambda_3\lambda_5\lambda_9\lambda_3\big]\lambda_7$
$$= \lambda_5\lambda_3\lambda_5\lambda_7^2 = \delta(\lambda_9\lambda_5\lambda_7^2).$$

(3.10.13)* $h_0p_1 = 0$ by $\overline{h_0p_1} = (\lambda_{29}\lambda_0 + \lambda_{27}\lambda_2 + \lambda_{23}\lambda_6 + \lambda_{15}\lambda_{14})\lambda_{11}\lambda_{15}^2$
$$= \delta(\lambda_{30}\lambda_{11}\lambda_{15}^2).$$

(3.10.14)* $h_1p_0 = 0$ by $\overline{h_1p_0} = \lambda_{11}\lambda_4\lambda_5\lambda_7^2 + \lambda_7\lambda_8\lambda_5\lambda_7^2 + \lambda_7\lambda_1\lambda_0\lambda_{19}\lambda_7$
$= \delta(\lambda_{14}\lambda_7^3 + \lambda_7\lambda_2\lambda_{19}\lambda_7)$.

(3.10.15)* $h_2p_0 = 0$ by $\bar{p}_0(3)\lambda_3 = \lambda_0(\lambda_{19}\lambda_7^2 + \lambda_7\lambda_{11}\lambda_{15})\lambda_3$
$= \delta[\lambda_0(\lambda_{23}\lambda_7^2 + \lambda_{19}\lambda_7\lambda_{11} + \lambda_{11}\lambda_3\lambda_{23} + \lambda_7\lambda_{11}\lambda_{19})]$.

(3.10.16)* $h_4p_0 = 0$ by $\bar{p}_0(2)\lambda_{15} = (\lambda_{14}\lambda_5\lambda_7^2 + \lambda_7\lambda_0\lambda_{19}\lambda_7)\lambda_{15} = 0$.

(3.10.17)* $h_5p_0 = 0$ by $\bar{p}_0(3)\lambda_{31} = \lambda_0(\lambda_{19}\lambda_7^2 + \lambda_7\lambda_{11}\lambda_{15})\lambda_{31}$
$= \lambda_0\lambda_{19}\lambda_{15}^3 = \delta(\lambda_0\lambda_{35}\lambda_{15}^2)$.

(3.10.18)* $h_0D_3(1) = 0$ by $\lambda_0\overline{D}_3(1)(3) = \lambda_0\lambda_1\lambda_{47}\lambda_{15}\lambda_{63} = 0$.

(3.10.19)* $h_0D_3 = 0$ by $\lambda_0\overline{D}_3(0)(3) = \lambda_0^2\lambda_{23}\lambda_7\lambda_{31}$
$$= \delta\begin{bmatrix} \lambda_0^2\lambda_{47}\lambda_{15} + (\lambda_9\lambda_3^2 + \lambda_3\lambda_5\lambda_7)\lambda_{47} \\ + \lambda_{21}\lambda_{11}\lambda_{15}^2 + \lambda_7\lambda_1\lambda_{23}\lambda_{31} \end{bmatrix}.$$

(3.10.20)* $h_5D_3 = 0$ by $\overline{D}_3(0)(3)\lambda_{31} = \lambda_0\lambda_{23}\lambda_7\lambda_{31}^2 = 0$.

(3.10.21)* $h_6D_3 = 0$ by $\overline{D}_3(0)(3)\lambda_{63} = \lambda_0\lambda_{23}\lambda_7\lambda_{31}\lambda_{63} = 0$.

(3.10.22)* $h_0p_1' = 0$ by $\lambda_0\bar{p}_1'(2) = \lambda_0\lambda_1(\lambda_{79}\lambda_{31}^2 + \lambda_{31}\lambda_{47}\lambda_{63}) = 0$.

(3.10.23)* $h_2p_0' = 0$ by $\lambda_3\bar{p}_0'(2) = \lambda_3\lambda_0(\lambda_{39}\lambda_{15}^2 + \lambda_{15}\lambda_{23}\lambda_{31})$
$= \delta[\lambda_4(\lambda_{39}\lambda_{15}^2 + \lambda_{15}\lambda_{23}\lambda_{31}) + \lambda_0\lambda_{11}\lambda_{31}^2]$.

(3.10.24)* $h_3p_0' = 0$ by $\bar{p}_0'(2)\lambda_7 = \lambda_0(\lambda_{39}\lambda_{15}^2 + \lambda_{15}\lambda_{23}\lambda_{31})\lambda_7$
$= \delta[\lambda_0(\lambda_{47}\lambda_{15}^2 + \lambda_{39}\lambda_{15}\lambda_{23} + \lambda_{23}\lambda_7\lambda_{47} + \lambda_{15}\lambda_{23}\lambda_{39})]$.

(3.10.25)* $h_6p_0' = 0$ by $\bar{p}_0'(2)\lambda_{63} = \lambda_0(\lambda_{39}\lambda_{15}^2 + \lambda_{15}\lambda_{23}\lambda_{31})\lambda_{63}$
$= \lambda_0\lambda_{39}\lambda_{31}^3 = \delta(\lambda_0\lambda_{71}\lambda_{31}^2)$.

(3.10.26)* $h_4h_1c_0 = h_3e_0$ by $\bar{e}_0(1)\lambda_7 = [\lambda_8\lambda_3^3 + \lambda_4(\lambda_5^2\lambda_3 + \lambda_7\lambda_3^2) + \lambda_2\lambda_3\lambda_5\lambda_7]\lambda_7$
$= \lambda_2\lambda_3\lambda_5\lambda_7^2 = \lambda_1\lambda_4\lambda_5\lambda_7^2 = \lambda_1\lambda_2\lambda_7^3$
$= \lambda_1\lambda_2\lambda_3^2\lambda_{15} = \lambda_1c_0^*\lambda_{15}$.

(3.10.27)* $h_4h_0c_3 = h_5p_0'$ by $\bar{p}_0'(2)\lambda_{31} = \lambda_0(\lambda_{39}\lambda_{15}^2 + \lambda_{15}\lambda_{23}\lambda_{31})\lambda_{31}$
$= \lambda_0\lambda_{15}\lambda_{23}\lambda_{31}^2 = \lambda_0\lambda_{15}c_3^*$.

(3.10.28)* $h_5^2c_0 = h_1p_0'$ by $\lambda_1\bar{p}_0'(2) = \lambda_1\lambda_0(\lambda_{39}\lambda_{15}^2 + \lambda_{15}\lambda_{23}\lambda_{31})$
$= \delta[\lambda_2(\lambda_{39}\lambda_{15}^2 + \lambda_{15}\lambda_{23}\lambda_{31})] + (\lambda_2\lambda_3^2\lambda_{31}^2 = c_0^*\lambda_{31}^2)$.

(3.10.29)* $h_0d_2 = h_3D_3$ by $\lambda_0\bar{d}_2(2) + \lambda_7\overline{D}_3(0)(3)$
$= \lambda_0(\lambda_{27}\lambda_{11}\lambda_{15}^2 + \lambda_{15}\lambda_3^2\lambda_{47}) + \lambda_7\lambda_0\lambda_{23}\lambda_7\lambda_{31}$
$= \delta(\lambda_{16}\lambda_3^2\lambda_{47} + \lambda_{14}\lambda_1\lambda_{39}\lambda_{15} + \lambda_7\lambda_0\lambda_{47}\lambda_{15})$.

(3.10.30)* $h_1d_1 = h_0p_0$ by $\overline{h_1d_1} = \lambda_{11}\lambda_3\lambda_5\lambda_7^2 + \lambda_7^2\lambda_5\lambda_7^2 + \lambda_7\lambda_0^2\lambda_{19}\lambda_7 = \overline{h_0p_0}$.

(3.10.31)* $h_2d_1 = h_4g_1$ by $\bar{g}_1(1)\lambda_{15} + \lambda_3\bar{d}_1(1)$
$= [\lambda_6\lambda_0\lambda_7^2 + \lambda_5(\lambda_9\lambda_3^2 + \lambda_3\lambda_5\lambda_7) + \lambda_3\lambda_5\lambda_9\lambda_3]\lambda_{15}$
$+ \lambda_3(\lambda_{13}\lambda_5\lambda_7^2 + \lambda_9^2\lambda_7^2 + \lambda_5\lambda_9\lambda_{11}\lambda_7)$
$= \lambda_5\lambda_9\lambda_7^3 + \lambda_3\lambda_1\lambda_{17}\lambda_7^2 = \delta(\lambda_5\lambda_{17}\lambda_7^2)$.

(3.10.32)* $h_2d_0 = h_0e_0$ by $\overline{h_0e_0} + \lambda_3\bar{d}_0(1)$
$= \lambda_6\lambda_2\lambda_3^3 + \lambda_3(\lambda_1\lambda_5^2\lambda_3 + \lambda_0\lambda_8\lambda_3^2) + \lambda_0^2\lambda_7^2\lambda_3$
$+ \lambda_3(\lambda_6\lambda_2\lambda_3^2 + \lambda_4^2\lambda_3^2 + \lambda_2\lambda_4\lambda_5\lambda_3)$
$= \delta(\lambda_6\lambda_2\lambda_7\lambda_3 + \lambda_5\lambda_1^2\lambda_{11} + \lambda_3^2\lambda_5\lambda_7 + \lambda_0^2\lambda_7\lambda_{11})$.

(3.10.33)* $h_1e_0 = h_0f_0$ by $\lambda_0\overline{f}_0(1) + \lambda_1\bar{e}_0(1)$

$$\begin{aligned}&= \lambda_0[\lambda_4\lambda_0\lambda_7^2 + \lambda_5\lambda_7\lambda_3^2 + \lambda_3^2\lambda_5\lambda_7 + \lambda_2\lambda_4\lambda_5\lambda_7]\\ &\quad + \lambda_1[\lambda_8\lambda_3^3 + \lambda_4\lambda_5^2\lambda_3 + \lambda_4\lambda_7\lambda_3^2 + \lambda_2\lambda_3\lambda_5\lambda_7]\\ &= \lambda_3\lambda_1\lambda_0\lambda_7^2 + \lambda_6\lambda_3^4 + \lambda_5\lambda_4\lambda_3^3\\ &= \delta(\lambda_{10}\lambda_3^3 + \lambda_7\lambda_0\lambda_5\lambda_7 + \lambda_3\lambda_2\lambda_7^2).\end{aligned}$$

(3.10.34)* $h_2e_1 = h_1f_1 = h_0^2c_2$ by $(\lambda_0^2c_2^* = \lambda_0^2\lambda_{11}\lambda_{15}^2) + \lambda_1\overline{f}_1(1)$

$$= \lambda_9\lambda_{11}\lambda_7^3 + \lambda_9\lambda_1^2\lambda_{15}^2 = \delta(\lambda_9\lambda_7\lambda_{11}\lambda_{15}).$$

(3.10.35)* $h_2e_0 = h_0g_1$ by $\lambda_0\bar{g}_1(1) + \lambda_3\bar{e}_0(1)$

$$\begin{aligned}&= \lambda_0\big[\lambda_6\lambda_0\lambda_7^2 + \lambda_5(\lambda_9\lambda_3^2 + \lambda_3\lambda_5\lambda_7) + \lambda_3\lambda_5\lambda_9\lambda_3\big]\\ &\quad + \lambda_3\big[\lambda_8\lambda_3^3 + \lambda_4(\lambda_5^2\lambda_3 + \lambda_7\lambda_3^2) + \lambda_2\lambda_3\lambda_5\lambda_7\big]\\ &= \delta(\lambda_5\lambda_2\lambda_7^2 + \lambda_4\lambda_5\lambda_1\lambda_{11} + \lambda_3^2\lambda_8\lambda_7).\end{aligned}$$

(3.10.36)* $h_0f_2 = h_4p_0'$ by $\lambda_0\overline{f}_2(1)$

$$\begin{aligned}&= \lambda_0\big[\lambda_{19}\lambda_3\lambda_{31}^2 + \lambda_{15}(\lambda_{39}\lambda_{15}^2 + \lambda_{15}\lambda_{23}\lambda_{31}) + \lambda_{11}^2\lambda_{31}^2\big]\\ &= \lambda_{16}\lambda_3^2\lambda_{31}^2 + \lambda_0\lambda_{15}(\lambda_{39}\lambda_{15}^2 + \lambda_{15}\lambda_{23}\lambda_{31})\\ &= \delta\big[\lambda_{16}(\lambda_{39}\lambda_{15}^2 + \lambda_{15}\lambda_{23}\lambda_{31})\big] + \lambda_{15}\overline{p}_0'(2).\end{aligned}$$

(3.10.37)* $h_0f_1 = h_3p_0$ by $\lambda_0\overline{f}_1(1)$

$$\begin{aligned}&= \lambda_0\big[\lambda_9\lambda_1\lambda_{15}^2 + \lambda_7(\lambda_{19}\lambda_7^2 + \lambda_7\lambda_{11}\lambda_{15}) + \lambda_5^2\lambda_{15}^2\big]\\ &= \delta\big[\lambda_8(\lambda_{19}\lambda_7^2 + \lambda_7\lambda_{11}\lambda_{15})\big] + \lambda_7\overline{p}_0(3).\end{aligned}$$

(3.10.38)* $h_2f_0 = h_1g_1$ by $\overline{h_1g_1} + \lambda_3\overline{f}_0(1)$

$$\begin{aligned}&= \lambda_6\lambda_1\lambda_0\lambda_7^2 + \lambda_5\lambda_1\lambda_9\lambda_3^2 + \lambda_5\lambda_3\lambda_5^2\lambda_3\\ &\quad + \lambda_3(\lambda_1\lambda_5\lambda_9\lambda_3 + \lambda_1\lambda_{11}\lambda_3^2 + \lambda_2\lambda_4\lambda_5\lambda_7)\\ &= \delta(\lambda_6\lambda_2\lambda_7^2 + \lambda_5^2\lambda_1\lambda_{11}).\end{aligned}$$

(3.10.39)* $h_3g_2 = h_5g_1$ by $\bar{g}_1(1)\lambda_{31} + \lambda_7\bar{g}_2(1)$

$$\begin{aligned}&= \big[\lambda_6\lambda_0\lambda_7^2 + \lambda_5(\lambda_9\lambda_3^2 + \lambda_3\lambda_5\lambda_7) + \lambda_3\lambda_5\lambda_9\lambda_3\big]\lambda_{31}\\ &\quad + \lambda_7(\lambda_{13}\lambda_1\lambda_{15}^2 + \lambda_{11}\lambda_{19}\lambda_7^2 + \lambda_7\lambda_{11}\lambda_{19}\lambda_7 + \lambda_{11}\lambda_7\lambda_{11}\lambda_{15})\\ &= \delta(\lambda_5\lambda_9\lambda_{15}\lambda_{23}).\end{aligned}$$

To complete the proof of Theorem 1.4 we have to show (3.11) which is restated more precisely as (3.11)* below. From the relations in (1.3)*, described before the proof of (3.10) above, and also the relations in $R(1.4)$, which is the set of the relations (1.4)(1) through (1.4)(39) in Section 1, we see that to prove (3.11) is equivalent to proving the following (3.11)* where the class $h_0^5 \in Ext_A^{5,5}$ is excluded, that is, only elements in $Ext_A^{s,t}$ with $t - s > 0$ are considered.

(3.11)* The set of the monomials (1) through (23) below is a $\mathbb{Z}/2$-base for $\bar{E} \cap Ext_A^{5,*}$, the $\mathbb{Z}/2$-submodule of $Ext_A^{5,*}$ generated by the decomposable elements.

(1) $h_i^3h_j^2$ for $i \geqslant j+5$, (2) $h_i^3h_jh_k$ for $i \geqslant j+4 \geqslant k+6$,
(3) $h_i^2h_j^3$ for $i \geqslant j+4$, (4) $h_i^2h_j^2h_k$ for $i \geqslant j+4 \geqslant k+7$,
(5) $h_i^2h_jh_k^2$ for $i \geqslant j+3 \geqslant k+6$,
(6) $h_i^2h_jh_kh_l$ for $i \geqslant j+3 \geqslant k+5 \geqslant l+7$,
(7) $h_ih_j^3h_k$ for $i \geqslant j+3 \geqslant k+7$,
(8) $h_ih_j^2h_k^2$ for $i \geqslant j+3 \geqslant k+7$,
(9) $h_ih_j^2h_kh_l$ for $i \geqslant j+3 \geqslant k+6 \geqslant l+8$,
(10) $h_ih_jh_k^3$ for $i \geqslant j+2 \geqslant k+5$,

(11) $h_i h_j h_k h_l^2$ for $i \geqslant j+2 \geqslant k+4 \geqslant l+7$,
(12) $h_i h_j h_k^2 h_l$ for $i \geqslant j+2 \geqslant k+5 \geqslant l+8$,
(13) $h_i h_j h_k h_l h_m$ for $i \geqslant j+2 \geqslant k+4 \geqslant l+6 \geqslant m+8$,
(14) $h_i h_0^4$ for $i \geqslant 4$,
(15) $c_i h_j^2$ for $j \leqslant i-3$ or $j \geqslant i+5$,
(16) $c_i h_j h_k$ for (i) $k \geqslant i+4$, $k-1 > j$ and $j \neq i-1, i, i+2, i+3$ or (ii) $j < k-1 < i-2$ or (iii) $k = i+1 > j-1$,
(17) $d_i h_j$ for $j \neq i-2, i-1, i+3$ and $i+4$,
(18) $e_i h_j$ for $j \neq i-1, i, i+3$ and $i+4$,
(19) $f_i h_j$ for $j \leqslant i-3$ or $j = i+2$ or $j \geqslant i+5$,
(20) $g_{i+1} h_j$ for $j \leqslant i-1$ or $j = i+2$ or $j \geqslant i+6$,
(21) $p_i h_j$ for $j \leqslant i-2$ or $j = i+3$ or $j \geqslant j+6$,
(22) $D_3(i) h_j$ for $j \neq i-1, i, i+5$ and $i+6$,
(23) $p_i' h_j$ for $j \leqslant i-2$ or $j = i$ or $j = i+4$ or $j \geqslant i+7$.

Let $\widetilde{B}$ be the set of the classes (3.11)*(1) through (3.11)*(23). To prove (3.11)* it suffices to show that $\widetilde{B}$ is a linearly independent set in $Ext_A^{5,*}$ as $\widetilde{B}$ spans the $\mathbb{Z}/2$-module $\bar{E} \cap Ext_A^{5,*}$ which is easy to see from the relations in (1.3)* and in $R(1.4)$.

The method of proving (3.11)* will be the same as that for $\overline{(1.3)}'$(ii), which was proved via the result (3.8), and this is roughly described as follows. We are going to show that for each class α in (3.11)* either there is a correspondence

$$\alpha = \{\overline{\lambda_j \lambda(j)}\} \to e_j\{\lambda(j)\} = \bar{\phi}(\alpha)$$

of type (f) or there is a correspondence

$$\alpha = \{\overline{\lambda_j \lambda(j)}\} \to e_k\{\lambda(k)\} = \widetilde{\phi}(\alpha)$$

of type (h) such that the resulting collection I of these $\bar{\phi}(\alpha)$ or $\widetilde{\phi}(\alpha)$ is linearly independent in $E_\infty^{*,4,*}$ of the spectral sequence $\{E_r^{*,*,*}\}_{r \geqslant 1}$ for $Ext_A^{*,*}(P)$, and then apply Proposition 3.7. To show I is linearly independent we shall resort to the differentials $E_r^{*,3,*} \xrightarrow{d_r} E_r^{*,4,*}$ described in (2.5) of Section 2.

The result of these correspondences for classes in (3.11)* is described in (3.14) below from (3.14.1) to (3.14.51), analogous to those in (3.8) for $\overline{(1.3)}'$(ii) discussed earlier. The correspondences (3.14.1) through (3.14.43) are of type (f) which are listed for the classes in (3.11)*(1) through (3.11)*(23) roughly in that order with several consecutive $(3.14.n)'^{s}$ for each such (3.11)*(l). The remaining eight correspondences (3.14.44) through (3.14.51) are of type (h). Proofs for these correspondences will be given afterwards. In stating these correspondences we still use h_j^* and c_j^* to stand for the cycles λ_{2^j-1} and $\lambda_{2^{j+1}+2^j-1}\lambda_{2^{j+2}-1}^2$, respectively. We will also use the cycle representations $\bar{d}_i(1)$, $\bar{e}_i(1)$, $\bar{f}_i(1)$, $\bar{g}_{i+1}(1)$, $\bar{p}_i(1)$, $\overline{D}_3(i)(1)$ and $\bar{p}_i'(1)$ described in (3.9) for the classes d_i, e_i, f_i, g_{i+1}, p_i, $D_3(i)$ and p_i', respectively. These cycle representations will simply be denoted by $d_i^*, e_i^*, \ldots, D_3^*(i)$ and $(p_i')^*$. From (3.9) we also note that $p_i^* = \bar{p}_i(1) = \bar{p}_i(3)$, $D_3^* = \overline{D}_3(i)(1) = \overline{D}_3(i)(3)$ and $(p_i')^* = \bar{p}_i'(1) = \bar{p}_i'(2)$. In addition to these, we will also consider the cycle representation $\bar{d}_i(2)$ in (3.9.1) for the class d_i.

(3.14.1) $h_i^3 h_j^2 = \{\overline{\lambda_{2^i-2^{j+1}-1}(h_i^*)^2(h_{j+1}^*)^2}\}$
$\to e_{2^i-2^{j+1}-1} h_i^2 h_{j+1}^2 = \bar{\phi}(h_i^3 h_j^2)$ for $i \geqslant j+5$.

(3.14.2) $h_i^3 h_j h_k = \{\overline{\lambda_{2^i-2^j-2^k-1}(h_i^*)^2 h_{j+1}^* h_{k+1}^*}\}$
$\to e_{2^i-2^j-2^k-1} h_i^2 h_{j+1} h_{k+1} = \bar{\phi}(h_i^3 h_j h_k)$ for $i \geqslant j+4 \geqslant k+6$.

(3.14.3) $h_i^2 h_j^3 = \{\overline{\lambda_{2^i-2^{j+1}-2^j-1} h_i^*(h_{j+1}^*)^3}\}$
$\to e_{2^i-2^{j+1}-2^j-1} h_i h_{j+1}^3 = \bar{\phi}(h_i^2 h_j^3)$ for $i \geqslant j+4$.

(3.14.4) $h_i^2 h_j^2 h_k = \{\overline{\lambda_{2^i-2^{j+1}-2^k-1} h_i^*(h_{j+1}^*)^2 h_{k+1}^*}\}$
$\to e_{2^i-2^{j+1}-2^k-1} h_i h_{j+1}^2 h_{k+1} = \bar{\phi}(h_i^2 h_j^2 h_k)$ for $i \geqslant j+4 \geqslant k+7$.

(3.14.5) $h_i^2 h_j h_k^2 = \{\overline{\lambda_{2^i-2^j-2^{k+1}-1} h_i^* h_{j+1}^* (h_{k+1}^*)^2}\}$
$\to e_{2^i-2^j-2^{k+1}-1} h_i h_{j+1} h_{k+1}^2 = \overline{\phi}(h_i^2 h_j h_k^2)$ for $i \geqslant j+3 \geqslant k+6$.

(3.14.6) $h_i^2 h_j h_k h_l = \{\overline{\lambda_{2^i-2^j-2^k-2^l-1} h_i^* h_{j+1}^* h_{k+1}^* h_{l+1}^*}\}$
$\to e_{2^i-2^j-2^k-2^l-1} h_i h_{j+1} h_{k+1} h_{l+1} = \overline{\phi}(h_i^2 h_j h_k h_l)$
for $i \geqslant j+3 \geqslant k+5 \geqslant l+7$.

(3.14.7) $h_i h_j^3 h_k = \{\overline{\lambda_{2^i-2^{j+1}-2^j-2^k-1} (h_{j+1}^*)^3 h_{k+1}^*}\}$
$\to e_{2^i-2^{j+1}-2^j-2^k-1} h_{j+1}^3 h_{k+1} = \overline{\phi}(h_i h_j^3 h_k)$ for $i \geqslant j+3 \geqslant k+7$.

(3.14.8) $h_i h_j^2 h_k^2 = \{\overline{\lambda_{2^i-2^{j+1}-2^{k+1}-1} (h_{j+1}^*)^2 (h_{k+1}^*)^2}\}$
$\to e_{2^i-2^{j+1}-2^{k+1}-1} h_{j+1}^2 h_{k+1}^2 = \overline{\phi}(h_i h_j^2 h_k^2)$ for $i \geqslant j+3 \geqslant k+7$.

(3.14.9) $h_i h_j^2 h_k h_l = \{\overline{\lambda_{2^i-2^{j+1}-2^k-2^l-1} (h_{j+1}^*)^2 h_{k+1}^* h_{l+1}^*}\}$
$\to e_{2^i-2^{j+1}-2^k-2^l-1} h_{j+1}^2 h_{k+1} h_{l+1} = \overline{\phi}(h_i h_j^2 h_k h_l)$
for $i \geqslant j+3 \geqslant k+6 \geqslant l+8$.

(3.14.10) $h_i h_j h_k^3 = \{\overline{\lambda_{2^i-2^j-2^{k+1}-2^k-1} h_{j+1}^* (h_{k+1}^*)^3}\}$
$\to e_{2^i-2^j-2^{k+1}-2^k-1} h_{j+1} h_{k+1}^3 = \overline{\phi}(h_i h_j h_k^3)$ for $i \geqslant j+2 \geqslant k+5$.

(3.14.11) $h_i h_j h_k h_l^2 = \{\overline{\lambda_{2^i-2^j-2^k-2^{l+1}-1} h_{j+1}^* h_{k+1}^* (h_{l+1}^*)^2}\}$
$\to e_{2^i-2^j-2^k-2^{l+1}-1} h_{j+1} h_{k+1} h_{l+1}^2 = \overline{\phi}(h_i h_j h_k h_l^2)$
for $i \geqslant j+2 \geqslant k+4 \geqslant l+7$.

(3.14.12) $h_i h_j h_k^2 h_l = \{\overline{\lambda_{2^i-2^j-2^{k+1}-2^l-1} h_{j+1}^* (h_{k+1}^*)^2 h_{l+1}^*}\}$
$\to e_{2^i-2^j-2^{k+1}-2^l-1} h_{j+1} h_{k+1}^2 h_{l+1} = \overline{\phi}(h_i h_j h_k^2 h_l)$
for $i \geqslant j+2 \geqslant k+5 \geqslant l+8$.

(3.14.13) $h_i h_j h_k h_l h_m = \{\overline{\lambda_{2^i-2^j-2^k-2^l-2^m-1} h_{j+1}^* h_{k+1}^* h_{l+1}^* h_{m+1}^*}\}$
$\to e_{2^i-2^j-2^l-2^l-2^m-1} h_{j+1} h_{k+1} h_{l+1} h_{m+1} = \overline{\phi}(h_i h_j h_k h_l h_m)$
for $i \geqslant j+2 \geqslant k+4 \geqslant l+6 \geqslant m+8$.

(3.14.14) $h_i h_0^4 = \{\overline{\lambda_{2^i-8} h_3^* (h_0^*)^3}\} \to e_{2^i-8} h_3 h_0^3 = \overline{\phi}(h_i h_0^4)$ for $i \geqslant 4$.

(3.14.15) $c_i h_j^2 = \{\overline{\lambda_{2^{i+1}+2^i-2^{j+1}-1} (h_{i+2}^*)^2 (h_{j+1}^*)^2}\}$
$\to e_{2^{i+1}+2^i-2^{j+1}-1} h_{i+2}^2 h_{j+1}^2 = \overline{\phi}(c_i h_j^2)$ for $i \geqslant j+3$.

(3.14.16) $c_i h_j^2 = h_j^2 c_i = \{\overline{\lambda_{2^j-2^{i+3}-2^{i+1}-2^i-1} h_j^* c_{i+1}^*}\}$
$\to e_{2^j-2^{i+3}-2^{i+1}-2^i-1} h_j c_{i+1} = \overline{\phi}(h_j^2 c_i)$ for $j \geqslant i+5$.

(3.14.17) $c_i h_j h_k = h_k c_i h_j = \{\overline{\lambda_{2^k-2^{i+3}-2^{i+1}-2^i-2^j-1} c_{i+1}^* h_{j+1}^*}\}$
$\to e_{2^k-2^{i+3}-2^{i+1}-2^i-2^j-1} c_{i+1} h_{j+1} = \overline{\phi}(h_k c_i h_j)$
for $k \geqslant i+4 \geqslant j+6$.

(3.14.18) $c_i h_j h_k = h_k h_j c_i = \{\overline{\lambda_{2^k-2^j-2^{i+3}-2^{i+1}-2^i-1} h_{j+1}^* c_{i+1}^*}\}$
$\to e_{2^k-2^j-2^{i+3}-2^{i+1}-2^i-1} h_{j+1} c_{i+1} = \overline{\phi}(h_k h_j c_i)$
for $k > j+1 \geqslant i+5$ or $k-2 > j+1 = i$.

(3.14.19) $c_i h_j h_{i+1} = h_{i+1} c_i h_j = \{\overline{\lambda_{2^{i+1}-2^j-1} c_i^* h_{j+1}^*}\}$
$\to e_{2^{i+1}-2^j-1} c_i h_{j+1} = \overline{\phi}(h_{i+1} c_i h_j)$ for $i \geqslant j+3$.

(3.14.20) $c_i h_j h_k = c_i h_k h_j = \{\overline{\lambda_{2^{i+1}+2^i-2^k-2^j-1}(h_{i+2}^*)^2 h_{k+1}^* h_{j+1}^*}\}$
$\to e_{2^{i+1}+2^i-2^k-2^j-1} h_{i+2}^2 h_{k+1} h_{j+1} = \bar{\phi}(c_i h_k h_j)$
for $i \geqslant k+2 \geqslant j+4$.

(3.14.21) $d_i h_j = \{\overline{\lambda_{2^{i+2}+2^{i+1}+2^i-2^j-1} c_i^* h_{j+1}^*}\}$
$\to e_{2^{i+2}+2^{i+1}+2^i-2^j-1} c_i h_{j+1} = \bar{\phi}(d_i h_j)$ for $i \geqslant j+3$.

(3.14.22) $d_i h_i = \{\overline{\lambda_{2^{i+2}+2^{i+1}-1} h_{i+1}^* c_i^*}\}$
$\to e_{2^{i+2}+2^{i+1}-1} h_{i+1} c_i = \bar{\phi}(d_i h_i)$ for $i \geqslant 0$.

(3.14.23) $d_i h_{i+2} = h_{i+2} d_i = \{\overline{\lambda_{2^{i+2}-1} d_i^*}\}$
$\to e_{2^{i+2}-1} d_i = \bar{\phi}(d_i h_{i+2})$ for $i \geqslant 0$.

(3.14.24) $d_i h_j = h_j d_i = \{\overline{\lambda_{2^j-2^{i+4}-2^{i+1}-1} d_{i+1}^*}\}$
$\to e_{2^j-2^{i+4}-2^{i+1}-1} d_{i+1} = \bar{\phi}(h_j d_i)$ for $j \geqslant i+5$.

(3.14.25) $e_i h_j = \{\overline{\lambda_{2^{i+3}+2^i-2^j-1}(h_{i+1}^*)^3 h_{j+1}^*}\}$
$\to e_{2^{i+3}+2^i-2^j-1} h_{i+1}^3 h_{j+1} = \bar{\phi}(e_i h_j)$ for $i \geqslant j+3$.

(3.14.26) $e_{i+2} h_i = \{\overline{\lambda_{2^{i+5}-2^i-1} c_{i+1}^* h_{i+5}^*}\}$
$\to e_{2^{i+5}-2^i-1} c_{i+1} h_{i+5} = \bar{\phi}(e_{i+2} h_i)$ for $i \geqslant 0$.

(3.14.27) $e_i h_j = h_j e_i = \{\overline{\lambda_{2^j-2^{i+4}-2^{i+2}-2^i-1} e_{i+1}^*}\}$
$\to e_{2^j-2^{i+4}-2^{i+2}-2^i-1} e_{i+1} = \bar{\phi}(h_j e_i)$ for $j \geqslant i+5$.

(3.14.28) $f_i h_{i+2} = h_{i+2} f_i = \{\overline{\lambda_{2^{i+2}-1} f_i^*}\}$
$\to e_{2^{i+2}-1} f_i = \bar{\phi}(h_{i+2} f_i)$ for $i \geqslant 0$.

(3.14.29) $f_i h_j = h_j f_i = \{\overline{\lambda_{2^j-2^{i+4}-2^{i+2}-2^{i+1}-1} f_{i+1}^*}\}$
$\to e_{2^j-2^{i+4}-2^{i+2}-2^{i+1}-1} f_{i+1} = \bar{\phi}(h_j f_i)$ for $j \geqslant i+5$.

(3.14.30) $g_{i+1} h_j = \{\overline{\lambda_{2^{i+2}+2^{i+1}+2^i-2^j-1}(h_{i+3}^*)^2 h_i^* h_{j+1}^*}\}$
$\to e_{2^{i+2}+2^{i+1}+2^i-2^j-1} h_{i+3}^2 h_i h_{j+1} = \bar{\phi}(g_{i+1} h_j)$ for $i \geqslant j+3$.

(3.14.31) $g_{i+1} h_{i-2} = \{\overline{\lambda_{2^{i+2}+2^{i+1}-1}(p_{i-2}')^*}\}$
$\to e_{2^{i+2}+2^{i+1}-1} p_{i-2}' = \bar{\phi}(g_{i+1} h_{i-2})$ for $i \geqslant 2$.

(3.14.32) $g_{i+1} h_{i-1} = \{\overline{\lambda_{2^{i+2}+2^{i+1}-1} p_{i-1}^*}\}$
$\to e_{2^{i+2}+2^{i+1}-1} p_{i-1} = \bar{\phi}(g_{i+1} h_{i-1})$ for $i \geqslant 1$.

(3.14.33) $g_{i+1} h_{i+2} = h_{i+2} g_{i+1} = \{\overline{\lambda_{2^{i+2}-1} g_{i+1}^*}\}$
$\to e_{2^{i+2}-1} g_{i+1} = \bar{\phi}(h_{i+2} g_{i+1})$ for $i \geqslant 0$.

(3.14.34) $g_{i+1} h_j = h_j g_{i+1} = \{\overline{\lambda_{2^j-2^{i+4}-2^{i+3}-1} g_{i+2}^*}\}$
$\to e_{2^j-2^{i+4}-2^{i+3}-1} g_{i+2} = \bar{\phi}(h_j g_{i+1})$ for $j \geqslant i+6$.

(3.14.35) $p_i h_j = \{\overline{\lambda_{2^{i+4}-2^i-2^j-1} c_{i+1}^* h_{j+1}^*}\}$
$\to e_{2^{i+4}-2^i-2^j-1} c_{i+1} h_{j+1} = \bar{\phi}(p_i h_j)$ for $i \geqslant j+2$.

(3.14.36) $p_i h_{i+3} = h_{i+3} p_i = \{\overline{\lambda_{2^{i+3}-1} p_i^*}\}$
$\to e_{2^{i+3}-1} p_i = \bar{\phi}(h_{i+3} p_i)$ for $i \geqslant 0$.

(3.14.37) $p_i h_j = h_j p_i = \{\overline{\lambda_{2^j-2^{i+5}-2^{i+2}-2^i-1} p_{i+1}^*}\}$
$\to e_{2^j-2^{i+5}-2^{i+2}-2^i-1} p_{i+1} = \bar{\phi}(h_j p_i)$ for $j \geqslant i+6$.

(3.14.38) $D_3(i)h_j = \{\overline{\lambda_{2^{i+4}+2^{i+3}-2^i-2^j-1} h^*_{i+5} h^*_{i+3} h^*_{i+1} h^*_{j+1}}\}$
$\to e_{2^{i+4}+2^{i+3}-2^i-2^j-1} h_{i+5} h_{i+3} h_{i+1} h_{j+1} = \bar\phi(D_3(i)h_j)$ for $i \geqslant j+2$.

(3.14.39) $D_3(i)h_{i+1} = \{\overline{\lambda_{2^{i+3}+2^{i+2}+2^i-1} c^*_{i+1} h^*_{i+5}}\}$
$\to e_{2^{i+3}+2^{i+2}+2^i-1} c_{i+1} h_{i+5} = \bar\phi(D_3(i)h_{i+1})$ for $i \geqslant 0$.

(3.14.40) $D_3(i)h_j = h_j D_3(i) = \{\overline{\lambda_{2^j-2^{i+6}-2^i-1} D^*_3(i+1)}\}$
$\to e_{2^j-2^{i+6}-2^i-1} D_3(i+1) = \bar\phi(h_j D_3(i))$ for $j \geqslant i+7$.

(3.14.41) $p'_i h_j = \{\overline{\lambda_{2^{i+5}+2^{i+3}-2^i-2^j-1} (h^*_{i+4})^2 h^*_{i+1} h^*_{j+1}}\}$
$\to e_{2^{i+5}+2^{i+3}-2^i-2^j-1} h^2_{i+4} h_{i+1} h_{j+1} = \bar\phi(p'_i h_j)$ for $i \geqslant j+2$.

(3.14.42) $p'_i h_i = \{\overline{\lambda_{2^{i+5}-2^{i+1}-1} f^*_{i+1}}\}$
$\to e_{2^{i+5}-2^{i+1}-1} f_{i+1} = \bar\phi(p'_i h_i)$ for $i \geqslant 0$.

(3.14.43) $p'_i h_j = h_j p'_i = \{\overline{\lambda_{2^j-2^{i+6}-2^{i+3}-2^i-1} (p'_{i+1})^*}\}$
$\to e_{2^j-2^{i+6}-2^{i+3}-2^i-1} p'_{i+1} = \bar\phi(h_j p'_i)$ for $j \geqslant i+7$.

(3.14.44) $d_i h_{i+1} = h_{i+1} d_i = \{\overline{\lambda_{2^{i+1}+2^i-1} \lambda_{2^{i+2}+2^i-1} (h^*_{i+2})^3}\}$
$\to e_{2^{i+1}-1} d_i = \tilde\phi(h_{i+1} d_i)$ for $i \geqslant 0$.

(3.14.45) $e_i h_{i+1} = h_{i+1} e_i = \{\overline{\lambda_{2^{i+2}+2^i-1} \lambda_{2^{i+2}+2^{i+1}-1} (h^*_{i+2})^3}\}$
$\to e_{2^{i+1}-1} e_i = \tilde\phi(h_{i+1} e_i)$ for $i \geqslant 0$.

(3.14.46) $e_i h_{i+2} = h_{i+2} e_i = \{\overline{\lambda_{2^{i+2}+2^i-1} h^*_{i+3} (h^*_{i+2})^3}\}$
$\to e_{2^{i+2}-1} e_i = \tilde\phi(h_{i+2} e_i)$ for $i \geqslant 0$.

(3.14.47) $f_i h_j = \{\overline{\lambda_{2^{i+2}+2^{i+1}-1} h^*_{i+3} (h^*_{i+2})^2 h^*_j}\}$
$\to e_{2^{i+2}+2^i-2^j-1} h^2_{i+3} h_i h_{j+1} = \tilde\phi(f_i h_j)$ for $i \geqslant j+3$.

(3.14.48) $D_3(i)h_{i+2} = h_{i+2} D_3(i) = \{\overline{\lambda_{2^{i+3}+2^{i+1}+2^i-1} \lambda_{2^{i+3}+2^{i+1}-1} (h^*_{i+4})^3}\}$
$\to e_{2^{i+2}-1} D_3(i) = \tilde\phi(h_{i+2} D_3(i))$ for $i \geqslant 0$.

(3.14.49) $D_3(i)h_{i+3} = h_{i+3} D_3(i) = \{\overline{\lambda_{2^{i+3}+2^i-1} (h^*_{i+4})^4}\}$
$\to e_{2^{i+3}-1} D_3(i) = \tilde\phi(h_{i+3} D_3(i))$ for $i \geqslant 0$.

(3.14.50) $D_3(i)h_{i+4} = \{\overline{\lambda_{2^{i+4}-2^i-1} \lambda_{2^{i+4}+2^{i+1}-1} (h^*_{i+4})^3}\}$
$\to e_{2^{i+3}-1} p'_i = \tilde\phi(D_3(i)h_{i+4})$ for $i \geqslant 0$.

(3.14.51) $p'_i h_{i+4} = h_{i+4} p'_i = \{\overline{\lambda_{2^{i+4}+2^{i+3}-2^i-1} h^*_{i+5} (h^*_{i+4})^2 h^*_{i+1}}\}$
$\to e_{2^{i+4}-1} p'_i = \tilde\phi(h_{i+4} p'_i)$ for $i \geqslant 4$.

Let I be the set of all the infinite cycles $\bar\phi(\alpha)$ or $\tilde\phi(\alpha)$ in $(3.14) = \bigcup_{n=1}^{51}(3.14.n)$. Let $B(4)$ be the set of all the boundaries in the differentials (2.5.1) through (2.5.74) in Section 2. It is not difficult to check that $B(4) \cap I = \phi$. This implies that I is linearly independent in $E^{*,4,*}_\infty$ of the spectral sequence $\{E^{*,*,*}_r\}_{r\geqslant 1}$ for $Ext^{*,*}_A(P)$. Since the set of the classes of $Ext^{5,*}_A$ listed in (3.14) is precisely the set $\widetilde{B}$ of the classes in (3.11)*, by Proposition 3.7, we see this in turn implies that $\widetilde{B}$ is a linearly independent subset of $Ext^{5,*}_A$. This will complete the proof of Theorem 1.4 once (3.14) is proved.

The proof of (3.14) will be parallel to that for (3.8) (which is to prove $\overline{(1.3)}'$(ii)). We are going to give detailed proofs only for

(i) (3.4.n) with $n = 1, 16, 23, 24, 26, 31, 32, 39$ and 42 which are of type (f).

and also for

(ii) (3.14.m) for $44 \leqslant m \leqslant 51$ which are of type (h)

since the proofs of some of the $(3.14.n)'^{s}$ in (i) are prototypes for those of the remaining (3.14.l). (iii) below lists these remaining (3.14.l). We will write $(3.14.l) \in (3.14.n)$ to mean that the proof of (3.14.n) is a prototype for a proof of (3.14.l). Then we have the following.

(iii) (1) $(3.14.l_1) \in (3.14.1)$ for $2 \leqslant l_1 \leqslant 15$ and also for $l_1 = 20, 21, 22, 25, 30, 35, 38$ and 41,
(2) $(3.14.l_2) \in (3.14.16)$ for $l_2 = 17, 18, 19$,
(3) $(3.14.l_3) \in (3.14.23)$ for $l_3 = 28, 33, 36$,
(4) $(3.14.l_4) \in (3.14.24)$ for $l_4 = 27, 29, 34, 37, 40, 43$.

We begin with the proof of (3.14.1). The class $h_i^3 h_j^2 = h_j^2 h_i^3$ is represented by the cycle $\lambda^2_{2^j-1}\lambda^3_{2^i-1} = (h_j^*)^2(h_i^*)^3$ which is inadmissible since $i \geqslant j+5$. In the remainder of this proof for (3.14.1) and also in the proof of (3.14.16) that follows we will use H_k to denote $h_k^* = \lambda_{2^k-1}$ for $k \geqslant 0$. We have

$$H_j^2 H_i^3 = \lambda^2_{2^j-1}\lambda^3_{2^i-1} \equiv \lambda_{2^i-2^{j+1}-1} H_{j+1}^2 H_i^2 \bmod \Lambda(2^i - 2^{j+1} - 2).$$

Let $T = \lambda_{2^i+2^{j+1}-1}$. We have $\delta(T) = H_{j+1}H_i + H_i H_{j+1}$ in Λ. It is easy to see that

$$\begin{aligned}&\delta\big[\lambda_{2^i-2^{j+1}-1}(H_{j+1}TH_i + TH_{j+1}H_i + H_iH_{j+1}T + H_iTH_{j+1})\big]\\ &\quad \equiv \lambda_{2^i-2^{j+1}-1}(H_{j+1}^2H_i^2 + H_i^2H_{j+1}^2) \bmod \Lambda(2^i - 2^{j+1} - 2).\end{aligned}$$

So $h_i^3h_j^2 = \{\overline{\lambda_{2^i-2^{j+1}-1}H_i^2H_{j+1}^2}\}$ since $\lambda_{2^i-2^{j+1}-1}H_i^2H_{j+1}^2$ is admissible. The condition $i \geqslant j+5$ insures that $h_i^2h_{j+1}^2 = \{H_i^2H_{j+1}^2\}$ is non-zero. This proves (3.14.1).

Proof of (3.14.16). The class $h_j^2c_i = c_ih_j^2$ is represented by the cycle $c_i^*H_j^2 = \lambda_{2^{i+1}+2^i-1}\lambda^2_{2^{i+2}-1}\lambda^2_{2^j-1}$ which is inadmissible since $j \geqslant i+5$. We have

$$c_i^*H_j^2 \equiv \lambda_{2^j-2^{i+3}-2^{i+1}-2^i-1}c_{i+1}^*H_j \bmod \Lambda(2^j - 2^{i+3} - 2^{i+1} - 2^i - 2).$$

Since $c_{i+1} = \{c_{i+1}^*\}$, $h_j = \{H_j\}$ and $c_{i+1}h_j = h_jc_{i+1}$ it follows that there is a $T_1 \in \Lambda^{3,*}$ such that $\delta(T_1) = c_{i+1}^*H_j + H_jc_{i+1}^*$ in Λ. As $j \geqslant i+5$, it is not difficult to see that

$$\delta(\lambda_kT_1) \equiv \lambda_k(c_{i+1}^*H_j + H_jc_{i+1}^*) \bmod \Lambda(k-1)$$

where $k = 2^j - 2^{i+3} - 2^{i+1} - 2^i - 1$. So $c_ih_j^2 = \{\overline{\lambda_{2^j-2^{i+3}-2^{i+1}-2^i-1}H_jc_{i+1}^*}\}$ since $\lambda_{2^j-2^{i+3}-2^{i+1}-2^i-1}H_jc_{i+1}^*$ is admissible. The condition $j \geqslant i+5$ insures that $h_jc_{i+1} = \{H_jc_{i+1}^*\}$ is non-zero. This proves (3.14.16). □

Proof of (3.14.23). The class $d_ih_{i+2} = h_{i+2}d_i$ is represented by the cycle $\lambda_{2^{i+2}-1}d_i^* = \lambda_{2^{i+2}-1}(Sq^0)^i(\lambda_6\lambda_2\lambda_3^2 + \lambda_4^2\lambda_3^2 + \lambda_2\lambda_4\lambda_5\lambda_3)$. The monomial $\lambda_{2^{i+2}-1}(Sq^0)^i(\lambda_6\lambda_2\lambda_3^2) = \lambda_{2^{i+2}-1}\lambda_{2^{i+3}-2^i-1}\lambda_{2^{i+1}+2^i-1}\lambda^2_{2^{i+2}-1}$ is admissible, and this implies $\lambda_{2^{i+2}-1}(Sq^0)^i(\lambda_4^2\lambda_3^2)$ and $\lambda_{2^{i+2}-1}(Sq^0)^i(\lambda_2\lambda_4\lambda_5\lambda_3)$ are also admissible. So $h_{i+2}d_i = \{\overline{\lambda_{2^{i+2}-1}d_i^*}\}$. This proves (3.14.23). □

Proof of (3.14.24). The class $d_ih_j = h_jd_i$ is represented by the cycle $d_i^*\lambda_{2^j-1} = (Sq^0)^i[(\lambda_6\lambda_2\lambda_3^2 + \lambda_4^2\lambda_3^2 + \lambda_2\lambda_4\lambda_5\lambda_3)\lambda_{2^{j-i}-1}]$ which is a sum of inadmissible monomials since $j \geqslant i+5$. We have

$$\begin{aligned}&(\lambda_6\lambda_2\lambda_3^2 + \lambda_4^2\lambda_3^2 + \lambda_2\lambda_4\lambda_5\lambda_3)\lambda_{2^{j-i}-1}\\ &\quad \equiv \lambda_{2^{j-i}-18-1}\big[(\lambda_{13}\lambda_5\lambda_7^2 + \lambda_9^2\lambda_7^2 + \lambda_5\lambda_9\lambda_{11}\lambda_7) = d_1^*\big] \bmod \Lambda(2^{j-i} - 20).\end{aligned}$$

This implies $d_i^*\lambda_{2^i-1} \equiv \lambda_{2^j-2^{i+4}-2^{i+1}-1}d_{i+1}^* \bmod \Lambda(2^j - 2^{i+4} - 2^{i+1} - 2)$. Since $\lambda_{2^j-2^{i+4}-2^{i+1}-1}d_{i+1}^*$ is a sum of admissible monomials (as $j \geqslant i+5$) it follows that $d_ih_j = \{\overline{\lambda_{2^j-2^{i+4}-2^{i+1}-1}d_{i+1}^*}\}$. This proves (3.14.24). □

Proof of (3.14.26). The class $e_2h_0 = h_0e_2$ is represented by the cycle $\lambda_0e_2^* = \lambda_0(\lambda_{35}\lambda_{15}^3 + \lambda_{19}\lambda_{23}^2\lambda_{15} + \lambda_{19}\lambda_{31}\lambda_{15}^2 + \lambda_{11}\lambda_{15}\lambda_{23}\lambda_{31})$. We have

$$\lambda_0e_2^* \equiv \lambda_0\lambda_{35}\lambda_{15}^3 \equiv \lambda_{30}\lambda_5\lambda_{15}^3 = \lambda_{30}\lambda_9\lambda_{11}\lambda_{15}^2 \bmod \Lambda(29).$$

Since $\lambda_{30}\lambda_9\lambda_{11}\lambda_{15}^2$ is admissible and $\lambda_9\lambda_{11}\lambda_{15}^2 = \lambda_5\lambda_7^2\lambda_{31} = c_1^*h_5^*$ it follows that $h_0e_2 = \{\overline{\lambda_{30}c_1^*h_5^*}\}$. Applying $(Sq^0)^i$ we get $h_ie_{i+2} = e_{i+2}h_i = \{\overline{\lambda_{2^{i+5}-2^i-1}c_{i+1}^*h_{i+5}^*}\}$. This proves (3.14.26). □

Proof of (3.14.31). The class $g_3h_0 = h_0g_3$ is represented by the cycle $\lambda_0g_3^* = \lambda_0(\lambda_{27}\lambda_3\lambda_{31}^2 + \lambda_{23}\lambda_{39}\lambda_{15}^2 + \lambda_{23}\lambda_{15}\lambda_{23}\lambda_{31} + \lambda_{15}\lambda_{23}\lambda_{39}\lambda_{15})$. We have

$$\lambda_0g_3^* \equiv \lambda_0\lambda_{27}\lambda_3\lambda_{31}^2 \equiv \lambda_{24}\lambda_3^2\lambda_{31}^2 \bmod \Lambda(22)$$

and $\delta[\lambda_{24}(\lambda_{39}\lambda_{15}^2 + \lambda_{15}\lambda_{23}\lambda_{31})] \equiv \lambda_{24}\lambda_3^2\lambda_{31}^2 + \lambda_{23}(p_0')^* \bmod \Lambda(22)$. Here we recall that

$$(p_0')^* = \overline{p}_0'(1) = \overline{p}_0'(2) = \lambda_0(\lambda_{39}\lambda_{15}^2 + \lambda_{15}\lambda_{23}\lambda_{31}) = \overline{\lambda_{38}(\lambda_1\lambda_{15}^2 = \lambda_{13}\lambda_{11}\lambda_7)}.$$

Thus $g_3h_0 = \{\overline{\lambda_{23}(p_0')^*}\}$ since $\lambda_{23}(p_0')^*$ is a sum of admissible monomials. Applying $(Sq^0)^i$ we get $g_{i+3}h_i = \{\overline{\lambda_{2^{i+4}+2^{i+3}-1}(p_i')^*}\}$. This proves (3.14.31). □

Proof of (3.14.32). The class $g_2h_0 = h_0g_2$ is represented by the cycle $\lambda_0g_2^* = \lambda_0(\lambda_{13}\lambda_1\lambda_{15}^2 + \lambda_{11}\lambda_{19}\lambda_7^2 + \lambda_{11}\lambda_7\lambda_{11}\lambda_{15} + \lambda_7\lambda_{11}\lambda_{19}\lambda_7)$. We have

$$\lambda_0g_2^* \equiv \lambda_0\lambda_{13}\lambda_1\lambda_{15}^2 \equiv \lambda_{12}\lambda_1^2\lambda_{15}^2 \bmod \Lambda(10)$$

and $\delta[\lambda_{12}(\lambda_{19}\lambda_7^2 + \lambda_7\lambda_{11}\lambda_{15})] \equiv \lambda_{12}\lambda_1^2\lambda_{15}^2 + \lambda_{11}p_0^* \bmod \Lambda(10)$. Here we recall that

$$p_0^* = \overline{p}_0(1) = \overline{p}_0(3) = \lambda_0(\lambda_{19}\lambda_7^2 + \lambda_7\lambda_{11}\lambda_{15}) = \lambda_{14}\lambda_5\lambda_7^2 + \lambda_{10}\lambda_9\lambda_7^2 + \lambda_6\lambda_9\lambda_{11}\lambda_7.$$

Thus $g_2h_0 = \{\overline{\lambda_{11}p_0^*}\}$ since $\lambda_{11}p_0^*$ is a sum of admissible monomials. Applying $(Sq^0)^i$ we get $g_{i+2}h_i = \{\overline{\lambda_{2^{i+3}+2^{i+2}-1}p_i^*}\}$. This proves (3.14.32). □

Proof of (3.14.39). The class $D_3(0)h_1 = h_1D_3(0)$ is represented by the cycle $\lambda_1D_3(0)^* = \lambda_1\overline{D}_3(0)(1) = \lambda_1\overline{D}_3(0)(3) = \lambda_1\lambda_0\lambda_{23}\lambda_7\lambda_{31}$. We have

$$\delta(\lambda_2\lambda_{23}\lambda_7\lambda_{15}) + \lambda_1\lambda_0\lambda_{23}\lambda_7\lambda_{31} = \lambda_2\lambda_{15}\lambda_7^2\lambda_{31} \equiv \lambda_{12}\lambda_5\lambda_7^2\lambda_{31} \bmod \Lambda(10).$$

Since $\lambda_{12}\lambda_5\lambda_7^2\lambda_{31} = \lambda_{12}\lambda_5\lambda_{15}^3 = \lambda_{12}\lambda_9\lambda_{11}\lambda_{15}^2$ is admissible it follows that $D_3(0)h_1 = \{\overline{\lambda_{12}\lambda_5\lambda_7^2\lambda_{31} = \lambda_{12}c_1^*h_5^*}\}$. Applying $(Sq^0)^i$ we get $D_3(i)h_{i+1} = \{\overline{\lambda_{2^{i+3}+2^{i+2}+2^i-1}c_{i+1}^*h_{i+5}^*}\}$. This proves (3.14.39). □

Proof of (3.14.42). Let $\overline{h_0p_0'}$ be the exotic cycle representation for the class $h_0p_0' = p_0'h_0$ obtained by applying Proposition 3.12 to the cycle

$$(p_0')^* = \overline{p}'(1) = \lambda_{38}\lambda_1\lambda_{15}^2 + \lambda_{30}\lambda_9\lambda_{15}^2 + \lambda_{28}\lambda_{11}\lambda_{15}^2 + \lambda_{22}\lambda_{17}\lambda_{15}^2 + \lambda_{20}\lambda_{19}\lambda_{15}^2 + \lambda_{14}\lambda_1\lambda_{23}\lambda_{31} + \lambda_{12}\lambda_{19}\lambda_{23}\lambda_{15}$$

for p_0' that comes from the cycle

$$(\widehat{p}_0')^* = e_{38}\lambda_1\lambda_{15}^2 + e_{30}\lambda_9\lambda_{15}^2 + e_{28}\lambda_{11}\lambda_{15}^2 + e_{22}\lambda_{17}\lambda_{15}^2 + e_{20}\lambda_{19}\lambda_{15}^2 + e_{14}\lambda_1\lambda_{23}\lambda_{31} + e_{12}\lambda_{19}\lambda_{23}\lambda_{15}$$

for the class $\widehat{p}_0'$ as in (2.6)(7). We have

$$\overline{h_0p_0'} \equiv (\lambda_{38}\lambda_0\lambda_1\lambda_{15}^2 = 0) + \lambda_{37}\lambda_1^2\lambda_{15}^2 + \lambda_{35}\lambda_3\lambda_1\lambda_{15}^2 + \lambda_{30}(\lambda_8\lambda_1\lambda_{15}^2 = \lambda_0\lambda_9\lambda_{15}^2) + \lambda_{29}(\lambda_9\lambda_1\lambda_{15}^2 = \lambda_9\lambda_{13}\lambda_{11}\lambda_7) + \lambda_{30}\lambda_0\lambda_9\lambda_{15}^2 + \lambda_{29}\lambda_1\lambda_9\lambda_{15}^2 \bmod \Lambda(28).$$

Let $T_2 = \lambda_{37}(\lambda_{19}\lambda_7^2 + \lambda_7\lambda_{11}\lambda_{15}) + \lambda_{35}\lambda_5\lambda_{15}^2 + \lambda_{31}\lambda_1\lambda_{15}\lambda_{23} \in \Lambda^{3,70}$. By direct calculations we find that

$$\delta(T_2) \equiv \lambda_{37}\lambda_1^2\lambda_{15}^2 + \lambda_{35}\lambda_3\lambda_1\lambda_{15}^2 + \lambda_{29}(\lambda_{11}\lambda_{15}\lambda_7^2 + \lambda_7^2\lambda_{11}\lambda_{15}) \bmod \Lambda(28).$$

So

$$\begin{aligned}\delta(T_2) &\equiv \overline{h_0 p_0'} + \lambda_{29}\big[\lambda_{11}\lambda_{15}\lambda_7^2 + \lambda_9\lambda_{13}\lambda_{11}\lambda_7 + \lambda_7^2\lambda_{11}\lambda_{15} + (\lambda_1\lambda_9\lambda_{15}^2 = \lambda_5\lambda_9\lambda_{11}\lambda_{15})\big]\\ &\equiv \overline{h_0 p_0'} + \lambda_{29} f_1^* \bmod \Lambda(28).\end{aligned}$$

Thus $h_0 p_0' = \{\overline{\lambda_{29} f_1^*}\}$ since $\lambda_{29} f_1^*$ is a sum of admissible monomials. Applying $(Sq^0)^i$ we get $p_i' h_i = h_i p_i' = \{\overline{\lambda_{2^{i+5}-2^{i+1}-1} f_{i+1}^*}\}$. This proves (3.14.42). □

Proof of (3.14.44). The class $h_1 d_0 = d_0 h_1$ is represented by the cycle

$$\lambda_1\bar{d}_0(2) = \lambda_1(\lambda_6\lambda_2\lambda_3^2 + \lambda_3\lambda_0^2\lambda_{11}) = \lambda_1\lambda_6\lambda_2\lambda_3^2 = \lambda_4\lambda_3\lambda_2\lambda_3^2 + \lambda_3\lambda_4\lambda_2\lambda_3^2.$$

Straightforward calculations show that

$$\delta(\lambda_5\lambda_1\lambda_7\lambda_3 + \lambda_4\lambda_6\lambda_3^2 + \lambda_2\lambda_4\lambda_7\lambda_3) = \lambda_1\bar{d}_0(2) + \lambda_2\lambda_4\lambda_3^3 + \lambda_1\lambda_2\lambda_4\lambda_5\lambda_3.$$

So $h_1 d_0 = \{\lambda_2\lambda_4\lambda_3^3 + \lambda_1\lambda_2\lambda_4\lambda_5\lambda_3\}$. Now apply the chain map $\Lambda \xrightarrow{\phi} \widetilde{H}_*(P) \otimes \Lambda$ in (3.5), (3.6)(1) to the admissibles $\lambda_2\lambda_4\lambda_3^3$ and $\lambda_1\lambda_2\lambda_4\lambda_5\lambda_3$. From (3.5) we find that $\phi(\lambda_2\lambda_4\lambda_3^3 + \lambda_1\lambda_2\lambda_4\lambda_5\lambda_3) = e_2\lambda_4\lambda_3^3 + e_1\lambda_5\lambda_3^3 + e_1\lambda_2\lambda_4\lambda_5\lambda_3$. We have the following equation in $\widetilde{H}_*(P) \otimes \Lambda$:

$$\begin{aligned}\delta(e_2\lambda_8\lambda_3^2 + e_1\lambda_1\lambda_{11}\lambda_3) &= \phi(\lambda_2\lambda_4\lambda_3^3 + \lambda_1\lambda_2\lambda_4\lambda_5\lambda_3)\\ &\quad + e_1\big[d_0^* = \bar{d}_0(1) = \lambda_6\lambda_2\lambda_3^2 + \lambda_4^2\lambda_3^2 + \lambda_2\lambda_4\lambda_5\lambda_3\big].\end{aligned}$$

This shows that $h_1 d_0 = \{\overline{\lambda_2\lambda_4\lambda_3^3 = \lambda_2\lambda_4(h_2^*)^3}\} \to e_1 d_0 = \widetilde{\phi}(h_1 d_0)$. Applying $(Sq^0)^i$ we get $h_{i+1} d_i = \{\overline{\lambda_{2^{i+1}+2^i-1}\lambda_{2^{i+2}+2^i-1}(h_{i+2}^*)^3}\} \to e_{2^{i+1}-1} d_i = \widetilde{\phi}(h_{i+1} d_i)$. This proves (3.14.44). □

For the remaining proofs for (3.14.45) through (3.14.51) to be given in what follows, the map $\Lambda \xrightarrow{\phi} \widetilde{H}_*(P) \otimes \Lambda$ always refers to the chain map in (3.5), (3.6)(1) as is used in the proof of (3.14.44) above. Also, it suffices to prove these (3.14.n) for $i = 0$ (and also $j = 0$ for (3.14.47)) because we can apply $(Sq^0)^i$ for arbitrary $i > 0$ as we did in the proof of (3.14.44) above.

Proof of (3.14.45). By (1.4)*(33), $h_1 e_0 = h_0 f_0$ which is represented by the cycle $\lambda_0 f_0^* = \lambda_4\lambda_5\lambda_3^3 + \lambda_3\lambda_1\lambda_0\lambda_7^2 + \lambda_2\lambda_3\lambda_5^2\lambda_3 + \lambda_1\lambda_2\lambda_3\lambda_5\lambda_7$. Since $\delta(\lambda_3\lambda_2\lambda_7^2) = \lambda_3\lambda_1\lambda_0\lambda_7^2$ it follows that

$$\lambda_0 f_0^* \sim z = \lambda_4\lambda_5\lambda_3^3 + \lambda_2\lambda_3\lambda_5^2\lambda_3 + \lambda_1\lambda_2\lambda_3\lambda_5\lambda_7.$$

We have $\phi(z) = e_4\lambda_5\lambda_3^3 + e_2\lambda_3\lambda_5^2\lambda_3 + e_1\lambda_2\lambda_3\lambda_5\lambda_7$ and

$$\begin{aligned}&\delta\big[e_4\lambda_9\lambda_3^2 + e_2(\lambda_5\lambda_9\lambda_3 + \lambda_{11}\lambda_3^2) + e_1(\lambda_6\lambda_1\lambda_{11} + \lambda_4\lambda_7^2)\big]\\ &\quad = \phi(z) + e_1(e_0^* = \lambda_8\lambda_3^3 + \lambda_4\lambda_5^2\lambda_3 + \lambda_4\lambda_7\lambda_3^2 + \lambda_2\lambda_3\lambda_5\lambda_7).\end{aligned}$$

Thus $h_1 e_0 = \{\overline{\lambda_4\lambda_5(h_2^*)^3}\} \to e_1 e_0 = \widetilde{\phi}(h_1 e_0)$. This proves (3.14.45). □

Proof of (3.14.46). $e_0 h_2 = h_2 e_0$ is represented by the cycle $\lambda_3 e_0^* = \lambda_4\lambda_7\lambda_3^3 + \lambda_3\lambda_4\lambda_5^2\lambda_3 + \lambda_3\lambda_4\lambda_7\lambda_3^2 + \lambda_3\lambda_2\lambda_3\lambda_5\lambda_7$ on which ϕ is given by $\phi(\lambda_3 e_0^*) = e_4\lambda_7\lambda_3^3 + e_2\lambda_9\lambda_3^3 + e_3(\lambda_4\lambda_5^2\lambda_3 + \lambda_4\lambda_7\lambda_3^2 + \lambda_2\lambda_3\lambda_5\lambda_7)$. We have $\delta(e_4\lambda_{11}\lambda_3^2 + e_3\lambda_0\lambda_7\lambda_{11}) = e_4\lambda_7\lambda_3^3 + e_3\lambda_8\lambda_3^3 + e_2\lambda_9\lambda_3^3$ and this implies

$$\begin{aligned}&\delta(e_4\lambda_{11}\lambda_3^2 + e_3\lambda_0\lambda_7\lambda_{11})\\ &\quad = \phi(\lambda_3 e_0^*) + e_3\big[e_0^* = \lambda_8\lambda_3^3 + \lambda_4(\lambda_5^2\lambda_3 + \lambda_7\lambda_3^2) + \lambda_2\lambda_3\lambda_5\lambda_7\big].\end{aligned}$$

Thus $h_2 e_0 = \{\overline{\lambda_4 h_3^*(h_2^*)^3}\} \to e_3 e_0 = \widetilde{\phi}(h_2 e_0)$. This proves (3.14.46). □

Proof of (3.14.47). The class $f_i h_0$ is represented by the cycle

$$f_i^*\lambda_0 = \big[(Sq^0)^i(f_0^* = \lambda_5\lambda_7\lambda_3^2 + \lambda_4(\lambda_6\lambda_5\lambda_3 = \lambda_0\lambda_7^2) + \lambda_3^2\lambda_5\lambda_7 + \lambda_2\lambda_4\lambda_5\lambda_7)\big]\lambda_0$$

which has

$$f_i^*\lambda_0 \equiv \lambda_{2^{i+2}+2^{i+1}-1}h_{i+3}^*(h_{i+2}^*)^2h_0^* + \lambda_{2^{i+2}+2^i-1}h_i^*(h_{i+3}^*)^2h_0^* \bmod \Lambda(2^{i+2}-1)$$

where $i \geqslant 3$. We have $\phi(\lambda_5\lambda_7\lambda_3^2) = e_5\lambda_7\lambda_3^2 + e_1\lambda_{11}\lambda_3^2$ and $\phi(\lambda_4\lambda_6\lambda_5\lambda_3) = e_4(\lambda_6\lambda_5\lambda_3 = \lambda_0\lambda_7^2) + e_1\lambda_9\lambda_5\lambda_3$. These imply

$$\phi(f_i^*\lambda_0) \equiv e_{2^{i+2}+2^{i+1}-1}h_{i+3}^*(h_{i+2}^*)^2h_0^* + e_{2^{i+2}+2^i-1}h_i^*(h_{i+3}^*)^2h_0^* \bmod F(2^{i+2}-1).$$

We have

$$\delta(e_{2^{i+2}+2^{i+1}-1}\lambda_{2^{i+3}+2^{i+2}-1}h_{i+2}^*h_0^*) \equiv e_{2^{i+2}+2^{i+1}-1}h_{i+3}^*(h_{i+2}^*)^2h_0^* \bmod F(2^{i+2}-1).$$

It is not difficult to see that

$$e_{2^{i+2}+2^i-1}h_i^*(h_{i+3}^*)^2h_0^* \sim e_{2^{i+2}+2^i-2}(h_{i+3}^*)^2h_i^*h_1^* \bmod F(2^{i+2}+2^i-3).$$

So $h_ih_0 = \{\overline{\lambda_{2^{i+2}+2^{i+1}-1}h_{i+3}^*(h_{i+2}^*)^2h_0^*}\} \to e_{2^{i+2}+2^i-2}h_{i+3}^2h_ih_1 = \widetilde{\phi}(f_ih_0)$. This proves (3.14.47). □

Proof of (3.14.48). The class $D_3(0)h_2 = h_2D_3(0)$ is represented by the cycle $\lambda_3(D_3^*(0) = \overline{D}_3(0)(1) = \overline{D}_3(0)(3) = \lambda_0\lambda_{23}\lambda_7\lambda_{31})$. We have

$$\delta(\lambda_4\lambda_{23}\lambda_7\lambda_{31} + \lambda_0\lambda_{35}\lambda_{15}^2) = \lambda_3D_3^*(0) + \lambda_{10}\lambda_9\lambda_{15}^3 + \lambda_8\lambda_{11}\lambda_{15}^3.$$

So $h_2D_3(0) = \{\lambda_{10}\lambda_9\lambda_{15}^3 + \lambda_8\lambda_{11}\lambda_{15}^3\}$. We have $\phi(\lambda_{10}\lambda_9\lambda_{15}^3) = e_{10}\lambda_9\lambda_{15}^3$ and $\phi(\lambda_8\lambda_{11}\lambda_{15}^3) = e_8\lambda_{11}\lambda_{15}^3 + e_2\lambda_{17}\lambda_{15}^2$. Thus

$$\phi\big(\lambda_2D_3^*(0)\big) \sim \phi(\lambda_{10}\lambda_9\lambda_{15}^3 + \lambda_8\lambda_{11}\lambda_{15}^3) \equiv e_{10}\lambda_9\lambda_{15}^3 + e_8\lambda_{11}\lambda_{15}^3 \bmod F(2).$$

By direct calculations we find

$$\begin{aligned}&\delta(e_{10}\lambda_1\lambda_{23}\lambda_{31} + e_8\lambda_3\lambda_{23}\lambda_{31} + e_6\lambda_5\lambda_{23}\lambda_{31} + e_4\lambda_{23}\lambda_7\lambda_{31})\\ &\quad\equiv e_{10}\lambda_9\lambda_{15}^3 + e_8\lambda_{11}\lambda_{15}^3 + e_3\big(D_3^*(0) = \lambda_0\lambda_{23}\lambda_7\lambda_{31}\big) \bmod F(2).\end{aligned}$$

Thus $h_2D_3(0) = \{\overline{\lambda_{10}\lambda_9(h_4^*)^3}\} \to e_3D_3(0) = \widetilde{\phi}(h_2D_3(0))$. This proves (3.14.48). □

Proof of (3.14.49). $D_3(0)h_3 = h_3D_3(0)$ is represented by the cycle $\lambda_7D_3^*(0) = \lambda_7\lambda_0\lambda_{23}\lambda_7\lambda_{31}$ which is inadmissible. It is easy to see

$$\delta(\lambda_8\lambda_{23}\lambda_7\lambda_{31} + \lambda_0\lambda_{15}\lambda_{23}\lambda_{31}) = \lambda_7D_3^*(0) + \lambda_8\lambda_{15}^4.$$

So $h_3D_3(0) = \{\lambda_8\lambda_{15}^4\}$ with $\lambda_8\lambda_{15}^4$ admissible. We have

$$\phi(\lambda_8\lambda_{15}^4) = e_8\lambda_{15}^4 + e_6\lambda_{17}\lambda_{15}^3 + e_4\lambda_{19}\lambda_{15}^3 \equiv e_8\lambda_{15}^4 \bmod F(6)$$

and

$$\delta(e_8\lambda_{23}\lambda_7\lambda_{31}) \equiv e_8\lambda_{15}^4 + e_7\big(\lambda_0\lambda_{23}\lambda_7\lambda_{31} = D_3^*(0)\big) \bmod F(6).$$

So $h_3D_3(0) = \{\overline{\lambda_8(h_4^*)^4}\} \to e_7D_3(0) = \widetilde{\phi}(h_3D_3(0))$. This proves (3.14.49). □

Proof of (3.14.50). $D_3(0)h_4$ is represented by the cycle $D_3^*(0)\lambda_{15} = \lambda_0\lambda_{23}\lambda_7\lambda_{31}\lambda_{15}$. We have

$$\delta(\lambda_0\lambda_{23}\lambda_7\lambda_{47}) = D_3^*(0)\lambda_{15} + \lambda_{14}\lambda_{17}\lambda_{15}^3 + \lambda_{12}\lambda_{19}\lambda_{15}^3 + \lambda_8\lambda_{15}^2\lambda_{23}\lambda_{15}$$

and

$$\begin{aligned}
&\phi(\lambda_{14}\lambda_{17}\lambda_{15}^3+\lambda_{12}\lambda_{19}\lambda_{15}^3+\lambda_8\lambda_{15}^2\lambda_{23}\lambda_{15})\\
&\quad = e_{14}\lambda_{17}\lambda_{15}^3+e_2\lambda_{29}\lambda_{15}^3+e_{12}\lambda_{19}\lambda_{15}^3+e_6\lambda_{25}\lambda_{15}^3+e_4\lambda_{27}\lambda_{15}^3\\
&\qquad + e_2\lambda_{29}\lambda_{15}^3+e_8\lambda_{15}^2\lambda_{23}\lambda_{15}+e_6\lambda_{17}\lambda_{15}\lambda_{23}\lambda_{15}+e_4\lambda_{19}\lambda_{15}\lambda_{23}\lambda_{15}\\
&\quad \equiv e_{14}\lambda_{17}\lambda_{15}^3+e_{12}\lambda_{19}\lambda_{15}^3+e_8\lambda_{15}^2\lambda_{23}\lambda_{15} \bmod F(6).
\end{aligned}$$

So $\phi(D_2^*(0)\lambda_{15})\sim e_{14}\lambda_{17}\lambda_{15}^3+e_{12}\lambda_{19}\lambda_{15}^3+e_8\lambda_{15}^2\lambda_{23}\lambda_{15} \bmod F(6)$. By direct calculations we find that

$$\begin{aligned}
&\delta\begin{bmatrix} e_{14}(\lambda_{33}\lambda_{15}^2+\lambda_{31}\lambda_{17}\lambda_{15}+\lambda_{47}\lambda_1\lambda_{15})+e_{13}\lambda_0\lambda_{49}\lambda_{15}\\ \quad + e_{12}(\lambda_3\lambda_{31}^2+\lambda_{35}\lambda_{15}^2)+e_{11}\lambda_2\lambda_{49}\lambda_{15}+e_8\lambda_{15}\lambda_{23}\lambda_{31}+e_7\lambda_6\lambda_{49}\lambda_{15}\end{bmatrix}\\
&\quad \equiv e_{14}\lambda_{17}\lambda_{15}^3+e_{12}\lambda_{19}\lambda_{15}^3+e_8\lambda_{15}^2\lambda_{23}\lambda_{15}+e_7\big[(p_0')^*=\overline{p}_0'(1)\big] \bmod F(6).
\end{aligned}$$

Thus $D_3(0)h_4=\{\overline{\lambda_{14}\lambda_{17}(h_4^*)^3}\}\to e_7p_0'=\widetilde{\phi}(D_3(0)h_4)$. This proves (3.14.50). $\square$

Proof of (3.14.51). The class p_0' is represented by the cycle

$$\begin{aligned}
(p_0')^*=\overline{p}_0'(1) &= \lambda_{38}\lambda_{13}\lambda_{11}\lambda_7+\lambda_{30}\lambda_9\lambda_{15}^2+\lambda_{28}\lambda_{11}\lambda_{15}^2+\lambda_{22}\lambda_{17}\lambda_{15}^2\\
&\quad +\lambda_{20}\lambda_{19}\lambda_{15}^2+\lambda_{14}\lambda_{21}\lambda_{27}\lambda_7+\lambda_{14}\lambda_{21}\lambda_{19}\lambda_{15}+\lambda_{14}\lambda_{17}\lambda_{23}\lambda_{15}\\
&\quad +\lambda_{14}\lambda_{13}\lambda_{19}\lambda_{23}+\lambda_{12}\lambda_{19}\lambda_{23}\lambda_{15}.
\end{aligned}$$

Each monomial in this sum is admissible. Let R be the sum of these monomials except $\lambda_{38}\lambda_{13}\lambda_{11}\lambda_7$ so that

(1) $(p_0')^*=\lambda_{38}\lambda_{13}\lambda_{11}\lambda_7+R$.

The class $p_0'h_4=h_4p_0'$ is represented by the cycle

(2) $\lambda_{15}(p_0')^*=\lambda_{15}\lambda_{38}\lambda_{13}\lambda_{11}\lambda_7+\lambda_{15}R$, where

$$\begin{aligned}
\lambda_{15}R &= \lambda_{15}\lambda_{30}\lambda_9\lambda_{15}^2+\lambda_{15}\lambda_{28}\lambda_{11}\lambda_{15}^2+\lambda_{15}\lambda_{22}\lambda_{17}\lambda_{15}^2\\
&\quad +\lambda_{15}\lambda_{20}\lambda_{19}\lambda_{15}^2+\lambda_{15}\lambda_{14}\lambda_{21}\lambda_{27}\lambda_7+\lambda_{15}\lambda_{14}\lambda_{21}\lambda_{19}\lambda_{15}\\
&\quad +\lambda_{15}\lambda_{14}\lambda_{17}\lambda_{23}\lambda_{15}+\lambda_{15}\lambda_{14}\lambda_{13}\lambda_{19}\lambda_{23}+\lambda_{15}\lambda_{12}\lambda_{19}\lambda_{23}\lambda_{15}.
\end{aligned}$$

Each monomial in the sum $\lambda_{15}R$ is admissible and begins with λ_{15}. So

(3) $\phi(\lambda_{15}R)\equiv e_{15}R \bmod F(14)$.

We have the admissible expansion

$$\lambda_{15}\lambda_{38}\lambda_{13}\lambda_{11}\lambda_7=\lambda_{22}\lambda_{31}\lambda_{13}\lambda_{11}\lambda_7+\lambda_{21}\lambda_{32}\lambda_{13}\lambda_{11}\lambda_7+\lambda_{19}\lambda_{34}\lambda_{13}\lambda_{11}\lambda_7$$

and we find that

(4) $$\begin{aligned}\phi(\lambda_{15}\lambda_{38}\lambda_{13}\lambda_{11}\lambda_7) &\equiv e_{22}\lambda_{31}\lambda_{13}\lambda_{11}\lambda_7+e_{21}\lambda_{32}\lambda_{13}\lambda_{11}\lambda_7+e_{19}\lambda_{34}\lambda_{13}\lambda_{11}\lambda_7\\ &\equiv e_{22}\lambda_{31}\lambda_1\lambda_{15}^2+e_{21}\lambda_{32}\lambda_1\lambda_{15}^2+e_{19}\lambda_{34}\lambda_1\lambda_{15}^2 \bmod F(14).\end{aligned}$$

Straightforward calculations show that

(5) $$\begin{aligned}&\delta\big[e_{22}(\lambda_{33}\lambda_{15}^2+\lambda_1\lambda_{47}\lambda_{15})+e_{20}\lambda_3\lambda_{31}^2\big]\\ &\quad\equiv e_{22}\lambda_{31}\lambda_1\lambda_{15}^2+e_{21}\lambda_{32}\lambda_1\lambda_{15}^2+e_{19}\lambda_{34}\lambda_1\lambda_{15}^2+e_{15}\lambda_{38}\lambda_{13}\lambda_{11}\lambda_7 \bmod F(14).\end{aligned}$$

From (1), (2), (3), (4) and (5) we deduce that

$$\phi\big[\lambda_{15}(p_0')^*\big]\sim e_{15}\big[(p_0')^*=\lambda_{38}\lambda_{13}\lambda_{11}\lambda_7+R\big] \bmod F(14).$$

Thus

$$h_4 p_0' = \{\overline{\lambda_{22}\lambda_{31}\lambda_{13}\lambda_{11}\lambda_7 = \lambda_{22}\lambda_{31}\lambda_1\lambda_{15}^2}\}$$
$$= \{\overline{\lambda_{22}\lambda_{31}\lambda_{15}^2\lambda_1}\} = \{\overline{\lambda_{22}h_5^*(h_4^*)^2h_1^*}\} \to e_{15}p_0' = \widetilde{\phi}(h_4 p_0').$$

Here $\lambda_{22}\lambda_{31}\lambda_1\lambda_{15}^2 \sim \lambda_{22}\lambda_{31}\lambda_{15}^2\lambda_1 \bmod \Lambda(21)$ is easy to see. This proves (3.14.51). $\square$

This completes the proof of Theorem 1.4.

The final work of this section is to prove Proposition 3.12. We are going to prove a more general result that covers Proposition 3.12. Let K be a locally finite graded left module over the Steenrod algebra A. Given a class $\alpha \in Ext_A^{s,*}(K)$ and suppose $\sum_q k_q \otimes \lambda(q)$ is a cycle in $K_* \otimes \Lambda^{s,*}$ representing α. We refer to the beginning of Section 2 for the $\mathbb{Z}/2$-dual K_* of K, which is a right A-module, and also for the differential right Λ-module $K_* \otimes \Lambda$. In particular we recall that αh_i is represented by the cycle $\sum_q k_q \otimes \lambda(q)\lambda_{2^i-1}$ for any $i \geqslant 0$. The following result says that αh_i can also be represented by some "exotic" cycle.

Proposition 3.12*. *The chain $\sum_q (\sum_{\nu \geqslant 0} k_q Sq^\nu \otimes \lambda_{2^i+\nu-1}\lambda(q))$ in $K_* \otimes \Lambda^{s+1,*}$ is also a cycle and represents αh_i.*

It is clear that this result implies Proposition 3.12. The remainder of this section is devoted to proving (3.12)*.

Let L be another locally finite left module over A. Consider $N_* = K_* \otimes L_*$ with diagonal A-action. From the differential formula (1) in the beginning of Section 2 we see the differential δ on $N_* \otimes \Lambda$ is given by

$$(*)\quad \delta\big[(k \otimes l)\lambda_I\big] = (k \otimes l)\delta(\lambda_I) + \sum_{j \geqslant 0}(k \otimes l)Sq^{j+1}\lambda_j\lambda_I$$
$$= (k \otimes l)\delta(\lambda_I) + \sum_{j \geqslant 0}\left(\sum_{\nu \geqslant 0} kSq^\nu \otimes lSq^{j+1-\nu}\right)\lambda_j\lambda_I.$$

Now consider the A-module $L = (\mathbb{Z}/2 = L^0) \oplus (\Sigma^{2^i}\mathbb{Z}/2 = L^{2^i})$ with $\mathbb{Z}/2$-generators $\bar{x}$, $\bar{y}$ for L^0, L^{2^i}, respectively, such that $Sq^{2^i}\bar{x} = \bar{y}$. Then $L_* = (\mathbb{Z}/2 = L_0) \oplus (\Sigma^{2^i}\mathbb{Z}/2 = L_{2^i})$ with $\mathbb{Z}/2$-generators x and y, respectively, such that $ySq^{2^i} = x$. From this point on, L will be this A-module.

We have a short exact sequence of A-modules

$$0 \to K_* = K_* \otimes \mathbb{Z}/2 \xrightarrow{j} K_* \otimes L_* \xrightarrow{p} K_* \otimes \Sigma^{2^i}\mathbb{Z}/2 = \Sigma^{2^i}K \to 0$$

which gives rise to a short exact sequence of differential Λ-modules

$$0 \to K_* \otimes \Lambda \xrightarrow{j'} (K_* \otimes L_* = N_*) \otimes \Lambda \xrightarrow{p'} \Sigma^{2^i}K_* \otimes \Lambda \to 0$$

resulting in a long exact sequence of *Ext* groups

$$(**)\quad \cdots \to Ext_A^{s,t}(K) \xrightarrow{j_*} Ext_A^{s,t}(K \otimes L) \xrightarrow{p_*} Ext_A^{s,t}(\Sigma^{2^i}K) = Ext_A^{s,t-2^i}(K)$$
$$\xrightarrow{\delta_*} Ext_A^{s+1,t}(K) \to \cdots.$$

It is well known that δ_* is given by $\delta_*(\beta) = \beta h_i$ for any β in $Ext_A^{s,t-2^i}(K)$ (see [2]). The differential δ in $(*)$, when applied to $L_* = \mathbb{Z}/2(x) \oplus \Sigma^{2^i}\mathbb{Z}/2(y)$, becomes

$$(*)'\ \text{(i)}\ \delta\big[(k \otimes x)\lambda_I\big] = (k \otimes x)\delta(\lambda_I) + \sum_{j \geqslant 0}(kSq^{j+1} \otimes x)\lambda_j\lambda_I,$$

$$\text{(ii)}\ \delta\big[(k \otimes y)\lambda_I\big] = (k \otimes y)\delta(\lambda_I) + \sum_{j \geqslant 0}(kSq^{j+1} \otimes y)\lambda_j\lambda_I$$
$$+ \sum_{\nu \geqslant 0}(kSq^\nu \otimes x)\lambda_{2^i+\nu-1}\lambda_I$$

since $xSq^0 = x$, $xSq^l = 0$ for $l > 0$, $ySq^0 = y$, $ySq^{2^i} = x$ and $ySq^l = 0$ for $l \neq 0, 2^i$. From $(*)'$(i), (ii) one easily sees that the map $\Sigma^{2^i} K_* \otimes \Lambda \to \Sigma K_* \otimes \Lambda$, given by $k\lambda_I \to \sum_{\nu \geqslant 0} kSq^\nu \lambda_{2^i+\nu-1}\lambda_I$ is a chain map and induces the boundary homomorphism δ_* in $(**)$. This implies the conclusion in $(3.12)^*$. This complete the proof of Proposition 3.12.

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