# BOX PRODUCT OF MACKEY FUNCTORS IN TERMS OF MODULES 

ZHULIN LI


#### Abstract

The box product of Mackey functors has been studied extensively in Lewis's notes. As shown in Thevenaz and Webb's paper, a Mackey functor may be identified with a module over a certain algebra, called the Mackey algebra. We aim at describing the box product, in the sense of Mackey algebra modules. For a cyclic $p$-group $G$, we recover a result from Mazur's thesis. We generalize it to a general finite group $G$ in this article.


## Introduction

A Mackey functor is an algebraic structure, related to many natural constructions from finite groups, such as group cohomology and the algebraic K-theory of group rings. The study of Mackey functor in abstract began in 1980s. Dress 2] and Green 1 first gave the axiomatic formulation of Mackey functors. Several equivalent descriptions of Mackey functors were given by Dress [2], Lindner [3, Lewis 4] and Thevenaz [5]. Specifically, Lewis [4] introduces box product and Thevenaz [5] describes a Mackey functor as a module over the Mackey algebra $\mu_{R}(G)$.

In this article, we give an inductive description of the box product of two left $\mu_{R}(G)$-modules. The main goal is to construct this box product explicitly and to prove that it is equivalent with the box product of two Mackey functors. When $G$ is a cyclic $p$-group, we recover the formula of Mazur [7.

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## 1. Mackey functors

Throughout this article, we assume that $G$ is a finite group and that $R$ is a unital, commutative ring. There are several equivalent definitions of Mackey functors and we concentrate on two of them here. Before giving the definitions, we introduce an auxiliary category $\Omega_{R}(G)$ and the Mackey algebra $\mu_{R}(G)$.
1.1. The category $\Omega_{R}(G)$. We recall the definition of the category $\Omega_{R}(G)$ from [4]. A finite group $G$ gives rise to a category $\omega(G)$ whose objects are finite $G$-sets and where the morphisms from $X$ to $Y$ are the equivalence classes of diagrams of $G$-sets $X \leftarrow V \rightarrow Y$. Two such diagrams are said to be equivalent if there is a commutative diagram

where $\sigma$ is an isomorphism of $G$-sets. To define the composition of morphisms, we consider a morphism from $X$ to $Y$ represented by a diagram $X \leftarrow V \rightarrow Y$ and a morphism from $Y$ to $Z$ represented by a diagram $Y \leftarrow W \rightarrow Z$. We form the pullback

which defines a diagram $X \leftarrow U \rightarrow Z$, hence a morphism from $X$ to $Z$. Such a pullback always exists and we can express it explicitly as

$$
U=\{(v, w) \in V \times W: f(v)=g(w)\}
$$

where $f^{\prime}(v, w)=v$ and $g^{\prime}(v, w)=w$. By defining addition in $\operatorname{Hom}_{\omega(G)}(X, Y)$ as

$$
(X \stackrel{\alpha}{\leftarrow} V \xrightarrow{\beta} Y)+\left(X \stackrel{\alpha^{\prime}}{\leftarrow} V^{\prime} \xrightarrow{\beta^{\prime}} Y\right):=\left(X \stackrel{\alpha+\alpha^{\prime}}{\longleftrightarrow} V \sqcup V^{\prime} \xrightarrow{\beta+\beta^{\prime}} Y\right)
$$

$\operatorname{Hom}_{\omega(G)(X, Y)}$ becomes a free abelian monoid, as shown in 5. Extending the scalars to the ring $R$, we get a free $R$-module

$$
\operatorname{Hom}_{\Omega_{R}(G)}(X, Y):=R \operatorname{Hom}_{\omega(G)}(X, Y)
$$

on the same basis as that of free monoid $\operatorname{Hom}_{\omega(G)}(X, Y)$. Let $\Omega_{R}(G)$ be a category with the same objects as $\omega(G)$ and hom-set $\operatorname{Hom}_{\Omega_{R}(G)}(X, Y)$ defined as above. $\Omega_{R}(G)$ is an $R$-additive category.
1.2. The Mackey algebra $\mu_{R}(G)$. We define the Mackey algebra as

$$
\mu_{R}(G):=\bigoplus_{H, K \leq G} \operatorname{Hom}_{\Omega_{R}(G)}(G / H, G / K)
$$

where the multiplication is defined on the components in the direct sum by composition of morphisms in the category $\Omega_{R}(G)$, or zero if two morphisms cannot be composed.

For convenience, we recall the notation from [6].
Definition 1.1. Let $f: X \rightarrow Y$ be a $G$-equivariant map of finite $G$-sets. Then two spans

are called the restriction $r_{f}$ and transfer $t_{f}$, respectively, of $f$.
Definition 1.2. Let $K \leq H \leq G$ and $g \in G$. Then define $R_{K}^{H}, I_{K}^{H}$ and $C_{g, H}$ as

$$
\begin{aligned}
& R_{K}^{H}=r_{\pi_{K}^{H}}:\left(G / K=G / K \stackrel{\pi_{K}^{H}}{\longleftrightarrow} G / H\right) \\
& I_{K}^{H}=t_{\pi_{K}^{H}}:\left(G / H \stackrel{\pi_{K}^{H}}{\longleftarrow} G / K=G / K\right) \\
& C_{g, H}=t_{c_{g, H}}=r_{c_{g-1, g_{H}}}:\left(G /{ }^{g} H \stackrel{c_{g, H}}{\leftrightarrows} G / H=G / H\right),
\end{aligned}
$$

where $\pi_{K}^{H}: G / K \rightarrow G / H$ denotes the canonical quotient map, mapping $g K$ to $g H$, and $c_{g, H}: G / H \rightarrow G /{ }^{g} H$ denotes the conjugation map, mapping $k H$ to $\left(k g^{-1}\right)^{g} H$. Observe that $R_{K}^{H}, I_{K}^{H}$ and $C_{g, H}$ are all elements in Mackey algebra $\mu_{R}(G)$.

For conjuation, the notation $C_{g}$ is preferred to $C_{g, H}$ for simplicity when there is no ambiguity. It is easy to check that the following identities hold:
(0) $R_{H}^{H}=I_{H}^{H}=C_{h, H}$ for all $H \leq G$ and $h \in H$
(1) $R_{J}^{K} R_{K}^{H}=R_{J}^{H}$ for all subgroups $J \leq K \leq H$
(2) $I_{K}^{H} I_{J}^{K}=I_{J}^{H}$ for all subgroups $J \leq K \leq H$
(3) $C_{g} C_{h}=C_{g h}$ for all $g, h \in G$
(4) $C_{g} R_{K}^{H}=R_{g}^{g}{ }_{K} C_{g}$ for all subgroups $K \leq H$ and $g \in G$
(5) $C_{g} I_{K}^{H}=I_{g_{K}^{g}}^{H_{g}} C_{g}$ for all subgroups $K \leq H$ and $g \in G$
(6) $R_{J}^{H} I_{K}^{H}=\sum_{g \in[J \backslash H / K]} I_{g}^{J}{ }_{K \cap J} C_{g} R_{K \cap J g}^{K}$ for all subgroups $J, K \leq H$
(7) $\sum_{H \leq G} I_{H}^{H}$ serves as the unit in $\mu_{R}(G)$.

We use $[J \backslash H / K]$ to denote the set of representatives of the double coset $J \backslash H / K$. In fact, the structure of Mackey algebra $\mu_{R}(G)$ is rather simple:
Lemma 1.1 (from [5]). Hom-set $\operatorname{Hom}_{\Omega_{R}(G)}(G / K, G / H)$ is a free $R$-module, with basis represented by the diagrams

$$
I_{g_{L}}^{K} C_{g, L} R_{L}^{H}=\left(G / K \stackrel{\pi_{g_{L}}^{K} c_{g, L}}{\longleftrightarrow} G / L \stackrel{\pi_{L}^{H}}{\longrightarrow} G / H\right)
$$

where $g \in[K \backslash G / H]$ and $L$ is a subgroup of $H \cap K^{g}$ taken up to $H \cap K^{g}$-conjugation.
In other words, the Mackey algebra $\mu_{R}(G)$ is generated by $R_{K}^{H}, I_{K}^{H}$ and $C_{g, H}$ 's as an $R$-algebra.

### 1.3. Definition of Mackey functors.

Definition 1.3 (from [2]). A Mackey functor is a $R$-additive functor $M: \Omega_{R}(G)^{\text {op }} \rightarrow$ $R$-mod. They form a category with natural transformations as morphisms and we denote this category by $\operatorname{Mack}_{R}(G)$.

It is shown in 5 that the category $\mu_{R}(G)$-mod of left $\mu_{R}(G)$-modules is equivalent to $\operatorname{Mack}_{R}(G)$ via the following equivalence of categories

$$
\begin{aligned}
\Phi: \operatorname{Mack}_{R}(G) & \longleftrightarrow \mu_{R}(G)-\bmod \\
M & \longmapsto \oplus_{H \leq G} M(G / H) \\
\left(G / H \mapsto I_{H}^{H} N\right) & \longleftrightarrow N .
\end{aligned}
$$

Since $\sum_{H \leq G} I_{H}^{H}$ is the unit in $\mu_{R}(G)$, a left $\mu_{R}(G)$-module $N$ can be graded into $N=\oplus_{H \leq G} I_{H}^{H} N$. Observe that the multiplication by $R_{K}^{H}$ maps $I_{H}^{H} N$ to $I_{K}^{K} N$ and the other grades to zero. Similarly, multiplication by $I_{K}^{H} \operatorname{maps} I_{K}^{K} N$ to $I_{H}^{H} N$ and the other grades to zero. Multiplication by $C_{g, H}$ maps $I_{H}^{H} N$ to $I_{g}^{g}{ }_{H}^{H} N$ and the other grades to zero. Also note that $I_{H}^{H} \Phi M=M(G / H)$ for later use.

## 2. The Box product in $\operatorname{Mack}_{R}(G)$

The box product is a symmetric monoidal structure on $\operatorname{Mack}_{R}(G)$. The box product in $\operatorname{Mack}_{R}(G)$ has been studied in [4] and we summarize it in this section. The result we are more interested in is that maps from the box product of $M$ and $N$ to $P$ can be characterized by Dress parings.

Given two Mackey functors $M, N \in \operatorname{Mack}_{R}(G)$, we can form the exterior product

$$
\begin{aligned}
M \bar{\square} N: \Omega_{R}(G)^{\mathrm{op}} \times \Omega_{R}(G)^{\mathrm{op}} & \longrightarrow R-\bmod \\
(X, Y) & \longmapsto M(X) \otimes N(Y) .
\end{aligned}
$$

Definition 2.1 (Box product in $\operatorname{Mack}_{R}(G)$ ). The box product $M \square N$ is defined to be the left Kan extension of $M \bar{\square} N$ along the Cartesian product functor $\times$ : $\Omega_{R}(G)^{\mathrm{op}} \times \Omega_{R}(G)^{\mathrm{op}} \longrightarrow \Omega_{R}(G)^{\mathrm{op}}$.


If a Mackey functor $M \in \operatorname{Mack}_{R}(G)$ is implicit, we use $r_{f}$ for both a morphism $(X=X \xrightarrow{f} Y)$ in $\Omega_{R}(G)$ and its value $M(Y) \rightarrow M(X)$ under $M$. Similarly for $t_{f}$.
Lemma 2.1 (from [4). A map $\theta: M \square N \rightarrow P$ determines and is determined by $a$ collection of $R$-module homomorphisms

$$
\theta_{X}: M(X) \otimes N(X) \rightarrow P(X)
$$

for every finite $G$-set $X$, such that the following three diagrams commute for each $G$-equivariant map $f: X \rightarrow Y$.


A good exposition and proof of this lemma can be found in [6]. The data in this lemma is called a Dress paring. The natural transformations from $M \square N$ to another Mackey functor $P$ are the same as the Dress parings from $M$ and $N$ to $P$, via the natural bijection

$$
\operatorname{Hom}_{\operatorname{Mack}_{R}(G)}(M \square N, P) \cong \operatorname{Dress}(M, N ; P)
$$

that maps a map $\theta: M \square N \rightarrow P$ to $\theta_{X}: M(X) \otimes N(X) \rightarrow P(X \times X) \xrightarrow{r \Delta} P(X)$. The first map comes from the Kan adjunction and $r \Delta$ is the restriction associated to the diagonal map of $G$-sets $X \rightarrow X \times X$.

In $\operatorname{Mack}_{R}(G)$, the Burnside ring Mackey functor

$$
B^{G}(-):=\operatorname{Hom}_{\Omega_{R}(G)}(-, G / G)
$$

is the unit for box product, as shown in [4]. In this way, $\left(\operatorname{Mack}_{R}(G), B^{G}(-), \square\right)$ is a symmetric monoidal category.

## 3. The Box Product in $\mu_{R}(G)$-mod

Given two left $\mu_{R}(G)$-modules $M$ an $N$, we can form an $R$-module $A_{H}$ for each $H \leq G$ by induction on the cardinality of subgroups of $G$

$$
\begin{aligned}
A_{e} & :=\left(I_{e}^{e} M\right) \otimes_{R}\left(I_{e}^{e} N\right) \\
A_{H} & :=\left(\left(I_{H}^{H} M\right) \otimes_{R}\left(I_{H}^{H} N\right)\right) \oplus \bigoplus_{K<H} A_{K}
\end{aligned}
$$

We say $A_{H}$ is of grade $H$. By combining all the grades together, we get $A:=$ $\oplus_{H \leq G} A_{H}$. Note that the component $A_{K}$ in grade $H$ is distinct from the grade $K$. Since $\mu_{R}(G)$ is generated by $R, I, C$ 's, we can endow $A$ with a left $\mu_{R}(G)$-module structure by giving actions of $R, I, C^{\prime}$ 's on $A$. The maps $C_{g, H}, I_{H}^{L}, R_{J}^{H}$ map the grade $H$ to grades ${ }^{g} H, L, J$ respectively, and map the other grades to zero. Their action on the grade $H$ is described as follows:
(1) $I_{H}^{L}$ action on the grade $H$ :

If $H=L, I_{H}^{L}$ acts as the identity on grade $H$.
If $H<L, I_{H}^{L}$ maps an element in $A_{H}$ to its corresponding copy $A_{H}$ in grade $L$. To distinguish $A_{H}$ from its copy in grade $L$, we write its copy in grade $L$ as $I_{H}^{L} A_{H}$ from now on. That is, grade $H$ is written as

$$
A_{H}=\left(\left(I_{H}^{H} M\right) \otimes_{R}\left(I_{H}^{H} N\right)\right) \oplus \bigoplus_{K<H} I_{K}^{H} A_{K}
$$

(2) $C_{g, H}$ action on the grade $H$ :

We define the action of $C_{g, H}$ by induction on the cardinality of $H$ as follows. For $m \otimes n \in\left(I_{H}^{H} M\right) \otimes\left(I_{H}^{H} N\right)$ and $x \in A_{K}$, where $K<H$,

$$
\begin{aligned}
C_{g, H}(m \otimes n) & :=\left(C_{g, H} m\right) \otimes\left(C_{g, H} n\right) \in A_{g_{H}} \\
C_{g, H} I_{K}^{H}(x) & :=I_{g_{K}^{g} H} C_{g, K}(x) \in A_{g_{H}} .
\end{aligned}
$$

(3) $R_{J}^{H}$ action on the grade $H$ :

We also define the action of $R_{J}^{H}$ by induction on the cardinality of $H$ as follows. For $m \otimes n \in\left(I_{H}^{H} M\right) \otimes\left(I_{H}^{H} N\right)$ and $x \in A_{K}$, where $K<H$,

$$
\begin{aligned}
R_{J}^{H}(m \otimes n) & :=\left(R_{J}^{H} m\right) \otimes\left(R_{J}^{H} n\right) \in A_{J} \\
R_{J}^{H} I_{K}^{H}(x) & :=\sum_{g \in[J \backslash H / K]} I_{g}^{J}{ }_{K \cap J} C_{g} R_{K \cap J^{g}}^{K}(x) \in A_{J} .
\end{aligned}
$$

Definition 3.1 (Box product in $\mu_{R}(G)$-mod). Based on the left $\mu_{R}(G)$-module $\oplus_{H \leq G} A_{H}$, we can define $M \square N$ as

$$
M \square N:=\left(\oplus_{H \leq G} A_{H}\right) / F R,
$$

where $F R$ is a submodule, called the Frobenius reciprocity submodule, generated by elements of the form

$$
a \otimes\left(I_{K}^{H} b\right)-I_{K}^{H}\left(\left(R_{K}^{H} a\right) \otimes b\right)
$$

and

$$
\left(I_{K}^{H} c\right) \otimes d-I_{K}^{H}\left(c \otimes\left(R_{K}^{H} d\right)\right)
$$

for all $K<H, a \in I_{H}^{H} M, b \in I_{K}^{K} N, c \in I_{K}^{K} M$ and $d \in I_{H}^{H} N$.
Naturally, the image of $A_{H}$ under the quotient is called the grade $H$ of $M \square N$.
Proposition 3.1. $\oplus_{H \leq G} \operatorname{Hom}_{\Omega_{R}(G)}(G / H, G / G)$ is the unit for box product in $\mu_{R}(G)$-mod.

Proof. Let $M$ be a left $\mu_{R}(G)$-module and $N=\oplus_{H \leq G} \operatorname{Hom}_{\Omega_{R}(G)}(G / H, G / G)$. Then $I_{H}^{H} N$ is an $R$-module generated by $G / H \stackrel{\pi_{L}^{H}}{\leftarrow} G / L \rightarrow G / G=I_{L}^{H} R_{L}^{H} n_{H}$, where

$$
n_{H}:=(G / H=G / H \rightarrow G / G)
$$

Thus, $I_{H}^{H} M \otimes I_{H}^{H} N$ is a $R$-module generated by $m \otimes I_{L}^{H} R_{L}^{H} n_{H}=I_{L}^{H}\left(R_{L}^{H} m \otimes n_{H}\right)$. Then we get a natural bijection between each grade of $M$ and $M \square N$

$$
\begin{aligned}
F: I_{H}^{H} M & \longleftrightarrow I_{H}^{H}(M \square N) \\
m & \longmapsto m \otimes n_{H} \\
I_{L}^{H} R_{L}^{H} m & \longleftrightarrow I_{L}^{H}\left(R_{L}^{H} m \otimes n_{H}\right) \in I_{H}^{H} M \otimes I_{H}^{H} N \\
I_{K}^{H} F^{-1}(x) & \longleftrightarrow I_{K}^{H} x \in I_{K}^{H} I_{K}^{K}(M \square N) .
\end{aligned}
$$

Here $F^{-1}$ is defined by induction on the grade. It is easy to check that this is a bijection and that it preserves the $\mu_{R}(G)$-module structure. Thus, $M \square N$ is isomorphic to $M$ as a $\mu_{R}(G)$-module and $N$ is the unit for box product in $\mu_{R}(G)$-mod.

In this way, $\left(\mu_{R}(G)\right.$-mod, $\left.\oplus_{H \leq G} \operatorname{Hom}_{\Omega_{R}(G)}(G / H, G / G), \square\right)$ is a symmetric monoidal category.

## 4. Equivalence of box Product in $\operatorname{Mack}_{R}(G)$ and $\mu_{R}(G)$-mod

Theorem 4.1. The equivalence of categories $\Phi$ : $\operatorname{Mack}_{R}(G) \stackrel{\cong}{\leftrightarrows} \mu_{R}(G)-\bmod$ is a symmetric monoidal equivalence. In other words, there are a natural isomorphism $\Phi M \square \Phi N \cong \Phi(M \square N)$ for any two Mackey functors $M, N \in \operatorname{Mack}_{R}(G)$, and $a$ compatible natural isomorphism $B^{G}(-) \xrightarrow{\cong} \operatorname{Hom}_{\Omega_{R}(G)}(G / H, G / G)$.
Lemma 4.2. For any Mackey functors $M, N, P \in \operatorname{Mack}_{R}(G)$, there is a natural bijection

$$
\operatorname{Dress}(M, N ; P) \cong \operatorname{Hom}_{\mu_{R}(G)-\bmod }(\Phi M \square \Phi N, \Phi P)
$$

Proof. Given $\beta \in \operatorname{Hom}_{\mu_{R}(G)-\bmod }(\Phi M \square \Phi N, \Phi P)$, we map it to $\theta \in \operatorname{Dress}(M, N ; P)$ defined as follows. For each $H \leq G, \beta$ maps grade $I_{H}^{H}(\Phi M \square \Phi N)$ to grade $I_{H}^{H} \Phi P$, because $\beta(x)=\beta\left(I_{H}^{H} x\right)=I_{H}^{H} \beta(x) \in I_{H}^{H} \Phi P$ for each $x \in I_{H}^{H}(\Phi M \square \Phi N)$. Since

$$
I_{H}^{H}(\Phi M \square \Phi N)=(M(G / H) \otimes N(G / H)) \oplus \bigoplus_{K<H} I_{K}^{H} A_{K} / F R
$$

and

$$
I_{H}^{H} \Phi P=P(G / H)
$$

$\beta$ induces an $R$-module homomorphism $\theta_{G / H}$

$$
\theta_{G / H}: M(G / H) \otimes N(G / H) \rightarrow P(G / H)
$$

by restricting to the first summand. For a general finite $G$-set $X=\sqcup_{i=1}^{p} G / H_{i}, \theta_{X}$ is defined as

$$
\begin{aligned}
\theta_{X}: \bigoplus_{i, j=1}^{p} M\left(G / H_{i}\right) \otimes N\left(G / H_{j}\right) & \longrightarrow \bigoplus_{i=1}^{p} P\left(G / H_{i}\right) \\
m_{i} \otimes n_{j} & \longmapsto\left\{\begin{array}{r}
\theta_{G / H_{i}}\left(m_{i} \otimes n_{i}\right), \text { if } i=j \\
0, \text { otherwise. }
\end{array}\right.
\end{aligned}
$$

Having constructed $\theta$, we now proceed to show that $\theta \in \operatorname{Dress}(M, N ; P)$. Given finite $G$-sets $X, Y$ and a $G$-equivariant map $f: X \rightarrow Y$, it is sufficient to show that the three diagrams in Lemma 2.1 commute.

If both $X$ and $Y$ are orbits, say $X=G / K$ and $Y=G / H$, observe that a $G$ equivariant map $f$ from $G / K$ to $G / H$ must be of the the form $f=\pi_{g}^{H} c_{g, K}$ for some $g \in G$. By composition of commuting diagrams, we only need to consider the case where $f=c$ and $f=\pi$.

When $f=c_{g, H}: G / H \rightarrow G /{ }^{g} H$, we have that $r_{f}=C_{g^{-1}, g_{H}}$ and $t_{f}=C_{g, H}$. The first diagram commutes because

$$
\begin{aligned}
r_{f} \theta_{Y}(m \otimes n) & =C_{g^{-1, g_{H}}} \beta(m \otimes n)=\beta\left(C_{g^{-1}, g_{H}}(m \otimes n)\right) \\
& \left.=\beta\left(C_{g^{-1}, g_{H}} m \otimes C_{g^{-1}, g_{H}} n\right)\right)=\theta_{X}\left(r_{f} m \otimes r_{f} n\right) .
\end{aligned}
$$

The second diagram commutes because

$$
\begin{aligned}
t_{f} \theta_{X}\left(m \otimes r_{f} n\right) & =C_{g, H} \beta\left(m \otimes C_{g^{-1}, g_{H}} n\right)=\beta\left(C_{g, H}\left(m \otimes C_{g^{-1,}, g_{H}} n\right)\right) \\
& =\beta\left(C_{g, H} m \otimes n\right)=\theta_{Y}\left(t_{f} m \otimes n\right)
\end{aligned}
$$

The third diagram commutes similarly.
When $f=\pi_{K}^{H}: G / K \rightarrow G / H$, we have that $r_{f}=R_{K}^{H}$ and $t_{f}=I_{K}^{H}$. The first diagram commutes because

$$
\begin{aligned}
r_{f} \theta_{Y}(m \otimes n) & =R_{K}^{H} \beta(m \otimes n)=\beta\left(R_{K}^{H}(m \otimes n)\right) \\
& \left.=\beta\left(R_{K}^{H} m \otimes R_{K}^{H} n\right)\right)=\theta_{X}\left(r_{f} m \otimes r_{f} n\right)
\end{aligned}
$$

The second diagram commutes because

$$
\begin{aligned}
t_{f} \theta_{X}\left(m \otimes r_{f} n\right) & =I_{K}^{H} \beta\left(m \otimes R_{K}^{H} n\right)=\beta\left(I_{K}^{H}\left(m \otimes R_{K}^{H} n\right)\right) \\
& =\beta\left(I_{K}^{H} m \otimes n\right)=\theta_{Y}\left(t_{f} m \otimes n\right) .
\end{aligned}
$$

The third diagram commutes similarly.
Now, if $Y$ is an orbit $G / H$ and $X=\sqcup_{i=1}^{p} G / K_{i}$ is a general $G$-set, denote the restriction of $f$ to $G / K_{i}$ by $f_{i}: G / K_{i} \rightarrow G / H$. Thus, $r_{f}: M(G / H) \rightarrow \oplus_{i=1}^{p} M\left(G / K_{i}\right)$ is the sum of $r_{f_{i}}: M(G / H) \rightarrow M\left(G / K_{i}\right)$. Similarly, $t_{f}: \oplus_{i=1}^{p} M\left(G / K_{i}\right) \rightarrow$ $M(G / H)$ is determined by components $t_{f_{i}}: M\left(G / K_{i}\right) \rightarrow M(G / H)$. The first diagram commutes because

$$
\begin{aligned}
r_{f} \theta_{Y}(m \otimes n) & =\sum_{i=1}^{p} r_{f_{i}} \theta_{Y}(m \otimes n)=\sum_{i=1}^{p} \theta_{X}\left(r_{f_{i}} m \otimes r_{f_{i}} n\right) \\
& =\theta_{X}\left(\sum_{i=1}^{p} r_{f_{i}} m \otimes r_{f_{i}} n\right)=\theta_{X}\left(\sum_{i=1}^{p} r_{f_{i}} m \otimes \sum_{i=1}^{p} r_{f_{i}} n\right) \\
& =\theta_{X}\left(r_{f} m \otimes r_{f} n\right) .
\end{aligned}
$$

The second diagram commutes because

$$
\begin{aligned}
t_{f} \theta_{\mathbf{b}}\left(m \otimes r_{f} n\right) & =t_{f} \theta_{X}\left(\sum_{i=1}^{p} m_{i} \otimes \sum_{i=1}^{p} r_{f_{i}} n\right)=t_{f} \theta_{X}\left(\sum_{i=1}^{p} m_{i} \otimes r_{f_{i}} n\right) \\
& =\sum_{i=1}^{p} t_{f} \theta_{X}\left(m_{i} \otimes r_{f_{i}} n\right)=\sum_{i=1}^{p} t_{f_{i}} \theta_{G / K_{i}}\left(m_{i} \otimes r_{f_{i}} n\right) \\
& =\sum_{i=1}^{p} \theta_{Y}\left(t_{f_{i}} m_{i} \otimes n\right)=\theta_{Y}\left(\sum_{i=1}^{p} t_{f_{i}} m_{i} \otimes n\right) \\
& =\theta_{Y}\left(t_{f} m \otimes n\right)
\end{aligned}
$$

for $m=\sum_{i=1}^{p} m_{i}$, where $m_{i} \in M\left(G / K_{i}\right)$. The third diagram commutes similarly.
Lastly, the general case: $X=\sqcup_{i=1}^{p} X_{i}, Y=\sqcup_{i=1}^{p} G / H_{i}$, and $f$ maps $X_{i}$ to $G / H_{i}$. Thus, $r_{f}$ maps $M\left(G / H_{i}\right)$ to $M\left(X_{i}\right)$ and $t_{f}$ maps $M\left(X_{i}\right)$ to $G / H_{i}$. For the first diagram, take $m \in G / H_{i}$ and $n \in G / H_{j}$. Thus, $r_{f} m \in M\left(X_{i}\right)$ and $r_{f} n \in M\left(X_{j}\right)$. If $i \neq j$, then $\theta_{Y}(m \otimes n)=0$ and $\theta_{X}\left(r_{f} m \otimes r_{f} n\right)=0$. If $i=j$, then this is the case when $Y$ is an orbit. For the second diagram, take $m \in M\left(X_{i}\right)$ and $n \in G / H_{j}$. Thus, $t_{f} m \in M\left(G / H_{i}\right)$ and $r_{f} n \in M\left(X_{j}\right)$. If $i \neq j$, then $\theta_{X}\left(m \otimes r_{f} n\right)=0$ and $\theta_{Y}\left(t_{f} m \otimes n\right)=0$. If $i=j$, it reduces to the case when $Y$ is an orbit. The third diagram commutes similarly.

Given a Dress pairing $\theta \in \operatorname{Dress}(M, N ; P)$, we define $\beta$, a map from $\Phi M \square \Phi N$ to $P$ grade by grade, as follows:

$$
\begin{aligned}
& \beta_{H}: I_{H}^{H}(\Phi M \square \Phi N) \longrightarrow I_{H}^{H} \Phi P=P(G / H) \\
& M(G / H) \otimes N(G / H) \ni m \otimes n \longmapsto \theta_{G / H}(m \otimes n) \\
& I_{K}^{H} A_{K} \ni I_{K}^{H}(x) \longmapsto I_{K}^{H}\left(\beta_{K}(x)\right)
\end{aligned}
$$

Having constructed $\beta$, we now proceed to show that $\beta$ is indeed a map of $\mu_{R}(G)$ modules. $\beta$ is linear in multiplication by elements in $\mu_{R}(G)$ : It is enough to check this for the generators $R, I, C$ 's. Say $K<H \leq G$ and take $m \otimes n \in M(G / H) \otimes$ $N(G / H)$. Then

$$
\begin{aligned}
R_{K}^{H} \beta(m \otimes n) & =R_{K}^{H} \theta_{G / H}(m \otimes n)=\theta_{G / K}\left(R_{K}^{H} m \otimes R_{K}^{H} n\right) \\
& =\beta\left(R_{K}^{H}(m \otimes n)\right) \\
C_{g, H} \beta(m \otimes n) & =C_{g, H} \theta_{G / H}(m \otimes n)=\theta_{G / g_{H}}\left(C_{g, H} m \otimes C_{g, H} n\right) \\
& =\beta\left(C_{g, H}(m \otimes n)\right) .
\end{aligned}
$$

Take $x \in A_{K}$ and $S \in \mu_{R}(G)$. Since $S I_{K}^{H}$ acts on the grade $K, S I_{K}^{H} \beta(x)=\beta\left(S I_{K}^{H} x\right)$ by induction. Therefore,

$$
S \beta\left(I_{K}^{H} x\right)=S I_{K}^{H}(x)=\beta\left(S I_{K}^{H} x\right)
$$

Let us show that $\beta$ maps the Frobenius reciprocity submodule $F R_{H}$ to zero for each $H \leq G$. Take $K<H, a \in M(G / H)$ and $b \in N(G / K)$. By the second commuting diagram in lemma 2.1, we have

$$
\begin{aligned}
\beta\left(I_{K}^{H}\left(R_{K}^{H} a \otimes b\right)\right) & =I_{K}^{H} \beta\left(R_{K}^{H} a \otimes b\right)=I_{K}^{H} \theta_{G / K}\left(R_{K}^{H} a \otimes b\right) \\
& =\theta_{G / H}\left(a \otimes I_{K}^{H} b\right)=\beta\left(a \otimes I_{K}^{H} b\right)
\end{aligned}
$$

Take $K<H, c \in M(G / K)$ and $d \in N(G / H)$. By the third commuting diagram in lemma 2.1 we have

$$
\begin{aligned}
\beta\left(I_{K}^{H}\left(c \otimes R_{K}^{H} d\right)\right) & =I_{K}^{H} \beta\left(c \otimes R_{K}^{H} d\right)=I_{K}^{H} \theta_{G / K}\left(c \otimes R_{K}^{H} d\right) \\
& =\theta_{G / H}\left(I_{K}^{H} c \otimes d\right)=\beta\left(I_{K}^{H} c \otimes d\right)
\end{aligned}
$$

Lastly, it is easy to see the composition of those two maps above is identity in either way. For instance, the map

$$
\operatorname{Dress}(M, N ; P) \rightarrow \operatorname{Hom}_{\mu_{R}(G)-\bmod }(\Phi M \square \Phi N, \Phi P) \rightarrow \operatorname{Dress}(M, N ; P)
$$

, which maps $\theta \mapsto \beta \mapsto \theta^{\prime}$, is identity because $\theta_{G / H}^{\prime}$ equals to the restriction of $\beta$ to the first summand of the grade $H$, which in turn equals to the maps $\theta_{G / H}$ according to the constructions above. Thus, $\theta^{\prime}=\theta$.

Proof for Theorem 4.1. Fix three Mackey functors $M, N, P \in \operatorname{Mack}_{R}(G)$. By Lemma 4.2, there is a natural bijection

$$
\operatorname{Dress}(M, N ; P) \cong \operatorname{Hom}_{\mu_{R}(G)-\bmod }(\Phi M \square \Phi N, \Phi P)
$$

By Lemma 2.1, there is a natural bijection

$$
\operatorname{Hom}_{\operatorname{Mack}_{R}(G)}(M \square N, P) \cong \operatorname{Dress}(M, N ; P)
$$

Since $\Phi: \operatorname{Mack}_{R}(G) \stackrel{\cong}{\cong} \mu_{R}(G)$-mod is an equivalence of categories, there is a natural bijection

$$
\operatorname{Hom}_{\operatorname{Mack}_{R}(G)}(M \square N, P) \cong \operatorname{Hom}_{\mu_{R}(G)-\bmod }(\Phi(M \square N), \Phi P)
$$

Therefore, we get a natural bijection

$$
\operatorname{Hom}_{\mu_{R}(G)-\bmod }(\Phi M \square \Phi N, \Phi P) \cong \operatorname{Hom}_{\mu_{R}(G)-\bmod }(\Phi(M \square N), \Phi P)
$$

Therefore, $\Phi M \square \Phi N$ is naturally isomorphic to $\Phi(M \square N)$ as a left $\mu_{R}(G)$-module.
Moreover, the equality $\Phi\left(B^{G}(-)\right)=\oplus_{H \leq G} B^{G}(G / H)=\oplus_{H \leq G} \operatorname{Hom}_{\Omega_{R}(G)}(G / H, G / G)$
verifies the correspondence of units for box products in $\operatorname{Mack}_{R}(G)$ and $\mu_{R}(G)$-mod. Thus, the equivalence $\Phi$ is monoidal.

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[^0]:    Date: September 24, 2015.

