# BOX PRODUCT OF MACKEY FUNCTORS IN TERMS OF MODULES

## ZHULIN LI

ABSTRACT. The box product of Mackey functors has been studied extensively in Lewis's notes. As shown in Thevenaz and Webb's paper, a Mackey functor may be identified with a module over a certain algebra, called the Mackey algebra. We aim at describing the box product, in the sense of Mackey algebra modules. For a cyclic *p*-group *G*, we recover a result from Mazur's thesis. We generalize it to a general finite group *G* in this article.

## INTRODUCTION

A Mackey functor is an algebraic structure, related to many natural constructions from finite groups, such as group cohomology and the algebraic K-theory of group rings. The study of Mackey functor in abstract began in 1980s. Dress[2] and Green[1] first gave the axiomatic formulation of Mackey functors. Several equivalent descriptions of Mackey functors were given by Dress[2], Lindner[3], Lewis[4] and Thevenaz[5]. Specifically, Lewis[4] introduces box product and Thevenaz[5] describes a Mackey functor as a module over the Mackey algebra  $\mu_R(G)$ .

In this article, we give an inductive description of the box product of two left  $\mu_R(G)$ -modules. The main goal is to construct this box product explicitly and to prove that it is equivalent with the box product of two Mackey functors. When G is a cyclic *p*-group, we recover the formula of Mazur[7].

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#### CONTENTS

Acknowledgement11. Mackey functors21.1. The category $\Omega_R(G)$ 21.2. The Mackey algebra $\mu_R(G)$ 31.3. Definition of Mackey functors42. The Box product in Mack_R(G)4
1.1.The category $\Omega_R(G)$ 21.2.The Mackey algebra $\mu_R(G)$ 31.3.Definition of Mackey functors4
1.2. The Mackey algebra $\mu_R(G)$ 31.3. Definition of Mackey functors4
1.3. Definition of Mackey functors 4
·
2. The Box product in $Mack_{P}(C)$
2. The box product in $Mack_R(G)$ 4
3. The Box Product in $\mu_R(G)$ -mod 5

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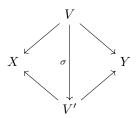
#### ZHULIN LI

4. Equivalence of box product in  $\operatorname{Mack}_R(G)$  and  $\mu_R(G)$ -mod References 10

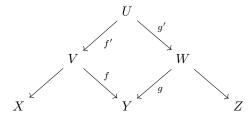
# 1. Mackey functors

Throughout this article, we assume that G is a finite group and that R is a unital, commutative ring. There are several equivalent definitions of Mackey functors and we concentrate on two of them here. Before giving the definitions, we introduce an auxiliary category  $\Omega_R(G)$  and the Mackey algebra  $\mu_R(G)$ .

1.1. The category  $\Omega_R(G)$ . We recall the definition of the category  $\Omega_R(G)$  from [4]. A finite group G gives rise to a category  $\omega(G)$  whose objects are finite G-sets and where the morphisms from X to Y are the equivalence classes of diagrams of G-sets  $X \leftarrow V \rightarrow Y$ . Two such diagrams are said to be equivalent if there is a commutative diagram



where  $\sigma$  is an isomorphism of G-sets. To define the composition of morphisms, we consider a morphism from X to Y represented by a diagram  $X \leftarrow V \rightarrow Y$  and a morphism from Y to Z represented by a diagram  $Y \leftarrow W \rightarrow Z$ . We form the pullback



which defines a diagram  $X \leftarrow U \rightarrow Z$ , hence a morphism from X to Z. Such a pullback always exists and we can express it explicitly as

$$U = \{ (v, w) \in V \times W : f(v) = g(w) \},\$$

where f'(v, w) = v and g'(v, w) = w. By defining addition in  $\operatorname{Hom}_{\omega(G)}(X, Y)$  as

$$(X \xleftarrow{\alpha} V \xrightarrow{\beta} Y) + (X \xleftarrow{\alpha'} V' \xrightarrow{\beta'} Y) := (X \xleftarrow{\alpha+\alpha'} V \sqcup V' \xrightarrow{\beta+\beta'} Y),$$

 $\operatorname{Hom}_{\omega(G)(X,Y)}$  becomes a free abelian monoid, as shown in [5]. Extending the scalars to the ring R, we get a free R-module

$$\operatorname{Hom}_{\Omega_R(G)}(X,Y) := R \operatorname{Hom}_{\omega(G)}(X,Y)$$

7

on the same basis as that of free monoid  $\operatorname{Hom}_{\omega(G)}(X,Y)$ . Let  $\Omega_R(G)$  be a category with the same objects as  $\omega(G)$  and hom-set  $\operatorname{Hom}_{\Omega_B(G)}(X,Y)$  defined as above.  $\Omega_R(G)$  is an *R*-additive category.

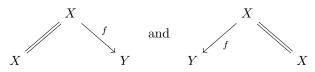
1.2. The Mackey algebra  $\mu_R(G)$ . We define the Mackey algebra as

$$\mu_R(G) := \bigoplus_{H,K \le G} \operatorname{Hom}_{\Omega_R(G)}(G/H, G/K),$$

where the multiplication is defined on the components in the direct sum by composition of morphisms in the category  $\Omega_R(G)$ , or zero if two morphisms cannot be composed.

For convenience, we recall the notation from [6].

**Definition 1.1.** Let  $f: X \to Y$  be a *G*-equivariant map of finite *G*-sets. Then two spans



are called the restriction  $r_f$  and transfer  $t_f$ , respectively, of f.

**Definition 1.2.** Let  $K \leq H \leq G$  and  $g \in G$ . Then define  $R_K^H$ ,  $I_K^H$  and  $C_{g,H}$  as

$$\begin{aligned} R_K^H &= r_{\pi_K^H} : \left( G/K = G/K \xrightarrow{\pi_K^H} G/H \right) \\ I_K^H &= t_{\pi_K^H} : \left( G/H \xleftarrow{\pi_K^H} G/K = G/K \right) \\ C_{g,H} &= t_{c_{g,H}} = r_{c_{g^{-1},g_H}} : \left( G/^g H \xleftarrow{c_{g,H}} G/H = G/H \right) \end{aligned}$$

where  $\pi_K^H: G/K \to G/H$  denotes the canonical quotient map, mapping gK to gH, and  $c_{q,H}: G/H \to G/{}^{g}H$  denotes the conjugation map, mapping kH to  $(kg^{-1})^{g}H$ . Observe that  $R_K^H, I_K^H$  and  $C_{g,H}$  are all elements in Mackey algebra  $\mu_R(G)$ .

For conjustion, the notation  $C_g$  is preferred to  $C_{g,H}$  for simplicity when there is no ambiguity. It is easy to check that the following identities hold:

- (0)  $R_{H}^{H} = I_{H}^{H} = C_{h,H}$  for all  $H \leq G$  and  $h \in H$ (1)  $R_{J}^{K}R_{K}^{H} = R_{J}^{H}$  for all subgroups  $J \leq K \leq H$ (2)  $I_{K}^{H}I_{J}^{K} = I_{J}^{H}$  for all subgroups  $J \leq K \leq H$
- (3)  $C_q C_h = C_{qh}$  for all  $g, h \in G$

- (5)  $C_g C_h = C_{gh}$  for all  $g, h \in G$ (4)  $C_g R_K^H = R_{gK}^{g} C_g$  for all subgroups  $K \leq H$  and  $g \in G$ (5)  $C_g I_K^H = I_{gK}^{g} C_g$  for all subgroups  $K \leq H$  and  $g \in G$ (6)  $R_J^H I_K^H = \sum_{g \in [J \setminus H/K]} I_{gK \cap J}^J C_g R_{K \cap J^g}^K$  for all subgroups  $J, K \leq H$ (7)  $\sum_{H \leq G} I_H^H$  serves as the unit in  $\mu_R(G)$ .

We use  $[J \setminus H/K]$  to denote the set of representatives of the double coset  $J \setminus H/K$ . In fact, the structure of Mackey algebra  $\mu_R(G)$  is rather simple:

**Lemma 1.1** (from [5]). Hom-set  $\operatorname{Hom}_{\Omega_R(G)}(G/K, G/H)$  is a free R-module, with basis represented by the diagrams

$$I_{g_L}^K C_{g,L} R_L^H = \left( G/K \xleftarrow{\pi_{g_L}^K c_{g,L}} G/L \xrightarrow{\pi_L^H} G/H \right)$$

where  $g \in [K \setminus G/H]$  and L is a subgroup of  $H \cap K^g$  taken up to  $H \cap K^g$ -conjugation.

In other words, the Mackey algebra  $\mu_R(G)$  is generated by  $R_K^H$ ,  $I_K^H$  and  $C_{g,H}$ 's as an *R*-algebra.

# 1.3. Definition of Mackey functors.

**Definition 1.3** (from [2]). A Mackey functor is a *R*-additive functor  $M : \Omega_R(G)^{\text{op}} \to R$ -mod. They form a category with natural transformations as morphisms and we denote this category by  $\operatorname{Mack}_R(G)$ .

It is shown in [5] that the category  $\mu_R(G)$ -mod of left  $\mu_R(G)$ -modules is equivalent to Mack<sub>R</sub>(G) via the following equivalence of categories

$$\Phi: \operatorname{Mack}_{R}(G) \longleftrightarrow \mu_{R}(G) \operatorname{-mod} M \longmapsto \bigoplus_{H \leq G} M(G/H)$$
$$(G/H \mapsto I_{H}^{H}N) \longleftrightarrow N.$$

Since  $\sum_{H \leq G} I_H^H$  is the unit in  $\mu_R(G)$ , a left  $\mu_R(G)$ -module N can be graded into  $N = \bigoplus_{H \leq G} I_H^H N$ . Observe that the multiplication by  $R_K^H$  maps  $I_H^H N$  to  $I_K^K N$  and the other grades to zero. Similarly, multiplication by  $I_K^H$  maps  $I_K^H N$  to  $I_g^H N$  and the other grades to zero. Multiplication by  $C_{g,H}$  maps  $I_H^H N$  to  $I_{gH}^g N$  and the other grades to zero. Also note that  $I_H^H \Phi M = M(G/H)$  for later use.

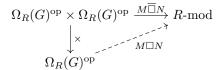
# 2. The Box product in $Mack_R(G)$

The box product is a symmetric monoidal structure on  $\operatorname{Mack}_R(G)$ . The box product in  $\operatorname{Mack}_R(G)$  has been studied in [4] and we summarize it in this section. The result we are more interested in is that maps from the box product of M and N to P can be characterized by Dress parings.

Given two Mackey functors  $M, N \in Mack_R(G)$ , we can form the exterior product

$$\begin{split} M \Box N : \Omega_R(G)^{\mathrm{op}} \times \Omega_R(G)^{\mathrm{op}} &\longrightarrow R\text{-}\mathrm{mod}\\ (X, Y) &\longmapsto M(X) \otimes N(Y). \end{split}$$

**Definition 2.1** (Box product in  $\operatorname{Mack}_R(G)$ ). The box product  $M \Box N$  is defined to be the left Kan extension of  $M \overline{\Box} N$  along the Cartesian product functor  $\times : \Omega_R(G)^{\operatorname{op}} \times \Omega_R(G)^{\operatorname{op}} \longrightarrow \Omega_R(G)^{\operatorname{op}}$ .



If a Mackey functor  $M \in \operatorname{Mack}_R(G)$  is implicit, we use  $r_f$  for both a morphism  $\left(X = X \xrightarrow{f} Y\right)$  in  $\Omega_R(G)$  and its value  $M(Y) \to M(X)$  under M. Similarly for  $t_f$ .

**Lemma 2.1** (from [4]). A map  $\theta : M \Box N \to P$  determines and is determined by a collection of *R*-module homomorphisms

$$\theta_X: M(X) \otimes N(X) \to P(X)$$

for every finite G-set X, such that the following three diagrams commute for each G-equivariant map  $f: X \to Y$ .

$$M(Y) \otimes N(Y) \xrightarrow{\theta_{Y}} P(Y)$$

$$\downarrow^{r_{f} \otimes r_{f}} \qquad \downarrow^{r_{f}}$$

$$M(X) \otimes N(X) \xrightarrow{\theta_{X}} P(X)$$

$$M(X) \otimes N(Y) \xrightarrow{id \otimes r_{f}} P(X)$$

$$M(X) \otimes N(Y) \xrightarrow{t_{f} \otimes id} \qquad \downarrow^{t_{f}}$$

$$M(Y) \otimes N(Y) \xrightarrow{\theta_{Y}} P(Y)$$

$$M(X) \otimes N(X) \xrightarrow{\theta_{X}} P(X)$$

$$M(Y) \otimes N(X) \xrightarrow{id \otimes t_{f}} \qquad \downarrow^{t_{f}}$$

$$M(Y) \otimes N(Y) \xrightarrow{\theta_{Y}} P(Y)$$

A good exposition and proof of this lemma can be found in [6]. The data in this lemma is called a Dress paring. The natural transformations from  $M \square N$  to another Mackey functor P are the same as the Dress parings from M and N to P, via the natural bijection

$$\operatorname{Hom}_{\operatorname{Mack}_R(G)}(M\Box N, P) \cong \operatorname{Dress}(M, N; P),$$

that maps a map  $\theta: M \Box N \to P$  to  $\theta_X: M(X) \otimes N(X) \to P(X \times X) \xrightarrow{r\Delta} P(X)$ . The first map comes from the Kan adjunction and  $r\Delta$  is the restriction associated to the diagonal map of *G*-sets  $X \to X \times X$ .

In  $Mack_R(G)$ , the Burnside ring Mackey functor

$$B^G(-) := \operatorname{Hom}_{\Omega_B(G)}(-, G/G)$$

is the unit for box product, as shown in [4]. In this way,  $(\operatorname{Mack}_R(G), B^G(-), \Box)$  is a symmetric monoidal category.

# 3. The Box Product in $\mu_R(G)$ -mod

Given two left  $\mu_R(G)$ -modules M an N, we can form an R-module  $A_H$  for each  $H \leq G$  by induction on the cardinality of subgroups of G

$$A_e := (I_e^e M) \otimes_R (I_e^e N)$$
$$A_H := ((I_H^H M) \otimes_R (I_H^H N)) \oplus \bigoplus_{K < H} A_K.$$

We say  $A_H$  is of grade H. By combining all the grades together, we get  $A := \bigoplus_{H \leq G} A_H$ . Note that the component  $A_K$  in grade H is distinct from the grade K. Since  $\mu_R(G)$  is generated by R, I, C's, we can endow A with a left  $\mu_R(G)$ -module structure by giving actions of R, I, C's on A. The maps  $C_{g,H}, I_H^L, R_J^H$  map the grade H to grades  ${}^{g}H, L, J$  respectively, and map the other grades to zero. Their action on the grade H is described as follows:

IN TERMS OF MODULES

(1)  $I_H^L$  action on the grade H:

If H = L,  $I_H^L$  acts as the identity on grade H. If H < L,  $I_H^L$  maps an element in  $A_H$  to its corresponding copy  $A_H$  in grade L. To distinguish  $A_H$  from its copy in grade L, we write its copy in grade L as  $I_H^L A_H$  from now on. That is, grade H is written as

$$A_H = \left( (I_H^H M) \otimes_R (I_H^H N) \right) \oplus \bigoplus_{K < H} I_K^H A_K.$$

(2)  $C_{g,H}$  action on the grade H:

We define the action of  $C_{g,H}$  by induction on the cardinality of H as follows. For  $m \otimes n \in (I_H^H M) \otimes (I_H^H N)$  and  $x \in A_K$ , where K < H,

$$C_{g,H}(m \otimes n) := (C_{g,H}m) \otimes (C_{g,H}n) \in A_{g_H}$$
$$C_{g,H}I_K^H(x) := I_{gK}^{g_H}C_{g,K}(x) \in A_{g_H}.$$

(3)  $R_J^H$  action on the grade H:

We also define the action of  $R_J^H$  by induction on the cardinality of H as follows. For  $m \otimes n \in (I_H^H M) \otimes (I_H^H N)$  and  $x \in A_K$ , where K < H,

$$R_J^H(m \otimes n) := (R_J^H m) \otimes (R_J^H n) \in A_J$$
$$R_J^H I_K^H(x) := \sum_{g \in [J \setminus H/K]} I_{gK \cap J}^J C_g R_{K \cap J^g}^K(x) \in A_J$$

**Definition 3.1** (Box product in  $\mu_R(G)$ -mod). Based on the left  $\mu_R(G)$ -module  $\bigoplus_{H \leq G} A_H$ , we can define  $M \square N$  as

$$M\Box N := \left(\oplus_{H \le G} A_H\right) / FR_{H}$$

where FR is a submodule, called the Frobenius reciprocity submodule, generated by elements of the form

$$a \otimes (I_K^H b) - I_K^H ((R_K^H a) \otimes b)$$

and

 $(I_K^H c) \otimes d - I_K^H (c \otimes (R_K^H d))$ for all  $K < H, a \in I_H^H M, b \in I_K^K N, c \in I_K^K M$  and  $d \in I_H^H N.$ 

Naturally, the image of  $A_H$  under the quotient is called the grade H of  $M \Box N$ .

**Proposition 3.1.**  $\oplus_{H \leq G} \operatorname{Hom}_{\Omega_R(G)}(G/H, G/G)$  is the unit for box product in  $\mu_R(G)$ -mod.

*Proof.* Let M be a left  $\mu_R(G)$ -module and  $N = \bigoplus_{H \leq G} \operatorname{Hom}_{\Omega_R(G)}(G/H, G/G)$ . Then  $I_H^H N$  is an R-module generated by  $G/H \xleftarrow{\pi_L^H} G/L \to G/G = I_L^H R_L^H n_H$ ,

Then  $I_H^H N$  is an *R*-module generated by  $G/H \xleftarrow{} G/L \rightarrow G/G = I_L^H R_L^H n_H$ where

$$n_H := (G/H = G/H \to G/G) \,.$$

Thus,  $I_H^H M \otimes I_H^H N$  is a *R*-module generated by  $m \otimes I_L^H R_L^H n_H = I_L^H (R_L^H m \otimes n_H)$ . Then we get a natural bijection between each grade of M and  $M \square N$ 

$$F: I_{H}^{H} M \longleftrightarrow I_{H}^{H} (M \Box N)$$
  

$$m \longmapsto m \otimes n_{H}$$
  

$$I_{L}^{H} R_{L}^{H} m \longleftrightarrow I_{L}^{H} (R_{L}^{H} m \otimes n_{H}) \in I_{H}^{H} M \otimes I_{H}^{H} N$$
  

$$I_{K}^{H} F^{-1}(x) \longleftrightarrow I_{K}^{H} x \in I_{K}^{H} I_{K}^{K} (M \Box N).$$

 $\mathbf{6}$ 

Here  $F^{-1}$  is defined by induction on the grade. It is easy to check that this is a bijection and that it preserves the  $\mu_R(G)$ -module structure. Thus,  $M \square N$  is isomorphic to M as a  $\mu_R(G)$ -module and N is the unit for box product in  $\mu_R(G)$ -mod.  $\square$ 

In this way,  $(\mu_R(G)-\text{mod}, \oplus_{H \leq G} \text{Hom}_{\Omega_R(G)}(G/H, G/G), \Box)$  is a symmetric monoidal category.

4. Equivalence of box product in  $Mack_R(G)$  and  $\mu_R(G)$ -mod

**Theorem 4.1.** The equivalence of categories  $\Phi : Mack_R(G) \xrightarrow{\cong} \mu_R(G)$ -mod is a symmetric monoidal equivalence. In other words, there are a natural isomorphism  $\Phi M \Box \Phi N \cong \Phi(M \Box N)$  for any two Mackey functors  $M, N \in Mack_R(G)$ , and a compatible natural isomorphism  $B^G(-) \xrightarrow{\cong} \operatorname{Hom}_{\Omega_R(G)}(G/H, G/G)$ .

**Lemma 4.2.** For any Mackey functors  $M, N, P \in Mack_R(G)$ , there is a natural bijection

$$Dress(M, N; P) \cong \operatorname{Hom}_{\mu_B(G) \operatorname{-mod}}(\Phi M \Box \Phi N, \Phi P).$$

*Proof.* Given  $\beta \in \operatorname{Hom}_{\mu_R(G)\operatorname{-mod}}(\Phi M \Box \Phi N, \Phi P)$ , we map it to  $\theta \in \operatorname{Dress}(M, N; P)$  defined as follows. For each  $H \leq G$ ,  $\beta$  maps grade  $I_H^H(\Phi M \Box \Phi N)$  to grade  $I_H^H \Phi P$ , because  $\beta(x) = \beta(I_H^H x) = I_H^H \beta(x) \in I_H^H \Phi P$  for each  $x \in I_H^H(\Phi M \Box \Phi N)$ . Since

$$I_{H}^{H}(\Phi M \Box \Phi N) = (M(G/H) \otimes N(G/H)) \oplus \bigoplus_{K < H} I_{K}^{H} A_{K} / FR$$

and

$$I_H^H \Phi P = P(G/H),$$

 $\beta$  induces an *R*-module homomorphism  $\theta_{G/H}$ 

$$\theta_{G/H}: M(G/H) \otimes N(G/H) \to P(G/H)$$

by restricting to the first summand. For a general finite G-set  $X = \sqcup_{i=1}^p G/H_i, \, \theta_X$  is defined as

$$\begin{aligned} \theta_X : \bigoplus_{i,j=1}^p M(G/H_i) \otimes N(G/H_j) &\longrightarrow \bigoplus_{i=1}^p P(G/H_i) \\ m_i \otimes n_j &\longmapsto \begin{cases} \theta_{G/H_i}(m_i \otimes n_i), \text{ if } i = j \\ 0, \text{ otherwise.} \end{cases} \end{aligned}$$

Having constructed  $\theta$ , we now proceed to show that  $\theta \in Dress(M, N; P)$ . Given finite G-sets X, Y and a G-equivariant map  $f: X \to Y$ , it is sufficient to show that the three diagrams in Lemma 2.1 commute.

If both X and Y are orbits, say X = G/K and Y = G/H, observe that a Gequivariant map f from G/K to G/H must be of the form  $f = \pi_{gK}^H c_{g,K}$  for some  $g \in G$ . By composition of commuting diagrams, we only need to consider the case where f = c and  $f = \pi$ .

When  $f = c_{g,H} : G/H \to G/{}^{g}H$ , we have that  $r_f = C_{g^{-1},gH}$  and  $t_f = C_{g,H}$ . The first diagram commutes because

$$r_f \theta_Y(m \otimes n) = C_{g^{-1}, g_H} \beta(m \otimes n) = \beta(C_{g^{-1}, g_H}(m \otimes n))$$
$$= \beta(C_{g^{-1}, g_H}m \otimes C_{g^{-1}, g_H}n)) = \theta_X(r_f m \otimes r_f n).$$

The second diagram commutes because

$$t_f \theta_X(m \otimes r_f n) = C_{g,H} \beta(m \otimes C_{g^{-1},gH} n) = \beta(C_{g,H}(m \otimes C_{g^{-1},gH} n))$$
$$= \beta(C_{g,H} m \otimes n) = \theta_Y(t_f m \otimes n).$$

The third diagram commutes similarly.

When  $f = \pi_K^H : G/K \to G/H$ , we have that  $r_f = R_K^H$  and  $t_f = I_K^H$ . The first diagram commutes because

$$r_f \theta_Y(m \otimes n) = R_K^H \beta(m \otimes n) = \beta(R_K^H(m \otimes n))$$
$$= \beta(R_K^H m \otimes R_K^H n)) = \theta_X(r_f m \otimes r_f n).$$

The second diagram commutes because

$$t_f \theta_X(m \otimes r_f n) = I_K^H \beta(m \otimes R_K^H n) = \beta(I_K^H(m \otimes R_K^H n))$$
$$= \beta(I_K^H m \otimes n) = \theta_Y(t_f m \otimes n).$$

The third diagram commutes similarly.

Now, if Y is an orbit G/H and  $X = \bigsqcup_{i=1}^p G/K_i$  is a general G-set, denote the restriction of f to  $G/K_i$  by  $f_i : G/K_i \to G/H$ . Thus,  $r_f : M(G/H) \to \bigoplus_{i=1}^p M(G/K_i)$  is the sum of  $r_{f_i} : M(G/H) \to M(G/K_i)$ . Similarly,  $t_f : \bigoplus_{i=1}^p M(G/K_i) \to M(G/H)$  is determined by components  $t_{f_i} : M(G/K_i) \to M(G/H)$ . The first diagram commutes because

$$r_f \theta_Y(m \otimes n) = \sum_{i=1}^p r_{f_i} \theta_Y(m \otimes n) = \sum_{i=1}^p \theta_X(r_{f_i} m \otimes r_{f_i} n)$$
$$= \theta_X(\sum_{i=1}^p r_{f_i} m \otimes r_{f_i} n) = \theta_X(\sum_{i=1}^p r_{f_i} m \otimes \sum_{i=1}^p r_{f_i} n)$$
$$= \theta_X(r_f m \otimes r_f n).$$

The second diagram commutes because

$$t_f \theta_{\mathbf{b}}(m \otimes r_f n) = t_f \theta_X(\sum_{i=1}^p m_i \otimes \sum_{i=1}^p r_{f_i} n) = t_f \theta_X(\sum_{i=1}^p m_i \otimes r_{f_i} n)$$
$$= \sum_{i=1}^p t_f \theta_X(m_i \otimes r_{f_i} n) = \sum_{i=1}^p t_{f_i} \theta_{G/K_i}(m_i \otimes r_{f_i} n)$$
$$= \sum_{i=1}^p \theta_Y(t_{f_i} m_i \otimes n) = \theta_Y(\sum_{i=1}^p t_{f_i} m_i \otimes n)$$
$$= \theta_Y(t_f m \otimes n)$$

for  $m = \sum_{i=1}^{p} m_i$ , where  $m_i \in M(G/K_i)$ . The third diagram commutes similarly.

Lastly, the general case:  $X = \bigsqcup_{i=1}^{p} X_i$ ,  $Y = \bigsqcup_{i=1}^{p} G/H_i$ , and f maps  $X_i$  to  $G/H_i$ . Thus,  $r_f$  maps  $M(G/H_i)$  to  $M(X_i)$  and  $t_f$  maps  $M(X_i)$  to  $G/H_i$ . For the first diagram, take  $m \in G/H_i$  and  $n \in G/H_j$ . Thus,  $r_f m \in M(X_i)$  and  $r_f n \in M(X_j)$ . If  $i \neq j$ , then  $\theta_Y(m \otimes n) = 0$  and  $\theta_X(r_f m \otimes r_f n) = 0$ . If i = j, then this is the case when Y is an orbit. For the second diagram, take  $m \in M(X_i)$  and  $n \in G/H_j$ . Thus,  $t_f m \in M(G/H_i)$  and  $r_f n \in M(X_j)$ . If  $i \neq j$ , then  $\theta_X(m \otimes r_f n) = 0$  and  $\theta_Y(t_f m \otimes n) = 0$ . If i = j, it reduces to the case when Y is an orbit. The third diagram commutes similarly. Given a Dress pairing  $\theta \in \text{Dress}(M, N; P)$ , we define  $\beta$ , a map from  $\Phi M \Box \Phi N$  to P grade by grade, as follows:

$$\beta_H : I_H^H(\Phi M \Box \Phi N) \longrightarrow I_H^H \Phi P = P(G/H)$$
$$M(G/H) \otimes N(G/H) \ni m \otimes n \longmapsto \theta_{G/H}(m \otimes n)$$
$$I_K^H A_K \ni I_K^H(x) \longmapsto I_K^H(\beta_K(x))$$

Having constructed  $\beta$ , we now proceed to show that  $\beta$  is indeed a map of  $\mu_R(G)$ modules.  $\beta$  is linear in multiplication by elements in  $\mu_R(G)$ : It is enough to check this for the generators R, I, C's. Say  $K < H \leq G$  and take  $m \otimes n \in M(G/H) \otimes N(G/H)$ . Then

$$\begin{aligned} R_K^H \beta(m \otimes n) &= R_K^H \theta_{G/H}(m \otimes n) = \theta_{G/K}(R_K^H m \otimes R_K^H n) \\ &= \beta(R_K^H(m \otimes n)) \\ C_{g,H}\beta(m \otimes n) &= C_{g,H}\theta_{G/H}(m \otimes n) = \theta_{G/^gH}(C_{g,H}m \otimes C_{g,H}n) \\ &= \beta(C_{g,H}(m \otimes n)). \end{aligned}$$

Take  $x \in A_K$  and  $S \in \mu_R(G)$ . Since  $SI_K^H$  acts on the grade K,  $SI_K^H\beta(x) = \beta(SI_K^Hx)$  by induction. Therefore,

$$S\beta(I_K^H x) = SI_K^H(x) = \beta(SI_K^H x).$$

Let us show that  $\beta$  maps the Frobenius reciprocity submodule  $FR_H$  to zero for each  $H \leq G$ . Take K < H,  $a \in M(G/H)$  and  $b \in N(G/K)$ . By the second commuting diagram in lemma 2.1, we have

$$\beta(I_K^H(R_K^H a \otimes b)) = I_K^H \beta(R_K^H a \otimes b) = I_K^H \theta_{G/K}(R_K^H a \otimes b)$$
$$= \theta_{G/H}(a \otimes I_K^H b) = \beta(a \otimes I_K^H b).$$

Take K < H,  $c \in M(G/K)$  and  $d \in N(G/H)$ . By the third commuting diagram in lemma 2.1, we have

$$\beta(I_K^H(c \otimes R_K^H d)) = I_K^H \beta(c \otimes R_K^H d) = I_K^H \theta_{G/K}(c \otimes R_K^H d)$$
$$= \theta_{G/H}(I_K^H c \otimes d) = \beta(I_K^H c \otimes d).$$

Lastly, it is easy to see the composition of those two maps above is identity in either way. For instance, the map

 $\mathrm{Dress}(M,N;P) \to \mathrm{Hom}_{\mu_R(G)\operatorname{-mod}}(\Phi M \Box \Phi N, \Phi P) \to \mathrm{Dress}(M,N;P)$ 

, which maps  $\theta \mapsto \beta \mapsto \theta'$ , is identity because  $\theta'_{G/H}$  equals to the restriction of  $\beta$  to the first summand of the grade H, which in turn equals to the maps  $\theta_{G/H}$  according to the constructions above. Thus,  $\theta' = \theta$ .

Proof for Theorem 4.1. Fix three Mackey functors  $M, N, P \in \operatorname{Mack}_R(G)$ . By Lemma 4.2, there is a natural bijection

$$Dress(M, N; P) \cong Hom_{\mu_B(G)-mod}(\Phi M \Box \Phi N, \Phi P).$$

By Lemma 2.1, there is a natural bijection

$$\operatorname{Hom}_{\operatorname{Mack}_{\mathcal{B}}(G)}(M\Box N, P) \cong \operatorname{Dress}(M, N; P).$$

Since  $\Phi$  : Mack<sub>R</sub>(G)  $\xrightarrow{\cong} \mu_R(G)$ -mod is an equivalence of categories, there is a natural bijection

$$\operatorname{Hom}_{\operatorname{Mack}_R(G)}(M\Box N, P) \cong \operatorname{Hom}_{\mu_R(G)\operatorname{-mod}}(\Phi(M\Box N), \Phi P).$$

Therefore, we get a natural bijection

# $\operatorname{Hom}_{\mu_R(G)\operatorname{-mod}}(\Phi M \Box \Phi N, \Phi P) \cong \operatorname{Hom}_{\mu_R(G)\operatorname{-mod}}(\Phi(M \Box N), \Phi P).$

Therefore,  $\Phi M \Box \Phi N$  is naturally isomorphic to  $\Phi(M \Box N)$  as a left  $\mu_R(G)$ -module.

Moreover, the equality  $\Phi(B^G(-)) = \bigoplus_{H \leq G} B^G(G/H) = \bigoplus_{H \leq G} \operatorname{Hom}_{\Omega_R(G)}(G/H, G/G)$ verifies the correspondence of units for box products in  $\operatorname{Mack}_R(G)$  and  $\mu_R(G)$ -mod. Thus, the equivalence  $\Phi$  is monoidal.  $\Box$ 

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