THE RO(G)-GRADED EQUIVARIANT ORDINARY COHOMOLOGY OF COMPLEX PROJECTIVE SPACES WITH LINEAR $\mathbb{Z}/p$ ACTIONS

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INTRODUCTION. If X is a CW complex with cells only in even dimensions and R is a ring, then, by an elementary result in cellular cohomology theory, the ordinary cohomology $H^*(X;R)$ of X with R coefficients is a free, $\mathbb{Z}$-graded R-module. Since this result is quite useful in the study of well-behaved complex manifolds like projective spaces or Grassmannians, it would be nice to be able to generalize it to equivariant ordinary cohomology. The result does generalize in the following sense. Let G be a finite group, X be a G-CW complex (in the sense of [MAT, LMSM]), and R be a ring-valued contravariant coefficient system [ILL]. Then the G-equivariant ordinary Bredon cohomology $H^*(X;R)$ of X with R coefficients may be regarded as a coefficient system. If the cells of X are all even dimensional, then $H^*(X;R)$ is a free module over R in the sense appropriate to coefficient systems. Unfortunately, this theorem does not apply to complex projective spaces or complex Grassmannians with any reasonable nontrivial G-action because these spaces do not have the right kind of G-CW structure. In fact, if G is $\mathbb{Z}/p$, for any prime p, and η is a nontrivial irreducible complex G-representation, then the theorem does not apply to S^η, the one-point compactification of η. Moreover, the $\mathbb{Z}$-graded Bredon cohomology of S^η with coefficients in the Burnside ring coefficient system is quite obviously not free over the coefficient system.

The purpose of this paper is to provide an equivariant generalization of the "freeness" theorem which does apply to an interesting class of G-spaces and to use this result to describe the equivariant ordinary cohomology of complex projective spaces with linear $\mathbb{Z}/p$ actions. These results are obtained by regarding equivariant ordinary cohomology as a Mackey functor-valued theory graded on the real representation ring RO(G) of G [LMM, LMSM]. To obtain such a theory, we take the Burnside ring Mackey functor as our coefficient ring. Instead of using cells of the form $G/H \times e^n$, where H runs over the subgroups of G, we use the unit disks of real G-representations as cells. Our main theorem, Theorem 2.6, then has roughly the following form.

THEOREM. Let G be $\mathbb{Z}/p$ and let X be a G-CW complex constructed from the unit disks of real G-representations. If these disks are all even dimensional and are attached in the proper order, then the equivariant ordinary cohomology $\underline{H}_G^*X$ of X is a free RO(G)-graded module over the equivariant ordinary cohomology of a point.

To show that this theorem is not without applications, we prove in Theorem 3.1 that if V is a complex G-representation and P(V) is the associated complex projective space with the induced linear G-action, then P(V) has the required type of cell structure. Theorems 4.3 and 4.9, which describe the ring structure of $\underline{H}_G^*P(V)$, follow from the freeness of $\underline{H}_G^*P(V)$. As a sample of these results, assume that $p = 2$ and V is a complex G-representation consisting of countably many copies of both the (complex) one-dimensional sign representation λ and the one dimensional trivial representation 1. Then P(V) is the classifying space for G-equivariant complex line bundles. As an RO(G)-graded ring, $\underline{H}_G^*P(V)$ is generated by an element c in dimension λ and an element C in dimension $1+\lambda$. The second generator is a polynomial generator; the first satisfies the single relation

$$c^2 = \epsilon^2 c + \xi C,$$

where ϵ and ξ are elements in the cohomology of a point. If, instead, V contains an equal, but finite, number of copies of λ and 1, then the only change in $\underline{H}_G^*P(V)$ is that the polynomial generator C is truncated in the appropriate dimension. If the number of copies of 1 in V is different from the number of copies of λ in V, or if p is odd, then the ring structure of $\underline{H}_G^*P(V)$ is more complex.

Equivariant ordinary Bredon cohomology with Burnside ring coefficients is just the part of RO(G)-graded equivariant ordinary cohomology with Burnside ring coefficients that is indexed on the trivial representations. All of the generators of $\underline{H}_G^*P(V)$ occur in dimensions corresponding to nontrivial representations of G. This behavior of the generators offers a partial explanation of the difficulties encountered in trying to compute Bredon cohomology. All that can been seen of $\underline{H}_G^*P(V)$ with $\mathbb{Z}$-graded Bredon cohomology is some junk connected to the RO(G)-graded cohomology of a point whose presence in $\underline{H}_G^*P(V)$ is forced by the unseen generators in the nontrivial dimensions.

Using $\underline{H}_G^*P(V)$, It is possible to give an alternative proof of the homotopy rigidity of linear $\mathbb{Z}/p$ actions on complex projective spaces [LIU]. Moreover, the "freeness" theorem should apply to complex Grassmannians with linear $\mathbb{Z}/p$ actions, and it should be possible to compute the ring structure of the equivariant ordinary cohomology of these spaces. Of course, it would be nice to extend the main theorem to groups other than $\mathbb{Z}/p$. Unfortunately, the obvious generalization of this theorem fails for groups other than $\mathbb{Z}/p$. The counterexamples have some interesting connections with the equivariant Hurewicz theorem [LE1]. All of these topics are being investigated.

All of the results in this paper depend on the observation that equivariant cohomology theories are Mackey functor-valued. Therefore, the first section of this paper contains a discussion of Mackey functors for the group $\mathbb{Z}/p$. In the second section, we discuss the RO(G)-graded cohomology of a point, precisely define what we mean by a G-CW complex, and prove our "freeness" theorem. The G-cell structure of complex projective spaces with linear $\mathbb{Z}/p$ actions is discussed in section 3. There the cohomology of these spaces is shown to be free over the cohomology of a point. Section 4 is devoted to the multiplicative structure of the cohomology of a point. The multiplicative structure of the cohomology of complex projective spaces is discussed in section 5. The results stated in this section are proved in section 6. The results on the cohomology of a point stated in sections 2 and 4 are proved in the appendix.

A few comments on notational conventions are necessary. Hereafter, all homology and cohomology is reduced. If X is a G-space and we wish to work with the unreduced cohomolgy of X, then we take the reduced cohomology of X^+, the disjoint union of X and a G-trivial basepoint. In particular, instead of speaking of the cohomology of a point, hereafter we speak of the cohomology of S^0, which always has trivial G action. If V is a G-representation, then SV and DV are the unit sphere and unit disk of V with respect to some G-invariant norm. The one-point compactification of V is denoted S^V and the point at infinity is taken as the basepoint. If X is a based G-space, then $\Sigma^V X$ denotes the smash product of X and S^V. Unless otherwise noted, all spaces, maps, homotopies, etc., are G-spaces, G-maps, and G-homotopies, etc. We will shift back and forth between real and complex G-representations; in general, real representations will be used for grading our cohomology groups and complex representations will be used in discussions of the structure of projective spaces. If the virtual representation α is represented by the difference $V - W$ of representations V and W, then $|\alpha| = \dim V - \dim W$ is the real virtual dimension of α and $\alpha^G = V^G - W^G$ is the fixed virtual representation associated to α. The trivial virtual representation of real dimension n is denoted by n. Recall that the set of irreducible complex representations of G forms a group under tensor product. If η is an irreducible complex representation, then η^{-1} denotes the inverse of η in this group. The tensor product of η and any representation V is denoted ηV. Many of our formulas contain terms of the form A/p, where A is some integer-valued espression. The claim that A is divisible by p is implicitly included in the use of such a term.

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Equivariant cohomology theories graded on RO(G) are not universally familiar objects, so a few remarks about what this paper assumes of its readers seem appropriate. Equivariant ordinary cohomology with Burnside ring coefficients assigns to each virtual representation α in RO(G) a contravariant functor $\underline{H}_G^\alpha$ from the homotopy category of based G-spaces to the category of Mackey functors. It also assigns a suspension natural isomorphism

$$\underline{H}_G^{\alpha+V}(\Sigma^V X) \cong \underline{H}_G^\alpha(X)$$

to each pair (α, V) consisting of a virtual representation α and an actual representation V. The isomorphisms associated to the three pairs (α, V), (α, W), and $(\alpha, V+W)$ are required to satisfy a coherence condition. The functors $\underline{H}_G^\alpha$ are required to be exact in the sense that they convert cofibre sequences into long exact sequences. The dimension axiom requires that $\underline{H}_G^0 S^0$ be the Burnside ring Mackey functor and that $\underline{H}_G^n S^0$ be zero if $n \in \mathbb{Z}$ and $n \neq 0$. If α is a nontrivial virtual representation, then $\underline{H}_G^\alpha S^0$ need not be zero, but it is uniquely determined by the axioms. Note that because $\underline{H}_G^* S^0$ is nonzero in dimensions other than zero, the assertion that the cohomology of certain spaces is free over the cohomology of S^0 is very different from the assertion that the cohomology is free over the coefficient ring. Our cohomology theory is ring valued; that is, any pair of elements drawn from $\underline{H}_G^\alpha X$

and $\underline{H}_G^{\beta}X$ have a cup product which is in $\underline{H}_G^{\alpha+\beta}X$. We will also work with RO(G)-graded, Mackey functor-valued, reduced equivariant ordinary homology with Burnside ring coefficients. This homology theory satisfies the obvious analogs of the cohomology axioms. Also, it has a Hurewicz map, which we use to convert various space level maps into homology classes. Finally, we assume that S^0 and the free orbit G^+ satisfy equivariant Spanier-Whitehead duality [WIR, LMSM]; that is, for any α in RO(G) there are isomorphisms

$$\underline{H}_G^{\alpha}S^0 \cong \underline{H}_{-\alpha}^G S^0 \quad \text{and} \quad \underline{H}_G^{\alpha}G^+ \cong \underline{H}_{-\alpha}^G G^+.$$

The proofs of all our results flow from these basic assumptions. In fact, most of the proofs are simple long exact sequence arguments which would be left to the reader in a paper dealing with a $\mathbb{Z}$-graded, abelian group-valued, nonequivariant cohomology. One of the points of this paper is that these simple techniques work perfectly well in RO(G)-graded, Mackey functor-valued, equivariant cohomology theories and yield useful results. The one serious demand made of the reader is a willingness to work with Mackey functors. When the group is $\mathbb{Z}/p$, these are really very simple objects. Section one is intended as a tutorial on them.

1. MACKEY FUNCTORS FOR $\mathbb{Z}/p$.

Since the language of Mackey functors pervades this paper, this section contains a brief introduction to Mackey functors for the groups $\mathbb{Z}/p$. For any finite group G, a G-Mackey functor M is a contravariant additive functor from the Burnside category B(G) of G to the category Ab of abelian groups [DRE, LE2, LIN]. However, since we are only concerned with $G = \mathbb{Z}/p$, rather than describing B(G) in detail, we simply note that a $\mathbb{Z}/p$-Mackey functor M is determined by two abelian groups, M(G/G) and M(G/e); two maps, a restriction map

$$\rho : M(G/G) \to M(G/e)$$

and a transfer map

$$\tau : M(G/e) \to M(G/G);$$

and an action of G on M(G/e). The trace of this action and the composite $\rho\tau$ are required to be equal by the definition of the composition of maps in B(G); that is, if $x \in M(G/e)$, then

$$\rho\tau(x) = \sum_{g \in G} gx.$$

The abelian groups M(G/G) and M(G/e) are the values of the Mackey functor M at the trivial orbit and the free orbit; or, if one prefers to think in terms of subgroups instead of orbits, the values of M at the group and at the trivial subgroup. For convenience, we abbreviate G/G to 1 and write M(e) for M(G/e). Frequently the G-action on M(e) is trivial; in these cases the composite $\rho\tau$ is just multiplication by p.

A map $f : M \to N$ between Mackey functors consists of homomorphisms

$$f(1) : M(1) \to N(1) \quad \text{and} \quad f(e) : M(e) \to N(e)$$

which commute with ρ and τ in the obvious sense. The map f(e) must also be G-equivariant. The category $\mathfrak{M}$ of Mackey functors is a complete and cocomplete abelian category. The limit or colimit of a diagram in $\mathfrak{M}$ is formed by taking the limit or colimit of the corresponding two diagrams consisting of the abelian groups associated to G/G and to G/e. The maps ρ and τ and the group action on the limit or colimit are the obvious induced maps and action.

We will describe Mackey functors diagramatically in the form

$$\begin{array}{c} M(1) \\ \rho \downarrow \quad \uparrow \tau \\ M(e) \\ \circlearrowleft \\ \theta \end{array}$$

where M(1) and M(e) will be replaced by the appropriate abelian groups, ρ and τ may be replaced by explicit descriptions of the restriction and transfer maps, and θ may be replaced by an explicit description of the group action. If ρ or τ is replaced by a number (usually 0, 1, or p), then the map is just multiplication by that number. If θ is omitted or replaced by 1, then the group action on M(e) is trivial. If $p = 2$ and θ is replaced by -1, then the generator of $G = \mathbb{Z}/2$ acts by multiplication by -1.

EXAMPLES 1.1 The following Mackey functors and maps appear repeatedly in our cohomology computations.

(a) The Burnside ring Mackey functor A is given by

$$\begin{array}{c} \mathbb{Z}\oplus\mathbb{Z} \\ (1,p) \downarrow \quad \uparrow (0,1) \\ \mathbb{Z} \end{array}$$

where the notation (1,p) means that the restriction map ρ is the identity on the first component and multiplication by p on the second. Similarly, (0,1) means that the transfer map is the inclusion into the second factor. For any Mackey functor M, there is a one-to-one correspondence between maps $f : A \to M$ and elements of M(1). The correspondence relates the map f to the element f(1)((1,0)) of M(1). It follows from this correspondence that A is a projective Mackey functor.

(b) The d-twisted Burnside ring Mackey functor A[d] is given by

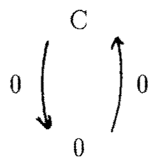
$$\begin{array}{c} \mathbb{Z}\oplus\mathbb{Z} \\ (d,p) \downarrow \quad \uparrow (0,1) \\ \mathbb{Z} \end{array}$$

where d $\in \mathbb{Z}$. Note that A = A[1]. If $d \equiv \pm d' \bmod p$, then there is an isomorphism $f : A[d] \cong A[d']$ of Mackey functors. The map f(e) is the identity and if $d' = \pm d + np$, then

$$f(1)(1,0) = (\pm 1,n) \in \mathbb{Z} \oplus \mathbb{Z}$$
$$f(1)(0,1) = (0,1).$$

If $d \equiv 0 \bmod p$, then A[d] decomposes as the sum of two other Mackey functors; thus A[d] is only of interest when $d \not\equiv 0 \bmod p$. In this case, it is a projective Mackey functor. An alternative $\mathbb{Z}$-basis for A[d](1) will be used in some of our cohomology calculations. To distinguish the two bases, we denote (1,0) and (0,1) in the present basis by μ and τ respectively. Select integers a and b such that $ad + bp = 1$. The alternative $\mathbb{Z}$-basis consists of $\sigma = a\mu + b\tau$ and $\kappa = p\mu - d\tau$. Note that $\rho(\sigma) = 1$, $\rho(\kappa) = 0$, and $\tau(1) = \tau$. In fact, κ generates the kernel of ρ, and τ generates the image of the map τ for which it is named. Of course, σ depends on the choice of a and b; in our applications, these choices will always be specified.

(c) If C is any abelian group, then we use $\langle C \rangle$ to denote the Mackey functor described by the diagram



(d) If d_1 and d_2 are integers prime to p, then there is an isomorphism

$$g_{12} : A[d_1] \oplus \langle \mathbb{Z} \rangle \to A[d_2] \oplus \langle \mathbb{Z} \rangle.$$

Let μ_i and τ_i be the standard generators for $A[d_i]$, and let z_1 and z_2 be generators of $\langle \mathbb{Z} \rangle(1)$ in the domain and range of g_{12}. Select integers a_i and b_i such that $a_i d_i + b_i p = 1$, for $i = 1$ or 2. The map $g_{12}(e) : \mathbb{Z} \to \mathbb{Z}$ is the identity map, and the map $g_{12}(1)$ is given by

$$g_{12}(1)(\mu_1) = d_1(a_2\mu_2 + b_2\tau_2) + (b_1 + b_2 - b_1 b_2 p) z_2$$
$$g_{12}(1)(\tau_1) = \tau_2$$

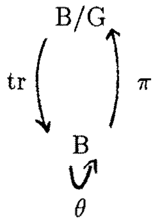
and

$$g_{12}(1)(z_1) = p\mu_2 - d_2\tau_2 - a_1 d_2 z_2.$$

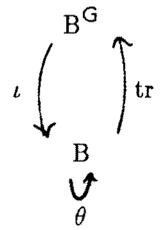
The inverse of g_{12} is just g_{21}. The existence of this isomorphism will explain an apparent inconsistency in our description of the equivariant cohomology of projective spaces.

(e) Associated to an abelian group B with a G-action, we have the Mackey functors L(B) and R(B) given by

L(B)



R(B)

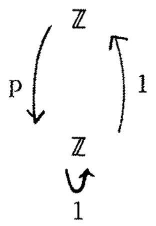


Here, $\iota : B^G \to B$ is the inclusion of the fixed point subgroup and $\pi : B \to B/G$ is the projection onto the orbit group. The two maps tr are variants of the trace map. The map $\mathrm{tr} : B \to B^G$ takes $x \in B$ to $\sum_{g\epsilon G} gx \in B^G$. If $x \in B$ and $[x]$ is the associated equivalence class in B/G, then $\mathrm{tr} : B/G \to B$ is given by

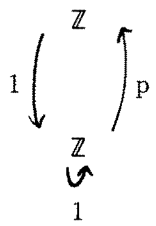
$$\mathrm{tr}([x]) = \sum_{g\epsilon G} gx \in B.$$

These two constructions give functors from the category of $\mathbb{Z}[G]$-modules to the category of Mackey functors. These functors are the left and right adjoints to the obvious forgetful functor from the category of Mackey functors to the category of $\mathbb{Z}[G]$-modules. We will encounter these functors most often when B is $\mathbb{Z}$ with the trivial action or, if $p = 2$, with the sign action. Denote the resulting Mackey functors by L, R, L_-, and R_-. These functors are described by the diagrams

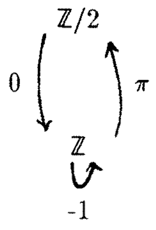
L



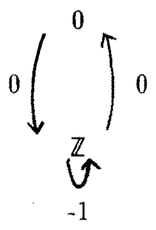
R



L_-



R_-



If C is any abelian group, there is an obvious permutation action of G on C^p, the direct sum of p copies of C. Unless otherwise indicated, this action is assumed when we refer to $L(C^p)$ or $R(C^p)$. These two functors are isomorphic and are described by the diagram

$$\begin{array}{c} C \\ \Delta \Big\downarrow \Big\uparrow \nabla \\ C^P \\ \circlearrowleft \\ \theta \end{array}$$

where Δ is the diagonal map, ∇ is the folding map, and θ is the permutation action.

(f) If M is a Mackey functor, then $L(M(e)^P) \cong R(M(e)^P)$ is denoted M_G. There are two reasonable choices of a G action on $M(e)^P$, the permutation action or the composite of the permutation action and the given action of G on each factor M(e). These actions yield isomorphic $\mathbb{Z}[G]$-modules, so the choice is not important. The simple permutation action is always assumed here. The assignment of M_G to M is a special case of an important construction in induction theory [DRE, LE2] that assigns a Mackey functor M_b to each object b of B(G) and each Mackey functor M.

The restriction map $\rho: M(1) \to M(e) \cong M_G(1)$ and the diagonal map $\Delta: M(e) \to M(e)^P \cong M_G(e)$ form a natural transformation ρ from M to M_G. Similarly, $\tau: M_G(1) \cong M(e) \to M(1)$ and the folding map $\nabla: M_G(e) \cong M(e)^P \to M(e)$ form a natural transformation $\tau: M_G \to M$. The Mackey functor $A_G = L(\mathbb{Z}^P)$ is characterized by the fact that, for any Mackey functor M, there is a one-to-one correspondence between maps $f: A_G \to M$ and elements of M(e). This correspondence relates the map f to the element $f(e)((1,0,0, \ldots ,0))$ of M(e). It follows that A_G is a projective Mackey functor.

(g) If Y is a G-space, M is a Mackey functor, $\alpha \in RO(G)$, and $H_G^\alpha(Y;M)$ and $H_\alpha^G(Y;M)$ denote the abelian group-valued equivariant ordinary cohomology and homology of Y with coefficients M in dimension α, then the Mackey functor valued cohomology $\underline{H}_G^\alpha(Y;M)$ and homology $\underline{H}_\alpha^G(Y;M)$ are described by the diagrams

$$\begin{array}{c} H_G^\alpha(Y;M) \\ \pi^* \Big\downarrow \Big\uparrow \pi_! \\ H_G^\alpha(G \times Y;M) \\ \circlearrowleft \\ \theta \end{array} \qquad \text{and} \qquad \begin{array}{c} H_\alpha^G(Y;M) \\ \pi^! \Big\downarrow \Big\uparrow \pi_* \\ H_\alpha^G(G \times Y;M) \\ \circlearrowleft \\ \theta \end{array}$$

where the maps π^* and π_* are induced by the projection $\pi : G \times Y \to Y$, and the maps $\pi_!$ and $\pi^!$ are the transfer maps arising from regarding the projection π as a covering space. The group $H_G^\alpha(G \times Y;M)$ is isomorphic to the nonequivariant cohomology group $H^{|\alpha|}(Y;M(e))$. If α is represented by the difference $V - W$ of representations V and W, then, under this isomorphism, the action of an element g of

G on $H^{\alpha}_{G}(G \times Y; M)$ may be described as the composite of multiplication by the degrees of the maps $g: S^V \to S^V$ and $g: S^W \to S^W$ and the actions of g on $H^{|\alpha|}(Y; M(e))$ induced by the action of g on M(e) and the action of g^{-1} on Y. Similar remarks apply in homology. If no coefficient Mackey functor M is indicated in equivariant cohomology or homology, then Burnside ring coefficients are intended.

(h) For any Mackey functor M and any abelian group B, the Mackey functor $M \otimes B$ has value $M(G/H) \otimes B$ for the orbit G/H and the obvious restriction, transfer, and action by G. If M^* is an RO(G)-graded G-Mackey functor and B^* is a $\mathbb{Z}$-graded abelian group, then $M^* \otimes B^*$ is the RO(G)-graded G-Mackey functor defined by

$$(M^* \otimes B^*)^{\alpha} = \sum_{\beta + n = \alpha} M^{\beta} \otimes B^n.$$

If a CW complex Y with cells only in even dimensions is regarded as a G-space by assigning it the trivial G-action, then there is an isomorphism of RO(G)-graded Mackey functors

$$\underline{H}^*_G Y \cong \underline{H}^*_G S^0 \otimes \underline{H}^*(Y; \mathbb{Z})$$

which preserves cup products.

For any finite group G, there is a box product operation $\Box$ on the category $\mathfrak{M}$ of G-Mackey functors which behaves like the tensor product on the category of abelian groups. In particular, $\mathfrak{M}$ is a symmetric monoidal closed category under the box product. The Burnside ring Mackey functor A is the unit for $\Box$. If $G = \mathbb{Z}/p$, then the box product $M \Box N$ of Mackey functors M and N is described by the diagram

$$\begin{array}{c} \big[M(1) \otimes N(1) \,\oplus\, M(e) \otimes N(e)\big]/\approx \\ \rho \big\downarrow \quad \big\uparrow \tau \\ M(e) \otimes N(e) \\ \circlearrowleft \\ \theta \end{array}$$

The equivalence relation $\approx$ is given by

$$x \otimes \tau y \approx \rho x \otimes y \quad \text{for } x \in M(1) \text{ and } y \in N(e)$$

$$\tau v \otimes w \approx v \otimes \rho w \quad \text{for } v \in M(e) \text{ and } w \in N(1).$$

The action θ of G on $M(e) \otimes N(e)$ is just the tensor product of the actions of G on M(e) and N(e). The map τ is derived from the inclusion of $M(e) \otimes N(e)$ as a summand of the direct sum used to define $M \Box N(1)$. The map ρ is induced by $\rho \otimes \rho$ on the first summand and the trace map of the action θ on the second.

EXAMPLES 1.2(a) For any integers d_1 and d_2, there is an isomorphism

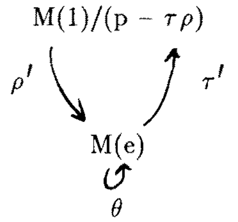
$$A[d_1] \Box A[d_2] \cong A[d_1 d_2]$$

of Mackey functors.

(b) If B is a $\mathbb{Z}[G]$-module and M is any Mackey functor, then there is an isomorphism

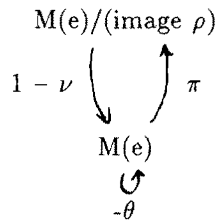
$$L(B) \square M \;\cong\; L(B \otimes M(e)).$$

(c) For any Mackey functor M, the product $R \square M$ is described by the diagram



where $M(1)/(p - \tau\rho)$ is the cokernel of the difference between the multiplication by p map and the composite $\tau\rho$. The maps ρ' and τ' are induced by the restriction and transfer maps for M. In particular, if $M = R(B)$ for some $\mathbb{Z}[G]$-module B, then $R \square R(B) \cong R(B)$. Also, for any abelian group C, $R \square <C> \cong <C/pC>$.

(d) If $p = 2$, then for any Mackey functor M, the product $R_- \square M$ is described by the diagram



Here $\pi: M(e) \to M(e)/(\text{image } \rho)$ is the projection onto the cokernel of the restriction map and $\nu: M(e) \to M(e)$ describes the action of the nontrivial element of G on M(e). The action $-\theta$ is the composite of the given action θ of G on M(e) and the sign action of G on M(e). In particular, $R_- \square R_- \cong L$.

(e) For any abelian group C and any Mackey functor M,

$$<C> \square M \;\cong\; <C \otimes (M(1)/\text{image } \tau)>.$$

A Mackey functor ring (or Green functor [DRE, LE2]) is a Mackey functor S together with a multiplication map $\mu : S \square S \to S$ and a unit map $\eta : A \to S$ making the appropriate diagrams commute. A module over S is just a Mackey functor M together with an action map $\xi : S \square M \to M$ making the appropriate diagrams commute. Since the Burnside ring Mackey functor A is the unit for $\square$, it is a Mackey functor ring whose multiplication is the isomorphism $A \square A \to A$ and whose unit is the identity map $A \to A$. Every Mackey functor is a module over A with action map the isomorphism $A \Box M \to M$. Note that if S is a Mackey functor ring and R is a ring, then the Mackey functor $S \otimes R$ of Examples 1.1(h) is a Mackey functor ring. Similar remarks apply in the graded case. The cohomology of any G-space Y with coefficients a Mackey functor ring S is an RO(G)-graded Mackey functor ring whose multiplication is given by maps

$$\underline{H}_G^\alpha(Y;S) \Box \underline{H}_G^\beta(Y;S) \to \underline{H}_G^{\alpha+\beta}(Y;S),$$

for α and β in RO(G).

The following result characterizes maps out of box products and allows us to describe a Mackey functor ring S in terms of S(1) and S(e). This is the approach to Mackey functor rings used in our discussion of the ring structure of the cohomology of complex projective spaces.

PROPOSITION 1.3 For any Mackey functors M, N and P, there is a one-to-one correspondence between maps $h : M \Box N \to P$ and pairs $H = (H_1, H_e)$ of maps

$$H_1 : M(1) \otimes N(1) \to P(1)$$

$$H_e : M(e) \otimes N(e) \to P(e)$$

such that, for $x \in M(1)$, $y \in N(1)$, $z \in M(e)$, and $w \in N(e)$,

$$\begin{aligned} H_e(\rho x \otimes \rho y) &= \rho H_1(x \otimes y) \\ H_1(\tau z \otimes y) &= \tau H_e(z \otimes \rho y) \\ H_1(x \otimes \tau w) &= \tau H_e(\rho x \otimes w). \end{aligned}$$

The second and third of these equations are called the Frobenius relations.

PROOF. The maps H_e and h are related by $H_e = h(e)$. Given h, H_1 is derived in an obvious way from h(1) using the definition of $M \Box N$. Given H_1 and H_e, h(1) is constructed from the maps H_1 and τH_e on the two summands used to define $M \Box N(1)$.

It follows easily from the proposition that, if S is a Mackey functor ring, then S(1) and S(e) are rings, $\rho : S(1) \to S(e)$ is a ring homomorphism, and $\tau : S(e) \to S(1)$ is an S(1)-module map when S(e) is considered an S(1)-module via ρ. Moreover, if M is a Mackey functor module over S, then M(1) is an S(1)-module and M(e) is an S(e)-module. If we regard M(e) as an S(1)-module via $\rho : S(1) \to S(e)$, then the maps $\rho : M(1) \to M(e)$ and $\tau : M(e) \to M(e)$ are S(1)-module maps.

2. $\underline{H}_G^* S^0$ AND SPACES WITH FREE COHOMOLOGY.

Here, we recall Stong's unpublished description of the additive structure of the RO(G)-graded equivariant ordinary cohomology of S^0. We use this to show that if X is a generalized G-cell complex constructed from suitable even-dimensional cells, then $\underline{H}_G^* X$ and $\underline{H}_*^G X$ are free over $\underline{H}_G^* S^0$. The additive structure of the cohomology $\underline{H}_G^* G^+$ of the free orbit is also described. This is used to show that $\underline{H}_G^* X$ and $\underline{H}_*^G X$ are projective over $\underline{H}_G^* S^0$

when X is constructed from a slightly more general class of even-dimensional cells.

Since $\mathbb{Z}/2$ has only one nontrivial irreducible representation, $\underline{\mathrm{H}}_{\mathrm{G}}^{*}\mathrm{S}^0$ is very easy to describe when $\mathrm{G} = \mathbb{Z}/2$.

THEOREM 2.1. If $\mathrm{G} = \mathbb{Z}/2$ and $\alpha \in \mathrm{RO(G)}$, then

$$\underline{\mathrm{H}}_{\mathrm{G}}^{\alpha}\mathrm{S}^0 = \begin{cases} \mathrm{A}, & \text{if } |\alpha| = |\alpha^{\mathrm{G}}| = 0, \\ \mathrm{R}, & \text{if } |\alpha| = 0,\ |\alpha^{\mathrm{G}}| < 0, \text{ and } |\alpha^{\mathrm{G}}| \text{ is even}, \\ \mathrm{R}_{-}, & \text{if } |\alpha| = 0,\ |\alpha^{\mathrm{G}}| \leq 1, \text{ and } |\alpha^{\mathrm{G}}| \text{ is odd}, \\ \mathrm{L}, & \text{if } |\alpha| = 0,\ |\alpha^{\mathrm{G}}| > 0, \text{ and } |\alpha^{\mathrm{G}}| \text{ is even}, \\ \mathrm{L}_{-}, & \text{if } |\alpha| = 0,\ |\alpha^{\mathrm{G}}| > 1, \text{ and } |\alpha^{\mathrm{G}}| \text{ is odd}, \\ \langle\mathbb{Z}\rangle, & \text{if } |\alpha| \neq 0 \text{ and } |\alpha^{\mathrm{G}}| = 0, \\ \langle\mathbb{Z}/2\rangle, & \text{if } |\alpha| > 0,\ |\alpha^{\mathrm{G}}| < 0, \text{ and } |\alpha^{\mathrm{G}}| \text{ is even}, \\ \langle\mathbb{Z}/2\rangle, & \text{if } |\alpha| < 0,\ |\alpha^{\mathrm{G}}| > 1, \text{ and } |\alpha^{\mathrm{G}}| \text{ is odd}, \\ 0, & \text{otherwise}. \end{cases}$$

The most effective way to visualize $\underline{\mathrm{H}}_{\mathrm{G}}^{*}\mathrm{S}^0$ is to display $\underline{\mathrm{H}}_{\mathrm{G}}^{\alpha}\mathrm{S}^0$ for various α on a coordinate plane in which the horizontal and vertical coordinates specify $|\alpha^{\mathrm{G}}|$ and $|\alpha|$, respectively. In such a plot, given as Table 2.2 below, the zero values of $\underline{\mathrm{H}}_{\mathrm{G}}^{*}\mathrm{S}^0$ are indicated by blanks. The only values in this plot with odd horizontal coordinate are the R_{-} and L_{-} on the horizontal axis and the $\langle\mathbb{Z}/2\rangle$ in the fourth quadrant.

	⋮		⋮		⋮		⋮							
⋯	$\langle\mathbb{Z}/2\rangle$		$\langle\mathbb{Z}/2\rangle$		$\langle\mathbb{Z}/2\rangle$		$\langle\mathbb{Z}\rangle$							
⋯	$\langle\mathbb{Z}/2\rangle$		$\langle\mathbb{Z}/2\rangle$		$\langle\mathbb{Z}/2\rangle$		$\langle\mathbb{Z}\rangle$							
⋯	$\langle\mathbb{Z}/2\rangle$		$\langle\mathbb{Z}/2\rangle$		$\langle\mathbb{Z}/2\rangle$		$\langle\mathbb{Z}\rangle$							
⋯	$\langle\mathbb{Z}/2\rangle$		$\langle\mathbb{Z}/2\rangle$		$\langle\mathbb{Z}/2\rangle$		$\langle\mathbb{Z}\rangle$							
⋯	R	R_{-}	R	R_{-}	R	R_{-}	A	R_{-}	L	L_{-}	L	L_{-}	L	⋯
							$\langle\mathbb{Z}\rangle$			$\langle\mathbb{Z}/2\rangle$		$\langle\mathbb{Z}/2\rangle$		⋯
							$\langle\mathbb{Z}\rangle$			$\langle\mathbb{Z}/2\rangle$		$\langle\mathbb{Z}/2\rangle$		⋯
							$\langle\mathbb{Z}\rangle$			$\langle\mathbb{Z}/2\rangle$		$\langle\mathbb{Z}/2\rangle$		⋯
							$\langle\mathbb{Z}\rangle$			$\langle\mathbb{Z}/2\rangle$		$\langle\mathbb{Z}/2\rangle$		⋯
							⋮			⋮		⋮		

TABLE 2.2. $\mathrm{H}_{\mathrm{G}}^{*}\mathrm{S}^0$ for $\mathrm{p} = 2$.

Even though the representation ring of G is much more complicated when $\mathrm{p} \neq 2$, $\underline{\mathrm{H}}_{\mathrm{G}}^{\alpha}\mathrm{S}^0$ is completely determined by the integers $|\alpha|$ and $|\alpha^{\mathrm{G}}|$ except in the special case where $|\alpha| = |\alpha^{\mathrm{G}}| = 0$. In this special case, $\underline{\mathrm{H}}_{\mathrm{G}}^{\alpha}\mathrm{S}^0$ is A[d] for some integer d which depends on α. Unfortunately, because of the isomorphism described in Examples 1.1(b), d is only determined up to a multiple of p. The major source of unpleasantness in the description of the multiplicative structure of the equivariant cohomology of a point and of complex projective spaces is this lack of a canonical choice for d. To explain the relation between α and d, we introduce several relatives of the representation ring. Let R(G) be the complex representation ring of G and RSO(G) be the ring of SO-isomorphism classes of SO-representations of G. Since any real representation of G is also an SO-representation, the difference between RO(G) and RSO(G) is that, in RSO(G), equivalences between representations are required to preserve underlying nonequivariant orientations on the representation spaces. The difference between R(G) and RSO(G) is that elements of RSO(G) may contain an odd number of copies of the trivial one-dimensional real representation of G. Let $R_0(G)$, $RO_0(G)$, and $RSO_0(G)$ denote the subrings of R(G), RO(G), and RSO(G) containing those virtual representations α with $|\alpha| = |\alpha^G| = 0$. Note that $R_0(G) = RSO_0(G)$. Let $\tilde{R}_0(G)$ be the free abelian monoid generated by the formal differences $\phi - \eta$ of complex isomorphism classes of nontrivial irreducible complex representations. Note that $R_0(G)$ is the quotient of $\tilde{R}_0(G)$ obtained by allowing the obvious cancellations and that $RO_0(G)$ is the quotient of $R_0(G)$ obtained by identifying conjugate representations. Let λ be the irreducible complex representation which sends the standard generator of $\mathbb{Z}/p$ to $e^{2\pi i/p}$. The monoid $\tilde{R}_0(G)$ is generated by elements of the form $\lambda^m - \lambda^n$, where $1 \leq m, n \leq p-1$. Define a homomorphism from $\tilde{R}_0(G)$ to $\mathbb{Z}$, regarded as a monoid under multiplication, by sending the generator $\lambda^m - \lambda^n$ to $m(n^{-1})$, where n^{-1} denotes the unique integer such that $1 \leq n^{-1} \leq p-1$ and $n(n^{-1}) \equiv 1 \bmod p$. Define functions from $RSO_0(G)$ and $RO_0(G)$ into $\mathbb{Z}$ by composing this homomorphism with sections of the projections from $\tilde{R}_0(G)$ to $RSO_0(G)$ or $RO_0(G)$. Let d_α denote the integer assigned to the virtual representation α by either map. The sections can not be chosen to be homomorphisms, so the assignment of d_α to α will not be a homomorphism from $RSO_0(G)$ or $RO_0(G)$ to the multiplicative monoid $\mathbb{Z}$. However, the assignment of d_α to α does give a homomorphism from $R_0(G)$ to the group of units $(\mathbb{Z}/p)^*$ of $\mathbb{Z}/p$ and a homomorphism from $RO_0(G)$ to the quotient $(\mathbb{Z}/p)^*/\{\pm 1\}$ of $(\mathbb{Z}/p)^*$. For later convenience, we select our sections so that d_0 is 1.

Stong's description of the additive structure of $\underline{H}_G^* S^0$ can now be translated into the Mackey functor language of section one.

THEOREM 2.3. If p is odd, then

$$\underline{H}_G^\alpha S^0 = \begin{cases} A[d_\alpha] & \text{if } |\alpha| = |\alpha^G| = 0 \\ R & \text{if } |\alpha| = 0 \text{ and } |\alpha^G| < 0 \\ L & \text{if } |\alpha| = 0 \text{ and } |\alpha^G| > 0 \\ \langle \mathbb{Z} \rangle & \text{if } |\alpha| \neq 0 \text{ and } |\alpha^G| = 0 \\ \langle \mathbb{Z}/p \rangle & \text{if } |\alpha| > 0,\ |\alpha^G| < 0, \text{ and } |\alpha^G| \text{ is an even integer} \\ \langle \mathbb{Z}/p \rangle & \text{if } |\alpha| < 0,\ |\alpha^G| > 1, \text{ and } |\alpha^G| \text{ is an odd integer} \\ 0 & \text{otherwise} \end{cases}$$

As in the case $p = 2$, $\underline{H}^*_G S^0$ is best visualized by plotting it on a coordinate plane whose horizontal and vertical axes specify $|\alpha^G|$ and $|\alpha|$ respectively. In this plot, given as Table 2.4 below, the zero values of $\underline{H}^*_G S^0$ are indicated by blanks. The vertical and horizontal coordinates of all the nonzero values, except the $\langle \mathbb{Z}/p \rangle$ values in the fourth quadrant, are even. Notice in the plots for both the odd primes and 2 that the vanishing of $\underline{H}^*_G S^0$ on the vertical line $|\alpha^G| = 1$ (for $|\alpha| \neq 0$ if $p = 2$) is unlike its behavior on the vertical lines corresponding to the other odd positive values for $|\alpha^G|$. These unusual zeroes for $\underline{H}^\alpha_G S^0$ are the key to our freeness and projectivity results. When $G = \mathbb{Z}/p^n$ for $n > 1$, the corresponding values are not zero, so our techniques do not extend to these groups.

Hereafter, we will often describe elements in $\underline{H}^*_G S^0$ by their position in these plots. For example, we may refer to the torsion in the fourth quadrant or the copies of $\langle \mathbb{Z} \rangle$ on the positive vertical axis.

	⋮	⋮	⋮	⋮						
···	$\langle \mathbb{Z}/p \rangle$	$\langle \mathbb{Z}/p \rangle$	$\langle \mathbb{Z}/p \rangle$	$\langle \mathbb{Z} \rangle$						
···	$\langle \mathbb{Z}/p \rangle$	$\langle \mathbb{Z}/p \rangle$	$\langle \mathbb{Z}/p \rangle$	$\langle \mathbb{Z} \rangle$						
···	$\langle \mathbb{Z}/p \rangle$	$\langle \mathbb{Z}/p \rangle$	$\langle \mathbb{Z}/p \rangle$	$\langle \mathbb{Z} \rangle$						
···	R	R	R	$A[d_\alpha]$	L		L		L	···
						$\langle \mathbb{Z}/p \rangle$		$\langle \mathbb{Z}/p \rangle$		···
				$\langle \mathbb{Z} \rangle$						
						$\langle \mathbb{Z}/p \rangle$		$\langle \mathbb{Z}/p \rangle$		···
				$\langle \mathbb{Z} \rangle$						
						$\langle \mathbb{Z}/p \rangle$		$\langle \mathbb{Z}/p \rangle$		···
				$\langle \mathbb{Z} \rangle$						
				⋮		⋮		⋮		

TABLE 2.4. $\underline{H}^*_G S^0$ for p odd.

Recall, from Examples 1.1(f), the new Mackey functor M_G which can be derived from any Mackey functor M, and the observation that $A_G = L(\mathbb{Z}^p) = R(\mathbb{Z}^p)$. It is easy to check that $\underline{H}^\alpha_G G^+$ is $\underline{H}^\alpha_G(S^0)_G$, and from this, to compute $\underline{H}^*_G G^+$.

COROLLARY 2.5. For any prime p,

$$\underline{H}_G^* G^+ = \begin{cases} A_G & \text{if } |\alpha| = 0 \\ 0 & \text{otherwise} \end{cases}$$

Proposition 4.12 tells us that $\underline{H}_G^* G^+$ is an RO(G)-graded projective module over $\underline{H}_G^* S^0$, and that any map

$$f: \underline{H}_G^* G^+ \to M^*$$

of RO(G)-graded modules over $\underline{H}_G^* S^0$ is completely determined by the image of $(1,0,0,\ldots,0) \in \mathbb{Z}^p = \underline{H}_G^0(G^+)(e)$ in $M^0(e)$.

A generalized G-cell complex X is a G-space X together with an increasing sequence of subspaces X_n of X such that X_0 is a single orbit, $X = \cup X_n$, X has the colimit (or weak) topology from the X_n, and X_{n+1} is formed from X_n by attaching G-cells. We will allow two types of G-cells. If V is a G-representation and DV and SV are the unit disk and sphere of V, then the first type of allowed cell is a copy of DV attached to X_n by a G-map from SV to X_n. The second type of cell is a copy of $G \times e^m$, where e^m is the unit m-disk with trivial G action, attached to X_n by a G-map from $G \times S^{m-1}$ to X_n. For each n, we let J_{n+1} denote the set of cells added to X_n to form X_{n+1}. Regard a cell DV of the first type as even-dimensional if $|V|$ and $|V^G|$ are even. Regard a cell $G \times e^m$ as even dimensional if m is even.

THEOREM 2.6. Let X be a generalized G-cell complex with only even-dimensional cells.

(a) Assume that $X_0 = *$ and all the cells of X are of the first type; that is, disks DV of G-representations V. Assume also that $|V^G| \geq |W^G|$ whenever $DV \in J_n$, $DW \in J_k$, $1 \leq k < n$, and $|V| > |W|$. Then $\underline{H}_G^* X^+$ is a free RO(G)-graded module over $\underline{H}_G^* S^0$ with one generator in dimension 0 and one generator in dimension V for each $DV \in J_n$, $n \geq 1$. The homology $\underline{H}_*^G X^+$ of X is also a free RO(G)-graded module over $\underline{H}_G^* S^0$ with generators in the same dimensions.

(b) If X contains cells of both types and all the cells of X of the first type satisfy the condition in part (a), then $\underline{H}_G^* X^+$ is a projective RO(G)-graded module over $\underline{H}_G^* S^0$. Moreover, $\underline{H}_G^* X^+$ is the sum of one copy of $\underline{H}_G^* X_0^+$, which is $\underline{H}_G^* S^0$ or $\underline{H}_G^* G^+$, in dimension 0, one copy of $\underline{H}_G^* S^0$ in dimension V for each $DV \in J_n$, and one copy of $\underline{H}_G^* G^+$ in dimension 2k for each $G \times e^{2k} \in J_n$, $n \geq 1$. The homology $\underline{H}_*^G X^+$ of X is also a projective RO(G)-graded module over $\underline{H}_G^* S^0$ and decomposes into the same summands.

PROOF. Abusing notation, we let J_{n+1} denote both the set of cells to be added to X_n and the space consisting of the disjoint union of those cells. Let ∂J_{n+1} denote the space consisting of the disjoint union of the boundaries of the cells in J_{n+1}. Associated to the cofibre sequence

$$X_n^+ \to X_{n+1}^+ \to J_{n+1}/\partial J_{n+1},$$

we have the long exact sequences

$$\cdots \to \underline{H}_\alpha^G X_{n+1}^+ \to \underline{H}_\alpha^G(J_{n+1}/\partial J_{n+1}) \xrightarrow{\partial} \underline{H}_{\alpha-1}^G X_n^+ \to \cdots$$

and

$$\cdots \to \underline{H}_G^\alpha X_{n+1}^+ \to \underline{H}_G^\alpha X_n^+ \xrightarrow{\partial} \underline{H}_G^{\alpha+1}(J_{n+1}/\partial J_{n+1}) \to \cdots.$$

The space $J_{n+1}/\partial J_{n+1}$ is a wedge of one copy of S^V for each $DV \in J_{n+1}$ and one copy of $G^+ \wedge S^{2k}$ for each $G \times e^{2k} \in J_{n+1}$. Thus, $\underline{H}_G^*(J_{n+1}/\partial J_{n+1})$ and $\underline{H}_*^G(J_{n+1}/\partial J_{n+1})$ are projective modules over $\underline{H}_G^* S^0$ with generators in dimensions corresponding to the cells added to X_n to form X_{n+1}. Moreover, if J_{n+1} contains only cells of the first type, then $\underline{H}_G^*(J_{n+1}/\partial J_{n+1})$ and $\underline{H}_*^G(J_{n+1}/\partial J_{n+1})$ are free modules over $\underline{H}_G^* S^0$. The space X_0 is either a point or the free orbit G, so $\underline{H}_G^* X_0^+$ and $\underline{H}_*^G X_0^+$ are projective, and perhaps free, modules over $\underline{H}_G^* S^0$ generated by single elements in dimension 0.

We will show inductively that the boundary maps ∂ in both long exact sequences are zero. The long exact sequences must then break up into short exact sequences which split by the projectivity of $\underline{H}_*^G(J_{n+1}/\partial J_{n+1})$ and $\underline{H}_G^* X_n^+$. Thus, by induction, $\underline{H}_G^* X_n^+$ and $\underline{H}_*^G X_n^+$ are free or projective, as appropriate, over $\underline{H}_G^* S^0$, with the indicated generators. It follows by the usual colimit argument that $\underline{H}_*^G X^+$ is free, or projective, with the appropriate generators. Since the map

$$\underline{H}_G^\alpha X_{n+1}^+ \to \underline{H}_G^\alpha X_n^+$$

is always a surjection, the appropriate $\lim^1$ term vanishes, and the cohomology of X, being the limit of the cohomologies of the X_n, is free (or projective) with the appropriate generators.

The graded Mackey functors $\underline{H}_G^*(J_{n+1}/\partial J_{n+1})$, $\underline{H}_*^G(J_{n+1}/\partial J_{n+1})$, $\underline{H}_G^* X_0^+$ and $\underline{H}_*^G X_0^+$ are sums of copies of $\underline{H}_G^* S^0$ and $\underline{H}_G^* G^+$ in various dimensions. By induction, we may assume that $\underline{H}_G^* X_n^+$ and $\underline{H}_*^G X_n^+$ are also of this form. To show that the maps ∂ are zero, it therefore suffices to show that they are zero from each summand of the domain to each summand of the range. For the cohomology sequence, the four possibilities for the summands and the map between them are:

$$\underline{H}_G^{*-2k}G^+ \cong \underline{H}_G^*(G^+ \wedge S^{2k}) \to \underline{H}_G^{*+1}(G^+ \wedge S^{2m}) \cong \underline{H}_G^{*+1-2m}G^+$$

$$\underline{H}_G^{*-W}S^0 \cong \underline{H}_G^* S^W \to \underline{H}_G^{*+1}(G^+ \wedge S^{2m}) \cong \underline{H}_G^{*+1-2m}G^+$$

$$\underline{H}_G^{*-2k}G^+ \cong \underline{H}_G^*(G^+ \wedge S^{2k}) \to \underline{H}_G^{*+1}S^V \cong \underline{H}_G^{*+1-V}S^0$$

and

$$\underline{H}_G^{*-W}S^0 \cong \underline{H}_G^* S^W \to \underline{H}_G^{*+1}S^V \cong \underline{H}_G^{*+1-V}S^0.$$

Here, we use $\underline{H}_G^*(G^+ \wedge S^{2k})$ and $\underline{H}_G^* S^W$ to denote summands of $\underline{H}_G^* X_n^+$ isomorphic to $\underline{H}_G^* G^+$ in dimension 2k or $\underline{H}_G^* S^0$ in dimension W. The four maps above are all maps of RO(G)-graded modules over $\underline{H}_G^* S^0$. Any such map out of $\underline{H}_G^* S^0$ is determined by the image of $1 \in A(1) = \underline{H}_G^0(S^0)(1)$. By Proposition 4.12, such a map out of $\underline{H}_G^* G^+$ is determined by the image of $(1,0,0, \dots ,0) \in \mathbb{Z}^p = \underline{H}_G^0(G^+)(e)$. Thus, to show that the four maps are zero, it suffices to show that the groups $\underline{H}_G^{2k+1-2m}(G^+)(e)$, $\underline{H}_G^{W+1-2m}(G^+)(1)$, $\underline{H}_G^{2k+1-V}(S^0)(e)$, and $\underline{H}_G^{W+1-V}(S^0)(1)$ are zero. The integers $|2k+1-2m|$ and $|W+1-2m|$ are odd and $\underline{H}_G^\alpha G^+$ vanishes whenever $|\alpha|$ is odd, so the first two groups are zero. The integer $|2k+1-V|$ is odd and $\underline{H}_G^\alpha(S^0)(e)$ vanishes when $|\alpha|$ is odd, so the third group is zero. For the fourth group, if $|V| \leq |W|$, then $\underline{H}_G^{W+1-V}S^0$ is zero because $|W^G+1-V^G|$ is odd and $|W+1-V|$ is positive. Otherwise, $|V^G| \geq |W^G|$, and $\underline{H}_G^{W+1-V}S^0$ is zero because $|W^G+1-V^G|$ is at most one. An analogous proof shows that the map ∂ in the homology sequence is zero. Note that if $|V| > |W|$ and $|V^G| = |W^G|$, then the vanishing of $\underline{H}_G^{W+1-V}S^0$ is a result of the anomalous zeroes on the $|\alpha^G| = 1$ line in the graph of $\underline{H}_G^\alpha S^0$.

In order to compute the ring structure of the equivariant cohomology of X, we must compare it with more familiar objects, such as the nonequivariant ordinary cohomology of X and X^G. If X is a generalized G-cell complex satisfying the conditions of either part of Theorem 2.6, then so is X^G. Thus, Examples 1.1(h) describes $\underline{H}_G^*(X^G)^+$ in terms of the nonequivariant cohomology of X^G. Since the group $\underline{H}_G^*(X^+)(e)$ is just the nonequivariant ordinary cohomology of X with $\mathbb{Z}$ coefficients, the map

$$\rho \oplus i^* : \underline{H}_G^\alpha(X^+)(1) \to \underline{H}_G^\alpha(X^+)(e) \oplus \underline{H}_G^\alpha((X^G)^+)(1)$$

offers a comparison between $\underline{H}_G^\alpha(X^+)(1)$ and two more easily understood cohomology rings. This map does not detect the torsion in $\underline{H}_G^*(X^+)(1)$ coming from the fourth quadrant torsion in $\underline{H}_G^* S^0$. Moreover, the torsion in $\underline{H}_G^*((X^G)^+)(1)$ makes it hard to compute the image of $\rho \oplus i^*$. These difficulties suggest that we also consider the image of $\underline{H}_G^*(X^+)(1)/\text{torsion}$ in $(\underline{H}_G^\alpha(X^+)(e) \oplus \underline{H}_G^\alpha((X^G)^+)(1))/\text{torsion}$. Since $\underline{H}_G^*(X^+)(e)$ contains no torsion, in the range we are only collapsing out the torsion in $\underline{H}_G^*((X^G)^+)(1)$. The most useful comparison map is produced by also collapsing out the image of the transfer map τ from $\underline{H}_G^*((X^G)^+)(e)$. The quotient

$$\underline{H}_G^*((X^G)^+)(1)/(\text{torsion} \oplus \text{ im } \tau)$$

consists of copies of $\mathbb{Z}$ in various dimensions; there is one $\mathbb{Z}$ in the quotient for each A[d] or $\langle \mathbb{Z} \rangle$ which appears in $\underline{H}_G^*((X^G)^+)(1)$.

For many spaces, including complex projective spaces with linear actions, the cells can be ordered so that $|V| \geq |W|$ whenever $DV \in J_n$, $DW \in J_k$, and $k < n$. When the cells can be so ordered, there is no torsion in $\underline{H}_G^*(X^+)(1)$ in the dimensions of the generators of $\underline{H}_G^* X^+$ as a module over $\underline{H}_G^* S^0$. Therefore, the collapsing we have done causes a minimal loss of information. The following result describes the extent to which $\underline{H}_G^*(X^+)(1)$ is detected by $\rho \oplus i^*$.

COROLLARY 2.7. Let X be a generalized G-cell complex satisfying the conditions of either part of Theorem 2.6 and let $i\colon X^G \to X$ be the inclusion of the fixed point set. Then, for any $\alpha \in RO(G)$ with $|\alpha|$ even, the map

$$\rho \oplus i^* : \underline{H}_G^\alpha(X^+)(1) \to \underline{H}_G^\alpha(X^+)(e) \oplus \underline{H}_G^\alpha((X^G)^+)(1)$$

is a monomorphism. Moreover, for any $\alpha \in RO(G)$, the map

$$\rho \oplus i^* : (\underline{H}_G^\alpha(X^+)(1))/\text{torsion} \to \underline{H}_G^\alpha(X^+)(e) \oplus (\underline{H}_G^\alpha((X^G)^+)(1))/(\text{torsion} \oplus \text{ im } \tau)$$

is a monomorphism.

PROOF. Since the equivariant cohomology of X is the limit of the cohomologies of the X_n, it suffices to show that the result holds for every X_n. It is easy to check the second part for X_0. Assume the second part for X_n, and let x be an element of $\underline{H}_G^\alpha(X_{n+1}^+)(1)/\text{torsion}$ vanishing under the map into

$$\underline{H}_G^\alpha(X_{n+1}^+)(e) \oplus (\underline{H}_G^\alpha((X_{n+1}^G)^+)(1))/(\text{torsion} \oplus \text{ im } \tau)$$

induced by $\rho \oplus i^*$. We must show that x is zero. The group $\underline{H}_G^\alpha(X_{n+1}^+)(1)$ is the direct sum of the groups $\underline{H}_G^\alpha(J_{n+1}/\partial J_{n+1})(1)$ and $\underline{H}_G^\alpha(X_n^+)(1)$, and this decomposition is respected by the map $\rho \oplus i^*$. Thus, x is the sum of classes y and z in $\underline{H}_G^\alpha(J_{n+1}/\partial J_{n+1})(1)/\text{torsion}$ and $\underline{H}_G^\alpha(X_n^+)(1)/\text{torsion}$, respectively, which vanish under the analogous maps. By our inductive hypothesis, z is zero. Since $J_{n+1}/\partial J_{n+1}$ is a wedge of copies of S^V and $G^+ \wedge S^{2k}$ for various V and k, y vanishes by our remark about X_0. Thus, x is zero. The proof of the first part is similar. For this part, we must assume that $|\alpha|$ is even because the map $\rho \oplus i^*$ does not detect the torsion in the fourth quadrant of $\underline{H}_G^*(S^0)(1)$.

3. THE COHOMOLOGY OF COMPLEX PROJECIVE SPACES. As an application of the results from section two, we show that the cohomology of a complex projective space with a linear action is free over $\underline{H}_G^* S^0$. Let V be a finite or countably infinite dimensional complex G-representation and let $\mathbb{C}^*$ be $\mathbb{C} - \{0\}$. The complex projective space P(V) with linear G-action associated to V is the quotient G-space $(V - \{0\})/\mathbb{C}^*$. Note that if $W \subset V$, then P(W) may be regarded as a subspace of P(V). If V is infinite dimensional, then we topologize V as the colimit of its finite dimensional subspaces W; the quotient topology on P(V) is then the same as the colimit topology from the associated subspaces P(W). To describe the cohomology of P(V), we must write V as the sum $\sum_{i=0}^{n} \phi_i$ of irreducible complex representations (including possibly the trivial complex representation). Of course, if V is infinite dimensional, then $n = \infty$. Points in P(V) will be described by homogeneous coordinates of the form

$$\langle x_0, x_1, x_2, \dots, x_n \rangle, \qquad x_i \in \phi_i$$

with the conventions that not all of the x_i are zero, and if V is infinite dimensional, that all but finitely many of the x_i are zero. Each element of the group G acts on each homogeneous coordinate of P(V) by multiplication by a complex number. Therefore, if all the irreducibles in V are isomorphic, then the action of G on P(V) is trivial. Moreover, if η is any irreducible complex representation, then P(V) and $P(\eta V)$ are isomorphic G-spaces. If η and ϕ are irreducible complex representations, then $P(\eta)$ is just a point and $P(\eta \oplus \phi)$ is G-homeomorphic to the one-point compactification of either $\eta^{-1}\phi$ or $\eta\phi^{-1}$.

Since we have selected a colimit topology on P(V) when V is infinite, to show that P(V) is a generalized G-cell complex for any G-representation V, it suffices to show this when V is finite dimensional. Let V_k be the representation $\sum_{i=0}^{k-1} \phi_i$ and let W be the representation $\phi_n^{-1} V_{n-1}$. Describe points in the unit disk DW by complex coordinates $(x_0, x_1, \dots, x_{n-1})$, with $x_i \in \phi_n^{-1}\phi_i$. Define a map $f\colon DW \to P(V)$ by

$$f((x_0, x_1, \dots, x_{n-1})) \;=\; \langle x_0, x_1, x_2, \dots, x_{n-1}, 1 - \sum_{i=0}^{n-1} |x_i|^2 \rangle.$$

The tensor product with ϕ_n^{-1} is inserted in the definition of W to ensure that the map f is equivariant. The image of SW in P(V) lies in the subspace $P(V_{n-1})$ of P(V), and f is a homeomorphism from $DW - SW$ to its image in P(V). Thus P(V) is formed from $P(V_{n-1})$ by adjoining the G-cell DW along the map $f|SW : SW \to P(V_{n-1})$. Working backwards through the sequence of representations V_k, we conclude that P(V) is a generalized G-cell complex with cells the unit disks of the representations $\phi_k^{-1} V_k$ for $1 \leq k \leq n$.

In order to show that the equivariant cohomology of P(V) is free over $\underline{H}_G^* S^0$, we must show that the cells of P(V) can be attached in an order satisfying the condition in Theorem 2.6(a). This proper ordering of cells is derived from a careful ordering of the set Φ of irreducible summands of V. Since the remainder of our discussion focuses on Φ, we write $P(\Phi)$ for P(V). An ordering ϕ_0, ϕ_1, ϕ_2, ... of the elements of Φ is said to be proper if the number of irreducibles in the set $\{\phi_i\}_{0 \leq i \leq k-1}$ isomorphic to ϕ_k is a nondecreasing function of k. For example, if ϕ and η are distinct complex irreducibles and Φ consists of two copies of ϕ and one of η, then η, ϕ, ϕ and ϕ, η, ϕ are proper orderings of Φ, but ϕ, ϕ, η is not. The dimension of the fixed subrepresentation of the representation $\phi_k^{-1} \sum_{i=0}^{k-1} \phi_i$ is the number of irreducibles in the set $\{\phi_i\}_{0 \leq i \leq k-1}$ isomorphic to ϕ_k. Thus, if Φ is properly ordered, then the cell structure described above satisfies the conditions of Theorem 2.6.(a).

PROPOSITION 3.1. If ϕ_0, ϕ_1, ϕ_2, ... is any ordering of the elements of a set Φ of irreducible representations, then $P(\Phi)$ is a generalized G-cell complex with cells the unit disks of the G-representations $\phi_k^{-1} \sum_{i=0}^{k-1} \phi_i$, for $k \geq 1$. Moreover, $\underline{H}_G^* P(\Phi)^+$ and $\underline{H}_*^G P(\Phi)^+$ are free RO(G)-graded modules over $\underline{H}_G^* S^0$. If the ordering of Φ is proper, then the homology and cohomology of $P(\Phi)$ are each generated by one element in dimension zero and one in each of the dimensions $\phi_k^{-1} \sum_{i=0}^{k-1} \phi_i$, for $k \geq 1$.

The G-fixed subspace of $P(\Phi)$ is a disjoint union of complex projective spaces, one for each isomorphism class of irreducibles in Φ. The (complex) dimension of the complex projective space in $P(\Phi)^G$ associated to the irreducible ϕ is one less than the multiplicity of ϕ in Φ. Thus, the effect of properly ordering the irreducibles is that the maximal dimension of the components of the G-fixed subspace of $P(\{\phi_i\}_{0 \leq i \leq k})$ increases as slowly as possible with increasing k.

REMARKS 3.2. Our description of the cohomology of $P(\Phi)$ contains one apparent anomaly. Suppose that ζ, η, and ϕ are distinct complex irreducible representations and $\Phi = \{\zeta, \eta, \phi\}$. If we assign the proper ordering ζ, η, ϕ to Φ, then we find that the generators of $\underline{H}_G^* P(\Phi)^+$ are in dimensions 0, $\eta^{-1}\zeta$, and $\phi^{-1}(\zeta \oplus \eta)$. However, if we select the proper ordering ϕ, ζ, η, we find that the generators are in dimensions 0, $\zeta^{-1}\phi$, and $\eta^{-1}(\phi \oplus \zeta)$. In particular, the cohomology in dimension $\eta^{-1}\zeta$ must be $A \oplus \langle \mathbb{Z} \rangle \oplus \langle \mathbb{Z} \rangle$ if we use the first set of generators, and $A[d] \oplus \langle \mathbb{Z} \rangle \oplus \langle \mathbb{Z} \rangle$ if we use the second, where d is the integer associated to the element $\eta^{-1}\zeta - \zeta^{-1}\phi$ of $RO_0(G)$. There is no contradiction in these two claims about the cohomology in dimension

$\eta^{-1}\zeta$ because these two Mackey functors are isomorphic by Examples 1.1.(d). The apparent difficulties in all the other dimensions are resolved in exactly the same way.

This example illustrates the latitude that one has in selecting the dimensions of the generators of the cohomology of $P(\Phi)$ for almost any set Φ of irreducibles. This latitude is necessary because, for most Φ, there are many proper orderings and a choice of a proper ordering corresponds to a selection of the dimensions of the generators.

It would be nice to have some simple cohomology invariants of $P(\Phi)$ which could be used for problems like comparing the cohomology of projective spaces with different G-actions. The fact that the dimensions for the cohomology generators don't provide such an invariant is a disappointment. However, one invariant related to the dimensions of the generators is readily available. Select a proper ordering of Φ and plot the dimensions α of the resulting set of generators of $\underline{H}_G^* P(\Phi)^+$ on a coordinate plane whose horizontal and vertical axes indicate $|\alpha^G|$ and $|\alpha|$, respectively. The dimensions lie on a stair-step pattern whose foot is at the origin. This plot is an invariant of $P(\Phi)$. The height of the steps in the plot decreases, or remains constant, as one goes up the steps (that is, moves in the direction of increasing $|\alpha^G|$ and $|\alpha|$). The height remains constant only if irreducible types appearing in Φ have equal multiplicity. The step-like structure of the plot reflects a filtration on Φ which plays an important role in our discussion of the ring structure of $\underline{H}_G^* P(\Phi)^+$. An increasing filtration

$$\emptyset = \Phi(0),\ \Phi(1),\ \Phi(2),\ \dots,\ \Phi(r),\ \dots$$

of the set Φ is said to be proper if, for every r and every complex irreducible ϕ, the number of irreducibles in $\Phi(r)$ isomorphic to ϕ is the lesser of r and the number of irreducibles in Φ isomorphic to ϕ. Any two proper filtrations of Φ differ only by an interchange of isomorphic irreducible complex representations, so there is essentially only one proper filtration of Φ. The steps in the plot of the dimensions of the generators are in a one-to-one correspondence with the stages in the filtration of Φ. The height of the step corresponding to filtration level r is the number of elements in $\Phi(r) - \Phi(r-1)$.

4. CUP PRODUCTS IN $\underline{H}_G^* S^0$. Here we describe the multiplicative structure of $\underline{H}_G^* S^0$. We begin with the case $p = 2$, which is due to Stong.

DEFINITIONS 4.1. Let ζ be the real one-dimensional sign representation of $G = \mathbb{Z}/2$. The identity element 1 in $A(1) = \underline{H}_G^0(S^0)(1)$ is the identity element of the RO(G)-graded Mackey functor ring $\underline{H}_G^* S^0$. Let $\kappa \in \underline{H}_G^0(S^0)(1)$ be $2 - \tau\rho(1)$. Observe that $\kappa^2 = 2\kappa$. Let $\epsilon \in \underline{H}_G^{\zeta}(S^0)(1)$ be the Euler class; that is, the image of $1 \in \underline{H}_G^0(S^0)(1)$ under the map induced by the inclusion $S^0 \subset S^{\zeta}$. Select a nonequivariant identication of S^ζ with S^1 and let $\iota_{1-\zeta} \in \underline{H}_G^{1-\zeta}(S^0)(e) \cong \underline{H}_G^1(S^\zeta)(e)$ and $\iota_{\zeta-1} \in \underline{H}_G^{\zeta-1}(S^0)(e) \cong \underline{H}_G^\zeta(S^1)(e)$ be the images of $\rho(1) \in \underline{H}_G^0(S^0)(e) \cong \underline{H}_G^1(S^1)(e)$ under the maps induced by this identification. Let $\xi \in \underline{H}_G^{2\zeta-2}(S^0)(1)$ be the unique element with $\rho(\xi) = \iota_{\zeta-1}^2$. The elements 1 and κ generate the abelian group $\underline{H}_G^0(S^0)(1)$ and the Mackey functor $\underline{H}_G^0 S^0$. Each of the elements ϵ^m, ξ^m, and $\epsilon^m \xi^n$, for m, n ≥ 1, generates the abelian group $\underline{H}_G^\alpha(S^0)(1)$ and the Mackey functor $\underline{H}_G^\alpha S^0$ in the appropriate dimension α. For m ≥ 1, the element $\iota_{1-\zeta}^m$ or $\iota_{\zeta-1}^m$ generates the abelian group $\underline{H}_G^\alpha(S^0)(e)$ in the appropriate dimension and $\iota_{1-\zeta}^m$ generates the Mackey functor $\underline{H}_G^* S^0$ in the appropriate dimension. For m ≥ 2, $\tau(\iota_{1-\zeta}^m)$ generates the abelian group $\underline{H}_G^*(S^0)(1)$ in the appropriate dimension.

LEMMA 4.2. The class $\kappa \in \underline{H}_G^0(S^0)(1)$ and, for n ≥ 1, the classes

$$\tau(\iota_{1-\zeta}^{2n+1}) \in \underline{H}_G^{(2n+1)(1-\zeta)}(S^0)(1)$$

are infinitely divisible by $\epsilon \in \underline{H}_G^\zeta(S^0)(1)$; that is, for m ≥ 1, there are unique elements

$$\epsilon^{-m}\kappa \in \underline{H}_G^{-m\zeta}(S^0)(1)$$

and

$$\epsilon^{-m}\tau(\iota_{1-\zeta}^{2n+1}) \in \underline{H}_G^{2n+1-(2n+m+1)\zeta}(S^0)(1)$$

such that

$$\epsilon^m(\epsilon^{-m}\kappa) = \kappa \quad \text{and} \quad \epsilon^m(\epsilon^{-m}\tau(\iota_{1-\zeta}^{2n+1})) = \tau(\iota_{1-\zeta}^{2n+1}).$$

Moreover, each of the elements $\epsilon^{-m}\kappa$ or $\epsilon^{-m}\tau(\iota^{2n+1})$ generates the abelian group $\underline{H}_G^*(S^0)(1)$ and the Mackey functor $\underline{H}_G^* S^0$ in its dimension.

THEOREM 4.3. The elements

$$\begin{aligned}
\epsilon &\in \underline{H}_G^\zeta(S^0)(1)\\
\iota_{1-\zeta} &\in \underline{H}_G^{1-\zeta}(S^0)(e)\\
\iota_{\zeta-1} &\in \underline{H}_G^{\zeta-1}(S^0)(e)\\
\xi &\in \underline{H}_G^{2\zeta-2}(S^0)(1)\\
\epsilon^{-m}\kappa &\in \underline{H}_G^{-m\zeta}(S^0)(1), \qquad \text{for m} \geq 1,
\end{aligned}$$

and

$$\epsilon^{-m}\tau(\iota_{1-\zeta}^{2n+1}) \in \underline{H}_G^{2n+1-(2n+m+1)\zeta}(S^0)(1), \qquad \text{for m, n} \geq 1,$$

generate $\underline{H}_G^* S^0$ as an RO(G)-graded Mackey functor algebra over the Burnside Mackey functor ring A. The only relations among these elements, other than those forced by the Frobenius relations or the vanishing of $\underline{H}_G^* S^0$ in various dimensions, are generated by the relations

$$\begin{aligned}
\rho(\epsilon) &= 0 \\
\iota_{1-\zeta}\,\iota_{\zeta-1} &= \rho(1) \\
\tau(\iota_{1-\zeta}) &= 0 \\
\tau(\iota_{\zeta-1}^{2m+1}) &= 0, && \text{for } m \geq 0, \\
\tau(\iota_{\zeta-1}^{2m}) &= 2\xi^m, && \text{for } m \geq 1, \\
\tau(\iota_{1-\zeta}^{m})\,\tau(\iota_{1-\zeta}^{n}) &= \begin{cases} 0, & \text{if m or n is odd,} \\ 2\tau(\iota_{1-\zeta}^{m+n}), & \text{if m and n are even,} \end{cases} \\
\rho(\xi) &= \iota_{\zeta-1}^{2} \\
2\,\epsilon\,\xi &= 0 \\
\rho(\epsilon^{-m}\kappa) &= 0, && \text{for } m \geq 0, \\
\epsilon\,(\epsilon^{-m}\kappa) &= \epsilon^{1-m}\kappa, && \text{for } m \geq 1, \\
(\epsilon^{-m}\kappa)(\epsilon^{-n}\kappa) &= 2\epsilon^{-(m+n)}\kappa, && \text{for } m, n \geq 0, \\
2\epsilon^{-m}\,\tau(\iota_{1-\zeta}^{2n+1}) &= 0, && \text{for } m \geq 0 \text{ and } n \geq 1, \\
\rho(\epsilon^{-m}\,\tau(\iota_{1-\zeta}^{2n+1})) &= 0, && \text{for } m \geq 0 \text{ and } n \geq 1, \\
\epsilon\,(\epsilon^{-m}\,\tau(\iota_{1-\zeta}^{2n+1})) &= \epsilon^{1-m}\,\tau(\iota_{1-\zeta}^{2n+1}), && \text{for } m, n \geq 1, \\
(\epsilon^{-m}\,\tau(\iota_{1-\zeta}^{2n+1}))\,(\epsilon^{-q}\kappa) &= 0, && \text{for } m, q \geq 0 \text{ and } n \geq 1,
\end{aligned}$$

and

$$\xi\,(\epsilon^{-m}\,\tau(\iota_{1-\zeta}^{2n+1})) = \epsilon^{-m}\,\tau(\iota_{1-\zeta}^{2n-1}), \qquad \text{for } m \geq 0 \text{ and } n \geq 2.$$

REMARKS 4.4. **(a)** The last relation indicates that, for $m \geq 0$ and $n \geq 1$, $\epsilon^{-m}\,\tau(\iota_{1-\zeta}^{2n+1})$ is infinitely divisible by ξ. Thus, we can think of all the elements in the fourth quadrant of the graph of $\underline{H}_G^*(S^0)$ as being derived from $\tau(\iota_{1-\zeta}^{3})$ via division by powers of ϵ and ξ. One mnemonic for the effect of ϵ and ξ on the elements in the fourth quadrant is to denote the nonzero element in $\underline{H}_G^{1-m\zeta-2n(\zeta-1)}(S^0)(1)$, for $m \geq 2$ and $n \geq 1$, by $\epsilon^{-m}\,\xi^{-n}\,\omega$, where ω is regarded as a fictitious element in dimension 1. The reason for selecting a fictitious element in dimension 1, instead of the actual element in dimension $3-3\zeta$, is discussed in Remarks 4.10(b).

(b) For $p = 2$, the elements $\pm(1-\tau\rho(1))$ in A(1) are units, and $1-\tau\rho(1)$ appears in the formula describing the anticommutativity of cup products. For any G-space X, if $a \in \underline{H}_G^{i+j\zeta}X^+$ and $b \in \underline{H}_G^{m+n\zeta}X^+$, then

$$ab = (-1)^{im}(1-\tau\rho(1))^{jn}\, ba.$$

The generators $\iota_{1-\zeta}$, $\iota_{\zeta-1}$, ϵ, $\epsilon^{-n}\kappa$, and $\epsilon^{-m}\tau(\iota_{1-\zeta}^{2n+1})$ are in dimensions where the behavior of this nontrivial unit matters. Of course, since $\epsilon^{-m}\tau(\iota_{1-\zeta}^{2n+1})$ has order 2, any unit acts trivially on it. It is easy to check that

$$(1-\tau\rho(1))\,\iota_{1-\zeta} = -\iota_{1-\zeta} \quad \text{and} \quad (1-\tau\rho(1))\,\iota_{\zeta-1} = -\iota_{\zeta-1}.$$

This action of $1-\tau\rho(1)$ on $\iota_{1-\zeta}$ and $\iota_{\zeta-1}$ never affects cup products in $\underline{H}_G^*S^0$ because it is always balanced by the $(-1)^{im}$ term in the commutativity formula. However, there are algebras over $\underline{H}_G^*S^0$ where the effects of this unit on $\iota_{1-\zeta}$ and $\iota_{\zeta-1}$ are visible. The unit $1-\tau\rho(1)$ acts trivially on ϵ and $\epsilon^{-n}\kappa$. This shows up dramatically in $\underline{H}_G^*S^0$. The elements ϵ and $\epsilon^{-2n+1}\kappa$ are odd-dimensional, so our intuition about graded algebras from the nonequivariant context suggests that their squares should vanish, or at least be 2-torsion. In fact, the squares are not torsion elements, an apparent anomaly possible only because the action of $1-\tau\rho(1)$ is trivial. The overall effect of the actions of the units of A on the generators of $\underline{H}_G^*S^0$ is that $\underline{H}_G^*S^0$ is commutative in both the graded and the ungraded sense.

When p is odd, several complications in the multiplicative structure of $\underline{H}_G^*S^0$ arise from the greater complexity of RO(G). The most obvious are a host of sign problems coming from the identification of representations with their complex conjugates. Initially, we resolve these sign problems by grading $\underline{H}_G^*S^0$ on RSO(G) instead of RO(G). In Remark 4.11, we explain steps which must be taken to pass back to an RO(G)-grading. The most serious complication arises from the misbehavior of the integers d_α associated to the virtual representations α in $RSO_0(G)$. One way to deal with this complication is to avoid it. This can be done very nicely if one is only interested in $\underline{H}_G^*S^0$. Because of the intuition this approach offers, we outline it as an introduction to the odd primes case.

The stable homotopy groups $\underline{\pi}_\beta^G S^0$, for $\beta \in RSO_0(G)$, have been studied extensively by tom Dieck and Petrie [tDP], and the stable Hurewicz map

$$h: \underline{\pi}_{-\beta}^G S^0 \to \underline{H}_{-\beta}^G S^0 \cong \underline{H}_G^\beta S^0.$$

is an isomorphism [LE1] if $\beta \in RSO_0(G)$. Thus, many of their results can be applied to homology in the appropriate dimensions. They have shown that the multiplication map

$$\underline{\pi}_\beta^G S^0 \,\square\, \underline{\pi}_\gamma^G S^0 \to \underline{\pi}_{\beta+\gamma}^G S^0$$

is an isomorphism for any $\beta \in RSO_0(G)$ and any $\gamma \in RSO(G)$. By similar reasoning, the multiplication map

$$\underline{H}_G^\beta S^0 \,\square\, \underline{H}_G^\gamma S^0 \to \underline{H}_G^{\beta+\gamma} S^0$$

is an isomorphism under the same conditions on β and γ. Thus, to understand all of $\underline{H}_G^*S^0$, it suffices to understand the part of $\underline{H}_G^*S^0$ which tom Dieck and Petrie have already described and the part indexed on some subset of RSO(G) complementary to $RSO_0(G)$. Recall that λ is the irreducible complex representation that takes the standard generator of $\mathbb{Z}/p$ to $e^{2\pi i/p}$. Let $RSO_\lambda(G)$ be the additive subgroup of $RSO(G)$ generated by 1 and λ. As an additive group, $RSO(G)$ is the internal direct sum of $RSO_0(G)$ and $RSO_\lambda(G)$. To complete our description of $\underline{H}^*_G S^0$, it suffices to describe that part of it indexed on $RSO_\lambda(G)$. This part is almost identical to $\underline{H}^*_G S^0$ for $G = \mathbb{Z}/2$. Consider the description given above of that part of $\underline{H}^*_G S^0$ for $p = 2$ indexed on the additive subgroup of $RO(\mathbb{Z}/2)$ generated by 1 and 2ζ. Replace 2ζ by λ and all the other 2's by p's. The result is a description of the part of $\underline{H}^*_G S^0$ for p odd indexed on $RSO_\lambda(G)$. This approach describes $\underline{H}^*_G S^0$ as the graded box product of two subrings indexed on complementary subsets of $RSO(G)$. The unpleasant behavior of the integers d_α is buried in the computations of the box products.

Unfortunately, because of peculiarities in the dimensions of the algebra generators of $\underline{H}^*_G P(V)^+$, this description of $\underline{H}^*_G S^0$ as the box product of two subrings can not be used to describe the ring structure of the cohomology of complex projective spaces. Thus, we offer an alternative description of the ring structure of $\underline{H}^*_G S^0$ for p odd. In section 2, we defined a function from $RO_0(G)$ to $\mathbb{Z}$ using a section of the projection from $\tilde{R}_0(G)$ to $RO_0(G)$. Since we are now working with $RSO_0(G)$ instead of $RO_0(G)$, we define an analogous function from $RSO_0(G)$ to $\mathbb{Z}$ using a section $s\colon RSO_0(G) \to \tilde{R}_0(G)$ of the projection from $\tilde{R}_0(G)$ to $RO_0(G)$. We insist that $s(0) = 0$ and that our original section $RO_0(G) \to \tilde{R}_0(G)$ factor through s.

DEFINITIONS 4.5. (a) If $\alpha \in RSO_0(G)$ and $s(\alpha) = \sum_i \phi_i - \eta_i$, then we wish to define an equivariant map $\mu_\alpha\colon S^{\Sigma\eta_i} \to S^{\Sigma\phi_i}$ with nonequivariant degree d_α. If $\alpha = \lambda^m - \lambda^n$ with $0 < m, n < p$ and n^{-1} is the unique integer such that $1 \le n^{-1} \le p-1$ and $n n^{-1} \equiv 1 \bmod p$, then μ_α is the extension to one-point compactifications of the complex power map $z \to z^{m(n^{-1})}$, for $z \in \mathbb{C}$. In general, we identify $S^{\Sigma\phi_i}$ and $S^{\Sigma\eta_i}$ with $\bigwedge_i S^{\phi_i}$ and $\bigwedge_i S^{\eta_i}$, respectively, and take the smash product of the appropriate complex power maps to obtain the equivariant map μ_α from $S^{\Sigma\phi_i}$ to $S^{\Sigma\eta_i}$ with nonequivariant degree d_α. Also denote by μ_α the image of this map in $\underline{H}^\alpha_G(S^0)(1)$ under the Hurewicz map. Clearly, if the ϕ_i and the η_i were paired off in a different order, then a different map from $S^{\Sigma\phi_i}$ to $S^{\Sigma\eta_i}$ would be obtained. However, the maps coming from the two pairings would be equivariantly homotopic and so would give the same element in $\underline{H}^\alpha_G(S^0)(1)$.

(b) Let α be an element of $RSO(G)$ with $|\alpha| = 0$. Then α must be represented by a sum $\sum_i \phi_i - \eta_i$, where the ϕ_i and η_i are irreducible complex representations, some of which may be trivial. Since the ϕ_i and η_i are complex representations, they have canonical nonequivariant orientations. Combine these to produce a nonequivariant identification ι_α of $S^{\Sigma\phi_i}$ with $S^{\Sigma\eta_i}$ which is unique up to homotopy. Let ι_α also denote the image of this identification in $\underline{H}^\alpha_G(S^0)(e)$. The resulting cohomology classes ι_α are then independent of the ordering of the ϕ_i and the η_i. The abelian group $\underline{H}^\alpha_G(S^0)(e)$ is generated by ι_α. If $|\alpha^G| > 0$, then $\tau(\iota_\alpha)$ generates the abelian group $\underline{H}^\alpha_G(S^0)(1)$ and ι_α generates the Mackey functor $\underline{H}^\alpha_G S^0$.

(c) If $\alpha \in RSO_0(G)$, then in $\underline{H}^\alpha_G S^0$,

$$\rho(\mu_\alpha) = d_\alpha \, \iota_\alpha \quad \text{and} \quad \rho\tau(\iota_\alpha) = p \, \iota_\alpha.$$

We have already asserted that $\underline{H}^\alpha_G S^0$ is $A[d_\alpha]$. Under this identification, μ_α and $\tau(\iota_\alpha)$ become the elements μ and τ of $A[d_\alpha](1)$ and ι_α becomes $1 \in \mathbb{Z} = A[d_\alpha](e)$. There is a unique integer b_α such that $d_{-\alpha} d_\alpha + b_\alpha p = 1$. Let $\kappa_\alpha = p\,\mu_\alpha - d_\alpha \tau(\iota_\alpha)$ and $\sigma_\alpha = d_{-\alpha}\mu_\alpha + b_\alpha \tau(\iota_\alpha)$. Then, σ_α and κ_α form an alternative $\mathbb{Z}$-basis for $\underline{H}^\alpha_G(S^0)(1)$.

(d) Let β be an element of $RSO(G)$ with $|\beta| > 0$ and $|\beta^G| = 0$. There exist an α in $RSO_0(G)$ and a G-representation V such that $V^G = 0$ and $\beta = \alpha + V$. Let $\epsilon_\beta \in \underline{H}^\beta_G(S^0)(1)$ be the image of $\mu_\alpha \in \underline{H}^\alpha_G(S^0)(1)$ under the map from $\underline{H}^\alpha_G(S^0)(1)$ to $\underline{H}^\beta_G(S^0)(1)$ induced by the inclusion $S^0 \subset S^V$. In Lemma A.11, it is shown that this Euler class ϵ_β is independent of the choice of the decomposition of β into the sum of the representation V and the element α of $RSO_0(G)$. The class ϵ_β generates the abelian group $\underline{H}^\beta_G(S^0)(1)$ and the Mackey functor $\underline{H}^\beta_G S^0$.

(e) If $|\alpha| = 0$ and $|\alpha^G| < 0$, let ξ_α be the unique element of $\underline{H}^\alpha_G(S^0)(1)$ with $\rho(\xi_\alpha) = \iota_\alpha$; this class generates the abelian group $\underline{H}^\alpha_G(S^0)(1)$ and the Mackey functor $\underline{H}^\alpha_G S^0$.

When p is odd, it is harder to pick a multiplicative basis for the torsion in the fourth quadrant of the graph of $\underline{H}^*_G S^0$. In each dimension there is a choice of $p-1$ generators, instead of a single nonzero element. Moreover, since these torsion elements are not tied by an Euler class to elements on the positive horizontal axis, there is no way to base the choice of a generator on choices already made for the axis. The following lemma justifies the procedure we employ to select multiplicative generators for the fourth quadrant.

LEMMA 4.6. Let β be an element of $RSO_0(G)$ and let α, γ, and δ be elements of $RSO(G)$ such that

$$|\delta| = |\gamma^G| = 0,$$
$$|\alpha|, |\delta^G| < 0,$$
$$|\gamma| > 0,$$
$$|\alpha^G| \geq 3,$$

and

$$|\alpha^G| \text{ is odd.}$$

If x is any nonzero element in $\underline{H}_G^\alpha(S^0)(1)$, then $\mu_\beta x$ is a generator in $\underline{H}_G^{\alpha+\beta}(S^0)(1)$. Moreover, x and $\mu_\beta x$ are uniquely divisible by both ϵ_γ and ξ_δ.

DEFINITIONS 4.7. Select a generator in $\underline{H}_G^{3-2\lambda}(S^0)(1)$ and denote it by $\nu_{3-2\lambda}$. If $\alpha = 1 - m(\lambda - 2) - n\lambda$, for m, n ≥ 1, then let ν_α be the unique element in $\underline{H}_G^\alpha(S^0)(1)$ such that

$$\epsilon_{(n-1)\lambda}\,\xi_{(m-1)(\lambda-2)}\,\nu_\alpha \;=\; \nu_{3-2\lambda}.$$

For any $\alpha \in RSO(G)$, there are unique integers m, n, and q such that $q = 0$ or 1 and

$$\alpha - [q - m(\lambda - 2) - n\lambda] \;\in\; RO_0(G).$$

Denote by $<\alpha>$ the element $q - m(\lambda - 2) - n\lambda$ associated to α by these conditions. If $\alpha \in RSO(G)$ with $|\alpha| < 0$, $|\alpha^G| \geq 3$, $|\alpha^G|$ odd, and $\alpha \neq <\alpha>$, then define $\nu_\alpha \in \underline{H}_G^\alpha(S^0)(1)$ by

$$\nu_\alpha \;=\; \mu_{\alpha - <\alpha>}\,\nu_{<\alpha>}.$$

The element ν_α generates the abelian group $\underline{H}_G^\alpha(S^0)(1)$ and the Mackey functor $\underline{H}_G^\alpha S^0$.

LEMMA 4.8. If $\alpha \in RSO_0(G)$, then $\kappa_\alpha \in \underline{H}_G^\alpha(S^0)(1)$ is divisible by ϵ_β, for any $\beta \in RSO(G)$ with $|\beta| > 0$ and $|\beta^G| = 0$; that is, there is a unique element

$$\epsilon_\beta^{-1}\,\kappa_\alpha \;\in\; \underline{H}_G^{\alpha-\beta}(S^0)(1)$$

such that

$$\epsilon_\beta\,(\epsilon_\beta^{-1}\,\kappa_\alpha) \;=\; \kappa_\alpha.$$

The element $\epsilon_\beta^{-1}\,\kappa_\alpha$ generates the abelian group $\underline{H}_G^{\alpha-\beta}(S^0)(1)$ and the Mackey functor

$\underline{H}_G^{\alpha-\beta} S^0$.

THEOREM 4.9. The elements

$$\mu_\alpha \in \underline{H}_G^\alpha(S^0)(1), \qquad \text{for } \alpha = \pm(\lambda^n - \lambda), \text{ with } 1 < n < p,$$
$$\iota_\alpha \in \underline{H}_G^\alpha(S^0)(e), \qquad \text{for } \alpha = \pm(\lambda^n - \lambda), \text{ with } 1 < n < p,$$
$$\epsilon_\lambda \in \underline{H}_G^\lambda(S^0)(1)$$
$$\xi_{\lambda-2} \in \underline{H}_G^{\lambda-2}(S^0)(1)$$
$$\iota_{2-\lambda} \in \underline{H}_G^{2-\lambda}(S^0)(e)$$
$$\epsilon_{m\lambda}^{-1}\kappa_0 \in \underline{H}_G^{-m\lambda}(S^0)(1), \qquad \text{for } m \geq 1,$$

and

$$\nu_\alpha \in \underline{H}_G^\alpha(S^0)(1), \qquad \text{for } \alpha = 1 - m(\lambda - 2) - n\lambda, \text{ with } m, n \geq 1,$$

generate $\underline{H}_G^* S^0$ as an RSO(G)-graded Mackey functor algebra over the Burnside Mackey functor ring A. All of relations among the elements of $\underline{H}_G^* S^0$, other than those forced by the Frobenius relations or the vanishing of $\underline{H}_G^* S^0$ in various dimensions, are generated by the relations

$$\rho(\mu_\alpha) = d_\alpha \iota_\alpha, \qquad \text{for } \alpha \in RSO_0(G);$$

$$\mu_\alpha \mu_\beta = \mu_{\alpha+\beta} + \left[\frac{d_\alpha d_\beta - d_{\alpha+\beta}}{p}\right]\tau(\iota_{\alpha+\beta}), \qquad \text{for } \alpha, \beta \in RSO_0(G);$$

$$\rho(\epsilon_\beta) = 0, \qquad \text{for } |\beta| > 0 \text{ and } |\beta^G| = 0;$$

$$\epsilon_\alpha \epsilon_\beta = \epsilon_{\alpha+\beta}, \qquad \text{for } |\alpha|, |\beta| > 0 \text{ and } |\alpha^G| = |\beta^G| = 0;$$

$$\mu_\alpha \epsilon_\beta = \epsilon_{\alpha+\beta}, \qquad \text{for } \alpha \in RSO_0(G), |\beta| > 0, \text{ and } |\beta^G| = 0;$$

$$\rho(\xi_\alpha) = \iota_\alpha, \qquad \text{for } |\alpha| = 0 \text{ and } |\alpha^G| < 0;$$

$$\tau(\iota_\alpha) = p\,\xi_\alpha, \qquad \text{for } |\alpha| = 0 \text{ and } |\alpha^G| < 0;$$

$$\xi_\alpha \xi_\beta = \xi_{\alpha+\beta}, \qquad \text{for } |\alpha| = |\beta| = 0 \text{ and } |\alpha^G|, |\beta^G| < 0;$$

$$\mu_\alpha \xi_\beta = d_\alpha \xi_{\alpha+\beta}, \qquad \text{for } \alpha \in RSO_0(G), |\beta| = 0, \text{ and } |\beta^G| < 0;$$

$$p\,\epsilon_\beta\,\xi_\alpha = 0, \qquad \text{for } |\alpha| = |\beta^G| = 0,\ |\alpha^G| < 0, \text{ and } |\beta| > 0;$$

$$\epsilon_\beta\,\xi_\alpha = d_{\delta-\alpha}\,\epsilon_\gamma\,\xi_\delta, \qquad \text{for } |\alpha| = |\delta| = |\beta^G| = |\gamma^G| = 0,\ |\alpha^G|, |\delta^G| < 0,\ |\beta|, |\gamma| > 0, \text{ and } \alpha + \beta = \gamma + \delta;$$

$$\epsilon_\beta^{-1}\,\kappa_\alpha = \epsilon_\gamma^{-1}\,\kappa_\delta, \qquad \text{for } \alpha, \delta \in \mathrm{RSO}_0(G),\ |\beta^G| = |\gamma^G| = 0,\ |\beta|, |\gamma| > 0, \text{ and } \alpha + \gamma = \beta + \delta;$$

$$\rho(\epsilon_\beta^{-1}\,\kappa_\alpha) = 0, \qquad \text{for } \alpha \in \mathrm{RSO}_0(G),\ |\beta^G| = 0, \text{ and } |\beta| > 0;$$

$$\mu_\gamma(\epsilon_\beta^{-1}\,\kappa_\alpha) = \epsilon_\beta^{-1}\,\kappa_{\alpha+\gamma}, \qquad \text{for } \alpha, \gamma \in \mathrm{RSO}_0(G),\ |\beta^G| = 0, \text{ and } |\beta| > 0;$$

$$\epsilon_\beta(\epsilon_\beta^{-1}\,\kappa_\alpha) = \kappa_\alpha, \qquad \text{for } \alpha \in \mathrm{RSO}_0(G),\ |\beta^G| = 0, \text{ and } |\beta| > 0;$$

$$\epsilon_\gamma(\epsilon_\beta^{-1}\,\kappa_\alpha) = \epsilon_{\beta-\gamma}^{-1}\,\kappa_\alpha, \qquad \text{for } \alpha \in \mathrm{RSO}_0(G),\ |\beta^G| = |\gamma^G| = 0, \text{ and } |\beta| > |\gamma| > 0;$$

$$(\epsilon_\beta^{-1}\kappa_\alpha)(\epsilon_\gamma^{-1}\kappa_\delta) = p\,\epsilon_{\beta+\gamma}^{-1}\,\kappa_{\alpha+\delta}, \qquad \text{for } \alpha, \delta \in \mathrm{RSO}_0(G),\ |\beta^G| = |\gamma^G| = 0, \text{ and } |\beta|, |\gamma| > 0;$$

$$p\,\nu_\alpha = 0, \qquad \text{for } |\alpha| < 0,\ |\alpha^G| \geq 3, \text{ and } |\alpha^G| \text{ odd};$$

$$\rho(\nu_\alpha) = 0, \qquad \text{for } |\alpha| < 0,\ |\alpha^G| \geq 3, \text{ and } |\alpha^G| \text{ odd};$$

$$\mu_\beta\,\nu_\alpha = \nu_{\alpha+\beta}, \qquad \text{for } \beta \in \mathrm{RSO}_0(G),\ |\alpha| < 0,\ |\alpha^G| \geq 3, \text{ and } |\alpha^G| \text{ odd};$$

$$\epsilon_\beta \nu_\alpha = \nu_{\alpha+\beta}, \qquad \text{for } |\alpha+\beta| < 0,\ |\alpha^G| \geq 3,\ |\alpha^G| \text{ odd},\ |\beta| > 0, \text{ and } |\beta^G| = 0;$$

$$\xi_\beta \nu_\alpha = d_{<\beta>-\beta}\, \nu_{\alpha+\beta}, \qquad \text{for } |\alpha| < 0,\ |\alpha^G + \beta^G| \geq 3,\ |\alpha^G| \text{ odd},\ |\beta| = 0, \text{ and } |\beta^G| < 0;$$

$$(\epsilon_\beta^{-1} \kappa_\gamma)\, \nu_\alpha = 0, \qquad \text{for } \gamma \in \mathrm{RSO}_0(G),\ |\alpha| < 0,\ |\alpha^G| \geq 3,\ |\alpha^G| \text{ odd},\ |\beta^G| = 0, \text{ and } |\beta| > 0;$$

$$\iota_\alpha \iota_\beta = \iota_{\alpha+\beta}, \qquad \text{for } |\alpha| = |\beta| = 0.$$

REMARKS 4.10. (a) For p odd, the only units in A(1) are ± 1. The only generators in odd dimensions are the ν_α. Since $\nu_\alpha \nu_\beta$ is zero for any α and β, no sign problems occur in commuting products in $\underline{H}_G^* S^0$. Thus, $\underline{H}_G^* S^0$ is commutative in both the graded and ungraded senses.

(b) As an alternative to using the ν_α as a basis in the fourth quadrant, one may define elements $\epsilon_\beta^{-1} \xi_\alpha^{-1} \omega$ in $\underline{H}_G^{1-\alpha-\beta}(S^0)(1)$, for $|\alpha| = |\beta^G| = 0$, $|\alpha^G| < 0$, and $|\beta| > 0$, by

$$\epsilon_\beta^{-1} \xi_\alpha^{-1} \omega = d_{\alpha-<\alpha>}\, \nu_{1-\alpha-\beta}.$$

Here, ω is regarded as a fictitious element in dimension 1 which is divisible by any product $\xi_\alpha \epsilon_\beta$. We employ a fictitious element because there is no canonical choice for the dimension of an actual element. The relations satisfied by the elements $\epsilon_\beta^{-1} \xi_\alpha^{-1} \omega$ are

$$\epsilon_\gamma (\epsilon_\beta^{-1} \xi_\alpha^{-1} \omega) = \epsilon_{\beta-\gamma}^{-1} \xi_\alpha^{-1} \omega, \qquad \text{for } |\alpha| = |\beta^G| = |\gamma^G| = 0,\ |\beta| > |\gamma| > 0, \text{ and } |\alpha^G| < 0;$$

$$\xi_\gamma (\epsilon_\beta^{-1} \xi_\alpha^{-1} \omega) = \epsilon_\beta^{-1} \xi_{\alpha-\gamma}^{-1} \omega, \qquad \text{for } |\alpha| = |\gamma| = |\beta^G| = 0,\ |\alpha^G| < |\gamma^G| < 0, \text{ and } |\beta| > 0;$$

$$\mu_\gamma(\epsilon_\beta^{-1}\xi_\alpha^{-1}\omega) = \epsilon_{\beta-\gamma}^{-1}\xi_\alpha^{-1}\omega, \qquad \text{for } \gamma \in RSO_0(G),\ |\alpha| = |\beta^G| = 0,\ |\alpha^G| < 0, \text{ and } |\beta| > 0;$$

$$\mu_\gamma(\epsilon_\beta^{-1}\xi_\alpha^{-1}\omega) = d_{<\gamma>-\gamma}\,\epsilon_\beta^{-1}\xi_{\alpha-\gamma}^{-1}\omega, \qquad \text{for } \gamma \in RSO_0(G),\ |\alpha| = |\beta^G| = 0,\ |\alpha^G| < 0, \text{ and } |\beta| > 0.$$

The one difficulty with this alternative basis is that if $\alpha + \beta = \gamma + \delta$, then $\epsilon_\beta^{-1}\xi_\alpha^{-1}\omega$ and $\epsilon_\delta^{-1}\xi_\gamma^{-1}\omega$ are in the same dimension, but they need not be equal. In fact,

$$\epsilon_\beta^{-1}\xi_\alpha^{-1}\omega \;=\; d_{\alpha-\gamma-<\alpha-\gamma>}\,\epsilon_\delta^{-1}\xi_\gamma^{-1}\omega.$$

(c) Observe that in the formulas for the product of μ_α with any of ϵ_β, $\epsilon_\beta^{-1}\kappa_\gamma$, or ν_β there is no d_α, but there is such a constant in the formula for the product $\mu_\alpha\xi_\beta$. On the other hand, $\sigma_\alpha\xi_\beta = \xi_{\alpha+\beta}$, but there is a $d_{-\alpha}$ in the formula for the product of σ_α with any of ϵ_β, $\epsilon_\beta^{-1}\kappa_\gamma$, or ν_β. This difference in the behavior of the elements μ_α and σ_α of $\underline{H}_G^*(S^0)(1)$ reflects the fact that there is a conjugacy class of subgroups of G associated to any well chosen element of any G-Mackey functor M for any finite group G. This association is based on the splitting of M which occurs when M is localized away from the order of G. This splitting can not be observed directly before localization, but it can be seen indirectly in the association of subgroups to well chosen elements in the Mackey functor. The elements μ_α, ϵ_β, $\epsilon_\beta^{-1}\kappa_\gamma$, and ν_β are all associated to the subgroup G of G, and products of pairs of them behave nicely. The elements σ_α and ξ_β are associated to the trivial subgroup, and their product is nice. However, the product of elements associated to two different subgroups will either be zero or involve some fudge factor like a d_α. We have introduced both μ_α and σ_α so that, when one of these elements is needed in our description of the relations in $\underline{H}_G^*P(V)^+$, we can always choose the one that will give us the simpler formula.

REMARKS 4.11. In order to explain the passage from an RSO(G) grading on $\underline{H}_G^*S^0$ to an RO(G) grading, we must first clarify what is meant by the assertion that $\underline{H}_G^*S^0$ is RO(G)-graded. The assertion does not mean that, for $\alpha \in RO(G)$, $\underline{H}_G^\alpha S^0$ can be described without reference to a choice of a representative for α. Rather it means that if $V_1 - W_1$ and $V_2 - W_2$ are two representatives for α and $\underline{H}^1$ and $\underline{H}^2$ are the values of $\underline{H}_G^\alpha S^0$ obtained using these representatives, then we can construct an isomorphism between $\underline{H}^1$ and $\underline{H}^2$ in a natural way from any isomorphism $f\colon V_2 \oplus W_1 \to V_1 \oplus W_2$ of representations illustrating the equivalence of $V_1 - W_1$ and $V_2 - W_2$ in RO(G). This is exactly what we mean when we say that nonequivariant homology is $\mathbb{Z}$ graded. To define the nonequivariant homology group H^nX, we must pick a standard n-simplex. Different choices of the n-simplex lead to different groups, as anyone who has been embarrassed by an orientation mistake knows all too well.

Let $\beta = V_2 \oplus W_1 - V_1 \oplus W_2$ and let $\tilde{f}$ denote the image of f in $\underline{H}_G^{\beta}(S^0)(1)$. Then the isomorphism from $\underline{H}^1$ to $\underline{H}^2$ is just multiplication by $\tilde{f}$. To provide a means of computing the effect of this isomorphism, we write $\tilde{f}$ in terms of the standard generators of $\underline{H}_G^{\beta}(S^0)(1)$. The map f induces a map f^G between the fixed point subspaces of the representations. If nonequivariant orientations are choose for their domains and ranges, then the maps f and f^G have well-defined nonequivariant degrees. It follows from Lemma A.12 that

$$\tilde{f} = (\deg f^G)\,\mu_\beta + \frac{(\deg f) - (\deg f^G)d_\beta}{p}\,\tau(\iota_\beta).$$

The structure of $\underline{H}_G^* G^+$ as an algebra over $\underline{H}_G^* S^0$ follows easily from our results on $\underline{H}_G^* S^0$ and the description of the additive structure of $\underline{H}_G^* G^+$ given in section 2.

PROPOSITION 4.12. As an RO(G)-graded module over $\underline{H}_G^* S^0$, $\underline{H}_G^* G^+$ is generated by the single element $\psi = (1, 0, 0, \dots, 0)$ of $\underline{H}_G^0(G^+)(e) = \mathbb{Z}^p$. Moreover, for any RO(G)-graded module M^* over $\underline{H}_G^* S^0$, there is a one-to-one correspondence between maps $f\colon \underline{H}_G^* G^+ \to M^*$ of RO(G)-graded modules over $\underline{H}_G^* S^0$ and elements in $M^0(e)$. This correspondence associates the map f with the element $f(e)(\psi)$ of $M^0(e)$. Thus, $\underline{H}_G^* G^+$ is a projective RO(G)-graded module over $\underline{H}_G^* S^0$.

PROOF. Unless $|\alpha| = 0$, $\underline{H}_G^{\alpha}(G^+) = 0$. If $|\alpha| = 0$, then $\iota_\alpha \psi$ generates $\underline{H}_G^{\alpha} G^+$ as a module over A. Thus, ψ generates $\underline{H}_G^* G^+$ as an RO(G)-graded module over $\underline{H}_G^* S^0$, and any RO(G)-graded module map $f\colon \underline{H}_G^* G^+ \to M^*$ is determined by $f(\psi)$. On the other hand, recall the observation from Examples 1.1(f) that a map from A_G to any Mackey functor N can be specified by giving the image of $(1, 0, 0, \dots, 0) \in A_G(e)$ in N(e). Let m be an element of $M^0(e)$. For each $\alpha \in RO(G)$ with $|\alpha| = 0$, $\iota_\alpha m$ is in $M^\alpha(e)$ and there is a unique map $f^\alpha\colon \underline{H}_G^{\alpha} G^+ \to M^\alpha$ of Mackey functors sending $\iota_\alpha \psi \in \underline{H}_G^{\alpha}(G^+)(e)$ to $\iota_\alpha m \in M^\alpha(e)$. These maps fit together to form a map $f\colon \underline{H}_G^* G^+ \to M^*$ of RO(G)-graded modules over $\underline{H}_G^* S^0$. The projectivity of $\underline{H}_G^* G^+$ follows immediately.

5. THE MULTIPLICATIVE STRUCTURE OF $\underline{H}_G^*P(V)^+$. We assume that there are at least two distinct isomorphism classes of irreducibles in V; otherwise, the multiplicative structure of $\underline{H}_G^*P(V)^+$ is completely described in Examples 1.1.(h). As in section 3, we take Φ to be the set of irreducible summands of the complex representation V. Let $\Phi(0)$, $\Phi(1)$, $\Phi(2)$, ... be a proper filtration of Φ. Then $\Phi(1)$ consists of exactly one representative of each of the isomorphism classes of irreducibles that appears in Φ. Let $\phi_0, \phi_1, \phi_2, \ldots, \phi_m$ be an enumeration of the elements in $\Phi(1)$, and let n_i be the number of elements of Φ isomorphic to ϕ_i (with $n_i = \infty$ allowed). Arrange the enumeration of the elements of $\Phi(1)$ so that $n_0 \geq n_1 \geq \ldots \geq n_m$. Extend the ordering of $\Phi(1)$ to Φ by selecting the unique proper ordering of Φ which is consistent with the filtration and in which, for each $r \geq 1$, the ordering of the representations in $\Phi(r+1) - \Phi(r)$ is the same as the ordering of the corresponding representations in $\Phi(1)$. If the irreducibles which appear in Φ appear with equal multiplicity, then, regarded as an ordered set, Φ is a sequence of blocks, each of which is a copy of $\Phi(1)$. If the multiplicities are not equal, then Φ is still a sequence of blocks, but each block after the first will be either a copy of $\Phi(1)$ or of an initial segment of $\Phi(1)$. The lengths of the initial segments in the sequence can not increase. We will abuse notation by writing $\phi_i \in \Phi(r+1) - \Phi(r)$ to mean that $\Phi(r+1) - \Phi(r)$ contains an irreducible representation isomorphic to ϕ_i. We say that two sets of irreducible representations are equivalent if they contain the same number of irreducibles in each isomorphism class. Moreover, we sometimes identify equivalent sets of irreducibles.

Corollary 2.7 will be used to derive the multiplicative structure of $\underline{H}_G^*P(V)^+$ from the multiplicative structures of $\underline{H}_G^*(P(V)^+)(e)$ and $\underline{H}_G^*((P(V)^G)^+)(1)$. The group $\underline{H}_G^\alpha(P(V)^+)(e)$ is isomorphic to the nonequivariant cohomology group $H^{|\alpha|}(P(V)^+;\mathbb{Z})$, and we will think of the restriction map ρ as a map from $\underline{H}_G^\alpha(P(V)^+)(1)$ to $\underline{H}^{|\alpha|}(P(V)^+;\mathbb{Z})$. Select an algebra generator $x \in \underline{H}^2(P(V)^+;\mathbb{Z})$ for $\underline{H}^*(P(V)^+;\mathbb{Z})$. The fixed point space of P(V) is the disjoint union of the spaces $P(n_i\phi_i) \cong P(n_i)$. Let q_i denote both the inclusion of the subspace $P(n_i)$ into P(V) and the map $\underline{H}_G^*(P(V)^+)(1) \to \underline{H}_G^*(P(n_i)^+)(1)$ induced by this inclusion. By Examples 1.1.(h), $\underline{H}_G^*P(n_i)^+$ is a truncated polynomial algebra over $\underline{H}_G^*S^0$ generated by an element x_i in $\underline{H}_G^2(P(n_i)^+)(1)$. Let

$$\tilde{q}_i : \underline{H}_G^*(P(V)^+)(1) \to \underline{H}_G^*(P(n_i)^+)(1)/(\text{torsion} \oplus \text{im } \rho)$$

denote the composition of q_i and the projection onto the quotient. If y is in $\underline{H}_G^*(P(n_i)^+)(1)$, then [y] denotes its image in $\underline{H}_G^*(P(n_i)^+)(1)/(\text{torsion} \oplus \text{im } \rho)$.

Throughout this section, we will index $\underline{H}_G^*P(V)^+$ on RSO(G) to simplify the selection of the integers d_α. The comments in Remarks 4.11 on the passage from RSO(G)-grading to RO(G)-grading for $\underline{H}_G^*S^0$ apply equally well to $\underline{H}_G^*P(V)^+$. Recall that λ is the irreducible complex representation that sends the standard generator of

$\mathbb{Z}/p$ to $e^{2\pi i/p}$ and that ζ is the real one-dimensional sign representation of $\mathbb{Z}/2$. If p is 2, then λ, regarded as a real representation, is just 2ζ.

We begin with the case $p = 2$. Any complex irreducible representation is isomorphic to either the complex one-dimensional trivial representation or the complex one-dimensional sign representation λ. Since $P(V)$ and $P(\lambda V)$ are G-homeomorphic, we may assume that there are at least as many copies of the trivial representation in Φ as there are copies of the sign representation. Thus, we may take ϕ_0 to be the trivial representation and ϕ_1 to be the sign representation.

By Theorem 3.1, $\underline{H}^*_G P(V)^+$, regarded as a module over $\underline{H}^*_G S^0$, has one generator in each of the dimensions

$$2k + 2k\zeta \quad \text{and} \quad 2k + 2(k+1)\zeta,$$

for $0 \le k < n_1$, and one in each of the dimensions

$$2k + 2n_1\zeta,$$

for $n_1 \le k < n_0$. If one assumes $n_0 = n_1$, or ignores the generators special to the case $n_0 > n_1$, then one might guess that, as an algebra, $\underline{H}^*_G P(V)^+$ had an exterior generator in dimension 2ζ and a truncated polynomial generator in dimension $2(1+\zeta)$. Except for the fact that the generator in dimension 2ζ is not quite an exterior generator and for some difficulties in the higher dimensions when $n_0 > n_1$, this guess is a good description of $\underline{H}^*_G P(V)^+$. However, in order to describe the behavior in the higher dimensions as simply as possible, we adopt a notation that does not immediately suggest this.

THEOREM 5.1. (a) As an algebra over $\underline{H}^*_G S^0$, $\underline{H}^*_G P(V)^+$ is generated by an element c of $\underline{H}^*_G(P(V)^+)(1)$ in dimension 2ζ and elements $C(k)$ of $\underline{H}^*_G(P(V)^+)(1)$ in dimensions $2k + 2\min(k, n_1)\zeta$, for $1 \le k < n_0$.

(b) For any positive integer k, let $\bar{k}$ denote the minimum of k and n_1. Then the generators c and $C(k)$ are uniquely determined by

$$\tilde{q}_0(c) = [0]$$
$$\tilde{q}_1(c) = [\epsilon^2]$$
$$\rho(c) = x \in H^2(P(V)^+; \mathbb{Z})$$
$$\tilde{q}_0(C(k)) = \left[\epsilon^{2\bar{k}} x_0^k\right]$$
$$\tilde{q}_1(C(k)) = \left[\epsilon^{2\bar{k}} x_1^k\right]$$

and

$$\rho(C(k)) = x^{k+\bar{k}}.$$

Moreover,

$$q_0(c) = \xi x_0 \in \underline{H}_G^{2\zeta}(P(n_0)^+)(1)$$
$$q_1(c) = \epsilon^2 + \xi x_1 \in \underline{H}_G^{2\zeta}(P(n_1)^+)(1)$$
$$q_0(C(k)) = x_0^k(\epsilon^2 + \xi x_0)^{\bar{k}} \in \underline{H}_G^{2(k+\bar{k}\zeta)}(P(n_0)^+)(1)$$

and

$$q_1(C(k)) = x_1^k(\epsilon^2 + \xi x_1)^{\bar{k}} \in \underline{H}_G^{2(k+\bar{k}\zeta)}(P(n_1)^+)(1).$$

If n_i is finite, then $x_i^{n_i} = 0$ and some of the terms in the last two sums above may vanish.

(c) The generators c and C(k) satisfy the relations

$$c^2 = \epsilon^2 c + \xi C(1),$$
$$c\,C(k) = \xi C(k+1), \qquad \text{for } k \geq n_1,$$

and

$$C(j)\,C(k) = \begin{cases} C(j+k), & \text{for } j+k \leq n_1, \\ \sum_{i=0}^{\bar{j}+\bar{k}-n_1} \binom{\bar{j}+\bar{k}-n_1}{i} \epsilon^{2(\bar{j}+\bar{k}-n_1-i)} \xi^i C(j+k+i), & \text{for } j+k > n_1. \end{cases}$$

In these relations, we take C(i) to be zero if $i \geq n_0$.

REMARKS 5.2. (a) By iteratively applying the third relation, we obtain

$$C(k) = (C(1))^k, \qquad \text{for } k \leq n_1,$$

so that below the dimensions where we run short of copies of the sign representation, $\underline{H}_G^* P(V)^+$ is generated by c and C(1). Moreover, in these dimensions, C(1) acts like a polynomial generator.

(b) If $n_0 = n_1$, then $\underline{H}_G^* P(V)^+$ is generated by c and C(1). The only relations satisfied by these two generators are the relation

$$c^2 = \epsilon^2 c + \xi C(1)$$

and, if $n_0 < \infty$, the relation

$$C(1)^{n_0} = 0.$$

REMARKS 5.3. Notice that the maps $\tilde{q}_0$ and $\tilde{q}_1$ behave differently on the generator c. The element $\tilde{c} = c + \epsilon^2 - \kappa c$ of $\underline{H}_G^{2\zeta} P(V)^+$ may be used as a generator in the place of c and its behavior with respect to $\tilde{q}_0$ and $\tilde{q}_1$ is exactly the reverse of the behavior of c. To understand the geometric relation between these elements, observe that c and $\tilde{c}$ can be detected in the cohomology of any subspace $P(1+\lambda)$ of P(V) arising from an inclusion $1+\lambda \subset V$. The space $P(1+\lambda)$ is G-homeomorphic to S^λ, but unlike S^λ, it lacks a canonical basepoint. Either choice for the basepoint of $P(1+\lambda)$ determines a splitting of $\underline{H}_G^* P(1+\lambda)^+$ into the direct sum of one copy of

$\underline{H}^*_G S^0$ and one copy of $\underline{H}^*_G S^\lambda$. The canonical generator of $\underline{H}^*_G S^\lambda$ in dimension 2ζ is identified with c by one of the two splittings and with $\tilde{c}$ by the other.

When p is 2, the multiplicative structure of $\underline{H}^*_G P(V)^+$ does not really exhibit any complexities beyond those one might experience in a $\mathbb{Z}$-graded ring. However, when p is odd, there are quirks in the multiplicative structure of $\underline{H}^*_G P(V)^+$ which are only possible because of the RSO(G)-grading. For the odd prime case, recall the stairstep diagram obtained by plotting the dimensions α of the generators of $\underline{H}^*_G P(V)^+$ in terms of $|\alpha|$ and $|\alpha^G|$. Looking at this diagram in the special case where the irreducibles appearing in V appear with equal multiplicity, one might guess that $\underline{H}^*_G P(V)^+$ was generated by two truncated polynomial generators, one in a dimension α with $|\alpha| = 2$ and $|\alpha^G| = 0$ and one in a dimension β with $|\beta| = 2m + 2$ and $|\beta^G| = 2$. Unfortunately, such a guess would badly underestimate the complexity of $\underline{H}^*_G P(V)^+$. The set of dimensions for a full set of additive generators must generate a larger additive subgroup of RSO(G) than can be accounted for by a pair of truncated polynomial generators. For example, recall that the first two additive generators of $\underline{H}^*_G P(V)^+$ are in dimensions $\phi_1^{-1}\phi_0$ and $\phi_2^{-1}(\phi_0 + \phi_1)$. If the additive generator in dimension $\phi_1^{-1}\phi_0$ were to serve as a truncated polynomial generator, then the additive generator in the next higher dimension would need to be in dimension $2\phi_1^{-1}\phi_0$ instead of $\phi_2^{-1}(\phi_0 + \phi_1)$. Any replacement of these two generators by an element and its square requires the introduction of further generators in some other dimensions inconsistent with a simple truncated polynomial structure. To provide a better feeling for the multiplicative structure of $\underline{H}^*_G P(V)^+$, we give two sets of multiplicative generators. The first is a natural set with a great deal of symmetry. It does not exhibit a preference for any one ordering of Φ. Unfortunately, this set is much too large. By selecting an ordering on Φ, we are able to construct a much smaller, but very asymmetrical, set of algebra generators.

In order to describe the effect of the maps q_i on our algebra generators, we must introduce more notation related to the integers d_α.

DEFINITIONS 5.4. (a) For any two distinct integers i and j with $0 \le i, j \le m$, let β_{ij} denote the irreducible representation $\phi_i^{-1}\phi_j$, and let d^{ij}_{rs} denote the integer d_α, for $\alpha = \beta_{ij} - \beta_{rs}$. Note that d^{ij}_{ij} is 1 for any pair of distinct integers i and j. For any integer i and any distinct pair of integers r and s such that $0 \le i, r, s \le m$, let d^{ii}_{rs} be zero. The integers d^{ij}_{rs} satisfy the relations

$$d^{ij}_{rs} d^{rs}_{uv} \equiv d^{ij}_{uv} \qquad \text{mod } p,$$

$$d^{ij}_{rs} + d^{jk}_{rs} \equiv d^{ik}_{rs} \qquad \text{mod } p,$$

and

$$d^{ij}_{rs} d^{tu}_{vw} \equiv d^{tu}_{rs} d^{ij}_{vw} \qquad \text{mod } p.$$

(b) If $\phi_i \in \Phi(r+1)-\Phi(r)$, then let $\alpha_i(r)$ denote the representation $\phi_i^{-1}\sum_{\phi\in\Phi(r)}\phi$, and let $\tilde{d}^r_{ij}$ be d_α, for $\alpha=\alpha_i(r)-\alpha_j(r)$. Note that, if $\phi_i \in \Phi(r+1)-\Phi(r)$, $\tilde{d}^r_{ii}=1$. If either ϕ_i or ϕ_j is not in $\Phi(r+1)-\Phi(r)$, then let $\tilde{d}^r_{ij}$ be zero. If ϕ_i, ϕ_j, and ϕ_k are in $\Phi(r+1)-\Phi(r)$, then the integers $\tilde{d}^r_{ij}$ satisfy the relations

$$\tilde{d}^r_{ij}\tilde{d}^r_{jk} \equiv \tilde{d}^r_{ik} \qquad \text{mod } p$$

and, if $i \neq j$,

$$\tilde{d}^r_{ij} \equiv (d^{ij}_{ji})^r \prod_{\substack{0\le k\le m\\ k\neq i,j}} (d^{ik}_{jk})^{a_k} \qquad \text{mod } p,$$

where a_k is the multiplicity of ϕ_k in $\Phi(r)$.

THEOREM 5.5. (a) If i and j are distinct integers with $0\le i, j\le m$, then there is a unique element c_{ij} in $\underline{H}_G^{\beta_{ij}}(P(V)^+)(1)$ such that

$$\tilde{q}_k(c_{ij}) = \left[d^{kj}_{ij}\,\epsilon_{\beta_{ij}}\right], \qquad \text{for } 0\le k\le m,$$

and

$$\rho(c_{ij}) = x.$$

If $r\ge 0$ and $\phi_j \in \Phi(r+1)-\Phi(r)$, then there is a unique element $C_j(r)$ in $\underline{H}_G^{\alpha_j(r)}(P(V)^+)(1)$ such that

$$\tilde{q}_k(C_j(r)) = \left[\tilde{d}^r_{kj}\,\epsilon_{\alpha_j(r)-r}\,x^r_k\right], \qquad \text{for } 0\le k\le m,$$

and

$$\rho(C_j(r)) = x^{|\alpha_j(r)|/2}.$$

The elements c_{ij}, for $0\le i,j\le m$ and $i\neq j$, and the elements $C_k(r)$, for $r\ge 1$ and $\phi_k \in \Phi(r+1)-\Phi(r)$, generate $\underline{H}_G^*P(V)^+$ as an algebra over $\underline{H}_G^*S^0$.

(b) For $0\le i,j,k\le m$ and $i\neq j$,

$$q_k(c_{ij}) = d^{kj}_{ij}\,\epsilon_{\beta_{ij}} + \xi_{\beta_{ij}-2}\,x_k.$$

(c) For $r\ge 1$ and $\phi_k \in \Phi(r+1)-\Phi(r)$,

$$q_k(C_k(r)) = x_k^r \left[\prod_{\substack{\phi_i \in \Phi(r) \\ i \neq k}} (\epsilon_{\beta_{ki}} + \xi_{\beta_{ki}-2} x_k) \right].$$

If $\phi_j \in \Phi(r+1) - \Phi(r)$ and $j \neq k$, then

$$q_k(C_j(r)) = x_k^r \left(d_{jk}^{kj} \epsilon_{\beta_{jk}} + \xi_{\beta_{jk}-2} x_k \right)^r \left[\prod_{\substack{\phi_i \in \Phi(r) \\ i \neq j,k}} (d_{ji}^{ki} \epsilon_{\beta_{ji}} + \xi_{\beta_{ji}-2} x_k) \right] +$$

$$\left[\tilde{d}_{kj}^r - (d_{jk}^{kj})^r \prod_{\substack{\phi_i \in \Phi(r) \\ i \neq j,k}} (d_{ji}^{ki}) \right] \epsilon_{\alpha_j(r)-r} x_k^r .$$

If $\phi_k \notin \Phi(r+1) - \Phi(r)$, then $q_k(C_j(r))$ is zero.

(d) For $1 \leq j \leq m$, let γ_j be the representation $\phi_j^{-1} \sum_{i=0}^{j-1} \phi_i$ and let D_j be the element $\prod_{i=0}^{j-1} c_{ji}$ in $\underline{H}_G^{\gamma_j}(P(V)^+)(1)$. Then the elements D_j, for $1 \leq j \leq m$, the elements $C_0(r)$, for $r \geq 1$ and $\phi_0 \in \Phi(r+1) - \Phi(r)$, and the elements $D_j C_j(r)$, for $r \geq 1$ and $\phi_j \in \Phi(r+1) - \Phi(r)$, generate $\underline{H}_G^* P(V)^+$ as an algebra over $\underline{H}_G^* S^0$.

REMARKS 5.6. In order to simplify our indexing, we define D_0 and $C_j(0)$, for $0 \leq j \leq m$, to be $1 \in \underline{H}_G^0(P(V)^+)(1)$. We also define γ_0 and $\alpha_j(0)$ to be 0. Our second set of generators for $\underline{H}_G^* P(V)^+$ is then just the set of elements $D_j C_j(r)$, for $r \geq 0$ and $\phi_j \in \Phi(r+1) - \Phi(r)$. This set of elements of $\underline{H}_G^*(P(V)^+)(1)$ is also a set of additive generators of $\underline{H}_G^* P(V)^+$ as a module over $\underline{H}_G^* S^0$. One might hope that a set of multiplicative generators could be much smaller than a set of additive generators, but if the various irreducibles in Φ appear with very different multiplicities, then small sets of multiplicative generators do not exist.

We will order the set of generators $D_j C_j(r)$ by the dictionary order on r and then j. On the stairstep plot of the dimensions of these generators, moving in the direction of increasing order corresponds to moving up and to the right.

REMARKS 5.7. Nothing that has been said in the discussion of the odd prime case actually depends on p being odd; rather, mod 2 arithmetic is so simple that most of the technicalities necessary when p is odd are unnecessary when $p = 2$. The elements c and $\tilde{c}$ in the case $p = 2$ are c_{10} and c_{01}. The element C(j) is $C_0(j)$.

In order to describe the relations among the generators in $\underline{H}_G^* P(V)^+$ in a palatable form, we must introduce one more batch of elements in $\underline{H}_G^*(P(V)^+)(1)$.

DEFINITION 5.8. Observe that, for $1 \leq j \leq m$, κD_j is divisible by ϵ_{γ_j}. Moreover, $\rho(\epsilon_{\gamma_j}^{-1} \kappa D_j) = 0$, and

$$\tilde{q}_k(\epsilon_{\gamma_j}^{-1} \kappa D_j) = \left[p \prod_{i=0}^{j-1} d_{ji}^{ki} \right] \in \underline{H}_G^0(P(n_k \phi_k)^+)/(\text{torsion} \oplus \text{im } \tau).$$

Since $\prod_{i=0}^{j-1} d_{ji}^{ki}$ is zero if $k < j$ and 1 if $k = j$, the coefficients $p \prod_{i=0}^{j-1} d_{ji}^{ki}$ which appear in the $\tilde{q}_k(\epsilon_{\gamma_j}^{-1} \kappa D_j)$ form a matrix which is p times an upper triangular matrix with 1's on the main diagonal. Applying the obvious analog of the process for diagonalizing an upper triangular matrix to the elements $\epsilon_{\gamma_j}^{-1} \kappa D_j$ produces elements $\hat{\kappa}_j$ of $\underline{H}_G^0(P(V)^+)(1)$ characterized by the conditions

$$\rho(\hat{\kappa}_j) = 0,$$

and

$$\tilde{q}_k(\hat{\kappa}_j) = \begin{cases} [p], & \text{if } k = j, \\ 0, & \text{otherwise.} \end{cases}$$

These elements can be described inductively by the equations

$$\hat{\kappa}_m = \epsilon_{\gamma_m}^{-1} \kappa D_m$$

and, for $1 \leq j < m$,

$$\hat{\kappa}_j = \epsilon_{\gamma_j}^{-1} \kappa D_j - \sum_{k=j+1}^{m} \left(\prod_{i=0}^{j-1} d_{ji}^{ki} \right) \hat{\kappa}_k .$$

Define $\hat{\kappa}_0 \in \underline{H}_G^0(P(V)^+)(1)$ to be $\kappa - \sum_{j=1}^{m} \hat{\kappa}_j$. The equations above characterizing $\hat{\kappa}_j$ for $j \neq 0$ then also characterize $\hat{\kappa}_0$. Moreover,

$$q_k(\hat{\kappa}_j) = \begin{cases} p, & \text{if } k = j, \\ 0, & \text{otherwise.} \end{cases}$$

For $r \geq 1$ and $\phi_j \in \Phi(r+1) - \Phi(r)$, define $\hat{\kappa}_j(r) \in \underline{H}_G^{\alpha_j(r)}(P(V)^+)(1)$ to be $\hat{\kappa}_j C_j(r)$. These elements $\hat{\kappa}_j(r)$ are characterized by the equations

$$\rho(\hat{\kappa}_j(r)) = 0,$$

and

$$\tilde{q}_k(\hat{\kappa}_j(r)) = \begin{cases} \left[p\, \epsilon_{\alpha_j(r) - r} x_k^r \right], & \text{if } k = j, \\ 0, & \text{otherwise.} \end{cases}$$

Moreover,

$$q_k(\hat{\kappa}_j(r)) = \begin{cases} p\,\epsilon_{\alpha_j(r)-r}\,x_k^r, & \text{if } k=j, \\ 0, & \text{otherwise.} \end{cases}$$

For convenience, we define $\hat{\kappa}_j(0)$ to be $\hat{\kappa}_j$. Observe that, for $r \geq 1$, the elements $\hat{\kappa}_j(r)$ can also be constructed from the elements $\kappa D_j C_j(r)$ in the same way that the elements $\hat{\kappa}_j$ are constructed from the κD_j.

We begin our list of relations with the relation between any two of the c_{ij} and the relation between any two of the $C_j(r)$.

PROPOSITION 5.9. (a) Let i, j, r, and s be integers with $0 \leq i, j, r, s \leq m$ and $i \neq j$, $r \neq s$. Then

$$c_{ij} = \sigma_{\beta_{ij}-\beta_{rs}} c_{rs} + d_{ij}^{sj}\,\epsilon_{\beta_{ij}} + \sum_{k\neq s} \frac{d_{ij}^{kj} - d_{ij}^{sj} - d_{ij}^{rs}\,d_{rs}^{ks}}{p}\,\epsilon_{\beta_{ij}}\,\hat{\kappa}_k .$$

(b) Let $r \geq 1$ and let i and j be integers such that ϕ_i and ϕ_j are in $\Phi(r+1) - \Phi(r)$. Then

$$C_i(r) = \sigma_{\alpha_i(r)-\alpha_j(r)} C_j(r) + \sum_{k\neq j} \frac{\tilde{d}_{ki}^{\,r} - \tilde{d}_{kj}^{\,r}\,\tilde{d}_{ji}^{\,r}}{p}\,\mu_{\alpha_i(r)-\alpha_k(r)}\,\hat{\kappa}_k(r).$$

An obvious initial response to this result is to assume that $\underline{H}_G^* P(V)^+$ can be generated as an algebra over $\underline{H}_G^* S^0$ by any one of the c_{ij} and, for each r with $\Phi(r+1) - \Phi(r)$ nonemepty, any one of the $C_j(r)$. The $\hat{\kappa}_k$ and $\hat{\kappa}_k(r)$ in the formulas spoil this simplification, especially since they are defined in terms of precisely the generators one would hope to omit. Solving this by taking the elements $\hat{\kappa}_k$ and $\hat{\kappa}_k(r)$ as part of a generating set is hardly satisfactory since, from a Mackey functor point of view, these are torsion elements (because $\rho(\hat{\kappa}_k)$ and $\rho(\hat{\kappa}_k(r))$ are zero).

The remaining results in this section describe the products of pairs of elements from either of the generating sets in terms of the smaller generating set. All of the relations in $\underline{H}_G^* P(V)^+$ follow from the relations in Proposition 5.9 and the relations below. If V is finite, then some of the elements appearing on the right hand side of these relations may not appear in the list of generators of $\underline{H}_G^* P(V)^+$. Any such element is to be regarded as zero. We begin with the products which land in dimensions where there is no torsion. These are easily computed using the maps $\tilde{q}_k$ and ρ.

PROPOSITION 5.10. (a) Let i, j, r, and s be integers with $0 \leq i,j,r,s \leq m$ and $i \neq j$, $r \neq s$. If $m \geq 2$, then

$$c_{ij} c_{rs} = d_{ij}^{0j}\,d_{rs}^{0s}\,\epsilon_{\beta_{ij}+\beta_{rs}} + (d_{ij}^{1j}\,d_{rs}^{1s} - d_{ij}^{0j}\,d_{rs}^{0s})\,\epsilon_{\beta_{ij}+\beta_{rs}-\beta_{10}}\,c_{10} + \sigma_\alpha D_2 +$$

$$\sum_{k=2}^{m} \frac{d_{ij}^{kj}\,d_{rs}^{ks} - d_{ij}^{0j}\,d_{rs}^{0s} - (d_{ij}^{1j}\,d_{rs}^{1s} - d_{ij}^{0j}\,d_{rs}^{0s})\,d_{10}^{k0} - d_{20}^{k0}\,d_{21}^{k1}\,d_{-\alpha}}{p}\,\epsilon_{\beta_{ij}+\beta_{rs}}\,\hat{\kappa}_k ,$$

where $\alpha = \beta_{ij} + \beta_{rs} - \gamma_2$.

If m = 1, then

$$c_{ij} c_{rs} = d_{ij}^{0j} d_{rs}^{0s} \epsilon_{\beta_{ij}+\beta_{rs}} + (d_{ij}^{1j} d_{rs}^{1s} - d_{ij}^{0j} d_{rs}^{0s}) \epsilon_{\beta_{ij}+\beta_{rs}-\beta_{10}} c_{10} + \xi_{\beta_{ij}+\beta_{rs}-\alpha_0(1)} C_0(1).$$

(b) Let i, j, and r be integers with $0 \leq i, j \leq m$, $i \neq j$, and $1 \leq r < m$. Then

$$c_{ij} D_r = d_{ij}^{rj} \epsilon_{\beta_{ij}} D_r + \sigma_\alpha D_{r+1} + \sum_{k=r+1}^{m} \frac{(d_{ij}^{kj} - d_{ij}^{rj}) \prod_{s=0}^{r-1} d_{rs}^{ks} - d_{-\alpha} \prod_{s=0}^{r} d_{r+1,s}^{ks}}{p} \epsilon_{\beta_{ij}+\gamma_r} \hat{\kappa}_k,$$

where $\alpha = \beta_{ij} + \gamma_r - \gamma_{r+1}$.

(c) Let i, j be integers with $0 \leq i, j \leq m$ and $i \neq j$. Then

$$c_{ij} D_m = d_{ij}^{mj} \epsilon_{\beta_{ij}} D_m + \xi_{\beta_{ij}+\gamma_m-\alpha_0(1)} C_0(1).$$

(d) Let i, j, r, and s be integers with $0 \leq i, j, s \leq m$, $i \neq j$, $r \geq 1$, and $\phi_s \in \Phi(r+1) - \Phi(r)$. If $\phi_1 \in \Phi(r+1) - \Phi(r)$, then

$$c_{ij} C_s(r) = d_{ij}^{0j} \tilde{d}_{0s}^{r} \epsilon_{\beta_{ij}+\alpha_s(r)-\alpha_0(r)} C_0(r) + \sigma_\alpha D_1 C_1(r) + \sum_{\substack{k \geq 1 \\ \phi_k \in \Phi(r+1)-\Phi(r)}} \frac{d_{ij}^{kj} \tilde{d}_{ks}^{r} - d_{ij}^{0j} \tilde{d}_{0s}^{r} \tilde{d}_{k0}^{r} - d_{10}^{k0} \tilde{d}_{k1}^{r} d_{-\alpha}}{p} \epsilon_{\beta_{ij}+\alpha_s(r)-\alpha_k(r)} \hat{\kappa}_k(r),$$

where $\alpha = \beta_{ij} + \alpha_s(r) - \gamma_1 - \alpha_1(r)$.

If $\phi_1 \notin \Phi(r+1) - \Phi(r)$, then

$$c_{ij} C_s(r) = d_{ij}^{0j} \tilde{d}_{0s}^{r} \epsilon_{\beta_{ij}} C_0(r) + \xi_{\beta_{ij}+\alpha_0(r)-\alpha_0(r+1)} C_0(r+1).$$

(e) Let i, j, r, and s be integers with $0 \leq i, j, s \leq m$, $i \neq j$, $r \geq 1$, and $\phi_s \in \Phi(r+1) - \Phi(r)$. If $\phi_{s+1} \in \Phi(r+1) - \Phi(r)$, then

$$c_{ij} D_s C_s(r) = d_{ij}^{sj} \epsilon_{\beta_{ij}} D_s C_s(r) + \sigma_\alpha D_{s+1} C_{s+1}(r) + \sum_{\substack{k \geq s+1 \\ \phi_k \in \Phi(r+1)-\Phi(r)}} \frac{\tilde{d}_{ks}^{r} (d_{ij}^{kj} - d_{ij}^{sj}) \prod_{t=0}^{s-1} d_{st}^{kt} - \tilde{d}_{k,s+1}^{r} d_{-\alpha} \prod_{t=0}^{s} d_{s+1,t}^{kt}}{p} \epsilon_{\delta_k} \hat{\kappa}_k(r),$$

where $\alpha = \beta_{ij} + \gamma_s + \alpha_s(\mathrm{r}) - \gamma_{s+1} - \alpha_{s+1}(\mathrm{r})$ and $\delta_k = \beta_{ij} + \gamma_s + \alpha_s(\mathrm{r}) - \alpha_k(\mathrm{r})$.

If $\phi_{s+1} \notin \Phi(\mathrm{r}+1) - \Phi(\mathrm{r})$, then

$$\mathrm{c}_{ij}\,\mathrm{D}_s\,\mathrm{C}_s(\mathrm{r}) = \mathrm{d}_{ij}^{sj}\,\epsilon_{\beta_{ij}}\mathrm{D}_s\,\mathrm{C}_s(\mathrm{r}) + \xi_{\beta_{ij}+\gamma_s+\alpha_s(r)-\alpha_0(r+1)}\mathrm{C}_0(\mathrm{r}+1).$$

(f) Let r, $\mathrm{s} \geq 1$ and assume that $1 \leq \mathrm{j} \leq \mathrm{m}$. If the irreducibles that appear in $\Phi(\mathrm{r}+\mathrm{s})$ appear with equal multiplicities, then

$$\mathrm{C}_j(\mathrm{r})\,\mathrm{C}_j(\mathrm{s}) = \mathrm{C}_j(\mathrm{r}+\mathrm{s}) + \sum_{\substack{k \neq j \\ \phi_k \in \Phi(r+s+1)-\Phi(r+s)}} \frac{\tilde{\mathrm{d}}_{kj}^{r}\,\tilde{\mathrm{d}}_{kj}^{s} - \tilde{\mathrm{d}}_{kj}^{r+s}}{\mathrm{p}}\,\mu_{\alpha_j(r+s)-\alpha_k(r+s)}\,\hat{\kappa}_k(\mathrm{r}+\mathrm{s}).$$

Moreover, the integers $\tilde{\mathrm{d}}_{kj}^{r+s}$ may be selected to be the products $\tilde{\mathrm{d}}_{kj}^{r}\,\tilde{\mathrm{d}}_{kj}^{s}$ so that the $\hat{\kappa}_k(\mathrm{r}+\mathrm{s})$ correction terms are not needed.

Since the elements $\hat{\kappa}_k(\mathrm{r})$ appear in so many formulas, we include a description of products involving them.

LEMMA 5.11. Let i, j, k, r, and s be integers with $0 \leq \mathrm{i,j,k} \leq \mathrm{m}$, $\mathrm{r, s} \geq 0$, and $\phi_k \in \Phi(\mathrm{s}+1) - \Phi(\mathrm{s})$.

(a) If $\mathrm{i} \neq \mathrm{j}$, then

$$\mathrm{c}_{ij}\,\hat{\kappa}_k(\mathrm{s}) = \mathrm{d}_{ij}^{kj}\,\epsilon_{\beta_{ij}}\,\hat{\kappa}_k(\mathrm{s}).$$

(b) If $\phi_j \in \Phi(\mathrm{r}+1) - \Phi(\mathrm{r})$ and $\phi_k \in \Phi(\mathrm{r}+\mathrm{s}+1) - \Phi(\mathrm{r}+\mathrm{s})$, then

$$\mathrm{C}_j(\mathrm{r})\,\hat{\kappa}_k(\mathrm{s}) = \tilde{\mathrm{d}}_{kj}^{r}\,\epsilon_{\alpha_j(r)+\alpha_k(r)-\alpha_k(r+s)}\,\hat{\kappa}_k(\mathrm{r}+\mathrm{s})$$

and

$$\mathrm{D}_j\,\mathrm{C}_j(\mathrm{r})\,\hat{\kappa}_k(\mathrm{s}) = \tilde{\mathrm{d}}_{kj}^{r}\left[\prod_{t=0}^{j-1}\mathrm{d}_{jt}^{kt}\right]\epsilon_{\gamma_j+\alpha_j(r)+\alpha_k(s)-\alpha_k(r+s)}\,\hat{\kappa}_k(\mathrm{r}+\mathrm{s}).$$

In the formula for $\mathrm{C}_j(\mathrm{r})\,\hat{\kappa}_k(\mathrm{s})$, replace $\epsilon_{\alpha_j(r)+\alpha_k(s)-\alpha_k(r+s)}$ by $\mu_{\alpha_j(r)+\alpha_k(s)-\alpha_k(r+s)}$ if $|\alpha_j(\mathrm{r}) + \alpha_k(\mathrm{s}) - \alpha_k(\mathrm{r}+\mathrm{s})|$ is zero.

(c) If $\phi_j \in \Phi(\mathrm{r}+1) - \Phi(\mathrm{r})$ and $\phi_k \notin \Phi(\mathrm{r}+\mathrm{s}+1) - \Phi(\mathrm{r}+\mathrm{s})$, then $\mathrm{C}_j(\mathrm{r})\,\hat{\kappa}_k(\mathrm{s})$ and $\mathrm{D}_j\,\mathrm{C}_j(\mathrm{r})\,\hat{\kappa}_k(\mathrm{s})$ are zero.

To complete our description of the multiplicative structure of $\underline{\mathrm{H}}_{\mathrm{G}}^{*}\mathrm{P}(\mathrm{V})^{+}$ we need to describe the products of various pairs made from elements of the types $\mathrm{C}_i(\mathrm{r})$,

$D_jC_j(r)$, and D_k. If we use the convention that $D_0 = C_j(0) = 1$, then the products we must describe are all special cases of the general product $(D_{i'}C_i(r))(D_{j'}C_j(s))$, where $r, s \geq 0$, $\phi_i \in \Phi(r+1) - \Phi(r)$, $\phi_j \in \Phi(s+1) - \Phi(s)$, i' is 0 or i, and j' is 0 or j. We may assume that $i' \geq j'$. Recall the formula given in Theorem 5.1(c) for the product $C(j)C(k)$ when $p = 2$ and $j + k > n_1$. Observe that this formula may be obtained from the binomial expansion of $(\epsilon^2 + \xi x)^{\bar{j}+\bar{k}-n_1}$ by replacing the powers of x by various generators C(t). The formula for our general product is related in a similar way to the expansion of an expression of the form $\prod_{i=0}^{n}(a_i + b_i x)$. The summands in this expansion are indexed on the subsets of the set $\{0, 1, \ldots, n\}$. The summand corresponding to the subset I is

$$\Big(\prod_{i \notin I} a_i\Big)\Big(\prod_{i \in I} b_i\Big)x^{|I|},$$

where $|I|$ denotes the number of elements in I. To describe the analogous part of our formula for $(D_{i'}C_i(r))(D_{j'}C_j(s))$, we must specify the indexing set which replaces $\{0, 1, \ldots, n\}$, the factors which replace $\prod a_i$ and $\prod b_i$, and the procedure for replacing the powers of x by the appropriate $D_k C_k(t)$.

In the $p = 2$ case, describing how the powers of x are to be replaced by the generators C(j) is very simple because, if $j \geq n_1$, then the next generator after C(j) is always $C(j+1)$. However, when p is odd, the generator after $D_k C_k(r)$ may be either $D_{k+1}C_{k+1}(r)$ or $C_0(r+1)$. To handle this complication, we introduce two functions f and g from the nonnegative integers to the nonnegative integers. These functions are to be chosen so that, for any $i \geq 0$, $C_{f(i+1)}(g(i+1))$ is the generator immediately following $C_{f(i)}(g(i))$ in our stairstep ordering. If $C_{f(n)}(g(n))$ is the last generator in $\underline{H}^*_G P(\Phi)^+$, then we define $f(i) = 0$ and $g(i) = g(n) + i - n$ for $i > n$ and use the convention that $D_j C_j(r)$ is to be regarded as zero if it does not appear in the list of generators of $\underline{H}^*_G P(\Phi)^+$. Each time we use this notation, the initial values, f(0) and g(0), of the functions will be specified to suit the particular application.

The indexing set which replaces the set $\{0, 1, \ldots, n\}$ is related to the difference in dimension between the product $(D_{i'}C_i(r))(D_{j'}C_j(s))$ and the lowest dimensional generator $D_{i'}C_i(r+s)$ which should appear in its description. If $r \geq 0$

and $0 \leq j \leq m$, then define the subset $\Phi_j(r)$ of $\Phi(r+1)$ by

$$\Phi_j(r) = \Phi(r) \cup \{\phi_i : i < j \text{ and } \phi_i \in \Phi(r+1) - \Phi(r)\}.$$

Let $\Phi_{i'}(r) \sqcup \Phi_{j'}(s)$ denote the disjoint union of the sets $\Phi_{i'}(r)$ and $\Phi_{j'}(s)$. Our replacement for the set $\{0, 1, \dots, n\}$ is the set Ψ obtained by deleting from $\Phi_{i'}(r) \sqcup \Phi_{j'}(s)$ a subset equivalent to the set $\Phi_{i'}(r+s)$. We abuse notation by writing Ψ as $\Phi_{i'}(r) \sqcup \Phi_{j'}(s) - \Phi_{i'}(r+s)$. Observe that $\Phi_{j'}(s)$ is equivalent to the disjoint union of Ψ and $\Phi_{i'}(r+s) - \Phi_{i'}(r)$. Let u be $|\Psi| - 1$ and number the elements of Ψ from 0 to u. Let h be a function from the set $\{0, 1, \dots, u\}$ to the set $\{0, 1, \dots, m\}$ such that the i^{th} element of Ψ is isomorphic to the irreducible representation $\phi_{h(i)}$.

One of the coefficients appearing in our formula is determined by a certain element α of RSO(G) with $|\alpha| = 0$ and $|\alpha^G| \leq 0$. This coefficient will be ξ_α if $|\alpha^G| < 0$ or σ_α if $|\alpha^G| = 0$. To simplify our notation, we write χ_α for either of these, relying on $|\alpha^G|$ to indicate whether ξ_α or σ_α is intended. Another coefficient will depend on a certain element β of RSO(G) with $|\beta^G| = 0$ and $|\beta| \geq 0$. This coefficient will be ϵ_β if $|\beta| > 0$ and μ_β if $|\beta| = 0$. We write θ_β for either of these, relying on $|\beta|$ to indicate which is intended.

PROPOSITION 5.12. Let i, i′, j, j′, r, and s be integers with $r, s \geq 0$, $\phi_i \in \Phi(r+1) - \Phi(r)$, $\phi_j \in \Phi(s+1) - \Phi(s)$, $i' = 0$ or i, $j' = 0$ or j, and $i' \geq j'$. Let $\Psi = \Phi_{i'}(r) \sqcup \Phi_{j'}(s) - \Phi_{i'}(r+s)$. Initialize the functions f and g by

$$f(0) = \begin{cases} i', & \text{if } \phi_{i'} \in \Phi(r+s+1), \\ 0, & \text{otherwise,} \end{cases}$$

and

$$g(0) = \begin{cases} r+s, & \text{if } \phi_{i'} \in \Phi(r+s+1), \\ r+s+1, & \text{otherwise.} \end{cases}$$

Let $u = |\Psi| - 1$ and number the elements of Ψ from 0 to u. Let $\Delta \subset \Psi$ and let s′ and s'' be the number of elements isomorphic to ϕ_j in Δ and $\Phi_{i'}(r+s) - \Phi_{i'}(r)$, respectively. If the subset Δ of Ψ contains the elements numbered $j_0, j_1, \ldots, j_w$, with $j_0 < j_1 < \ldots < j_w$, then let

$$d_\Delta^k = \left[\prod_{\substack{t=0 \\ h(j_t)\neq j}}^{w} d_{j,h(j_t)}^{f(j_t - t),h(j_t)}\right]\left[\prod_{\substack{t=0 \\ h(j_t)=j}}^{w} d_{jk}^{f(j_t-t),j}\right],$$

$$\epsilon_\Delta^k = \left[\prod_{\substack{t=0 \\ h(j_t)\neq j}}^{w} \epsilon_{\beta_{j,h(j_t)}}\right]\left[\prod_{\substack{t=0 \\ h(j_t)=j}}^{w} \epsilon_{\beta_{jk}}\right],$$

and

$$\chi_\Delta = \chi_\alpha,$$

where

$$\alpha = \phi_j^{-1}\left[\sum_{\substack{t=0 \\ h(j_t)\neq j}}^{w} \phi_{h(j_t)} + \sum_{\substack{\phi_t \in \Phi_{i'}(r+s)-\Phi_{i'}(r) \\ t\neq 0,j}} \phi_t\right] +$$

$$\phi_i^{-1}\left[\sum_{\phi_t \in \Phi_{i'}(r)} \phi_t\right] + \left[(s' + s'')\,\phi_j^{-1}\phi_0\right]_{j\neq 0} +$$

$$2s - \phi_{f(|\Delta|)}^{-1}\left[\sum_{\phi_t \in \Phi_{f(|\Delta|)}(g(|\Delta|))} \phi_t\right].$$

The tag $j \neq 0$ on the bracket about the $(s' + s'')\,\phi_j^{-1}\phi_0$ indicates that this term is present only if $j \neq 0$. The 2s term in α indicates 2s copies of the real one-dimensional trivial representation. If $\alpha \in RSO_0(G)$, then let

$$\hat{d}_\Delta = d_\alpha .$$

If $\Delta = \emptyset$, then let d_Δ^k, ϵ_Δ^k, $\hat{d}_\Delta$, and χ_Δ be 1.

If $i' \leq k \leq m$ and $\phi_k \in \Phi(r+s+1) - \Phi(r+s)$, let

$$\theta_k = \theta_\beta,$$

where

$$\beta \;=\; \alpha_i(r) + \alpha_j(s) + \gamma_{i'} + \gamma_{j'} - \alpha_k(r+s),$$

and let A_k be

$$\frac{1}{p}\left[\tilde{d}^{\,r}_{ki}\tilde{d}^{\,s}_{kj}\Big(\prod_{t=0}^{i'-1} d^{kt}_{i't}\Big)\Big(\prod_{t=0}^{j'-1} d^{kt}_{j't}\Big) - \sum_{v=i'}^{k}\left[\tilde{d}^{\,r+s}_{kv}\Big(\prod_{t=0}^{v-1} d^{kt}_{vt}\Big)\sum_{\substack{\Delta\subset\Psi\\|\Delta|=v-i'}} d^{0}_{\Psi-\Delta}\,\hat{d}_{\Delta}\right]\right].$$

Then

$$(D_{i'}C_i(r))(D_{j'}C_j(s)) \;=\; \sum_{\Delta\subset\Psi} d^{0}_{\Psi-\Delta}\,\epsilon^{0}_{\Psi-\Delta}\,\chi_{\Delta}\,D_{f(|\Delta|)}C_{f(|\Delta|)}(g(|\Delta|)) \;+$$

$$\sum_{\substack{k=i'\\ \phi_k\in\Phi(r+s+1)-\Phi(r+s)}}^{m} A_k\,\theta_k\,\hat{\kappa}_k(r+s).$$

REMARKS 5.13. (a) Let $r \geq 1$. If $\Phi(r)$ contains r copies of every irreducible complex G-representation, then $\alpha_i(r)$ is independent of i and it is easy to see that $C_i(r) = C_j(r)$ for every i and j such that $\phi_i, \phi_j \in \Phi(r+1) - \Phi(r)$. Moreover, $C_j(r) = C_j(1)^r$. Thus, if Φ contains every irreducible complex G-representation and these representations appear with equal multiplicities in Φ, then $C_i(r)$ generates a polynomial, or truncated polynomial, subalgebra of $\underline{H}^*_G P(\Phi)^+$. In this case, the elements D_j, for $1 \leq j \leq m$, and $C_i(1)$, for any i, generate $\underline{H}^*_G P(\Phi)^+$ as an algebra over $\underline{H}^*_G S^0$.

(b) If $p = 3$, then we may choose the integers d_α so that $d_\alpha = \pm 1$ for every α in $RSO_0(G)$. When this is done, the assignment of d_α to α is a homomorphism from the additive group of $RSO_0(G)$ to the multiplicative group $\{\pm 1\}$. With this choice of the integers d_α, all the relations among the d^{ij}_{rs} and the $\tilde{d}^{\,r}_{ij}$ given in Definitions 5.4, except the one involving a sum, hold in $\mathbb{Z}$ as well as in $\mathbb{Z}/3$. If $r \geq 1$ and $\phi_i, \phi_j \in \Phi(r+1)$, then

$$C_i(r) = \sigma_{\alpha_i(r)-\alpha_j(r)}\,C_j(r).$$

Thus, the only elements of the form $C_j(r)$ needed to generate $\underline{H}^*_G P(\Phi)^+$ as an algebra over $\underline{H}^*_G S^0$ are the elements $C_0(r)$ for $r \geq 1$. Also, a pair of elements c_{ij} and c_{rs} will generate D_1 and D_2 if $\tilde{q}_k(c_{ij}c_{rs})$ is nonzero for only one value of k. In particular, c_{01} and c_{10} generate D_1 and D_2. When all three irreducible complex G-representations of $\mathbb{Z}/3$ appear in Φ with equal multiplicities, c_{01}, c_{10}, and $C_0(1)$ generate $\underline{H}^*_G P(\Phi)^+$ as an algebra over $\underline{H}^*_G S^0$.

6. PROOFS. The results stated in section 5 are proved here. As indicated in Remark 5.7, our results for $p = 2$ are a special case of the results asserted for odd

primes. They have been presented separately only because they can be stated so simply. The proofs given here are independent of whether p is 2 or odd. We begin by construct the elements c_{ij} and $C_j(r)$. We then show that they generate $\underline{H}_G^*P(V)^+$ as an algebra over $\underline{H}_G^*S^0$. Finally, the relations stated at the end of section 5 are verified. Throughout this section, Φ is a set of irreducible complex representations of $\mathbb{Z}/p$ and $\Phi(0)$, $\Phi(1)$, ... is a proper filtration of Φ. We order the elements of Φ in the standard proper ordering introduced in section 5. Recall the maps q_i and $\tilde{q}_i$ and the cohomology classes x and x_i from the introductory remarks in section 5 and the representations $\alpha_i(r)$, β_{ij}, and γ_j from Definitions 5.4 and Theorem 5.5(d). If $\Delta \subset \Psi$, then x also denotes the image of $x \in \underline{H}_G^2(P(\Phi)^+)(e)$ in $\underline{H}_G^2(P(\Delta)^+)(e)$; thus, the powers of x are thought of as the standard additive generators for the nonequivariant cohomology of all the sub-projective spaces of $P(\Psi)$. For each integer j with $0 \leq j \leq m$, let $P_j(\Phi)$ be the component of the fixed point space of $P(\Phi)$ associated to the irreducible representation ϕ_j.

The classes c_{ij} and $C_j(r)$ are constructed by defining them on the smallest possible projective space and then inductively lifting them to larger projective spaces.

CONSTRUCTION 6.1. (a) Let i and j be distinct integers with $0 \leq i, j \leq m$. The space $P(\{\phi_j\})$ is just a point and the space $P(\{\phi_i, \phi_j\})$ is G-homeomorphic to $S^{\beta_{ij}}$. The inclusion of $P(\{\phi_j\})$ into $P(\{\phi_i, \phi_j\})$ induces the cofibre sequence

$$P(\{\phi_j\})^+ \xrightarrow{q_j} P(\{\phi_i, \phi_j\})^+ \xrightarrow{\pi} S^{\beta_{ij}}.$$

Let $c_{ij} \in \underline{H}_G^{\beta_{ij}}(P(\{\phi_i, \phi_j\})^+)(1)$ be the image of $1 \in A(1) \cong \underline{H}_G^{\beta_{ij}}(S^{\beta_{ij}})(1)$ under π^*. Then $q_j(c_{ij}) = 0$ by exactness and $q_i(c_{ij}) = \epsilon_{\beta_{ij}}$ by the commutativity of the diagram

$$\begin{array}{ccc} P(\{\phi_i\})^+ & \xrightarrow{q_i} & P(\{\phi_i, \phi_j\})^+ \\ \downarrow & & \downarrow \\ S^0 & \xrightarrow{\epsilon_{\beta_{ij}}} & S^{\beta_{ij}}. \end{array}$$

These are the correct values for $q_i(c_{ij})$ and $q_j(c_{ij})$ because x_i and x_j are zero. Since the map $\pi^*: \underline{H}_G^{\beta_{ij}}(S^{\beta_{ij}})(e) \to \underline{H}_G^{\beta_{ij}}(P(\{\phi_i, \phi_j\})^+)(e)$ is an isomorphism in dimension β_{ij}, $\rho(c_{ij}) = x$.

Let Ψ be a subset of Φ which properly contains the set $\{\phi_i, \phi_j\}$ and assume that, for every proper subset Δ of Ψ containing $\{\phi_i, \phi_j\}$, c_{ij} has been defined in $\underline{H}_G^{\beta_{ij}}(P(\Delta)^+)(1)$ and has the proper images under the maps q_k and ρ. Pick an irreducible representation ϕ_t which appears in Ψ at least as often as any other irreducible. If no irreducible appears more than once in Ψ, then we may also insist that $t \neq i, j$. Let $\Delta = \Psi - \{\phi_t\}$, and let V be the representation $\phi_t^{-1} \sum_{\phi \in \Delta} \phi$. The inclusion of Δ into Ψ induces the cofibre sequences

$$P(\Delta)^+ \xrightarrow{\theta} P(\Psi)^+ \to S^V$$

and

$$P_t(\Delta)^+ \xrightarrow{\theta_t} P_t(\Psi)^+ \to S^{V^G}.$$

We will lift the class $c_{ij} \in \underline{H}_G^{\beta_{ij}}(P(\Delta)^+)(1)$ along the map

$$\theta^*(1): \underline{H}_G^{\beta_{ij}}(P(\Psi)^+)(1) \to \underline{H}_G^{\beta_{ij}}(P(\Delta)^+)(1)$$

induced by θ. To distinguish the class c_{ij} and its lifting, we will denote the class in $\underline{H}_G^{\beta_{ij}}(P(\Delta)^+)(1)$ by $\hat{c}_{ij}$. The maps q_k, for $k \neq t$, factor through $\theta^*(1)$, so any lifting of $\hat{c}_{ij}$ along $\theta^*(1)$ will have the right image under q_k, for $k \neq t$. Moreover, since $\theta^*(e)$ is an isomorphism in dimension β_{ij}, any lifting of $\hat{c}_{ij}$ will also have the right image under ρ.

It remains to show that we can choose a lifting of c_{ij} with the correct image under q_t. We have chosen t so that the long exact cohomology sequences associated to our cofibre sequences have zero boundary maps. If $|V^G| \geq 2$, then $\underline{H}_G^{\beta_{ij}}(S^V)(1) = 0$ and we take c_{ij} to be the unique lifting of $\hat{c}_{ij}$. If $|V^G| > 2$, then θ_t induces a cohomology isomorphism in dimension β_{ij} and this lifting of $\hat{c}_{ij}$ along $\theta^*(1)$ must have the correct image under q_t. If $|V^G| = 2$, then the short exact sequence

$$0 \to \underline{H}_G^{\beta_{ij}} S^2 \to \underline{H}_G^{\beta_{ij}} P_t(\Psi)^+ \xrightarrow{\theta_t^*} \underline{H}_G^{\beta_{ij}} P_t(\Delta)^+ \to 0$$

splits. The end terms are

$$\underline{H}_G^{\beta_{ij}} S^2 \cong R \qquad \text{and} \qquad \underline{H}_G^{\beta_{ij}} P_t(\Delta)^+ \cong \langle \mathbb{Z} \rangle.$$

The image of $1 \in \mathbb{Z} = R(1)$ in $\underline{H}_G^{\beta_{ij}} P_t(\Psi)^+$ is $\xi_{\beta_{ij}-2} x_t$. By our induction hypothesis,

$$\theta_t^*(1) q_t(c_{ij}) = q_t(\hat{c}_{ij}) = d_{ij}^{tj} \epsilon_{\beta_{ij}}.$$

Since $\rho(c_{ij}) = x$, $\rho\, q_t(c_{ij})$ is the generator of $\underline{H}_G^{\beta_{ij}}(P_t(\Psi)^+)(e)$. It follows that $q_t(c_{ij}) = d_{ij}^{tj} \epsilon_{\beta_{ij}} + \xi_{\beta_{ij}-2} x_t$.

If $|V^G| = 0$, then no irreducible appears more than once in Ψ and we have selected ϕ_t so that $t \neq i, j$. In the diagram

$$\begin{array}{ccccccccc}
0 & \to & \underline{H}_G^{\beta_{ij}} S^V & \to & \underline{H}_G^{\beta_{ij}} P(\Psi)^+ & \to & \underline{H}_G^{\beta_{ij}} P(\Delta)^+ & \to & 0 \\
 & & \downarrow \epsilon_V & & \downarrow q_t & & \downarrow q_t & & \\
0 & \to & \underline{H}_G^{\beta_{ij}} S^0 & \to & \underline{H}_G^{\beta_{ij}} P_t(\Psi)^+ & \to & 0 & &
\end{array}$$

comparing the cohomology sequences of our two cofibre sequences, we have that $\underline{H}_G^{\beta_{ij}}S^V$ and $\underline{H}_G^{\beta_{ij}}S^0$ are $\langle \mathbb{Z} \rangle$ and the map ϵ_V is multiplication by p. Thus, if z is a lifting of $\hat{c}_{ij}$, then by adding elements from the image of $\underline{H}_G^{\beta_{ij}}S^V$ to z, we can adjust $q_t(z)$ by any multiple of p. It now suffices to show that there is a lifting z with $q_t(z) \equiv d_{ij}^{tj}\epsilon_{\beta_{ij}}$ mod p. The lifting problems for $P(\Psi)$ and $P(\{\phi_i, \phi_j, \phi_t\})$ can be compared via the cohomology maps induced by the inclusion of $\{\phi_i, \phi_j, \phi_t\}$ into Ψ. This comparison indicates that it suffices to show that the lifting problem can be solved when $\Psi = \{\phi_i, \phi_j, \phi_t\}$. In this case, consider the diagram

$$\begin{array}{ccccccccc} 0 & \to & \underline{H}_G^{\beta_{ij}}S^V & \to & \underline{H}_G^{\beta_{ij}}P(\Psi)^+ & \xrightarrow{\theta^*} & \underline{H}_G^{\beta_{ij}}P(\triangle)^+ & \to & 0 \\ & & \downarrow \epsilon & & \downarrow q & & \downarrow q_j & & \\ 0 & \to & \underline{H}_G^{\beta_{ij}}S^{\beta_{tj}} & \xrightarrow{\gamma} & \underline{H}_G^{\beta_{ij}}P(\{\phi_j, \phi_t\})^+ & \xrightarrow{q_j} & \underline{H}_G^{\beta_{ij}}P(\{\phi_j\})^+ & \to & 0 \end{array}$$

comparing the cohomology exact sequences for the pairs $(P(\Psi), P(\triangle))$ and $(P(\{\phi_j, \phi_t\}), P(\{\phi_j\}))$. Let $\alpha = \beta_{ij} - \beta_{tj}$. If z is a lifting of $\hat{c}_{ij}$ along $\theta^*(1)$, then $q_j(z) = q_j q(z) = 0$. Thus, $q(z) = \gamma(y)$ for some $y \in \underline{H}_G^{\beta_{ij}}(S^{\beta_{tj}})(1)$. Since $\rho\, q(z)$ is the generator x of $\underline{H}_G^{\beta_{ij}}(P(\{\phi_j, \phi_t\})^+)(e)$, $\rho(y)$ must generate $\underline{H}_G^{\beta_{ij}}S^{\beta_{tj}}(e)$, and y must be $\sigma_\alpha + n\kappa_\alpha$ for some integer n. The diagram

$$\begin{array}{ccc} \underline{H}_G^{\beta_{ij}}S^{\beta_{tj}} & \xrightarrow{\gamma} & \underline{H}_G^{\beta_{ij}}P(\{\phi_j, \phi_t\})^+ \\ \downarrow \epsilon & & \downarrow q_t \\ \underline{H}_G^{\beta_{ij}}S^0 & \overset{\cong}{\underset{=}{\to}} & \underline{H}_G^{\beta_{ij}}P(\{\phi_t\})^+ \end{array}$$

commutes and gives that $q_t(z) = q\, q_t(z) = \epsilon(y) \equiv \epsilon(\sigma_\alpha)$ mod p. By the definition of σ_α, $\epsilon(\sigma_\alpha) = d_{ij}^{tj}\epsilon_{\beta_{ij}}$.

(b) Let $r \geq 1$ and let $\phi_j \in \Phi(r+1)$. The cofibre sequence associated to the inclusion of $P(\Phi(r))$ into $P(\Phi(r) \cup \{\phi_j\})$ is

$$P(\Phi(r))^+ \to P(\Phi(r) \cup \{\phi_j\})^+ \xrightarrow{\pi} S^{\alpha_j(r)}.$$

Define $C_j(r) \in \underline{H}_G^{\alpha_j(r)}(P(\Phi(r) \cup \{\phi_j\})^+)(1)$ to be the image under $\pi^*(1)$ of $1 \in A(1) = \underline{H}_G^{\alpha_j(r)}(S^{\alpha_j(r)})(1)$. Since π^* is an isomorphism in dimension $\alpha_j(r)$, $\rho(C_j(r)) = x^{|\alpha_j(r)|/2}$. The cohomology diagram in dimension $\alpha_j(r)$ induced by the diagram

$$\begin{array}{ccc} P_j(\Phi(r) \cup \{\phi_j\})^+ & \xrightarrow{q_j} & P(\Phi(r) \cup \{\phi_j\})^+ \\ \Big\downarrow \pi_j & & \Big\downarrow \pi \\ S^r & \xrightarrow{\epsilon} & S^{\alpha_j(r)} \end{array}$$

indicates that $q_j(C_j(r)) = \epsilon_{\alpha_j(r)-r} x_j^r$. If $k \neq j$, $q_k(C_j(r)) = 0$ for dimensional reasons. As we did with the definition of c_{ij} in part (a), we extend the definition of $C_j(r)$ to $\underline{H}_G^* P(\Phi)^+$ by working inductively along a sequence of subsets of Φ between $\Phi(r) \cup \{\phi_j\}$ and Φ. The only difference between the argument given for c_{ij} and the one which should be used for $C_j(r)$ is that the liftings of $C_j(r)$ should be chosen to behave properly with respect to ρ and $\tilde{q}_k$ instead of ρ and q_k. This change is necessary because $q_k(C_j(r))$ is more complicated than $q_k(c_{ij})$. The behavior of the $C_j(r)$ with respect to the maps q_k is established in the lemma below.

LEMMA 6.2. Let $r \geq 1$ and $\phi_k \in \Phi(r+1) - \Phi(r)$. Then

$$q_k(C_k(r)) = x_k^r \left[\prod_{\substack{\phi_i \in \Phi(r) \\ i \neq k}} (\epsilon_{\beta_{ki}} + \xi_{\beta_{ki}-2} x_k) \right].$$

If $\phi_j \in \Phi(r+1) - \Phi(r)$ and $j \neq k$, then

$$q_k(C_j(r)) = x_k^r \left(d_{jk}^{kj} \epsilon_{\beta_{jk}} + \xi_{\beta_{jk}-2} x_k \right)^r \left[\prod_{\substack{\phi_i \in \Phi(r) \\ i \neq j,k}} (d_{ji}^{ki} \epsilon_{\beta_{ji}} + \xi_{\beta_{ji}-2} x_k) \right] +$$

$$\left[\tilde{d}_{kj}^r - (d_{jk}^{kj})^r \prod_{\substack{\phi_i \in \Phi(r) \\ i \neq j,k}} (d_{ji}^{ki}) \right] \epsilon_{\alpha_j(r)-r} x_k^r.$$

If $\phi_k \notin \Phi(r+1) - \Phi(r)$, then $q_k(C_j(r))$ is zero.

PROOF. If $\phi_k \notin \Phi(r+1) - \Phi(r)$, then $q_k(C_j(r))$ vanishes for dimensional reasons. Therefore, assume that $\phi_j, \phi_k \in \Phi(r+1) - \Phi(r)$. Let

$$\Psi = \Phi(r) \cup \{\phi : \phi \in \Phi - \Phi(r) \text{ and } \phi \cong \phi_k\}.$$

The image of the class $C_j(r)$ in $\underline{H}_G^* P(\Phi)^+$ under the map

$$\underline{H}_G^* P(\Phi)^+ \to \underline{H}_G^* P(\Psi \cup \{\phi_j\})^+$$

may be computed using the maps ρ and $\tilde{q}_i$. It is the class $C_j(r)$ in $\underline{H}_G^* P(\Psi \cup \{\phi_j\})^+$. The image of this class under the map

$$\underline{H}_G^* P(\Psi \cup \{\phi_j\})^+ \to \underline{H}_G^* P(\Psi)^+$$

is the class $\sigma_{\alpha_j(r)-\alpha_k(r)}C_k(r)$. Thus,

$$q_k(C_j(r)) = q_k(\sigma_{\alpha_j(r)-\alpha_k(r)}C_k(r)) = \sigma_{\alpha_j(r)-\alpha_k(r)}q_k(C_k(r)),$$

since $P_k(\Phi) = P_k(\Psi)$ and the map q_k for $P(\Phi)$ factors as the composite of the map $\underline{H}^*_G P(\Phi)^+ \to \underline{H}^*_G P(\Psi)^+$ and the map q_k for Ψ. Observe that

$$\sigma_{\alpha_j(r)-\alpha_k(r)} = (\sigma_{\beta_{jk}-\beta_{kj}})^r \left[\prod_{\substack{\phi_i \in \Phi(r) \\ i \neq j,k}} \sigma_{\beta_{ji}-\beta_{ki}} \right] + a\kappa_{\alpha_j(r)-\alpha_k(r)}$$

for some integer a. With this description of $\sigma_{\alpha_j(r)-\alpha_k(r)}$, it is easy to derive the formula for $q_k(C_j(r))$ from the formula for $q_k(C_k(r))$. The formula for $q_k(C_k(r)$ is derived using an iterative procedure. Let $s \geq r$ and pick $\phi_t \in \Psi$ with $t \neq k$. The image of $C_k(s) \in \underline{H}^*_G(P(\Psi)^+)(1)$ under the map $\underline{H}^*_G P(\Psi)^+ \to \underline{H}^*_G P(\Psi - \{\phi_t\})^+$ is

$$\epsilon_{\beta_{kt}} C_k(s) + \xi_{\beta_{kt}-2} C_k(s+1).$$

Iterating this process to eliminate from Ψ all the irreducible representations not isomorphic to ϕ_k, we move from $\underline{H}^*_G P(\Psi)^+$ to $\underline{H}^*_G P(n_k \phi_k)^+ \cong \underline{H}^*_G P_k(\Psi)^+$ and from $C_k(r)$ to the expansion of

$$x_k^r \left[\prod_{\substack{\phi_i \in \Phi(r) \\ i \neq k}} (\epsilon_{\beta_{ki}} + \xi_{\beta_{ki}-2} x_k) \right].$$

On the other hand, the image of $C_k(r)$ under this sequence of transformations must be $q_k(C_k(r))$.

Now that we have defined the classes c_{ij} and $C_j(r)$, we must show that they generate $\underline{H}^*_G P(\Phi)^+$ as an algebra over $\underline{H}^*_G S^0$.

PROPOSITION 6.3. The classes c_{ij}, for ϕ_i, $\phi_j \in \Phi(1)$, and the classes $C_j(r)$, for $r \geq 1$ and $\phi_j \in \Phi(r+1) - \Phi(r)$, generate $\underline{H}^*_G P(\Phi)^+$ as an algebra over $\underline{H}^*_G S^0$.

PROOF. If Φ is infinite, then, by the proof of Theorem 2.6, $\underline{H}^*_G P(\Phi)^+$ is the limit of the $\underline{H}^*_G P(\Delta)^+$ where Δ runs over the finite subsets of Φ. Thus, it suffices to prove the result for Φ finite. Recall the functions f and g and the subsets $\Phi_j(r)$ of Φ defined in the remarks preceding Proposition 5.12. For this proof, initialize f and g by $f(0) = 0$ and $g(0) = 0$. We will show, by induction on n, that the classes c_{ij} and $C_j(r)$ which are defined in $\underline{H}^*_G P(\Phi_{f(n)}(g(n)))^+$ generate that Mackey functor as an algebra over $\underline{H}^*_G S^0$. The result is obvious for $n = 1$, since $\Phi_{f(1)}(g(1)) = \{\phi_0\}$ and

$P(\{\phi_0\})$ is a point. Assume the result for n. Denote $\alpha_{f(n+1)}(g(n+1)) + \gamma_{f(n+1)}$ by α. The boundary map is zero in the cohomology long exact sequence associated to the cofibre sequence

$$P(\Phi_{f(n)}(g(n)))^+ \xrightarrow{\theta} P(\Phi_{f(n+1)}(g(n+1)))^+ \to S^\alpha.$$

Thus, we have a split short exact sequence

$$0 \to \underline{H}^*_G S^\alpha \to \underline{H}^*_G P(\Phi_{f(n+1)}(g(n+1)))^+ \xrightarrow{\theta^*} \underline{H}^*_G P(\Phi_{f(n)}(g(n)))^+ \to 0.$$

All of the classes c_{ij} and $C_j(r)$ which are defined in $\underline{H}^*_G P(\Phi_{f(n)}(g(n)))^+$ are also defined in $\underline{H}^*_G P(\Phi_{f(n+1)}(g(n+1)))^+$. Moreover, θ^* takes these classes in $\underline{H}^*_G P(\Phi_{f(n+1)}(g(n+1)))^+$ to the corresponding classes in $\underline{H}^*_G P(\Phi_{f(n)}(g(n)))^+$. Thus, to generate $\underline{H}^*_G P(\Phi_{f(n+1)}(g(n+1)))^+$ as an algebra over $\underline{H}^*_G S^0$, it suffices to add to these classes the image z of the canonical generator of $A(1) = \underline{H}^\alpha_G(S^\alpha)(1)$. Clearly, $\rho(z)$ is the generator of $\underline{H}^\alpha_G(P(\Phi_{f(n+1)}(g(n+1)))^+)(e)$. Moreover, for $k \neq f(n+1)$, $\tilde{q}_k(z) = 0$ since $\tilde{q}_k$ factors through $\underline{H}^*_G P(\Phi_{f(n)}(g(n)))^+$. Finally,

$$\tilde{q}_{f(n+1)}(z) = \left[\epsilon_{\alpha - g(n+1)}\left(x_{f(n+1)}\right)^{g(n+1)}\right]$$

since the diagram

$$\begin{array}{ccc} P_{f(n+1)}(\Phi_{f(n+1)}(g(n+1)))^+ & \xrightarrow{q_{f(n+1)}} & P(\Phi_{f(n+1)}(g(n+1)))^+ \\ \downarrow \pi_j & & \downarrow \pi \\ S^{g(n+1)} & \xrightarrow{\epsilon} & S^\alpha \end{array}$$

commutes. The elements z and $D_{f(n+1)}C_{f(n+1)}(g(n+1))$ must be equal since they have the same image under the maps $\tilde{q}_k$ and ρ.

The equations in Propositions 5.9 and 5.10 describe elements in dimensions where there is no torsion. As a result, these equations can be checked easily by applying the maps ρ and $\tilde{q}_k$ to both sides. The equations in Lemma 5.11 are easily checked using the maps ρ and q_k because the images of the classes $\hat{\kappa}_j(r)$ under the maps q_k are so simple. However, the formula in Proposition 5.12 is more difficult to verify.

PROOF OF PROPOSITION 5.12. We may assume that $|\Phi| \geq |\Phi_{i'}(r)| + |\Phi_{j'}(s)|$ so that all of the $D_{f(|\Delta|)}C_{f(|\Delta|)}(g(|\Delta|))$ on the right hand side of the equation are nonzero. If $|\Phi|$ is too small, then form a sufficiently large set Φ' by adding enough copies of ϕ_0 to Φ. The proof below applies to Φ'; the result for Φ is obtained using the cohomology map induced by the inclusion of Φ into Φ'. We show the equality of the images of the two sides of the equation under the maps ρ and q_k. Since the map ρ preserves products, $\rho(D_{i'}C_i(r)D_{j'}C_j(s))$ is the generator of $\underline{H}^*_G(P(\Phi)^+)(e)$ in the appropriate dimension. The only term on the right hand side of the equation in Proposition 5.12 which is not in the kernel of ρ is the summand corresponding to Ψ regarded as a subset of itself. This term is $\chi_{\Psi}D_{f(u)}C_{f(u)}(g(u))$ and its image under ρ is the generator of $\underline{H}^*_G(P(\Phi)^+)(e)$ in the same dimension. Thus, the expressions on the two sides of the equation have the same image under ρ.

Let k be an integer with $0 \le k \le m$. If $\phi_k \notin \Phi(r+s+1) - \Phi(r+s)$, then both sides of the equation vanish under q_k. If $\phi_k \in \Phi(r+s+1) - \Phi(r+s)$, then expand the polynomial obtained by applying q_k to $D_{i'}C_i(r)D_{j'}C_j(s)$. Each term in the expansion consists of the product of an integer, a power of x_k, and an element of the form ϵ_β, ξ_α, or $\epsilon_\beta\xi_\alpha$ from $\underline{H}^*_G S^0$. We classify these terms according to the factor from $\underline{H}^*_G S^0$. There is exactly one term with a ξ_α; its integer coefficient is one. There is exactly one term with an ϵ_β; its integer coefficient may be zero. This term is exactly the part of q_k which is detected by $\tilde{q}_k$. There may be any number, including zero, of terms containing a product $\epsilon_\beta\xi_\alpha$. These terms are all torsion elements of order p.

Expand the polynomial obtained by applying q_k to the right hand side of the equation and observe that the same three types of terms appear. The summand indexed on Ψ regarded as a subset of itself is the only source of a ξ_α. It is easy to see that this ξ_α term exactly matches the corresponding term from the left hand side of the equation. If $i' > k$, then the expansion of the image of the right hand side under q_k will contain no ϵ_β term. In this case, $\tilde{q}_k(D_{i'})$ is zero and the image of the left hand side under q_k also lacks an ϵ_β term. If $i' \le k$, then numerous summands contribute to the ϵ_β term of the left hand side, but the coefficient of the $\hat{\kappa}_k(r+s)$ term is explicitly designed to ensure that the ϵ_β terms of the expansions of both sides match. The only problem here is that it is not obvious that the coefficient A_k of $\hat{\kappa}_k(r+s)$ is an integer. To show that A_k is an integer, it suffices to show that, modulo p, the image under q_k of the left hand side is equal to the image of the part of the right hand side indexed on the subsets of Ψ. Since the $\epsilon_\beta\xi_\alpha$ terms are all torsion of order p and the $\hat{\kappa}_t(r+s)$ summands on the right hand side contribute nothing to them, proving the equation

$$q_k(D_{i'}C_i(r)D_{j'}C_j(s)) \equiv q_k\Big(\sum_{\Delta\subset\Psi} d_{\Psi-\Delta}\,\epsilon_{\Psi-\Delta}\,\chi_\Delta D_{f(|\Delta|)}C_{f(|\Delta|)}(g(|\Delta|))\Big) \mod p$$

also shows that the $\epsilon_\beta\xi_\alpha$ terms of the two sides agree and so completes the proof of the proposition.

We prove this equation modulo p by transforming the right hand side into the left. In Theorem 5.5(c), $q_k(C_j(r))$ is described as a sum of two terms when $j \neq k$. The second term can be ignored in this transformation process because it vanishes modulo p. Recall that each χ_Δ is a χ_α for some virtual representation α. We accomplish our transformation by writing α as a sum of differences $\eta - \phi$ of irreducible complex representations. We then rewrite $\chi_\Delta = \chi_\alpha$ as the product of the elements $\chi_{\eta-\phi}$. To see that such a rewriting is justified, recall that if β and γ in RSO(G) are chosen so that the elements below are defined, then in $\underline{H}^*_G(S^0)(1)$

$$\xi_\beta \xi_\gamma = \xi_{\beta+\gamma} \qquad \xi_\beta \kappa_\gamma = 0 \qquad \epsilon_\beta \kappa_\gamma = p\,\epsilon_{\beta+\gamma}$$

and

$$\sigma_\beta \sigma_\gamma = \sigma_{\beta+\gamma} + A\,\kappa_{\beta+\gamma},$$

where A is some integer depending on β and γ. Now observe that every summand in the expansion of $q_k(D_{f(|\Delta|)} C_{f(|\Delta|)}(g(|\Delta|)))$ contains either an ϵ_β or a ξ_β. Thus, the $\kappa_{\beta+\gamma}$ error terms that might arise in the rewriting of χ_Δ as the product of the $\chi_{\eta-\phi}$ are killed by the ϵ_β and ξ_β from $q_k(D_{f(|\Delta|)} C_{f(|\Delta|)}(g(|\Delta|)))$.

We perform our transformation of the left hand side in four stages. During the first three stages, we think of the left hand side as a sum indexed on the subsets of Ψ and work on each summand separately. Therefore, fix a subset Δ of Ψ and let α be the virtual representation such that $\chi_\Delta = \chi_\alpha$. Recall that s′ and s″ are the number of elements isomorphic to ϕ_j in Δ and $\Phi_{i'}(r+s) - \Phi_{i'}(r)$, respectively. Recall that $u = |\Psi| - 1$, that the elements of Ψ are numbered from 0 to u, and that h is a function from the set $\{0, 1, \ldots, u\}$ to the set $\{0, 1, \ldots, m\}$ such that the i^{th} element in Ψ is isomorphic to $\phi_{h(i)}$. Assume that the elements of Ψ numbered $j_0, j_1, \ldots, j_w$, with $j_0 < j_1 < \ldots < j_w$, are in Δ and that the elements numbered $i_0, i_1, \ldots, i_v$, with $i_0 < i_1 < \ldots < i_v$, are in $\Psi - \Delta$. For any integers q and t, with $0 \leq q, t \leq m$, abbreviate $\epsilon_{\beta_{qt}}$ and $\xi_{\beta_{qt-2}}$ by ϵ_{qt} and ξ_{qt}. Define the elements α_1, α_2, and α_3 of RSO(G) by

$$\alpha_1 = \left(\phi_i^{-1} - \phi_{f(|\Delta|)}^{-1}\right)\left[\sum_{\substack{\phi_t \in \Phi_{i'}(r) \\ t \neq f(|\Delta|), i, k}} \phi_t\right] + \left[r\,(\phi_i^{-1}\phi_k - \phi_{f(|\Delta|)}^{-1}\phi_i)\right]_{i \neq f(|\Delta|),\, k} +$$

$$\left[(r+\delta)\,(\phi_i^{-1}\phi_{f(|\Delta|)} - \phi_{f(|\Delta|)}^{-1}\phi_k)\right]_{f(|\Delta|) \neq i,k} +$$

$$\left[\phi_i^{-1}\phi_k - \phi_{f(|\Delta|)}^{-1}\phi_k\right]_{f(|\Delta|)>i>k} + \left[\phi_i^{-1}\phi_k - 2\right]_{i>f(|\Delta|),\,k}$$

$$\alpha_2 = \left(\phi_j^{-1} - \phi_{f(|\Delta|)}^{-1}\right)\left[\sum_{\substack{\phi_t \in \Phi_{i'}(r+s) - \Phi_{i'}(r) \\ t \neq f(|\Delta|),j,k}} \phi_t\right] +$$

$$\left[(s - s' - s'')\left(\phi_j^{-1}\phi_k - \phi_j^{-1}\phi_0\right)\right]_{0 \neq j,\,k} +$$

$$\left[s''\left(\phi_j^{-1}\phi_k - \phi_{f(|\Delta|)}^{-1}\phi_j\right)\right]_{j \neq f(|\Delta|),\,k} +$$

$$\left[s\left(\phi_j^{-1}\phi_{f(|\Delta|)} - \phi_{f(|\Delta|)}^{-1}\phi_k\right)\right]_{f(|\Delta|) \neq j,k}$$

and

$$\alpha_3 = \alpha - \alpha_1 - \alpha_2,$$

where

$$\delta = \begin{cases} 1, & \text{if } i' > f(|\Delta|), \\ 0, & \text{otherwise.} \end{cases}$$

In the first stage of our transformation, χ_{α_1} is used to convert

$$d^0_{\Psi-\Delta}\,\epsilon^0_{\Psi-\Delta}\,\chi_{\Delta}\,q_k\Big(D_{f(|\Delta|)}\,C_{f(|\Delta|)}(g(|\Delta|))\Big)$$

into the product of

$$d^0_{\Psi-\Delta}\,\epsilon^0_{\Psi-\Delta}\,\chi_{\alpha_2+\alpha_3}\,q_k\Big(D_{i'}\,C_i(r)\Big)$$

and

$$x_k^{g(|\Delta|)-r-\delta'}\left[\prod_{\substack{\phi_t \in \Phi_{f(|\Delta|)}(g(|\Delta|)) - \Phi_{i'}(r) \\ t\neq f(|\Delta|),k}} \left(d^{kt}_{f(|\Delta|),t}\,\epsilon_{f(|\Delta|),t} + \xi_{f(|\Delta|),t}\,x_k\right)\right] \cdot$$

$$\left[\left(d^{k,f(|\Delta|)}_{f(|\Delta|),k}\,\epsilon_{f(|\Delta|),k} + \xi_{f(|\Delta|),k}\,x_k\right)^{g(|\Delta|)-r-\delta}\right]_{f(|\Delta|)\neq k}\left[\xi_{f(|\Delta|),k}\,x_k\right]_{\substack{i' > f(|\Delta|) > k \text{ or } \\ f(|\Delta|) > k \geq i'}} \cdot$$

Here, δ is as in the definition of α_1 and

$$\delta' = \begin{cases} 1, & \text{if } i' > f(|\Delta|),\ k; \\ 0, & \text{otherwise.} \end{cases}$$

In the second stage of the transformation, χ_{α_2} is used to convert this product into the product of $d^k_{\Psi-\Delta}\,\epsilon^k_{\Psi-\Delta}\,\chi_{\alpha_3}\,q_k\left(D_{i'}, C_i(r)\right)$ with the three factors

$$x_k^s\left[\prod_{\substack{\phi_t \in \Phi_{i'}(r+s) - \Phi_{i'}(r) \\ t\neq j,k}} \left(d^{kt}_{jt}\,\epsilon_{jt} + \xi_{jt}\,x_k\right)\right]\left[\left(d^{kj}_{jk}\,\epsilon_{jk} + \xi_{jk}\,x_k\right)^{s''}\right]_{j\neq k},$$

$$x_k^{g(|\Delta|)-r-s-\delta'}\left[\prod_{\substack{t=0 \\ f(t)\neq f(|\Delta|),k}}^{w} \left(d^{k,f(t)}_{f(|\Delta|),f(t)}\,\epsilon_{f(|\Delta|),f(t)} + \xi_{f(|\Delta|),f(t)}\,x_k\right)\right],$$

and

$$\left[\left(d^{k,f(|\Delta|)}_{f(|\Delta|),k}\,\epsilon_{f(|\Delta|),k} + \xi_{f(|\Delta|),k}\,x_k\right)^{g(|\Delta|)-r-s-\delta}\right]_{f(|\Delta|)\neq k}\left[\xi_{f(|\Delta|),k}\,x_k\right]_{\substack{i' > f(|\Delta|) > k \text{ or } \\ f(|\Delta|) > k \geq i'}} \cdot$$

Observe that the $d^0_{\Psi-\Delta}\,\epsilon^0_{\Psi-\Delta}$ factor has been transformed into a $d^k_{\Psi-\Delta}\,\epsilon^k_{\Psi-\Delta}$ factor. This is accomplished by the $\left[(s - s' - s'')(\phi_j^{-1}\phi_k - \phi_j^{-1}\phi_0)\right]_{0\neq j,k}$ summand in α_2. If $k = 0$, then obviously no such transformation is needed. If $j = 0$, then there will not be any elements of Ψ isomorphic to ϕ_j, and the value of $d^k_{\Psi-\Delta}\,\epsilon^k_{\Psi-\Delta}$ will not depend on k. In the description of the factor above indexed on t, for $0 \leq t \leq w$, and throughout the third stage of the transformation, the set $\Phi_{f(|\Delta|)}(g(|\Delta|)) - \Phi_{i'}(r+s)$ is identified with the set $\{\phi_{f(t)} : 0 \leq t \leq w\}$. By this identification, constructions that would naturally be indexed on $\Phi_{f(|\Delta|)}(g(|\Delta|)) - \Phi_{i'}(r+s)$ may be indexed on t. The description of the set $\{\phi_{f(t)} : 0 \leq t \leq w\}$ involves our usual abuse of notation in that, whenever $q \neq t$ and $f(q) = f(t)$, the representations $\phi_{f(q)}$ and $\phi_{f(t)}$ are intended to be distinct, but isomorphic, elements of the set.

The factor

$$q_k\Big(D_{i'}, C_i(r)\Big) x_k^s \left[\prod_{\substack{\phi_t \in \Phi_{i'}(r+s) - \Phi_{i'}(r) \\ t \neq j,k}} \Big(d_{jt}^{kt} \epsilon_{jt} + \xi_{jt} x_k \Big) \right] \left[\Big(d_{jk}^{kj} \epsilon_{jk} + \xi_{jk} x_k \Big)^{s''} \right]_{j \neq k}$$

appears in every summand of the transformation of the right hand side of the equation. We therefore factor it out of the sum and ignore it for the rest of the transformation. Observe that this factor consists of $q_k\Big(D_{i'}, C_i(r)\Big)$ and that part of $q_k\Big(D_{j'}, C_j(s)\Big)$ which is associated with the set $\Phi_{i'}(r+s) - \Phi_{i'}(r)$ when $\Phi_{i'}(s)$ is regarded as the disjoint union of Ψ and $\Phi_{i'}(r+s) - \Phi_{i'}(r)$. Thus, we must transform what remains of the sum after this factor is removed into the part of $q_k\Big(D_{j'}, C_j(s)\Big)$ coming from Ψ.

In the third stage of the transformation, χ_{α_3} is used to transform the remaining part of the Δ summand into

$$d_{\Psi-\Delta}^{k} \epsilon_{\Psi-\Delta}^{k} \left[\prod_{\substack{t=0 \\ h(j_t) \neq j}}^{w} \Big(d_{j,h(j_t)}^{k,f(t)} \epsilon_{j,h(j_t)} + \xi_{j,h(j_t)} x_k \Big) \right] \left[\prod_{\substack{t=0 \\ h(j_t) = j}}^{w} \Big(d_{jk}^{k,f(t)} \epsilon_{jk} + \xi_{jk} x_k \Big) \right].$$

For the fourth stage of the transformation, consider the subsets Δ of Ψ that contain the last element $\phi_{h(u)}$ of Ψ. The summands indexed on Δ and $\Delta - \{\phi_{h(u)}\}$ contain the common factor

$$\left[\prod_{\substack{t=0 \\ h(i_t) \neq j}}^{v-1} d_{j,h(i_t)}^{f(i_t - t), h(i_t)} \right] \left[\prod_{\substack{t=0 \\ h(i_t) = j}}^{v-1} d_{jk}^{f(i_t - t), j} \right] \left[\prod_{\substack{t=0 \\ h(i_t) \neq j}}^{v-1} \epsilon_{j,h(i_t)} \right] \left[\prod_{\substack{t=0 \\ h(i_t) = j}}^{v-1} \epsilon_{jk} \right].$$

$$\left[\prod_{\substack{t=0\\ \mathsf{h}(j_t)\neq j}}^{w-1}\left(d^{k,\mathsf{f}(t)}_{j,\mathsf{h}(j_t)}\epsilon_{j,\mathsf{h}(j_t)}+\xi_{j,\mathsf{h}(j_t)}x_k\right)\right]\left[\prod_{\substack{t=0\\ \mathsf{h}(j_t)=j}}^{w-1}\left(d^{k,\mathsf{f}(t)}_{jk}\epsilon_{jk}+\xi_{jk}x_k\right)\right],$$

which we have written down using the i_t and j_t numbering of the elements in $\Psi-\Delta$ and Δ. Each of the two summands contains exactly one term not in this common factor. If $h(u)\neq j$, then these terms are

$$d^{\mathsf{f}(w),\mathsf{h}(u)}_{j,\mathsf{h}(u)}\epsilon_{j,\mathsf{h}(u)}+d^{k,\mathsf{f}(w)}_{j,\mathsf{h}(u)}\epsilon_{j,\mathsf{h}(u)}+\xi_{j,\mathsf{h}(u)}x_k = d^{k,\mathsf{h}(u)}_{j,\mathsf{h}(u)}\epsilon_{j,\mathsf{h}(u)}+\xi_{j,\mathsf{h}(u)}x_k.$$

If $h(u)=j$, then these terms are

$$d^{\mathsf{f}(w),j}_{j,k}\epsilon_{j,k}+d^{k,\mathsf{f}(w)}_{j,k}\epsilon_{j,k}+\xi_{j,k}x_k = d^{k,j}_{j,k}\epsilon_{j,k}+\xi_{j,k}x_k.$$

In either case, the result is independent of Δ and may be factored out of the sum. Moreover, this factor is exactly the contribution that $\phi_{\mathsf{h}(u)}$ should make to $q_k\Big(D_{j'},C_j(s)\Big)$ when $\phi_{\mathsf{h}(u)}$ is regarded as an element of $\Phi_{j'}(s)$ under the identification of $\Phi_{j'}(s)$ with the disjoint union of Ψ and $\Phi_{i'}(r+s)-\Phi_{i'}(r)$.

The sum that remains after the factor associated to $\phi_{\mathsf{h}(u)}$ is removed may be regarded as one indexed on the subsets Δ of $\Psi-\{\phi_{\mathsf{h}(u)}\}$. We now pair the summand indexed on a subset Δ containing the last element $\phi_{\mathsf{h}(u-1)}$ of $\Psi-\{\phi_{\mathsf{h}(u)}\}$ with the summand indexed on $\Delta-\{\phi_{\mathsf{h}(u-1)}\}$ to obtain the factor of $q_k\Big(D_{j'},C_j(s)\Big)$ associated to $\phi_{\mathsf{h}(u-1)}$. Repeating this process until the elements of Ψ are exhausted, we recover the part of $q_k\Big(D_{j'},C_j(s)\Big)$ associated with Ψ.

APPENDIX. Computing $\underline{H}^*_G S^0$. Here, we outline the calculation of $\underline{H}^*_G S^0$. The computation of the additive structure and, for $G=\mathbb{Z}/2$ or $\mathbb{Z}/3$, the computation of the multiplicative structure are unpublished work of Stong.

Three cofibre sequences suffice for the computation of the additive structure of $\underline{H}^*_G(S^0)$. Recall that ζ is the real 1-dimensional sign representation of $\mathbb{Z}/2$. Let η be a nontrivial irreducible complex representation of $G=\mathbb{Z}/p$, for any prime p. Let $G^+\to S\eta^+$ be the inclusion of an orbit and let $S\eta^+\to S^0$ and $S\zeta^+\to S^0$ be the maps collapsing the unit spheres $S\eta$ and $S\zeta$ to the non-basepoint in S^0. The cofibre sequences associated to these maps are

$$G^+\to S\eta^+\to \Sigma G^+$$

$$S\eta^+\to S^0\xrightarrow{\epsilon} S^\eta$$

and

$$G^+ \cong S\zeta^+ \to S^0 \xrightarrow{\epsilon} S^\zeta.$$

The first step in the computation is obtaining the values of $\underline{H}^G_* S\eta^+$ and $\underline{H}^*_G S\eta^+$ from the first cofibre sequence.

LEMMA A.1. For any nontrivial irreducible complex representation η of G,

$$\underline{H}^G_\alpha S\eta^+ = \begin{cases} L, & \text{if } |\alpha| = 0 \text{ and } |\alpha^G| \text{ is even,} \\ L_-, & \text{if } |\alpha| = 0 \text{ and } |\alpha^G| \text{ is odd,} \\ R, & \text{if } |\alpha| = 1 \text{ and } |\alpha^G| \text{ is odd,} \\ R_-, & \text{if } |\alpha| = 1 \text{ and } |\alpha^G| \text{ is even,} \\ 0, & \text{otherwise,} \end{cases}$$

$$\underline{H}_G^\alpha S\eta^+ = \begin{cases} R, & \text{if } |\alpha| = 0 \text{ and } |\alpha^G| \text{ is even,} \\ R_-, & \text{if } |\alpha| = 0 \text{ and } |\alpha^G| \text{ is odd,} \\ L, & \text{if } |\alpha| = 1 \text{ and } |\alpha^G| \text{ is odd,} \\ L_-, & \text{if } |\alpha| = 1 \text{ and } |\alpha^G| \text{ is even,} \\ 0, & \text{otherwise.} \end{cases}$$

PROOF. The next map $\Sigma G^+ \to \Sigma G^+$ in the first cofibre sequence is $1-g$, the difference of the identity map and the multiplication by g map, for some element g of G which depends on η. The homology and cohomology long exact sequences associated to the first cofibre sequence have the form

$$\dots \to \underline{H}^G_\alpha G^+ \to \underline{H}^G_\alpha G^+ \to \underline{H}^G_\alpha S\eta^+ \to \underline{H}^G_{\alpha-1} G^+ \to \underline{H}^G_{\alpha-1} G^+ \to \dots$$

and

$$\dots \to \underline{H}_G^{\alpha-1} G^+ \to \underline{H}_G^{\alpha-1} G^+ \to \underline{H}_G^\alpha S\eta^+ \to \underline{H}_G^\alpha G^+ \to \underline{H}_G^\alpha G^+ \to \dots .$$

The Mackey functor $\underline{H}^G_\alpha G^+$ may be identified with the Mackey functor $(\underline{H}^G_\alpha S^0)_G$ defined in Examples 1.1(f). The difference $1-g$ may be regarded as a map in B(G). Under the identification of $\underline{H}^G_\alpha G^+$ with $(\underline{H}^G_\alpha S^0)_G$, the first map in the part of the homology long exact sequence displayed above becomes the map from $(\underline{H}^G_\alpha S^0)_G$ to $(\underline{H}^G_\alpha S^0)_G$ induced by the map $1-g$ in B(G). It follows that the cokernel of the map $(1-g)_*: \underline{H}^G_\alpha G^+ \to \underline{H}^G_\alpha G^+$ is the Mackey functor $L(\underline{H}^G_\alpha(S^0)(e))$ defined in Examples 1.1(e). Similar observations reduce the homology and cohomology long exact sequences of the first cofibre sequence to the short exact sequences

$$0 \to L(\underline{H}^G_\alpha(S^0)(e)) \to \underline{H}^G_\alpha S\eta^+ \to R(\underline{H}^G_{\alpha-1}(S^0)(e)) \to 0$$

and

$$0 \to L(\underline{H}_G^{\alpha-1}(S^0)(e)) \to \underline{H}_G^\alpha S\eta^+ \to R(\underline{H}_G^\alpha(S^0)(e)) \to 0.$$

Since $\underline{H}^G_\alpha(S^0)(e) \cong H_{|\alpha|}(S^0;\mathbb{Z})$, $L(\underline{H}^G_\alpha(S^0)(e))$ is zero if $|\alpha| \neq 0$. If $|\alpha| = 0$, then $L(\underline{H}^G_\alpha(S^0)(e))$ is $L(\mathbb{Z})$ for some action of G on $\mathbb{Z}$. This action is the sign action of $\mathbb{Z}/2$ on $\mathbb{Z}$ when $p = 2$ and α contains an odd number of copies of ζ; otherwise, the action is trivial. Similar remarks apply to $L(\underline{H}_G^{\alpha-1}(S^0)(e))$, $R(\underline{H}^G_{\alpha-1}(S^0)(e))$, and $R(\underline{H}_G^\alpha(S^0)(e))$.

Notice the frequency with which $\underline{H}_G^\alpha S\eta^+$ and $\underline{H}^G_\alpha S\eta^+$ vanish. From the dimension axiom, we also obtain that $\underline{H}_G^\alpha G^+ = \underline{H}^G_\alpha G^+ = 0$ if $|\alpha| \neq 0$. These vanishing results determine most of the homological and cohomological behavior of the maps ϵ in our second and the third cofibre sequences.

LEMMA A.2. Let $\alpha \in RSO(G)$.

(a) The map $\epsilon^* : \underline{H}_G^{\alpha-\eta} S^0 \cong \underline{H}_G^\alpha(S^\eta) \to \underline{H}_G^\alpha(S^0)$

$$\text{is } \begin{cases} \text{mono} & \text{for } |\alpha| \neq 1,\ 2, \\ \text{epi} & \text{for } |\alpha| \neq 0,\ 1, \\ \text{iso} & \text{for } |\alpha| \neq 0,\ 1,\ 2. \end{cases}$$

(b) If $p = 2$, then the map $\epsilon^* : \underline{H}_G^{\alpha-\zeta} S^0 \cong \underline{H}_G^\alpha(S^\zeta) \to \underline{H}_G^\alpha(S^0)$

$$\text{is } \begin{cases} \text{mono} & \text{for } |\alpha| \neq 1, \\ \text{epi} & \text{for } |\alpha| \neq 0, \\ \text{iso} & \text{for } |\alpha| \neq 0,\ 1. \end{cases}$$

The divisibility results involving Euler classes in Lemmas 4.2, 4.6, and 4.8 follow from this lemma. Moreover, from this lemma and the vanishing of $\underline{H}_G^n S^0$, for $n \in \mathbb{Z}$ and $n \neq 0$, one can derive all of the zeroes in the first and third quadrants of our standard plot of $\underline{H}_G^* S^0$.

LEMMA A.3. Let $\alpha \in RSO(G)$. Then $\underline{H}_G^\alpha S^0 = 0$ if $|\alpha|$ and $|\alpha^G|$ are both positive or both negative.

Lemma A.2 indicates that all of $\underline{H}_G^* S^0$ can be determined from the values of $\underline{H}_G^\alpha S^0$ for the α in RSO(G) with $-2 \leq |\alpha| \leq 2$. If $p = 2$, it suffices to know $\underline{H}_G^\alpha S^0$ for the α in RSO(G) with $-1 \leq |\alpha| \leq 1$. The next lemma describes $\underline{H}_G^* S^0$ on the edges of these two ranges of values for $|\alpha|$.

LEMMA A.4. Let $\alpha \in RSO(G)$ and let η be any nontrivial irreducible complex representation of G.

(a) If $|\alpha| = 2$, then

$$\underline{H}_G^\alpha S^0 \cong \operatorname{coker}(\tau : \underline{H}_G^{\alpha-\eta} G^+ \to \underline{H}_G^{\alpha-\eta} S^0).$$

(b) If $|\alpha| = -2$, then

$$\underline{H}_G^{\alpha} S^0 \;\cong\; \ker\,(\rho\colon \underline{H}_G^{\alpha+\eta} S^0 \to \underline{H}_G^{\alpha+\eta} G^+).$$

(c) If $p = 2$ and $|\alpha| = 1$, then

$$\underline{H}_G^{\alpha} S^0 \;\cong\; \operatorname{coker}\,(\tau\colon \underline{H}_G^{\alpha-\zeta} G^+ \to \underline{H}_G^{\alpha-\zeta} S^0).$$

(d) If $p = 2$ and $|\alpha| = -1$, then

$$\underline{H}_G^{\alpha} S^0 \;\cong\; \ker\,(\rho\colon \underline{H}_G^{\alpha+\zeta} S^0 \to \underline{H}_G^{\alpha+\zeta} G^+).$$

Moreover, in all four cases, $\underline{H}_G^{\alpha}(S^0)(e) = 0$.

PROOF. Part (d) follows immediately from the cohomology long exact sequence associated to the third cofibre sequence. Part (c) follows via duality from the homology long exact sequence associated to the third cofibre sequence. For part (b), consider the diagram

$$\begin{array}{ccccccc} 0 & \to & \underline{H}_G^{\alpha} S^0 & \to & \underline{H}_G^{\alpha+\eta} S^0 & \xrightarrow{f} & \underline{H}_G^{\alpha+\eta} S\eta^+ \\ & & & & & & \downarrow h \\ & & & & & & \underline{H}_G^{\alpha+\eta} G^+ \end{array}$$

in which the row is from the cohomology exact sequence of the second cofibre sequence and the vertical arrow comes from the inclusion of an orbit G into $S\eta$. Clearly, $\underline{H}_G^{\alpha} S^0 \cong \ker f$. By our computation of $\underline{H}_G^{*} S\eta^+$, the map h is mono, so $\ker f \cong \ker hf$. The composite hf is just ρ. The proof for part (a) is similar, but uses the homology long exact sequence to describe $\underline{H}^G_{-\alpha} S^0$ as the cokernel of the map $\underline{H}^G_{\eta-\alpha} G^+ \to \underline{H}^G_{\eta-\alpha} S^0$ induced by the collapse map $G^+ \to S^0$. Dualizing the homology Mackey functors to cohomology Mackey functors gives the result since the transfer is the dual of the collapse map. In all four cases, the group $\underline{H}_G^{\alpha}(S^0)(e)$ is zero either because $\tau(e)$ is surjective or because $\rho(e)$ is injective.

Most of the values of $\underline{H}_G^{\alpha} S^0$ for $|\alpha| = 0$ and $|\alpha^G| \neq 0$ follow immediately from the cohomology long exact sequence of the second cofibre sequence and Lemmas A.1 and A.3.

LEMMA A.5. Let $\alpha \in RSO(G)$ with $|\alpha| = 0$. Then

$$\underline{H}_G^{\alpha} S^0 \;=\; \begin{cases} R, & \text{if } |\alpha^G| \le -2 \text{ and } |\alpha^G| \text{ is even,} \\ R_-, & \text{if } |\alpha^G| \le -1 \text{ and } |\alpha^G| \text{ is odd,} \\ L, & \text{if } |\alpha^G| \ge 2 \text{ and } |\alpha^G| \text{ is even,} \\ L_-, & \text{if } |\alpha^G| \ge 3 \text{ and } |\alpha^G| \text{ is odd.} \end{cases}$$

PROOF. Let η be any nontrivial irreducible complex representation. If $|\alpha^G| < 0$, then consider the portion

$$\underline{H}_G^{\alpha-\eta}S^0 \cong \underline{H}_G^{\alpha}S^{\eta} \to \underline{H}_G^{\alpha}S^0 \to \underline{H}_G^{\alpha}S\eta^+ \to \underline{H}_G^{\alpha+1}S^{\eta} \cong \underline{H}_G^{\alpha+1-\eta}S^0$$

of the cohomology long exact sequence of the second cofibre sequence. The left hand term is zero by Lemma A.3 and the right hand term is zero by the same lemma unless $|\alpha^G|$ is -1. If $|\alpha^G| = -1$, then $p = 2$, $\alpha = \zeta - 1$, $\underline{H}_G^{\alpha}S\eta^+$ is R_- by Lemma A.1, and $\underline{H}_G^{\alpha+1-\eta}S^0$ is $\langle \mathbb{Z} \rangle$ by Lemma A.4. The last identification is based on the observations that η must be 2ζ and $\underline{H}_G^0 S^0$ is A. By inspection, there are no nontrivial maps from R_- to $\langle \mathbb{Z} \rangle$. Thus, if $|\alpha^G| < 0$, the middle arrow must be an isomorphism.

If $|\alpha^G| \geq 2$, then consider the portion

$$\underline{H}_G^{\alpha+\eta-1}S^0 \to \underline{H}_G^{\alpha+\eta-1}S\eta^+ \to \underline{H}_G^{\alpha+\eta}S^{\eta} \cong \underline{H}_G^{\alpha}S^0 \to \underline{H}_G^{\alpha+\eta}S^0$$

of the cohomology long exact sequence for the second cofibre sequence. The left and right hand terms in this portion of the sequence must be zero by Lemma A.3. Therefore, the middle arrow is an isomorphism.

If $p = 2$, then the results above reduce the computation of $\underline{H}_G^* S^0$ to the determination of $\underline{H}_G^0 S^0$, which is A by the dimension axiom, and $\underline{H}_G^{1-\zeta}S^0$, which is given by the following lemma.

LEMMA A.6. If $p = 2$, then $\underline{H}_G^{1-\zeta}S^0 \cong R_-$.

PROOF. Consider the portion

$$\underline{H}_G^0 S^0 \to \underline{H}_G^0 G^+ \to \underline{H}_G^1 S^{\zeta} \cong \underline{H}_G^{1-\zeta}S^0 \to \underline{H}_G^1 S^0$$

of the cohomology long exact sequence of the third cofibre sequence. By the dimension axiom, the right hand term is zero and the first two terms from the left are A and A_G, respectively. The value of $\underline{H}_G^{1-\zeta}S^0$ follows by computation.

If $p \neq 2$, then we must still determine the value of $\underline{H}_G^{\alpha}S^0$ when $|\alpha| = \pm 1$ or $\alpha \in RSO_0(G)$. The next three lemmas dispose of the α with $|\alpha| = \pm 1$ which are not already covered by Lemma A.3.

LEMMA A.7. Let M be a Mackey functor and $f\colon L \to M$ be a map. If f(e) is a monomorphism, then so is f.

PROOF. The composite $f(e)\,\rho$ is a monomorphism and $\rho\, f(1) = f(e)\,\rho$.

LEMMA A.8. If $p \neq 2$, $\alpha \in RSO(G)$, $|\alpha| = 1$, and $|\alpha^G| < 0$, then $\underline{H}_G^\alpha S^0 = 0$.

PROOF. Consider the portion

$$\underline{H}_G^{\alpha-\eta} S^0 \cong \underline{H}_G^\alpha S^\eta \to \underline{H}_G^\alpha S^0 \to \underline{H}_G^\alpha S\eta^+ \xrightarrow{f} \underline{H}_G^{\alpha+1} S^\eta \cong \underline{H}_G^{\alpha+1-\eta} S^0$$

of the cohomology long exact sequence associated to the second cofibre sequence. The left hand term must be zero by Lemma A.3. By Lemma A.1, $\underline{H}_G^\alpha S\eta^+ \cong L$. Since $|\alpha + 1 - \eta| = 0$, $\underline{H}_G^{\alpha+1-\eta}(S^0)(e)$ is $\mathbb{Z}$. The map $f: \underline{H}_G^\alpha S\eta^+ \to \underline{H}_G^{\alpha+1-\eta} S^0$ is induced by the geometric map $S^\eta \to \Sigma S\eta^+$ which identifies the points 0 and ∞ in S^η. From this description, it follows that f(e) is an isomorphism. By the lemma above, f is a monomorphism. Therefore, $\underline{H}_G^\alpha S^0$ must be zero.

LEMMA A.9. Assume that $p \neq 2$, $\alpha \in RSO(G)$, $|\alpha| = -1$, and $|\alpha^G| > 0$. Then for any nontrivial irreducible complex representation η,

$$\underline{H}_G^\alpha S^0 \cong \operatorname{coker} (\underline{H}_G^{\alpha+\eta-1} S^0 \to \underline{H}_G^{\alpha+\eta-1} S\eta^+).$$

Moreover, if $|\alpha^G| > 1$,

$$\underline{H}_G^\alpha S^0 \cong \langle \mathbb{Z}/p \rangle.$$

PROOF. Consider the portion

$$\underline{H}_G^{\alpha+\eta-1} S^0 \xrightarrow{h} \underline{H}_G^{\alpha+\eta-1} S\eta^+ \to \underline{H}_G^{\alpha+\eta} S^\eta \cong \underline{H}_G^\alpha S^0 \to \underline{H}_G^{\alpha+\eta} S^0$$

of the cohomology long exact sequence for the second cofibre sequence. The right hand term must be zero by Lemma A.3. The first part of the lemma follows immediately. By Lemma A.1, $\underline{H}_G^{\alpha+\eta-1} S\eta^+ \cong R$. The map h is induced by the collapse map $S\eta^+ \to S^0$. Since $|\alpha + \eta - 1| = 0$,

$$\underline{H}_G^{\alpha+\eta-1}(S^0)(e) = \underline{H}_G^{\alpha+\eta-1}(S\eta^+)(e) = \mathbb{Z}.$$

The map h(e) is an isomorphism by an obvious computation in nonequivariant cohomology. If $|\alpha^G| > 1$, then by Lemma A.5, $\underline{H}_G^{\alpha+\eta-1} S^0 \cong L$. The only two maps h from L to R with h(e) an isomorphism have cokernel $\langle \mathbb{Z}/p \rangle$.

If $d \not\equiv 0 \bmod p$, then the only maps $h: A[d] \to R$ with h(e) an isomorphism are surjective. Therefore, once we have shown that $\underline{H}_G^\beta S^0$ is $A[d_\beta]$ when $\beta \in RSO_0(G)$, it will follow from the lemma above that $\underline{H}_G^\alpha S^0 = 0$ when $|\alpha| = -1$ and $|\alpha^G| = 1$.

Lemma 4.6 follows from Lemma A.9.

PROOF OF LEMMA 4.6. Let α and β be elements of RSO(G) with $|\alpha| = -1$, $|\alpha^G| > 0$, $|\beta| = 0$, and $|\beta^G| \leq 0$. Let η be a nontrivial irreducible complex representation. Consider the diagram

$$\begin{array}{ccccc} R \cong \underline{H}_G^{\alpha+\eta-1}S^0 & \to & \underline{H}_G^{\alpha}S^0 & \to & 0 \\ \downarrow & & \downarrow & & \\ R \cong \underline{H}_G^{\alpha+\beta+\eta-1}S^0 & \to & \underline{H}_G^{\alpha+\beta}S^0 & \to & 0 \end{array}$$

in which the vertical arrows are given by multiplication by ξ_β or μ_β. The rows of this diagram are exact by the proof of Lemma A.9. Let $y \in \underline{H}_G^{\alpha+\eta-1}(S^0)(1)$ be a generator and let $x \in \underline{H}_G^{\alpha}(S^0)(1)$ be its image. Since ρ preserves products, $\rho(\xi_\beta y)$ must be a generator. Thus, $\xi_\beta y$ must be a generator and so must $\xi_\beta x$. Similarly, $\rho(\mu_\beta y)$ is d_β times a generator, so $\mu_\beta y$ is d_β times a generator. It follows that $\mu_\beta x$ is a generator. This proves Lemma 4.6 in the special case where $|\alpha| = -1$ and $|\alpha^G| > 0$. The general case follows from the special case and Lemma A.2.

Let α be an element of $RSO_0(G)$. The main difficulty in identifying $\underline{H}_G^{\alpha}S^0$ with $A[d_\alpha]$ is that we must select a representative for α in $\tilde{R}_0(G)$ in order to define μ_α and d_α. To circumvent this difficulty, we work primarily with elements of $\tilde{R}_0(G)$ instead of elements of $RSO_0(G)$ in the remainder of our discussion of the additive structure of $\underline{H}_G^{*}S^0$. If α is in $\tilde{R}_0(G)$, we write $\underline{H}_G^{\alpha}S^0$ for the cohomology Mackey functor associated to the image of α in $RSO(G)$. To work with elements of $\tilde{R}_0(G)$, we must introduce variants of Definitions 4.5(a) and 4.5(d).

DEFINITION A.10. Observe that the procedure used to produce the element μ_α in Definitions 4.5(a) actually associates a map $\mu: S^{\Sigma\eta_i} \to S^{\Sigma\phi_i}$ to any element $\sum\phi_i - \eta_i$ of $\tilde{R}_0(G)$. If α is a nonzero element of $\tilde{R}_0(G)$, denote this map, and its image in $\underline{H}_G^{\alpha}(S^0)(1)$, by $\tilde{\mu}_\alpha$. Let $\tilde{\mu}_0$ denote the identity map of S^0 and $1 \in \underline{H}_G^{0}(S^0)(1)$. If ϕ is a nontrivial irreducible complex representation, then let $\epsilon_{\alpha,\phi}: S^{\Sigma\eta_i} \to S^{\phi+\Sigma\phi_i}$ denote the smash product of the map $\epsilon: S^0 \to S^\phi$ and the map $\tilde{\mu}_\alpha$. We also use $\epsilon_{\alpha,\phi}$ to denote the corresponding element in $\underline{H}_G^{\alpha+\phi}(S^0)(1)$.

If α and β are elements in $\tilde{R}_0(G)$ which represent the same element in $RSO_0(G)$, then $\tilde{\mu}_\alpha$ and $\tilde{\mu}_\beta$ need not be the same class in $\underline{H}_G^{\alpha}(S^0)(1)$. However, the class $\epsilon_{\alpha,\phi}$ in $\underline{H}_G^{\alpha+\phi}(S^0)(1)$ is uniquely determined by the sum $\alpha+\phi$ in $RSO(G)$. This uniqueness can be exploited to resolve the problems caused by dependence of $\tilde{\mu}_\alpha$ on α.

LEMMA A.11. Let α and β be in $\tilde{R}_0(G)$ and let ϕ and η be nontrivial irreducible complex representations such that $\alpha+\phi$ and $\beta+\eta$ represent the same element in $RSO(G)$. Then the cohomology classes $\epsilon_{\alpha,\phi}$ and $\epsilon_{\beta,\eta}$ in $\underline{H}_G^{\alpha+\phi}(S^0)(1)$ are equal.

PROOF. We establish the result for three special cases and then argue that the general case follows from them. Let η, η_1, η_2, ϕ, ϕ_1, and ϕ_2 be nontrivial irreducible complex representations and let $c\colon S^{\phi_1+\phi_2} \to S^{\phi_2+\phi_1}$ be the switch map. Regard $\alpha_1 = \phi_1 - \eta$, $\alpha_2 = \phi_2 - \eta$, and $\alpha = \phi_1 + \phi_2 - 2\eta$ as elements of $\tilde{R}_0(G)$. Let $\epsilon\colon S^0 \to S^\eta$ be the usual Euler class. The two maps $1 \wedge \epsilon$ and $\epsilon \wedge 1$ from S^η to $S^{\eta+\eta}$ are obviously equivariantly homotopic. On the level of maps,

$$\epsilon_{\alpha_2,\phi_1} = \tilde{\mu}_\alpha(\epsilon \wedge 1) \quad \text{and} \quad \epsilon_{\alpha_1,\phi_2} = c\,\tilde{\mu}_\alpha(1 \wedge \epsilon).$$

Therefore, $\epsilon_{\alpha_2,\phi_1}$ and $c\,\epsilon_{\alpha_1,\phi_2}$ are equivariantly homotopic. Thus, $\epsilon_{\alpha_2,\phi_1}$ and $\epsilon_{\alpha_1,\phi_2}$, regarded as cohomology classes, are equal. Here, the map c is, of course, absorbed in the passage to an RSO(G)-grading for $\underline{H}_G^* S^0$.

If η and ϕ_1 are equal and $\epsilon'\colon S^0 \to S^{\phi_2}$ is the inclusion, then the trick used above can also be used to show that $1 \wedge \epsilon'\colon S^\eta \to S^{\phi_1+\phi_2}$ is equivariantly homotopic to $\epsilon_{\alpha_2,\phi_1}$. Thus, if $\alpha_3 = \phi_1 - \phi_1 \in \tilde{R}_0(G)$, then ϵ' and $\epsilon_{\alpha_3,\phi_2}$ are equal in $\underline{H}_G^{\phi_2}(S^0)(1)$.

Regard $\beta_1 = (\phi_1 - \eta_1) + (\phi_2 - \eta_2)$ and $\beta_2 = (\phi_1 - \eta_2) + (\phi_2 - \eta_1)$ as elements of $\tilde{R}_0(G)$. By three applications of the result just proved for $\epsilon_{\alpha_2,\phi_1}$ and $\epsilon_{\alpha_1,\phi_2}$, it is possible to show that $\epsilon_{\beta_1,\phi}$ and $\epsilon_{\beta_2,\phi}$ are equal in $\underline{H}_G^{\beta_1+\phi}(S^0)(1)$.

If α and β are in $\tilde{R}_0(G)$ and ϕ and η are nontrivial irreducible complex representations such that $\alpha + \phi$ and $\beta + \eta$ represent the same element in RSO(G), then we can convert the pair (α, ϕ) into the pair (β, η) by some combination of the three basic transformations for which the lemma has already been proved. Thus, $\epsilon_{\alpha,\phi}$ and $\epsilon_{\beta,\eta}$ must be equal in $\underline{H}_G^{\alpha+\phi}(S^0)(1)$.

This lemma establishes that the element ϵ_β of Definition 4.5(d) does not depend on the choice of α and V used in its definition.

LEMMA A.12. If $\alpha \in RSO_0(G)$, then $\underline{H}_G^\alpha S^0 \cong A[d_\alpha]$. Moreover, if η is any nontrivial irreducible complex representation, then μ_α is the unique element of $\underline{H}_G^\alpha(S^0)(1)$ such that $\epsilon_\eta\,\mu_\alpha = \epsilon_{\alpha+\eta}$ and $\rho(\mu_\alpha) = d_\alpha\,\iota_\alpha$.

PROOF. Recall the map $s\colon RSO_0(G) \to \tilde{R}_0(G)$ introduced in section 2. Let $\alpha \in RSO_0(G)$ and assume that $s(\alpha) = \sum_{i=1}^{n} \phi_i - \eta_i$. Let α_0 be $0 \in \tilde{R}_0(G)$ and, for

$1 \leq k \leq n$, let α_k be the element $\sum_{i=1}^{k} \phi_i - \eta_i$ of $\tilde{R}_0(G)$. Denote by $d(\alpha_k)$ the integer associated to α_k by our homomorphism from $\tilde{R}_0(G)$ to $\mathbb{Z}$. For $0 \leq k \leq n$, let β_k be the element $\alpha_k + \phi_{k+1}$ of RSO(G). We will show by induction on k that

i) $\underline{H}_G^{\alpha_k} S^0$ is isomorphic to $A[d(\alpha_k)]$,

ii) $\tilde{\mu}_{\alpha_k}$ and $\tau(\iota_{\alpha_k})$ generate $\underline{H}_G^{\alpha_k}(S^0)(1)$,

iii) $\underline{H}_G^{\beta_k} S^0$ is isomorphic to $\langle \mathbb{Z} \rangle$, and

iv) ϵ_{β_k} generates $\underline{H}_G^{\beta_k} S^0$.

By the dimension axiom and Lemma A.4, these statements are true for $k = 0$. Consider the portion

$$\underline{H}_G^{\beta_k - 1}(S\eta_{k+1})^+ \to \underline{H}_G^{\beta_k} S^{\eta_{k+1}} \cong \underline{H}_G^{\alpha_{k+1}} S^0 \to \underline{H}_G^{\beta_k} S^0 \to \underline{H}_G^{\beta_k}(S\eta_{k+1})^+$$

of the cohomology long exact sequence of the second cofibre sequence. By Lemma A.1, The left hand term is isomorphic to L and the right hand term is zero. By Lemma A.7, the left hand arrow is a monomorphism. Thus, we have a short exact sequence

$$0 \to L \xrightarrow{f} \underline{H}_G^{\alpha_{k+1}} S^0 \to \underline{H}_G^{\beta_k} S^0 \to 0.$$

Assume that the assertions above hold for some integer k. The element μ_{k+1} in $\underline{H}_G^{\alpha_{k+1}}(S^0)(1)$ hits the generator ϵ_{β_k} in $\underline{H}_G^{\beta_k}(S^0)(1)$ by Lemma A.11. Since f(e) is an isomorphism, we may assume that f(e) takes the generator $1 \in \mathbb{Z} = L(e)$ to the generator $\iota_{\alpha_{k+1}}$ of $\underline{H}_G^{\alpha_{k+1}}(S^0)(e)$. It follows that $\tilde{\mu}_{\alpha_{k+1}}$ and $\tau(\iota_{\alpha_{k+1}})$ generate $\underline{H}_G^{\alpha_{k+1}}(S^0)(1)$. Since

$$\rho(\mu_{\alpha_{k+1}}) = d(\alpha_{k+1})\,\iota_{\alpha_{k+1}} \quad \text{and} \quad \rho\,\tau(\iota_{\alpha_{k+1}}) = p\iota_{\alpha_{k+1}},$$

$\underline{H}_G^{\alpha_{k+1}} S^0$ is isomorphic to $A[d(\alpha_{k+1})]$. By Lemma A.4, $\underline{H}_G^{\beta_{k+1}} S^0$ is isomorphic to $\langle \mathbb{Z} \rangle$ and is generated by $\epsilon_{\beta_{k+1}}$. Since $\tilde{\mu}_{\alpha n} = \mu_\alpha$ and $d(\alpha_n) = d_\alpha$, $\underline{H}_G^{\alpha} S^0$ is isomorphic to $A[d_\alpha]$.

Replacing α_{k+1} by α, η_{k+1} by η, and β_k by $\alpha + \eta$ in the cohomology long exact sequence above, we obtain the short exact sequence

$$0 \to L \to \underline{H}_G^{\alpha} S^0 \xrightarrow{h} \underline{H}_G^{\alpha+\eta} S^0 \to 0.$$

Our characterization of μ_α in terms of $\epsilon_\eta\,\mu_\alpha = h(\mu_\alpha)$ and $\rho(\mu_\alpha)$ follows directly from this sequence.

Two general observations suffice for the proofs of many of the multiplicative relations. Any product involving at least one element in the image of the transfer map τ is easily computed using the Frobenius property

$$x\,\tau(y) = \tau(\rho(x)\,y).$$

Any relation involving an element, like $\epsilon^{-m}\kappa$, obtained by divided some other element by an Euler class may be checked by eliminating the division by the Euler class and checking the resulting relation. The original relation then follows by Lemma A.2.

PROOF OF THEOREM 4.1. We will describe the individual Mackey functors $\underline{H}^{\alpha}_{G}S^0$ of $\underline{H}^{*}_{G}S^0$ by their positions in our standard plot of $\underline{H}^{*}_{G}S^0$. Since $\underline{H}^{\alpha}_{G}(S^0)(e) \cong H^{|\alpha|}(S^0;\mathbb{Z})$, it is easy to check that the elements $\iota_{1-\zeta}$ and $\iota_{\zeta-1}$ generate $\underline{H}^{*}_{G}(S^0)(e)$ and satisfy no relations in $\underline{H}^{*}_{G}(S^0)(e)$ other than the obvious relation $\iota_{1-\zeta}\,\iota_{\zeta-1} = \rho(1)$. It follows immediately from the structure of the Mackey functors R_-, L, and L_- that the elements $\tau(\iota^{n}_{1-\zeta})$, for $n \geq 1$, generate the part of $\underline{H}^{*}_{G}(S^0)(1)$ on the positive horizontal axis. For any positive integer n, $\rho(\xi^n) = \iota^{2n}_{\zeta-1}$. Therefore, ξ^n must generate $\underline{H}^{2n(\zeta-1)}_{G}(S^0)(1)$. The relation $\tau(\iota^{m}_{\zeta-1}) = 2\,\xi^m$ follows from the additive structure. No other relations involving only ξ and $\iota_{\zeta-1}$ are permitted by the additive structure. Lemmas A.2 and A.4 ensure that the powers of ϵ generate the part of $\underline{H}^{*}_{G}(S^0)(1)$ on the positive vertical axis. These two lemmas also indicate that the elements $\epsilon^m\,\xi^n$, for m, n $\geq$ 1, generate the part of $\underline{H}^{*}_{G}(S^0)(1)$ in the second quadrant. The same two lemmas indicate that the elements $\epsilon^{-m}\kappa$ and the elements $\epsilon^{-m}\,\tau(\iota^{2n+1}_{1-\zeta})$ generate the parts of $\underline{H}^{*}_{G}(S^0)(1)$ on the negative vertical axis and in the fourth quadrant, respectively. The relations not already verifed follow easily from the additive structure of $\underline{H}^{*}_{G}S^0$ or from our general observations. The additive structure of $\underline{H}^{*}_{G}S^0$ eliminates the possibility of any unlisted relations involving a single element. Since we have described every possible nonzero product of a pair of generators in terms of the generators, no further relations involving products are possible.

PROOF OF THEOREM 4.9. Again, we describe the individual Mackey functors $\underline{H}^{\alpha}_{G}S^0$ in terms of their positions in our plot of $\underline{H}^{*}_{G}S^0$. Since $\underline{H}^{\alpha}_{G}(S^0)(e) \cong H^{|\alpha|}(S^0;\mathbb{Z})$, it is easy to check that the relation $\iota_\alpha\,\iota_\beta = \iota_{\alpha+\beta}$ holds for any α, $\beta \in$ RSO(G) with $|\alpha| = |\beta| = 0$ and that no other relations in $\underline{H}^{*}_{G}(S^0)(e)$ hold among the ι_α. Therefore, for any $\beta \in$ RSO(G) with $|\beta| = 0$, ι_β can be written as a product of the ι_α included in the proposed list of generators of $\underline{H}^{*}_{G}S^0$. The elements ι_β, for $\beta \in$ RSO(G) with $|\beta| = 0$, generate $\underline{H}^{*}_{G}(S^0)(e)$ and the elements $\tau(\iota_\beta)$, for $\beta \in$ RSO(G) with $|\beta| = 0$ and $|\beta^G| > 0$, generate the part of $\underline{H}^{*}_{G}(S^0)(1)$ on the positive horizontal axis.

Let α and β be in $RSO_0(G)$ and let γ be an element of RSO(G) such that $|\gamma| > 0$ and $|\gamma^G| = 0$. The relation $\mu_\alpha\,\epsilon_\gamma = \epsilon_{\alpha+\gamma}$ follows from Lemma A.11. The relation

$$\mu_\alpha \mu_\beta = \mu_{\alpha+\beta} + [(d_\alpha d_\beta - d_{\alpha+\beta})/p]\, \tau(\iota_{\alpha+\beta})$$

follows from our characterization in Lemma A.12 of $\mu_{\alpha+\beta}$ as an element of $\underline{H}_G^{\alpha+\beta}(S^0)(1)$. From this relation, it follows that all of the elements μ_α can be constructed from the μ_β and ι_β in our proposed list of generators. By Lemma A.12, the elements μ_α and ι_α generate all of the $\underline{H}_G^\alpha S^0$ which are plotted at the origin. The relation $\mu_\alpha \epsilon_\gamma = \epsilon_{\alpha+\gamma}$ indicates that we can construct all the elements ϵ_γ from our proposed list of generators. By Lemmas A.2 and A.4, these elements generate all of the $\underline{H}_G^\alpha S^0$ on the positive vertical axis.

Let $\alpha \in RSO_0(G)$ and $\beta, \gamma \in RSO(G)$ with $|\beta| = |\gamma| = 0$ and $|\beta^G|$, $|\gamma^G| < 0$. The element σ_α can be obtained from μ_α and ι_α. The relations

$$\rho(\mu_\alpha \xi_\beta) = d_\alpha \iota_{\alpha+\beta} = \rho(d_\alpha \xi_{\alpha+\beta}),$$

$$\rho(\sigma_\alpha \xi_\beta) = \iota_{\alpha+\beta} = \rho(\xi_{\alpha+\beta}),$$

and

$$\rho(\xi_\beta \xi_\gamma) = \iota_{\beta+\gamma} = \rho(\xi_{\beta+\gamma})$$

follow from the fact that ρ is a ring homomorphism. They imply the relations $\mu_\alpha \xi_\beta = d_\alpha \xi_{\alpha+\beta}$, $\sigma_\alpha \xi_\beta = \xi_{\alpha+\beta}$, and $\xi_\beta \xi_\gamma = \xi_{\beta+\gamma}$ since ρ is a monomorphism in dimensions $\alpha + \beta$ and $\beta + \gamma$. These relations indicate that all of the elements ξ_β can be produced from our proposed list of generators. These elements generate the part of $\underline{H}_G^* S^0$ on the negative horizontal axis. By Lemmas A.2 and A.4, the elements $\epsilon_\delta \xi_\beta$ generate the part of $\underline{H}_G^* S^0$ in the second quadrant.

The relations $\mu_\gamma(\epsilon_\beta^{-1} \kappa_\alpha) = \epsilon_\beta^{-1} \kappa_{\alpha+\beta}$ and $\epsilon_\beta^{-1} \kappa_\alpha = \epsilon_\gamma^{-1} \kappa_\delta$, for $\alpha + \gamma = \beta + \delta$, may be checked by our general procedure for relations involving division by an Euler class. Together, these relations indicate that our proposed set of generators suffices to construct all of the elements $\epsilon_\beta^{-1} \kappa_\alpha$ and therefore to generate the part of $\underline{H}_G^* S^0$ on the negative vertical axis.

Let $\beta \in RSO_0(G)$ and let $\alpha \in RSO(G)$ with $|\alpha| < 0$ and $|\alpha^G| > 0$. Recall the class ν_α and the virtual representation $<\alpha>$ from Definitions 4.7. By definition, $<\alpha + \beta> = <\alpha>$, and by the Frobenius relation, $\nu_{<\alpha>} \tau(\iota_{\alpha+\beta}) = 0$. Therefore,

$$\begin{aligned} \mu_\beta \nu_\alpha &= \mu_\beta \mu_{\alpha-<\alpha>} \nu_{<\alpha>} \\ &= \mu_{\alpha+\beta-<\alpha>} \nu_{<\alpha>} \\ &= \nu_{\alpha+\beta}. \end{aligned}$$

This relation indicates that our proposed set of generators suffices to produce all of the elements ν_α and therefore the part of $\underline{H}_G^* S^0$ in the fourth quadrant.

We have now shown that our proposed set of generators does generate $\underline{H}_G^* S^0$. Seven of the relations we have not already established deserve comments. The relation $\epsilon_\alpha \epsilon_\beta = \epsilon_{\alpha+\beta}$ follows easily from the definition of the Euler classes, the Frobenius relation and the product relation for the classes μ_γ. The relation $\epsilon_\beta \xi_\alpha = d_{\delta-\alpha} \epsilon_\gamma \xi_\delta$, for $\alpha + \beta = \gamma + \delta$, follows from the sequence of equations

$$\epsilon_\beta \xi_\alpha = \mu_{\beta-\gamma} \epsilon_\gamma \xi_\alpha$$

$$= \epsilon_\gamma \mu_{\delta-\alpha} \xi_\alpha$$
$$= d_{\delta-\alpha} \epsilon_\gamma \xi_\delta .$$

The relations $\kappa_\alpha \kappa_\delta = p\,\kappa_{\alpha+\delta}$ and $\kappa_\gamma \nu_\alpha = 0$ can be confirmed from the definitions, the Frobenius property, and the relations which have already been established. Given these equations, the relations

$$\epsilon_\gamma (\epsilon_\beta^{-1} \kappa_\alpha) = \epsilon_{\beta-\gamma}^{-1} \kappa_\alpha ,$$
$$(\epsilon_\beta^{-1} \kappa_\alpha)(\epsilon_\gamma^{-1} \kappa_\delta) = p\,\epsilon_{\beta+\gamma}^{-1} \kappa_{\alpha+\delta} ,$$

and

$$(\epsilon_\beta^{-1} \kappa_\gamma)\, \nu_\alpha = 0$$

follow from our general procedure for checking relations involving classes divided by Euler classes. For the relations $\epsilon_\beta \nu_\alpha = \nu_{\alpha+\beta}$ and $\xi_\beta \nu_\alpha = d_{<\beta>-\beta} \nu_{\alpha+\beta}$, observe that ξ_β can be written as $\sigma_{\beta-<\beta>} \xi_{<\beta>}$ and that ϵ_β can be written as $\mu_\gamma \epsilon_{n\lambda}$, for some $\gamma \in RSO_0(G)$ and some positive integer n. The relations now follow by straightforward computations using the definitions, the Frobenius property, and the previously established relations. All of the remaining relations in the theorem follow directly from the definitions or the additive structure of $\underline{H}_G^* S^0$. The additive structure of $\underline{H}_G^* S^0$ eliminates the possibility of any unlisted relations involving a single element. Since we have described every possible nonzero product of a pair of generators in terms of the generators, no further relations involving products are possible.

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