THE ADAMS-NOVIKOV SPECTRAL SEQUENCE AND VOEVODSKY'S SLICE TOWER

MARC LEVINE

Abstract. We show that the spectral sequence converging to the stable homotopy groups of spheres, induced by the Betti realization of the slice tower for the motivic sphere spectrum, agrees with the Adams-Novikov spectral sequence, after a suitable re-indexing. The proof relies on a partial extension of Deligne's "décalage" construction to the Tot-tower of a cosimplicial spectrum.

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Introduction

Voevodsky has defined a natural tower in the motivic stable homotopy category $\mathcal{SH}(k)$ over a field k, called the *slice tower* (see [27, 28]). Relying on the computation of the slices of MGL by Hopkins-Morel [13], complete proofs of which have been recently made available through the work of Hoyois [10], we have filled in the details of a proof of the conjecture of Voevodsky [28], identifying the slices of the motivic sphere spectrum with a motive built out of the E_2-complex in the classical Adams-Novikov spectral sequence for the stable homotopy groups of spheres (*cf.* [1]). In addition, we have shown that the Betti realization of the slice tower yields a tower over the classical sphere spectrum $\mathbb{S}$, and the resulting spectral sequence strongly

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converges to the homotopy groups of $\mathbb{S}$. Furthermore, we have also shown that the resulting comparison map from the homotopy sheaves $\pi_{n,0}$ of the slice tower, evaluated on any algebraically closed subfield of $\mathbb{C}$, to the homotopy groups of the Betti realization, is an isomorphism. For all these results, we refer the reader to [18].

Putting all this together, we have a spectral sequence, strongly converging to $\pi_*\mathbb{S}$, of "motivic origin" and whose E_2-term agrees with the E_2-term in the Adams-Novikov spectral sequence, after a reindexing. The question thus arises: are these two spectral sequences the same, again after reindexing? The main result of this paper is an affirmative answer to this question, more precisely:

Theorem 1. *Let k be an algebraically closed field of characteristic zero. Consider the Adams-Novikov spectral sequence*

$$E_2^{p,q}(AN) = \mathrm{Ext}^{p,-q}_{MU_*(MU)}(MU_*, MU_*) \Longrightarrow \pi_{-p-q}(\mathbb{S})$$

and the "Atiyah-Hirzebruch" spectral sequence for $\Pi_{*,0}\mathbb{S}_k(k)$ associated to the slice tower for $\mathbb{S}_k$,

$$E_1^{p,q}(AH) = \pi_{-p-q,0}(s_p\mathbb{S}_k)(k) \Longrightarrow \pi_{-p-q,0}\mathbb{S}_k(k) = \pi_{-p-q}(\mathbb{S}_k).$$

Then there is an isomorphism

$$\gamma_1^{p,q} : E_1^{p,q}(AH) \cong E_2^{3p+q,-2p}(AN)$$

which induces a sequence of isomorphisms of complexes

$$\oplus_{p,q}\gamma_r^{p,q} : (\oplus_{p,q}E_r^{p,q}(AH), d_r) \to (\oplus_{p,q}E_{2r+1}^{3p+q,-2p}(AN), d_{2r+1})$$

Note that the fact that the homotopy groups of MU^* are concentrated in even degree implies that the Adams-Novikov differentials d_{2r} are all zero, and so $E_{2r}^{p,q}(AN) = E_{2r+1}^{p,q}(AN)$.

Remark. The Atiyah-Hirzebruch spectral sequence is often presented as an E_2-spectral sequence:

$$E_2^{p,q}(AH; \mathcal{E}, X)' := H^{p-q}(X, \pi^\mu_{-q}(n-q)) \Longrightarrow \mathcal{E}^{p+q,n}(X).$$

Here $\pi^\mu_n(\mathcal{E})$ is the *homotopy motive* of $\mathcal{E}$, that is, a canonically determined object of $DM(k)$ with $EM_{\mathbb{A}^1}(\pi^\mu_n(\mathcal{E})(n)[2n]) \cong s^t_n\mathcal{E}$. Thus, for $\mathcal{E} = \mathbb{S}_k$, $X = \mathrm{Spec}\, k$, this gives

$$E_2^{p,q}(AH)' = E_1^{-q,p+2q}(AH)$$

and theorem 1 yields the isomorphism

$$E_r^{p,q}(AH)' \cong E_{2r-1}^{p-q,2q}(AN),$$

answering affirmatively the question raised in [18, Introduction].

In the first four sections, we collect and review some material on various structures arising from functor categories with values in a model category. This material is to a varying degree quite well known; we include it here to aid the reader who may not be so familiar with this material and to fix notation.

In section 1, we review two constructions of a model category structure on the functor category, the projective model structure and the Reedy model structure. We apply this material to give constructions of slice towers and Betti realizations for motivic homotopy categories associated to functor categories. In section 2 we specialize to the case of the category Δ of finite ordered sets, and recall the Bousfield-Kan functor Tot and the associated tower and spectral sequence. In section 3 we describe how the Tot-tower can be described using cubical constructions, which are technically easier to handle. As an application, we show how applying the slice tower termwise to the truncated cosimplicial objects arising in of the motivic Adams-Novikov tower give approximations to the slice tower for the motivic sphere spectrum (proposition 3.5).

We then turn to some new material. In section 4 we adapt Deligne's décalage construction to the setting of cosimplicial objects in a stable model category that admits a t-structure and associated Postnikov tower, this latter construction replacing the canonical truncation of a complex. The main comparison result is achieved in proposition 4.3. This is the technical tool that enables us to compare the Atiyah-Hirzebruch and Adams-Novikov spectral sequences. The treatment of this topic is less than optimal, as one should expect a more general extension of Deligne's décalage construction to some version of filtered objects in a model category. In section 5 we examine the Adams-Novikov spectral sequence, both in the motivic as well as the classical setting, and relate this to the slice tower for the motivic sphere spectrum. With the help of recent work on Betti realizations and the slices for MGL, it is rather easy to show that the Betti realization of the slice tower for the motivic sphere spectrum agrees with the décalage tower associated to the classical Adams-Novikov tower. We then apply our results on the décalage construction to achieve the desired comparison.

1. Constructions in functor categories

It is convenient to perform constructions, such as Postnikov towers in various settings, or realization functors, in functor categories. This can be accomplished in a number of ways. The Postnikov towers may be constructed via cofibrant replacements associated to a right Bousfield localization; by making the cofibrant replacement functorial, this extends immediately to functor categories. The Betti realization is similarly accomplished as the left derived functor of a left Quillen functor, so again, applying this functor to a functorial cofibrant replacement extends the Betti realization to a realization functor between functor categories. However, it is often useful to have more control over these constructions, for which a full extension to the appropriate model category structure on the functor category is useful; we give some details of this approach here. None of this material is new; it is assembled from [3, 8, 12] and collected here for the reader's convenience.

1.1. **Model structures on functor categories.** Let $\mathcal{S}, \mathcal{T}$ be small categories, $\mathcal{M}$ a complete and cocomplete category, $\mathcal{M}^{\mathcal{S}}$ the category of functors $\mathcal{X} : \mathcal{S} \to \mathcal{M}$. For $f : \mathcal{T} \to \mathcal{S}$ a functor, we have the restriction functor $f_* : \mathcal{M}^{\mathcal{S}} \to \mathcal{M}^{\mathcal{T}}$, $f_*\mathcal{X} := \mathcal{X} \circ f$, with left adjoint f^* and right adjoint $f^!$. For $\mathcal{X} \in \mathcal{M}^{\mathcal{T}}$, $f^*\mathcal{X}$, resp., $f^!\mathcal{X}$, is the left, resp. right, Kan extension in the diagram

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{\mathcal{X}} & \mathcal{M} \\ {\scriptstyle f}\downarrow & & \\ \mathcal{S}. & & \end{array}$$

In particular, for $s \in \mathcal{S}$, we have $i_s : pt \to \mathcal{S}$, the inclusion functor with value s, inducing the evaluation functor $i_{s*} : \mathcal{M}^{\mathcal{S}} \to \mathcal{M}$, the left adjoint $i_s^* : \mathcal{M} \to \mathcal{M}^{\mathcal{S}}$, and right adjoint $i_s^! : \mathcal{M} \to \mathcal{M}^{\mathcal{S}}$.

We take $\mathcal{M}$ to be a simplicial model category and consider two model structures on $\mathcal{M}^{\mathcal{S}}$. If $\mathcal{M}$ is cofibrantly generated, we may give $\mathcal{M}^{\mathcal{S}}$ the projective model structure, that is, weak equivalences and fibrations are defined pointwise, and cofibrations are characterized by having the left lifting property with respect to trivial fibrations.

In case $\mathcal{S}$ is a Reedy category, one can also give $\mathcal{M}^{\mathcal{S}}$ the Reedy model structure. We first recall the definition of a Reedy category $\mathcal{S}$: There is an ordinal λ, a function (called *degree*) $d : \mathrm{Obj}\mathcal{S} \to \lambda$ and two subcategories $\mathcal{S}_+$, $\mathcal{S}_-$, such that all non-identity morphisms in $\mathcal{S}_+$ increase the degree, all non-identity morphisms in $\mathcal{S}_-$ decrease the degree, and each morphism f in $\mathcal{S}$ admits a unique factorization $f = a \circ b$ with $a \in \mathcal{S}_+$, $b \in \mathcal{S}_-$. For $s \in \mathcal{S}$, we let $\mathcal{S}_-^s$ be the category of non-identity morphisms $s \to t$ in $\mathcal{S}_-$, and let $\mathcal{S}_+^s$ be the category of non-identity morphisms $t \to s$ in $\mathcal{S}_+$. Given an object $\mathcal{X} \in \mathcal{M}^{\mathcal{S}}$, and $s \in \mathcal{S}$, we have the *latching space* $L^s\mathcal{X}$ and *matching space* $M^s\mathcal{X}$:

$$L^s\mathcal{X} := \varinjlim_{t \to s \in \mathcal{S}_+^s} \mathcal{X}(t), \ M^s\mathcal{X} := \varprojlim_{s \to t \in \mathcal{S}_-^s} \mathcal{X}(t),$$

with the canonical morphisms $L^s\mathcal{X} \to \mathcal{X}(s)$, $\mathcal{X}(s) \to M^s\mathcal{X}$.

The Reedy model structure on $\mathcal{M}^{\mathcal{S}}$ has weak equivalences the maps $f : \mathcal{X} \to \mathcal{Y}$ such that $f(s) : \mathcal{X}(s) \to \mathcal{Y}(s)$ is a weak equivalence in $\mathcal{M}$ for all $s \in \mathcal{S}$, fibrations the maps $f : \mathcal{X} \to \mathcal{Y}$ such that $\mathcal{X}(s) \to \mathcal{Y}(s) \times_{M^s\mathcal{Y}} M^s\mathcal{X}$ is a fibration in $\mathcal{M}$ for all $s \in \mathcal{S}$ and cofibrations the maps $f : \mathcal{X} \to \mathcal{Y}$ such that $\mathcal{X}(s) \amalg_{L^s\mathcal{X}} L^s\mathcal{Y} \to \mathcal{Y}(s)$ is a cofibration for all $s \in \mathcal{S}$. This makes $\mathcal{M}^{\mathcal{S}}$ a model category without any additional conditions on $\mathcal{M}$.

In each of these two model structures, the evaluation functor i_{s*} preserves fibrations, cofibrations and weak equivalences, and admits i_s^* as left Quillen functor and $i_s^!$ as right Quillen functor.

Remark 1.1. Suppose $\mathcal{M}$ is cofibrantly generated. If $\mathcal{S}$ is a direct category, these two model structures agree; if $\mathcal{S}$ is a general Reedy category, the weak equivalences in the two model structures agree, every fibration for the Reedy model structure is a fibration in the projective model structure, and thus every cofibration in the projective model structure is a cofibration in the Reedy model structure. Furthermore, the projective model structure is also cofibrantly generated, and is cellular, resp. combinatorial, if $\mathcal{M}$ is cellular, resp. combinatorial; we refer the reader to [8, theorem 11.6.1, theorem 12.1.5], [3, theorem 2.14] for proofs of these assertions. The Reedy model structure likewise inherits the combinatorial property from $\mathcal{M}$ [3, lemma 3.33].

Left and right properness are similarly passed on from $\mathcal{M}$ to the projective model structure on $\mathcal{M}^{\mathcal{S}}$ [3, proposition 2.18]. For the Reedy model structure, the inheritance of left and right properness is proven in [3, lemma 3.24].

Example 1.2. The classical example of a Reedy category is the category of finite ordered sets. Let Δ denote the category with objects the finite ordered sets $[n] := \{0, \ldots, n\}$, with the standard order, $n = 0, 1, \ldots$. For a category $\mathcal{C}$ the functor categories $\mathcal{C}^{\Delta}$, $\mathcal{C}^{\Delta^{\mathrm{op}}}$ are as usual called the category of cosimpliicial, resp. simplicial objects in $\mathcal{C}$.

We let Δ_{inj}, Δ_{surj} denote the subcategories of Δ with the same objects, and with morphisms the injective, resp. surjective order-preserving maps. Taking $\Delta_+ := \Delta_{inj}$, $\Delta_- := \Delta_{surj}$ and $d : \Delta \to \mathbb{N}$ the function $d([n]) = n$ makes Δ a Reedy category. We have the standard co-face maps $d^j : [n] \to [n+1]$, $j = 0, \ldots, n+1]$ and co-degeneracy maps $s_i : [n] \to [n-1]$, $i = 0, \ldots, n-1$.

Let **Spc** denote the category of simplicial sets, $\mathbf{Spc}_\bullet$ the category of pointed simplicial sets, each with the standard model structures, *cf.* [12, §3.2]. Note that this is not the Reedy model structure!

Let $\Delta[n]$ be the representable simplicial set, $\Delta[n] := \mathrm{Hom}_\Delta(-, [n])$ and let $\Delta[*] : \Delta \to \mathbf{Spc}$ the cosimplicial space $n \mapsto \Delta[n]$.

1.2. **Simplicial structure.** We consider a small category $\mathcal{S}$ and a simplicial model category $\mathcal{M}$ satisfying the conditions discussed in the previous section. Both model structures for $\mathcal{M}^\mathcal{S}$ discussed above yield simplicial model categories: for a simplicial set A and a functor $\mathcal{X} : \mathcal{S} \to \mathcal{M}$, the product $\mathcal{X} \otimes A$ and Hom-object $\mathcal{H}om(A, \mathcal{X})$ are the evident functors $(\mathcal{X} \otimes A)(s) := \mathcal{X}(s) \otimes A$ and $\mathcal{H}om(A, \mathcal{X})(s) := \mathcal{H}om(A, \mathcal{X}(s))$. The simplicial Hom-object $\mathrm{Map}_{\mathcal{M}^\mathcal{S}}(\mathcal{X}, \mathcal{Y})$ is given as the simplicial set

$$n \mapsto \mathrm{Hom}_{\mathcal{M}^\mathcal{S}}(\mathcal{X} \otimes \Delta[n], \mathcal{Y}),$$

or equivalently, as the equalizer

$$\mathrm{Map}_{\mathcal{M}^\mathcal{S}}(\mathcal{X}, \mathcal{Y}) \to \prod_{s \in \mathcal{S}} \mathrm{Map}_\mathcal{M}(\mathcal{X}(s), \mathcal{Y}(s)) \underset{\prod g_*}{\overset{\prod g^*}{\rightrightarrows}} \prod_{g: s \to s'} \mathrm{Map}_\mathcal{M}(\mathcal{X}(s), \mathcal{Y}(s')).$$

Together with the evident adjunction $\mathrm{Hom}(\mathcal{X}, \mathcal{H}om(A, \mathcal{Y})) \cong \mathrm{Hom}(\mathcal{X} \otimes A, \mathcal{Y})$, this makes $\mathcal{M}^\mathcal{S}$ into a simplicial model category (see below).

We have as well the object $\mathcal{H}om(\mathcal{A}, X)$ in $\mathcal{M}^\mathcal{S}$ for $\mathcal{A} \in \mathbf{Spc}^{\mathcal{S}^{\mathrm{op}}}$, $X \in \mathcal{M}$, with $\mathcal{H}om(\mathcal{A}, X)(s) := \mathcal{H}om(\mathcal{A}(s), X)$ and the object $X \otimes \mathcal{A}$ in $\mathcal{M}^\mathcal{S}$ for $\mathcal{A} \in \mathbf{Spc}^\mathcal{S}$, $X \in \mathcal{M}$, with $(X \otimes \mathcal{A})(s) := X \otimes \mathcal{A}(s)$.

For $\mathcal{A} \in \mathbf{Spc}^\mathcal{S}$, $\mathcal{X} \in \mathcal{M}^\mathcal{S}$, we have $\mathcal{H}om^\mathcal{S}(\mathcal{A}, \mathcal{X})$ in $\mathcal{M}$ defined as the equalizer

$$\mathcal{H}om^\mathcal{S}(\mathcal{A}, \mathcal{X}) \to \prod_{s \in \mathcal{S}} \mathcal{H}om(\mathcal{A}(s), \mathcal{X}(s)) \underset{\prod g_*}{\overset{\prod g^*}{\rightrightarrows}} \prod_{g: s \to s'} \mathcal{H}om(\mathcal{A}(s), \mathcal{X}(s')).$$

Similarly, for $\mathcal{A} \in \mathbf{Spc}^{\mathcal{S}^{\mathrm{op}}}$, $\mathcal{X} \in \mathcal{M}^\mathcal{S}$, we have $\mathcal{X} \otimes^\mathcal{S} \mathcal{A}$ in $\mathcal{M}$, defined as the coequalizer

$$\amalg_{g: s' \to s} \mathcal{X}(s') \otimes \mathcal{A}(s) \underset{\prod \mathrm{id} \otimes \mathcal{A}(g)}{\overset{\prod \mathcal{X}(g) \otimes \mathrm{id}}{\rightrightarrows}} \amalg_{s \in \mathcal{S}} \mathcal{X}(s) \otimes \mathcal{A}(s) \to \mathcal{X} \otimes^\mathcal{S} \mathcal{A}.$$

Besides the adjunction already mentioned, one has the adjunction, for $X \in \mathcal{M}$, $\mathcal{A} \in \mathbf{Spc}^\mathcal{S}$, $\mathcal{Y} \in \mathcal{M}^\mathcal{S}$,

$$\mathrm{Hom}_\mathcal{M}(X, \mathcal{H}om^\mathcal{S}(\mathcal{A}, \mathcal{Y})) \cong \mathrm{Hom}_{\mathcal{M}^\mathcal{S}}(X \otimes \mathcal{A}, \mathcal{Y})$$

and for $\mathcal{X} \in \mathcal{M}^\mathcal{S}$, $\mathcal{A} \in \mathbf{Spc}^{\mathcal{S}^{\mathrm{op}}}$, $Y \in \mathcal{M}$,

$$\mathrm{Hom}_{\mathcal{M}^\mathcal{S}}(\mathcal{X}, \mathcal{H}om(\mathcal{A}, Y)) \cong \mathrm{Hom}_\mathcal{M}(\mathcal{X} \otimes^\mathcal{S} \mathcal{A}, Y)$$

These all follow directly from the adjunctions for $\mathcal{H}om$ and $\otimes$.

Both adjunctions are Quillen adjunctions of two variables. In case $\mathcal{M}$ is cofibrantly generated and we use the projective model structure, this is [8, theorem 11.7.3]; if $\mathcal{S}$ is a Reedy category and we give $\mathcal{M}^\mathcal{S}$ the Reedy model structure, this is [3, lemma 3.24]. This gives $\mathcal{M}^{\mathcal{S}}$ the structure of a $\mathbf{Spc}^{\mathcal{S}}$ model category and a $\mathbf{Spc}^{\mathcal{S}^{\mathrm{op}}}$ model category.

1.3. **Monoidal structure.** We now suppose that $\mathcal{M}$ has a symmetric monoidal structure $\otimes_{\mathcal{M}}$, making $\mathcal{M}$ into a closed symmetric monoidal simplicial model category, with internal Hom $\mathcal{H}om_{\mathcal{M}}(-,-)$.

For $X \in \mathcal{M}$, $\mathcal{Y} \in \mathcal{M}^{\mathcal{S}}$, we have $X \otimes_{\mathcal{M}} \mathcal{Y}$ and $\mathcal{H}om_{\mathcal{M}}(X, \mathcal{Y})$ in $\mathcal{M}^{\mathcal{S}}$, with the adjunction, for $\mathcal{Y}, \mathcal{Z} \in \mathcal{M}^{\mathcal{S}}$, $X \in \mathcal{M}$,

$$\mathrm{Hom}_{\mathcal{M}^{\mathcal{S}}}(X \otimes_{\mathcal{M}} \mathcal{Y}, \mathcal{Z}) \cong \mathrm{Hom}_{\mathcal{M}^{\mathcal{S}}}(\mathcal{Y}, \mathcal{H}om_{\mathcal{M}}(X, \mathcal{Z}))$$

This extends to the adjunction on mapping spaces

$$\mathrm{Map}_{\mathcal{M}^{\mathcal{S}}}(X \otimes_{\mathcal{M}} \mathcal{Y}, \mathcal{Z}) \cong \mathrm{Map}_{\mathcal{M}^{\mathcal{S}}}(\mathcal{Y}, \mathcal{H}om_{\mathcal{M}}(X, \mathcal{Z})).$$

We define the $\mathcal{M}$-valued internal Hom

$$\mathcal{H}om^{\mathcal{S}}_{\mathcal{M}} : (\mathcal{M}^{\mathcal{S}})^{\mathrm{op}} \times \mathcal{M}^{\mathcal{S}} \to \mathcal{M}$$

as the equalizer

$$\mathcal{H}om^{\mathcal{S}}_{\mathcal{M}}(\mathcal{X}, \mathcal{Y}) \to \prod_{s \in \mathcal{S}} \mathcal{H}om_{\mathcal{M}}(\mathcal{X}(s), \mathcal{Y}(s)) \underset{\prod g_*}{\overset{\prod g^*}{\rightrightarrows}} \prod_{g: s \to s'} \mathcal{H}om_{\mathcal{M}}(\mathcal{X}(s), \mathcal{Y}(s')).$$

Similarly, for $\mathcal{X} \in \mathcal{M}^{\mathcal{S}}$, $\mathcal{Y} \in \mathcal{M}^{\mathcal{S}^{\mathrm{op}}}$, we have $\mathcal{X} \otimes^{\mathcal{S}}_{\mathcal{M}} \mathcal{Y}$ in $\mathcal{M}$, defined as the coequalizer

$$\amalg_{g: s' \to s} \mathcal{X}(s') \otimes_{\mathcal{M}} \mathcal{Y}(s) \underset{\prod \mathrm{id} \otimes \mathcal{Y}(g)}{\overset{\prod \mathcal{X}(g) \otimes \mathrm{id}}{\rightrightarrows}} \amalg_{s \in \mathcal{S}} \mathcal{X}(s) \otimes_{\mathcal{M}} \mathcal{Y}(s) \to \mathcal{X} \otimes^{\mathcal{S}}_{\mathcal{M}} \mathcal{A}.$$

We have the adjunctions, for $\mathcal{A} \in \mathbf{Spc}^{\mathcal{S}}$, $\mathcal{Y} \in \mathcal{M}^{\mathcal{S}}$, $X \in \mathcal{M}$,

$$\mathcal{H}om^{\mathcal{S}}(\mathcal{A}, \mathcal{H}om_{\mathcal{M}}(X, \mathcal{Y})) \cong \mathcal{H}om^{\mathcal{S}}_{\mathcal{M}}(X \otimes \mathcal{A}, \mathcal{Y}) \cong \mathcal{H}om_{\mathcal{M}}(X, \mathcal{H}om^{\mathcal{S}}(\mathcal{A}, \mathcal{Y})),$$

induced by the adjunctions

$$\mathcal{H}om(A, \mathcal{H}om_{\mathcal{M}}(X, Y)) \cong \mathcal{H}om_{\mathcal{M}}(X \otimes A, Y) \cong \mathcal{H}om_{\mathcal{M}}(X, \mathcal{H}om(A, Y))$$

for $X, Y \in \mathcal{M}$, $A \in \mathbf{Spc}$. The analogous constructions and statements hold in the pointed setting.

Lemma 1.3. *Give $\mathcal{M}^{\mathcal{S}}$ either the Reedy model structure or, in case $\mathcal{M}$ is cofibrantly generated, the projective model structure. Then the operations $\otimes_{\mathcal{M}}$ and $\mathcal{H}om^{\mathcal{S}}_{\mathcal{M}}$ are a Quillen adjunction of two variables, that is, these make $\mathcal{M}^{\mathcal{S}}$ into an $\mathcal{M}$-model category.*

Proof. For the projective model structure, the proof of [8, theorem 11.7.2] extends word for word to prove the result; the case of the Reedy model structure is proven in [3, lemma 3.36] □

$\mathcal{M}^{\mathcal{S}}$ is a closed symmetric monoidal category, with $(\mathcal{A} \otimes_{\mathcal{M}^{\mathcal{S}}} \mathcal{B})(s) := \mathcal{A}(s) \otimes_{\mathcal{M}} \mathcal{B}(s)$ for $\mathcal{A}, \mathcal{B} \in \mathcal{M}^{\mathcal{S}}$. The internal Hom is given as

$$\mathcal{H}om_{\mathcal{M}^{\mathcal{S}}}(\mathcal{A}, \mathcal{B})(s) := \mathcal{H}om^{s/\mathcal{S}}(s/\mathcal{A}, s/\mathcal{B})$$

where $s/\mathcal{A} \in \mathcal{M}^{s/\mathcal{S}}$ is the functor $s/\mathcal{A}(s \to t) := \mathcal{A}(t)$; for $f : s \to s'$, the induced map $\mathcal{H}om_{\mathcal{M}^{\mathcal{S}}}(\mathcal{A}, \mathcal{B})(s) \to \mathcal{H}om_{\mathcal{M}^{\mathcal{S}}}(\mathcal{A}, \mathcal{B})(s')$ is the map $\mathcal{H}om^{s/\mathcal{S}}(s/\mathcal{A}, s/\mathcal{B}) \to \mathcal{H}om^{s'/\mathcal{S}}(s'/\mathcal{A}, s'/\mathcal{B})$ induced by the functor $f^* : s/\mathcal{S} \to s'/\mathcal{S}$, noting that $(s'/\mathcal{A}) \circ f^* = s/\mathcal{A}$. The unit is the constant functor with value the unit in $\mathcal{M}$.

The question of when this gives $\mathcal{M}^{\mathcal{S}}$ the structure of a symmetric monoidal model category does not appear to have a simple answer. In the case of the Reedy model structure, Barwick proves the following result:

Proposition 1.4. *Let $\mathcal{S}$ be a Reedy category and give $\mathcal{M}^{\mathcal{S}}$ the Reedy model structure. Suppose that either*
a. all morphisms in $\mathcal{S}_-$ are epimorphisms and for each $s \in \mathcal{S}$ the category $\mathcal{S}_-^s$ is connected
or the dual
b. all morphisms in $\mathcal{S}_+$ are monomorphisms and for each $s \in \mathcal{S}$, the category $\mathcal{S}_+^s$ is connected
Then $\otimes_{\mathcal{M}^{\mathcal{S}}}$ and $\mathcal{H}om_{\mathcal{M}^{\mathcal{S}}}(-,-)$ is a Quillen adjunction of two variables, making $\mathcal{M}^{\mathcal{S}}$ a symmetric monoidal model category.

The condition (a) is satisfied for $\mathcal{S} = \Delta$ and the dual (b) is satisfied for $\mathcal{S} = \Delta^{\mathrm{op}}$, so the categories of cosimplicial or simplicial objects in a symmetric monoidal model category have the structure of a symmetric monoidal model category. As another example of an $\mathcal{S}$ satisfying (a), one can take for $\mathcal{S}$ the category associated to a finite poset having a final object, with Reedy structure $\mathcal{S} = \mathcal{S}_-$; a finite poset with initial object similarly satisfies (b) if one takes $\mathcal{S} = \mathcal{S}_+$.

1.4. **Bousfield localization.** We suppose that $\mathcal{M}$ is cellular and right proper. Let K be a set of cofibrant objects in $\mathcal{M}$. We have the right Bousfield localization $R_K\mathcal{M}$ with associated functorial cofibrant replacement $Q_K \to \mathrm{id}$ (see [8, theorem 5.1.1]). Let $K^{\mathcal{S}}$ be the set of cofibrant objects i_s^*a, $a \in K$, $s \in \mathcal{S}$, and let $R_{K^{\mathcal{S}}}\mathcal{M}^{\mathcal{S}}$ be the right Bousfield localization of $\mathcal{M}^{\mathcal{S}}$ with respect to $K^{\mathcal{S}}$ (as noted in remark 1.1, $\mathcal{M}^{\mathcal{S}}$ inherits celluarity and right properness from $\mathcal{M}$).

Lemma 1.5. *Suppose that $\mathcal{M}$ is cellular and right proper, and give $\mathcal{M}^{\mathcal{S}}$ the projective model structure. Let K be a set of cofibrant objects in $\mathcal{M}$.*
1. The right Bousfield localization $R_{K^{\mathcal{S}}}\mathcal{M}^{\mathcal{S}}$ is the same as the projective model structure on $(R_K\mathcal{M})^{\mathcal{S}}$.
2. Take $x \in \mathcal{M}^{\mathcal{S}}$ and let $Qx \to x$ be a cofibrant replacement in $R_{K^{\mathcal{S}}}\mathcal{M}^{\mathcal{S}}$. Then for all $s \in \mathcal{S}$, $i_{s*}Qx \to i_{s*}x$ is a cofibrant replacement of $i_{s*}x$ in $R_K\mathcal{M}$.

Proof. Right Bousfield localization leaves the fibrations unchanged, hence $R_{K^{\mathcal{S}}}\mathcal{M}^{\mathcal{S}}$ and $(R_K\mathcal{M})^{\mathcal{S}}$ have the same fibrations. The weak equivalences in a right Bousfield localization with respect to a set of objects K are the K-colocal weak equivalences, that is, maps $X \to Y$ that induce a weak equivalence on the Hom spaces $\mathcal{H}om(a, RX) \to \mathcal{H}om(a, RY)$ for all $a \in K$, where RX, RY are fibrant replacements. From this it follows that $X \to Y$ is a weak equivalence in $R_{K^{\mathcal{S}}}\mathcal{M}^{\mathcal{S}}$ if and only if $i_{s*}X \to i_{s*}Y$ is a weak equivalence in $R_K\mathcal{M}$ for all s, that is, the weak equivalences in $R_{K^{\mathcal{S}}}\mathcal{M}^{\mathcal{S}}$ and $(R_K\mathcal{M})^{\mathcal{S}}$ agree.

(2) follows from (1), noting that i_{s*} preserves cofibrations, fibrations and weak equivalences (for the projective model structure). □

Examples 1.6. 1. *"Topological" Postnikov towers.* We recall a functorial construction of the $n-1$-connected cover $f_n\mathcal{X} \to \mathcal{X}$ of a pointed space. Fix an integer $n \geq 0$ and let K_n be the set of spaces of the form $\Sigma^m X$, with X in $\mathbf{Spc}_\bullet$ and $m \geq n$. $\mathbf{Spc}_\bullet$ is a right proper cellular simplicial model category, hence by [8, theorem 5.1.1], the right Bousfield localization $R_{K_n}\mathbf{Spc}_\bullet$ of $\mathbf{Spc}_\bullet$ with respect to the K_n-colocal maps exists. In addition, there is a cofibrant replacement functor

$f_n : R_{K_n}\mathbf{Spc}_\bullet \to R_{K_n}\mathbf{Spc}_\bullet$. By the definition of right Bousfield localization ([8, definition 3.3.1], see also [18, theorem 2.5]) $f_n\mathcal{X} \to \mathcal{X}$ in $\mathbf{HoSpc}_\bullet$ is universal for maps from $n-1$-connected $\mathcal{Y}$ to $\mathcal{X}$; by obstruction theory, it follows that $f_n\mathcal{X} \to \mathcal{X}$ is an $n-1$-connected cover of $\mathcal{X}$. Using lemma 1.5, we may form the $n-1$-connected cover $f_n^{\mathcal{S}}\mathcal{X} \to \mathcal{X}$ in the functor category $\mathbf{Spc}_\bullet^{\mathcal{S}}$ as the cofibrant replacement with respect to the right Bousfield localization $R_{K_n^{\mathcal{S}}}\mathbf{Spc}_\bullet$.

Varying n and noting that $K_n \subset K_m$ if $n \geq m$ gives the tower of cofibrant replacement functors

$$\ldots \to f_{n+1}^{\mathcal{S}} \to f_n^{\mathcal{S}} \to \ldots \to f_0^{\mathcal{S}} = \mathrm{id}.$$

Let $\mathbf{Spt}$ be the category of S^1-spectra in $\mathbf{Spt}_\bullet$, with stable model structure as defined in [11]. We have the nth evaluation functor $ev_n : \mathbf{Spt} \to \mathbf{Spt}_\bullet$, $ev_n(S_0, S_1, \ldots) := S_n$, and its left adjoint $F_n : \mathbf{Spc}_\bullet \to \mathbf{Spt}$,

$$F_n(S) := (pt, \ldots, pt, S, \Sigma S, \Sigma^2 S, \ldots).$$

We repeat the construction of the Postnikov tower, with $\mathbf{Spt}$ replacing $\mathbf{Spc}_\bullet$ and taking K_n to be the set of objects $F_a\Sigma^b X$, with $X \in \mathbf{Spc}_\bullet$, $b - a \geq n$, $n \in \mathbb{Z}$. This gives us the Postnikov tower in the functor category $\mathbf{Spt}^{\mathcal{S}}$ (with $n \in \mathbb{Z}$)

$$\ldots \to f_{n+1}^{\mathcal{S}} \to f_n^{\mathcal{S}} \to \ldots \to \mathrm{id}.$$

We may extend these constructions to other model categories. Rather than attempting an axiomatic discussion, we content ourselves with the examples arising in motivic homotopy theory. Let S be a noetherian separated base-scheme and let $\mathbf{Spc}_\bullet(S)$ be the category of pointed spaces over S, that is, $\mathbf{Spc}_\bullet$-valued presheaves on the category $\mathbf{Sm}/S$ of smooth S of finite type. We give $\mathbf{Spc}_\bullet(S)$ the motivic model structure; this gives $\mathbf{Spc}_\bullet(S)$ the structure of a proper combinatorial symmetric monoidal simplicial model category (for details see [9, corollary 1.6], [14, §1, theorem 1.1], [15, Appendix A] and [25, theorem 2.3.2]). Letting $K_n(S)$ be the set of objects of the form $\Sigma^m\mathcal{X}$, with $\mathcal{X} \in \mathbf{Spc}_\bullet(S)$ and $m \geq n$, we have the right Bousfield localization, $R_{K_n(S)}\mathbf{Spc}_\bullet(S)$ and the cofibrant replacement functor f_n, with universal property for maps with source in the $K_n(S)$-cellular objects of $\mathbf{Spc}_\bullet(S)$. These turn out to be the $n-1$-connected objects in $\mathbf{Spc}_\bullet(S)$, that is, those objects with vanishing homotopy sheaves π_m for $m < n$ (see e.g. [24]. See [18, theorem 3.1, remark 3.3] for a discussion of the stable case and an indication of how this construction works in the unstable case).

We may also use categories of S^1 or $\mathbb{P}^1$ spectra, $\mathbf{Spt}_{S^1}(S)$, $\mathbf{Spt}_{\mathbb{P}^1}(S)$, with the respective motivic model structures (see [15] for a description of the model structures and e.g. [25, theorem 2.5.4] for the fact that these are cellular). For S^1 spectra, replace K_n with $K_n^{S^1}(S) := \{F_q^{S^1}\Sigma^p\mathcal{X}, \mathcal{X} \in \mathbf{Spc}_\bullet(S), p - q \geq n\}$. Here $F_q^{S^1} : \mathbf{Spc}_\bullet(S) \to \mathbf{Spt}_{S^1}(S)$ is given by using the functor $F_q : \mathbf{Spc}_\bullet \to \mathbf{Spt}$, that is,

$$F_q^{S^1}(\mathcal{X})(T) := F_q(\mathcal{X}(T))$$

for each $T \to S$ in $\mathbf{Sm}/S$. Again, the $K_n^{S^1}(S)$-cellular objects are those $E \in \mathbf{Spt}_{S^1}(S)$ with stable homotopy sheaves $\pi_n E$ zero for $n < m$. Suppose $S = \mathrm{Spec}\, k$, k a perfect field. Then in this stable model category, the subcategory $\mathcal{SH}_{S^1}(S)_{\leq 0} := \mathbf{Ho}R_{K_0^{S^1}(S)}\mathbf{Spt}_{S^1}(S)$ of the homotopy category $\mathcal{SH}_{S^1}(S)$ of $\mathbf{Spt}_{S^1}(S)$ is half of a t-structure with heart the strictly $\mathbb{A}^1$-invariant Nisnevich sheaves on $\mathbf{Sm}/S$. and with $\mathcal{SH}_{S^1}(S)_{\geq 0}$ the full subcategory of the E with $\pi_n E = 0$ for $n > 0$. This all follows from results of Morel, *cf.* [20, theorem 4.3.4, lemma 4.3.7].

For $\mathbf{Spt}_{\mathbb{P}^1}(S)$, we use $K_n^{\mathbb{P}^1}(S) := \{F_q^{\mathbb{P}^1}\Sigma_{S^1}^p\mathcal{X}, \mathcal{X} \in \mathbf{Spc}_\bullet(S), p - q \geq n\}$, with $F_q^{\mathbb{P}^1}\mathcal{X} := (F_q^{\mathbb{P}^1}\mathcal{X}_0, F_q^{\mathbb{P}^1}\mathcal{X}_1, \ldots)$, $F_q^{S^1}\mathcal{X}_n = pt$ for $n < q$, $F_q\mathcal{X}_n = \Sigma_{\mathbb{P}^1}^{n-q}\mathcal{X}$ for $n \geq q$ and identity bonding maps. The $K_n^{\mathbb{P}^1}(S)$-cellular objects are those $\mathcal{E} \in \mathbf{Spt}_{\mathbb{P}^1}(S)$ with stable homotopy sheaves $\pi_{n,q}\mathcal{E}$ zero for $n < m$, $q \in \mathbb{Z}$. Assuming that $S = \operatorname{Spec} k$, k a field, then in this stable model category, the subcategory $\mathcal{SH}(S)_{\leq 0} := \mathbf{Ho}R_{K_0^{\mathbb{P}^1}(S)}\mathbf{Spt}_{\mathbb{P}^1}(S)$ of the homotopy category $\mathcal{SH}(S)$ of $\mathbf{Spt}_{\mathbb{P}^1}(S)$ is half of a t-structure with $\mathcal{SH}(S)_{\geq 0}$ the full subcategory of the $\mathcal{E}$ with $\pi_{n,*}E = 0$ for $n > 0$. The other half is $\mathcal{SH}(S)_{\geq 0}$: the full subcategory of the $\mathcal{E}$ with $\pi_{n,*}E = 0$ for $n < 0$. The heart is Morel's category of "homotopy modules" [20, definition 5.2.4], see [20, theorem 5.2.3, theorem 5.2.6] for detailed statements.

2. *Slice towers.* This is modification of the construction in (1) in $\mathbf{Spc}_\bullet(S)$, using the set K_n^t of objects of the form $\Sigma_{S^1}^a\Sigma_{\mathbb{G}_m}^b\mathcal{X}$, with $b \geq n$. The S^1-stable version uses the set of objects of the form $F_m\Sigma_{S^1}^a\Sigma_{\mathbb{G}_m}^b\mathcal{X}$ with $b \geq n$ and the $\mathbb{P}^1$-stable version uses the set of objects of the form $F_m\Sigma_{S^1}^a\Sigma_{\mathbb{G}_m}^b\mathcal{X}$ with $b - m \geq n$. Varying n, the first two yield the slice tower

$$\ldots \to f_{n+1}^t\mathcal{X} \to f_n^t\mathcal{X} \to \ldots \to f_0^t\mathcal{X} = \mathcal{X}$$

while the $\mathbb{P}^1$-version gives us the doubly infinite tower

$$\ldots \to f_{n+1}^t\mathcal{E} \to f_n^t\mathcal{E} \to \ldots \to \mathcal{X}.$$

Replacing K_n^t with $K_n^{t,\mathcal{S}}$ gives the slice towers

$$\ldots \to f_{n+1}^{t,\mathcal{S}}\mathcal{X} \to f_n^{t,\mathcal{S}}\mathcal{X} \to \ldots \to f_0^{t,\mathcal{S}}\mathcal{X} = \mathcal{X}$$

and

$$\ldots \to f_{n+1}^{t,\mathcal{S}}\mathcal{E} \to f_n^{t,\mathcal{S}}\mathcal{E} \to \ldots \to \mathcal{E}.$$

in $\mathbf{Spc}_\bullet(S)^{\mathcal{S}}$, $\mathbf{Spt}_{S^1}(S)^{\mathcal{S}}$ and $\mathbf{Spt}_{\mathbb{P}^1}(S)^{\mathcal{S}}$. There are similarly defined versions in categories of T-spectra ($T = \mathbb{A}^1/\mathbb{A}^1 \setminus \{0\}$) or the various flavors of symmetric spectra. As above, we refer the reader to [24] and [18, theorem 3.1, remark 3.3] for details.

3. *Betti realizations.* Betti realizations are left derived functors of functors of a left Quillen functor An^*, either on categories of spaces over k, or the various spectrum categories, where An^* is a left Kan extension of the functor sending a smooth k-scheme X to the topological space of its $\mathbb{C}$-points (with respect to a fixed embedding $k \hookrightarrow \mathbb{C}$) or if one prefers $\mathbf{Spc}$ as target category, the singular complex of this space. As a left derived functor of a left Quillen functor, the result Betti realization functor on the appropriate homotopy category is constructed by applying An^* (or some allied construction, in the case of spectra) to a cofibrant resolution for a suitable (cellular) model structure. Thus, we may form a Betti resolution for functor categories by noting that An^* extends by applying it pointwise to a left Quillen functor between functor categories, and by taking the cofibrant resolution in the domain functor category.

Fix an embedding $\sigma : k \to \mathbb{C}$. We use the Betti realization of Panin-Pimenov-Röndigs [22], modified to pass to $\mathbf{Spc}$ instead of locally compact Hausdorff spaces.

This functor arises from the left Quillen functor

$$\mathrm{An}^* : \mathbf{Spc}_\bullet(k) \to \mathbf{Spc}_\bullet$$

which is the Kan extension of the functor sending $X \in \mathbf{Sm}/k$ to the singular complex of X^{an}, this latter being the topological space of $\mathbb{C}$-points of X^σ, with the classical topology.

One extends to $\mathbb{P}^1$-spectra using the fact that $(\mathbb{P}^1)^{\mathrm{an}} \cong S^2$, An^* is symmetric monoidal and using an equivalence of **Spt** and S^2-spectra. Glossing over this latter equivalence, we have the isomorphism (in $\mathcal{SH}$)

$$Re_B(\mathrm{MGL}) \cong \mathrm{MU}$$

There is a similar version from symmetric $\mathbb{P}^1$-spectra to symmetric S^2-spectra, inducing an equivalent functor on the homotopy categories.

Finally, the Betti realization functor extends to a left Quillen functor

$$\mathrm{An}^{\mathcal{S},*} : \mathbf{Spt}_T(S)^{\mathcal{S}} \to \mathbf{Spt}^{\mathcal{S}}_{S^2},$$

with a natural isomorphism $i_s^* \circ \mathrm{An}^{\mathcal{S},*} \cong \mathrm{An}^* \circ i_{s*}$; note that one needs to use a different model structure on $\mathbf{Spt}_T(S)$ than the one we have been using, see [22, §A4] and [18, §5] for details. For other versions of the Betti realization, see [2, definition 2.1], [26] and [29, §4].

We still use the projective model structure on $\mathbf{Spt}_T(S)^{\mathcal{S}}$, but with respect to the Panin-Pimenov-Röndigs model structure on $\mathbf{Spt}_T(S)$.

We let

$$Re_B^{\mathcal{S}} : \mathbf{HoSpt}_T(S)^{\mathcal{S}} \to \mathbf{HoSpt}^{\mathcal{S}}$$

be the left derived functor of $\mathrm{An}^{\mathcal{S},*}$ composed with the equivalence $\mathbf{HoSpt}^{\mathcal{S}}_{S^2} \cong \mathbf{HoSpt}^{\mathcal{S}}$.

In what follows, $\mathcal{M}$ will be either $\mathbf{Spc}_\bullet$ or $\mathbf{Spc}_\bullet(S)$. We will also consider the corresponding spectrum categories, **Spt**, $\mathbf{Spt}_{S^1}(S)$ or $\mathbf{Spt}_{\mathbb{P}^1}(S)$. We will refer to an object in any one of these latter categories as a "spectrum", an object in the underlying model category $\mathcal{M}$ will be referred to as a "space". In the unstable motivic setting, $\mathbf{Spc}_\bullet(S)$, π_n will be the Nisnevich sheaf of homotopy groups (sets for $n = 0$). Similarly, for a Nisnevich sheaf of abelian groups A on $\mathbf{Sm}/S$, we have the associated motivic Eilenberg-MacLane space $K(A, n) \in \mathbf{Spc}_\bullet(S)$, with $\pi_n K(A, n) = A$, $\pi_m K(A, n) = 0$ for $m \neq n$. At least for $n \geq 2$, $K(A, n)$ is $\mathbb{A}^1$-local exactly when A is strictly $\mathbb{A}^1$-invariant.

Lemma 1.7. *Take $\mathcal{E} \in \mathbf{Spt}^{\mathcal{S}}$ such that $i_{s*}\mathcal{E}$ is $n-1$-connected for each $s \in \mathcal{S}$. Then $f_n^{\mathcal{S}}\mathcal{E} \to \mathcal{E}$ is a weak equivalence.*

Proof. Since $i_{s*}f_n^{\mathcal{S}}\mathcal{E} \cong f_n i_{s*}\mathcal{E}$, our assumption on $\mathcal{E}$ implies that $i_{s*}f_n^{\mathcal{S}}\mathcal{E} \to i_{s*}\mathcal{E}$ is a weak equivalence for each $s \in \mathcal{S}$, and thus $f_n^{\mathcal{S}}\mathcal{E} \to \mathcal{E}$ is a weak equivalence. □

Definition 1.8. Recall that a $\mathbb{P}^1$-spectrum $\mathcal{E}$ is said to be *topologically c-connected* if the homotopy sheaf $\pi_{n+m,m}\mathcal{E}$ is zero for all $n \leq c$ and all $m \in \mathbb{Z}$.

Lemma 1.9. *Take $\mathcal{E} \in (\mathbf{Spt}^{\Sigma}_{\mathbb{P}^1}(k)^{\mathcal{S}}$ and suppose that $i_{s*}\mathcal{E}$ is topologically -1 connected for each $s \in \mathcal{S}$. Then there is a canonical morphism*

$$\gamma_n(\mathcal{E}) : Re_B(f_n^{t,\mathcal{S}}\mathcal{E}) \to f_n^{\mathcal{S}} Re_B\mathcal{E}$$

in $\mathbf{HoSpt}^{\mathcal{S}}$.

Proof. We have canonical isomorphisms

$$i_{s*}f_n^{\mathcal{S}}Re_B\mathcal{E} \cong f_nRe_Bi_{s*}\mathcal{E},\ i_{s*}Re_B(f_n^{t,\mathcal{S}}\mathcal{E} \cong Re_B(f_n^ti_{s*}\mathcal{E}.$$

By [18, theorem 5.2], $Re_B(f_n^ti_{s*}\mathcal{E})$ is $n-1$-connected for each n, hence by lemma 1.7, the canonical map

$$f_n^{\mathcal{S}}Re_B(f_n^{t,\mathcal{S}}\mathcal{E}) \to Re_B(f_n^{t,\mathcal{S}}\mathcal{E})$$

is a weak equivalence. Inverting this map in $\mathbf{HoSpt}^{\mathcal{S}}$ and using the commutative diagram

$$\begin{array}{ccc}
f_n^{\mathcal{S}}Re_B(f_n^{t,\mathcal{S}}\mathcal{E}) & \xrightarrow{\rho_n(Re_B(f_n^{t,\mathcal{S}}\mathcal{E}))} & Re_B(f_n^{t,\mathcal{S}}\mathcal{E}) \\
{\scriptstyle f_n^{\mathcal{S}}Re_B(\rho_n^t(\mathcal{E})))}\downarrow & & \downarrow{\scriptstyle Re_B(\rho_n^t(\mathcal{E}))} \\
f_n^{\mathcal{S}}Re_B\mathcal{E} & \xrightarrow[\rho_n(Re_B\mathcal{E})]{} & Re_B\mathcal{E}
\end{array}$$

gives the desired map. □

2. Cosimplicial objects in a model category

We will work in a fairly general setting, letting $\mathcal{M}$ be a pointed closed symmetric monoidal simplicial model category. The reader can keep in mind the example $\mathcal{M} = \mathbf{Spc}_\bullet$, the category of *pointed spaces*, that is, pointed simplicial sets. We will eventually require $\mathcal{M}$ to be a stable model category, such as spectra.

This material, as well as much of the material in the next section, may be found in the beginning portions of [4]

We have the functor category $\mathcal{M}^\Delta$ of cosimplicial objects in $\mathcal{M}$. For $\mathcal{X} : \Delta \to \mathcal{M}$, we often write $\mathcal{X}^n$ for $\mathcal{X}([n])$. We let $\mathcal{X}^{-1}$ denote the maximal augmentation of $\mathcal{X}$, that is, the equalizer

$$\mathcal{X}^{-1} \to \mathcal{X}^0 \underset{d^1}{\overset{d^0}{\rightrightarrows}} \mathcal{X}^1.$$

We give $\mathcal{M}^\Delta$ the Reedy model structure. In fact, a map $f : \mathcal{X} \to \mathcal{Y}$ is a cofibration if and only if $f^n : \mathcal{X}^n \to \mathcal{Y}^n$ is a cofibration for each $n \geq 0$, and the map $f^{-1} : \mathcal{X}^{-1} \to \mathcal{Y}^{-1}$ is an isomorphism.

Remark 2.1. The unit for the monoidal structure on $\mathcal{M}^\Delta$ is the constant cosimplicial object on the unit $\mathbf{1}$ in $\mathcal{M}$; this is not a cofibrant object in $\mathcal{M}^\Delta$.

If A is an object in $\mathcal{M}$, write cA for the constant cosimplicial object. The functor c does not preserve cofibrations, however, if $i : A \to B$ is a cofibration in $\mathcal{M}$ and $p : \mathcal{X} \to \mathcal{Y}$ is a fibration in $\mathcal{M}^\Delta$ with $\mathcal{Y}$ (and hence $\mathcal{X}$) fibrant, then

$$\mathcal{H}om(cB, \mathcal{X}) \to \mathcal{H}om(cA, \mathcal{X}) \times_{\mathcal{H}om(cA,\mathcal{Y})} \mathcal{H}om(cB, \mathcal{Y})$$

is a fibration, and is a trivial fibration if either i or p is a weak equivalence.

We may also consider the full subcategory $\Delta^{\leq n}$ of Δ, with objects $[k]$, $k = 0, \ldots, n$; $\Delta^{\leq n}$ is also a Reedy category with the evident $+$ and $-$ subcategories. We usually give $\mathcal{M}^{\Delta^{\leq n}}$ the Reedy model category structure.

For $T \in \mathcal{M}$, we write Ω_T for the functor $\mathcal{H}om(T, -) : \mathcal{M} \to \mathcal{M}$, right adjoint to Σ_T, $\Sigma_T(X) = X \wedge T$. We also write Ω_T for the functor $\mathcal{H}om(cT, -) : \mathcal{M}^\Delta \to \mathcal{M}^\Delta$, leaving the context to determine the precise meaning. Similarly, we may use the $\mathbf{Spc}_\bullet$-structure to define $\Omega_K := \mathcal{H}om(K, -) : \mathcal{M} \to \mathcal{M}$, right adjoint to Σ_K,

$\Sigma_K(X) = X \wedge K$, and also $\Omega_K := \mathcal{H}om(K, -) : \mathcal{M}^\Delta \to \mathcal{M}^\Delta$. We write Ω and Σ for Ω_{S^1} and Σ_{S^1}.

2.1. **The total complex and associated towers.** We recall the construction of towers associated to cosimplicial objects, recapping the construction of [5] for cosimplicial spaces, generalized to cosimplicial objects in a simplicial model category in [4].

Let $\mathcal{X}$ be a cosimplicial object in $\mathcal{M}$. We have the associated total object $\mathrm{Tot}\mathcal{X} := \mathcal{H}om^\Delta(\Delta[*], \mathcal{X})$ in $\mathcal{M}$; note that $\Delta[*]$ is a cofibrant object in $\mathbf{Spc}^\Delta$, hence the functor $\mathrm{Tot} : \mathcal{M}^\Delta \to \mathcal{M}$ is a right Quillen functor with left adjoint $\mathcal{A} \mapsto \mathcal{A} \times \Delta[*]$. We make the analogous definition in the pointed setting.

For $T \in \mathcal{M}$, $\mathcal{X} \in \mathcal{M}^\Delta$, the adjoint property of $\mathcal{H}om$ gives the isomorphism in $\mathcal{M}$

$$\begin{aligned}\mathcal{H}om^{\mathcal{M}}(T, \mathrm{Tot}\mathcal{X}) &\cong \mathcal{H}om^{\mathcal{M}^\Delta}(T \times \Delta[*], \mathcal{X})\\ &\cong \mathcal{H}om^{\mathcal{M}}(\Delta[*], \mathcal{H}om^{\mathcal{M}}(T, \mathcal{X})) = \mathrm{Tot}(\mathcal{H}om^{\mathcal{M}}(T, \mathcal{X})).\end{aligned}$$

Remark 2.2. Suppose $\mathcal{M}$ is a category of T-spectra in some model category $\mathcal{M}_0$. For $\mathcal{E} : \Delta \to \mathbf{Spt}_T^{\mathcal{M}_0}$ a cosimplicial T-spectrum,

$$\mathcal{E} = (\mathcal{E}_0, \ldots, \mathcal{E}_n, \ldots),$$

with bonding maps $\epsilon_n : \mathcal{E}_n \to \Omega_T\mathcal{E}_{n+1}$, $\mathrm{Tot}\mathcal{E}$ is the spectrum $(\mathrm{Tot}\mathcal{E}_0, \mathrm{Tot}\mathcal{E}_1, \ldots)$ with bonding maps $\mathrm{Tot}\epsilon_n : \mathrm{Tot}\mathcal{E}_n \to \mathrm{Tot}\Omega_T\mathcal{E}_{n+1} \cong \Omega_T\mathrm{Tot}\mathcal{E}_{n+1}$.

Let $i_k : \Delta^{\leq k} \to \Delta$ be the inclusion functor, and let $\mathbf{Spc}^{(k)}$ be the category of preheaves of sets on $\Delta^{\leq k}$. Restricting via i_k, gives the functor $i_{k*} : \mathbf{Spc} \to \mathbf{Spc}^{(k)}$, which admits the left adjoint $i_k^* : \mathbf{Spc}^{(k)} \to \mathbf{Spc}$; the k-skeleton functor sk_k is the composition $i_k^* \circ i_{k*}$, and co-unit $\mathrm{sk}_k \to \mathrm{id}$. We write $A^{(k)}$ for $\mathrm{sk}_k A$. We have the canonical natural transformations $\iota_{m,k} : \mathrm{sk}_k \to \mathrm{sk}_m$, for $0 \leq k \leq m$, with $\iota_{n,m} \circ \iota_{m,k} = \iota_{n,k}$ for $k \leq m \leq n$.

Let $\iota_k : \Delta[*]^{(k)} \to \Delta[*]$ be the k-skeleton of $\Delta[*]$, that is, the cosimplicial simplicial set $n \mapsto \Delta[n]^{(k)}$. For $\mathcal{X}$ a cosimplicial object of $\mathcal{M}$, let $\mathrm{Tot}_{(k)}\mathcal{X} := \mathcal{H}om^{\mathcal{M}}(\Delta[*]^{(k)}, \mathcal{X})$. The sequence of inclusions

$$\emptyset := \Delta^{(-1)} \hookrightarrow \Delta[*]^{(0)} \hookrightarrow \Delta[*]^{(1)} \hookrightarrow \ldots \hookrightarrow \Delta[*]^{(k)} \hookrightarrow \ldots \Delta[*]$$

thus gives the tower in $\mathcal{M}$

$$\mathrm{Tot}\mathcal{X} \to \ldots \to \mathrm{Tot}_{(k)}\mathcal{X} \to \ldots \to \mathrm{Tot}_{(1)}\mathcal{X} \to \mathrm{Tot}_{(0)}\mathcal{X} \to \mathrm{Tot}_{(-1)}\mathcal{X} := pt \tag{2.1}$$

which is a tower of fibrations if $\mathcal{X}$ is fibrant.

We let $\mathrm{Tot}^{(k)}\mathcal{X} \to \mathrm{Tot}\mathcal{X}$ be the homotopy fiber of $\mathcal{X} \to \mathrm{Tot}_{(k-1)}\mathcal{X}$, giving the tower in $\mathcal{M}$

$$\ldots \to \mathrm{Tot}^{(k+1)}\mathcal{X} \to \mathrm{Tot}^{(k)}\mathcal{X} \to \ldots \to \mathrm{Tot}^{(1)}\mathcal{X} \to \mathrm{Tot}^{(0)}\mathcal{X} = \mathrm{Tot}\mathcal{X}. \tag{2.2}$$

For $m \geq k$, let $\mathrm{Tot}_{(m/k)}\mathcal{X}$ be the homotopy fiber of $\mathrm{Tot}_{(m)}\mathcal{X} \to \mathrm{Tot}_{(k)}\mathcal{X}$. We assume we have a chosen fibrant cosimplicial object $\mathcal{Y}$ and chosen isomorphism $\mathcal{X} \cong \Omega^2\mathcal{Y}$ in $\mathbf{Ho}\mathcal{M}^\Delta$. Via this data, we have an isomorphism of the tower (2.1) for $\mathcal{X}$ with Ω^2 applied to the tower (2.1) for $\mathcal{Y}$. For $m \geq k$, let $\mathrm{Tot}^{(k/m)}\mathcal{X} = \Omega\mathrm{hofib}(\mathrm{Tot}^{(m)}\mathcal{Y} \to \mathrm{Tot}^{(k)}\mathcal{Y})$, giving us the homotopy fiber sequence

$$\mathrm{Tot}^{(m)}\mathcal{X} \to \mathrm{Tot}^{(k)}\mathcal{X} \to \mathrm{Tot}^{(m/k)}\mathcal{X}.$$

In case $\mathcal{M}$ is a stable model category, the loops functor Ω is invertible in the homotopy category, so this assumption is automatically satisfied, and we just define $\mathrm{Tot}^{(m/k)}\mathcal{X}$ as the homotopy fiber of $\tilde{\Sigma}\mathrm{Tot}_{(m)}\mathcal{X} \to \tilde{\Sigma}\mathrm{Tot}_{(k)}\mathcal{X}$, where $\tilde{\Sigma}$ mean the functorial fibrant model of the suspension.

To unify the notation, we define $\mathrm{Tot}^{(0)} := \mathrm{Tot} =: \mathrm{Tot}_{(\infty)}$, and $\mathrm{Tot}^{(m/\infty)} := \mathrm{Tot}^{(m)}$. We let $\mathrm{Tot}_{(\infty/k)}$ denote the homotopy fiber of $\mathrm{Tot} \to \mathrm{Tot}_{(k)}$.

We fix a homotopy functor π_* on $\mathcal{M}$. Rather than try to give an axiomatic treatment, we list the examples of interest:

(1) $\mathcal{M} = \mathbf{Spc}_\bullet$, π_* the usual direct sum of the homotopy groups (set for $* = 0$).
(2) $\mathcal{M} = \mathbf{Spc}_\bullet(S)$, π_* the Nisnevich sheaf of $\mathbb{A}^1$-homotopy groups (sets for $* = 0$).
(3) $\mathcal{M} = \mathbf{Spc}_\bullet(S)$, $\pi_n := \oplus_{m\geq 0}\pi_{n+m,m}$.

These all have the property that a map $f : X \to Y$ in $\mathcal{M}$ is a weak equivalence if and only if f induces an isomorphism on π_* for all choices of base-point in X. For the case of a stable model category, we will assume that π_* is the graded truncation functor associated to a non-degenerate t-structure on $\mathbf{Ho}\mathcal{M}$ and again that a map $f : X \to Y$ in $\mathcal{M}$ is a weak equivalence if and only if f induces an isomorphism on π_*. Our main examples of interest are

(1) $T = S^1$, $\mathcal{M} = \mathbf{Spt}_{S^1}(S)$, π_n the stable homotopy sheaf.
(2) $T = S^1$, $\mathcal{M} = \mathbf{Spt}_{S^1}(S)$, $\pi_n := \oplus_{m\geq 0}\pi_{n+m,m}$, $n \in \mathbb{Z}$, with $\pi_{a,b}$ the bi-graded stable homotopy sheaf.
(3) $T = \mathbb{P}^1, \mathbb{A}^1/\mathbb{A}^1 \setminus \{0\}$ or some other convenient model of $\mathbb{P}^1$, $\mathcal{M} = \mathbf{Spt}_T(S)$, $\pi_n := \oplus_{m\in\mathbb{Z}}\pi_{n+m,m}$, $n \in \mathbb{Z}$,

For a cosimplicial abelian group $n \mapsto A^n$, we have the associated complex A^* with differential the alternating sum of the co-face maps. We also have the quasi-isomorphic normalized subcomplex NA^* with $NA^n := \cap_{i=0}^{n-1} \ker s_i$. Consider the following condition on a cosimplicial pointed space $\mathcal{X}$:

(2.3)

(1) There is a fibrant cosimplicial object $\mathcal{Y}$ in $\mathcal{M}^\Delta$ and an isomorphism $\mathcal{X} \cong \Omega^2\mathcal{Y}$ in $\mathbf{Ho}\mathcal{M}^\Delta$.
(2) Given an integer $i \geq 0$, there is an integer N_i such that $(N\pi_j\mathcal{X})^n = 0$ for $n \geq N_i$, $j \leq i + n$.

In the stable case, we have the analog of these conditions for $\mathcal{X} \in \mathcal{M}^\Delta$, namely,

(2.4)

(1) $\mathcal{X}$ is fibrant.
(2) Given an integer i, there is an integer N_i such that $(N\pi_j\mathcal{X})^n = 0$ for $n \geq N_i$, $j \leq i + n$.

For a cosimplicial object $\mathcal{X} \in \mathcal{M}^\Delta$, let $N\mathcal{X}^n$ be the fiber of $s^n : \mathcal{X}^n \to M^n(\mathcal{X})$ (over the base-point).

Lemma 2.3. *There is a natural isomorphism of* $\Omega_{\Delta[n]/\partial\Delta[n]}N\mathcal{X}^n$ *with the fiber of the map* $\mathrm{Tot}_{(n)}\mathcal{X} \to \mathrm{Tot}_{(n-1)}\mathcal{X}$. *If* $\mathcal{X}$ *if fibrant, the induced map* $\Omega_{\Delta[n]/\partial\Delta[n]}N\mathcal{X}^n \to \mathrm{Tot}_{(n/n-1)}\mathcal{X}$ *gives rise to an isomorphism*

$$\Omega^n N\mathcal{X}^n \cong \mathrm{Tot}_{(n/n-1)}\mathcal{X}. \tag{2.5}$$

in $\mathbf{Ho}\mathcal{M}$. *In particular, we have an isomorphism* $\pi_j N\mathcal{X}^n \cong (N\pi_j\mathcal{X})^n$.

Proof. The fiber of $\mathrm{Tot}_{(n)}\mathcal{X} \to \mathrm{Tot}_{(n-1)}\mathcal{X}$ is equal to $\mathcal{H}om^{\mathcal{M}}(\Delta[*]^{(n)}/\Delta[*]^{(n-1)}, \mathcal{X})$. This in turn is isomorphic to the equalizer

$$\mathcal{H}om^{\mathcal{M}}(\Delta[*]^{(n)}/\Delta[*]^{(n-1)}, \mathcal{X}) \to \mathcal{H}om(\Delta[n]/\partial\Delta[n], \mathcal{X}^n) \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \prod_{i=0}^{n-1} \mathcal{X}^{n-1}$$

where $\alpha(f) = \prod_i s_i \circ f$ and β is the map to the base-point. This gives the asserted identification of $\mathcal{H}om^{\mathcal{M}}(\Delta[*]^{(n)}/\Delta[*]^{(n-1)}, \mathcal{X})$ with the fiber of $\mathrm{Tot}_{(n)}\mathcal{X} \to \mathrm{Tot}_{(n-1)}\mathcal{X}$.

As $\Delta[*]^{(n-1)}, \to \Delta[*]^{(n)}$ is a cofibration, the map $\mathrm{Tot}_{(n)}\mathcal{X} \to \mathrm{Tot}_{(n-1)}\mathcal{X}$ is a fibration, hence the induced map

$$\Omega_{\Delta[n]/\partial\Delta[n]} N\mathcal{X}^n \to \mathrm{Tot}_{(n/n-1)}\mathcal{X}$$

is a weak equivalence. If $\mathcal{X}$ is fibrant, so is $N\mathcal{X}^n$, hence a weak equivalence $(S^1)^{\wedge n} \to \Delta[n]/\partial\Delta[n]$ induces a weak equivalence $\Omega_{\Delta[n]/\partial\Delta[n]} N\mathcal{X}^n \to \Omega^n N\mathcal{X}^n$. □

In other words, under the assumption (2.3)(1), the condition (2.3)(2) is equivalent to

$$\pi_j \mathrm{Tot}_{(n/n-1)}\mathcal{X} = 0 \text{ for } j \le i, n \ge N_i.$$

2.2. **Spectral sequences and convergence.** Suppose that $\mathcal{X} \in \mathcal{M}_\bullet^\Delta$ is fibrant. The tower of fibrations (2.1) gives the spectral sequence

$$(2.6) \qquad {}_*E_1^{p,q}(\mathcal{X}) = \pi_{-p-q}\mathrm{Tot}_{(p/p-1)}\mathcal{X} \Longrightarrow \pi_{-p-q}\mathrm{Tot}_{(A/B)}\mathcal{X}; B < p \le A,$$

for $-1 \le B \le A \le \infty$. Note that we use a different indexing convention than that of [5].

Under the assumption (2.3)(1) or (2.4)(1), we have canonical isomorphisms in the respective homotopy categories $\mathrm{Tot}^{(k/m)}\mathcal{X} \cong \mathrm{Tot}_{(m-1/k-1)}\mathcal{X}$. In addition, the spectral sequence (2.6) is isomorphic to the spectral sequences of the tower (2.2):

$$(2.7) \qquad E_1^{p,q}(\mathcal{X}) = \pi_{-p-q}\mathrm{Tot}^{(p/p+1)}\mathcal{X} \Longrightarrow \pi_{-p-q}^{(A/B)}\mathrm{Tot}\mathcal{X}; A \le p < B,$$

for $0 \le A < B \le \infty$. Furthermore, using (2.5), the E_1-terms are

$$E_1^{p,q}(\mathcal{X}) = N\pi_{-q}\mathcal{X}^p.$$

Lemma 2.4. *1. If $\mathcal{X} \in \mathcal{M}_\bullet^\Delta$ satisfies* (2.3)*(1), (resp.* (2.4)*(1) if $\mathcal{M}$ is a stable model category), then the spectral sequence* (2.6) *is strongly convergent if $A < \infty$ and the spectral sequences* (2.7)*, is strongly convergent if $B < \infty$.*
2. Suppose $\mathcal{X} \in \mathcal{M}_\bullet^\Delta$ satisfies (2.3) *(resp.* (2.4) *if $\mathcal{M}$ is a stable model category). Then the spectral sequences* (2.6) *(for $A = \infty$) and* (2.7) *(for $B = \infty$) are strongly convergent.*

Proof. It suffices to give the proof in the unstable case. (1) follows easily, as in all cases, the associated tower is finite.

For (2), since $\mathcal{X} \cong \Omega^2\mathcal{Y}$, there are no low dimensional subtleties, and all the statements we will be using from [5] make sense and are valid for π_1 and π_0.

We first show that for each (p, q), there is an r_0 such that $E^r_{p,q} = E^{r+1}_{p,q}$ for all $r \ge r_0$. Indeed, $E^1_{p,q} = 0$ for $p > 0$, and if $p + q = i$, then $E^1_{p-r,q+r-1} = 0$ for $r - p \ge N_{i-1}$. Thus we have $E^r_{p,q} = E^{r+1}_{p,q}$ for $r \ge r_0 := \max(p + N_{i-1}, -p, 0) + 1$.

Thus, the terms $\{E^r\}$ are "Mittag-Leffler in dimension i" for all $i \ge 0$ [5, IX, §5, pg. 264] and hence, by [5, IX, proposition 5.7] the spectral sequence converges completely to $\pi_*\mathrm{Tot}\mathcal{X}$. Fix an integer $n \geq 0$. Then $E^\infty_{p,q} = 0$ for $p+q = n$, $p > 0$ or $p \leq N_n$, and so the filtration of $\pi_n\mathrm{Tot}\mathcal{X}$ induced by the spectral sequence is finite, giving the strong convergence. □

Lemma 2.5. *Suppose $\mathcal{X}^n$ is $c-1$-connected for all n. Then for all $r \geq 0$, $m \geq 0$ with $c - m \leq r \leq \infty$:*
1. The map $\mathrm{Tot}^{(c-m/r)}\mathcal{X} \to \mathrm{Tot}^{(0/r)}\mathcal{X}$ induces a surjection

$$\pi_m\mathrm{Tot}^{(c-m/r)}\mathcal{X} \to \pi_m\mathrm{Tot}^{(0/r)}\mathcal{X}.$$

2. The map $\mathrm{Tot}^{(c-m-1/r)}\mathcal{X} \to \mathrm{Tot}^{(0/r)}\mathcal{X}$ induces a isomorphism

$$\pi_m\mathrm{Tot}^{(c-m-1/r)}\mathcal{X} \to \pi_m\mathrm{Tot}^{(0/r)}\mathcal{X}.$$

Proof. We have the strongly convergent spectral sequence

$$E_1^{p,q}(\mathcal{X}) = \pi_{-p-q}\mathrm{Tot}^{(p/p+1)}\mathcal{X} \Longrightarrow \pi_{-p-q}\mathrm{Tot}^{(0/b)}\mathcal{X}; \quad 0 \leq p \leq b-1.$$

By (2.5), $E_1^{p,q} = \pi_{-q}N\mathcal{X}^p \subset \pi_{-q}\mathcal{X}^p$, so $E_1^{p,q} = 0$ for $-q < c$. Since $E_1^{p,q} = 0$ for $p > b-1$ this implies that $E_1^{p,q} = 0$ for $-p-q \leq c-b$ and thus $\pi_s\mathrm{Tot}^{(0/b)}\mathcal{X} = 0$ for $s \leq c-b$. Using the homotopy fiber sequence

$$\mathrm{Tot}^{(c-m/r)}\mathcal{X} \to \mathrm{Tot}^{(0/r)}\mathcal{X} \to \mathrm{Tot}^{(0/c-m)}\mathcal{X}$$

proves (1). Similarly, $\pi_s\mathrm{Tot}^{(0/c-m-1)}\mathcal{X} = 0$ for $s \leq m+1$, and (2) follows by a similar argument. □

3. Cosimplices and cubes

The functors $\mathrm{Tot}_{(n)}$ are complicated by the mixture of codegeneracies and coface maps in Δ; in this section we discuss the reduction of $\mathrm{Tot}_{(n)}$ to a homotopy limit over an associated direct category, namely, a punctured $n+1$-cube.

As above, we have the full subcategory $\Delta^{\leq n}$ of Δ with objects $[\ell]$, $0 \leq \ell \leq n$, and inclusion $\iota_n : \Delta^{\leq n} \to \Delta$. We have the restriction functor $\iota_{n*} : \mathbf{Spc}_\bullet^\Delta \to \mathbf{Spc}_\bullet^{\Delta^{\leq n}}$ with left adjoint ι_n^*. We have as well the representable simplicial sets $\Delta[n] := \mathrm{Hom}_\Delta(-,[n])$ and the cosimplicial space $\Delta[*]$, $[n] \mapsto \Delta[n]$.

Throughout this section we fix a pointed simplicial model category $\mathcal{M}$; we will eventually restrict to the case of a stable model category, but the intial portions of this section do not require this.

Lemma 3.1. *Take $\mathcal{X}$ in $\mathcal{M}^\Delta$. There is a natural isomorphism*

$$\mathrm{Tot}_{(n)}\mathcal{X} \cong \mathcal{H}om(\iota_{n*}\Delta[*], \iota_{n*}\mathcal{X});$$

if $\mathcal{X}$ is fibrant, there is a natural weak equivalence

$$\operatorname*{holim}_{\Delta^{\leq n}} \iota_{n*}\mathcal{X} \to \mathrm{Tot}_{(n)}\mathcal{X}.$$

Proof. We note that we have a canonical isomorphism of cosimplicial spaces

$$\mathrm{sk}_n\Delta[*] \cong \iota_n^*\iota_{n*}\Delta[*].$$

Indeed, $(\mathrm{sk}_n\Delta[m])([k])$ is the colimit over $[k] \to [\ell] \in ([k]/\Delta^{\leq n})^{\mathrm{op}}$ of $\mathrm{Hom}_\Delta([\ell],[m])$, while $(\iota_n^*\iota_{n*}\Delta[*])[m]([k])$ is the colimit over $[\ell] \to [m] \in \Delta^{\leq n}/[m]$ of $\mathrm{Hom}_\Delta([k],[\ell])$. Both colimits are equal to the subset of $\mathrm{Hom}_\Delta([k],[m])$ consisting of maps that admit a factorization $[k] \to [\ell] \to [m]$ with $\ell \leq n$.

This gives us the isomorphism in $\mathcal{M}$

$$\mathrm{Tot}_n\mathcal{X} := \mathcal{H}om(\mathrm{sk}_n\Delta[*], \mathcal{X}) \cong \mathcal{H}om(\iota_{n*}\Delta[*], \iota_{n*}\mathcal{X}).$$

For $0 \leq k \leq n$, the nerve of $\Delta^{\leq n}/[k]$ is the barycentric subdivision of $\Delta[k]$ and sending the non-degenerate k-simplex of $\Delta[k]$ to the k-simplex

$$\begin{array}{ccccccc} \{0\} & \hookrightarrow & \{0,1\} & \hookrightarrow & \ldots & \hookrightarrow & \{0,\ldots,k\} \\ & \searrow & & \searrow & & \swarrow & \\ & & & \{0,\ldots,k\} & & & \end{array}$$

in $\Delta^{\leq n}/[k]$ gives an acyclic cofibration $\alpha : \iota_{n*}\Delta[*] \to [[k] \mapsto \mathcal{N}\Delta^{\leq n}/[k]]$ in $\mathbf{Spc}_\bullet^{\Delta^{\leq n}}$. As $\mathcal{X}$ is fibrant in $\mathcal{M}^{\Delta}$, it follows that $\iota_{n*}\mathcal{X}$ is fibrant in $\mathcal{M}^{\Delta^{\leq n}}$, so α induces the desired weak equivalence

$$\alpha^* : \operatorname*{holim}_{\Delta^{\leq n}} \iota_{n*}\mathcal{X} \to \mathrm{Tot}_{(n)}\mathcal{X}.$$

□

Let $\square^n$ be the category associated to the set of subsets of $\{1,\ldots,n\}$, with morphisms being inclusions of subsets, and let $\square^n_0$ the full subcategory of non-empty subsets. Letting $i_{I,J} : J \to I$ denote the morphism associated to an inclusion $I \subset J$, the *split n-cube* $\square^n_s$ is formed by adjoining morphisms $p_{J,I} : I \to J$ for each inclusion $J \subset I$, with $p_{K,J} \circ p_{J,I} = p_{K,I}$ for $K \subset J \subset I$ and with $p_{J,I\cup J} \circ i_{I\cup J,I} = i_{J,I\cap J} \circ p_{I\cap J,I}$ for $I, J \subset \{1,\ldots,n\}$.

$\square^n$ and $\square^{n+1}_0$ are both direct categories and $\square^n_s$ is a Reedy category with $(\square^n_s)_+ = \square^n$ and $(\square^n_s)_-$ the subcategory with morphisms $p_{J,I}$. For a model category $\mathcal{M}$ and for $\mathcal{C} = \square^{n+1}_0, \square^n, \square^n_s$, we give $\mathcal{M}^{\mathcal{C}}$ the Reedy model structure; as $\square^{n+1}_0$ and $\square^n$ are direct categories, this agrees with the projective model structure in these cases.

Give $\{1,\ldots,n\}$ the *opposite* of the standard order. The maps $i_{I,J}$ are clearly order-preserving, so sending I to the ordered set $[|I|-1]$ by the unique order-preserving bijection defines a functor

$$\varphi^{n+1}_0 : \square^{n+1}_0 \to \Delta^{\leq n}$$

Similarly, sending I to $[|I|]$ by the unique order-preserving injection which avoids 0 defines a functor

$$\psi^n : \square^n \to \Delta^{\leq n}$$

We may extend ψ^n to

$$\psi^n_s : \square^n_s \to \Delta^{\leq n}$$

as follows: given an inclusion $J = \{j_1 < \ldots < j_r\} \subset I = \{i_1 < \ldots < i_s\} \subset \{1,\ldots,n\}$, define $\psi^n_s(p_{J,I}) : [|I|] \to [|J|]$ by sending j to ℓ if $j_\ell \leq i_j < j_{\ell+1}$ and to 0 if $i_j < j_1$. These are all functors of Reedy categories.

Take an integer $n \geq 1$. We decompose $\square^{n+1}_0$ into three pieces, by defining $\square^{n-}_0$ to be the full subcategory with objects I, $n \notin I$, $\square^{n+}_0$ the full subcategory with objects I, $n \in I$, $I \neq \{n\}$ and $pt_n := \{n\}$ (with identity morphism). We have the isomorphisms $j^-_n : \square^n_0 \to \square^{n-}_0$, $j^+_n : \square^n_0 \to \square^{n+}_0$: let $j_n : \{1,\ldots,n\} \to \{1,\ldots,n+1\}$ be the inclusion $j_n(i) = i$ for $1 \leq i < n$, $j_n(n) = n+1$. j^-_n is just the functor induced by j_n, and $j^+_n(I) = j_n(I) \cup \{n\}$. Let $i^+_n : \square^n_0 \to \square^{n+1}_0$ and $i^-_n : \square^n_0 \to \square^{n+1}_0$ be the inclusions induced by j^+_n and j^-_n.

The inclusions $I \subset I \cup \{n\}$ defines a natural transformation $\alpha_n : i_n^- \to i_n^+$, and the inclusions $\{n\} \subset I$, $I \in \square_0^{n+}$, define the morphisms $\beta_I : \{n\} \to i_n^+(I)$. For each $\mathcal{X} \in \mathcal{M}^{\square_0^{n+1}}$, we thus have the diagram in $\mathcal{M}$

$$\begin{array}{ccc} \mathrm{holim}_{\square_0^n} i_{n*}^- \mathcal{X} & \xrightarrow{\alpha_n} & \mathrm{holim}_{\square_0^n} i_{n*}^+ \mathcal{X} \\ & & \uparrow{\scriptstyle \beta_*} \\ & & \mathcal{X}(\{n\}) \end{array}$$

This defines a functor

$$\mathrm{holim}_{n+1}^{+,-} : \mathcal{M}^{\square_0^{n+1}} \to \mathcal{M}^{\square_0^2}$$

and we have a natural isomorphism in $\mathcal{M}$

$$\operatorname*{holim}_{\square_0^{n+1}} \mathcal{X} \cong \operatorname*{holim}_{\square_0^2} \mathrm{holim}_{n+1}^{+,-}(\mathcal{X}).$$

In case $\mathcal{X}(\{n\}) = pt$, we have the natural isomorphisms

$$\operatorname*{holim}_{\square_0^{n+1}} \mathcal{X} \cong \operatorname*{holim}_{\square_0^2} \mathrm{holim}_{n+1}^{+,-}(\mathcal{X}) \cong \mathrm{hofib}(\alpha_n : \operatorname*{holim}_{\square_0^n} i_{n*}^- \mathcal{X} \to \operatorname*{holim}_{\square_0^n} i_{n*}^+ \mathcal{X}) \tag{3.1}$$

Let $\rho_n^+ : \square^n \to \square_0^{n+1}$ be the functor $\rho_n^+(I) := I \cup \{n+1\}$, giving the restriction functor

$$\rho_{n*}^+ : \mathcal{M}^{\square_0^{n+1}} \to \mathcal{M}^{\square^n}$$

and the left adjoint $\rho_n^{+*} : \mathcal{M}^{\square^n} \to \mathcal{M}^{\square_0^{n+1}}$. Explicitly, for $\mathcal{X} \in \mathcal{M}^{\square^n}$, $\rho_n^{+*}\mathcal{X} \in \mathcal{M}^{\square_0^{n+1}}$ is given by $\rho_n^{+*}\mathcal{X}(\rho_n^+(I)) = \mathcal{X}(I)$ and $\rho_n^{+*}\mathcal{X}(J) = pt$ for $J \subset \{1, \ldots, n\}$.

The inclusion $\rho_n^- : \{1, \ldots, n\} \to \{1, \ldots, n+1\}$ induces the restriction functor

$$\rho_{n*}^- : \mathcal{M}^{\square_0^{n+1}} \to \mathcal{M}^{\square_0^n}$$

with right adjoint $\rho_n^{-!} : \mathcal{M}^{\square_0^n} \to \mathcal{M}^{\square_0^{n+1}}$ given by $\rho_n^{-!}\mathcal{X}(\rho_n^-(I)) = \mathcal{X}(I)$ and $\rho_n^{-!}\mathcal{X}(J) = pt$ if $n+1 \in J$.

Finally, the inclusion functor $pt_{n+1} : pt \to \square_0^{n+1}$, $pt_{n+1}(pt) = \{1, \ldots, n+1\}$ induces the restriction functor

$$pt_{n+1*} : \mathcal{M}^{\square_0^{n+1}} \to \mathcal{M}$$

with left adjoint $pt_{n+1}^* : \mathcal{M} \to \mathcal{M}^{\square_0^{n+1}}$ given explicitly by

$$pt_{n+1}^*(X)(I) = \begin{cases} X & \text{for } I = \{1, \ldots, n+1\} \\ pt & \text{else.} \end{cases}$$

Clearly, pt_{n+1} factors through ρ_n^+, giving us the commutative diagram of natural transformations

$$\begin{array}{ccc} pt_{n+1}^* pt_{n+1*} & \longrightarrow & \rho_n^{+*}\rho_{n*}^+ \\ & \searrow & \downarrow \\ & & \mathrm{id} \end{array}$$

We have the canonical isomorphisms

$$\operatorname*{holim}_{\square_0^n} \rho_{n*}^- \mathcal{X} \cong \operatorname*{holim}_{\square_0^{n+1}} \rho_n^{-!} \rho_{n*}^- \mathcal{X}$$

and a morphism

$$\beta_n : \operatorname*{holim}_{\square_0^{n+1}} pt_{n+1}^*(X) \to \Omega^n X.$$

that is a weak equivalence if X is fibrant. Indeed, $\mathcal{N}(\square_0^{n+1}/\{1,\ldots,n+1\})$ has geometric realization homeomorphic to $[0,1]^n$, where the boundary of $[0,1]^n$ is the geometric realization of $\varinjlim_I |\mathcal{N}(\square_0^{n+1}/I)|$, the colimit being over I with $\emptyset \neq I \subsetneq \{1,\ldots,n+1\}$. Since

$$\operatorname*{holim}_{\square_0^{n+1}} pt_{n+1}^*(X) = \mathcal{H}om_{\mathcal{M}}(\mathcal{N}(\square_0^{n+1}/\{1,\ldots,n+1\})/\varinjlim_I |\mathcal{N}(\square_0^{n+1}/I)|, X)$$

a choice of a non-degenerate n-simplex in $\mathcal{N}(\square_0^{n+1}/\{1,\ldots,n+1\})$ defines a weak equivalence in $\mathbf{Spc}_\bullet$

$$\pi : \mathcal{N}(\square_0^{n+1}/\{1,\ldots,n+1\})/\varinjlim_I |\mathcal{N}(\square_0^{n+1}/I)| \to \Delta[n]/\partial\Delta[n];$$

this defines the morphism

$$\pi^* : \mathcal{H}om_{\mathcal{M}}(\mathcal{N}(\square_0^{n+1}/\{1,\ldots,n+1\})/\varinjlim_I |\mathcal{N}(\square_0^{n+1}/I)|, X) \to \Omega^n X,$$

which is a weak equivalence if X is fibrant. In addition, the morphisms π and π^* are independent up to homotopy of the choice of non-degenerate n-simplex.

For a fibrant $\mathcal{X} \in \mathcal{M}^{\square_0^{n+1}}$ we have the fiber sequence

$$\rho_n^{+*}\rho_{n*}^+\mathcal{X} \to \mathcal{X} \to \rho_n^{-!}\rho_{n*}^-\mathcal{X}$$

inducing the homotopy fiber sequence

$$\operatorname*{holim}_{\square_0^{n+1}} \rho_n^{+*}\rho_{n*}^+\mathcal{X} \to \operatorname*{holim}_{\square_0^{n+1}} \mathcal{X} \to \operatorname*{holim}_{\square_0^{n}} \rho_{n*}^-\mathcal{X} \tag{3.2}$$

via the evident isomorphism $\operatorname{holim}_{\square_0^n} \rho_{n*}^-\mathcal{X} \cong \operatorname{holim}_{\square_0^{n+1}} \rho_n^{-!}\rho_{n*}^-\mathcal{X}$. All this extends to spectra in the evident manner.

Let $\mathcal{X}$ be in $\mathcal{M}^{\square_s^n}$. Define $N\mathcal{X}^n$ to be the fiber (over the base-point) of the map

$$\mathcal{X}(\{1,\ldots,n\}) \xrightarrow{(\ldots,p_{I\subset\{1,\ldots,n\}}\ldots)} \prod_{I\subset\{1,\ldots,n\},|I|=n-1} \mathcal{X}(I)$$

Let $\mathcal{X}_s$ denote the restriction of $\mathcal{X}$ to $\mathcal{M}^{\square^n}$; we sometimes write $\mathcal{X}$ for $\mathcal{X}_s$ when the context makes the meaning clear.

The inclusion $N\mathcal{X}^n \subset \mathcal{X}(\{1,\ldots,n\})$ gives us the morphism

$$\xi_n : \operatorname*{holim}_{\square_0^{n+1}} pt_{n+1}^* N\mathcal{X}^n \to \operatorname*{holim}_{\square_0^{n+1}} \rho_n^{+*}\mathcal{X}_s$$

defined as the composition

$$\begin{aligned}\operatorname*{holim}_{\square_0^{n+1}} pt_{n+1}^* N\mathcal{X}^n &\to \operatorname*{holim}_{\square_0^{n+1}} pt_{n+1}^*\mathcal{X}(\{1,\ldots,n\}) \\ &= \operatorname*{holim}_{\square_0^{n+1}} \rho_n^{+*} pt_n^* pt_{n*}\mathcal{X}_s \to \operatorname*{holim}_{\square_0^{n+1}} \rho_n^{+*}\mathcal{X}_s,\end{aligned}$$

where $pt_n : pt \to \square^n$ is the inclusion with value $\{1,\ldots,n\}$.

Lemma 3.2. *Suppose $\mathcal{M}$ is a stable model category and let $\mathcal{E}$ be in $\mathcal{M}^{\square_s^n}$. Then the map*

$$\tilde{\xi}_n : \operatorname*{holim}_{\square_0^{n+1}} pt_{n+1}^* N\mathcal{E}^n \to \operatorname*{holim}_{\square_0^{n+1}} \rho_n^{+*}\mathcal{E}_s$$

is a weak equivalence in $\mathcal{M}$ and induces an isomorphism

$$\xi_n : \Omega^n N\mathcal{E}^n \to \operatorname*{holim}_{\square_0^{n+1}} \rho_n^{+*}\mathcal{E}_s$$

in $\mathbf{Ho}\mathcal{M}$.

Proof. We may assume that $\mathcal{E}$ is a fibrant. We proceed by induction on n. In case $n = 0$, the map $\tilde{\xi}_0$ is just the identity map on $\mathcal{E}(\emptyset)$. In the general case, let $\tau^+_{n-1}, \tau^-_{n-1} : \square^{n-1}_s \to \square^n_s$ be the "top" and "bottom" inclusion functors, given by the same formulas as ρ^+_{n-1} and ρ^-_{n-1}. We have the natural transformations $\beta_{n-1} : \tau^-_{n-1} \to \tau^+_{n-1}$ and $\gamma_{n-1} : \tau^+_{n-1} \to \tau^-_{n-1}$, defined by $\beta_{n-1}(I) := i_{\tau^-_{n-1}(I)\subset\tau^+_{n-1}(I)}$ and $\gamma_n(I) := p_{\tau^-_{n-1}(I)\subset\tau^+_{n-1}(I)}$.

We have identities

$$\begin{aligned} &i^-_{n*}\rho^*_n\mathcal{E} = \rho^{+*}_{n-1}\tau^-_{n-1*}\mathcal{E}\\ &i^+_{n*}\rho^*_n\mathcal{E} = \rho^{+*}_{n-1}\tau^+_{n-1*}\mathcal{E}\\ &(\tilde{\rho}^*_n\mathcal{E})(\{1,\ldots,n+1\}) = pt. \end{aligned}$$

and the commutative diagram

$$\begin{array}{ccc} \rho^{+*}_{n-1}\tau^-_{n-1*}\mathcal{E} & \xrightarrow{\rho^{+*}_{n-1}(\beta_{n-1})} & \rho^{+*}_{n-1}\tau^+_{n-1*}\mathcal{E} \\ \| & & \| \\ i^-_{n*}\tilde{\rho}^*_n\mathcal{E} & \xrightarrow[\alpha_n]{} & i^+_{n*}\tilde{\rho}^*_n\mathcal{E}. \end{array}$$

As $\rho^{+*}_n\mathcal{E}(\{n\}) = pt$, this gives us, via (3.1), the isomorphism

$$\mathrm{hofib}(\rho^{+*}_{n-1}(\beta_{n-1})) \cong \operatorname*{holim}_{\square^{n+1}_0} \tilde{\rho}^*_n\mathcal{E}.$$

Write $\tilde{\Omega}^n E$ for $\mathrm{holim}_{\square^{n+1}_0} pt^*_{n+1}E$. Assuming the result for $n-1$, we have the commutative diagram

$$\begin{array}{ccccc} \mathrm{hofib}(\tilde{\Omega}^{n-1}N\beta_{n-1}) & \longrightarrow & \tilde{\Omega}^{n-1}N(\tau^-_{n-1*}\mathcal{E})^{n-1} & \xrightarrow{\tilde{\Omega}^{n-1}N\beta_{n-1}} & \tilde{\Omega}^{n-1}N(\tau^+_{n-1*}\mathcal{E})^{n-1} \\ \downarrow & & \downarrow{\scriptstyle \tilde{\xi}_{n-1}} & & \downarrow{\scriptstyle \tilde{\xi}_{n-1}} \\ \operatorname*{holim}_{\square^{n+1}_0} \rho^{+*}_n\mathcal{E} & \longrightarrow & \operatorname*{holim}_{\square^n_0} \rho^{+*}_{n-1}\tau^-_{n-1*}\mathcal{E} & \xrightarrow[\rho^{+*}_{n-1}(\beta_{n-1})]{} & \operatorname*{holim}_{\square^n_0} \rho^{+*}_{n-1}\tau^+_{n-1*}\mathcal{E}, \end{array}$$

with rows homotopy fiber sequences, inducing a weak equivalence

$$\mathrm{hofib}(\tilde{\Omega}^{n-1}N\alpha_n) \to \operatorname*{holim}_{\square^{n+1}_0} \rho^{+*}_n\mathcal{E}.$$

Thus, we just need to see that the inclusion $i : N\mathcal{E}^n \to N(\tau^+_{n-1*}\mathcal{E})^{n-1}$ induces a weak equivalence $\tilde{\Omega}^n N\mathcal{E}^n \to \mathrm{hofib}(\tilde{\Omega}^{n-1}N\alpha_n)$, making the relevant diagram commute.

For this, we note that the map $N\beta_{n-1} : N(\tau^-_{n-1*}\mathcal{E})^{n-1} \to N(\tau^+_{n-1*}\mathcal{E})^{n-1}$ is the restriction of $i_{\{1,\ldots,n-1\}\subset\{1,\ldots,n\}}$, hence is split by $p := p_{\{1,\ldots,n-1\}\subset\{1,\ldots,n\}}$. Also, $N\mathcal{E}^n$ is the fiber over the base-point of the map $p : N(\tau^+_{n-1*}\mathcal{E})^{n-1} \to N(\tau^-_{n-1*}\mathcal{E})^{n-1}$; since $\mathcal{E}$ is fibrant, the sequence

$$N\mathcal{E}^n \xrightarrow{i} N(\tau^+_{n-1*}\mathcal{E})^{n-1} \xrightarrow{p} N(\tau^-_{n-1*}\mathcal{E})^{n-1}$$

is a homotopy fiber sequence, with splitting given by $N\beta_{n-1}$. This gives rise to the isomorphism in the triangulated category $\mathbf{Ho}\mathcal{M}$

$$N(\tau^-_{n-1*}\mathcal{E})^{n-1} \oplus N\mathcal{E}^n \xrightarrow{(N\beta_{n-1}, i)} N(\tau^+_{n-1*}\mathcal{E})^{n-1}$$

and thereby the desired isomorphism $\mathrm{hofib}(\tilde{\Omega}^{n-1}N\alpha_n) \cong \Omega^n N\mathcal{E}^n$. By construction, the composition $\tilde{\Omega}^n N\mathcal{E}^n \to \Omega^n N\mathcal{E}^n \to \mathrm{holim}_{\square_0^{n+1}} \rho_n^{+*}\mathcal{E}$ is equal to $\tilde{\xi}_n$ (in $\mathbf{Ho}\mathcal{M}$), completing the proof. □

Proposition 3.3. *Suppose $\mathcal{M}$ is a stable model category and take $\mathcal{E}$ in $\mathcal{M}^\Delta$. The map*

$$\varphi_0^{n+1*} : \operatorname*{holim}_{\Delta^{\leq n}} \iota_{n*}\mathcal{E} \to \operatorname*{holim}_{\square_0^{n+1}} \varphi_{0*}^{n+1}\iota_{n*}\mathcal{E}$$

induced by the functor $\varphi_0^{n+1} : \square_0^{n+1} \to \Delta^{\leq n}$ is a weak equivalence.

Proof. As replacing $\mathcal{E}$ with a fibrant model induces a weak equivalence on the respective homotopy limits, we may assume that $\mathcal{E}$ is fibrant.

We note that we have natural isomorphisms

$$\operatorname*{holim}_{\square_0^{n+1}} \rho_n^{-*}\varphi_{0*}^{n+1}\iota_{n*}\mathcal{E} \cong \operatorname*{holim}_{\square_0^{n}} \varphi_{0*}^{n}\iota_{n-1*}\mathcal{E}$$
$$\psi_*^n\mathcal{E} \cong \rho_{n*}^+\varphi_{0*}^{n+1}\iota_{n*}\mathcal{E}.$$

Applying the homotopy fiber sequence arising from lemma 2.3, the isomorphism of lemma 3.1, and the homotopy fiber sequence (3.2) gives us the commutative diagram (in $\mathbf{HoSpt}_T^{\mathcal{M}}$), with rows homotopy fiber sequences

$$\begin{array}{ccccc}
\Omega^n N\mathcal{E}^n & \longrightarrow & \mathrm{holim}_{\Delta^{\leq n}} \iota_{n*}\mathcal{E} & \longrightarrow & \mathrm{holim}_{\Delta^{\leq n-1}} \iota_{n-1*}\mathcal{E} \\
\downarrow{\scriptstyle \xi_n} & & \downarrow{\scriptstyle \varphi_0^{n+1*}} & & \downarrow{\scriptstyle \varphi_0^{n*}} \\
\mathrm{holim}_{\square_0^{n+1}} \rho_n^{+*}\psi_*^n\mathcal{E} & \longrightarrow & \mathrm{holim}_{\square_0^{n+1}} \varphi_{0*}^{n+1}\iota_{n*}\mathcal{E} & \longrightarrow & \mathrm{holim}_{\square_0^{n}} \varphi_{0*}^{n}\iota_{n-1*}\mathcal{E}
\end{array}$$

The map $\xi_n : \Omega^n N\mathcal{E}^n \to \mathrm{holim}_{\square_0^{n+1}} \rho_n^{+*}\psi^n\mathcal{E}$ is a weak equivalence by lemma 3.2. the result then follows by induction. □

Example 3.4. We let $\mathcal{M}_0$ be one of the model categories discussed below in §4 (4.1) and apply the above results to the stable model category $\mathcal{M}$ of symmetric T-spectra $\mathbf{Spt}_T^{\Sigma,\mathcal{M}_0}$, with $T = S^1$ or some model of $\mathbb{P}^1$. Let $\mathcal{E}$ be a commutative monoid in $\mathbf{Spt}_T^{\Sigma,\mathcal{M}_0}$. Form the cosimplicial (symmetric) spectrum $n \mapsto \mathcal{E}^{\wedge n+1}$, with coface maps given by the appropriate multiplication maps and codegeneracies by unit maps. Letting $\tilde{\mathcal{E}}^{\wedge *+1}$ be a fibrant model, we have the isomorphisms in $\mathbf{HoSpt}_T^{\Sigma,\mathcal{M}_0} \cong \mathbf{HoSpt}_T^{\mathcal{M}_0}$

$$\mathrm{Tot}_n\tilde{\mathcal{E}}^{\wedge *+1} \cong \operatorname*{holim}_{\Delta^{\leq n}} \iota_{n*}\mathcal{E}^{\wedge *+1} \cong \operatorname*{holim}_{\square_0^{n+1}} \varphi_0^{n+1}\mathcal{E}^{\wedge *+1}.$$

Let $\mathbb{S} \in \mathbf{Spt}_T^{\Sigma,\mathcal{M}_0}$ be the unit. We have as well the map $\mathbb{S} \cong \mathrm{Tot}_n c\mathbb{S} \to \mathrm{Tot}_n\tilde{\mathcal{E}}^{\wedge *+1}$, induced by the unit map $c\mathbb{S} \to \tilde{\mathcal{E}}^{\wedge *+1}$. Letting $\bar{\mathcal{E}}$ be the homotopy cofiber of the unit map $\mathbb{S} \to \mathcal{E}$, we claim there is a natural isomorphism in $\mathcal{SH}$

$$\Omega^n\bar{\mathcal{E}}^{\wedge n+1} \cong \mathrm{hocofib}(\mathbb{S} \to \mathrm{Tot}_n\tilde{\mathcal{E}}^{\wedge *+1}).$$

Indeed, let $[\mathbb{S}\to\mathcal{E}]^{\wedge n+1}$ be the evident $n+1$ cube in spectra. The distinguished triangle $\mathbb{S}\to\mathcal{E}\to\bar{\mathcal{E}}\to\mathbb{S}[1]$ gives the isomorphism

$$\Omega^{n+1}\bar{\mathcal{E}}^{\wedge n+1}\cong \operatorname*{holim}_{\square_0^{n+2}}\rho^{+*}_{n+1}[\mathbb{S}\to\mathcal{E}]^{\wedge n+1}$$

in $\mathbf{HoSpt}_T^{\Sigma,\mathcal{M}_0}$. On the other hand, fill in the punctured $n+1$-cube $\varphi_0^{n+1}\mathcal{E}^{\wedge *+1}$ to an $n+1$-cube $\tilde{\varphi}_0^{n+1}\mathcal{E}^{\wedge *+1}$ by inserting pt at the entry $\emptyset$, and similarly extend $\mathbb{S}$ to an $n+1$-cube $\tilde{\mathbb{S}}$ with value $\mathbb{S}$ at $\emptyset$ and value pt at $I\neq\emptyset$. This gives us the homotopy fiber sequence in $(\mathbf{Spt}_T^{\Sigma,\mathcal{M}_0})^{\square_0^{n+2}}$

$$\rho^{+*}_{n+1}\tilde{\varphi}_0^{n+1}\mathcal{E}^{\wedge *+1}\to\rho^{+*}_{n+1}[\mathbb{S}\to\mathcal{E}]^{\wedge n+1}\to\rho^{+*}_{n+1}\tilde{\mathbb{S}}.$$

Applying $\operatorname{holim}_{\square_0^{n+2}}$ and noting the isomorphisms (in $\mathbf{HoSpt}_T^{\Sigma,\mathcal{M}}$)

$$\Omega\operatorname*{holim}_{\square_0^{n+1}}\varphi_0^{n+1}\mathcal{E}^{\wedge *+1}\cong\operatorname*{holim}_{\square_0^{n+2}}\rho^{+*}_{n+1}\tilde{\varphi}_0^{n+1}\mathcal{E}^{\wedge *+1}$$
$$\Omega^{n+1}\bar{\mathcal{E}}^{\wedge n+1}\cong\operatorname*{holim}_{\square_0^{n+2}}\rho^{+*}_{n+1}[\mathbb{S}\to\mathcal{E}]^{\wedge n+1}$$
$$\mathbb{S}\cong\operatorname*{holim}_{\square_0^{n+2}}\tilde{\mathbb{S}}$$

gives the distinguished triangle in $\mathbf{HoSpt}_T^{\Sigma,\mathcal{M}_0}$

$$\Omega\operatorname*{holim}_{\square_0^{n+1}}\varphi_0^{n+1}\mathcal{E}^{\wedge *+1}\to\Omega^{n+1}\bar{\mathcal{E}}^{\wedge n+1}\to\mathbb{S}\to\operatorname*{holim}_{\square_0^{n+1}}\varphi_0^{n+1}\mathcal{E}^{\wedge *+1},$$

which yields the desired result.

We consider the case of $\mathcal{E}=\mathrm{MGL}$ in $\mathbf{Spt}_T^{\Sigma}(S)$. For the construction of MGL we refer the reader to [27]; for the structure as a symmetric monoidal object in $\mathbf{Spt}_T^{\Sigma}(S)$, we cite [23, §2.1]. Applying the above example, we have the distinguished triangle in $\mathcal{SH}(S)$

$$\mathbb{S}_S\xrightarrow{i_n}\operatorname*{holim}_{\square_0^{n+1}}\varphi_0^{n+1}\mathrm{MGL}^{\wedge *+1}\to\Omega^n\overline{\mathrm{MGL}}^{\wedge n+1}\to\mathbb{S}_S[1].\tag{3.3}$$

Since f^t_m is an exact functor, we have the isomorphism in $\mathcal{SH}(S)$

$$\operatorname*{holim}_{\square_0^{n+1}}f_m^{t,\square_0^{n+1}}\varphi_0^{n+1}\mathrm{MGL}^{\wedge *+1}\cong f^t_m\operatorname*{holim}_{\square_0^{n+1}}\varphi_0^{n+1}\mathrm{MGL}^{\wedge *+1}$$

Proposition 3.5. *1. The morphism i_n induces an isomorphism*

$$f^t_{m/N}\mathbb{S}_S\to\operatorname*{holim}_{\square_0^{n+1}}f_{m/M}^{t,\square_0^{n+1}}\varphi_0^{n+1}MGL^{\wedge *+1}$$

for all $m\leq M\leq n+1$.
2. There is a natural isomorphism

$$\xi_{m/N,n}:f^t_{m/N}\mathbb{S}_S\to\mathrm{Tot}_n\tilde{f}^{t,\Delta}_{m/N}MGL^{\wedge *+1}$$

for $m\leq N\leq n+1$, compatible with the maps in the Tot_n-tower (for fixed m,N and varying n) and the maps in the slice tower (for fixed n and varying m,N).

Proof. The map $\mathbb{S}_S\to\mathrm{MGL}$ induces an isomorphism $s^t_0\mathbb{S}_S\to s^t_0\mathrm{MGL}$, and hence $s^t_0\overline{\mathrm{MGL}}=0$. As both $\mathbb{S}_S$ and MGL are in $\mathcal{SH}^{eff}(S)$, it follows that $f^t_1\overline{\mathrm{MGL}}=\overline{\mathrm{MGL}}$, and thus $f^t_{n+1}\Omega^n\overline{\mathrm{MGL}}^{\wedge n+1}\cong\Omega^n\overline{\mathrm{MGL}}^{\wedge n+1}$. From this follows

$$f^t_{m/N}\Omega^n\overline{\mathrm{MGL}}^{\wedge n+1}=0 \text{ for } m\leq N\leq n+1.$$

Applying $f_{m/N}$ to the distinguished triangle (3.3) completes the proof of (1).

For (2), the restriction $\iota_{n*}\tilde{f}^{t,\Delta}_{m/N}\mathrm{MGL}^{\wedge *+1}$ is fibrant in $\mathbf{HoSpt}^{\Sigma}_{\mathbb{P}^1}(S)^{\Delta^{\leq n}}$ since $\tilde{f}^{t,\Delta}_{m/N}\mathrm{MGL}^{\wedge *+1}$ is a fibrant object in $\mathbf{HoSpt}^{\Sigma}_{\mathbb{P}^1}(S)^{\Delta}$. In addition, we have an isomorphism in $\mathbf{HoSpt}^{\Sigma}_{\mathbb{P}^1}(S)^{\Delta^{\leq n}}$

$$\iota_{n*}\tilde{f}^{t,\Delta}_{m/N}\mathrm{MGL}^{\wedge *+1} \cong \tilde{f}^{t,\Delta^{\leq n}}_{m/N}\iota_{n*}\mathrm{MGL}^{\wedge *+1}.$$

Thus, we have a canonical isomorphism in $\mathcal{SH}(S)$

$$\operatorname*{holim}_{\Delta^{\leq n}} f^{t,\Delta^{\leq n}}_{m/M}\iota_{n*}\mathrm{MGL}^{\wedge *+1} \cong \mathrm{Tot}_{(n)}\tilde{f}^{t,\Delta}_{m/N}\mathrm{MGL}^{\wedge *+1}$$

Similarly, by proposition 3.3, we have the isomorphism

$$\operatorname*{holim}_{\Delta^{\leq n}} f^{t,\Delta^{\leq n}}_{m/M}\iota_{n*}\mathrm{MGL}^{\wedge *+1} \cong \operatorname*{holim}_{\square_0^{n+1}} f^{t,\square_0^{n+1}}_{m/M}\varphi_0^{n+1}\mathrm{MGL}^{\wedge *+1}$$

in $\mathcal{SH}(S)$; together with (1), these isomorphisms yield (2). □

4. Décalage

Deligne's décalage operation [6, (1.3.3)] constructs a new filtration Dec$F$ on a complex K from a given filtration F on K; this change of filtration has the effect of accelerating the associated spectral sequence associated to the filtered complex K. Here we replace the filtered complex K with a cosimplicial spectrum object together with the tower $\mathrm{Tot}^{(*)}$. The tower replacing Dec$F$ turns out to arise from a suitable Postnikov tower, where the nth term is formed by applying the functor of the $n-1$-connected cover termwise to the given cosimplicial object. Our main result in this section is the exact analog of Deligne's comparison of the spectral sequences for (K,F) and $(K,\mathrm{Dec}F)$ [6, proposition 1.3.4].

For the application of this construction to the comparison of the slice and Adams-Novikov spectral sequence, we need only consider the model categories of simplicial sets and suspension spectra. However, with an eye to possible future applications, we will present this section in a somewhat more general setting. We were not able to formulate a good axiomatic description of the appropriate setting for this construction, rather, we give a list of examples, which we hope will cover enough ground to be useful.

We take $\mathcal{M}_0$ to be one of the following pointed closed symmetric monoidal simplicial model categories:

(4.1)

1. $\mathbf{Spc}_\bullet$, the category of pointed simplicial sets, with the usual model structure
2. Take $\mathcal{C}$ to be a small category, τ a Grothendieck topology on $\mathcal{C}$ and $\mathcal{M}$ the category of $\mathbf{Spc}_\bullet$-valued presheaves on $\mathcal{C}$ with the injective model structure.
3. B a scheme, $\mathcal{C} = \mathbf{Sm}/B$, the category of smooth quasi-projective B-schemes and $\mathcal{M}$ the category $\mathbf{Spc}_\bullet(B)$ with the motivic model structure, that is, the left Bousfield localization of example (2) with $\mathcal{C} = \mathbf{Sm}/B$, τ the Nisnevich topology, and the localization with respect to maps $\mathcal{X}\wedge(\mathbb{A}^1,0)\to pt$. As a variant, one can replace the Nisnevich topology with the étale topology; we denote this model category by $\mathbf{Spc}^{\text{ét}}_\bullet(B)$

We note that these are all cofibrantly generated, cellular and combinatorial model categories. In case (2), we recall that the weak equivalences are given via the τ-homotopy sheaves $\pi_n^\tau(\mathcal{X})$, this being the τ-sheaf associated to the presheaf $U \mapsto [\Sigma^n U_+, \mathcal{X}]_{\mathbf{Ho}\mathcal{M}}$, and in case (3), the weak equivalences are given via the $\mathbb{A}^1$-homotopy sheaves $\pi_n^{\mathbb{A}^1}(\mathcal{X})$, these being similarly defined as the Nisnevich (resp. étale) sheaf associated to the presheaf $U \mapsto [\Sigma^n U_+, \mathcal{X}]_{\mathbf{Ho}\mathcal{M}}$.

For the stable model categories $\mathcal{M} := \mathbf{Spt}_T\mathcal{M}_0$ we will use the model structure induced from $\mathcal{M}_0$ by the construction given in [12, chapter 7]. We take in case (1) $T = S^1$, giving us the category of suspension spectra, with weak equivalences the stable weak equivalences. In (2), we take again the category of suspension spectra, where now $T = S^1$ acts through the simplicial structure. We assume that the weak equivalences are the stable weak equivalences, that is, maps that induce an isomorphism on the stable homotopy sheaves $\pi_n^s(\mathcal{E}) := \varinjlim_N \pi^\tau_{n+N}(\mathcal{E}_N)$ if $\mathcal{E} = (\mathcal{E}_0, \mathcal{E}_1, \ldots)$. In case (3), we may take $T = S^1$, giving the category of S^1-spectra $\mathbf{Spt}_{S^1}(B)$ or for the étale version $\mathbf{Spt}^{\text{ét}}_{S^1}(B)$. Here the weak equivalences are the stable weak equivalences, using the $\mathbb{A}^1$ homotopy sheaves $\pi_n^{\mathbb{A}^1}$ in place of π_n^τ. These are all cofibrantly generated, cellular, combinatorial stable simplicial $\mathcal{M}$ model categories. If at some point we require the stable category to have a monoidal model category structure, we will replace the spectrum category with symmetric spectra.

In all cases, one has for $\mathcal{X}$ homotopy objects $\pi_n(\mathcal{X})$, $n = 0, 1, \ldots$, with π_n an abelian group object for $n \geq 2$, and a group object for $n = 1$, so that the $\{\pi_n, n \geq 0\}$ detects weak equivalences, a loops functor $\mathcal{X} \to \Omega\mathcal{X}$ with $\pi_n(\Omega\mathcal{X}) = \pi_{n+1}(\mathcal{X})$, so that a homotopy fiber sequence induces a long exact sequence in the π_n in the usual extended sense, a functorial (left) Postnikov tower

$$\ldots \to f_{n+1}\mathcal{X} \to f_n\mathcal{X} \to \ldots \to f_0\mathcal{X} = \mathcal{X}$$

with $f_n\mathcal{X} \to \mathcal{X}$ inducing an isomorphism on π_m for $m \geq n$ and with $\pi_m f_n\mathcal{X} = \{*\}$ for $m < n$. Furthermore, for an integer $n \geq 2$, there is an Eilenberg-MacLane space $K(A, n)$ associated to an abelian group (in case (1)) or τ-sheaf of abelian groups (in case (2)) or strictly $\mathbb{A}^1$-invariant sheaf of abelian groups (in case (3)), which is determined up to unique isomorphism in $\mathbf{Ho}\mathcal{M}$ by the vanishing of $\pi_m K(A, n)$ for $m \neq n$ and the choice of an isomorphism $A \cong \pi_n K(A, n)$.

For the spectrum categories, stabilizing the π_n gives the collection of stable homotopy objects $\{\pi_n, n \in \mathbb{Z}\}$ which detect weak equivalences and which are abelian group objects for all n, one has a functorial (left) Postnikov tower

$$\ldots \to f_{n+1}\mathcal{E} \to f_n\mathcal{E} \to \ldots \to \mathcal{E}$$

and Eilenberg-MacLane spectrum $EM(A, n)$ for A an abelian group object as above, and $n \in \mathbb{Z}$.

In the sequel, we will treat all these cases simultaneously; we will usually not need to distinguish between the stable and unstable setting, and will refer to the model category at hand as $\mathcal{M}$, whether stable or unstable. We will retain the notation $K(A, n)$ for the Eilenberg-MacLane space in the unstable setting, and write $K(A, n)$ for the Eilenberg-MacLane spectrum $EM(A, n)$ in the stable case.

We apply the Postnikov tower construction in functor categories, as described in example 1.6(1), to an object $\mathcal{X} \in \mathcal{M}^\Delta$, giving the cosimplicial object $f_n\mathcal{X} \in \mathcal{M}^\Delta$:

$$f_n\mathcal{X} := [m \mapsto f_n\mathcal{X}^m]$$

and the resulting tower

$$\ldots \to f_{n+1}\mathcal{X} \to f_n\mathcal{X} \to \ldots \to \mathcal{X}.$$

As the notation suggests, this tower has the property that evaluation at some $[m] \in \Delta$ yields the Postnikov tower for $\mathcal{X}^m$.

We will assume that we have a double de-looping $\mathcal{Y}$ of $\mathcal{X}$, that is, a weak equivalence of cosimplicial spaces $\mathcal{X} \to \Omega^2\mathcal{Y}$; we will simply replace $\mathcal{X}$ with $\Omega^2\mathcal{Y}$, so we may assume that this weak equivalence is an identity. This assumption is of course fulfilled for all $\mathcal{X}$ if we are in the stable case.

Definition 4.1. Fix an integer A and an extended integer B, with $0 \le A < B \le \infty$. Let $\mathcal{X}$ be in $\mathcal{M}^\Delta$. Applying the functor $\mathrm{Tot}^{(A/B)}$ to the Postnikov tower for $\mathcal{X}$ gives the *tower décalé* of spaces

$$\ldots \to \mathrm{Tot}^{(A/B)}(f_{n+1}\mathcal{X}) \to \mathrm{Tot}^{(A/B)}(f_n\mathcal{X}) \to \ldots \to \mathrm{Tot}^{(A/B)}(\mathcal{X}) \tag{4.2}$$

Using our chosen de-looping $\mathcal{X} = \Omega^2\mathcal{Y}$, let $f_{k/m}\mathcal{X} := \Omega\mathrm{hofib}(f_{m+2}\mathcal{Y} \to f_{k+2}\mathcal{Y})$. This gives us the homotopy fiber sequences

$$f_m\mathcal{X} \to f_k\mathcal{X} \to f_{k/m}\mathcal{X}.$$

The tower (4.2) gives rise to the spectral sequence

$$E_1^{p,q}(\mathrm{Dec}, \mathcal{X}) = \pi_{-p-q}\mathrm{Tot}^{(A/B)}f_{(p/p+1)}\mathcal{X} \Longrightarrow \pi_{-p-q}\mathrm{Tot}^{(A/B)}\mathcal{X} \tag{4.3}$$

for $0 \le A < B \le \infty$.

The constructions $\mathrm{Tot}^{(m/k)}$, f_q are strictly functorial and natural. In particular, we have the commutative diagram of natural transformations (for $0 \le m < N \le \infty$, $0 \le p$)

$$\begin{array}{ccc}
\mathrm{Tot}^{(m+1/N)}(f_{p+1}(-)) & \longrightarrow & \mathrm{Tot}^{(m+1/N)}(f_p(-)) \\
\downarrow & & \downarrow \\
\mathrm{Tot}^{(m/N)}(f_{p+1}(-)) & \longrightarrow & \mathrm{Tot}^{(m/N)}(f_p(-))
\end{array}$$

Define $F^{m/m+1}_{p/p+1}(\mathcal{Y})$ to the the homotopy fiber of the map

$$\mathrm{Tot}^{(m+1/N)}(f_{p+3}(\mathcal{Y})) \to \mathrm{Tot}^{(m/N)}(f_{p+2}(\mathcal{Y})).$$

Note that, up to weak equivalence, $F^{m/m+1}_{p/p+1}(\mathcal{Y})$ is, as the notation suggests, independent of the choice of N. For $\mathcal{X} = \Omega^2\mathcal{Y}$, define

$$\mathrm{Tot}^{(m/m+1)}_{p/p+1}(\mathcal{X}) := \Omega F^{m/m+1}_{p/p+1}(\mathcal{Y}).$$

As $f_n \circ \Omega$ is isomorphic to $\Omega \circ f_{n+1}$ as natural transformations to $\mathbf{Ho}\mathcal{M}$, this gives us the commutative diagram

$$\begin{array}{ccccc}
\mathrm{Tot}^{(\frac{m+1}{N})}(f_{p+1}(\mathcal{X})) & \to & \mathrm{Tot}^{(\frac{m+1}{N})}(f_p(\mathcal{X})) & \to & \mathrm{Tot}^{(\frac{m+1}{N})}(f_{\frac{p}{p+1}}(\mathcal{X})) \\
\downarrow & \searrow & \downarrow & & \downarrow \\
\mathrm{Tot}^{(\frac{m}{N})}(f_{p+1}(\mathcal{X})) & \longrightarrow & \mathrm{Tot}^{(\frac{m}{N})}(f_p(\mathcal{X})) & \longrightarrow & \mathrm{Tot}^{(\frac{m}{N})}(f_{\frac{p}{p+1}}(\mathcal{X})) \\
\downarrow & & \downarrow & \searrow & \uparrow\alpha \\
\mathrm{Tot}^{(\frac{m}{m+1})}(f_{p+1}(\mathcal{X})) & \to & \mathrm{Tot}^{(\frac{m}{m+1})}(f_p(\mathcal{X})) & \xleftarrow[\beta]{} & \mathrm{Tot}^{(\frac{m}{m+1})}_{\frac{p}{p+1}}(\mathcal{X})
\end{array} \tag{4.4}$$

with top two rows, the two left-hand columns and the diagonal all homotopy fiber sequences.

Lemma 4.2. *Let p, q be integers with $p \geq 0$, and $-2p \leq q \leq -p$. Take N with $2p+q+1 \leq N \leq \infty$, and consider the diagram*

$$\begin{array}{ccc} & & \pi_{-p-q}\mathrm{Tot}^{(\frac{2p+q}{N})}(f_{\frac{p}{p+1}}(\mathcal{X})) \\ & & \uparrow{\scriptstyle\alpha} \\ \pi_{-p-q}\mathrm{Tot}^{(\frac{2p+q}{2p+q+1})}(f_p(\mathcal{X})) & \xleftarrow[\beta]{} & \pi_{-p-q}\mathrm{Tot}^{(\frac{2p+q}{2p+q+1})}_{\frac{p}{p+1}}(\mathcal{X}) \end{array}$$

Then the map α is an isomorphism and the map β is injective.

Proof. Let us first consider the map α. The diagram (4.4) gives us the homotopy fiber sequence

$$\mathrm{Tot}^{(\frac{2p+q}{2p+q+1})}(f_{p+1}(\mathcal{X})) \to \mathrm{Tot}^{(\frac{2p+q}{2p+q+1})}_{\frac{p}{p+1}}(\mathcal{X}) \xrightarrow{\alpha} \mathrm{Tot}^{(\frac{2p+q}{N})}(f_{\frac{p}{p+1}}(\mathcal{X})).$$

Using the canonical isomorphism in $\mathcal{H}_\bullet$, $f_n(\Omega(\mathcal{T})) \cong \Omega f_{n+1}(\mathcal{T})$, the isomorphism $\Omega \circ \mathrm{Tot}^{(a/b)} \cong \mathrm{Tot}^{(a/b)} \circ \Omega$, and the de-looping $\mathcal{X} \cong \Omega^2\mathcal{Y}$ gives us the isomorphism in $\mathbf{Ho}\mathcal{M}$,

$$\Omega^2\mathrm{Tot}^{(\frac{2p+q}{2p+q+1})}(f_{p+3}(\mathcal{Y})) \cong \mathrm{Tot}^{(\frac{2p+q}{2p+q+1})}(f_{p+1}(\mathcal{X}))$$

which allows us to extend α to a homotopy fiber sequence in $\mathbf{Ho}\mathcal{M}$

$$\mathrm{Tot}^{(\frac{2p+q}{2p+q+1})}_{\frac{p}{p+1}}(\mathcal{X}) \xrightarrow{\alpha} \mathrm{Tot}^{(\frac{2p+q}{N})}(f_{\frac{p}{p+1}}(\mathcal{X})) \to \Omega\mathrm{Tot}^{(\frac{2p+q}{2p+q+1})}(f_{p+3}(\mathcal{Y})).$$

We have by (2.5)

$$\Omega\mathrm{Tot}^{(\frac{2p+q}{2p+q+1})}(f_{p+3}(\mathcal{Y})) \cong \Omega^{2p+q+1}Nf_{p+3}(\mathcal{Y}^{2p+q}),$$

hence $\pi_{-p-q+\epsilon}\Omega\mathrm{Tot}^{(\frac{2p+q}{2p+q+1})}(f_{p+3}(\mathcal{Y}))$ is a subgroup of $\pi_{p+1+\epsilon}f_{p+3}(\mathcal{Y}^{2p+q})$. As $\pi_{p+1+\epsilon}f_{p+3}(\mathcal{Y}^{2p+q}))$ is zero for $\epsilon = 0, 1$, α is an isomorphism.

For β, we have the homotopy fiber sequence

$$\mathrm{Tot}^{(\frac{2p+q+1}{N})}(f_{\frac{p}{p+1}}(\mathcal{X})) \to \mathrm{Tot}^{(\frac{2p+q}{2p+q+1})}_{\frac{p}{p+1}}(\mathcal{X}) \xrightarrow{\beta} \mathrm{Tot}^{(\frac{2p+q}{2p+q+1})}(f_{\frac{p}{p-1}}(\mathcal{X})).$$

The cosimplicial object $f_{\frac{p}{p+1}}(\mathcal{X})$ is weakly equivalent to the cosimplicial Eilenberg-MacLane object

$$n \mapsto K(\pi_p(\mathcal{X}^n), p)$$

hence $\pi_t\mathrm{Tot}^{(\frac{2p+q+1}{N})}(f_{\frac{p}{p+1}}(\mathcal{X}))$ is the cohomology in degree $-t$ of the complex

$$N\pi_p(\mathcal{X}^{2p+q+1}) \to N\pi_p(\mathcal{X}^{2p+q+2}) \to \ldots \to N\pi_p(\mathcal{X}^{N-1}),$$

concentrated in degrees $[p+q+1, N-p-1]$. Thus $\pi_{-p-q}\mathrm{Tot}^{(\frac{2p+q+1}{N})}(f_{\frac{p}{p+1}}(\mathcal{X})) = 0$ and β is injective. □

We consider the spectral sequences (2.7) and (4.3) for $A = 0$ and $0 < B \leq \infty$. Take integers p, q with $0 \leq -p$ and $0 \leq 2p+q < B$. We have

$$E_1^{m,-p}(\mathcal{X}) = N\pi_p\mathcal{X}^m;$$

the E_1-complex $E_1^{*,-p}(\mathcal{X})$ is the (truncated) normalized complex (shifted to be supported in degrees d, $-p \leq d < B-p$)

$$\sigma_{<B}N\pi_p sX^* := \pi_p\mathcal{X}^0 \to \ldots \to N\pi_p\mathcal{X}^{2p+q} \to N\pi_p\mathcal{X}^{2p+q+1} \to \ldots \to \pi_p\mathcal{X}^{B-1}$$

and $E_2^{2p+q,-p} = H^{p+q}(E_1^{*,-p}(\mathcal{X}))$.

As $f_{p/p+1}(\mathcal{X})$ is weakly equivalent to the cosimplicial object

$$m \mapsto K(\pi_p(\mathcal{X}^m), p)$$

it follows that $E_1^{p,q}(\mathrm{Dec}, \mathcal{X}) := \pi_{-p-q}\mathrm{Tot}^{(0/B)} f_{p/p+1}(\mathcal{X})$ is H^{p+q} of the complex (shifted to be supported in degrees d, $-p \le d < B - p$)

$$\sigma_{<B} N\pi_p\mathcal{X}^* := N\pi_p\mathcal{X}^0 \to \ldots, \to N\pi_p\mathcal{X}^{2p+q} \to N\pi_p\mathcal{X}^{2p+q+1} \to \ldots \to N\pi_p\mathcal{X}^{B-1}.$$

As this complex is equal to $E_1^{*,-p}(\mathcal{X})$, the identity maps on $N\pi_p\mathcal{X}^*$ induce the isomorphism

$$\gamma_1^{p,q} : E_1^{p,q}(\mathrm{Dec}, \mathcal{X}) \to E_2^{2p+q,-p}(\mathcal{X}). \tag{4.5}$$

Proposition 4.3. *Take $A = 0$, $0 < B \le \infty$. The maps* (4.5) *give rise to an isomorphism of complexes*

$$\gamma_1^{*,q} : E_1^{*,q}(Dec, \mathcal{X}) \to E_2^{2*+q,-*}(\mathcal{X})$$

and inductively a sequence of isomorphisms

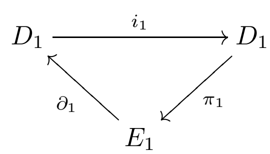
$$\gamma_r^{p,q} : E_r^{p,q}(Dec, \mathcal{X}) \to E_{r+1}^{2p+q,-p}(\mathcal{X}).$$

which give an isomorphism of complexes

$$\gamma_r^{*,*} : (\oplus_{p,q} E_r^{p,q}(Dec, \mathcal{X}), d_r) \to (\oplus_{p,q} E_{r+1}^{2p+q,-p}(\mathcal{X}), d_{r+1})$$

for each $r \ge 1$.

Proof. The spectral sequence (2.7) is the spectral sequence associated to the exact couple



with

$$D_1^{p,q} := \pi_{-p-q}\mathrm{Tot}^{(p/B)}(\mathcal{X}),\ \ E_1^{p,q} := \pi_{-p-q}\mathrm{Tot}^{(p/p+1)}(\mathcal{X}),$$

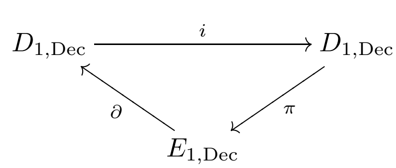
the maps $i_1^{p,q} : D_1^{p+1,q-1} \to D_1^{p,q}$ and $\pi_1^{p,q} : D_1^{p,q} \to E_1^{p,q}$ induced by the canonical morphisms

$$\mathrm{Tot}^{(p+1/B)}(\mathcal{X}) \to \mathrm{Tot}^{(p/B)}(\mathcal{X}),$$
$$\mathrm{Tot}^{(p/B)}(\mathcal{X}) \to \mathrm{Tot}^{(p/p+1)}(\mathcal{X}),$$

respectively, and with $\partial_1^{p,q} : E_1^{p,q} \to D_1^{p+1,q}$ the boundary map associated to the homotopy fiber sequence

$$\mathrm{Tot}^{(p+1/B)}(\mathcal{X}) \to \mathrm{Tot}^{(p/B)}(\mathcal{X}) \to \mathrm{Tot}^{(p/p+1)}(\mathcal{X}).$$

Similarly, the spectral sequence (4.3) arises from the exact couple



defined in a similar manner, where we replace $\mathrm{Tot}^{(p/B)}(\mathcal{X})$, $\mathrm{Tot}^{(p+1/B)}(\mathcal{X})$ and $\mathrm{Tot}^{(p/p+1)}(\mathcal{X})$ with $\mathrm{Tot}^{(0/B)}f_p(\mathcal{X})$, $\mathrm{Tot}^{(0/B)}f_{p+1}(\mathcal{X})$ and $\mathrm{Tot}^{(0/B)}f_{p/p+1}(\mathcal{X})$. To prove the result, it suffices to define maps

$$\delta_1^{p,q} : D_{1,\mathrm{Dec}}^{p,q} \to D_2^{2p+q,-p}$$

such that

$$\begin{pmatrix} \delta_1 & & \delta_1 \\ & \gamma_1 & \end{pmatrix} : \quad \begin{array}{ccc} D_{1,\mathrm{Dec}} & \xrightarrow{i_1} & D_{1,\mathrm{Dec}} \\ {\scriptstyle \partial_1}\nwarrow & & \swarrow{\scriptstyle \pi_1} \\ & E_{1,\mathrm{Dec}} & \end{array} \quad \to \quad \begin{array}{ccc} D_2 & \xrightarrow{i_2} & D_2 \\ {\scriptstyle \partial_2}\nwarrow & & \swarrow{\scriptstyle \pi_2} \\ & E_2 & \end{array}$$

defines a map of (reindexed) exact couples.

We recall that E_2 is the cohomology of the complex (E_1, d_1), with $d_1 = \pi_1 \circ \partial_1$. Let $Z_2 \subset E_1$ be the kernel of d_1 and note that $Z_2 \supset \pi_1(D_1)$. By definition, $D_2^{p,q} = i_1(D_1^{p,q}) \subset D_1^{p-1,q+1}$, $i_2 : D_2 \to D_2$ is the map induced by i_1, the map $\pi_2 : D_2 \to E_2$ is defined by the commutative diagram

$$\begin{array}{ccccc} D_2 & \xrightarrow[\pi_{1|D_2}]{} & Z_2 & \twoheadrightarrow & E_2 \\ \downarrow & & \downarrow & & \\ D_1 & \xrightarrow[\pi_1]{} & E_1 & & \end{array} \qquad (\pi_2 : D_2 \to Z_2 \twoheadrightarrow E_2)$$

and $\partial_2 : E_2 \to D_2$ is induced by restricting ∂_1 to Z_2, noting that this restriction sends Z_2 to $i_1(D_1) \subset D_1$, and descends to E_2.

Next, we note that the maps

$$\begin{aligned} &\pi_{-p-q}\mathrm{Tot}^{(2p+q/B)}f_p\mathcal{X} \to \pi_{-p-q}\mathrm{Tot}^{(0/B)}f_p\mathcal{X} \\ &\pi_{-p-q}\mathrm{Tot}^{(2p+q/B)}f_{p/p+1}\mathcal{X} \to \pi_{-p-q}\mathrm{Tot}^{(0/B)}f_{p/p+1}\mathcal{X} \\ &\pi_{-p-q-1}\mathrm{Tot}^{(2p+q+2/B)}f_{p+1}\mathcal{X} \to \pi_{-p-q-1}\mathrm{Tot}^{(0/B)}f_{p+1}\mathcal{X} \end{aligned}$$

are surjective and

$$\begin{aligned} &\pi_{-p-q}\mathrm{Tot}^{(2p+q-1/B)}f_p\mathcal{X} \to \pi_{-p-q}\mathrm{Tot}^{(0/B)}f_p\mathcal{X} \\ &\pi_{-p-q}\mathrm{Tot}^{(2p+q-1/B)}f_{p/p+1}\mathcal{X} \to \pi_{-p-q}\mathrm{Tot}^{(0/B)}f_{p/p+1}\mathcal{X} \\ &\pi_{-p-q-1}\mathrm{Tot}^{(2p+q+1/B)}f_{p+1}\mathcal{X} \to \pi_{-p-q-1}\mathrm{Tot}^{(0/B)}f_{p+1}\mathcal{X} \end{aligned}$$

are isomorphisms, by lemma 2.5. By considering the commutative diagram

$$\begin{array}{ccccc} \pi_{-p-q}\mathrm{Tot}^{(2p+q/B)}f_p\mathcal{X} & \twoheadrightarrow & \pi_{-p-q}\mathrm{Tot}^{(2p+q-1/B)}f_p\mathcal{X} & \xrightarrow{\sim} & \pi_{-p-q}\mathrm{Tot}^{(0/B)}f_p\mathcal{X} \\ \downarrow & & \downarrow & & \\ \pi_{-p-q}\mathrm{Tot}^{(2p+q/B)}\mathcal{X} & \longrightarrow & \pi_{-p-q}\mathrm{Tot}^{(2p+q-1/B)}\mathcal{X} & & \end{array}$$

we arrive at the well-defined map

$$\begin{aligned} D_{1,\mathrm{Dec}}^{p,q} = \pi_{-p-q}\mathrm{Tot}^{(0/B)}f_p\mathcal{X} &\xrightarrow{\delta_1^{p,q}} D_2^{2p+q,-p} \\ &= \mathrm{im}[\pi_{-p-q}\mathrm{Tot}^{(2p+q/B)}\mathcal{X} \to \pi_{-p-q}\mathrm{Tot}^{(2p+q-1/B)}\mathcal{X}]. \end{aligned}$$

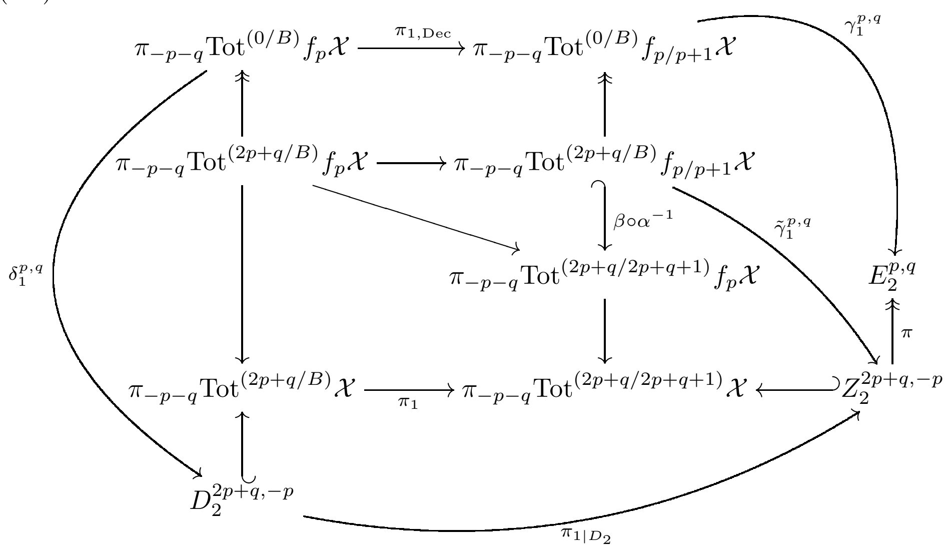
The identity

$$i_2 \circ \delta_1 = \delta_1 \circ i_{1,\mathrm{Dec}}$$

follows directly.

To show that $\pi_2 \circ \delta_1 = \gamma_1 \circ \pi_{1,\mathrm{Dec}}$, we consider the diagram (which is well-defined by lemma 4.2)

(4.6)



The right-hand column may be described explicitly as follows: let

$$\sigma_{<B} N\pi_p\mathcal{X}^* := [N\pi_p\mathcal{X}^0 \to N\pi_p\mathcal{X}^1 \to \ldots \to N\pi_p\mathcal{X}^{B-1}]$$

be the (truncated) normalized complex associated to the cosimplicial abelian group object $n \mapsto \pi_p\mathcal{X}^n$, shifted to be supported in degrees $[-p, B-p-1]$. Then the right-hand column is the sequence of evident maps

$$H^{p+q}(N\pi_p\mathcal{X}^*) \twoheadleftarrow Z^{p+q}(N\pi_p\mathcal{X}^*) \hookrightarrow N\pi_p\mathcal{X}^{2p+q} = N\pi_p\mathcal{X}^{2p+q}$$

The map $\tilde\gamma_1^{p,q}$ is the evident identification of $Z^{p+q}(\sigma_{<B}N\pi_p\mathcal{X}^*)$ with $Z_2^{2p+q,-p}$. The commutativity of (4.6) follows from this computation and the commutativity of diagram (4.4). Since $\pi_2 = \pi \circ \pi_{1|D_2}$, this shows that $\pi_2 \circ \delta_1 = \gamma_1 \circ \pi_{1,\mathrm{Dec}}$.

For the remaining identity $\partial_2 \circ \gamma_1 = \delta_1 \circ \partial_1$, we extract from the diagram (4.4) a commutative diagram (in $\mathcal{M}$) with rows being homotopy fiber sequences

$$\begin{array}{ccccc}
\mathrm{Tot}^{(\frac{2p+q+1}{B})} f_p\mathcal{X} & \longrightarrow & \mathrm{Tot}^{(\frac{2p+q}{B})} f_p\mathcal{X} & \longrightarrow & \mathrm{Tot}^{(\frac{2p+q}{2p+q+1})} f_p\mathcal{X} \\
\uparrow & & \| & & \uparrow \beta \\
\mathrm{Tot}^{(\frac{2p+q+1}{B})} f_{p+1}\mathcal{X} & \longrightarrow & \mathrm{Tot}^{(\frac{2p+q}{B})} f_p\mathcal{X} & \longrightarrow & \mathrm{Tot}^{(\frac{2p+q}{2p+q+1})}_{\frac{p}{p+1}}\mathcal{X} \\
\downarrow & & \| & & \sim\downarrow \alpha \\
\mathrm{Tot}^{(\frac{2p+q}{B})} f_{p+1}\mathcal{X} & \longrightarrow & \mathrm{Tot}^{(\frac{2p+q}{B})} f_p\mathcal{X} & \longrightarrow & \mathrm{Tot}^{(\frac{2p+q}{B})} f_{\frac{p}{p+1}}\mathcal{X}.
\end{array}$$

This gives us the commutative diagram

$$\begin{array}{ccc}
\pi_{-p-q}\mathrm{Tot}^{(\frac{2p+q}{2p+q+1})}\mathcal{X} & \xrightarrow{\partial} & \pi_{-p-q-1}\mathrm{Tot}^{(\frac{2p+q+1}{B})}\mathcal{X} \\
\uparrow{\scriptstyle\tilde{\gamma}} & & \uparrow{\scriptstyle\tilde{\delta}} \\
\pi_{-p-q}\mathrm{Tot}^{(\frac{2p+q}{2p+q+1})}f_p\mathcal{X} & \xrightarrow{\partial^{\frac{2p+q}{2p+q+1}}} & \pi_{-p-q-1}\mathrm{Tot}^{(\frac{2p+q+1}{B})}f_p\mathcal{X} \\
\uparrow{\scriptstyle\beta} & & \uparrow{\scriptstyle j} \\
\pi_{-p-q}\mathrm{Tot}^{(\frac{2p+q}{2p+q+1})}_{\frac{p}{p+1}}\mathcal{X} & \xrightarrow{\tilde{\partial}} & \pi_{-p-q-1}\mathrm{Tot}^{(\frac{2p+q+1}{B})}f_{p+1}\mathcal{X} \\
{\scriptstyle\alpha}\downarrow\wr & & \wr\downarrow{\scriptstyle\tilde{\alpha}} \\
\pi_{-p-q}\mathrm{Tot}^{(\frac{2p+q}{B})}f_{\frac{p}{p+1}}\mathcal{X} & \xrightarrow[\partial_{p/p+1}]{} & \pi_{-p-q-1}\mathrm{Tot}^{(\frac{2p+q}{B})}f_{p+1}\mathcal{X} \\
{\scriptstyle\varphi}\downarrow\wr & & \wr\downarrow{\scriptstyle\tilde{\varphi}} \\
\pi_{-p-q}\mathrm{Tot}^{(\frac{0}{B})}f_{\frac{p}{p+1}}\mathcal{X} & \xrightarrow[\partial_{1,\mathrm{Dec}}]{} & \pi_{-p-q-1}\mathrm{Tot}^{(\frac{0}{B})}f_{p+1}\mathcal{X}.
\end{array}$$

The map ∂_2 is induced from ∂, the map δ_1 is induced from $\tilde{\delta}\circ j\circ\tilde{\alpha}^{-1}\circ\tilde{\varphi}^{-1}$ (noting that this latter map has image in $D_2^{2p+q+2,-p-1}$), and $\gamma_1=\tilde{\gamma}\circ\beta\circ\alpha^{-1}\circ\varphi^{-1}$ (as we have noted above). This gives the identity $\partial_2\circ\gamma_1=\delta_1\circ\partial_{1,\mathrm{Dec}}$, completing the proof. □

Remark 4.4. Proposition 4.3 may be viewed as a homotopy-theoretic analog of a special case of Deligne's result [6, proposition 1.3.4]. Indeed, let K^{**} be a double complex and let K^* be the associated (extended) total complex

$$K^n := \prod_{a+b=n} K^{a,b}.$$

Give K^n the filtration by taking the stupid filtration in the first variable, that is, $(F^mK)^n := \prod_{a+b=n,a\geq m} K^{a,b}$. Then Deligne's filtration $\mathrm{Dec}^m K^*$ is given by $\mathrm{Dec}^m K^n = \prod_{a+b=n}\mathrm{Dec}^m K^{a,b}$ with

$$\mathrm{Dec}^m K^{a,b} = \begin{cases} K^{a,b} & \text{for } b<-m \\ 0 & \text{for } b>-m \\ \ker(\partial_2 : K^{a,-m}\to K^{a,-m+1}) & \text{for } b=m. \end{cases}$$

That is, $\mathrm{Dec}^m K^*$ is the extended total complex of the double complex

$$a\mapsto \tau^{can}_{\leq -m}(K^{a,*},\partial_2),$$

$\tau^{can}_{\leq -m}C^*$ being the canonical subcomplex of a complex C^*.

If $K^{a,b}=0$ for $a<0$, we may use the Dold-Kan correspondence to give a cosimplicial object in complexes

$$n\mapsto \tilde{K}^{n,*}$$

such that $K^{a,*}=N\tilde{K}^{a,*}$ as complexes, and the differential $\partial_1 : K^{a,b}\to K^{a+1,b}$ is the differential $N\tilde{K}^{a,*}\to N\tilde{K}^{a+1,*}$ given as the usual alternating sum of coface maps. If we let $EM\tilde{K}^{a,*}$ be the Eilenberg-MacLane spectrum associated to the complex $\tilde{K}^{a,*}$, then $\mathrm{Tot}[n\mapsto EM\tilde{K}^{n,*}]$ is the Eilenberg-MacLane spectrum associated to $\mathrm{Tot}K^*$, the tower $\mathrm{Tot}^{(*)}[n \mapsto EM\tilde{K}^{n,*}]$ is the tower associated to the filtration F^*K^*, and the tower $\mathrm{Tot}[n \mapsto f_*EM\tilde{K}^{n,*}]$ is associated to Dec^*K. Furthermore, the spectral sequences (2.7) and (4.3) are the same as the ones associated to the filtered complex F^*K and Dec^*K, respectively, and the isomorphism of Proposition 4.3 is the same as that of [6, proposition 1.3.4]. The proof given here is considerably more involved than that in [6], due to the fact that one could not simply compute with elements as was possible in the setting of filtered complexes.

5. The Adams-Novikov spectral sequence

For $\mathcal{E} = \mathbb{S}_k$ the motivic sphere spectrum and k an algebraically closed field with an embedding $\sigma : k \hookrightarrow \mathbb{C}$, the Betti realization of the slice tower for $\mathbb{S}_k$ gives a tower in $\mathcal{SH}$

$$\ldots \to Re_\sigma f^t_{n+1}\mathbb{S}_k \to Re_\sigma f^t_n\mathbb{S}_k \to \ldots \to Re_\sigma f^t_0\mathbb{S}_k = \mathbb{S},$$

where $\mathbb{S}$ is the usual sphere spectrum in $\mathcal{SH}$, with nth layer equal to $Re_\sigma s^t_n\mathbb{S}_k$. This gives the spectral sequence

$$E_2^{p,q}(AH)' = \pi_{-p-q}Re_\sigma s^t_{-q}\mathbb{S}_k(k) \cong H^{p-q}(k, \pi^\mu_{-q}(-q)) \Longrightarrow \pi_{-p-q}\mathbb{S};$$

we have shown in [16, theorem 4] that this spectral sequence is strongly convergent. In addition, we have identified $E_2^{p,q}(AH)'$ with an E_2 term of the Adams-Novikov spectral sequence for $\mathbb{S}$:

$$E_2^{p,q}(AH)' \cong E_2^{p-q,2q}(AN).$$

Our purpose in this section is to show that the spectral sequence $E(AH)$ agrees with the Adams-Novikov spectral sequence, after a suitable reindexing.

In principle, the argument should go like this: Let $\tilde{\mathrm{MU}}$ be a strict monoid object in symmetric spectra representing the usual MU in $\mathcal{SH}$. Let $\tilde{\mathrm{MU}}^{\wedge *+1}$ be the cosimplicial (symmetric) spectrum $n \mapsto \tilde{\mathrm{MU}}^{\wedge n+1}$ with the ith co-face map inserting the unit map in ith spot, and the ith co-degeneracy map taking the product of the ith and $i+1$st factors. The Adams-Novikov spectral sequence is just the spectral sequence (2.6) associated to the cosimplicial symmetric spectrum $\tilde{\mathrm{MU}}^{\wedge *+1}$. Let $\tilde{\mathrm{MGL}}^{\wedge *+1}$ be the motivic analog, giving us a cosimplicial T-spectrum $n \mapsto \tilde{\mathrm{MGL}}^{\wedge n+1}$, with co-face and co-degeneracy maps defined as for $\tilde{\mathrm{MU}}^{\wedge *}$. One could hope to have a "total T-spectrum functor" $\mathrm{Tot} : \mathbf{Spt}_T(k)^\Delta \to \mathbf{Spt}_T(k)$ and weak equivalences

$$\mathbb{S}_k \cong \mathrm{Tot}\tilde{\mathrm{MGL}}^{\wedge *+1}; \quad f^t_p\mathbb{S}_k \cong \mathrm{Tot}f^t_p\tilde{\mathrm{MGL}}^{\wedge *+1},$$

where $f^t_p\tilde{\mathrm{MGL}}^{\wedge *+1}$ is the cosimplicial spectrum $n \mapsto f^t_p\tilde{\mathrm{MGL}}^{\wedge n+1}$, using a suitable functorial model f^t_p in $\mathbf{Spt}^\Sigma_T(k)$.

The layers of $\tilde{\mathrm{MGL}}^{\wedge n+1}$ for the slice filtration are known, and one can show that the Betti realization $Re_\sigma s^t_p\tilde{\mathrm{MGL}}^{\wedge n+1}$ is just $f_{2p/2p+1}\mathrm{MU}^{\wedge n+1}$. Thus, one could hope to have an isomorphism in $\mathbf{HoSpt}^\Delta$

$$Re_\sigma f^t_p\tilde{\mathrm{MGL}}^{\wedge *+1} \cong f_{2p-1}\tilde{\mathrm{MU}}^{\wedge *+1} \cong f_{2p}\tilde{\mathrm{MU}}^{\wedge *+1}.$$

After changing the E_2 Atiyah-Hirzebruch spectral sequence to an E_1 spectral sequence

$$E_1^{p,q}(AH) := \pi_{-p-q,0}(s^t_p\mathbb{S}_k)(k) \Longrightarrow \pi_{-p-q,0}\mathbb{S},$$

we would then have an isomorphism

$$E_1^{p,q}(AH) \cong E_1^{2p,q-p}(\mathrm{Dec}, \tilde{\mathrm{MU}}^{\wedge *})$$

leading to the isomorphisms

$$E_r^{p,q}(AH) \cong E_{2r-1}^{2p,q-p}(\mathrm{Dec}, \tilde{\mathrm{MU}}^{\wedge *+1}) \cong E_{2r}^{2p,q-p}(\mathrm{Dec}, \tilde{\mathrm{MU}}^{\wedge *})$$

and corresponding isomorphisms of complexes.

Using proposition 4.3 (for spectra) would then give the sequence of isomorphisms

$$E_r^{p,q}(AH) \cong E_{2r+1}^{3p+q,-2p}(AN)$$

and corresponding isomorphisms of complexes. This would then give the isomorphisms

$$E_r^{p,q}(AH)' \cong E_{2r-1}^{p-q,2q}(AN).$$

for all $r \geq 2$.

We prefer to avoid the technical problems arising from the compatibility of the Betti realization with the functor Tot, and with verifying that $\mathbb{S} \to \mathrm{Tot}\tilde{\mathrm{MGL}}^{\wedge *+1}$ is an isomorphism; instead we work with the approximations $\mathrm{Tot}_{(n)}\tilde{\mathrm{MGL}}^{\wedge *+1}$ and $\mathrm{Tot}_{(n)}\tilde{\mathrm{MU}}^{\wedge *+1}$. These will suffice to give the desired isomorphisms of complexes, by simply taking n sufficiently large and using proposition 3.5 to show that the truncation $\mathrm{Tot}_{(n)}\tilde{\mathrm{MGL}}^{\wedge *+1}$ approximates $\mathbb{S}_k$ sufficiently well with respect to the slice tower. We drop the ˜ from the notation, considering both MU and MGL as objects in the appropriate category of symmetric spectra.

We have the cosimplicial objects

$$\mathrm{MGL}^{\wedge *+1} \in \mathbf{Spt}^{\Sigma}_{\mathbb{P}^1}(k)^{\Delta},\ \mathrm{MU}^{\wedge *+1} \in (\mathbf{Spt}^{\Sigma})^{\Delta},$$

giving us the punctured n-cubes

$$\varphi_{0*}^{n+1}\mathrm{MGL}^{\wedge *+1} \in \mathbf{Spt}^{\Sigma}_{\mathbb{P}^1}(k)^{\square_0^{n+1}},\ \varphi_{0*}^{n+1}\mathrm{MU}^{\wedge *+1} \in (\mathbf{Spt}^{\Sigma})^{\square_0^{n+1}}.$$

As the Betti realization of MGL is isomorphic to MU and Re_B is a monoidal functor, we have the isomorphism in $\mathbf{Ho}(\mathbf{Spt}^{\Sigma})^{\square_0^{n+1}}$

$$Re_B^{\square_0^{n+1}}\varphi_{0*}^{n+1}\mathrm{MGL}^{\wedge *+1} \cong \varphi_{0*}^{n+1}\mathrm{MU}^{\wedge *+1}.$$

Our main task is to identify the tower

$$\ldots \to Re_B^{\square_0^{n+1}} f_{n+1}^{\square_0^{n+1},t}\varphi_{0*}^{n+1}\mathrm{MGL}^{\wedge *+1} \to Re_B^{\square_0^{n+1}} f_{n}^{\square_0^{n+1},t}\varphi_{0*}^{n+1}\mathrm{MGL}^{\wedge *+1} \to \ldots \to Re_B^{\square_0^{n+1}}\varphi_{0*}^{n+1}\mathrm{MGL}^{\wedge *+1}.$$

As notation, for $\mathcal{E} \in \mathbf{Spt}_T(k)$, $I = (i_1, \ldots, i_r)$ an index with $0 \leq i_j \in \mathbb{Z}$, $b_I = b_1^{i_1} \cdot \ldots \cdot b_r^{i_r}$ a monomial, with b_j of degree n_j, we define $\mathcal{E} \cdot b_I := \Sigma_T^{|I|}\mathcal{E}$, where $|I| := \sum_{j=1}^r n_j \cdot i_j$. More generally, if $\{b^i_j\}$ is a set of variables, $i = 1, \ldots m$, with some assigned positive integral degrees, we let $\mathcal{E}[\{b^i_j\}]$ denote the coproduct of the $\mathcal{E}b^1_{I_1} \cdot \ldots \cdot b^m_{I_m}$.

Lemma 5.1. *We have an isomorphism of left MGL-modules*

$$MGL^{\wedge m+1} \cong MGL[b_\bullet^{(1)}, \ldots, b_\bullet^{(m+1)}]$$

where $b_\bullet^{(j)}$ is the collection of variables $b_1^{(j)}, b_2^{(j)}, \ldots$, with $b_n^{(j)}$ of degree n.

Proof. It clearly suffices to handle the case $m = 1$. For this, [21, lemma 6.2] gives us elements $b_n \in \pi_{2n,n}(\mathrm{MGL}\wedge\mathrm{MGL})$ giving rise to an isomorphism of left $\pi_{*,*}\mathrm{MGL}$-modules

$$\pi_{*,*}(\mathrm{MGL}\wedge\mathrm{MGL}) \cong \pi_{*,*}\mathrm{MGL}[b_1, b_2, \ldots].$$

For each monomial b_I in $b_1, b_2, \ldots$, we view $b_I \in \pi_{2|I|,|I|}(\mathrm{MGL}\wedge\mathrm{MGL})$ as a map $b_I : \Sigma_T^{|I|}\mathbb{S}_k \to \mathrm{MGL}\wedge\mathrm{MGL}$; using the product in MGL, this gives us the left MGL-map

$$\vartheta := \sum_I b_I : \oplus_I \Sigma_T^{|I|}\mathrm{MGL} \to \mathrm{MGL}\wedge\mathrm{MGL}.$$

Now, MGL is stably cellular [7, theorem 6.4] hence $\oplus_I \Sigma_T^{|I|}\mathrm{MGL}$ and $\mathrm{MGL}\wedge\mathrm{MGL}$ are stably cellular (the second assertion follows from [7, lemma 3.4]). Clearly ϑ induces an isomorphism on $\pi_{a,b}$ for all a, b, hence by [7, corollary 7.2] ϑ is an isomorphism in $\mathcal{SH}(k)$. □

Lemma 5.2. *1. For all $n, m \geq 0$, $Re_B(f_n^t MGL^{\wedge m+1})$ is $2n-1$ connected.*
2. The map

$$f_{2n}Re_B(f_n^t MGL^{\wedge m+1}) \to f_{2n}Re_B(MGL^{\wedge m+1})$$

induced by the natural transformation $f_n^t \to \mathrm{id}$ and the map

$$f_{2n}Re_B(f_n^t MGL^{\wedge m+1}) \to Re_B(f_n^t MGL^{\wedge m+1})$$

induced by the natural transformation $f_{2n} \to \mathrm{id}$ are weak equivalences.
3. The map

$$f_{2n}^{\square_0^{n+1}} Re_B^{\square_0^{n+1}}(f_n^{\square_0^{n+1},t}\varphi_{0*}^{n+1} MGL^{\wedge *+1}) \to f_{2n}^{\square_0^{n+1}} Re_B \varphi_{0*}^{n+1} MGL^{\wedge *+1}$$

induced by the natural transformation $f_n^{\square_0^{n+1},t} \to \mathrm{id}$ and the map

$$f_{2n}^{\square_0^{n+1}} Re_B^{\square_0^{n+1}}(f_n^{\square_0^{n+1},t}\varphi_{0*}^{n+1} MGL^{\wedge *+1}) \to Re_B^{\square_0^{n+1}}(f_n^{\square_0^{n+1},t}\varphi_{0*}^{n+1} MGL^{\wedge *+1})$$

induced by the natural transformation $f_{2n}^{\square_0^{n+1}} \to \mathrm{id}$ are weak equivalences.

Proof. It follows from Morel's $\mathbb{A}^1$-connectedness theorem [19] that $\pi_{a+b,b}\mathrm{MGL}_n = 0$ for $a < 2n$, $b \geq 0$. Thus the stable homotopy sheaves $\pi_{a+b,b}\mathrm{MGL}$ are zero for $a < 0$, that is, MGL is topologically -1 connected. By [17, proposition 3.2] $f_n^t\mathrm{MGL}$ is also topologically -1 connected, hence by [18, theorem 5.2] $Re_B(f_n^t\mathrm{MGL})$ is $n-1$-connected for all $n \geq 0$.

We have an isomorphism (of left MGL-modules)

$$\mathrm{MGL}^{\wedge m+1} \cong \oplus_{I=(i_1,\ldots,i_m}\Sigma_T^{|I|}\mathrm{MGL} \tag{5.1}$$

from which it follows that $f_n^t\mathrm{MGL}^{\wedge m+1}$ is topologically -1 connected and that $Re_B f_n^t\mathrm{MGL}^{\wedge m+1}$ is $n-1$ connected for all $n \geq 0$. Thus the tower

$$\ldots \to Re_B f_{N+1}^t\mathrm{MGL}^{\wedge m+1} \to Re_B f_N^t\mathrm{MGL}^{\wedge m+1} \to \ldots \to Re_B f_n^t\mathrm{MGL}^{\wedge m+1}$$

is strongly convergent. As both f_N^t and Re_B are exact functors, the ℓth layer in this tower is $Re_B s_{n+\ell}^t\mathrm{MGL}^{\wedge m+1}$, so to prove (1), it suffices to show that $Re_B s_{n+\ell}^t\mathrm{MGL}^{\wedge m+1}$ is $2n-1$-connected for all $\ell \geq 0$.

By the Hopkins-Morel-Hoyois theorem [10, 13] and the above computation of $\mathrm{MGL}^{\wedge m+1}$, $s_N\mathrm{MGL}^{\wedge m+1}$ is a finite coproduct of copies of $\Sigma_T^N M\mathbb{Z}$, where $M\mathbb{Z}$ is the motivic Eilenberg-MacLane spectrum representing motivic cohomology. In addition, $Re_B(M\mathbb{Z}) \cong EM(\mathbb{Z})$, hence $Re_B s_N \mathrm{MGL}^{\wedge m+1}$ is a finite coproduct of copies of $\Sigma^{2N} EM(\mathbb{Z})$, and is thus $2N-1$-connected.

For (2), applying Re_B to the decomposition (5.1) gives

$$Re_B(\mathrm{MGL}^{\wedge m+1}) \cong \oplus_I \Sigma^{2|I|} MU;$$

since $f_n^t \circ \Sigma_T \cong \Sigma_T \circ f_{n-1}^t$, and $f_m \circ \Sigma \cong \Sigma \circ f_{m-1}$, this reduces the proof of (2) to the case $m = 0$. Since

$$Re_B s_N^t \mathrm{MGL} \cong \Sigma^{2N} EM(\mathbb{Z}) \otimes MU^{-2N}$$

the strongly convergent spectral sequences

$$E_1^{p,q} = \pi_{-p-q} Re_B s_p^t \mathrm{MGL} \Longrightarrow \pi_{-p-q} Re_B \mathrm{MGL}$$

and

$$E_1^{p,q} = \pi_{-p-q} Re_B s_p^t \mathrm{MGL} \Longrightarrow \pi_{-p-q} Re_B f_n^t \mathrm{MGL}$$

degenerate at E_1 and show that $\pi_m Re_B f_n^t \mathrm{MGL} \to \pi_m Re_B \mathrm{MGL}$ is an isomorphism for $m \geq 2n$ and $\pi_m Re_B f_n^t \mathrm{MGL} = 0$ for $m < 2n$. Thus $Re_B f_n^t \mathrm{MGL} \to Re_B \mathrm{MGL} \cong MU$ is isomorphic (in $\mathcal{SH}$) to the $2n-1$-connected cover of MU, proving (2).

(3) follows immediately from (2), by the definition of the weak equivalences in the functor category $\mathcal{M}^{\mathcal{S}}$. □

We can now prove our main result:

Proof of theorem 1. Denote the spectral sequence (2.7) for fixed $A < B$ and cosimplicial spectrum $\mathcal{E}$ as $E(\mathcal{E}; A, B)$. The Adams-Novikov spectral sequence may be constructed as the spectral sequence associated to the cosimplicial spectrum

$$n \mapsto MU^{n+1},$$

that is, the spectral sequence $E(\mathrm{MU}^{\wedge *+1}; 0, \infty)$. For $A = 0$, $B < \infty$, we have $E_r^{p,q}(E; 0, B) = E_r^{p,q}(E; 0, \infty)$, and similarly for the differentials, in a range that goes to infinity in p, q, r as $B \to \infty$.

Letting $E(\mathrm{Dec}, \mathcal{E}; A, B)$ similarly denote the spectral sequence (4.3) for given values of $A < B$ and cosimplicial spectrum $\mathcal{E}$, we have a similar comparison statement for the spectral sequences $E(\mathrm{Dec}, \mathrm{MU}^{\wedge *+1}; A, B)$, $A < B \leq \infty$.

For $k \subset K$ an extension of algebraically closed fields, the base extension induces an isomorphism of spectral sequence $E(AH)$ for k and $E(AH)$ for K; this follows from e.g. [18, theorem 8.3]. Thus, we may assume that k admits an embedding into $\mathbb{C}$, giving the associated Betti realization functor

$$Re_B : \mathcal{SH}(k) \to \mathcal{SH}.$$

By lemma 5.2 and proposition 3.3, we have an isomorphism in $\mathcal{SH}$

$$Re_B(\mathrm{Tot}^{(0/B)} f_{a/b}^t \mathrm{MGL}^{\wedge *+1}) \cong \mathrm{Tot}^{(0/B)} f_{2a/2b} \mathrm{MU}^{\wedge *+1}$$

for all $a \leq b$, including $b = \infty$, compatible with respect to the maps in the slice tower for $\mathrm{MGL}^{\wedge *+1}$ and the Postnikov tower for $\mathrm{MU}^{\wedge *+1}$. Furthermore, by proposition 3.5, this gives us an isomorphism in $\mathcal{SH}$

$$Re_B(f_{a/b}^t \mathbb{S}_k) \cong \mathrm{Tot}^{(0/B)} f_{2a/2b} \mathrm{MU}^{\wedge *+1}$$

compatible with respect to change in a and b. Thus, we have an isomorphism of the spectral sequence associated to the tower

$$\ldots \to Re_B(f_{n+1}^t \mathbb{S}_k) \to Re_B(f_n^t \mathbb{S}_k) \to \ldots \to Re_B(f_0^t \mathbb{S}_k) \cong \mathbb{S}$$

and the one associated to the tower

$$\ldots \to \mathrm{Tot} f_{2n+2}\mathrm{MU}^{\wedge *+1} \to \mathrm{Tot} f_{2n}\mathrm{MU}^{\wedge *+1} \to \ldots \\ \to \mathrm{Tot} f_0\mathrm{MU}^{\wedge *+1} = \mathrm{Tot}\mathrm{MU}^{\wedge *+1}.$$

Since all the odd homotopy groups of $\mathrm{MU}^{\wedge m+1}$ vanish, this latter spectral sequence is just $E(\mathrm{Dec}, \mathrm{MU}^{\wedge *+1}, 0, \infty)$, except with a reindexing.

By [18, proposition 6.4], the functor Re_B induces an isomorphism

$$\pi_{n,0}(s^t_m\mathbb{S}_k))(k) \cong \pi_n(Re_B(s^t_m\mathbb{S}_k))$$

for all n and m. In addition, the tower

$$\ldots \to f^t_{m+1}\mathbb{S}_k \to f^t_m\mathbb{S}_k \to \ldots \to f^t_0\mathbb{S}_k = \mathbb{S}_k$$

and its Betti realization

$$\ldots \to Re_B f^t_{m+1}\mathbb{S}_k \to Re_B f^t_m\mathbb{S}_k \to \ldots \to Re_B f^t_0\mathbb{S}_k = \mathbb{S}$$

yield strongly convergent spectral sequences ([16, theorem 4], [18, proof of theorem 6.7])

$$E_1^{p,q} = \pi_{-p-q,0}(s^t_p\mathbb{S}_k)(k) \Longrightarrow \pi_{-p-q,0}(f^t_{a/b}\mathbb{S}_k)(k)$$

and

$$E_1^{p,q} = \pi_{-p-q}Re_B(s^t_p\mathbb{S}_k) \Longrightarrow \pi_{-p-q}Re_B(f^t_{a/b}\mathbb{S}_k)$$

and thus the functor Re_B induces an isomorphism

$$\pi_{n,0}(f^t_{a/b}\mathbb{S}_k)(k) \cong \pi_n(Re_B(f^t_{a/b}\mathbb{S}_k))$$

for all n and all $a < b \leq \infty$.

Putting these two pieces together, the Betti realization functor gives an isomorphism of the spectral sequence $E(AH)$ with the spectral sequence $E(\mathrm{Dec}, \mathrm{MU}^{\wedge *+1})$, after a suitable reindexing. Explicitly, this gives

$$E_1^{p,q}(AH) \cong E_1^{2p,q-p}(\mathrm{Dec}, \mathrm{MU}^{\wedge *+1}) = E_2^{2p,q-p}(\mathrm{Dec}, \mathrm{MU}^{\wedge *+1});$$

the terms $E_*^{p,q}(\mathrm{Dec}, \mathrm{MU}^{\wedge *+1})$ with p odd are all zero, and by induction, we have isomorphisms

$$E_r^{p,q}(AH) \cong E_{2r}^{2p,q-p}(\mathrm{Dec}, \mathrm{MU}^{\wedge *+1})$$

commuting with the differentials $d_r(AH)$ and $d_{2r}(\mathrm{Dec})$. Combined with the isomorphisms of proposition 4.3,

$$\gamma_r^{p,q} : E_r^{p,q}(\mathrm{Dec}, \mathcal{E}) \to E_{r+1}^{2p+q,-p}(\mathcal{E})$$

completes the proof. □

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Universität Duisburg-Essen, Fakultät Mathematik, Campus Essen, 45117 Essen, Germany

E-mail address: marc.levine@uni-due.de