# COHERENCE FOR DISTRIBUTIVITY

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## INTRODUCTION

A familiar situation in category theory is given by a category, $\underline{C}$, and two functors, $\otimes$, $\oplus : \underline{C} \times \underline{C} \longrightarrow \underline{C}$, that within natural isomorphisms are associative, commutative and such that $\otimes$ is distributive relative to $\oplus$. A coherence result for this situation is to characterize the diagrams whose commutativity is a consequence of the above structure and some suitable conditions on the natural isomorphisms. We are going to give a more precise description of this situation.

Let $\underline{C}$ be a category, $\otimes$, $\oplus : \underline{C} \times \underline{C} \longrightarrow \underline{C}$, two functors, U and N fixed objects of $\underline{C}$, called the unit and null objects. Suppose that we have natural isomorphisms,

$$\begin{array}{lll} \alpha_{A,B,C}: A\otimes(B\otimes C) \longrightarrow (A\otimes B)\otimes C, & \gamma_{A,B}: A\otimes B \longrightarrow B\otimes A, & \\ \alpha'_{A,B,C}: A\oplus(B\oplus C) \longrightarrow (A\oplus B)\oplus C, & \gamma'_{A,B}: A\oplus B \longrightarrow B\oplus A, & \\ \lambda_A: U\otimes A \longrightarrow A \quad , & \rho_A: A\otimes U \longrightarrow A, & (1) \\ \lambda'_A: N\oplus A \longrightarrow A \quad , & \rho'_A: A\oplus N \longrightarrow A, & \\ \lambda^*_A: N\otimes A \longrightarrow N \quad , & \rho^*_A: A\otimes N \longrightarrow N, & \end{array}$$

and natural monomorphisms,

$$\begin{array}{ll} \delta_{A,B,C}: A\otimes(B\oplus C) \longrightarrow (A\otimes B)\oplus(A\otimes C) & \\ & (2) \\ \delta^{\#}_{A,B,C}: (A\oplus B)\otimes C \longrightarrow (A\otimes C)\oplus(B\otimes C) & \end{array}$$

which are defined for any objects, A, B, C of $\underline{C}$.

A coherence result for the structure given to $\underline{C}$ by the family of isomorphisms, $\left\{\alpha_{A,B,C}: \lambda_A, \rho_A, \gamma_{A,B}\right\}$ was given by S. Mac Lane (see

[4] and [1]), and when we say that $\underline{C}$ is coherent for $\{\alpha_{A.B.C}, \lambda_A, \rho_A, \gamma_{A,B}\}$ or for $\{\alpha'_{A,B,C}, \lambda'_A, \rho'_A, \gamma'_{A,B}\}$ we want to refer to that result, although we are going to use the conditions in the form given by G. M. Kelly in [1].

We are going to give a coherence theorem for the above structure on $\underline{C}$, answering a question proposed in [5]. An announcement of this paper was given in [3]. Roughly speaking we intend to characterize the commutative diagrams that can be obtained by taking for vertices the combinations by $\otimes$ and $\oplus$ of objects of $\underline{C}$ and for arrows the combinations (also by $\otimes$ and $\oplus$) of the natural morphisms (1) and (2) with identities; to obtain a reasonable result we have to impose some conditions on $\underline{C}$ that are called the coherence conditions which hold in some usual situations.

The paper can be summarized in the following words: Let $X = \{x_1, x_2, \cdots, x_p, u, n\}$ be a set and construct the "free" category on the set X with functors $\otimes$ and $\oplus$ and with the natural morphisms (1) and (2); this is a category $\underline{C}(X)$ such that for any map, $m: X \longrightarrow \mathrm{Ob}\ \underline{C}$, such that $m(u) = U$, $m(n) = N$, there exists one and only one functor, $\tilde{m}: \underline{C}(X) \longrightarrow \underline{C}$, extending the map m and preserving $\otimes$, $\oplus$ and the morphisms (1) and (2). The objects of $\underline{C}(X)$ will be the elements of the free algebra with two operations, $\{.,+\}$, over X and the arrows will be all the elements generated by . and + over formal symbols of type (1), (2) and identities. The coherence result states that if $\underline{C}$ satisfies the coherence conditions detailed in §1 and is regular (in the definition given later) then the image by $\tilde{m}$ of the set $\underline{C}(X)(a,b)$ has at most one element.

We have to remark that the construction of the category $\underline{C}(X)$ will be given almost completely, but we are not going to use the concept of "free" category given above.

From now on $\underline{C}$ will be a category with the structure given above, whose objects will be denoted by capital letters. We shall use the parenthesis with the usual conventions on sums and products and the symbols ⊗ will be omitted as often as possible.

The core of this work was done in the Department of Mathematics of the University of Chicago where the author spent one year as Post-doctoral Visitor and he wants to thank Professor S. Mac Lane for his illuminating direction and patient revision of the different versions of this paper.

## §1. The Coherence Conditions

We will say that the category C is coherent when C is coherent in the sense of [1] separately for $\{\alpha, \gamma, \lambda, \rho\}$ and $\{\alpha', \gamma', \lambda', \rho'\}$ and the following types of diagrams are commutative for any vertices:

$$\begin{array}{ccc} A(B\oplus C) & \xrightarrow{\delta_{A,B,C}} & AB\oplus AC \\ \downarrow{\scriptstyle 1_A\cdot\gamma'_{B,C}} & & \downarrow{\scriptstyle \gamma'_{AB,AC}} \\ A(C\oplus B) & \xrightarrow{\delta_{A,C,B}} & AC\oplus AB \end{array} \quad , \tag{I}$$

$$(\gamma_{A,C}\oplus\gamma_{B,C})\delta^{\#}_{A,B,C} = \delta_{C,A,B}\gamma_{A\oplus B,C} : (A\oplus B)C \longrightarrow CA\oplus CB \quad , \tag{II}$$

$$\gamma'_{AC,BC}\delta^{\#}_{A,B,C} = \delta^{\#}_{B,A,C}(\gamma'_{A,B}\oplus 1_C) : (A\oplus B)C \longrightarrow BC\oplus AC \quad , \tag{III}$$

$$\begin{array}{ccccc} [A\oplus(B\oplus C)]D & \xrightarrow{\delta^{\#}_{A,B\oplus C,D}} & AD\oplus(B\oplus C)D & \xrightarrow{1_{AD}\oplus\delta^{\#}_{B,C,D}} & AD\oplus(BD\oplus CD) \\ \downarrow{\scriptstyle \alpha'_{A,B,C}\cdot 1_D} & & & & \downarrow{\scriptstyle \alpha'_{AD,BD,CD}} \\ [(A\oplus B)\oplus C]D & \xrightarrow{\delta^{\#}_{A\oplus B,C,D}} & (A\oplus B)D\oplus CD & \xrightarrow{\delta^{\#}_{A,B,D}\oplus 1_{CD}} & (AD\oplus BD)\oplus CD \end{array} \quad , \tag{IV}$$

$$\begin{array}{ccccc}
A[B\oplus(C\oplus D)] & \xrightarrow{\delta_{A,B,C\oplus D}} & AB\oplus A(C\oplus D) & \xrightarrow{1_{AB}\oplus\delta_{A,C,D}} & AB\oplus(AC\oplus AD) \\
\Big\downarrow{\scriptstyle 1_A\cdot\alpha'_{B,C,D}} & & & & \Big\downarrow{\scriptstyle \alpha'_{AB,AC,AD}} \\
A[(B\oplus C)\oplus D] & \xrightarrow{\delta_{A,B\oplus C,D}} & A(B\oplus C)\oplus AD & \xrightarrow{\delta_{A,B,C}\oplus 1_{AD}} & (AB\oplus AC)\oplus AD
\end{array} \qquad \text{(V)}$$

,

$$\begin{array}{ccccc}
A[B(C\oplus D)] & \xrightarrow{1_A\cdot\delta_{B,C,D}} & A(BC\oplus BD) & \xrightarrow{\delta_{A,BC,BD}} & A(BC)\oplus A(BD) \\
\Big\downarrow{\scriptstyle \alpha_{A,B,C\oplus D}} & & & & \Big\downarrow{\scriptstyle \alpha_{A,B,C}\oplus\alpha_{A,B,D}} \\
(AB)(C\oplus D) & & \xrightarrow{\delta_{AB,C,D}} & & (AB)C\oplus(AB)D
\end{array} \qquad \text{(VI)}$$

$$\begin{array}{ccccc}
(A\oplus B)(CD) & & \xrightarrow{\delta^{\#}_{A,B,CD}} & & A(CD)\oplus B(CD) \\
\Big\downarrow{\scriptstyle \alpha_{A\oplus B,C,D}} & & & & \Big\downarrow{\scriptstyle \alpha_{A,C,D}\oplus\alpha_{B,C,D}} \\
[(A\oplus B)C]D & \xrightarrow{\delta^{\#}_{A,B,C}\cdot 1_D} & (AC\oplus BC)D & \xrightarrow{\delta^{\#}_{AC,BC,D}} & (AC)D\oplus(BC)D
\end{array} \qquad \text{(VII)}$$

,

$$\begin{array}{ccccc}
A[(B\oplus C)D] & \xrightarrow{1_A\cdot\delta^{\#}_{B,C,D}} & A(BD\oplus CD) & \xrightarrow{\delta_{A,BD,CD}} & A(BD)\oplus A(CD) \\
\Big\downarrow{\scriptstyle \alpha_{A,B\oplus C,D}} & & & & \Big\downarrow{\scriptstyle \alpha_{A,B,D}\oplus\alpha_{A,C,D}} \\
[A(B\oplus C)]D & \xrightarrow{\delta_{A,B,C}\cdot D} & (AB\oplus AC)D & \xrightarrow{\delta^{\#}_{AB,AC,D}} & (AB)D\oplus(AC)D
\end{array} \qquad \text{(VIII)}$$

$$\begin{array}{ccc}
 & A(C\oplus D)\oplus B(C\oplus D) \xrightarrow{\delta_{A,C,D}\oplus\delta_{B,C,D}} & (AC\oplus AD)\oplus(BC\oplus BD) \\
 \nearrow \delta^{\#}_{A,B,C\oplus D} & & \downarrow \alpha'_{AC\oplus AD,BC,BD} \\
 & & [(AC\oplus AD)\oplus BC]\oplus BD \\
 & & \downarrow \alpha'^{-1}_{AC,AD,BC}\oplus 1_{BD} \\
 & & [AC\oplus(AD\oplus BC)]\oplus BD \\
(A\oplus B)(C\oplus D) & & \downarrow (1_{AC}\oplus\gamma'_{AD,BC})\oplus 1_{BD} \\
 & & [AC\oplus(BC\oplus AD)]\oplus BD \\
 & & \downarrow \alpha'_{AC,BC,AD}\oplus 1_{BD} \\
 \searrow \delta_{A\oplus B,C,D} & & [(AC\oplus BC)\oplus AD]\oplus BD \\
 & & \uparrow \alpha'_{AC\oplus BC,AD,BD} \\
 & (A\oplus B)C\oplus(A\oplus B)D \xrightarrow{\delta^{\#}_{A,B,C}\oplus\delta^{\#}_{A,B,D}} & (AC\oplus BC)\oplus(AD\oplus BD)
\end{array} \qquad (IX)$$

$$\lambda^*_N = \rho^*_N : N\ N \longrightarrow N \quad , \qquad (X)$$

$$\begin{array}{ccc}
N(A\oplus B) & \xrightarrow{\delta_{N,A,B}} & NA\oplus NB \\
\downarrow \lambda^*_{A\oplus B} & & \downarrow \lambda^*_A\oplus\lambda^*_B \\
N & \xleftarrow{\lambda'_N} & N\oplus N
\end{array} \qquad (XI)$$

$$\lambda'_N(\rho^*_A \oplus \rho^*_B)\delta^{\#}_{A,B,N} = \rho^*_{A\oplus B} : (A\oplus B)N \longrightarrow N\oplus N \quad , \tag{XII}$$

$$\rho_N = \lambda^*_N : NU \longrightarrow N \quad , \tag{XIII}$$

$$\lambda_N = \rho^*_N : UN \longrightarrow N \quad , \tag{XIV}$$

$$\rho^*_A = \lambda^*_A \gamma'_{A,N} : AN \longrightarrow N \quad , \tag{XV}$$

$$\begin{array}{ccc} N(AB) & \xrightarrow{\alpha_{N,A,B}} & (NA)B \\ \downarrow{\scriptstyle \lambda^*_{AB}} & & \downarrow{\scriptstyle \lambda^*_A \cdot 1_B} \\ N & \xleftarrow{\lambda^*_B} & NB \end{array} \quad , \tag{XVI}$$

$$\begin{array}{ccccc} A(NB) & & \xrightarrow{\alpha_{A,N,B}} & & (AN)B \\ \downarrow{\scriptstyle 1_A \cdot \lambda^*_B} & & & & \downarrow{\scriptstyle \rho^*_A \cdot 1_B} \\ AN & & & & NB \\ & \searrow{\scriptstyle \rho^*_A} & & \swarrow{\scriptstyle \lambda^*_B} & \\ & & N & & \end{array} \quad , \tag{XVII}$$

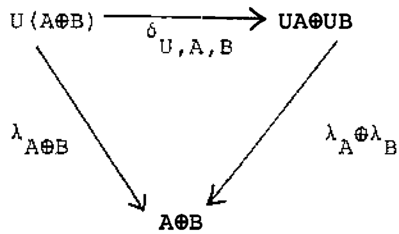
$$\rho^*_{AB}\alpha_{A,B,N} = \rho^*_A(1_A \otimes \rho^*_B) : A(BN) \longrightarrow N \quad , \tag{XVIII}$$

$$\begin{array}{ccc} A(N\oplus B) & \xrightarrow{\delta_{A,N,B}} & AN\oplus AB \\ \downarrow{\scriptstyle 1_A \cdot \lambda'_B} & & \downarrow{\scriptstyle \rho^*_A \oplus 1_{AB}} \\ AB & \xleftarrow{\lambda'_{AB}} & N\oplus AB \end{array} \quad , \tag{XIX}$$

$$\lambda'_{BA}(\lambda^{*}_{A}\oplus 1_{BA})\delta^{\#}_{N,B,A} = \lambda'_{B}\otimes 1_{A} : (N\oplus B)A \longrightarrow BA \quad , \qquad \text{(XX)}$$

$$\rho'_{AB}(1_{AB}\oplus\rho^{*}_{A})\delta_{A,B,N} = 1_{A}\otimes\rho'_{B} : A(B\oplus N) \longrightarrow AB \quad , \qquad \text{(XXI)}$$

$$\rho'_{AB}(1_{AB}\oplus\lambda^{*}_{B})\delta^{\#}_{A,N,B} = \rho'_{A}\otimes 1_{B} : (A\oplus N)B \longrightarrow AB \quad , \qquad \text{(XXII)}$$




(XXIII)

$$(\rho_{A}\oplus\rho_{B})\delta_{A,B,U} = \rho_{A\oplus B} : (A\oplus B)U \longrightarrow A\oplus B \quad , \qquad \text{(XXIV)}$$

The commutativity of some types of diagrams imply the commutativity of others, and we are going to indicate some of those relations. A detailed study of the minimal conditions assuring the coherence of $\underline{C}$ for $\{\alpha, \gamma, \lambda, \rho\}$ or $\{\alpha', \gamma', \lambda', \rho'\}$ is contained in [1].

We will prove the following set of relations, in which the number of the diagram denotes the condition of commutativity of all the diagrams of that type:

1) (II) $\Longrightarrow$ ( (I) $\Longleftrightarrow$ (III) ),

2) (II) $\Longrightarrow$ ( (IV) $\Longleftrightarrow$ (V) ),

3) Coherence of $\underline{C}$ for $\{\alpha, \gamma, \lambda, \rho\}$ $\wedge$ (II) $\Longrightarrow$ ( (VI) $\Longleftrightarrow$ (VII) ),

4) Coherence of $\underline{C}$ for $\{\alpha, \gamma, \lambda, \rho\}$ $\wedge$ (II) $\Longrightarrow$ ( (VI) $\Longrightarrow$ (VIII) ),

5) Coherence of $\underline{C}$ for $\{\alpha, \gamma, \lambda, \rho\}$ $\wedge$ (XV) $\Longrightarrow$ ( (XIII) $\Longleftrightarrow$ (XIV) ),

6) (II) $\wedge$ (XV) $\Longrightarrow$ ( (XI) $\Longleftrightarrow$ (XII) ) ,

7) Coherence of $\underline{C}$ for $\{\alpha', \gamma', \lambda', \rho'\}$ $\wedge$ (XV) $\Longrightarrow$

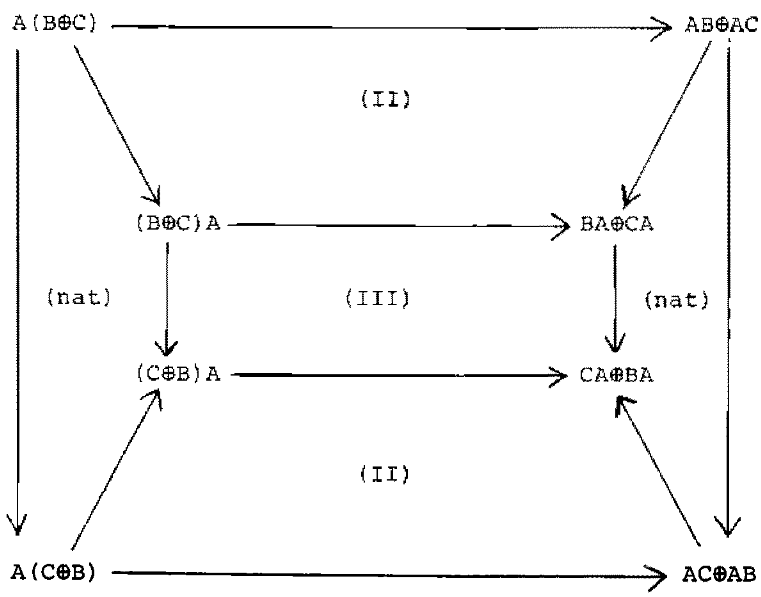
$\Longrightarrow$ Any two of $\{$(XVI), (XVII), (XVIII)$\}$ imply the other ,

8) (XV) $\wedge$ (I) $\wedge$ (II) $\Longrightarrow$ Each one of $\{$(XIX), (XX), (XXI), (XXII)$\}$ implies the others,

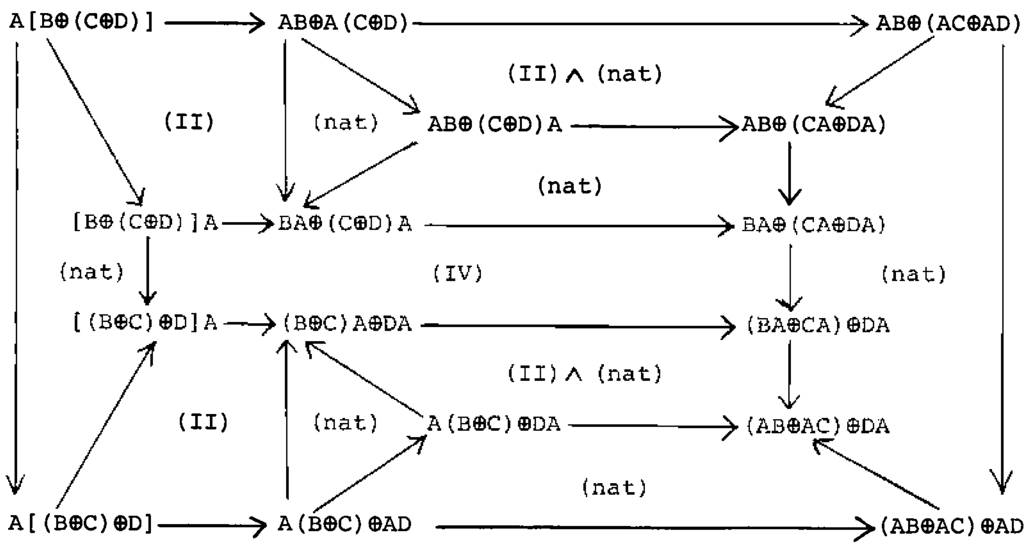
9) Coherence of $\underline{C}$ for $\{\alpha, \gamma, \lambda, \rho\} \wedge$ (II) $\Longrightarrow$ ( (XXIII) $\Longleftrightarrow$ (XXIV) ).

The proof of all the above relations uses the same method: the construction of a diagram in which the commutativity of all the sub-diagrams with the exception of two follows from the hypothesis of the relation so that the commutativity of any of these two diagrams are equivalent conditions. We are going to indicate the construction of these diagrams and to identify by its number each of the subdiagrams involved. The symbols (coh) and (nat) in the inside of a subdiagram will indicate that the reason for the commutativity is the coherence of $\underline{C}$ for $\{\alpha, \gamma, \lambda, \rho\}$ or the naturality of the elements involved in the construction of the subdiagram.

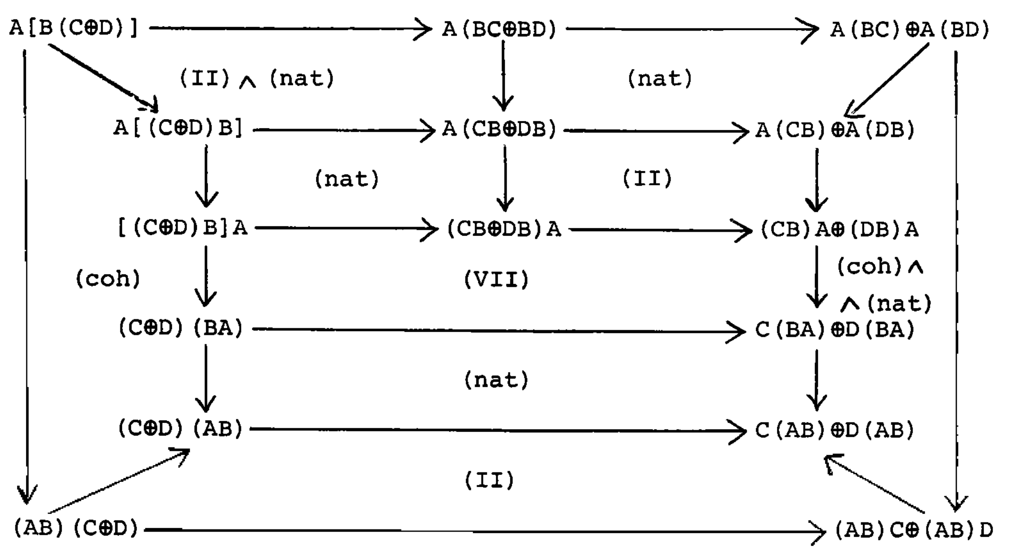
<u>Proof of 1)</u>: It is given by the following diagram in which the outside is of type (I)



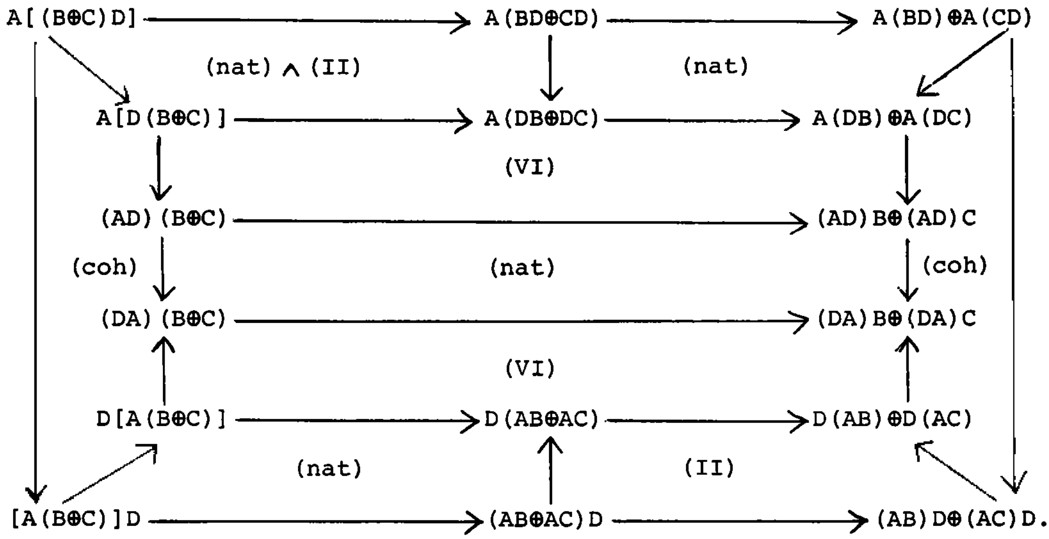
Proof of 2): It is given by the following diagram in which the outside is (V)



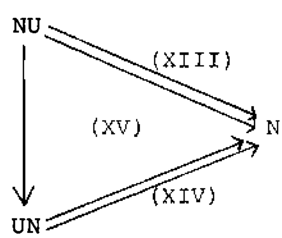

Proof of 3): It is given by the following diagram in which the outside is (VI)



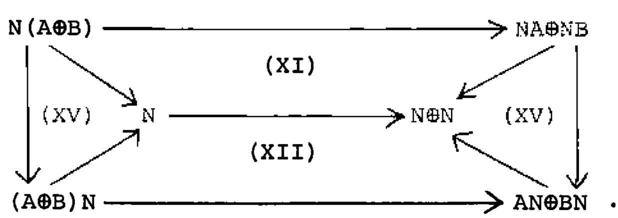
Proof of 4): It is given by the following diagram in which the outside is (VIII)



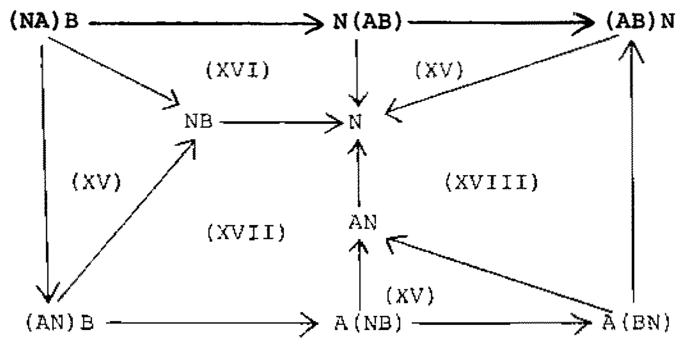

Proof of 5): It is given by the following diagram in which the outside diagram commutes by the coherence of C for $\{\alpha, \gamma, \lambda, \rho\}$



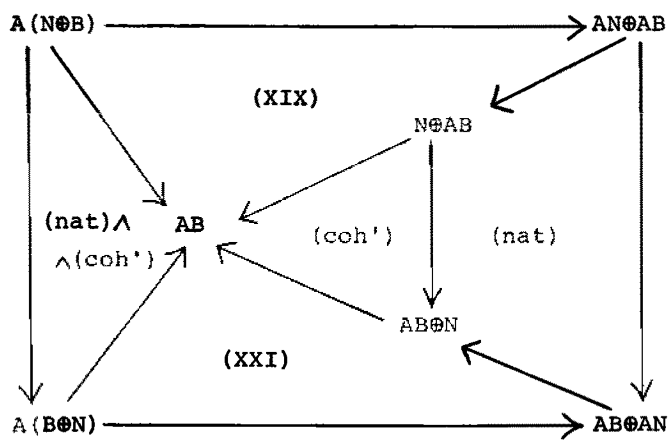

Proof of 6): It is given by the following diagram in which the outside is of type (II)

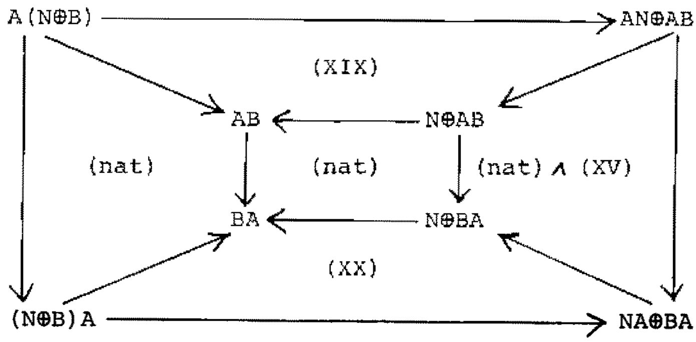


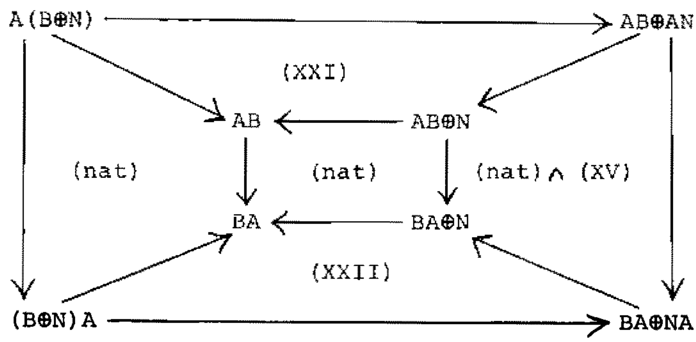
Proof of 7): It is given by the following diagram in which the outside is commutative by the coherence of C for $\{\alpha, \lambda, \rho, \gamma\}$



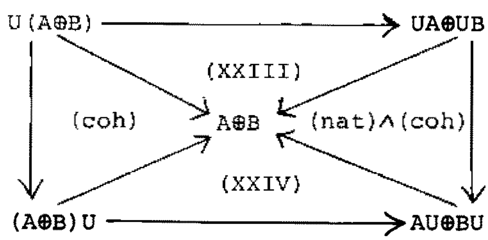

Proof of 8): It is given by the following diagrams in which the outside are of type (I), (II), and (II) respectively






Proof of 9): It is given by the following diagram in which the outside is (II)




An immediate consequence of the above relations is that for C to be coherent it is sufficient to check that C satisfies the following conditions:

1) C is coherent for $\{\alpha, \gamma, \lambda, \rho\}$ and for $\{\alpha', \gamma', \lambda', \rho'\}$ .

2) All the diagrams of type (II),(IX), (X) and (XV) are commutative.

3) For one type contained in each one of the sets, $\{(I), (III)\}$, $\{(IV), (V)\}$, $\{(VI), (VII)\}$, $\{(XI), (XII)\}$, $\{(XIII), (XIV)\}$, $\{(XIX), (XX), (XXI), (XXII)\}$, $\{(XXIII), (XXIV)\}$, all the diagrams are commutative.

3) For two of the types contained in $\{(XVI), (XVII), (XVIII)\}$ all the diagrams are commutative.

## §2. Definition and evaluation of the paths: Formulation of the coherence problem

Let X be the set $\{x_1, x_2, \cdots, x_p, n, u\}$, $\underline{A}$ the free $\{+,.\}$-algebra over X and $\underline{G}$ the graph consisting of all the following formal symbols, for x, y, z $\in \underline{A}$,

$$\alpha_{x,y,z}: x(yz) \longrightarrow (xy)z \quad , \quad \alpha'_{x,y,z}: x + (y + z) \longrightarrow (x + y) + z,$$

$$\lambda_x: ux \longrightarrow x \quad , \quad \lambda'_x: n + x \longrightarrow x \quad ,$$

$$\rho_x: xu \longrightarrow x \quad , \quad \rho'_x: x + n \longrightarrow x \quad ,$$

$$\gamma_{x,y}: xy \longrightarrow yx \quad , \quad \gamma'_{x,y}: x + y \longrightarrow y + x \quad ,$$

$$\lambda^*_x: nx \longrightarrow n \quad ,$$

$$\rho^*_x: xn \longrightarrow n \quad ,$$

their formal inverses, indicated by the upper index -1, and

$$\delta_{x,y,z}: x(y + z) \longrightarrow xy + xz \quad ,$$

$$\delta^{\#}_{x,y,z}: (x + y)z \longrightarrow xz + yz \quad ,$$

$$1_x: x \longrightarrow x \; .$$

Note that we use the symbol $\longrightarrow$ to indicate the edges of the graph to distinguish them from the arrows of the category denoted by $\longrightarrow$.

Let $\underline{H}$ be the free $\{+,1\}$-algebra over the edges of $\underline{G}$ and take on $\underline{H}$ the unique extension of the graph structure of $\underline{G}$ in which the projections are $\{+,.\}$-morphisms. An element of $\underline{H}$ is an <u>instantiation</u> if, with at most one exception, only elements of $\underline{G}$ of type $1_x$ are involved in its expression: the elements involving only elements of $\underline{G}$ of type $1_x$ are called instantiations of identities or simply identities. We will denote by $\underline{T}$ the graph consisting of all the instantiations of $\underline{H}$. We can define now the <u>paths</u> as the sequences,

$$y_1 \xrightarrow{\Pi_1} y_2 \xrightarrow{\Pi_2} \cdots \xrightarrow{\Pi_{m-1}} y_m \quad ,$$

where $\Pi_i \in \underline{T}$. We can speak of the existence of diagrams involving elements of $\underline{T}$, but not yet of the commutativity of such diagrams because the product of edges of $\underline{T}$ is not (and will not be) defined.

Fix now p objects, $0_1$, $0_2$, $\cdots$ , $0_p$ in $\underline{C}$ and let $g:\underline{T} \longrightarrow \underline{C}$ be the morphism of graphs defined on the vertices by the conditions, i) $gu = U$, $gn = N$, $gx_i = 0_i$, for $i = 1, 2, \cdots, p$, ii) $g(x + y) = gx \oplus gy$, $g(xy) = gx \otimes gy$, for $x,y \in \underline{A}$; on $\underline{G}$ by taking each formal symbol onto the arrow of $\underline{C}$ determined replacing each subscript by its image by g and such that for $x$, $y \in \underline{T}$, $g(x + y) = gx \oplus gy$, $g(xy) = gx \otimes gy$. This definition depends upon the $0_i$ and allows us to define the <u>value</u> of a path as the product of the value of the steps and to define that a diagram with elements in $\underline{T}$ is commutative if any two paths contained in the diagram and with the same origin and end have the same value.

An ideal coherence result would state that if $\underline{C}$ is coherent in the sense of $\S 1$, then for any choice of the $0_i$ any diagram of elements of $\underline{T}$ is commutative, that is, for any choice of the $0_i$ the value of any path only depends upon the origin and the end of the path. But this is not true in some simple cases; for instance if $\underline{C}$ is the category of unitary modules over a commutative ring, $\otimes$ the tensor product, $\oplus$ the direct sum and if $0_1$ is not the null module, then the value of $1_{x_1+x_1}:x_1 + x_1 \longrightarrow x_1 + x_1$ is the identity map of $0_1 \oplus 0_1$ and the value of $\gamma'_{x_1,x_1}:x_1 + x_1 \longrightarrow x_1 + x_1$ is the map defined by $\langle a,b\rangle \longrightarrow \langle b,a\rangle$ that is not the identiry. In this sense the coherence problem has a negative answer but we are going to prove that it is sufficient to impose a reasonable restriction on the vertices of the diagrams to get a coherence result that holds for any choice of the $0_i$.

Note that the free category generated by the graph $\underline{G}$ would be the free category $\underline{C}(X)$ referred to in the Introduction.

We shall use the symbol $\overset{\theta}{\longrightarrow}$ to indicate paths with steps in $\underline{T}$. The expression $a \overset{\theta}{\longrightarrow} b$ will denote the existence of a path from a to b.

## §3. Regularity and some preliminary concepts

We shall indicate by N the set of natural numbers and by $S^{[N]}$ the set of all finite sequences of elements of S. In general, we shall represent the elements of $S^{[N]}$ by putting into parenthesis the sequence of the elements, identifying the elements of S with the sequences of $S^{[N]}$ with only one element.

All the definitions included in this part, with the exception of the concept of regularity, are auxiliary tools to be used in the proof of the propositions.

The rank of the elements of $\underline{A}$ is defined by means of the map, $\text{rank}:\underline{A} \longrightarrow N$, uniquely determined by the following conditions,

i) For $x \in X$, rank $x = 2$ ,

ii) For $a, b \in \underline{A}$, rank $(a + b)$ = rank$(ab)$ = rank $a$ + rank $b$.

The size, $\text{siz}:\underline{A} \longrightarrow N$, is defined by the conditions,

i) For $x \in X$, siz $x = 2$,

ii) For $a,b \in \underline{A}$, siz$(a + b)$ = siz $a$ + siz $b$, siz$(ab)$ = (siz $a$)(siz $b$).

It is very easy to prove that for any element, y, of $\underline{A}$,

$$\text{rank } y \leqq \text{siz } y,$$

and that rank y = siz y, if and only if y is the sum of elements of that are products of elements of S.

The norm, $|\,|:\underline{A} \longrightarrow N$, is uniquely defined by the conditions,

i) For $x \in X$, $|X| = 1$,

ii) For $a, b \in \underline{A}$, $|a + b| = |ab| = |a| + |b|$.

The <u>additive decomposition</u>, Adec: $\underline{A} \longrightarrow \underline{A}^{[N]}$, is defined by the conditions,

i) For $x \in X$, Adec $x = x$,

ii) For $y, z \in \underline{A}$, Adec$(yz) = yz$,

iii) If Adec $a = (a_1,\cdots,a_r)$, Adec $b = (b_1,\cdots,b_s)$, then

$$\text{Adec}(a + b) = (a_1,\cdots,a_r,b_1,\cdots,b_s).$$

In a similar way, the <u>multiplicative decomposition</u>, Mdec:$\underline{A} \longrightarrow \underline{A}^{[N]}$, is defined by the conditions,

i) For $x \in X$, Mdec $x = x$,

ii) For $y, z \in \underline{A}$, Mdec$(y + z) = y + z$,

iii) If Mdec $a = (a_1,\cdots,a_r)$, Mdec $b = (b_1,\cdots,b_s)$, then

Mdec$(ab) = (a_1,\cdots,a_r,b_1,\cdots,b_s)$.

The <u>additive pattern of the top</u>, Apt:$\underline{A} \longrightarrow \underline{A}$, is defined by the conditions,

i) For $x, y \in \underline{A}$, Apt$(x + y)$ = Apt $x$ + Apt $y$,

ii) For $x \in \underline{A}$, if Adec $x = x$, then, Apt $x = x_1$.

In a similar way, the <u>multiplicative pattern of the top</u>, Mpt:$\underline{A} \longrightarrow \underline{A}$, is defined by the conditions,

i) For $x,y \in \underline{A}$, Mpt$(xy)$ = (Mpt $x$)(Mpt $y$),

ii) For $x \in \underline{A}$, if Mdec $x = x$, then, Mpt $x = x_1$.

<u>Proposition 1</u>

For any elements a and b of $\underline{A}$ we have the following relations:

i) Apt a = Apt b $\wedge$ Adec a = Adec b $\Longrightarrow$ a = b.

ii) Mpt a = Mpt b $\wedge$ Mdec a = Mdec b $\Longrightarrow$ a = b.

<u>Proof</u>:

It will be sufficient to prove one of the relations, say i). If Apt a = Apt b = $x_1$, then, a = Adec a = Adec b = b, and the relation is proved. Suppose now that,

$$\text{Apt } a = \text{Apt } b = x + y ,$$

$$\text{Adec } a = \text{Adec } b = (c_1, \cdots, c_t) .$$

Then it is immediate that if $|x| = r$, then, $a = a' + a''$, $b = b' + b''$ with

$$\text{Apt } a' = \text{Apt } b' = x, \quad \text{Apt } a'' = \text{Apt } b'' = y,$$

$$\text{Adec } a' = \text{Adec } b' = (c_1, \cdots, c_r)$$

$$\text{Adec } a'' = \text{Adec } b'' = (c_{r+1}, \cdots, c_t) .$$

From these facts, the proof of the proposition by induction on $|\text{Apt } a|$ is immediate.

Let $\underline{A}^*$ be the free $\{+,.\}$-algebra over X, with associativity and commutativity for . and +, distributivity of . relatively to +, null element n, identity element u, and the additional condition, $na = an = n$ for $a \in \underline{A}^*$. $\underline{A}^*$ is a strict algebra and the identity map of X defines a $\{+,.\}$-morphism, called the support, $\text{Supp}:\underline{A} \longrightarrow \underline{A}^*$. That means that the support is defined by the following conditions:

i) If $x \in X$, $\text{Supp } x = x \in \underline{A}^*$ ,

ii) If $x,y \in \underline{A}$, $\text{Supp}(x + y) = \text{Supp } x + \text{Supp } y$,

iii) If $x,y \in \underline{A}$, $\text{Supp}(xy) = (\text{Supp } x)(\text{Supp } y)$.

An element a of $\underline{A}$ is defined to be <u>regular</u> if Supp a can be expressed as a sum of different elements of $\underline{A}^*$ each of which is a product of different elements of X. In any concrete case this definition can be easily checked, but we shall present later (Proposition 3) another simple case in which the regularity of an element can immediately be asserted.

<u>Proposition 2</u>.

Suppose $a \xrightarrow{\theta} b$, that is, assume the existence of a path from a to b. Then, a is regular if and only if b is regular.

<u>Proof</u>:

It is easy to prove that, $a \longrightarrow b \Longrightarrow \text{Supp } a = \text{Supp } b$, and hence,

$a \xrightarrow{\theta} b \Longrightarrow$ Supp a = Supp b, and this relation immediately proves the proposition.

Define the elemental components, Ecomp:$\underline{A} \longrightarrow \mathcal{P}(X)$, the power set of X, by the conditions:

i) If $x \in X$, Ecomp $x = \{x\}$,

ii) For a, b $\in \underline{A}$, Ecomp(x + y) = Ecomp(xy) = Ecomp x $\cup$ Ecomp y.

PROPOSITION 3

Suppose that a is an element of $\underline{A}$ such that any element of X appears at most once in the expression of a. Then, a is regular.

Proof.

The first thing to prove is that if x and y are regular elements of $\underline{A}$ such that, Ecomp x $\cap$ Ecomp y = $\emptyset$, then xy and x + y are also regular elements and this is routine.This fact allows us to prove immediately the proposition by induction on $|a|$, because if a = xy or a = x + y, then, the proposition hypothesis implies, Ecomp x $\cap$ Ecomp y = $\emptyset$.

Observe that if a is not a regular element, it is possible to find a path, $a \xrightarrow{\theta} b$, where b involves a situation of type x + x or xx: as it has been noted in the counterexample included in §2, this type of element originates an "incoherent" diagram in some usual cases.

## §4. The concept of reduction

Let a be an element of $\underline{A}$. A reduction of a is a path $a \xrightarrow{\theta} a'$ such that,

i) Every step in the path is an instantiation of $\lambda^*$, $\rho^*$, $\lambda'$, $\rho'$ or an identity.

ii) a' = n or there is no occurrence of n in the expression of a'.

Note that the condition ii) is equivalent to say that a' is not the origin of an instantiation of $\lambda^*$, $\rho^*$, $\lambda'$, or $\rho'$. Intuitively speaking a reduction of a is any path obtained by elimination of n in a by means of $\lambda^*$, $\rho^*$, $\lambda'$ and $\rho'$.

PROPOSITION 4

Let a be an element of $\underline{A}$. Then, there exists a reduction $a \xrightarrow{\theta} a'$ of a, a' is uniquely determined by a and if $\underline{C}$ is coherent the value of the reduction is unique.

Proof:

The proof of the existence of a reduction of a can be done immediately by induction on rank a.

For the proof of the uniqueness of a' we have to state some preliminary relations:

1) Supp $a = n \Longleftrightarrow a' = n$.

It is clear that Supp a = Supp a'; hence if Supp a = n, then, a' = n because otherwise the expression of a' and also of Supp a' would involve no occurrence of n.

2) If $a = a_1 + a_2$ and $a_1 \xrightarrow{\theta} a_1'$, $a_2 \xrightarrow{\theta} a_2'$ are reductions we have,

$$a_1' \neq n \wedge a_2' \neq n \Longrightarrow a' = a_1' + a_2' \quad ,$$

$$a_1' = n \wedge a_2' \neq n \Longrightarrow a' = a_2' \quad ,$$

$$a_1' \neq n \wedge a_s' = n \Longrightarrow a' = a_1' \quad .$$

The proof of the above assertion can be done very easily by induction on rank a.

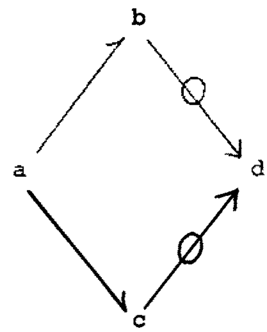
3) If $a = a_1 a_2$ and $a_1 \xrightarrow{\theta} a_2'$, $a_2 \xrightarrow{\theta} a_2'$ are reductions we have,

$$a_1' \neq n \wedge a_2' \neq n \Longrightarrow a' = a_1' a_2' \quad .$$

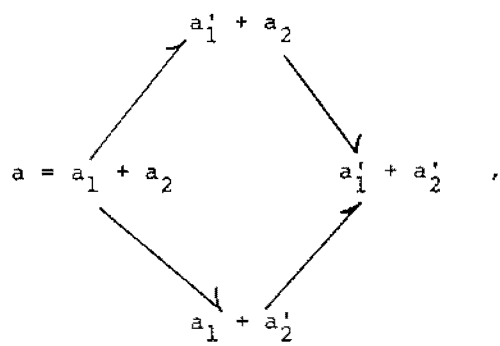
The proof is similar to the proof of 2).

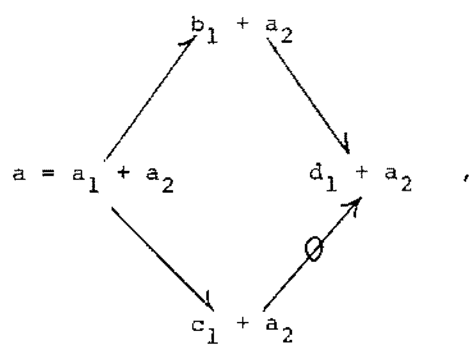
The above three assertions allow us to prove immediately the uniqueness of a' by induction on $|a|$.

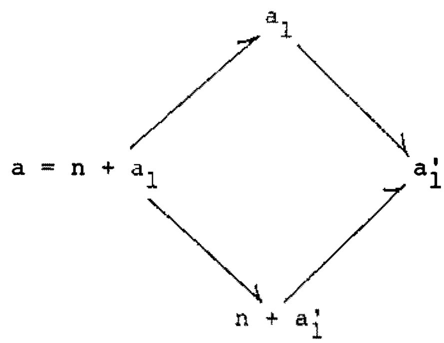
Suppose that $a \longrightarrow b$ and $a \longrightarrow c$ are instantiations of $\lambda^*$, $\rho^*$, $\lambda'$ or $\rho'$ and that $\underline{C}$ is coherent; as a preliminary step to end the proof of the proposition we need to prove the existence of a commutative diagram of type



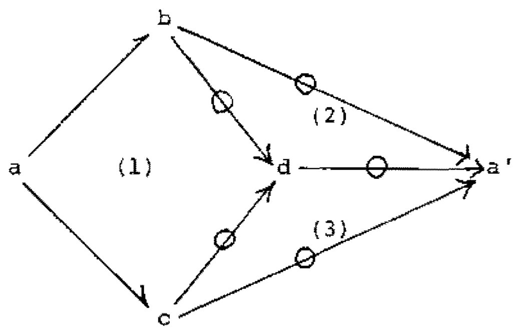
such that any step in b $\xrightarrow{\theta}$ d and c $\xrightarrow{\theta}$ d is an identity or an instantiation of $\lambda^*$, $\rho^*$, $\lambda'$, or $\rho'$. The proof is a routine induction on $|a|$ outlined by the following diagrams (and other analogous diagrams).








Now we can prove the uniqueness of the value of reduction of a by induction on $|a|$: if $a \longrightarrow b \xrightarrow{\theta} a'$ and $a \longrightarrow c \xrightarrow{\theta} a'$ are two reductions of a we can construct a commutative diagram

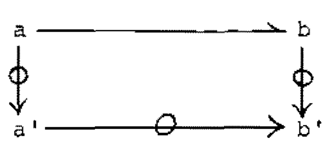



,

where $b \xrightarrow{\theta} a'$, $d \xrightarrow{\theta} a'$ and $c \xrightarrow{\theta} a'$ are reductions, (1) has been taken commutative following the above result and (2) and (3) are commutative by the induction hypothesis. If a reduction $a \xrightarrow{\theta} a'$ is a sequence of identities, the above argument does not apply, but in this case any reduction is a sequence of identities and the last part of the proposition is trivial.

<u>PROPOSITION 5</u>

Let a and b be elements of <u>A</u>, $a \longrightarrow b$ an edge of <u>T</u> and suppose <u>C</u> is coherent. Then, there exists a commutative diagram of type,



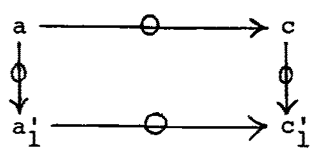

,

where $a \xrightarrow{\theta} a'$ and $b \xrightarrow{\theta} b'$ are reductions and no step in $a' \xrightarrow{\theta} b'$ is an instantiation of $\lambda^*$, $\rho^*$, $\lambda'$, $\rho'$ or their inverses.

<u>Proof</u>:

Remark that the proposition is immediate when $a \longrightarrow b$ is an identity or an instantiation of $\lambda^*$, $\rho^*$, $\lambda'$, $\rho'$ or their inverses because proposition 4 allows us to choose the reductions in the most suitable way for our purposes. If $a' = a$, that is, if an identity is a reduction of a and we are not in the preceding case, then an identity is also a reduction of b and the proposition is immediate.

For the general case we need to prove a preliminary statement: suppose that $a \xrightarrow{\theta} c$ is a path with no instantiation of $\lambda^*$, $\rho^*$, $\lambda'$, $\rho'$ or their inverses and such that an identity is not a reduction of a. Then, we are going to prove the existence of a commutative diagram of type




where $a \xrightarrow{\theta} a_1'$ is a sequence, with at least one element, of instantiations of $\lambda^*$, $\rho^*$, $\lambda'$ or $\rho'$, $c \xrightarrow{\theta} c_1'$ is a sequence of identities or instantiations of $\lambda^*$, $\rho^*$, $\lambda'$ or $\rho'$ and in $a_1' \xrightarrow{\theta} c_1'$ there are no instantiations of $\lambda^*$, $\rho^*$, $\lambda'$, $\rho'$ or their inverses.

Observe that one consequence of the conditions of the above statement is that $|a_1'| < |a|$ and that if an identity is a reduction of $a_1'$ then any vertex in the path $a_1' \xrightarrow{\theta} c_1'$ has an identity as a reduction (because in it there is no instantiation of $\lambda^*$, $\rho^*$, $\lambda'$, $\rho'$ or their inverses). This preliminary statement can be proved by induction on $|a|$ following the method outlined in the following diagrams and their analogons:

$$\begin{array}{ccc} a = x + y & \longrightarrow & x' + y = b \\ \phi\big\downarrow & & \big\downarrow\phi \\ x_1 + y & \longrightarrow & z_1 + y \end{array} \quad ,$$

$$\begin{array}{ccc} a = x + n & \longrightarrow & x' + n = b \\ \big\downarrow & & \big\downarrow \\ x & \longrightarrow & x' \end{array} \quad ,$$

$$\begin{array}{ccc} a = x + (y + z) & \longrightarrow & (x + y) + z \\ \big\downarrow & & \big\downarrow \\ x' + (y + z) & \longrightarrow & (x' + y) + z \end{array} \quad ,$$

$$\begin{array}{ccc} a = x(y + z) & \longrightarrow & xy + xz \\ \big\downarrow & & \big\downarrow\phi \\ x'(y' + z') & \longrightarrow & x'y' + x'z' \end{array} \quad .$$

From this it is immediate to prove by induction on $|a|$ that if in the path $a \xrightarrow{\theta} b$ there is no instantiation of $\lambda^*$, $\rho^*$, $\lambda'$, $\rho'$ or their inverses, then, for $a \xrightarrow{\theta} a'$ and $b \xrightarrow{\theta} b'$ reductions there is a commutative diagram of type

$$\begin{array}{ccc} a & \xrightarrow{\quad\theta\quad} & b \\ \phi\big\downarrow & & \big\downarrow\phi \\ a' & \xrightarrow{\quad\theta\quad} & b' \end{array} \quad ,$$

where in $a' \xrightarrow{\theta} b'$ there is no instantiation of $\lambda^*$, $\rho^*$, $\lambda'$, $\rho'$ or their inverses. This statement includes all the cases in which the proposition was not proved yet.

Note that we have not used the hypothesis that the arrows of distributivity are monomorphisms and that an immediate consequence of the above proposition is that if for some element, a of $\underline{A}$, Supp a = n, then the value of any path from a to b depends only upon a and b.

## §5. The concept of rappel

Let a be an element of A. A rappel of a is a path a $\xrightarrow{\Theta}$ a' such that,

i) Each step in a $\xrightarrow{\Theta}$ a' is an identity or an instantiation of $\delta$ or $\delta^{\#}$.

ii) There is no instantiation of $\delta$ or $\delta^{\#}$ with origin in a'.

Intuitively speaking, a rappel of a is a path with origin in a obtained by application, as many times as possible, of the distributive law. It is easy to check that condition ii) is equivalent to stating that a is the sum of elements that are product of elements of X.

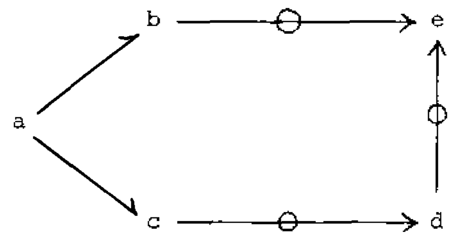
We have to remark that the end of a rappel is not uniquely determined by the origin: thus, it is easy to prove the existence of two rappels with origin in $(x_1 + x_2)(x_3 + x_4)$ ending in the elements $(x_1x_3 + x_2x_3) + (x_1x_4 + x_2x_4)$ and $(x_1x_3 + x_1x_4) + (x_2x_3 + x_2x_4)$. We will see that this difficulty is easy to handle.

In this paragraph we are going to use often induction on siz a-rank a. Note that this number is always non-negative and that any instantiation of $\alpha$, $\alpha'$, their inverses, $\gamma$, and $\gamma'$ preserves the size and rank and that any instantiation of $\delta$ or $\delta^{\#}$ preserves the size and increases the rank, that is, the value of siz a - rank a decreases by instantiations of $\delta$ or $\delta^{\#}$: this fact can be used to prove by induction on siz a - rank a the existence of a rappel for the element a.

For any element a of A it is easy to prove that an identity path a $\xrightarrow{\Theta}$ a is a rappel if and only if rank a = siz a, and that this is equivalent to stating that a is a sum of products of elements of X.

PROPOSITION 6.

Suppose that a $\longrightarrow$ b is not an instantiation of $\lambda^*$, $\rho^*$, $\lambda'$, $\rho'$ or their inverses and that a $\longrightarrow$ c is an instantiation of $\delta$ or $\delta^{\#}$. Then if C is coherent there exists a commutative diagram of type

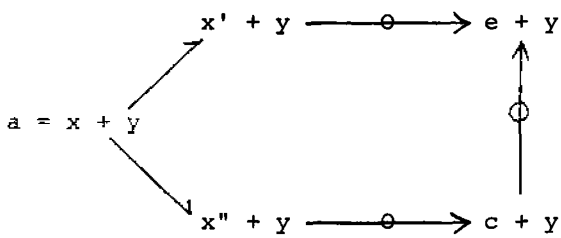


such that $d \xrightarrow{\Theta} e$ is a sequence of identities or instantiations of $\alpha$, $\alpha'$, $\lambda$, $\rho$, their inverses, $\gamma$, and $\gamma'$, while $b \xrightarrow{\Theta} e$, $c \xrightarrow{\Theta} d$ are sequences of instantiations of $\delta$ and $\delta^{\#}$. Moreover in $d \xrightarrow{\Theta} e$ there is some instantiation of $\lambda$, $\rho$, or their inverses if and only if $a \longrightarrow b$ is an instantiation of the same type.

<u>Proof</u>:

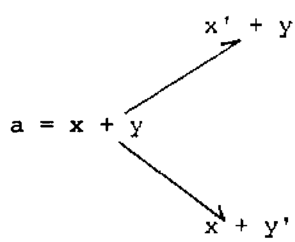
The proof can be done by induction on $|a|$ in the form outlined by the following diagrams.

1) In the case

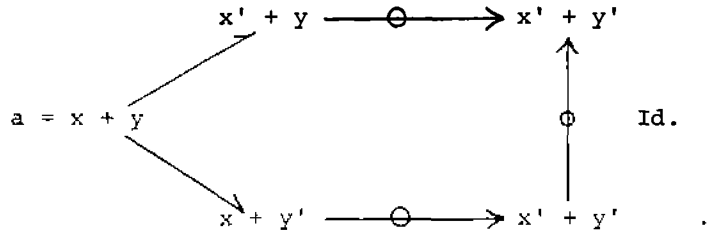


we use the induction hypothesis.

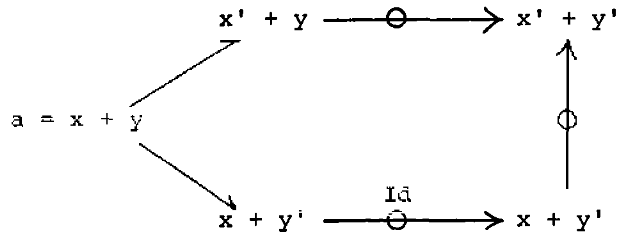
2) In the situation given by



there are two different cases. If $x + y \longrightarrow x' + y$ is an instantiation of $\delta$ or $\delta^{\#}$ we can use the construction given by

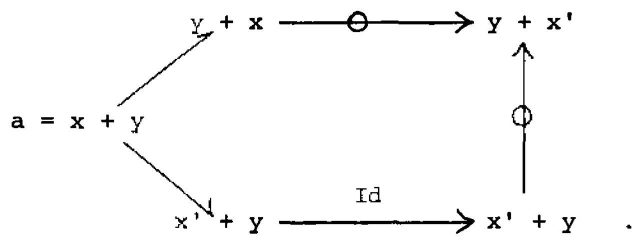



Otherwise, we can take the construction given by

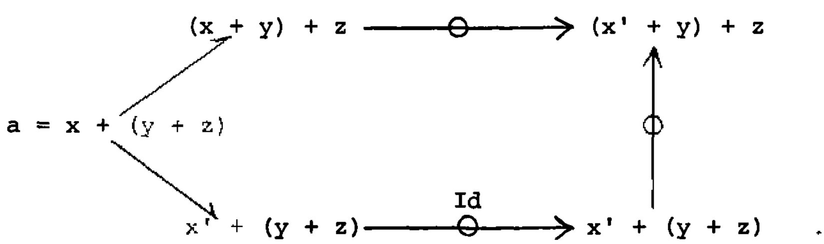



In both constructions we make use of the naturality of ⊕.

3) The naturality of γ allows us to make the construction given by




4) The naturality of γ allows the following construction




We omit the analogous cases for the product.

5) The commutativity of the diagrams of type (VIII) of the coherence conditions is used in the following construction

$$\begin{array}{ccccc} & & x[(y + z)w] & \xrightarrow{\theta} & x(yw) + x(zw) \\ & \nearrow & & & \uparrow \phi \\ a = [x(y + z)]w & & & & \\ & \searrow & & & \\ & & (xy + xz)w & \xrightarrow{\theta} & (xy)w + (xz)w \end{array} .$$

We omit the analogous cases in which we should use the commutativity of (VI) and (VII).

6) The commutativity of the diagrams of type (II) is used in the following construction

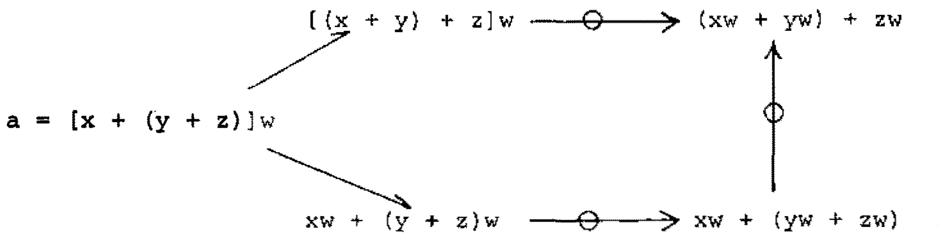
$$\begin{array}{ccccc} & & (y + z)x & \xrightarrow{\theta} & yx + zx \\ & \nearrow & & & \uparrow \phi \\ a = x(y + z) & & & & \\ & \searrow & & & \\ & & xy + xz & \xrightarrow{\mathrm{Id}} & xy + xz \end{array} .$$

We omit the analogous cases in which we should use the commutativity of (I) and (III).

7) The commutativity of the diagrams of type (IX) is used in the following construction

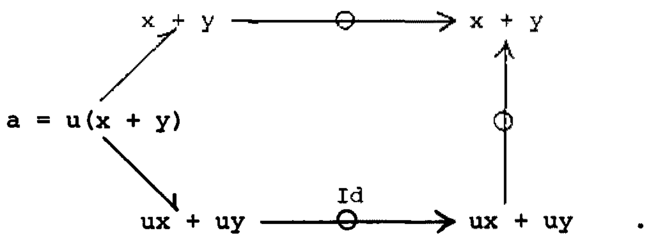
$$\begin{array}{ccccc} & & (x + y)z + (x + y)w & \xrightarrow{\theta} & (xz + yz) + (xw + yw) \\ & \nearrow & & & \uparrow \phi \\ a = (x + y)(z + w) & & & & \\ & \searrow & & & \\ & & x(z + w) + y(z + w) & \xrightarrow{\theta} & (xz + xw) + (yz + yw) \end{array} .$$

8) The commutativity of the diagrams of type (IV) is used in the following construction



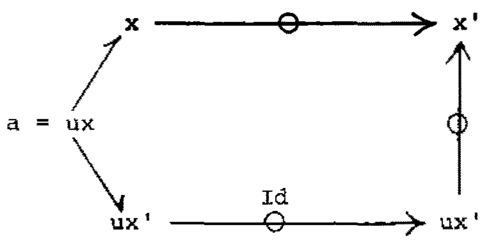

We omit the analogous cases in which we should use the commutativity of (V).

9) We use the commutativity of the diagrams of type (XXIII) in the following construction




We omit the analogous case in which we should use the commutativity of (XXIV).

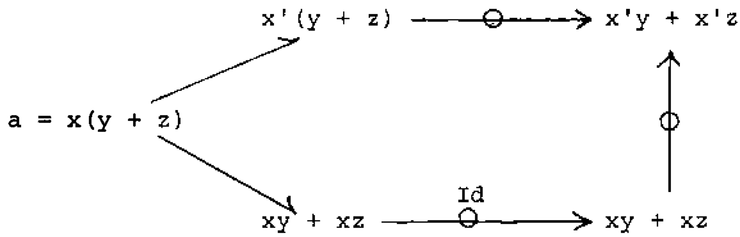
10) In the construction




We are using the naturality of $\lambda$.

We omit the analogous case in which we should use the naturality of $\rho$.

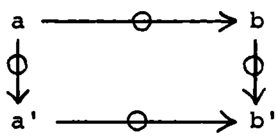
11) We use the naturality of $\delta$ in the construction given by




We omit the analogous cases in which we should use the naturality of $\delta$ and $\delta^{\#}$.

PROPOSITION 7

Suppose that $\underline{C}$ is coherent, that a $\xrightarrow{\circ}$ b is a path in whose vertices there is no occurrence of n and that a $\xrightarrow{\circ}$ a' and b $\xrightarrow{\circ}$ b' are rappels. Then there exists a commutative diagram of type

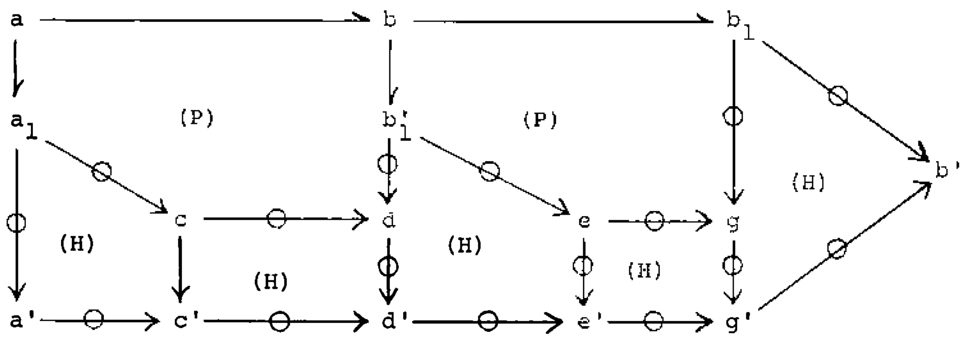



such that a' $\xrightarrow{\circ}$ b' is a sequence of identities or instantiations of $\alpha$, $\alpha'$, $\lambda$, $\rho$, their inverses, $\gamma$ and $\gamma'$.

Proof:

The exclusion of n in the vertices of a $\xrightarrow{\circ}$ b implies that in a $\xrightarrow{\circ}$ b there is no instantiation of $\lambda^*$, $\rho^*$, $\lambda'$, $\rho'$ or their inverses and then the form of the proposition allows us to reduce to the case in which a $\xrightarrow{\circ}$ b is an identity or an instantiation of $\lambda$, $\rho$, $\alpha$, $\alpha'$, their inverses, $\gamma$, $\gamma'$, $\delta$ or $\delta^{\#}$, and this will be done in three parts.

The first part will be proved by induction on siz a - rank a and studies the case in which a $\xrightarrow{\circ}$ b is an identity or an instantiation of $\alpha$, $\alpha'$, their inverses, $\gamma$ and $\gamma'$, in which case, siz a - rank a = siz b - rank b. The case siz a - rank a = siz b - rank b = 0 is trivial; otherwise we can use the diagram




in which $a \longrightarrow a_1 \xrightarrow{\Theta} a'$ and $b \longrightarrow b_1 \xrightarrow{\Theta} b'$ are the given rappels, the diagrams with the symbol (H) have been constructed by the induction hypothesis and the ones with the symbol (P) are constructed using proposition 6. Note that the possibility of the decomposition of the path $b \longrightarrow b_1' \xrightarrow{\Theta} d$ in proposition 6 is assured by the fact that siz a - rank a = siz b - rank b, which implies

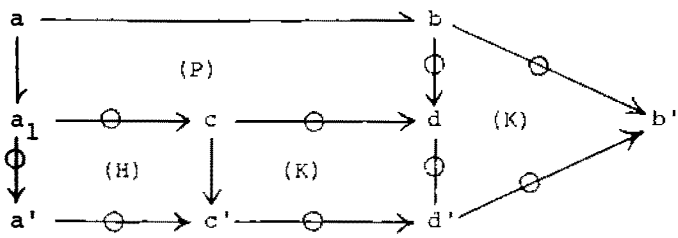
$$\text{siz } d - \text{rank } d \leq \text{siz } a_1 - \text{rank } z_1 < \text{siz } a - \text{rank } a.$$

Remark also that to do the induction we have to impose the additional condition that $a' \xrightarrow{\Theta} b'$ is a sequence of identities or instantiations of $\alpha$, $\alpha'$, their inverses, $\gamma$ and $\gamma'$.

The second part is going to be the proof of the proposition when $a \longrightarrow b$ is an instantiation of $\lambda$, $\rho$, $\delta$ or $\delta^{\#}$, and this will be done by induction on siz a - rank a. Remark that for any a of $\underline{A}$,

$$\text{siz(au)} - \text{rank(au)} = \text{(siz a)(siz u)} - \text{rank a} - \text{rank u} =$$
$$\text{siz a} - \text{rank a} + \text{siz a} - \text{rank u},$$

where siz a - rank u = 0 if and only if $a \in X$. Hence, in this case, within trivial exception, we can suppose that, siz a - rank a > siz b - rank b, and the proof is outlined in the following diagram




where the symbol (H) in the inside of a diagram means that the induction hypothesis is the reason for the commutativity, (K) the first part of this proposition and the induction hypothesis, and (P) the proposition 6.

The third part is going to be the proof of the proposition for the case in which $a \longrightarrow b$ is an instantiation of $\lambda^{-1}$ or $\rho^{-1}$ and this is an immediate consequence of the second part and the fact that the bottom path is a sequence of instantiations with formal inverse whose value is an isomorphism.

## §6. The concept of normalization

Let a be an element of <u>A</u>. A <u>normalization</u> of a is a path $a \xrightarrow{\ominus} a'$ satisfying the following conditions:

i) Any step in $a \xrightarrow{\ominus} a'$ is an identity or an instantiation of $\lambda$ or $\rho$.

ii) a' is not the origin of any instantiation of $\lambda$ or $\rho$.

Intuitively speaking a normalization is a path obtained by application, as many times as possible, of instantiations of $\lambda$ and $\rho$. It is easy to prove that if a is the end of a rappel the condition ii) is equivalent to the following: a is the sum of elements that are either u or the product of elements of X different from u. In the general case it is not possible to give a simple characterization of the elements satisfying ii).

The concept of normalization is similar to the concept of reduction or rappel, but it is only useful when applied to elements that are ends of rappels because in this case it eliminates almost completely the occurrences of u in the expression of the elements. In the general case one typical situation is the following: an identity is a normalization of the element $x_1(u + x_2)$, but $x_1(u + x_2) \longrightarrow x_1u + x_1x_2$ and an identity is not a normalization of $x_1u + x_1x_2$ for which a normalization is the path

$$x_1u + x_1x_2 \longrightarrow x_1 + x_1x_2$$

that in fact eliminates all the occurrences of u in the expression of the element.

<u>PROPOSITION 8</u>

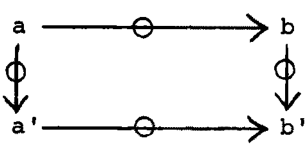
Suppose that $\underline{C}$ is coherent and that a is the end of a rappel. Then if $a \xrightarrow{\Theta} a'$ is a normalization, the element a' and the value of $a \xrightarrow{\Theta} a'$ are uniquely determined by a.

<u>Proof</u>:

The proof is similar to (and simpler than) the proof of proposition 4.

<u>PROPOSITION 9</u>

Let a and b be elements of $\underline{A}$ that are ends of rappels, $a \xrightarrow{\Theta} b$ a path whose steps are instantiations of $\alpha$, $\alpha'$, $\lambda$, $\rho$, their inverses, $\gamma$ and $\gamma'$ and $a \xrightarrow{\Theta} a'$, $b \xrightarrow{\Theta} b'$ normalizations of a and b respectively. If $\underline{C}$ is coherent, there exists a commutative diagram of type




such that $a' \xrightarrow{\Theta} b'$ is a sequence of identities and instantiations of $\alpha$, $\alpha'$, their inverses, $\gamma$ and $\gamma'$.

<u>Proof</u>:

It is analogous to (and simpler than) the proof of proposition 5.

Suppose that a is the end of a rappel and that $a \xrightarrow{\Theta} a'$ is a normalization. If a is regular so is a' and if Adec a = $(a_1', \cdots, a_r')$

then if $i \neq j$ the set of factors of $a_i$ is different from the set of factors of $a_j$ as is an immediate consequence of the definition of regularity, and, moreover, among the factors of any $a_i$ there is no repetition of elements, as can be also proved almost immediately.

## §7. The coherence theorem

We are going to use the results on coherence stated in the Theorem 4.2 of [4], but expressed in a more formal language. We omit a complete proof of the equivalence that is neither difficult nor specially illuminating: in fact it reduces to the same proof given in [4] that holds in the formulation we are going to give. We have to remark that this formulation is different from the one contained in §3 of [2].

Let $\underline{A}'$ be the subset of $\underline{A}$ generated additively by $X - \{n\}$: $\underline{A}'$ is the free $\{+\}$-algebra over $X - \{n\}$. The edges of $\underline{H}$ are a $\{+,.\}$-algebra and hence a $\{+\}$-algebra, and we take as $\underline{H}'$ the subgraph of $\underline{H}$ whose edges are all the elements of the + -subalgebra of the edges of $\underline{H}$ generated by all the elements of the form, $\alpha'_{x,y,z}, \alpha'^{-1}_{x,y,z}, \gamma'_{x,y}$ and $1_x$ for x,y,z elements of $\underline{A}'$. Suppose that $a \overset{\Theta}{\longrightarrow} b$ is a path whose vertices are in $\underline{A}'$ and whose steps are elements of $\underline{H}' \cap \underline{T}$, then the Theorem 4.2 of [4] states that if a is regular and $\underline{C}$ is coherent for $\{\alpha', \gamma', \lambda', \rho'\}$ the value of the path $a \overset{\Theta}{\longrightarrow} b$ only depends upon a and b. Note that an element of $\underline{A}'$ is regular if and only if it is the sum of different elements of X.

We are going to deduce some consequences of that result. Let a be a regular element of $\underline{A}$ in which there is no occurrence of n and suppose that $a \overset{\Theta}{\longrightarrow} c$ is a path whose steps are identities or instantiations of $\alpha'$, $\alpha'^{-1}$ or $\gamma'$. If Adec $a = (a_1, \cdots, a_r)$, the regularity of a implies the relation, $i \neq j \Longrightarrow a \neq a$ and from this and the coherence result above it follows that the value of $a \overset{\Theta}{\longrightarrow} c$ only depends upon a and c.

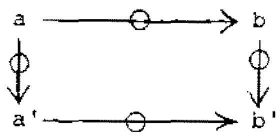
Similar consequences hold for the product.

PROPOSITION 10 (Coherence theorem)

If $\underline{C}$ is coherent and a is a regular element of $\underline{A}$, the value of any path a $\xrightarrow{\Theta}$ b depends only upon a and b.

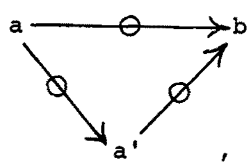
Proof:

Let a $\xrightarrow{\Theta}$ a' and b $\xrightarrow{\Theta}$ b' be reductions. By proposition 5 it is possible to find a commutative diagram of type



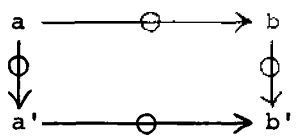

where the value of the columns are isomorphisms that only depend upon a and b, in a' $\xrightarrow{\Theta}$ b' there is no occurrence of instantiations of $\lambda'$, $\rho'$, $\lambda^*$, $\rho^*$ or their inverses, and where all the vertices are n or no n is in the vertices of a' $\xrightarrow{\Theta}$ b'. Hence we are reduced to proving the proposition when in a $\xrightarrow{\Theta}$ b there is no instantiation of $\lambda'$, $\rho'$, $\lambda^*$, $\rho^*$ or their inverses and where the symbol n is not involved in the expression of the vertices: from now on we are going to assume these hypotheses on a $\xrightarrow{\Theta}$ b.

Take now a rappel b $\xrightarrow{\Theta}$ b': the value of it is a monomorphism, hence we are reduced to prove the uniqueness of the value of any path a $\xrightarrow{\Theta}$ b $\xrightarrow{\Theta}$ b', that is, we can (and will) assume the additional hypothesis that b is the end of a rappel. Let a $\xrightarrow{\Theta}$ a' be a rappel: By proposition 7 there is a commutative diagram of type




where a' $\xrightarrow{\Theta}$ b is a path with no occurrence of instantiations of $\delta$ or $\delta^{\#}$, and we are reduced to prove the uniqueness of the value of a' $\xrightarrow{\Theta}$ b, that is, we are going to assume that a and b are ends of rappels.

Suppose now that a $\xrightarrow{\ominus}$ a' and b $\xrightarrow{\ominus}$ b' are normalizations. By propositions 8 and 9 there exists a commutative diagram of type

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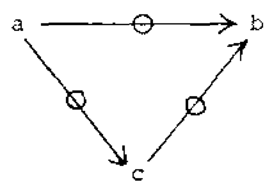
where a', b' and the values of a $\xrightarrow{\ominus}$ a' and b $\xrightarrow{\ominus}$ b' depend only upon a and b, and the fact that the values of the columns are isomorphisms allows us to reduce our considerations to the uniqueness of the value of the path a' $\xrightarrow{\ominus}$ b', that satisfies the conditions indicated in proposition 9. Hence, we are reduced to proving the proposition for the following conditions:

i) Every step in a $\xrightarrow{\ominus}$ b is an identity or an instantiation of $\alpha$, $\alpha'$, their inverses, $\gamma$ and $\gamma'$.

ii) Any vertex in the path a $\xrightarrow{\ominus}$ b is a sum of elements each of which is either or a product of elements of X different from u.

The naturality of $\otimes$ and $\oplus$ implies that any instantiation of $\alpha$, $\alpha^{-1}$ or $\gamma$ is commutative with any instantiation of $\alpha'$, $\alpha'^{-1}$ or $\gamma'$ and this proves the existence of a commutative diagram of type




such that in a $\xrightarrow{\ominus}$ c every step is an identity or an instantiation of $\alpha'$, $\alpha'^{-1}$ or $\gamma'$, and every step in c $\xrightarrow{\ominus}$ b is an identity or any instantiation of $\alpha$, $\alpha^{-1}$ or $\gamma$. Our next aim is to prove the uniqueness of c. For this, note the following relations:

1) If d $\longrightarrow$ e is an instantiation of $\alpha$, $\alpha^{-1}$ or $\gamma$, then Apt d = Apt e. This can be proved very easily by induction on $|d|$. Form this it follows that Apt c = Apt b.

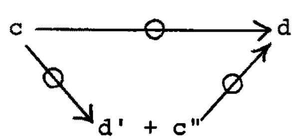
2) If $d \longrightarrow e$ is an instantiation of $\alpha'$, $\alpha'^{-1}$ or $\gamma$, and $d$ is the end of a rappel, and Adec $d = (d_1, d_2, \cdots, d_r)$, then, Adec $e = (d_{\sigma 1}, d_{\sigma 2}, \cdots, d_{\sigma r})$, with $\sigma \in S_r$. This can be proved by induction on $|d|$. From this it follows that if Adec $a = (a_1, a_2, \cdots, a_r)$, then, Adec $c = (a_{\sigma 1}, \cdots, a_{\sigma r})$ for some $\sigma \in S_r$ and, as we will see later, $b$ determines $\sigma$ uniquely.

3) If $c \xrightarrow{\Theta} b$ is a sequence of instantiations of $\alpha$, $\alpha^{-1}$ and $\gamma$, $c$ is the end of a rappel, Adec $c = (c_1, \cdots, c_r)$, and Adec $b = (b_1, \cdots, b_r)$, then for $i = 1, 2, \cdots, r$, there is a path $a_i \xrightarrow{\Theta} b_i$ whose steps are identities or instantiations of $\alpha$, $\alpha^{-1}$ or $\gamma$.

From this it follows that for $i = 1, 2, \cdots, r$, $a_{\sigma i} \xrightarrow{\Theta} b_i$, and hence, Supp $a_{\sigma i}$ = Supp $b_i$. But the regularity of $a$ imposes that, $i \neq j \Longrightarrow$ Supp $a_i \neq$ Supp $a_j$, and this proves that $\sigma i$ is uniquely determined by the condition, Supp $a_{\sigma i}$ = Supp $b_i$. Thus $b$ and $a$ determine uniquely Adec $c$ and Apt $c$ and by proposition 1 the element $c$ is uniquely defined.

The uniqueness of the value of $a \xrightarrow{\Theta} c$ has been stated in the remarks of the beginning of §7.

The only thing that remains to be proved is the uniqueness of the value of the path $c \xrightarrow{\Theta} d$ in which all the steps are instantiations of $\alpha$, $\alpha^{-1}$ and $\gamma$. Suppose that $c = c' + c''$, then it is very easy to prove the existence of a commutative diagram of type,




such that in $c \xrightarrow{\Theta} d' + c''$ all the steps are elements of type $\Pi + 1_{c''}$ for some step $\Pi$ and in $d' + c'' \xrightarrow{\Theta} d$ all the steps are of type $1_{d'} + \Pi$, and with a trivial induction on $|c|$ we are reduced to the case in which $c$ is the product of elements of $X$, and the proof in this case is analogous to (and easier than) the proof of the uniqueness of the value of path $a \xrightarrow{\Theta} c$.

## REFERENCES

[1] G. M. Kelly, "On Mac Lane's conditions for coherence of natural associativities, commutativities, etc.", J. Algebra, 4(1964), 397-402.

[2] G. M. Kelly and S. Mac Lane, "Coherence in closed categories", J. Pure Appl. Algebra, 1(1971), 97-140.

[3] M. Laplaza, "Coherence for categories with associativity, commutativity and distributivity", Bull. Amer. Math. Soc. (to appear).

[4] S. Mac Lane, "Natural associativity and commutativity", Rice Univ. Studies, 49(1963), 28-46.

[5] S. Mac Lane, "Coherence and canonical maps", Symposia Mathematica, IV(1970), 231-241.