Summer Seminar - Mike Hopkins - 7/8/09

θ_j exists iff $h_j^2 \in \underset{ASS}{\mathrm{Ext}_A}(\mathbb{Z}/2, \mathbb{Z}/2)$ is permanent cycle (Haynes' talk)

$$\begin{array}{llll} \text{ANSS} & b & \mathrm{Ext}_{MU_*MU}(MU_*, MU_*) & \Longrightarrow \text{easier SS} \\ & \Big\downarrow \text{many-to-one} & \Big\Downarrow & \\ \text{ASS} & h_j^2 & \mathrm{Ext}_A & \end{array}$$

~~ANSS~~

<u>ANSS</u> $MU \rightrightarrows MU \wedge MU \mathrel{\substack{\to\\\to\\\to}} MU \wedge MU \wedge MU \equiv \quad \Big\Downarrow \pi_*$

standard resolution

$A \rightrightarrows \Gamma \equiv \Gamma \underset{A}{\otimes} \Gamma \equiv \Gamma \underset{A}{\otimes} \Gamma \underset{A}{\otimes} \Gamma$ cosimplicial ab gp $\Downarrow$ ring homomorphisms into R $\mathrm{Ring}(-, R)$ (simplicial)

$\left\{\begin{matrix}\text{Formal gp} \\ \text{laws } /R\end{matrix}\right\} \leftleftarrows \left\{\begin{matrix} F_0 \xleftrightarrow{f} F_1 \\ \text{two f.g.l + iso}\end{matrix}\right\} \equiv \{F_0 \to F_1 \to F_2\}$

= nerve of category of f.g.l $/R$

$F \circlearrowleft G$ (group acting on F.G.L F) $\{F\}$. $\{F \circlearrowleft G\}$

$\downarrow$

R (cat w/ 1 obj, G worth of morph)

$$\Longleftrightarrow \begin{array}{cccc} A & \Gamma & \Gamma \underset{A}{\otimes} \Gamma & \Gamma \underset{A}{\otimes} \Gamma \underset{A}{\otimes} \Gamma \\ \downarrow & \downarrow & \downarrow & \\ R & R^G & R^{G \times G} & \end{array} \quad \Leftrightarrow \begin{array}{c} \mathrm{Ext}_{MU_*MU}(MU_*, MU_*) \\ \downarrow \\ H^*(G; R) \end{array}$$

(Modeled after what Ravenel does on the paper on nonexistance of Arf invariant...)

Ex $R = \mathbb{Z}/2$, $F = G_a$ $\left\{ \right.$ $F(x,y) = x+y+\cdots = x +_F y$ s.t

$F(x,y) = x+y$

$$x +_F y = y +_F x, \quad 0 +_F y = y, \quad (x +_F y) +_F z = x +_F (y +_F z)$$

$\varphi : F_0 \longrightarrow F_1$ is power series $\varphi(x)$ s.t $\varphi(x +_{F_0} y) = \varphi(x) +_{F_1} \varphi(y)$

most general $\varphi : G_a \circlearrowleft$ $\quad \varphi(x) = x + \sum \xi_n x^{2^n}$ $\quad (\varphi(x+y) = \varphi(x) + \varphi(y))$

$$R^G = \mathbb{F}_2[\xi_1, \xi_2, \ldots]$$

$R \qquad R^G \qquad R^{G\times G}$

is standard resolution for calculating $\mathrm{Ext}_A(\mathbb{Z}/2, \mathbb{Z}/2)$ and that is how we get the map

$$\underset{\text{ANSS}}{\mathrm{Ext}_{MU_*MU}} \Longrightarrow \underset{\text{ASS}}{\mathrm{Ext}_A}$$

Ex (Lubin-Tate)

$A = \mathbb{Z}_2[\zeta]$ $\quad \zeta$ is an 8-th root of unity

$\pi = 1 - \zeta$

maximal ideal is principal $\qquad A/(\pi) = \mathbb{Z}/2$

$$l(x) = x + \frac{x^p}{\pi} + \cdots + \frac{x^{p^n}}{\pi^n} + \cdots \qquad (p=2 \text{ in our case})$$

Thm There is a unique power series $F(x,y) \in A[[x,y]]$ s.t.

$$l(F(x,y)) = l(x) + l(y) \in \mathbb{Q} \otimes A[[x,y]]$$

F is a FGL $\qquad$ (what's important here is the integrality)

Fact: $F \circlearrowright \underset{\underset{\mathbb{Z}/8}{\cup}}{A^\times}$ (act as automorphisms)

$A[\omega^{\pm}]$ $|\omega| = 2$ (graded ring)

$\zeta^{-1}: \omega \longmapsto \xi\omega$

$$\mathrm{Ext}_{MU_*MU}(MU_*, MU_*) \longrightarrow H^*(\mathbb{Z}/8, A[\omega^{\pm 1}])$$

Detection thm Any $b \in \mathrm{Ext}^2_{MU_*MU}(MU_*, MU_*)$ hitting $h_j^2 \in \mathrm{Ext}_A(\mathbb{Z}/2, \mathbb{Z}/2)$ has non-zero image in $H^2(\mathbb{Z}/8; A[\omega^{\pm 1}])$

Editorial segway:

Use an intermediate step that corresponds to a cohomology thy E st

$$\pi_* E = \mathbb{Z}_{(2)}[x_1, x_2, x_3, x_4, \Delta^{-1}] \circlearrowright \mathbb{Z}/8 \qquad \Delta = x_1 x_2 x_3 x_4$$

$|x_i| = 2$

$$x_1 \xrightarrow{\sigma} x_2 \xrightarrow{\sigma} x_3 \xrightarrow{\sigma} x_4 \xrightarrow{\sigma} x_1^{-1} \quad \text{(action)}$$

$$\begin{array}{ccc} \mathrm{Ext}_{MU_*MU}(MU_*, MU_*) & \longrightarrow & H^*(\mathbb{Z}/8; \pi_* E) \\ & & \downarrow \\ & & H^*(\mathbb{Z}/8; A[\omega^{\pm 1}]) \end{array}$$

Another example: • $\mathbb{Z}/_{(2)}[x_1, \Delta^{-1}] \circlearrowright \mathbb{Z}/2$ $\Delta = x_1$, $x_1 \to x_1^{-1}$,

this is like $K_* \circlearrowright \psi_{-1}$

• $\mathbb{Z}/2[x_1, x_2, (x_1x_2)^{-1}] \circlearrowright \mathbb{Z}/4$ $x_1 \to x_2 \to x_1^{-1}$ $\rightsquigarrow tmf(5)$

(detection thm fails for these two)

Periodicity Thm $\pi_i E^{h\mathbb{Z}/8} = \pi_{i+256} E^{h\mathbb{Z}/8}$

Gap Thm $\pi_{-2} E^{h\mathbb{Z}/2} = 0$

These three together prove the Kervaire invariant problem.

They really use $\Delta^{-1} MU \wedge MU \wedge MU \wedge MU \circlearrowleft \mathbb{Z}/8$

$$(a, b, c, d) \longmapsto (\bar{d}, a, b, c)$$

$$\pi_* MU \wedge MU \wedge MU \wedge MU = \mathbb{Z}_{(2)}[\overset{x_1}{r_1}, \sigma_* r_1, \sigma^2 r_1, \ldots, \overset{x_2}{r_2}, \sigma r_2, \ldots, r_3, \ldots]$$

$$\sigma^4 r_i = -r_i, \quad |r_i| = 2i$$

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h_j^2

$$\mathbb{F}_2 \rightrightarrows \mathbb{F}_2[\xi_1, \xi_2, \ldots] \rightrightarrows \mathbb{F}_2[\xi_1, \xi_2 \ldots] \otimes \mathbb{F}_2[\xi_1, \xi_2 \ldots]$$

$$\xi_i \longrightarrow \xi_i \otimes 1$$

$$1 \otimes \xi_i$$

$$\xi_i \otimes \xi_j^{2i} \quad \text{(composition)}$$

$$i+j = n$$

(using $x + \xi_1 x^2 + \xi_2 x^4 + \ldots$)

ξ_1 $\quad$ $\xi_1 \otimes 1$, $1 \otimes \xi_1 + \xi_1 \otimes 1$, $\xi_2 1 \otimes \xi_1$) alternating sum $= 0$ $\quad h_j = [\xi_1^2]$

$h_j = |\xi_1^{2^j}| \in \mathrm{Ext}_A^{1}(\mathbb{Z}/2, \mathbb{Z}/2)$

$h_j^2 = \xi_1^{2^j} \otimes \xi_1^{2^j} = [\xi_1^{2^j} | \xi_1^{2^j}]$

In $MU \wedge MU$

$A \qquad \Gamma$

$\qquad\quad A[b_1, b_2, \ldots]$

(nonstandard notation) $\quad \varphi(x) = x + b_1 x^2 + b_2 x^3 + \cdots$

universal iso

$b_1 \qquad b_1 \longmapsto b_1 \otimes 1 \qquad h_{1,0} = [b_1] \in \mathrm{Ext}^1_{MU_*MU}$

$b_1 \otimes 1 + 1 \otimes b_1$

$1 \otimes b_1$

So $h_{1,0} \longmapsto h_1$

$b_1^2 \longmapsto b_1^2 \otimes 1 - (b_1 \otimes 1 + 1 \otimes b_1)^2 + 1 \otimes b_1^2$

$= -2\, b_1 \otimes b_1$

$\Rightarrow \underbrace{b_1 \otimes b_1}_{h_1^2}$ is a cocycle

$b_1^{2^n} \longmapsto -\sum \binom{2^n}{i} b_1^{2^n - i} b_1^i$

$\therefore \frac{1}{2} \sum \binom{2^n}{i} b_1^{2^n - i} b_1^i$ is a cocycle

$\equiv b_1^{2^{n-1}} \otimes b_1^{2^{n-1}} \mod 2 \qquad h_{n-1}^2$

$$B_j = \frac{1}{2}\left((b_1 \otimes 1 + 1 \otimes b_1)^{2^{j+1}} - b_1^{2^{j+1}} \otimes 1 - 1 \otimes b_1^{2^{j+1}} \right) \in \mathrm{Ext}^2_{MU_*MU}(MU_*, MU_*)$$

$B_j \longmapsto h_j^2$

Want to look at $B_j \rightsquigarrow \quad \in H^2(\mathbb{Z}/8; A[w^{\pm 1}])$

actually in $H^2(\mathbb{Z}/8, w^{2^{j+1}} A) = w^{2^{j+1}} A/(8)$

$j \geq 2$, then $2^{j+1} \equiv 0\ (8)$, so $\mathbb{Z}/8$ acts trivially

Thm $B_j \longmapsto 4\cdot\omega^{2^{j+1}} \in \omega^{2^{j+1}} A/(8)$

$$\begin{array}{ccccc} b_1^{2^{j+1}} & \in & \mathrm{Ext}^1_{MU_*MU}(MU_*, MU_*/2) & \xrightarrow{\text{main calculation}} & H^1(\mathbb{Z}/8;\, A[\omega^{\pm1}]_{/2}) \\ \downarrow & & \downarrow \delta & & \downarrow \delta \\ B_j & & \mathrm{Ext}^2_{MU_*MU}(MU_*, MU_*) & \longrightarrow & H^2(\mathbb{Z}/8,\, A[\omega^{\pm1}]) \end{array}$$

$$\begin{array}{ccc} MU_* & MU_*MU & b_1 \\ \downarrow & \downarrow & \downarrow \\ A & A^{\mathbb{Z}/8} & ? \end{array}$$

write the automorphism of F corresponding to $\sigma \in \mathbb{Z}/8$ as power series $x + \alpha_1 x^2 + \cdots$

map $\sigma \longmapsto \alpha_1$

Thm α_1 is a unit in $A[\omega^{\pm1}]$

(this works for K_*, tmf, and all the others...)

$v_i \in \pi_{2(2^i-1)} MU$, $x +_F x = 2x + \cdots + v_1 x^2 + \cdots + v_2 x^4 + \cdots + v_3 x^8 \cdots$

play an important role.

Thm (Mitchell, Shimamura) $\mathrm{Ext}^{2,2^{j+1}}_{MU_*MU}(MU_*, MU_*)$ has a basis

$$\{ \underset{\substack{\downarrow \\ h_j^2}}{B_j},\ \underbrace{\underset{\substack{\downarrow \\ 0}}{v_2^4 B_{j-2}},\ \underset{\substack{\downarrow \\ 0}}{v_2^8 B_{j-4}},\ \ldots}_{\text{basis for kernel of map to ASS}} \} \qquad (v_i \mapsto 0 \text{ in ASS})$$

(almost right) (these aren't really cocycles, and we are missing a couple of terms that don't really matter).

($v_i \mapsto 0$ in ASS b/c there $x +_F x = 0$, so $v_i \mapsto 0$.)

Need: Claim $v_2^4 B_{j-2}, \ldots \longmapsto 0$ in $H^2(\mathbb{Z}/8; A[\omega^{\pm1}])$

F FGL for "A" $\ell(x +_F x) = \ell(x) + \ell(x) = 2\sum \frac{x^{2^n}}{\pi^n}$

$\ell(2x + v_1 x^2 + \cdots)$

If I expand both of these out, get formulas for v_i's in terms of π.

$\Rightarrow v_1 = \frac{2}{\pi} = \pi^3 \varepsilon_1$ (ε_i unit)

$v_2 = \pi^2 \cdot \varepsilon_2$

$v_3 = \pi \cdot \varepsilon_3$ $\Longrightarrow$ theory is v_4-periodic.

$v_4 = \varepsilon_4$

$v_2^4 = \pi^8 \cdot \varepsilon = 4\varepsilon$

$v_2^4 B_{j-2} = 4B_{j-2} = 0$ (B_{j-2} is 2-torsion)

Stepping back

tmf doesn't work b/c powers of v_2 are not 0.

Any geometric theory sits between Ext_{MU_*MU} and $H^*(\mathbb{Z}/8, A[w^{\pm 1}])$

That pushes the period in the periodic thm, have to work w/ $\mathbb{Z}/8$.