ON THE $RO(G)$-GRADED COEFFICIENTS OF Σ_3-EQUIVARIANT COHOMOLOGY

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1. Introduction

A 1982 Northwestern conference problem asked for a complete calculation of the $RO(G)$-graded cohomology groups of a point for a non-trivial finite group G (see [10] for definitions and [9] for background). This question was quickly solved by Stong [12] for cyclic groups $\mathbb{Z}/p$ with p prime. Partial calculations for groups $\mathbb{Z}/(p^n)$ and $(\mathbb{Z}/p)^n$ were much more recently done in [3, 6, 4, 5, 8]. In a recent lecture, Peter May [11] emphasized the fact that no case of a non-abelian group was known to date.

The purpose of this brief note is to advance progress in this direction by completely calculating the $RO(G)$-equivariant coefficients of the homology of a point for $G = \Sigma_3$, both with Burnside ring $\underline{A}$ and constant $\underline{\mathbb{Z}}$ coefficients. The constant coefficients $\underline{\mathbb{Z}}$ are obtained by taking the quotient of the Burnside ring Mackey functor $\underline{A}$ by its augmentation ideal. Burnside ring coefficients are "universal" among ordinary $RO(G)$-graded cohomology theories in the same sense as $\mathbb{Z}$-coefficients are non-equivariantly (see [2]), and thus were of primary interest historically. However, for non-trivial groups, the Burnside ring is not a regular ring, and because of that, passage from Burnside ring to other coefficients is not immediate. In applications [7, 3], the use of constant coefficients, which are simpler, prevailed so far.

The group Σ_3 has two non-trivial irreducible real representation, namely the 1-dimensional sign representation, which we denote by α, and a 2-dimensional representation γ, obtained as the orthogonal complement of the trivial subrepresentation in the standard permutation module $\mathbb{R}^3$. While the reader will observe that the methods of this note apply to more general groups, the number of irreducible representations of finite groups grows rather quickly with the size of the group, and because of that, any generalization to an infinite class of non-abelian groups involves additional issues, and will thus be left to future work.

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To state our results, we will state the calculation with constant $\underline{\mathbb{Z}}$ coefficients here, and postpone the statement with the Burnside ring coefficients $\underline{A}$ till Section 3 below, since it essentially follows from the constant coefficient case after some algebraic discussion of the Burnside rings.

To discuss the $\underline{\mathbb{Z}}$-coefficient case, it is useful to recall the following calculation [7]. Denote, for $\ell \geq 0$,

$$B_\ell = \widetilde{H}_*^{\Sigma_3}(S^{\ell\alpha}, \underline{\mathbb{Z}}) = \widetilde{H}_*^{\mathbb{Z}/2}(S^{\ell\alpha}, \underline{\mathbb{Z}}), \tag{1}$$

$$B^\ell = \widetilde{H}^*_{\Sigma_3}(S^{\ell\alpha}, \underline{\mathbb{Z}}) = \widetilde{H}^*_{\mathbb{Z}/2}(S^{\ell\alpha}, \underline{\mathbb{Z}}). \tag{2}$$

Proposition 1. *We have*

$$B_{\ell,n} = \begin{cases} \mathbb{Z} & n = \ell \ \textit{even} \\ \mathbb{Z}/2 & 0 \leq n < \ell \ \textit{even} \\ 0 & \textit{else,} \end{cases}$$

$$B^{\ell,n} = \begin{cases} \mathbb{Z} & n = \ell \ \textit{even} \\ \mathbb{Z}/2 & 3 \leq n \leq \ell \ \textit{odd} \\ 0 & \textit{else.} \end{cases}$$

□

We also put

$$B_{\ell,n} = B^{-\ell,-n}, \ B^{\ell,n} = B_{-\ell,-n} \text{ for } \ell < 0.$$

Then (1) and (2) extend to $\ell < 0$ by Spanier-Whitehead duality.

Now define ${}_pA_q$ and ${}^pA^q$ by

$${}_pA_q = \begin{cases} \mathbb{Z}/3 & \text{when } 2p < n < 2q-1, \ n \equiv 3 \mod 4, \\ 0 & \text{else,} \end{cases} \tag{3}$$

$${}^pA^q = \begin{cases} \mathbb{Z}/3 & \text{when } 2p < n < 2q-1, \ n \equiv 0 \mod 4, \\ 0 & \text{else.} \end{cases} \tag{4}$$

A complete calculation of the $RO(G)$-graded (co)homology of a point with coefficients in $\underline{\mathbb{Z}}$ is given by the following

Theorem 2. *For $m > 0$, we have*

$$H_*(S^{m\gamma+\ell\alpha}, \underline{\mathbb{Z}}) = {}_{\ell-1}A_{\ell+m}[-\ell+1] \oplus B_{\ell+m}[m], \tag{5}$$

$$H^*(S^{m\gamma+\ell\alpha}, \underline{\mathbb{Z}}) = {}^{\ell}A^{\ell+m}[-\ell+1] \oplus B^{\ell+m}[m]. \tag{6}$$

Here $[k]$ denotes shift up by k in homology or cohomology, whichever groups we are considering. (Note that since it is often appropriate to identify the cohomological grading with the negative of homological, some authors prefer to define shifts in one grading only; in that case, there would be a negative sign in the square brackets of one of the formulas (5), (6).

Theorem 2 and Proposition 1 give a complete calculation of the $RO(G)$-graded cohomology of a point with $\underline{\mathbb{Z}}$ coefficients. We prove Theorem 2 in Section 2 below, and give the discussion of Burnside ring coefficients in Section 3 below.

2. Proof of the main theorem

Our main tool is constructing an explicit Σ_3-equivariant CW structure on $S(m\gamma)$, and its associated equivariant chain complex. This structure is obtained by "subdividing" the cells of the standard $\mathbb{Z}/3$-equivariant cell structure on $S(m\gamma)$. Identifying $S(m\gamma)$ with the unit sphere in $\mathbb{C}^m$ on which $\mathbb{Z}/3 = \{1, \zeta, \zeta^2\} \subset S^1$ (where $\zeta = \zeta_3$) acts by multiplication, all the $\mathbb{Z}/3$-equivariant cells are free, and can be written as $\mathbb{Z}/3$ times the following non-equivariant cells:

$$\{(z_1, \dots, z_k, 0 \dots, 0) \in S(m\gamma) \mid z_k \in [0,1]\}, \tag{7}$$

$$\{(z_1, \dots, z_k, 0, \dots, 0) \in S(m\gamma) \mid z_k \in [0,1] \cdot e^{\lambda i},\ 2\pi/3 \leq \lambda \leq 4\pi/3\}, \tag{8}$$

$1 \leq k \leq m$. Now we may let Σ_3 act on this space by identifying Σ_3 with the group generated by the above model of $\mathbb{Z}/3$, and complex conjugation, which we will denote by τ. The reason of our choice of (8) is that both (7), (8) are in fact stable under the action of τ. However, they cannot be called "cells", since τ acts non-trivially on them. In fact, they can be identified with unit disks of the representations

$$(k-1)\alpha + (k-1),\ k\alpha + (k-1), \tag{9}$$

respectively. This gives a guide on how to subdivide them into Σ_3-equivariant cells. Concretely, (7) subdivides into Σ_3-equivariant cells generated by

$$\begin{gathered}\{(z_1, \dots, z_k, 0 \dots, 0) \in S(m\gamma) \mid z_k \in [0,1],\ z_{\ell+1}, \dots, z_{k-1} \in [-1,1], \\ Im(z_\ell) \geq 0 \text{ if } \ell > 0\},\end{gathered} \tag{10}$$

We denote these Σ_3-cells by

$$a_{k,\ell},\ 0 \leq \ell \leq k-1,\ 1 \leq k \leq m.$$

The cell $a_{k,\ell}$ is of dimension $k+\ell-1$ and has isotropy $\mathbb{Z}/2\{\tau\}$ for $\ell=0$, and $\{e\}$ otherwise. The $\mathbb{Z}/3$-cell generated by (8) subdivides into Σ_3-equivariant cells generated by

(11)
$$\{(z_1,\dots,z_k,0\dots,0)\in S(m\gamma)\mid z_k\in[-1,0],\ z_{\ell+1},\dots,z_{k-1}\in[-1,1],\ Im(z_\ell)\geq 0 \text{ if } \ell>0\},$$

which we denote by

$$b_{k,\ell},\ 0\leq \ell<k,\ 1\leq k\leq m,$$

and

(12)
$$\{(z_1,\dots,z_k)\in S(m\gamma)\mid z_k\in[0,1]\cdot e^{i\lambda},\ 0\leq\lambda\leq\pi/3\},$$

which we denote by

$$c_k,\ 1\leq k\leq m.$$

The Σ_3-cell $b_{k,\ell}$ is of dimension $k+\ell-1$ and has isotropy $\mathbb{Z}/2\{\tau\}$ for $\ell=0$ and $\{e\}$ otherwise. The cell c_k is of dimension $2k-1$ and has isotropy $\{e\}$.

Lemma 3. *With respect to this CW-structure just described, choosing orientation suitably, the Σ_3-equivariant cell chain complex of $S(m\gamma)$ in the sense of Bredon [1] has differential*

$$\begin{aligned}
&da_{k,0}=a_{k-1,0}-b_{k-1,0}\\
&db_{k,0}=a_{k-1,0}-b_{k-1,0}\\
&da_{k,1}=a_{k-1,1}-b_{k-1,1}+(-1)^{k-1}a_{k,0}\\
&db_{k,1}=a_{k-1,1}-b_{k-1,1}+(-1)^{k-1}b_{k,0}\\
&da_{k,\ell}=a_{k-1,\ell}-b_{k-1,\ell}+(-1)^{k-\ell}a_{k,\ell-1}+(-1)^{k-1}\tau a_{k,\ell-1}\qquad 1<\ell<k-1\\
&db_{k,\ell}=a_{k-1,\ell}-b_{k-1,\ell}+(-1)^{k-\ell}b_{k,\ell-1}+(-1)^{k-1}\tau b_{k,\ell-1}\qquad 1<\ell<k-1\\
&da_{k,k-1}=-a_{k,k-2}+(-1)^{k-1}\tau a_{k,k-2}-c_{k-1}-\zeta c_{k-1}+(-1)^{k-2}\zeta\tau c_{k-1}\\
&db_{k,k-1}=-b_{k,k-2}+(-1)^{k-1}\tau b_{k,k-2}-c_{k-1}-\zeta c_{k-1}+(-1)^{k-2}\zeta\tau c_{k-1}\\
&dc_k=-a_{k,k-1}+(-1)^k\tau a_{k,k-1}+\zeta^2 b_{k,k-1}+(-1)^{k-1}\zeta^2\tau b_{k,k-1}\ \textit{for}\ k>1,\\
&dc_1=b_{1,0}-a_{1,0}.
\end{aligned}$$

Proof. A direct inspection. □

Thus, we can calculate the Σ_3-equivariant homology and cohomology of $S(m\gamma)$ simply directly algebraically as Bredon (co)homology. It is useful, however, to make the following observation: There is a cellular filtration on $S(m\gamma)$ by the $\mathbb{Z}/3$-equivariant cells generated by (7), (8) of dimension $\leq p$. As observed in (9), topologically, we have

(13)
$$F_pS(m\gamma)/F_{p-1}S(m\gamma)\cong\begin{cases}\Sigma^{k-1}S^{(k-1)\alpha} & \text{for } p=2k-1\\ \Sigma^{k-1}S^{k\alpha} & \text{for } p=2k.\end{cases}$$

Algebraically, in the associated graded complex of the cellular chain complex, the filtration degree p part is generated by $b_{k,\ell}, c_k$ for $p = 2k-1$, and by $a_{k+1,\ell}$ for $p = 2k$, $1 \le k \le m$. Thus, in the spectral sequence for homology with coefficients $\underline{\mathbb{Z}}$ associated with this filtration, the filtration degree p part of the E^1-term is

$$\begin{aligned} &B_{k-1}[k-1] \quad \text{for } p = 2k-1,\ 1 \le k \le m,\\ &B_k[k-1] \quad\quad \text{for } p = 2k,\ 1 \le k \le m. \end{aligned} \tag{14}$$

By Lemma 3, we see that the effect of the differential

$$d^1 : F_{2k+1}/F_{2k} \to F_{2k}/F_{2k-1}, 1 \le k < m$$

of our spectral sequence wipes out all of (14) except when there is a $\mathbb{Z}$ in the target (which is supported by c_k, k even, which then turns into $\mathbb{Z}/3$. The exception is filtration degree $2m$, where there is no differential with that target, and filtration degree 0, where there is no differential with that source. There is no room for higher differentials for dimensional reasons. In cohomology, the story is precisely mirrored, with all subscripts turned into superscripts and also arrows turned around, so the cocycle supported by c_k become the source, rather than the target of the differential $3 : \mathbb{Z} \to \mathbb{Z}$. Thus, we have proved the following

Proposition 4. *For $m > 0$, we have*

$$H_*^{\Sigma_3}(S(m\gamma), \underline{\mathbb{Z}}) = \mathbb{Z} \oplus {}_0A_m \oplus B_m[m-1],$$
$$H^*_{\Sigma_3}(S(m\gamma), \underline{\mathbb{Z}}) = \mathbb{Z} \oplus {}^0A^m \oplus B^m[m-1].$$

□

It may be tempting to try to use the same method for calculating the reduced Σ_3-equivariant (co)homology of $\Sigma^{\ell\alpha} \wedge S(m\gamma)_+$ for $\ell \in \mathbb{Z}$, but there are two difficulties. First, for $\ell > 0$, the chain complex we obtain by smashing the CW-complexes cell-wise grows with ℓ. More importantly, for $\ell < 0$, the method actually fails: the Bredon chain complex is not an equivariantly stable object, and actually does not exist for spectra obtained by desuspending by non-trivial representations. There is, of course, a concept of an equivariant CW-spectrum [9], but any chain complex in this stable context has to be built directly on the Mackey functor level.

However, there is a more direct method. The space-level filtration of $S(m\gamma)$ can certainly be suspended by $\ell\alpha$. In other words, we may consider the filtration

$$\Sigma^{\ell\alpha} F_p S(m\gamma)_+$$

of $\Sigma^{\ell\alpha} S(m\gamma)_+$, and just as for $\ell = 0$, it leads to a spectral sequence in Σ_3-equivariant (co)homology, whose d^1 (resp. d_1) is determined by the $\Sigma^{\ell\alpha}$-suspension of the connecting map of the cofibration sequence

$$F_{2k}S(m\gamma)/F_{2k-1}S(m\gamma) \to F_{2k+1}S(m\gamma)/F_{2k-1}S(m\gamma) \to F_{2k+1}S(m\gamma)/F_{2k}S(m\gamma). \tag{15}$$

This connecting map, stably, of course does not depend on ℓ, and desuspending by $(k+\ell)\alpha$, is a stable Σ_3-equivariant map

$$\Sigma_3/(\mathbb{Z}/2)_+ \to \Sigma_3/(\mathbb{Z}/2)_+. \tag{16}$$

Now by adjunction, stable maps (16) are the same thing as $\mathbb{Z}/2$-equivariant stable maps

$$S^0 \to \Sigma_3/\mathbb{Z}/2_+,$$

where $\mathbb{Z}/2$-equivariantly, the target is $S^0 \vee \mathbb{Z}/2_+$. From this point of view, then, maps (16) are classified by elements of

$$A[\mathbb{Z}/2] \oplus \mathbb{Z}\{t\} \tag{17}$$

where t is the transfer. Now to see which map arises as the connecting map of (15), we must carefully note that we are dealing with an unbased equivariant CW-complex, and therefore the connecting map will be visible as the canonical map from the cell $a_{k+1,k}$ to the unreduced suspension of c_k (an equivariant version of the canonical map from a sphere to the unreduced suspension of its boundary), which, in the nomenclature (17), has the name

$$(1, t). \tag{18}$$

In particular, we see that the connecting map, thought of as a stable Σ_3-equivariant map (16), does not in fact depend on k, either. Thus, all the connecting maps of the $\ell\alpha$-suspensions of the cofiber sequence (15) already arise in the CW-complex $S(m\gamma)_+$ and its dual, and thus we know their effect in (co)-homology, with the exception, curiously, of the case $k + \ell = 0$, in which case, however, it is easy to verify directly that the relevant map $\mathbb{Z} \to \mathbb{Z}$ is 3. Thus, we know the d^1 (resp. d_1) of the spectral sequence for the (co)homology of $\Sigma^{\ell\alpha} S(m\gamma)_+$ for all $m > 0$ and all $\ell \in \mathbb{Z}$, and again, there is no room for higher differentials for dimensional reasons. Thus, we have proved the following

Proposition 5. *For $m > 0$, we have*

$$H_*^{\Sigma_3}(\Sigma^{\ell\alpha} S(m\gamma)_+, \underline{\mathbb{Z}}) = B_\ell \oplus B_{\ell+m}[m-1] \oplus {}_\ell A_{\ell+m}[-\ell],$$

$$H^*_{\Sigma_3}(\Sigma^{\ell\alpha} S(m\gamma)_+, \underline{\mathbb{Z}}) = B^\ell \oplus B^{\ell+m}[m-1] \oplus {}^\ell A^{\ell+m}[-\ell].$$

Proof of Theorem 2: We have a cofibration sequence

$$\Sigma^{\ell\alpha} S(m\gamma)_+ \to S^{\ell\alpha} \to S^{\ell\alpha+m\gamma}. \tag{19}$$

We use the long exact sequence in (co)homology associated with the cofibration sequence (19). In homology, this corresponds to finding the map

$$B_\ell \to B_\ell$$

given by the induction

$$H_*^{\mathbb{Z}/2}(S^{\ell\alpha}, \underline{\mathbb{Z}}) \to H_*^{\Sigma_3}(S^{\ell\alpha}, \underline{\mathbb{Z}}),$$

coming from the attaching map of the bottom two $\mathbb{Z}/3$-cells (here we consider $S^{\ell\alpha}$ as a spectrum, so no "reduced homology" symbol is needed). This is multiplication by 3, so the map is an isomoprhism except in the top dimension, where it is injective with cokernel $\mathbb{Z}/3$ when ℓ is even. This gives (5).

In cohomology, the first map (19) leads to the restriction map

$$H^*_{\Sigma_3}(S^{\ell\alpha}, \underline{\mathbb{Z}}) \to H^*_{\mathbb{Z}/2}(S^{\ell\alpha}, \underline{\mathbb{Z}})$$

which is always an isomorphism, thus giving (6) (and, in particular, explaining the somewhat different behavior in the bottom dimension). $\square$

3. Burnside ring coefficients

To calculate the $RO(G)$-graded coefficients of Σ_3-equivariant (co)homology with coefficients in $\underline{A}$, we just need to repeat the above procedure with Burnside ring coefficients, and keep track of what changes. The changes are somewhat minor due to the fact that most of the cells of $S(m\gamma)$ are free. It is useful to introduce the notation $A[\mathbb{Z}/2] = \mathbb{Z}\{1, t_2\}$, $A[\mathbb{Z}/3] = \mathbb{Z}\{1, t_3\}$, $A[\Sigma_3] = \mathbb{Z}\{1, t_2, t_3, t_6\}$ where t_i denotes the orbit of cardinality i.

We start with an analogue of Proposition 1, which is proved analogously. Denote, for $\ell \geq 0$,

$$\mathscr{B}_\ell = \widetilde{H}_*^{\mathbb{Z}/2}(S^{\ell\alpha}, \underline{A}), \tag{20}$$

$$\mathscr{B}^\ell = \widetilde{H}^*_{\mathbb{Z}/2}(S^{\ell\alpha}, \underline{A}). \tag{21}$$

Denote by $I_{\mathbb{Z}/2}$ the kernel of the restriction $A[\mathbb{Z}/2] \to A[\{e\}]$ (i.e., the augmentation ideal), and by $J_{\mathbb{Z}/2}$ the cokernel of the induction $A[\{e\}] \to A[\mathbb{Z}/2]$. Both $I_{\mathbb{Z}/2}$ and $J_{\mathbb{Z}/2}$ are, of course, isomorphic to $\mathbb{Z}$.

Proposition 6. *We have*

$$\mathscr{B}_{\ell,n} = \begin{cases} J_{\mathbb{Z}/2} & n{=}0 \\ \mathbb{Z} & n = \ell \text{ even} \\ \mathbb{Z}/2 & 0 < n < \ell \text{ even} \\ 0 & \text{else,} \end{cases}$$

$$\mathscr{B}^{\ell,n} = \begin{cases} I_{\mathbb{Z}/2} & n{=}0 \\ \mathbb{Z} & n = \ell \text{ even} \\ \mathbb{Z}/2 & 3 \le n \le \ell \text{ odd} \\ 0 & \text{else.} \end{cases}$$

□

We also put

$$\mathscr{B}_{\ell,n} = \mathscr{B}^{-\ell,-n},\ \mathscr{B}^{\ell,n} = \mathscr{B}_{-\ell,-n} \text{ for } \ell < 0.$$

Then (20) and (21) extend to $\ell < 0$ by Spanier-Whitehead duality.

Using Lemma 3, we then immediately get analogously to Proposition 4 the following

Proposition 7. *For $m > 0$, we have*

$$H_*^{\Sigma_3}(S(m\gamma), \underline{A}) = A[\mathbb{Z}/2] \oplus {}_0A_m \oplus \mathscr{B}_m[m-1],$$

$$H^*_{\Sigma_3}(S(m\gamma), \underline{A}) = A[\mathbb{Z}/2] \oplus {}^0A^m \oplus \mathscr{B}^m[m-1].$$

□

To suspend by $\ell\alpha$, we need to additionally observe what the attaching map of $\mathbb{Z}/3$-cells (16) induces on $\underline{A}$. Using the notation (18), we find that this map

$$A[\mathbb{Z}/2] \to A[\mathbb{Z}/2]$$

sends

$$1 \mapsto 1 + t_2,$$

$$t_2 \mapsto 3t_2.$$

Thus, it is injective, and its cokernel is $\mathbb{Z}/3$, just as with $\underline{\mathbb{Z}}$ coefficients.

Proposition 8. *For $m > 0$, we have*

$$H_*^{\Sigma_3}(\Sigma^{\ell\alpha} S(m\gamma)_+, \underline{A}) = \mathscr{B}_\ell \oplus \mathscr{B}_{\ell+m}[m-1] \oplus {}_\ell A_{\ell+m}[-\ell],$$

$$H^*_{\Sigma_3}(\Sigma^{\ell\alpha} S(m\gamma)_+, \underline{A}) = \mathscr{B}^\ell \oplus \mathscr{B}^{\ell+m}[m-1] \oplus {}^\ell A^{\ell+m}[-\ell].$$

□

Similarly as in the case of constant coefficients, to get to the (co)homology of $S^{\ell\alpha+m\gamma}$, we need to use the cofibration sequence (19). To this end, we need to calculate

$$\mathscr{C}_\ell = \widetilde{H}_*^{\Sigma_3}(S^{\ell\alpha}, \underline{A}), \tag{22}$$

$$\mathscr{C}^\ell = \widetilde{H}^*_{\Sigma_3}(S^{\ell\alpha}, \underline{A}). \tag{23}$$

The key point here is that the Weyl group $\mathbb{Z}/2$ of $\mathbb{Z}/2 \subset \Sigma_3$ acts trivially on the Burnside ring $A[\mathbb{Z}/3]$. It also helps to denote by $J_{\Sigma_3}^{\mathbb{Z}/3}$ the cokernel of the induction $A[\mathbb{Z}/3] \to A[\Sigma_3]$, and by $I_{\Sigma_3}^{\mathbb{Z}/3}$ the kernel of the restriction $A[\Sigma_3] \to A[\mathbb{Z}/3]$. Of course, induction sends $1 \mapsto t_2$, $t_3 \mapsto t_6$, so both $I_{\Sigma_3}^{\mathbb{Z}/3}$ and $J_{\Sigma_3}^{\mathbb{Z}/3}$ are, as groups, isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. Therefore, we have

Proposition 9. *We have*

$$\mathscr{C}_{\ell,n} = \begin{cases} J_{\Sigma_3}^{\mathbb{Z}/3} & n=0 \\ A[\mathbb{Z}/3] & n = \ell \text{ even} \\ A[\mathbb{Z}/3]/2 & 0 < n < \ell \text{ even} \\ 0 & \text{else,} \end{cases}$$

$$\mathscr{C}^{\ell,n} = \begin{cases} I_{\Sigma_3}^{\mathbb{Z}/3} & n=0 \\ A[\mathbb{Z}/3] & n = \ell \text{ even} \\ A[\mathbb{Z}/3]/2 & 3 \le n < \ell \text{ odd} \\ 0 & \text{else.} \end{cases}$$

□

We also put

$$\mathscr{C}_{\ell,n} = \mathscr{C}^{-\ell,-n},\ \mathscr{C}^{\ell,n} = \mathscr{C}_{-\ell,-n} \text{ for } \ell < 0.$$

Note that then we have short exact sequences

$$0 \to \mathscr{B}_\ell \to \mathscr{C}_\ell \to \mathscr{B}_\ell \to 0$$

where the first map is induction from $\mathbb{Z}/2$ to Σ_3, and

$$0 \to \mathscr{B}^\ell \to \mathscr{C}^\ell \to \mathscr{B}^\ell \to 0$$

where the second map is restriction from Σ_3 to $\mathbb{Z}/2$. Therefore, we have

Theorem 10. *For $m > 0$, we have*

$$H_*(S^{m\gamma+\ell\alpha}, \underline{A}) = \mathscr{B}_\ell \oplus {}_{\ell-1}A_{\ell+m}[-\ell+1] \oplus \mathscr{B}_{\ell+m}[m], \tag{24}$$

$$H^*(S^{m\gamma+\ell\alpha}, \underline{A}) = \mathscr{B}^\ell \oplus {}_{\ell}A^{\ell+m}[-\ell+1] \oplus \mathscr{B}^{\ell+m}[m]. \tag{25}$$

□

References

[1] G.Bredon: *Equivariant cohomology theories*, Lecture Notes Math. 34, Springer Verlag, 1967

[2] J.P.C.Greenlees: Some remarks on projective Mackey functors, *J. Pure Appl. Algebra* 81 (1992) 17-38

[3] M. A. Hill, M. J. Hopkins, D. C. Ravenel: On the nonexistence of elements of Kervaire invariant one, *Annals of Mathematics* Volume 184 (2016), Issue 1, 1-262

[4] J.Holler, I.Kriz: On $RO(G)$-graded equivariant "ordinary" cohomology where G is a power of $\mathbb{Z}/2$, *Alg. Geom. Top.* 17 (2017) 741-763

[5] J.Holler, I.Kriz 2: Coefficients of $(\mathbb{Z}/p)^n$-equivariant geometric fixed points of $H\mathbb{Z}/p$, preprint 2017, www.math.lsa.umich.edu/~ikriz

[6] P.Hu, I.Kriz: Coefficients of the constant Mackey functor over cyclic p-groups, preprint, 2010

[7] P. Hu, I. Kriz: Real-oriented homotopy theory and an analogue of the Adams-Novikov spectral sequence, *Topology* 40(2) (2001) 317–399

[8] S.Kriz: Equivariant cohomology and the super reciprocal plane of a hyperplane arrangement, preprint, 2015

[9] L. G. Lewis, Jr., J. P. May, M. Steinberger, and J. E. McClure: *Equivariant stable homotopy theory*, Vol 1213 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1986

[10] G.Lewis, J.P.May, J.McClure: Ordinary $RO(G)$-graded cohomology, *Bull. AMS* 4 (1981) 208-212

[11] J.P.May: Glimpses of equivariant algebraic topology, *colloquium talk at the University of Michigan*, 2017

[12] R.E.Stong, private communication to J.P.May, 1982