CHAPTER 3: LOW DIMENSIONAL COMPUTATIONS

1. Introduction

In this chapter, we illustrate how our algorithm works by computing π_N^S for $0 \le N \le 8$. In addition we introduce notation and derive results that will be useful in our higher dimensional computations. Moreover, we are able to make gloabal computations of $E^4_{n,t}$ and $E^6_{n,t}$ for all n, t as well as global computations of $E^r_{n,t}$ for all r, n and $0 < t \le 8$.

In Section 2, we compute π_1^S and π_2^S and determine the behavior of the spectral sequence on the entire 1 and 2 rows. We also determine all d^2-differentials in the spectral sequence and give the global computation of E^4. In Section 3, we compute π_N^S and determine the behavior of the spectral sequence on the entire N row for $3 \le N \le 6$. We also determine all d^4 differentials in the spectral sequence and give the global computation of E^6. Then in Section 4, we compute π_7^S and π_8^S and determine the behavior of the spectral sequence on the entire 7 and 8 rows. We conclude Section 4 with a summary of some important notation which is introduced in this chapter. The results about π_*^S derived in this chapter are summarized in the initial parts of the tables in Appendices 1 to 4.

As the reader may have noticed, the computations of this chapter are in the range where the elements η, ν and σ of Hopf invariant one exist. That may explain why the spectral sequence has such a simple description in this range.

2. d^2 Differentials and the Determination of E^4

Consider the following diagram of E^2:

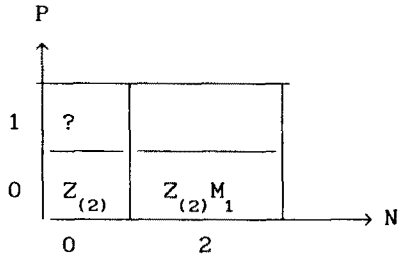

Figure 3.2.1: $E^2_{N,P}$

Now $E^4_{2,0} = E^\infty_{2,0} = \pi_2 BP = Z_{(2)}V_1 = Z_{(2)}(2M_1)$. Therefore $d^2(M_1)$ must be a nonzero element η of π_1^S of order two. There are no other possibilities for nonzero differentials to land in $E^r_{0,1}$. Therefore, $\pi_1^S = Z_2\eta$. Consider the following diagram of E^2.

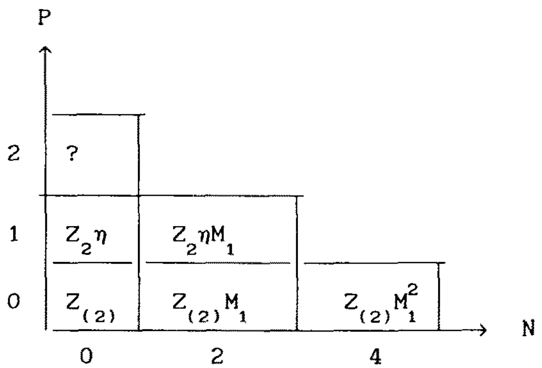

Figure 3.2.2: $E^2_{N,P}$

Now $E^4_{2,1} = E^\infty_{2,1} = 0$, and $d^2(M_1^2) = 0$. Thus, $d^2(\eta\cdot M_1) = \eta^2$ must be a nonzero element of π_2^S of order two. There are no other possibilities for nonzero differentials to land in $E^r_{0,2}$. Thus, $\pi_2^S = Z_2\eta^2$. We have thus proved the first part of the following theorem.

THEOREM 3.2.1 (a) $\pi_1^S = Z_2\eta$ and $\pi_2^S = Z_2\eta^2$.

(b) $\eta^2 \in \langle 2, \eta, 2\rangle$.

PROOF. (b) Consider the Atiyah-Hirzebruch spectral sequence:

$$'E^2_{N,p} = H_N(K(Z_2);\pi^S_p) \Longrightarrow \pi^S_{N+p}(K(Z_2))$$

Here $K(Z_2)$ is the Eilenberg-MacLane spectrum with $\pi^S_*(K(Z_2)) = \pi^S_0(K(Z_2)) = Z_2$. Let $F\colon BP \longrightarrow K(Z_2)$ such that $F^*(\iota) = 1$. Then F induces a map of spectral sequences $F^r\colon E^r_{N,p} \longrightarrow {}'E^r_{N,p}$. From [30, Theorem 3.2] we see that the cellular chains of $K(Z_2)$ can be written as $C_*K(Z_2) = Z[\xi_1, \bar{\xi}_k \mid k \geq 2]$ with $\partial(\xi_1) = 2$. Recall from [20] that $E^1_{N,p} = C_N K(Z_2) \otimes \pi^S_p$. Thus $'E^1_{1,1} = Z\xi_1$ and $d^1(\xi_1) = 2$. Now $F^2(M_1) = \xi_1^2$ and hence $d^2(\eta\xi_1^2) = \eta^2$. By Theorem 2.4.2, $\eta^2 \in \langle 2,\eta,2\rangle$. ∎

We repeat our argument one more time. Consider the following diagram of E^2.

P \ N	0	2	4
3	?		
2	$Z_2\eta^2$	$Z_2\eta^2M_1$	
1	$Z_2\eta$	$Z_2\eta M_1$	$Z_2\eta M_1^2$
0	$Z_{(2)}$	$Z_{(2)}M_1$	$Z_{(2)}M_1^2$

Figure 3.2.3: $E^2_{N,P}$

Now $0 = E^\infty_{2,2} = E^4_{2,2}$ and $d^2(\eta M_1^2) = 0$. Thus, $\eta^3 = d^2(\eta^2 M_1)$ is a nonzero element of order two in π^S_3. We now use Quillen operations to extend our computation to compute the d^2 differentials on the entire 1 and 2 rows.

THEOREM 3.2.2 (a) $d^2(M_N) = \eta M^2_{N-1}$ for $N \geq 1$.

(b) Let $\bar{M}_1 = M_1$, $\bar{M}_2 = 3M_2 - M_1^3$ and $\bar{M}_N = M_N - M_1M^2_{N-1}$ for $N \geq 3$. Then $\bar{M}_N$ is a d^2 cycle for $N \geq 2$.

PROOF. (a) The only nonzero Quillen operation of degree $2^{N+1}-2$ on M_N is

$r_{2\Delta_{N-1}}$ and $r_{2\Delta_{n-1}} \circ d^2(M_N) = d^2 \circ r_{2\Delta_{N-1}}(M_N) = d^2(M_1) = \eta$. Thus, $d^2(M_N) = \eta M_{N-1}^2$.

(b) $d^2(\bar{M}_2) = 3d^2(M_2) - d^2(M_1^3) = 0$. For $N \geq 3$, $d^2(\bar{M}_N) = d^2(M_N) - d^2(M_1 M_{N-1}^2)$ $= \eta M_{N-1}^2 - d^2(M_1)M_{N-1}^2 = 0$. ∎

Observe that $H_*BP = Z_{(2)}[\bar{M}_N \mid N \geq 1]$. Thus, we can use these new polynomial generators to describe the behavior of the d^2-differentials in the entire spectral sequence.

THEOREM 3.2.3 Let $\mu_q(\eta): \pi_q^S \longrightarrow \pi_{q+1}^S$ denote multiplication by η. For every $p \geq 1$, π_p^S can be written as a direct sum of cyclic groups $Z_{2^{N(k)}} X_k$, $1 \leq k \leq t$, such that:

(i) $Z_{2^{N(k)}} X_k \otimes H_*BP$ is a direct summand of the p row, $E^2_{*,p}$;

(ii) Kernel $\mu_p(\eta)$ is generated by $\{X_k \mid \eta X = 0\} \cup \{2X_k \mid \eta X \neq 0\}$;

(iii) Image $\mu_{p-1}(\eta)$ has a Z_2 basis $\{2^{N(k)-1} X_k \mid k \in A\}$ for some subset A of $\{1, \ldots, t\}$.

Define B<2> as the subalgebra $Z_{(2)}[M_1^2, \bar{M}_N \mid N \geq 2]$ of $E^4_{*,0}$. Then E^4 is the direct sum of the following summands:

(a) if $\eta X_k \neq 0$ and $k \notin A$ then

$$\left(Z_{2^{N(k)}} X_k \oplus Z_{2^{N(k)-1}}(2X_k M_1)\right) \otimes B\langle 2\rangle$$

is a direct summand of the p row of E^4;

(b) if $\eta X_k \neq 0$ and $k \in A$ then

$$\left(Z_{2^{N(k)-1}} X_k \oplus Z_{2^{N(k)-1}}(2X_k M_1)\right) \otimes B\langle 2\rangle$$

is a direct summand of the p row of E^4;

(c) if $\eta X_k = 0$ and $k \in A$ then

$$\left(Z_{2^{N(k)-1}} X_k \oplus Z_{2^{N(k)}} X_k M_1\right) \otimes B\langle 2\rangle$$

is a direct summand of the p row of E^4;

(d) if $\eta X_k = 0$ and $k \notin A$ then $Z_{2^{N(k)}} X_k \otimes H_*BP$ is a direct summand of the p row of E^4.

PROOF. The decomposition of π^S_p as a direct sum of cyclic groups with the required properties follows routinely from the fundamental theorem of abelian groups. Note that Image $d^2 = \eta\cdot\pi^S_* \otimes B\langle 2\rangle$ and Kernel $d^2 =$

$(\oplus_{\{k|\eta X_k=0\}} Z_{2^{N(k)}}X_k \otimes H_*BP) \oplus (\oplus_{\{k|\eta X_k\neq 0\}} (Z_{2^{N(k)-1}}2X_kM_1 \oplus Z_{2^{N(k)}}X_k) \otimes B\langle 2\rangle$.

The description of E^4 in (a)-(d) is a direct consequence of this observation.∎

COROLLARY 3.2.4 $E^4_{N,1} = E^4_{N,2} = 0$ for all $N \geq 0$.

PROOF. In the remarks preceeding Theorem 3.2.2 we saw that $\eta^3 \neq 0$. Thus, this corollary follows from Theorem 3.2.3(b).∎

COROLLARY 3.2.5 $E^4_{*,0} = (Z_{(2)} \oplus Z_{(2)}2M_1) \otimes B\langle 2\rangle$.

PROOF. This corollary follows from Theorem 3.2.3(a).∎

3. d^4 Differentials and the Determination of E^6

We continue our analysis of the spectral sequence by considering the following diagram of E^4.

P \ N	0	2	4	6
3	?			
2	0	0		
1	0	0	0	
0	$Z_{(2)}$	$Z_{(2)}2M_1$	$Z_{(2)}M_1^2$	$Z_{(2)}2M_1^3 \oplus Z_{(2)}\overline{M}_2$

Figure 3.3.1: $E^4_{N,P}$

Note that $Z_{(2)}(4M_1^2) = E^\infty_{4,0} = E^6_{4,0}$. Thus, $d^4(M_1^2)$ must be a nonzero element

$\bar{\nu} \in E^4_{0,3}$ of order 4. Thus we have the following composition series for π^S_3:

$$0 \longrightarrow Z_2\eta^3 \longrightarrow \pi^S_3 \longrightarrow Z_4\bar{\nu} \longrightarrow 0$$

Let $\nu \in \pi^S_3$ be a lifting of $\bar{\nu}$. Either $4\nu = \eta^3$ and $\pi^S_3 = Z_8$ or $4\nu = 0$ and $\pi^S_3 = Z_4\nu \oplus Z_2\eta^3$. By Theorem 2.4.2, we see that

$$2\nu \in \langle\eta, 2, \eta\rangle. \qquad [3.1]$$

By Theorem 3.2.1(b), $\eta^2 \in \langle 2, \eta, 2\rangle$. By Theorem 2.3.3(b), $4\nu \in 2\langle\eta, 2, \eta\rangle = \langle 2, \eta, 2\rangle\eta = \eta^3$. We have thus proved the following theorem.

THEOREM 3.3.1 $\pi^S_3 = Z_8\nu$ and $\eta^3 = 4\nu$.

We use Quillen operations to extend our computation of d^4-differentials to the entire 0 row.

THEOREM 3.3.2 (a) $d^4(\bar{M}_2) = \nu M_1$ and $d^4(2M_1^3) = 2\nu M_1$.

(b) $d^4(\bar{M}_N) = 5\nu M_1 M^4_{N-2} + 2\nu\bar{M}_{N-1}M^2_{N-2}$.

PROOF. (a) $r_1 \circ d^4(\bar{M}_2) = d^4 \circ r_1(\bar{M}_2) = -3\nu$ and $r_1 \circ d^4(2M_1^3) = d^4 \circ r_1(2M_1^3) = 6\nu$. Thus $d^4(\bar{M}_2) \equiv 5\nu M_1$ and $d^4(2M_1^3) \equiv 6\nu \mod (4)$ in $E^4_{2,3} = Z_8\nu M_1$. The Hazewinkel generator $V_2 = 2M_2 - 4M_1^3 = \frac{2}{3}\bar{M}_2 - \frac{5}{3}(2M_1^3)$ is a d^4-cycle. Thus $2\nu M_1 = 2d^4(\bar{M}_2) = 5d^4(2M_1^3)$ and $d^4(2M_1^3) = 2\nu M_1$. Note that our definition of ν in Theorem 3.3.1 was only made modulo (4). Thus, define ν so that $d^4(\bar{M}_2)$ is ν and not 5ν.

(b) The only nonzero Quillen operations of degree $2^{N+1} - 6$ on $\bar{M}_N$ are $r_{\Delta_1+4\Delta_{N-2}}(\bar{M}_N) = -M_1^2$ and $r_{\Delta_{N-1}+2\Delta_{N-2}}(\bar{M}_N) = -2M_1^2$. Thus, $d^4(\bar{M}_N) \equiv 3\nu M_1 M^4_{N-2} + 2\nu M_{N-1}M^2_{N-2} \equiv 2\nu\bar{M}_{N-1}M^2_{N-2} + \nu M_1 M^4_{N-2} \mod (4)$. Then $r_{4\Delta_{N-2}} \circ d^4(\bar{M}_N) = d^4 \circ r_{4\Delta_{N-2}}(\bar{M}_N) = d^4(\frac{1}{3}\bar{M}_2 - \frac{2}{3}M_1^3) = 5\nu M_1$. Note that $r_{4\Delta_{N-2}}(\bar{M}_{N-1}M^2_{N-2}) = 0$ and $r_{4\Delta_{N-2}}(M_1 M^4_{N-2}) = M_1$. Therefore, $d^4(\bar{M}_N) = 2\nu\bar{M}_{N-1}M^2_{N-2} + 5\nu M_1 M^4_{N-2}$. ∎

We use the preceding theorem to define several very important d^4-cycles.

COROLLARY 3.3.3 The following elements are all d^4-cylces.

(a) $\langle M_1^4\rangle = M_1^4 + 2M_1M_2$.

(b) $\langle M_2^2\rangle = M_2^2 + M_1^6$.

(c) $\langle M_3\rangle = \bar{M}_3 + 5\bar{M}_2M_1^4 + 14M_1^7$.

PROOF. It is straightforward to use the previous theorem to compute $d^4\langle M_1^4\rangle$, $d^4\langle M_2^2\rangle$, $d^4\langle M_3\rangle$ and to find that each of these differentials is zero. ∎

The following two lemmas will be used to compute E^6 and to prove the existence of polynomial generators $\langle M_N\rangle$, $N \geq 3$, which are d^4-cycles.

LEMMA 3.3.4 In the notation of Theorem 3.2.3, assume that νX_k has order 2^B where $B = 1$, 2 or 3. Let $\varepsilon = 1$ if $\eta X_k \neq 0$, $\varepsilon = 0$ if $\eta X_k = 0$ and $\beta = B - \delta_3^B$. Then the kernel of d^4 restricted to $Z_{2^{N(k)}}X_k \otimes Z[\langle M_1^4\rangle,\langle M_2^2\rangle]$ is the $Z[\langle M_1^4\rangle,\langle M_2^2\rangle]$- module spanned by

$$\{X_k,\ 2^{\varepsilon}M_1X_k,\ 2^{\beta}M_1^2X_k,\ 2^BM_1^3X_k,\ V_2X_k,\ V_1V_2X_k,\ 2^BM_1^2\bar{M}_2X_k,\ 2^BM_1^3\bar{M}_2X_k\}.$$

PROOF. Observe that $d^4(X_k) = 0$, $d^4(M_1^2X_k) = \nu X_k$, $d^4(\bar{M}_2X_k) = \nu M_1X_k$ and $d^4(M_1^2\bar{M}_2X_k) = \nu(\bar{M}_2+M_1^3)X_k$. If $\eta X_k = 0$ then $d^4(M_1X_k) = 0$, $d^4(M_1^3X_k) = \nu M_1X_k$, $d^4(M_1\bar{M}_2X_k) = \nu M_1^2X_k$ and $d^4(M_1^3\bar{M}_2X_k) = \nu(M_1\bar{M}_2+M_1^4)X_k$. If $\eta X_k \neq 0$ then $d^4(M_1X_k) = 0$, $d^4(2M_1^3X_k) = 2\nu M_1X_k$, $d^4(2M_1\bar{M}_2X_k) = 2\nu M_1^2X_k$ and $d^4(2M_1^3\bar{M}_2X_k) = 2\nu(M_1\bar{M}_2+M_1^4)X_k$. Since $\eta^3 = 4\nu$, $\nu M_1^A\bar{M}_2^CX_k$ has order 2^B if A is odd and has order 2^{β} if A is even. The conclusion of the lemma follows from these observations. ∎

LEMMA 3.3.5 In the notation of Theorem 3.2.3 and Lemma 3.3.4, assume that νX_k has order 2^B where B is 1, 2 or 3. Then the image of d^4 in $Z_{2^B}(\nu X_k) \otimes Z[\langle M_1^4\rangle,\langle M_2^2\rangle]$ is the $Z[\langle M_1^4\rangle,\langle M_2^2\rangle]$-module spanned by

$$\{\nu X_k,\ \nu M_1X_k,\ 2^{\varepsilon}\nu M_1^2X_k,\ \nu(\bar{M}_2+M_1^3)X_k,\ 2^{\varepsilon}\nu M_1\bar{M}_2X_k\}.$$

PROOF. This result follows from the computations of d^4-differentials in the proof of Lemma 3.3.4. ∎

Our first applications of these lemmas is to compute π_4^S and π_5^S.

THEOREM 3.3.6 $\pi_4^S = 0$ and $\pi_5^S = 0$.

PROOF. It follows from Lemma 3.3.5 that $E^6_{2,3} = 0$. It follows from Lemma 3.3.4 that $E^6_{6,0} = Z_{(2)}V_2 \oplus Z_{(2)}(8M_1^3) = E^{\infty}_{6,0}$. We saw in Corollary 3.2.4 that $E^4_{*,1} = E^4_{*,2} = 0$. Therefore, there are no possibilities for nonzero differentials to land in $E^r_{0,4}$ or $E^r_{0,5}$. Thus, $\pi_4^S = \pi_5^S = 0$. ∎

It follows from Lemma 3.3.5 that νM_1^2, $2\nu M_1^3$ and $4\nu M_1^3\bar{M}_2$ are not d^4-boundaries. This leads to the following theorem.

THEOREM 3.3.7 (a) Let $\nu^2 = d^4(\nu M_1^2)$. Then $\pi_6^S = Z_2\nu^2$.

(b) $A[8] = d^6(2\nu M_1^3)$ is a nonzero element of order 2 in π_8^S.

(c) $A[14] = d^{12}(4\nu M_1^3\bar{M}_2)$ is a nonzero element of π_{14}^S.

PROOF. (a) Since νM_1^2 is not a d^4-boundary and $E^{\infty}_{4,3} = 0$, $\nu^2 \neq 0$. Since ν has odd degree, $2\nu^2 = 0$. By Theorem 2.4.2,

$$\nu^2 \in \langle \eta, \nu, \eta \rangle \qquad [3.2]$$

(b) Since $2\nu M_1^3$ is not a d^4-boundary and $E^{\infty}_{6,3} = 0$, $A[8] \neq 0$. From Theorem 2.4.4(a), we see that

$$A[8] \in \langle \eta, \nu, 2\nu \rangle. \qquad [3.3]$$

Therefore $2A[8] \in 2\langle \eta, \nu, 2\nu \rangle = \langle 2, \eta, \nu \rangle 2\nu = 0$ because $\langle 2, \eta, \nu \rangle \in \pi_5^S = 0$.

(c) Since $4\nu M_1^3\bar{M}_2$ is not a d^4-boundary and $E^{\infty}_{12,3} = 0$, $A[14] \neq 0$. ∎

We next apply our lemmas is to produce the d^4-cylces $\langle M_N \rangle$, $N \geq 3$, which will be used in our computation of E^6.

THEOREM 3.3.8 For $N \geq 3$, there are polynomial generators $\langle M_N \rangle$ of H_*BP which survive to E^6.

PROOF. We construct the $\langle M_N \rangle$ by induction on $N \geq 3$. We already found $\langle M_3 \rangle$ in Theorem 3.3.4(c). Assume that $N \geq 4$ and that we have found $\langle M_k \rangle$ for $3 \leq k < N$. Then $d^4(\bar{M}_N) \in \{Z_8 \nu M_1 \bar{M}_2^{\delta} \oplus Z_4 \nu \bar{M}_2^{\delta} | \delta=0,1\} \otimes Z[M_1^2, \langle M_2^2 \rangle, \langle M_3 \rangle, \ldots, \langle \bar{M}_{N-1} \rangle]$ and $d^4(\bar{M}_N)$ is a d^4-cycle, a d^6-cycle and a d^{12}-cycle. The tables in Figure 3.3.2 shows that all such cycles are in the image of d^4 on $S = \{Z_{(2)} \oplus Z_{(2)} M_1\} \otimes Z_{(2)}[M_1^2, \bar{M}_2, \ldots, \bar{M}_{N-1}]$. In those tables an entry $\longleftarrow X$ in row $k\nu$, column M means that X hits $k\nu M$ and an entry $\longrightarrow Y$ in row $k\nu$, column M means that $k\nu M$ hits Y. Thus there is an element $U \in S$ such that $d^4(U) = d^4(\bar{M}_N)$. Let $\langle M_N \rangle = \bar{M}_N - U$. ∎

4ν	$\longleftarrow \eta^2 M_1$	$\longleftarrow 4\bar{M}_2$	$\longleftarrow \eta^2 M_1^3$	$\longleftarrow \eta^2 M_1 \bar{M}_2$	$\longleftarrow 4M_1^2\bar{M}_2$	$\longleftarrow 4(M_1^3\bar{M}_2 - M_1^6)$
2ν	$\longleftarrow 2M_1^2$	$\longleftarrow 2\bar{M}_2$	$\longleftarrow 2M_1\bar{M}_2$	$\longrightarrow A[8]$	$\longleftarrow 2M_1^2\bar{M}_2$	$\longleftarrow 2(M_1^3\bar{M}_2 - M_1^6)$
ν	$\longleftarrow M_1^2$	$\longleftarrow \bar{M}_2$	$\longrightarrow \nu^2$	$\longrightarrow \nu^2 M_1$	$\longleftarrow M_1^2\bar{M}_2$	$\longrightarrow \nu^2 M_1^2$
	1	M_1	M_1^2	$\bar{M}_2$	$M_1^3+\bar{M}_2$	$M_1\bar{M}_2$

4ν	$\longleftarrow \eta^2 M_1^3\bar{M}_2$	$\longrightarrow A[14]$
2ν	$\longrightarrow A[8]M_1^2$	$\longrightarrow A[8]\bar{M}_2$
ν	$\longrightarrow \nu^2 M_2$	$\longrightarrow \nu^2(M_1\bar{M}_2 + M_1^4)$
	$M_1^2\bar{M}_2$	$M_1^3\bar{M}_2$

Figure 3.3.2: d^4 on $E^4_{*,3}$

The final application of our two lemmas is to compute E^6. We begin by strengthening the two lemmas to obtain a global calculation of all d^4-cycles and all d^4-boundaries. First we introduce an important algebra.

DEFINITION 3.3.9 Let $B\langle 4)$ be the subalgebra $Z_{(2)}[\langle M_1^4\rangle, \langle M_2^2\rangle, \langle M_3\rangle, \ldots \langle M_N\rangle, \ldots]$ of $E^8_{*,0}$.

LEMMA 3.3.10 In the notation of Theorem 3.2.3, assume that νX_k has order 2^B where $B = 1$, 2 or 3. Let $\varepsilon = 1$ if $\eta X_k \neq 0$, let $\varepsilon = 0$ if $\eta X_k = 0$ and let $\beta = B - \delta_3^B$. Then the kernel of d^4 on $Z_{2^{N(k)}} X_k \otimes H_*BP$ is the $B\langle 4)$- module spanned by $\{X_k, 2^{\varepsilon}M_1X_k, 2^{\beta}M_1^2X_k, 2^BM_1^3X_k, V_2X_k, V_1V_2X_k, 2^BM_1^2\bar{M}_2X_k, 2^BM_1^3\bar{M}_2X_k\}$.
The image of d^4 in $Z_{2^B}\ \nu\ X_k \otimes H_*BP$ is the $B\langle 4\rangle$-module spanned by
$\{\nu X_k, \nu M_1X_k, 2^{\varepsilon}\nu M_1^2X_k, \nu(\bar{M}_2+M_1^3)X_k, 2^{\varepsilon}\nu M_1\bar{M}_2X_k\}$.
PROOF. Recall that π_*^S is the direct sum of the X_k and therefore E^2 is the direct sum of the $Z_{2^k} X_k \otimes Z_{(2)}\{M_1^e, M_2^f \mid 0 \le e \le 3,\ 0 \le f \le 1\} \otimes B\langle 4\rangle$. Since all the elements of $B\langle 4\rangle$ survive to E^8,

$$E^4 = H_*(\oplus Z_{2^k} X_k \otimes Z_{(2)}\{M_1^eM_2^f \mid 0 \le e \le 3,\ 0 \le f \le 1\}, d^2) \otimes B\langle 4\rangle.$$

Now $H\langle 2\rangle = H_*(\oplus Z_{2^k} X_k \otimes Z_{(2)}\{M_1^eM_2^f \mid 0 \le e \le 3,\ 0 \le f \le 1\}, d^2)$ is given by Theorem 3.2.3. Then $E^6 = H_*(H\langle 2\rangle, d^4) \otimes B\langle 4\rangle$. Thus, Kernel d^4 = (Kernel $d^4 | H\langle 2\rangle) \otimes B\langle 4\rangle$ and Image d^4 = (Image $d^4 | H\langle 2\rangle) \otimes B\langle 4\rangle$. Therefore, this lemma follows from Lemmas 3.3.5 and 3.3.6. ∎

THEOREM 3.3.11 Let $\mu_q(\nu): \pi_q^S \longrightarrow \pi_{q+3}^S$ denote multiplication by ν. In the notation of Theorem 3.2.3, assume that the X_k have the following two addditional properties:

(i) Kernel $\mu_p(\nu)$ is generated by $\{X_k \mid \nu X_k = 0\} \cup \{2^{B(k)}X_k \mid \nu X_k \neq 0\}$ where $\nu \cdot X_k$ has order $2^{B(k)}$;

(ii) Image $\mu_{p-3}(\nu)$ is the direct sum of $\{Z_{2^{N(k)-\gamma(k)}}\ 2^{N(k)-\gamma(k)}X_k \mid k \in S\}$ for some subset S of $\{1,\ldots,t\}$.

Abuse notation by denoting B(k) by B and $\gamma(k)$ by γ. Let $\varepsilon = 1$ if $\eta\cdot X_k \neq 0$ and let $\varepsilon = 0$ if $\eta\cdot X_k = 0$. Let $\delta = 1$ if $2^{N(k)-1}X_k$ is divisible by η and let $\delta = 0$ if $2^{N(k)-1}X_k$ is not divisible by η. Then E^4 is the direct sum of the following B<4>-modules X_k<4>.

(a) If $2^{N(k)-1}X_k$ is not divisible by ν and $\nu X_k = 0$ then

$$X_k\langle 4\rangle = [Z_{2^{N(k)-\delta}}\ X_k \oplus Z_{2^{N(k)-\varepsilon}}\ (2^{\varepsilon}XM_1)] \otimes B\langle 2\rangle.$$

(b) If $2^{N(k)-1}X_k$ is not divisible by ν and $\nu X_k \neq 0$ then

$$\begin{aligned} X_k\langle 4\rangle = {} & [Z_{2^{N(k)-\delta}}\ X_k \oplus Z_{2^{N(k)-\varepsilon}}\ 2^{\varepsilon}M_1X_k \oplus Z_{2^{N(k)-\beta-\delta}}\ 2^{\beta}M_1^2X_k \oplus Z_{2^{N(k)-B}}\ 2^{B}M_1^3X_k \\ & \oplus Z_{2^{N(k)-\delta}}\ (\bar{M}_2+M_1^3)X_k \oplus Z_{2^{N(k)-\beta}}\ 2^{\beta}M_1\bar{M}_2X_k \oplus Z_{2^{N(k)-\delta-B}}\ 2^{B}M_1^2\bar{M}_2X_k \\ & \oplus Z_{2^{N(k)-B}}\ 2^{B}M_1^3\bar{M}_2X_k] \otimes B\langle 4\rangle. \end{aligned}$$

(c) Assume that X_k is divisible by ν, $\nu Y = 2^{N(k)-\gamma}X_k$ and $\nu X_k = 0$.

Let $\lambda = 0$ if $\eta Y = 0$ and let $\lambda = 1$ if $\eta Y \neq 0$. Then

$$\begin{aligned} X_k\langle 4\rangle = {} & [Z_{2^{N(k)-\gamma}}\ X_k \oplus Z_{2^{N(k)-\gamma-\varepsilon}}\ 2^{\varepsilon}M_1X_k \oplus Z_{2^{N(k)-\gamma+\lambda}}\ M_1^2X_k \oplus Z_{2^{N(k)-\varepsilon}}\ 2^{\varepsilon}M_1^3X_k \\ & \oplus Z_{2^{N(k)-\gamma}}\ (\bar{M}_2+M_1^3)X_k \oplus Z_{2^{N(k)-\gamma-\varepsilon+\lambda}}\ 2^{\varepsilon}M_1\bar{M}_2X_k \oplus Z_{2^{N(k)-\delta}}\ 2^{\delta}M_1^2\bar{M}_2X_k \\ & \oplus Z_{2^{N(k)-\varepsilon}}\ 2^{\varepsilon}M_1^3\bar{M}_2X_k] \otimes B\langle 4\rangle. \end{aligned}$$

(d) Assume that X_k is divisible by ν, $\nu Y = 2^{N(k)-\gamma}X_k$, and $\nu X_k \neq 0$.

Define λ as in (c). Then

$$\begin{aligned} X_k\langle 4\rangle = {} & [Z_{2^{N(k)-\gamma}}X_k \oplus Z_{2^{N(k)-\gamma-\varepsilon}}2^{\varepsilon}M_1X_k \oplus Z_{2^{N(k)-\beta-\gamma+\lambda}}2^{\beta}M_1^2X_k \oplus Z_{2^{N(k)-B-\delta}}2^{B}\bar{M}_2X_k \\ & \oplus Z_{2^{N(k)-\gamma}}(\bar{M}_2+M_1^3)X_k \oplus Z_{2^{N(k)-\beta-\gamma+\lambda}}2^{\beta}M_1\bar{M}_2X_k \oplus Z_{2^{N(k)-B-\delta}}2^{B}M_1^2\bar{M}_2X_k \\ & \oplus Z_{2^{N(k)-B}}2^{B}M_1^3\bar{M}_2X_k] \otimes B\langle 4\rangle. \end{aligned}$$

PROOF. This theorem follows from considering the various cases of the preceding lemma and Theorem 3.2.3.∎

COROLLARY 3.3.12 $E^6_{*,0} = E^8_{*,0} = Z_{(2)}\{V_1, V_1^2, V_1^3, V_2, V_1V_2, V_1^2V_2, 8M_1^3M_2\} \otimes B\langle 4\rangle$.

PROOF. $E^6_{*,0}$ follows from Theorem 3.3.11(b). $E^8_{*,0} = E^6_{*,0}$ because $\pi^S_5 = 0$. ∎

In the next theorem, we analyze the behavior of the spectral sequence on the 3 row .

THEOREM 3.3.13 (a) $E^6_{*,3} = \{Z_2(2\nu\bar{M}_2) \oplus Z_2(2\nu M_1^2\bar{M}_2) \oplus Z_4(2\nu M_1^3\bar{M}_2)\} \otimes B\langle 4\rangle$.

(b) $d^6(E^6_{*,3}) = \{Z_2A[8] \oplus Z_2A[8]M_1^2 \oplus Z_2A[8]\bar{M}_2\} \otimes B\langle 4\rangle$.

(c) $E^8_{*,3} = Z_2(4\nu M_1^3\bar{M}_2) \otimes B\langle 4\rangle$.

(d) $d^{12}(E^{12}_{*,3}) = Z_2A[14] \otimes B\langle 4\rangle$.

(e) $E^{14}_{*,3} = 0$.

PROOF. (a) This follows from Theorem 3.3.11(d).

(b) This follows from Theorem 3.3.7(b) and Figure 3.3.2.

(c) This is an immediate consequence of (a) and (b).

(d) This follows from Theorem 3.3.7(c).

(e) This is an immediate consequence of (c) and (d). ∎

We conclude this section with the determination of the behavior of the spectral sequence on the 6 row. We will require a relation in π^S_9 which is a nontrivial extension. The relation will be deduced from the following lemma.

LEMMA 3.3.14 If $X \in \pi^S_*$ such that $2X = 0$ and $\eta^2X \neq 0$ then there is an element $Y \in \langle\eta, 2, X\rangle$ such that $2Y = \eta^2X$.

PROOF. By Theorem 3.2.1(b), $\eta^2 \in \langle 2,\eta,2\rangle$, and $\eta^2\cdot X \in \langle 2,\eta,2\rangle X = 2\langle\eta,2,X\rangle$. ∎

Although the proof of the following two theorems require several technical results which will be proved later, the results seem more appropriate to this chapter than to Chapter 5. Therefore, we record them here.

THEOREM 3.3.15 (a) $\nu^3 = \eta A[8]$ is a nonzero element of π_9^S of order two.

(b) $E^6_{*,6} = 0$.

PROOF. (a) $A[8]M_1$ is not a d^6-boundary and therfore can not bound because $E^4_{*,1} = 0$. Therefore $\eta\cdot A[8] = d^2(A[8]M_1)$ is nonzero. If $\eta^2A[8] \neq 0$ then by Lemma 3.3.14 there is $Y \in \langle\eta,2,A[8]\rangle$ such that $2Y = \eta^2A[8]$. Thus,

modulo $\nu\cdot\pi_7^S \oplus \eta\cdot\pi_9^S$, $Y \in \langle\eta,2,A[8]\rangle$

$\subset \langle\eta,2,\langle\eta,\nu,2\nu\rangle\rangle$ by the proof of Theorem 3.3.7(b)

$= \langle\langle\eta,2,\eta\rangle,\nu,2\nu\rangle + \langle\eta,\langle2,\eta,\nu\rangle,2\nu\rangle$ by Theorem 2.3.3

$= \langle2\nu,\nu,2\nu\rangle$ by Theorem 2.4.1, noting $\langle2,\eta,\nu\rangle \in \pi_5^S = 0$

$= \langle\nu,4\nu,\nu\rangle$ by Theorem 2.3.3(d),(e)

$= \langle\nu,\eta^3,\nu\rangle = \eta^2\langle\nu,\eta,\nu\rangle$ by Theorem 2.3.3(a),(d).

Therefore $2Y \in 2\eta^2\langle\nu,\eta,\nu\rangle = 0 \bmod 2\nu\cdot\pi_7^S$. As we shall see in the next section, $\pi_7^S = Z_{16}\sigma$ and $\nu\cdot\sigma = 0$. Thus $\eta^2A[8] = 2Y = 0$. Therefore $\eta A[8]M_1$ must be a boundary. Since $\pi_4^S = 0$ and $E^4_{*,2} = 0$, $\eta A[8]M_1$ can only bound from the 0 row or the 6 row. In the next section we shall see that $E^{10}_{12,0}$ $= Z_{(2)}V_2^2 \oplus Z_{(2)}16V_1^3V_2 \oplus Z_{(2)}16M_1^6$ and in Chapter 4 we will see that $16M_1^6$ survives to E^{12} and $d^{12}(16M_1^6) = \beta_1$. Thus $\eta A[8]M_1$ must bound by a d^4-differential from the 6 row. Since $\pi_6^S = Z_2\nu^2$, it follows that $\nu^3 = \eta A[8]$. By Theorem 2.4.2,

$$\eta A[8] \in \langle\eta,\nu^2,\eta\rangle \qquad [3.4]$$

(b) This result is now an immediate consequence of Theorem 3.3.11(d). ∎

THEOREM 3.3.16 $2A[14] = 0$

PROOF. $0 \in \langle\eta,2,A[8]\rangle$ because $\langle\eta,2,A[8]\rangle \in \pi_{10}^S$ which, as we shall see, equals $\eta\cdot\pi_9^S$. Also $\langle2,A[8],\nu\rangle \in \pi_{12}^S$ which, as we shall see, is zero. Note that

$$\nu A[8] \in \nu\langle\eta,2,\nu^2\rangle = \langle\nu,\eta,2\rangle\nu^2 = 0. \qquad [3.5]$$

Thus by Theorem 2.2.7, $\langle\eta,2,A[8],\nu\rangle$ is defined. Note that hypothesis (i) of Theorem 2.4.3 is satisfied. Therefore,

$$A[14] \in \langle\eta, 2, A[8], \nu\rangle \qquad [3.6]$$

Note that $\nu A[8] \in \nu\langle\eta, 2, \nu^2\rangle = \langle\nu, \eta, 2\rangle\nu^2 = 0$. By Theorem 2.3.6(a), $2A[14] \in 2\langle\eta, 2, A[8], \nu\rangle \subset \langle\langle 2, \eta, 2\rangle, A[8], \nu\rangle = \langle\eta^2, A[8], \nu\rangle \subset \langle\eta, \eta A[8], \nu\rangle$ $= \langle\eta, \nu^3, \nu\rangle \supset \langle\eta, \nu, \nu^3\rangle = \langle\eta, \nu, \eta A[8]\rangle \supset \langle\eta, \nu, \eta\rangle A[8] = \nu^2 A[8] = 0$. Thus, $2A[14] = 0$ modulo $\eta\cdot\pi^S_{13} + \nu\cdot\pi^S_{11}$ which as we shall see is zero. ∎

4. d^8 Differentials and the Seven Row

We continue the study of our spectral sequence with an analysis of $d^8 : E^8_{*,0} \longrightarrow E^8_{*,7}$ and an analysis of all the differentials which originate on the 7 row: a d^8-differential which defines $\sigma^2 \in \pi^S_{14}$, a d^{10}-differential which defines $A[16] \in \pi^S_{16}$, a d^{12}-differential which defines $C[18] \in \pi^S_{18}$ and a d^{24}-differential which defines $A[30] \in \pi^S_{30}$. In the process of this analysis, we construct polynomial generators $\{M_N\}$ of H_*BP for $N \geq 5$ which survive to E^{10}. We conclude the section with a complete analysis of the 8 row.

Observe that $E^8_{8,0} = Z_{(2)}\langle M_1^4\rangle$ and $E^{10}_{8,0} = E^\infty_{8,0} = Z_{(2)}(16M_1^4)$. Thus, $\sigma = d^8\langle M_1^4\rangle$ is a nonzero element of π^S_7 of order 16. Since $E^4_{*,2} = 0$, $\pi^S_4 = 0$ and $\eta\cdot\pi^S_6 = \eta\cdot\nu^2 = 0$, there are no other nonzero differentials which land in $E^r_{0,7}$. Thus, we have proven the first part of the following theorem.

THEOREM 3.4.1 $\pi^S_7 = Z_{16}\ \sigma$ and $\eta\cdot\sigma \neq 0$.

PROOF. To show that $\eta\cdot\sigma \neq 0$, it suffices to show that σM_1 is not a d^8-boundary. Observe that $E^8_{10,0} = Z_{(2)}(2M_1\langle M_1^4\rangle) \oplus Z_{(2)}(V_1^2V_2)$. Note that $d^8(2M_1\langle M_1^4\rangle) = 10\sigma M_1$ and $V_1^2V_2$ is an infinite cycle. Thus σM_1 does not bound. ∎

We next compute d^8 on several of the key elements of $E^8_{*,0}$.

LEMMA 3.4.2 (a) $d^8\langle M_1^4\rangle = \sigma$.

(b) $d^8\langle M_2^2\rangle = 15\sigma M_1^2$.

(c) $d^8\langle M_3\rangle = 5\sigma\overline{M}_2$.

(d) We can choose $\langle M_4\rangle$ such that $d^8\langle M_4\rangle \equiv 2\sigma M_1M_2M_3$ modulo (4σ).

PROOF. (a) This differential defines σ.

(b) $d^8\langle M_2^2\rangle = d^8(M_2^2+M_1^6) = 15\sigma M_1^2$, using the Quillen operation $r_{2\Delta_1}$.

(c) $d^8\langle M_3\rangle = d^8(\overline{M}_3+5M_1^4\overline{M}_2+14M_1^7) = 15\sigma M_2 + 11\sigma M_1^3 = 5\sigma\overline{M}_2$ using the Quillen operations r_{Δ_2} and $r_{3\Delta_1}$.

(d) Observe that we can choose $\langle M_4\rangle \equiv M_4-M_1M_3^2+\overline{M}_2M_2^4+2M_1^2M_2^2M_3+2M_1^5\overline{M}_2M_3+5M_1^8\langle M_3\rangle$ modulo (4). Then d^8 of such an $\langle M_4\rangle$ is $2\sigma\ M_1M_2M_3$ modulo (4σ).∎

Next we analyze the d^8-differentials which originate on the 7 row.

THEOREM 3.4.3 (a) $\sigma^2 = d^8(\sigma\ M_1^4)$ is a nonzero element of order 2 in π_{14}^S.

(b) $[Z_8(2\sigma) \otimes H_*BP] \oplus [d^8(E_{*,0}^8) \otimes Z_2] = \text{Kernel}\ [d^8\colon E_{*,7}^8 \longrightarrow E_{*,14}^8]$.

(c) $[Z_2\sigma^2 \otimes H_*BP]\ /\ \text{Image}\ [d^8\colon E_{*,7}^8 \longrightarrow E_{*,14}^8]$

$$= \left[Z_2\sigma^2\{M_1^3,\ M_1^2\overline{M}_2,\ M_1^3\overline{M}_2\} \otimes B\langle 4\rangle\right]$$

$$\oplus \left[Z_2\sigma^2\{\langle M_1^4\rangle\langle M_2^2\rangle,\ M_1\langle M_1^4\rangle\langle M_2^2\rangle,\ M_1^2\langle M_1^4\rangle\langle M_2^2\rangle,\ \overline{M}_2\langle M_1^4\rangle\langle M_2^2\rangle,\right.$$
$$\left. M_1^2\overline{M}_2\langle M_1^4\rangle\langle M_2^2\rangle\}\otimes Z[\langle M_1^4\rangle^2,\langle M_2^2\rangle^2,\langle M_3\rangle,\ldots,\langle M_N\rangle,\ldots]\right]$$

$$\oplus \left[Z_2\sigma^2\{\overline{M}_2\langle M_1^4\rangle,\ M_1^2\overline{M}_2\langle M_1^4\rangle\} \otimes Z[\langle M_1^4\rangle^2,\langle M_2^2\rangle,\langle M_3\rangle,\ldots,\langle M_N\rangle,\ldots]\right]$$

$$\oplus \left[Z_2\sigma^2(M_1^6\langle M_3\rangle)[\langle M_1^4\rangle^2,\langle M_2^2\rangle^2,\langle M_3\rangle^2,\langle M_4\rangle,\ldots,\langle M_N\rangle,\ldots]\right].$$

(d) For $N \geq 4$, there are polynomial generators $\{M_N\}'$ of H_*BP which survive to E^8 and $d^8\{M_N\}' \in (2\sigma)$. Let $B'\langle 8\rangle = Z[\langle M_1^4\rangle^2,\langle M_2^2\rangle^2\langle M_3\rangle^2,\{M_4\}',\ldots,\{M_N\}',\ldots]$.

(e) $E_{*,7}^{10} = [Z_8(2\sigma) \otimes H_*BP]\ /\ [d^8(E_{*,0}^8) \cap (Z_8(2\sigma) \otimes H_*BP)]$.

PROOF. (a) $E_{16,0}^8 = Z_{(2)}\langle M_1^4\rangle^2 \oplus Z_{(2)}V_1V_2\langle M_1^4\rangle \oplus Z_{(2)}V_1^2\langle M_2^2\rangle \oplus Z_{(2)}V_1\langle M_3\rangle$.

Clearly $d^8(E_{16,0}^8) \subset (2\sigma)$. Therefore, σM_1^4 is not a boundary and $\sigma^2 = d^8(\sigma M_1^4) \neq 0$. By Theorem 2.4.2,

$$\sigma^2 \in \langle \nu,\sigma,\nu\rangle \qquad [3.7]$$

(b), (c), (d) These parts of the theorem will follow from the table in Figure 3.4.1. The symbol ≡ in the right hand column of that table means "congruent modulo $d^8(E^8_{*,7})$".

$E^8_{*,0}$		$E^8_{*,7}$		$E^8_{*,14}$
$\langle M_1^4\rangle$	$\longrightarrow$	σ		
$\langle M_2^2\rangle$	$\longrightarrow$	σM_1^2		
$\langle M_3\rangle$	$\longrightarrow$	$\sigma\bar{M}_2$		
		$\sigma M_1^2\bar{M}_2$	$\longrightarrow$	$\sigma^2 M_1$
		$\sigma\langle M_1^4\rangle$	$\longrightarrow$	σ^2
		$\sigma M_1^2\langle M_1^4\rangle$	$\longrightarrow$	$\sigma^2 M_1^2$
		$\sigma\bar{M}_2\langle M_1^4\rangle$	$\longrightarrow$	$\sigma^2\bar{M}_2$
		$\sigma M_1^2\bar{M}_2\langle M_1^4\rangle$	$\longrightarrow$	$\sigma^2 M_1^2 M_2$
$\langle M_1^4\rangle\langle M_2^2\rangle$	$\longrightarrow$	$\sigma(\langle M_2^2\rangle+M_1^2\langle M_1^4\rangle)$		
		$\sigma M_1^2\langle M_2^2\rangle$	$\longrightarrow$	$\sigma^2 M_1^4$
		$\sigma\bar{M}_2\langle M_2^2\rangle$	$\longrightarrow$	$\sigma^2(M_1^2M_2+M_1^5) \equiv \sigma^2 M_1^5$
		$\sigma M_1^2\bar{M}_2\langle M_2^2\rangle$	$\longrightarrow$	$\sigma^2(M_1M_2^2+M_2M_1^4)$
$\langle M_1^4\rangle\langle M_3\rangle$	$\longrightarrow$	$\sigma(\langle M_3\rangle+\bar{M}_2\langle M_1^4\rangle)$		
$\langle M_2^2\rangle\langle M_3\rangle$	$\longrightarrow$	$\sigma(M_1^2\langle M_3\rangle+\bar{M}_2\langle M_2^2\rangle)$		
		$\sigma\bar{M}_2\langle M_3\rangle$	$\longrightarrow$	$\sigma^2(M_2^2+M_1^6) \equiv \sigma^2 M_1^6$
		$\sigma M_1^2\bar{M}_2\langle M_3\rangle$	$\longrightarrow$	$\sigma^2(M_1\langle M_3\rangle+M_1^2\langle M_2^2\rangle) \equiv \sigma^2 M_1\langle M_3\rangle$
		$\sigma\langle M_1^4\rangle\langle M_2^2\rangle$	$\longrightarrow$	$\sigma^2 M_2^2$
		$\sigma M_1^2\langle M_1^4\rangle\langle M_2^2\rangle$	$\longrightarrow$	$\sigma^2 M_1^2 M_2^2$
		$\sigma\bar{M}_2\langle M_1^4\rangle\langle M_2^2\rangle$	$\longrightarrow$	$\sigma^2\bar{M}_2 M_2^2$
		$\sigma M_1^2\bar{M}_2\langle M_1^4\rangle\langle M_2^2\rangle$	$\longrightarrow$	$\sigma^2(M_1^2\bar{M}_2M_2^2+M_1^5\langle M_2^2\rangle)$
		$\sigma\langle M_1^4\rangle\langle M_3\rangle$	$\longrightarrow$	$\sigma^2(\langle M_3\rangle+\bar{M}_2M_1^4)$
		$\sigma M_1^2\langle M_1^4\rangle\langle M_3\rangle$	$\longrightarrow$	$\sigma^2(M_1^2\langle M_3\rangle+M_1^6\bar{M}_2)$
		$\sigma\bar{M}_2\langle M_1^4\rangle\langle M_3\rangle$	$\longrightarrow$	$\sigma^2(\bar{M}_2\langle M_3\rangle+M_1^4\langle M_2^2\rangle)$

$$\begin{array}{lll}
 & \sigma M_1^2\bar{M}_2\langle M_1^4\rangle\langle M_3\rangle \longrightarrow & \sigma^2(M_1^2\bar{M}_2\langle M_3\rangle+M_1^6\langle M_2^2\rangle+M_1^5\langle M_3\rangle) \\
\langle M_1^4\rangle\langle M_2^2\rangle\langle M_3\rangle \longrightarrow & \sigma(\langle M_2^2\rangle\langle M_3\rangle+M_1^2\langle M_1^4\rangle\langle M_3\rangle+\bar{M}_2\langle M_1^4\rangle\langle M_2^2\rangle) & \\
 & \sigma M_1^2\langle M_2^2\rangle\langle M_3\rangle \longrightarrow & \sigma^2(M_1^4\langle M_3\rangle+M_1^2\bar{M}_2\langle M_2^2\rangle) \\
 & \sigma\bar{M}_2\langle M_2^2\rangle\langle M_3\rangle \longrightarrow & \sigma^2(\langle M_2^2\rangle^2+M_1^2\bar{M}_2\langle M_3\rangle) \equiv \sigma^2M_1^2\bar{M}_2\langle M_3\rangle \\
 & \sigma M_1^2\bar{M}_2\langle M_2^2\rangle\langle M_3\rangle \longrightarrow & \sigma^2(M_1\langle M_2^2\rangle\langle M_3\rangle+\bar{M}_2M_1^4\langle M_3\rangle+M_1^2\langle M_2^2\rangle^2) \\
 & & \equiv \sigma^2(M_1\langle M_2^2\rangle\langle M_3\rangle+\bar{M}_2M_1^4\langle M_3\rangle) \\
 & \sigma\langle M_1^4\rangle\langle M_2^2\rangle\langle M_3\rangle \longrightarrow & \sigma^2(\langle M_2^2\rangle\langle M_3\rangle+M_1^6\langle M_3\rangle+\bar{M}_2M_1^4\langle M_2^2\rangle) \\
 & \sigma M_1^2\langle M_1^4\rangle\langle M_2^2\rangle\langle M_3\rangle \longrightarrow & \sigma^2(M_1^2\langle M_2^2\rangle\langle M_3\rangle+M_1^8\langle M_3\rangle+M_1^6\bar{M}_2\langle M_2^2\rangle) \\
 & \sigma\bar{M}_2\langle M_1^4\rangle\langle M_2^2\rangle\langle M_3\rangle \longrightarrow & \sigma^2(\bar{M}_2\langle M_2^2\rangle\langle M_3\rangle+M_1^6\bar{M}_2\langle M_3\rangle+M_1^4\langle M_2^2\rangle^2) \\
 & & \equiv \sigma^2(\bar{M}_2\langle M_2^2\rangle\langle M_3\rangle+M_1^6\bar{M}_2\langle M_3\rangle) \\
 & \sigma M_1^2M_2\langle M_1^4\rangle\langle M_2^2\rangle\langle M_3\rangle \longrightarrow & \\
 & & \sigma^2(M_1^5\langle M_2^2\rangle\langle M_3\rangle+M_1^2M_2\langle M_2^2\rangle\langle M_3\rangle+M_2M_1^8\langle M_3\rangle+M_1^6\langle M_2^2\rangle^2) \\
 & & \equiv \sigma^2(M_1^5\langle M_2^2\rangle\langle M_3\rangle+M_1^2\bar{M}_2\langle M_2^2\rangle\langle M_3\rangle+\bar{M}_2M_1^8\langle M_3\rangle)
\end{array}$$

Figure 3.4.1: d^8-Boundaries

We prove that (b), (c) and (d) are valid for $E^8_{N,7}$ simultaneously by induction on N. Of course we only need to worry about (d) when N is of the form $2^p - 10$, and in that case we have to show that $\{M_p\}'$ exists. Clearly (b), (c) and (d) are valid for small values of N. Assume that $2^q-10 < N \le 2^{q+1}-10$ and that (b), (c), (d) are valid for $E^8_{K,7}$ if $k < N$. Let $A = Z[M_1^8, \langle M_2^2\rangle^2, \langle M_3\rangle^2, \{M_4\}', \ldots, \{M_q\}']$, and let B be the set of all monomials in the given polynomial generators of A. Let R be the set of elements in the right hand column of the above table, and let L be the set of elements in the left hand column of the above table. The elements of R are summarized in the table of Figure 3.4.2 below. In that table an entry * indicates that the given element bounds modulo other entries of the table. Note that this table is valid when tensored with $Z_2[\langle M_1^4\rangle^2, \langle M_2^2\rangle^2, \langle M_3\rangle^2, \langle M_4\rangle, \ldots, \langle M_N\rangle, \ldots]$. In particular, it is evident from the table that $\{r\cdot b | r \in R \text{ and } b \in B\}$ is linearly independent. Therefore $\{ l\cdot b \mid l \in L \text{ and } b \in B \}$ are showing how all the elements in the kernel of $d^8 : E^8_{t,7} \longrightarrow E^8_{t,14}$ are boundaries for $t \le N$. This proves (b). The image of $d^8 : E^8_{*,7} \longrightarrow E^8_{*,14}$ is the $Z_2[<M_1^4>^2, <M_2^2>^2, <M_3>^2, <M_4>, \ldots <M_N>, \ldots]$-module spanned by the elements in the right hand column of Figure 3.4.1. Examination of that column, as depicted in Figure 3.4.2, verifies that the assertion made in (c) is true. If $N = 2^{q+1}-10$ then $d^8<M_{q+1}>$ is a d^8-cycle. By our proof of (b), this d^8-cycle is in $[Z_8(2\sigma) \otimes H_*BP] \oplus d^8(Z_{(2)}[<M_1^4>, <M_2^2>, <M_3>, \{M_4\}', \ldots, \{M_q\}'])$. So write $d^8<M_{q+1}> = 2\sigma X + d^8(\gamma)$ for some $\gamma \in Z_{(2)}[<M_1^4>, <M_2^2>, <M_3>, \{M_4\}', \ldots, \{M_q\}']$. Let $\{M_{q+1}\}' = <M_{q+1}> - \gamma$. This completes the proof of the induction step.

(e) This statement is a consequence of (b). ∎

σ^2	1	$<M_1^4>$	$<M_2^2>$	$<M_3>$	$<M_1^4><M_2^2>$	$<M_1^4><M_3>$	$<M_2^2><M_3>$	$<M_1^4><M_2^2><M_3>$
1	*	*	*	*		*	*	
M_1	*	*	*	*		*	*	
M_1^2	*	*	*	*			*	
$\bar{M}_2$	*		*	*			*	
$M_1^2\bar{M}_2$	*		*	*			*	

Figure 3.4.2: Summary of d^8- Boundaries

We digress to prove a result from which it will follow that $\nu \cdot \sigma = 0$.

THEOREM 3.4.4 Let $\phi \in \pi_*^S$ such that ϕM_1^2 and $\phi \bar{M}_2$ are both boundaries. Then $\nu \cdot \phi = 0$.

PROOF. Since ϕM_1^2 bounds, $d^4(\phi M_1^2) = 0$. Therefore $\nu \cdot \phi$ is a d^2-boundary which means $\nu \cdot \phi$ is divisible by η. Then $d^4(\phi \bar{M}_2) = \nu \phi M_1$ is zero if and only if $\nu \cdot \phi = 0$ in π_*^S. Since $\phi \bar{M}_2$ bounds, it follows that $\nu \cdot \phi = 0$. ∎

COROLLARY 3.4.5 $\nu\cdot\sigma = 0$ and $\eta\cdot\sigma^2 = 0$.

PROOF. Note that $\sigma\, M_1^2 = d^8\langle M_2^2\rangle$ and $\sigma\, \bar{M}_2 = d^8\langle M_3\rangle$. By Theorem 3.4.4, $\nu\cdot\sigma = 0$. Now $\sigma^2 M_1 = d^8(\sigma M_1^2\bar{M}_2)$. Thus, $\eta\sigma^2 = d^2(\sigma^2 M_1) = 0$. ∎

We next prove that the analogue Theorem 3.4.3 which analyzes the d^{10}–differentials which originate on the 7 row.

THEOREM 3.4.6 (a) $A[16] = d^{10}(\sigma M_1^5)$ is a nonzero element of order 2 in π_{16}^S.

(b) $E_{*,7}^{12}$ consists of the $[d^8(E_{*,0}^8) \cap (Z_8(2\sigma)\otimes H_*BP)]$-cosets of

$$E(7,12) \equiv [Z_8(2\sigma)\{A\} \otimes B'\langle 8\rangle]$$
$$\oplus\ [Z_8\{2\sigma M_1^{2\alpha}\bar{M}_2^{\beta}\langle M_1^4\rangle^{\gamma}\langle M_2^2\rangle^{\delta}\langle M_3\rangle^{\varepsilon} \mid 0 \le \alpha,\beta,\gamma,\delta,\varepsilon \le 1,\ (\alpha,\beta) \ne (1,1) \text{ and } (\delta,\varepsilon) \ne (1,1)\} \otimes B'\langle 8\rangle].$$

Here A is the following set:

$$\{M_1^7\bar{M}_2+M_1\bar{M}_2\langle M_2^2\rangle,\quad M_1^3\bar{M}_2\langle M_3\rangle+M_1^7M_2^2,\quad M_1^6\bar{M}_2+M_1^3\langle M_2^2\rangle,$$
$$M_1^2\bar{M}_2\langle M_2^2\rangle+M_1\bar{M}_2\langle M_3\rangle,\quad M_2^2\bar{M}_3+M_1^{13},\quad M_2^2\langle M_3\rangle M_1^2+M_1^{12}\bar{M}_2,$$
$$\bar{M}_2M_2^2\bar{M}_3+M_1^6\bar{M}_2\langle M_3\rangle,\quad \langle M_1^4\rangle M_2^2\bar{M}_3+M_1^{11}\langle M_2^2\rangle,\quad M_1^6M_2^2\bar{M}_3+M_1^9\bar{M}_2\langle M_3\rangle,$$
$$M_1^6\bar{M}_2\langle M_3\rangle+M_1^3\langle M_2^2\rangle\langle M_3\rangle+M_1^7\bar{M}_2\langle M_2^2\rangle,\quad \bar{M}_2\langle M_1^4\rangle M_2^2\bar{M}_3+M_1^{10}\bar{M}_2\langle M_3\rangle\}.$$

(c) $d^{10}(E_{*,7}^{10}) \subset Z_2A[16] \otimes B\langle 2\rangle$ and

$$[Z_2A[16] \otimes B\langle 2\rangle] \,/\, \text{Image}\ [d^{10}: E_{*,7}^{10} \longrightarrow E_{*,16}^{10}]$$
$$= Z_2A[16]\{M_1^2\bar{M}_2\langle M_1^4\rangle,\ M_1^2\bar{M}_2\langle M_1^4\rangle\langle M_2^2\rangle,\ M_1^2\langle M_1^4\rangle\langle M_3\rangle,\ M_1^2\bar{M}_2\langle M_1^4\rangle\langle M_3\rangle,$$
$$M_1^2\langle M_1^4\rangle\langle M_2^2\rangle\langle M_3\rangle,\ \bar{M}_2\langle M_1^4\rangle\langle M_2^2\rangle\langle M_3\rangle,\ M_1^2\bar{M}_2\langle M_1^4\rangle\langle M_2^2\rangle\langle M_3\rangle\} \otimes B'\langle 8\rangle.$$

(d) For $N \ge 4$, there are polynomial generators $\{M_N\}''$ of H_*BP which survive to E^8 and $d^8\{M_N\}'' \in E(7,12)$.

Let $B''\langle 8\rangle = Z[\langle M_1^4\rangle^2, \langle M_2^2\rangle^2\langle M_3\rangle^2, \{M_4\}'', \ldots, \{M_N\}'', \ldots]$.

PROOF. (a) $E_{18,0}^8 = Z_{(2)}\{V_1^2\langle M_3\rangle, V_2\langle M_2^2\rangle, V_1^3\langle M_2^2\rangle, V_1^2V_2\langle M_1^4\rangle, V_1\langle M_1^4\rangle^2\}$. Therefore, $d^8(E_{18,0}^8) = Z_8(2\sigma M_1^2M_2) \oplus Z_4(4\sigma M_1^5)$ and $E_{10,7}^{10} = Z_2(2\sigma M_1^5)$. It follows that there must be a nonzero differential originating on $2\sigma M_1^5 = \sigma V_1\langle M_1^4\rangle$. Now $2\sigma M_1^5$ survives to E^{10} and $d^{10}(2\sigma M_1^5)$ defines a nonzero element $A[16]$ of π_{16}^S. By Theorem 2.4.2,

$$A[16] \in \langle \eta, 2, \sigma^2 \rangle. \qquad [3.8]$$

Then $2A[16] \in 2\langle \eta, 2, \sigma^2 \rangle = \langle 2, \eta, 2 \rangle \sigma^2 = \eta^2\sigma^2 = 0$. Hence $2A[16] = 0$.

(b), (c), (d) The proof of the remainder of this theorem is analogous to the proof of the last four parts of Theorem 3.4.3. We create a new table in Figure 3.4.3 from the table in Figure 3.4.1 by changing each σ^2 to $A[16]$ and multiplying each monomial in the first and second columns by V_1. In addition there are monomials μ in $Z[M_1^2, \bar{M}_2, \bar{M}_3, \{M_4\}', \ldots, \{M_N\}', \ldots] \otimes Z_8(2\sigma)$ which map to $2\sigma M_1^2 \bar{M}_2$ under a Quillen operation and hence $d^{10}(2\sigma\mu) \neq 0$. Such monomials μ must be divisible by $M_1^2\bar{M}_2$ or by $M_2^2\bar{M}_3$. We include the $2\sigma\mu$ of this form in the table in Figure 3.4.3. Note that all $2\sigma\mu$ which are not listed in Figure 3.4.3 are d^{10}-cycles.

$E^8_{*,0}$		$E^r_{*,7}$		$E^{10}_{*,16}$
$V_1\langle M_1^4\rangle$	$\longrightarrow$	$2\sigma M_1$		
$V_1\langle M_2^2\rangle$	$\longrightarrow$	$2\sigma M_1^3$		
$V_1\langle M_3\rangle$	$\longrightarrow$	$2\sigma M_1\bar{M}_2$		
$V_2\langle M_3\rangle+2\langle M_1^4\rangle\langle M_2^2\rangle$	$\longrightarrow$	$2\sigma(M_1^3\bar{M}_2+M_1^6)$		
		$2\sigma M_1\langle M_1^4\rangle$	$\longrightarrow$	$A[16]$
		$2\sigma M_1^3\langle M_1^4\rangle$	$\longrightarrow$	$A[16]M_1^2$
		$2\sigma M_1\bar{M}_2\langle M_1^4\rangle$	$\longrightarrow$	$A[16]\bar{M}_2$
		$2\sigma M_1^3\bar{M}_2\langle M_1^4\rangle$	$\longrightarrow$	$A[16]M_1^2\bar{M}_2$
$V_1\langle M_1^4\rangle\langle M_2^2\rangle$	$\longrightarrow$	$2\sigma M_1M_2^2$		
		$2\sigma M_1^3\langle M_2^2\rangle$	$\longrightarrow$	$A[16]M_1^4$
		$2\sigma(M_1\bar{M}_2\langle M_2^2\rangle+M_1^3\bar{M}_2\langle M_1^4\rangle)$	$\longrightarrow$	0
		$2\sigma M_1^3\bar{M}_2\langle M_2^2\rangle$	$\longrightarrow$	$A[16]\bar{M}_2M_1^4$
$V_1\langle M_1^4\rangle\langle M_3\rangle$	$\longrightarrow$	$2\sigma M_1(\langle M_3\rangle+\bar{M}_2\langle M_1^4\rangle)$		
$V_1\langle M_2^2\rangle\langle M_3\rangle$	$\longrightarrow$	$2\sigma M_1(M_1^2\langle M_3\rangle+\bar{M}_2\langle M_2^2\rangle)$		
		$2\sigma(M_1\bar{M}_2\langle M_3\rangle+M_1^5\langle M_2^2\rangle)$	$\longrightarrow$	$A[16]M_1^6$

$$2\sigma(M_1^3\bar{M}_2\langle M_3\rangle+M_1^7M_2^2) \longrightarrow 0$$

$$2\sigma M_1\langle M_1^4\rangle\langle M_2^2\rangle \longrightarrow A[16]M_2^2$$

$$2\sigma M_1^3\langle M_1^4\rangle\langle M_2^2\rangle \longrightarrow A[16]M_1^2M_2^2$$

$$2\sigma M_1\bar{M}_2\langle M_1^4\rangle\langle M_2^2\rangle \longrightarrow A[16]\bar{M}_2M_2^2$$

$$2\sigma M_1^3\bar{M}_2\langle M_1^4\rangle\langle M_2^2\rangle \longrightarrow A[16]M_1^2\bar{M}_2M_2^2$$

$$2\sigma M_1\langle M_1^4\rangle\langle M_3\rangle \longrightarrow A[16]\bar{M}_3$$

$$2\sigma M_1^3\langle M_1^4\rangle\langle M_3\rangle \longrightarrow A[16]M_1^2\bar{M}_3$$

$$2\sigma M_1\bar{M}_2\langle M_1^4\rangle\langle M_3\rangle \longrightarrow A[16](\bar{M}_2\langle M_3\rangle+M_1^{10})$$

$$2\sigma(M_1^3\bar{M}_2\langle M_1^4\rangle\langle M_3\rangle+M_1\bar{M}_2\langle M_2^2\rangle\langle M_3\rangle+M_1^5\langle M_2^2\rangle^2) \longrightarrow A[16]M_1^6M_2^2$$

$$V_1\langle M_1^4\rangle\langle M_2^2\rangle\langle M_3\rangle \longrightarrow 2\sigma M_1(M_2^2\langle M_3\rangle+\bar{M}_2\langle M_1^4\rangle\langle M_2^2\rangle)$$

$$2\sigma(M_1^3\langle M_2^2\rangle\langle M_3\rangle+M_1^3\bar{M}_2\langle M_1^4\rangle\langle M_2^2\rangle) \longrightarrow A[16]M_1^4\langle M_3\rangle$$

$$2\sigma(M_1\bar{M}_2\langle M_2^2\rangle\langle M_3\rangle+M_1^5\langle M_2^2\rangle^2+M_1^{11}\langle M_2^2\rangle) \longrightarrow A[16]M_1^2\bar{M}_2\langle M_3\rangle$$

$$2\sigma(M_1^3\bar{M}_2\langle M_2^2\rangle\langle M_3\rangle+M_1^7\langle M_2^2\rangle^2+M_1^9\bar{M}_2\langle M_3\rangle) \longrightarrow A[16]\bar{M}_2M_1^4\langle M_3\rangle$$

$$2\sigma M_1\langle M_1^4\rangle\langle M_2^2\rangle\langle M_3\rangle \longrightarrow A[16](M_2^2\langle M_3\rangle+\bar{M}_2M_1^4\langle M_2^2\rangle)$$

$$2\sigma M_1^3\langle M_1^4\rangle\langle M_2^2\rangle\langle M_3\rangle \longrightarrow A[16](M_1^2M_2^2\langle M_3\rangle+M_1^6\bar{M}_2\langle M_2^2\rangle)$$

$$2\sigma(M_1\bar{M}_2\langle M_1^4\rangle\langle M_2^2\rangle\langle M_3\rangle+M_1^3\langle M_2^2\rangle^3) \longrightarrow A[16]\bar{M}_2M_2^2\langle M_3\rangle$$

$$2\sigma(M_1^3\bar{M}_2\langle M_1^4\rangle\langle M_2^2\rangle\langle M_3\rangle+M_1\bar{M}_2\langle M_3\rangle\langle M_2^2\rangle^2+M_1^{23}) \longrightarrow A[16]M_1^2\bar{M}_2M_2^2\langle M_3\rangle$$

$$V_2\langle M_2^2\rangle \longrightarrow 2\sigma(M_1^2\bar{M}_2+M_1^5)$$

$$2\sigma(M_1^2\bar{M}_2\langle M_1^4\rangle+M_1^3\langle M_2^2\rangle) \longrightarrow 0$$

$$2\sigma(M_1^2\bar{M}_2\langle M_2^2\rangle+M_1\bar{M}_2\langle M_3\rangle) \longrightarrow 0$$

$$V_2\langle M_2^2\rangle\langle M_3\rangle \longrightarrow 2\sigma(M_1^2\bar{M}_2\langle M_3\rangle+M_1^5\langle M_3\rangle+\bar{M}_2\langle M_2^2\rangle)$$

$$2\sigma M_1^2\bar{M}_2\langle M_1^4\rangle\langle M_2^2\rangle \longrightarrow A[16]M_1^4\langle M_2^2\rangle$$

$$2\sigma(M_1^2\bar{M}_2\langle M_1^4\rangle\langle M_3\rangle+M_1^3\langle M_2^2\rangle\langle M_3\rangle+M_1^7\bar{M}_2\langle M_2^2\rangle) \longrightarrow 0$$

$$2\sigma(M_1^2\bar{M}_2\langle M_2^2\rangle\langle M_3\rangle+M_1^5\langle M_2^2\rangle\langle M_3\rangle) \longrightarrow A[16](M_1^6\langle M_3\rangle+M_1^4\bar{M}_2^3)$$

$$2\sigma M_1^2\bar{M}_2\langle M_1^4\rangle\langle M_2^2\rangle\langle M_3\rangle \longrightarrow A[16]\langle M_1^4\rangle\langle M_2^2\rangle\langle M_3\rangle$$

$$2\sigma(M_2^2\bar{M}_3+M_1^{13}) \longrightarrow 0$$

$$2\sigma(M_1^2M_2^2\bar{M}_3+M_1^6\bar{M}_2\langle M_2^2\rangle) \longrightarrow 0$$

$$2\sigma(\bar{M}_2M_2^2\bar{M}_3+M_1^6\bar{M}_2\langle M_3\rangle) \longrightarrow 0$$

$$2\sigma(\langle M_1^4\rangle M_2^2\overline{M}_3+M_1^{11}\langle M_2^2\rangle) \longrightarrow 0$$

$$2\sigma(M_1^2\langle M_1^4\rangle M_2^2\overline{M}_3+M_1^9\overline{M}_2\langle M_3\rangle) \longrightarrow 0$$

$$2\sigma(\overline{M}_2\langle M_1^4\rangle M_2^2\overline{M}_3+M_1^{10}\overline{M}_2\langle M_3\rangle) \longrightarrow 0$$

Figure 3.4.3: d^{10}-Boundaries

The analysis of the table in Figure 3.4.1 which we used to prove Theorem 3.4.3(b)-(e) applies to the table in Figure 3.4.3 to prove (b), (c) and (d) of our Theorem. The right hand column of the table in Figure 3.4.3 is summarized in the table of Figure 3.4.4 below to assist in verifying (c).∎

A[16]	1	$\langle M_1^4\rangle$	$\langle M_2^2\rangle$	$\langle M_3\rangle$	$\langle M_1^4\rangle\langle M_2^2\rangle$	$\langle M_1^4\rangle\langle M_3\rangle$	$\langle M_2^2\rangle\langle M_3\rangle$	$\langle M_1^4\rangle\langle M_2^2\rangle\langle M_3\rangle$
1	*	*	*	*	*	*	*	*
M_1^2	*	*	*	*	*		*	
$\overline{M}_2$	*	*	*	*	*	*	*	
$M_1^2\overline{M}_2$	*		*	*			*	

Figure 3.4.4: Summary of d^{10}- Boundaries

We continue our analysis of differentials which originate on the 7 row by considering the d^{12}-differentials which originate there. We will eventually see in Chapter 4 that they land in a direct sum of Z_8s and Z_4s. In the next theorem we analyze these d^{12}-differentials tensored with Z_2. We use the notation $\tilde{E}(S)$ to denote the reduced exterior algebra generated by the set S.

THEOREM 3.4.7 (a) There is an element $C[18] \in \pi_{18}^S$ such that the projection of C[18] into $E_{0,18}^{12}$ has order four and equals $d^{12}(2\sigma M_1^6)$.

(b) Let $K(12) = \{X \in E^8_{*,7} | X$ survives to E^{12} and

$X \in \text{Kernel } [d^{12}: E^{12}_{*,7} \longrightarrow E^{12}_{*,18}]\}$.

Then $K(12) / \{[d^8(E^8_{*,0}) \oplus Z_4(4\sigma) \otimes H_*BP] \cap K(12)]\} =$

$$K_2 \otimes Z[<M_1^4>^4, <M_2^2>^4, <M_3>^4, <M_4>^2, \{M_5\}', \ldots, \{M_N\}', \ldots\}]$$

where $K_2 = Z_2(2\sigma)\{<M_1^4>, M_1^2<M_2^2>, \bar{M}_2<M_3>, M_1M_2M_3\} \otimes \tilde{E}(<M_1^4>^2, <M_2^2>^2, <M_3>^2, <M_4>)$

modulo the following relations:

(i) $2\sigma<M_2^2> + 2\sigma M_1^2<M_1^4> = 0$;

(ii) $2\sigma<M_3> + 2\sigma\bar{M}_2<M_1^4> = 0$;

(iii) $2\sigma<M_4> + 2\sigma M_1M_2M_3<M_1^4> = 0$;

(iv) $2\sigma M_1^2<M_3> + 2\sigma\bar{M}_2<M_2^2> = 0$;

(v) $2\sigma M_1^2<M_4> + 2\sigma M_1M_2M_3<M_2^2> = 0$;

(vi) $2\sigma\bar{M}_2<M_4> + 2\sigma M_1M_2M_3<M_3> = 0$;

(vii) $2\sigma<M_2^2><M_3> + 2\sigma M_1^2<M_1^4><M_3> + 2\sigma\bar{M}_2<M_1^4><M_2^2> = 0$;

(viii) $2\sigma<M_2^2><M_4> + 2\sigma M_1^2<M_1^4><M_4> + 2\sigma M_1M_2M_3<M_1^4><M_2^2> = 0$;

(ix) $2\sigma<M_3><M_4> + 2\sigma\bar{M}_2<M_1^4><M_4> + 2\sigma M_1M_2M_3<M_1^4><M_3> = 0$;

(x) $2\sigma M_1^2<M_3><M_4> + 2\sigma\bar{M}_2<M_2^2><M_4> + 2\sigma M_1M_2M_3<M_2^2><M_3> = 0$;

(xi) $2\sigma<M_2^2><M_3><M_4> + 2\sigma M_1^2<M_1^4><M_3><M_4> + 2\sigma\bar{M}_2<M_1^4><M_2^2><M_4>$

$+ 2\sigma M_1M_2M_3<M_1^4><M_2^2><M_3> = 0$;

(c) Let $T = Z_2 \otimes [Z_8C[18] \otimes H_*BP]$. Then

$T / \{T \cap \text{Image } [d^{12}: E^{12}_{*,7} \longrightarrow E^{12}_{*,18}]\}$

$= Z_2\{M_1^\alpha M_2^\beta<M_1^4>^{e(1)}<M_2^2>^{e(2)}<M_3>^{e(3)}\cdots<M_N>^{e(N)}\cdots | 0\le\alpha\le3,\ 0\le\beta\le1, e(N)\ge0$ for $N\ge1$

and either (i) $\alpha+\beta\ge2$ or (ii) $e(1)e(2)$ is odd while $e(3)$ is even$\}$.

(d) For $N \ge 5$, there are polynomial generators $\{M_N\}''$ of H_*BP which survive to E^8 such that $d^8\{M_N\}'' \in (4\sigma)$.

PROOF. (a) $E^8_{20,0} = Z_{(2)}V_1^2<M_1^4>^2 \oplus Z_{(2)}(8M_1^3M_2<M_1^4>) \oplus Z_{(2)}<M_1^4><M_2^2>$ $\oplus Z_{(2)}V_1V_2<M_2^2> \oplus Z_{(2)}V_1^3<M_3> \oplus Z_{(2)}V_2<M_3>$. Thus, $d^8(E^8_{20,0}) =$ $Z_2(8\sigma M_1^6) \oplus Z_{16}(\sigma M_2^2) \oplus Z_8(2\sigma M_1^3M_2)$. Hence $E^8_{12,7} = Z_4(2\sigma M_1^6)$. Now $2\sigma M_1^2<M_1^4>$ survives to E^{10}. Since $2\sigma M_1^2<M_1^4> \in \text{Image } r_{\Delta_1}$, $2\sigma M_1^2<M_1^4>$ must be a d^{10}-cycle.

Therefore $d^{12}(2\sigma M_1^6)$ is a nonzero element of $E^{12}_{0,18}$ of order four. This element is the projection into $E^{12}_{0,18}$ of an element $C[18] \in \pi^S_{18}$. By Theorem 2.4.2,

$$C[18] \in \langle \nu, \sigma, 2\sigma \rangle. \tag{3.9}$$

(b)-(d) The proof will be analogous to the proof of Theorem 3.4.3(b)-(e) and will use the table in Figure 3.4.5 below. The entries in the middle column are all the $B''\langle 8\rangle$-module generators of $E^{12}_{*,7}$ according to Theorem 3.4.4(b). The only change that we have made is to replace $2\sigma(M_1^2\bar{M}_2\langle M_2^2\rangle+M_1\bar{M}_2\langle M_3\rangle)$ by $2M_1M_2M_3$ since they are equal modulo other entries in the table.

$E^8_{*,0}$		$E^r_{*,7}$		$E^{12}_{*,18}$
$2\langle M_1^4\rangle$	$\longrightarrow$	2σ		
$2\langle M_2^2\rangle$	$\longrightarrow$	$2\sigma M_1^2$		
$2\langle M_3\rangle$	$\longrightarrow$	$2\sigma\bar{M}_2$		
$\langle M_1^4\rangle^2$	$\longrightarrow$	$2\sigma\langle M_1^4\rangle$		
		$2\sigma M_1^2\langle M_1^4\rangle$	$\longrightarrow$	$C[18]$
		$2\sigma\bar{M}_2\langle M_1^4\rangle$	$\longrightarrow$	$C[18]M_1$
$V_2\langle M_1^4\rangle\langle M_2^2\rangle+V_1\langle M_1^4\rangle^3$	$\longrightarrow$	$2\sigma(M_1^2\bar{M}_2\langle M_1^4\rangle+M_1^3\langle M_2^2\rangle+\bar{M}_2\langle M_2^2\rangle)$		
$2\langle M_1^4\rangle\langle M_2^2\rangle$	$\longrightarrow$	$2\sigma\langle M_2^2\rangle+M_1^2\langle M_1^4\rangle$		
$\langle M_2^2\rangle^2$	$\longrightarrow$	$2\sigma M_1^2\langle M_2^2\rangle$		
		$2\sigma\bar{M}_2\langle M_2^2\rangle$	$\longrightarrow$	$C[18]M_2$
		$2\sigma(M_1^2\bar{M}_2\langle M_2^2\rangle+M_1\bar{M}_2\langle M_3\rangle)$	$\longrightarrow$	$C[18]M_1^5$
$2\langle M_1^4\rangle\langle M_3\rangle$	$\longrightarrow$	$2\sigma(\langle M_3\rangle+\bar{M}_2\langle M_1^4\rangle)$		
$2\langle M_2^2\rangle\langle M_3\rangle$	$\longrightarrow$	$2\sigma(M_1^2\langle M_3\rangle+\bar{M}_2\langle M_2^2\rangle)$		
$\langle M_3\rangle^2$	$\longrightarrow$	$2\sigma\bar{M}_2\langle M_3\rangle$		
		$2\sigma\langle M_1^4\rangle\langle M_2^2\rangle$	$\longrightarrow$	$C[18]M_1^4$
		$2\sigma M_1^2\langle M_1^4\rangle\langle M_2^2\rangle$	$\longrightarrow$	$C[18]\langle M_2^2\rangle$
		$2\sigma\bar{M}_2\langle M_1^4\rangle\langle M_2^2\rangle$	$\longrightarrow$	$C[18](M_1^4M_2+M_1\langle M_2^2\rangle)$
$\{M_4\}'$	$\longrightarrow$	$2\sigma M_1M_2M_3$		

$$\begin{array}{rll}
 & 2\sigma M_1^2\langle M_1^4\rangle\langle M_3\rangle & \longrightarrow C[18](\bar{M}_3+M_1^7) \\
 & 2\sigma\bar{M}_2\langle M_1^4\rangle\langle M_3\rangle & \longrightarrow C[18]M_1\langle M_3\rangle \\
 & 2\sigma(M_1^2\bar{M}_2\langle M_1^4\rangle\langle M_3\rangle+M_1^3\langle M_2^2\rangle\langle M_3\rangle+M_1^7\bar{M}_2\langle M_2^2\rangle) & \longrightarrow C[18]M_2\langle M_3\rangle \\
2\langle M_1^4\rangle\langle M_2^2\rangle\langle M_3\rangle \rightarrow & 2\sigma(\langle M_2^2\rangle\langle M_3\rangle+M_1^2\langle M_1^4\rangle\langle M_3\rangle+\bar{M}_2\langle M_1^4\rangle\langle M_2^2\rangle) & \\
 & 2\sigma M_1^2\langle M_2^2\rangle\langle M_3\rangle & \longrightarrow C[18]M_2\langle M_2^2\rangle \\
V_2V_3\langle M_1^4\rangle\langle M_2^2\rangle \longrightarrow & 2\sigma(M_2M_3\langle M_2^2\rangle+M_1^2M_2\langle M_1^4\rangle M_3) & \\
 & 2\sigma(\langle M_1^4\rangle M_2^2\bar{M}_3+M_1^{11}\langle M_2^2\rangle) & \longrightarrow C[18]M_1^4M_3 \\
 & 2\sigma(M_1^2\langle M_1^4\rangle M_2^2\bar{M}_3+M_1^9\bar{M}_2\langle M_3\rangle) & \longrightarrow C[18](\langle M_2^2\rangle\langle M_3\rangle+M_2\langle M_1^4\rangle\langle M_2^2\rangle) \\
 & 2\sigma(\bar{M}_2\langle M_1^4\rangle M_2^2\bar{M}_3+M_1^{10}\bar{M}_2\langle M_3\rangle) & \longrightarrow C[18](M_1^4\bar{M}_2M_3+M_1M_2^2M_3+M_1^{14}) \\
\langle M_3\rangle^2+V_1\langle M_2^2\rangle\langle M_3\rangle+V_2\langle M_1^4\rangle\langle M_3\rangle \rightarrow & 2\sigma(M_1^7\bar{M}_2+M_1\bar{M}_2\langle M_2^2\rangle+\langle M_1^4\rangle\langle M_2^2\rangle) & \\
 & 2\sigma(M_1^3\bar{M}_2\langle M_3\rangle+M_1^7M_2^2+M_1^2\langle M_1^4\rangle\langle M_3\rangle & \longrightarrow C[18]M_1^4M_2
\end{array}$$

Figure 3.4.5: d^{12}-Boundaries Mod Two

There is a new phenomenon in the table of Figure 3.4.5. The nonzero d^8-differentials on $\langle M_1^4\rangle^2$, $\langle M_2^2\rangle^2$, $\langle M_3\rangle^2$ and $\langle M_4\rangle$ extend to nonzero d^8-differentials on

$$\langle M_1^4\rangle^{2\alpha}\langle M_2^2\rangle^{2\beta}\langle M_3\rangle^{2\gamma}\langle M_4\rangle^{\delta}\langle M_1^4\rangle^{4e(1)}\langle M_2^2\rangle^{4e(2)}\langle M_3\rangle^{4e(3)}\langle M_4\rangle^{2e(4)}\{M_5\}^{e(5)}\cdots\{M_N\}^{e(N)}$$

for all α, β, γ, $\delta \in \{0,1\}$, $\alpha+\beta+\gamma+\delta > 0$ and $e(k) \geq 0$, $1 \leq k \leq N$. That is, the images of these d^8-differentials are the only d^8-boundaries in

$$Z_2\otimes \{Z_8(2\sigma)\{\langle M_1^4\rangle, M_1^2\langle M_2^2\rangle, \bar{M}_2\langle M_3\rangle, M_1M_2M_3\} \otimes E(\langle M_1^4\rangle^2, \langle M_2^2\rangle^2, \langle M_3\rangle^2, \langle M_4\rangle)$$
$$\otimes Z[\langle M_1^4\rangle^4, \langle M_2^2\rangle^4, \langle M_3\rangle^4, \langle M_4\rangle^2, \{M_5\}', \ldots, \{M_N\}', \ldots\}]\}.$$

This determines the 11 relations in $K(12)/\{[d^8(E^8_{*,0})\otimes Z_4(4\sigma)\otimes H_*BP]\cap K(12)]\}$. Now the proof of the remainder of this theorem is a direct analogue of the proof of Theorem 3.4.3(b)-(e). The right hand column of the table in Figure 3.4.5 is summarized in the table of Figure 3.4.6 to assist in the proof of (c). The only additional observation required to prove (d) is that all of the $d^8\{M_n\}''$, $N \geq 5$, must be elements of $[Z_4(4\sigma) \otimes H_*BP] \cap E^{14}_{*,7}$ because if that were not so then we could apply a Quillen operation to see that $2\sigma M_1^{12}$ or $2\sigma(M_2^3\bar{M}_3+M_1^9\langle M_3\rangle)$ is a d^8-boundary which we know is not the case. Thus, the $\{M_N\}'''$ exist.∎

C[18]	1	$<M_1^4>$	$<M_2^2>$	$<M_3>$	$<M_1^4><M_2^2>$	$<M_1^4><M_3>$	$<M_2^2><M_3>$	$<M_1^4><M_2^2><M_3>$
1	*	*	*	*		*	*	
M_1	*	*	*	*			*	
M_2	*	*	*	*				

Figure 3.4.6: Summary of d^{12}- Boundaries Modulo Two

In the previous theorem we analyzed the d^{12}-differentials which originate on the 7 row modulo two. In the next theorem we analyze these d^{12}-differentials modulo four. We continue with the notation of the previous theorem.

THEOREM 3.4.8 (a) $K(12) \,/\, \{d^8(E^8_{*,0}) \oplus Z_2(8\sigma)] \cap K(12)\} =$

$$\left[K_2 \otimes Z[<M_1^4>^4, <M_2^2>^4, <M_3>^4, <M_4>^2, \{M_5\}', \ldots, \{M_N\}', \ldots\}\right]$$
$$\oplus \left[Z_2(4\sigma(M_1<M_1^4><M_3>+M_1^3M_2<M_2^2>)) \otimes B'<8>\right].$$

(b) Let $U = Z_2 \otimes [Z_4(2\nu^*) \otimes H_*BP]$. Then $U \,/\, \{U \cap \text{Image } [d^{12}: E^{12}_{*,7} \longrightarrow E^{12}_{*,18}]\}$

$= Z_2\{2\nu^* M_1^{\alpha} M_2^{\beta} <M_1^4>^{e(1)} <M_2^2>^{e(2)} <M_3>^{e(3)} \cdots <M_N^{e(N)}> \cdots \mid 0 \le \alpha \le 3,\ 0 \le \beta \le 1,$

$e(N) \ge 0$ for $N \ge 1$, $\alpha + \beta \ge 3$ and if $e(3)$ is odd then either

(i) $e(1)$ is even or (ii) $e(1)$ is odd, $e(2)$ is even, $\alpha = 1$, $\beta = 0\}$.

(c) For $N \ge 5$, there are polynomial generators $\{M_N\}^{iv}$ of H_*BP which survive to E^8 such that $d^8\{M_N\}^{iv} \in (8\sigma)$.

PROOF. Again the proof is analogous to the proof of Theorem 3.4.3(b)-(e). It makes use of the table in Figure 3.4.7 below. In addition, observe that $d^{12}(K_2) \subset Z_2(4C[18]) \otimes H_*BP$.

$E^8_{*,0}$		$E^r_{*,7}$	$E^{12}_{*,18}$
$4<M_1^4>$	$\longrightarrow$	4σ	
$2V_1<M_1^4>$	$\longrightarrow$	$4\sigma M_1$	

$$4\langle M_2^2\rangle \longrightarrow 4\sigma M_1^2$$

$$2V_1\langle M_2^2\rangle \longrightarrow 4\sigma M_1^3$$

$$4\langle M_3\rangle \longrightarrow 4\sigma \bar{M}_2$$

$$2V_1\langle M_3\rangle \longrightarrow 4\sigma M_1\bar{M}_2$$

$$2V_2\langle M_2^2\rangle \longrightarrow 4\sigma M_1^2 M_2$$

$$V_1V_2\langle M_2^2\rangle \longrightarrow 4\sigma M_1^3 M_2$$

$$V_1^2\langle M_2^2\rangle \longrightarrow 4\sigma\langle M_1^4\rangle$$

$$V_1^2\langle M_3\rangle+2V_2\langle M_2^2\rangle \longrightarrow 4\sigma M_1\langle M_1^4\rangle$$

$$4\sigma M_1^2\langle M_1^4\rangle \longrightarrow 2C[18]$$

$$4\sigma M_1^3\langle M_1^4\rangle \longrightarrow 2C[18]M_1$$

$$V_1V_2\langle M_3\rangle+2V_1\langle M_1^4\rangle\langle M_2^2\rangle \longrightarrow 4\sigma M_2\langle M_1^4\rangle$$

$$4\sigma M_1M_2\langle M_1^4\rangle \longrightarrow 2C[18]M_1^2$$

$$4\sigma M_1^2M_2\langle M_1^4\rangle \longrightarrow 2C[18](\bar{M}_2+M_1^3)$$

$$4\sigma M_1^3M_2\langle M_1^4\rangle \longrightarrow 2C[18]M_1\bar{M}_2$$

$$4\langle M_1^4\rangle\langle M_2^2\rangle \longrightarrow 4\sigma(\langle M_2^2\rangle+M_1^2\langle M_1^4\rangle)$$

$$2V_1\langle M_1^4\rangle\langle M_2^2\rangle \longrightarrow 4\sigma(M_1\langle M_2^2\rangle+M_1^3\langle M_1^4\rangle)$$

$$V_1^2\langle M_1^4\rangle\langle M_2^2\rangle+\langle M_1^4\rangle^3 \longrightarrow 4\sigma M_1^2\langle M_2^2\rangle$$

$$4\langle M_2^2\rangle\langle M_3\rangle+2V_2\langle M_1^4\rangle\langle M_2^2\rangle+V_1^2\langle M_1^4\rangle\langle M_3\rangle+2V_1\langle M_1^4\rangle^3 \longrightarrow 4\sigma M_1^3\langle M_2^2\rangle$$

$$2V_2\langle M_1^4\rangle\langle M_2^2\rangle \longrightarrow 4\sigma(M_2\langle M_2^2\rangle+M_1^2M_2\langle M_1^4\rangle)$$

$$V_1V_2\langle M_1^4\rangle\langle M_2^2\rangle \longrightarrow 4\sigma(M_1M_2\langle M_2^2\rangle+M_1^3M_2\langle M_1^4\rangle)$$

$$4V_1^2\langle M_2^2\rangle\langle M_3\rangle \longrightarrow 4\sigma M_1^2\bar{M}_2\langle M_2^2\rangle+4\bar{M}_2\langle M_1^4\rangle\langle M_3\rangle$$

$$4\sigma M_1^3M_2\langle M_2^2\rangle \longrightarrow 2C[18]M_1^6$$

$$4\langle M_1^4\rangle\langle M_3\rangle \longrightarrow 4\sigma(\langle M_3\rangle+\bar{M}_2\langle M_1^4\rangle)$$

$$2V_1\langle M_1^4\rangle\langle M_3\rangle \longrightarrow 4\sigma(M_1\langle M_3\rangle+M_1\bar{M}_2\langle M_1^4\rangle)$$

$$4\langle M_2^2\rangle\langle M_3\rangle \longrightarrow 4\sigma(M_1^2\langle M_3\rangle+\bar{M}_2\langle M_2^2\rangle)$$

$$2V_1\langle M_2^2\rangle\langle M_3\rangle \longrightarrow 4\sigma(M_1^3\langle M_3\rangle+M_1\bar{M}_2\langle M_2^2\rangle)$$

$$2V_2\langle M_1^4\rangle\langle M_3\rangle \longrightarrow 4\sigma(M_2\langle M_3\rangle+M_2\bar{M}_2\langle M_1^4\rangle)$$

$$V_1V_3\langle M_3\rangle \longrightarrow 4\sigma M_1\bar{M}_2M_3$$

$$V_2V_3\langle M_2^2\rangle \longrightarrow 4\sigma M_1^2M_2M_3$$

$$V_1V_2\langle M_2^2\rangle\langle M_3\rangle+2V_1\langle M_1^4\rangle\langle M_2^2\rangle^2 \longrightarrow 4\sigma(M_1^3M_2\langle M_3\rangle+\bar{M}_2\langle M_1^4\rangle\langle M_2^2\rangle$$

$$4\sigma\langle M_1^4\rangle\langle M_2^2\rangle \longrightarrow 2C[18]M_1^4$$

$$4\sigma M_1\langle M_1^4\rangle\langle M_2^2\rangle \longrightarrow 2C[18]M_1^5$$

$$4\sigma M_1^2\langle M_1^4\rangle\langle M_2^2\rangle \longrightarrow 2C[18]\langle M_2^2\rangle$$

$$4\sigma M_1^3\langle M_1^4\rangle\langle M_2^2\rangle \longrightarrow 2C[18]M_1\langle M_2^2\rangle$$

$$4\sigma M_2\langle M_1^4\rangle\langle M_2^2\rangle \longrightarrow 2C[18](M_1^4\bar{M}_2+M_1^7)$$

$$4\sigma M_1M_2\langle M_1^4\rangle\langle M_2^2\rangle \longrightarrow 2C[18](M_1^5\bar{M}_2+M_1^2M_2^2+M_1^8)$$

$$4\sigma M_1^2M_2\langle M_1^4\rangle\langle M_2^2\rangle \longrightarrow 2C[18](\bar{M}_2\langle M_2^2\rangle+M_1^3\langle M_2^2\rangle)$$

$$4\sigma M_1^3M_2\langle M_1^4\rangle\langle M_2^2\rangle \longrightarrow 2C[18](M_1\bar{M}_2\langle M_2^2\rangle+M_1^{10})$$

$$V_1^2\langle M_2^2\rangle\langle M_3\rangle \longrightarrow 4\sigma(\langle M_1^4\rangle\langle M_3\rangle+M_1^2\bar{M}_2\langle M_2^2\rangle)$$

$$4\sigma(M_1\langle M_1^4\rangle\langle M_3\rangle+M_1^3M_2\langle M_2^2\rangle) \to 0$$

$$4\sigma M_1^2\langle M_1^4\rangle\langle M_3\rangle \longrightarrow 2C[18](\langle M_3\rangle+M_1^4\bar{M}_2+M_1^7)$$

$$4\sigma M_1^3\langle M_1^4\rangle\langle M_3\rangle \longrightarrow 2C[18](M_1\langle M_3\rangle+M_1^5\bar{M}_2+M_1^8)$$

$$4\sigma M_2\langle M_1^4\rangle M_3 \longrightarrow 2C[18]M_1^2M_2^2$$

$$4\sigma M_1M_2\langle M_1^4\rangle\langle M_3\rangle \longrightarrow 2C[18](M_1^2\langle M_3\rangle+M_1^9)$$

$$4\sigma M_1^2M_2\langle M_1^4\rangle\langle M_3\rangle \longrightarrow 2C[18](\bar{M}_2\langle M_3\rangle+M_1^3\langle M_3\rangle+M_1^4M_2^2)$$

$$4\sigma M_1^3M_2\langle M_1^4\rangle\langle M_3\rangle \longrightarrow 2C[18](M_1\bar{M}_2\langle M_3\rangle+M_1^5M_2^2+M_1^8\bar{M}_2)$$

$$4\langle M_1^4\rangle\langle M_2^2\rangle\langle M_3\rangle \longrightarrow 4\sigma(\langle M_2^2\rangle\langle M_3\rangle+M_1^2\langle M_1^4\rangle\langle M_3\rangle+\bar{M}_2\langle M_1^4\rangle\langle M_2^2\rangle)$$

$$2V_1\langle M_1^4\rangle\langle M_2^2\rangle\langle M_3\rangle \longrightarrow 4\sigma(M_1\langle M_2^2\rangle\langle M_3\rangle+M_1^3\langle M_1^4\rangle\langle M_3\rangle+M_1\bar{M}_2\langle M_1^4\rangle\langle M_2^2\rangle)$$

$$V_1^2\langle M_1^4\rangle\langle M_2^2\rangle\langle M_3\rangle \longrightarrow 4\sigma(M_1^2\langle M_2^2\rangle\langle M_3\rangle+\langle M_1^4\rangle^2\langle M_3\rangle+M_1^2\bar{M}_2\langle M_1^4\rangle\langle M_2^2\rangle)$$

$$4\sigma M_1^3\langle M_2^2\rangle\langle M_3\rangle \longrightarrow 2C[18](M_1\bar{M}_2\langle M_2^2\rangle+M_1^4\langle M_2^2\rangle)$$

$$2V_2\langle M_1^4\rangle\langle M_2^2\rangle\langle M_3\rangle \longrightarrow 4\sigma(M_2\langle M_2^2\rangle\langle M_3\rangle+M_1^2M_2\langle M_1^4\rangle\langle M_3\rangle+M_2\bar{M}_2\langle M_1^4\rangle\langle M_2^2\rangle)$$

$$V_1V_2\langle M_1^4\rangle\langle M_2^2\rangle\langle M_3\rangle \longrightarrow 4\sigma(M_1M_2\langle M_2^2\rangle\langle M_3\rangle+M_1^3M_2\langle M_1^4\rangle\langle M_3\rangle+M_1M_2\bar{M}_2\langle M_1^4\rangle\langle M_2^2\rangle)$$

$$4\sigma(M_1^2M_2\langle M_2^2\rangle\langle M_3\rangle+M_1^6M_2^4) \longrightarrow 2C[18]M_1^6\langle M_2^2\rangle$$

$$4\sigma M_1^3M_2\langle M_2^2\rangle\langle M_3\rangle \longrightarrow 2C[18](M_1^6\bar{M}_3+M_1M_2^4+\bar{M}_2\langle M_1^4\rangle M_2^2+M_1^7M_2^2)$$

$$4\sigma\langle M_1^4\rangle\langle M_2^2\rangle\langle M_3\rangle \longrightarrow 2C[18](M_1^4\langle M_3\rangle+M_1\langle M_1^4\rangle\langle M_2^2\rangle)$$

$$4\sigma M_1\langle M_1^4\rangle\langle M_2^2\rangle\langle M_3\rangle \longrightarrow 2C[18](M_1^5\langle M_3\rangle+M_1^6\langle M_2^2\rangle)$$

$$4\sigma M_1^2\langle M_1^4\rangle\langle M_2^2\rangle\langle M_3\rangle \longrightarrow 2C[18](\langle M_2^2\rangle\bar{M}_3+M_1^7\langle M_2^2\rangle)$$

$$4\sigma M_1^3\langle M_1^4\rangle\langle M_2^2\rangle\langle M_3\rangle \longrightarrow 2C[18](M_1\langle M_2^2\rangle\bar{M}_3+M_1^8\langle M_2^2\rangle)$$

$$4\sigma M_2\langle M_1^4\rangle\langle M_2^2\rangle\langle M_3\rangle \longrightarrow 2C[18](\bar{M}_2\langle M_1^4\rangle\langle M_3\rangle+M_1^3\langle M_1^4\rangle\langle M_3\rangle+M_1\bar{M}_2\langle M_1^4\rangle\langle M_2^2\rangle)$$

$$4\sigma M_1M_2\langle M_1^4\rangle\langle M_2^2\rangle\langle M_3\rangle \longrightarrow 2C[18](M_1^2\langle M_2^2\rangle\langle M_3\rangle+M_1^5\bar{M}_2\bar{M}_3)$$

$$4\sigma M_1^2\bar{M}_2\langle M_1^4\rangle\langle M_2^2\rangle\langle M_3\rangle \longrightarrow 2C[18](\bar{M}_2\langle M_2^2\rangle\bar{M}_3+M_1^3\langle M_2^2\rangle\langle M_3\rangle+M_1^{10}\langle M_2^2\rangle)$$

$$4\sigma M_1^3\bar{M}_2\langle M_1^4\rangle\langle M_2^2\rangle\langle M_3\rangle \longrightarrow 2C[18](M_1M_2M_2^2M_3+M_1^7M_2\langle M_3\rangle+M_1^{10}M_3+M_1^8M_2M_2^2)$$

$$2\sigma(M_2^3\bar{M}_3+M_1^9\langle M_3\rangle+2M_1M_2\langle M_1^4\rangle^3) \longrightarrow 0$$

Figure 3.4.7: d^{12} Boundaries Modulo Four

To prove (b) we summarize the boundaries of the right hand column of the table in Figure 3.4.7 in the table of Figure 3.4.8 below. The proofs of (b) and (c) are analogous to the proofs in the previous theorems. ▮

2C[18]	1	$\langle M_1^4\rangle$	$\langle M_2^2\rangle$	$\langle M_3\rangle$	$\langle M_1^4\rangle\langle M_2^2\rangle$	$\langle M_1^4\rangle\langle M_3\rangle$	$\langle M_2^2\rangle\langle M_3\rangle$	$\langle M_1^4\rangle\langle M_2^2\rangle\langle M_3\rangle$
1	*	*	*	*	*		*	
M_1	*	*	*	*	*	*	*	
M_1^2	*	*	*	*	*		*	
$\bar{M}_2$	*	*	*	*	*		*	
$M_1\bar{M}_2$	*	*	*	*	*		*	

Figure 3.4.8: Summary of d^{12}- Boundaries Modulo Four

We will see in Chapter 5 that C[18] has order eight. At this point, however, we can not yet eliminate the possibility that C[18] has order four. This accounts for the indeterminate symbol ξ in the following theorem. With the insight from Chapter 5, the reader can replace ξ by $4C[18]M_1$.

THEOREM 3.4.9 If $4C[18] \neq 0$ then let $r = 12$ and $\xi = 4C[18]M_1$. If $4C[18] = 0$ then $r = 14$ and there is $\xi \in \pi_{20}^S$ which projects to a nonzero element of $E_{0,20}^{14}$. In both cases, the following is true.

(a) $E_{*,7}^{2r+2} = Z_2(2\sigma)\{<M_1^4>^3, M_1^2<M_2^2>^3, \bar{M}_2<M_3>^3, M_1M_2M_3<M_4>, \bar{M}_2<M_3><M_1^4>^2<M_3>^2,$
$M_1M_2M_3<M_1^4><M_4>, M_1M_2M_3<M_2^2><M_4>, M_1M_2M_3<M_1^4><M_2^2><M_4>\}$
$\otimes Z[<M_1^4>^4, <M_2^2>^4, <M_3>^4, <M_4>^2, \{M_5\}', \ldots, \{M_N\}', \ldots]$.

(b) $[Z_2(\xi) \otimes H_*BP] / d^r[Z_2(8\sigma) \otimes H_*BP]$

$= [Z_2(\xi M_1) \otimes B<2>]$

$\oplus [Z_2(\xi M_1^2\bar{M}_2<M_4>) \otimes B<4> / (<M_2^2>^3<M_3><M_4>, <M_1^4>^2<M_2^2>^3<M_3><M_4>)]$

$\oplus [Z_2(\xi<M_1^4><M_2^2><M_3>\{1, M_1^2, \bar{M}_2\}) \otimes B''<8>] \oplus [Z_2(\xi M_1^2<M_1^4><M_3>) \otimes B''<8>]$.

(c) For $N \geq 5$, there are polynomial generators $\{M_N\}$ of H_*BP which survive to E^{10}.

PROOF. We begin by computing $d^8: E_{22,0}^8 \longrightarrow E_{14,7}^8$ to see that $E_{14,7}^{12} = Z_8(2\sigma)M_1^7$. Observe that $E_{22,0}^8 = Z_{(2)}<M_1^4><M_3> \oplus Z_{(2)}V_1V_2<M_3> \oplus Z_{(2)}V_1^2V_2<M_2^2>$ $\oplus Z_{(2)}V_1<M_1^4><M_2^2> \oplus Z_{(2)}V_2<M_1^4>^2 \oplus Z_{(2)}V_1^3<M_1^4>^2$. Thus, $d^8(E_{22,0}^8)$ $= Z_8(\sigma\bar{M}_3) \oplus Z_4(2\sigma M_1M_2^2) \oplus Z_2(4\sigma M_1^4M_2)$. Note that $d^8(\sigma M_1^4\bar{M}_2) = \sigma^2\bar{M}_2$ and $d^{10}(2\sigma M_1^4M_2) = \eta^*M_1^2$. Thus $E_{14,7}^{12} = Z_8(2\sigma)M_1^7$ as asserted. Therefore if $4\nu^* \neq 0$ then $d^{12}(8\sigma M_1^7) = 4\nu^*M_1 = \xi$. On the other hand, if $4\nu^* = 0$ then $E_{14,7}^{14} = Z_2(4\sigma M_1^7)$ and $d^{14}(4\sigma M_1^7)$ is a nonzero element ξ of $E_{0,20}^{14}$.

(a), (b), (c) Once again the proof is analogous to the proof of Theorem 3.4.3(b)-(e). It makes use of the table in Figure 3.4.9 below.

$E_{*,0}^8$		$E_{*,7}^r$		$E_{*,r+6}^r$
$V_1^3<M_3>+2V_1V_2<M_2^2>$	$\longrightarrow$	$8\sigma M_1^2<M_1^4>$		
		$8\sigma M_1^3<M_1^4>$	$\longrightarrow$	ξ
$V_1^2V_2<M_3>+2V_1^2<M_1^4><M_2^2>$	$\longrightarrow$	$8\sigma M_1M_2<M_1^4>$		
		$8\sigma M_1^2M_2<M_1^4>$	$\longrightarrow$	ξM_1^2
		$8\sigma M_1^3M_2<M_1^4>$	$\longrightarrow$	$\xi\bar{M}_2$
$V_1^3<M_2^2><M_3>+V_1V_3<M_1^4>^2+2V_1V_2<M_1^4>^3$	$\longrightarrow$	$8\sigma M_1^3M_2<M_2^2>$		

$$4V_1\langle M_2^2\rangle\langle M_3\rangle+V_1^3\langle M_1^4\rangle\langle M_3\rangle+2V_1V_2\langle M_1^4\rangle\langle M_2^2\rangle+\langle M_2^2\rangle\langle M_1^4\rangle^2 \longrightarrow 8\sigma\langle M_1^4\rangle\langle M_2^2\rangle$$

$$8\sigma M_1\langle M_1^4\rangle\langle M_2^2\rangle \longrightarrow \xi M_1^4$$

$$8(M_2^3M_3+M_1^2\overline{M}_3^2) \longrightarrow 8\sigma M_1^2\langle M_1^4\rangle\langle M_2^2\rangle$$

$$8\sigma M_1^3\langle M_1^4\rangle\langle M_2^2\rangle \longrightarrow \xi\langle M_2^2\rangle$$

$$8\sigma M_2\langle M_1^4\rangle\langle M_2^2\rangle \longrightarrow \xi M_1^6$$

$$8\sigma M_1M_2\langle M_1^4\rangle\langle M_2^2\rangle \longrightarrow \xi M_1^4\overline{M}_2$$

$$8\sigma M_1^2M_2\langle M_1^4\rangle\langle M_2^2\rangle \longrightarrow \xi M_1^2\langle M_2^2\rangle$$

$$8\sigma M_1^3M_2\langle M_1^4\rangle\langle M_2^2\rangle \longrightarrow \xi\overline{M}_2\langle M_2^2\rangle$$

$$4\sigma(M_1\langle M_1^4\rangle\langle M_3\rangle+M_1^3M_2\langle M_2^2\rangle) \longrightarrow \xi M_1^2\overline{M}_2$$

$$V_1^2V_3\langle M_3\rangle+2V_2V_3\langle M_2^2\rangle \longrightarrow 8\sigma M_1M_3\langle M_1^4\rangle$$

$$8(M_1M_2^3M_3+M_1^{11}M_2^2) \longrightarrow 8\sigma(M_1^2\langle M_1^4\rangle\langle M_3\rangle+M_2\langle M_1^4\rangle\langle M_2^2\rangle)$$

$$8\sigma M_1^3\langle M_1^4\rangle\langle M_3\rangle \longrightarrow \xi(\langle M_3\rangle+M_1^4\overline{M}_2)$$

$$8(M_1^2M_2^3M_3+M_1M_2M_3^2) \longrightarrow 8\sigma(M_2\langle M_1^4\rangle\langle M_3\rangle+M_1M_2\langle M_1^4\rangle\langle M_2^2\rangle)$$

$$8(M_1^3M_2^3M_3+M_1^6M_2^2M_3+M_1^4M_2^5+M_1^7M_2^4) \longrightarrow 8\sigma(M_1M_2\langle M_1^4\rangle\langle M_3\rangle+M_1^{15})$$

$$8\sigma M_1^2M_2\langle M_1^4\rangle\langle M_3\rangle \longrightarrow \xi M_1^2\langle M_3\rangle$$

$$8\sigma M_1^3M_2\langle M_1^4\rangle\langle M_3\rangle \longrightarrow \xi(\overline{M}_2\langle M_3\rangle+M_1^4M_2^2)$$

$$V_1^3\langle M_1^4\rangle\langle M_2^2\rangle\langle M_3\rangle+2V_1V_3\langle M_1^4\rangle^3+8M_1^3M_2\langle M_2^2\rangle\langle M_1^4\rangle^2+2V_2^2\langle M_2^2\rangle\langle M_1^4\rangle^2+8\langle M_1^4\rangle^5 \longrightarrow 8\sigma M_1^3\langle M_2^2\rangle\langle M_3\rangle$$

$$V_1^2V_2\langle M_1^4\rangle\langle M_2^2\rangle\langle M_3\rangle+V_2V_3\langle M_1^4\rangle^3+V_1^3\langle M_3\rangle\langle M_2^2\rangle^2+V_1V_2\langle M_2^2\rangle^3+8\langle M_2^2\rangle\langle M_1^4\rangle^4 \longrightarrow 8\sigma M_1^2M_2\langle M_2^2\rangle\langle M_3\rangle$$

$$8\sigma M_1^3M_2\langle M_2^2\rangle\langle M_3\rangle \longrightarrow \xi(M_2^4+M_1^6M_2^2)$$

$$8\sigma\langle M_1^4\rangle\langle M_2^2\rangle\langle M_3\rangle \longrightarrow \xi\langle M_1^4\rangle\langle M_2^2\rangle$$

$$8\sigma M_1\langle M_1^4\rangle\langle M_2^2\rangle\langle M_3\rangle \longrightarrow \xi M_1^4\langle M_3\rangle$$

$$8M_1^7M_2^3M_3 \longrightarrow 8\sigma(M_1^2\langle M_1^4\rangle\langle M_2^2\rangle\langle M_3\rangle+M_1^3M_2\langle M_2^2\rangle\langle M_3\rangle+M_1^3\langle M_1^4\rangle\langle M_2^2\rangle^2)$$

$$8\sigma M_1^3\langle M_1^4\rangle\langle M_2^2\rangle\langle M_3\rangle \longrightarrow \xi\langle M_2^2\rangle\overline{M}_3$$

$$8\sigma M_2\langle M_1^4\rangle\langle M_2^2\rangle\langle M_3\rangle \longrightarrow \xi(M_1^2\langle M_1^4\rangle\langle M_3\rangle+\overline{M}_2\langle M_1^4\rangle\langle M_2^2\rangle)$$

$$8\sigma M_1M_2\langle M_1^4\rangle\langle M_2^2\rangle\langle M_3\rangle \longrightarrow \xi M_1^4\overline{M}_2\overline{M}_3$$

$$8\sigma M_1^2\overline{M}_2\langle M_1^4\rangle\langle M_2^2\rangle\langle M_3\rangle \longrightarrow \xi M_1^2\langle M_2^2\rangle\langle M_3\rangle$$

$$8\sigma M_1^3\overline{M}_2\langle M_1^4\rangle\langle M_2^2\rangle\langle M_3\rangle \longrightarrow \xi(\overline{M}_2M_2^2\overline{M}_3+M_1^6\overline{M}_2\langle M_3\rangle)$$

$$2\sigma\langle M_1^4\rangle^3 \longrightarrow 0$$

$$2\sigma\langle M_1^4\rangle\langle M_2^2\rangle^2 \longrightarrow \xi M_1^2\bar{M}_2\langle M_1^4\rangle$$

$$2\sigma M_1^2\langle M_2^2\rangle^3 \longrightarrow 0$$

$$2\sigma\langle M_1^4\rangle\langle M_3\rangle^2 \longrightarrow \xi M_1^2\bar{M}_2\langle M_2^2\rangle$$

$$2\sigma M_1^2\langle M_2^2\rangle\langle M_3\rangle^2 \longrightarrow \xi M_1^2\bar{M}_2\langle M_1^4\rangle\langle M_2^2\rangle$$

$$2\sigma\bar{M}_2\langle M_3\rangle^3 \longrightarrow 0$$

$$2\sigma\langle M_1^4\rangle\langle M_4\rangle \longrightarrow \xi M_1^2\bar{M}_2\langle M_3\rangle$$

$$2\sigma M_1^2\langle M_2^2\rangle\langle M_4\rangle \longrightarrow \xi M_1^2\bar{M}_2\langle M_1^4\rangle\langle M_3\rangle$$

$$2\sigma\bar{M}_2\langle M_3\rangle\langle M_4\rangle \longrightarrow \xi M_1^2\bar{M}_2\langle M_2^2\rangle\langle M_3\rangle$$

$$2\sigma M_1M_2M_3\langle M_4\rangle \longrightarrow 0$$

$$2\sigma\langle M_1^4\rangle^3\langle M_2^2\rangle^2 \longrightarrow \xi M_1^2\bar{M}_2\langle M_1^4\rangle^3$$

$$2\sigma M_1^2\langle M_1^4\rangle^2\langle M_2^2\rangle^3 \longrightarrow \xi M_1^2\bar{M}_2\langle M_1^4\rangle\langle M_2^2\rangle^2$$

$$2\sigma\langle M_1^4\rangle^3\langle M_3\rangle^2 \longrightarrow \xi M_1^2\bar{M}_2\langle M_2^2\rangle\langle M_1^4\rangle^2$$

$$2\sigma M_1^2\langle M_1^4\rangle^2\langle M_2^2\rangle\langle M_3\rangle^2 \longrightarrow \xi M_1^2\bar{M}_2\langle M_1^4\rangle^3\langle M_2^2\rangle$$

$$2\sigma\bar{M}_2\langle M_1^4\rangle^2\langle M_3\rangle^3 \longrightarrow 0$$

$$2\sigma\langle M_1^4\rangle^3\langle M_4\rangle \longrightarrow \xi M_1^2\bar{M}_2\langle M_3\rangle\langle M_1^4\rangle^2$$

$$2\sigma M_1^2\langle M_1^4\rangle^2\langle M_2^2\rangle\langle M_4\rangle \longrightarrow \xi M_1^2\bar{M}_2\langle M_3\rangle\langle M_1^4\rangle^3$$

$$2\sigma\bar{M}_2\langle M_1^4\rangle^2\langle M_3\rangle\langle M_4\rangle \longrightarrow \xi M_1^2\bar{M}_2\langle M_2^2\rangle\langle M_3\rangle\langle M_1^4\rangle^2$$

$$2\sigma M_1M_2M_3\langle M_1^4\rangle^2\langle M_4\rangle \longrightarrow 0$$

$$2\sigma\langle M_1^4\rangle\langle M_2^2\rangle^2\langle M_3\rangle^2 \longrightarrow \xi M_1^2\bar{M}_2[\langle M_1^4\rangle\langle M_3\rangle^2+\langle M_2^2\rangle^3]$$

$$2\sigma M_1^2\langle M_2^2\rangle^3\langle M_3\rangle^2 \longrightarrow \xi M_1^2\bar{M}_2\langle M_1^4\rangle\langle M_2^2\rangle^3$$

$$2\sigma\bar{M}_2\langle M_2^2\rangle^2\langle M_3\rangle^3 \longrightarrow \xi M_1^2\bar{M}_2\langle M_1^4\rangle\langle M_2^2\rangle\langle M_3\rangle^2$$

$$2\sigma\langle M_1^4\rangle\langle M_2^2\rangle^2\langle M_4\rangle \longrightarrow \xi M_1^2\bar{M}_2[\langle M_3\rangle\langle M_2^2\rangle^2+\langle M_1^4\rangle\langle M_4\rangle]$$

$$2\sigma M_1^2\langle M_2^2\rangle^3\langle M_4\rangle \longrightarrow \xi M_1^2\bar{M}_2\langle M_1^4\rangle\langle M_3\rangle\langle M_2^2\rangle^2$$

$$2\sigma\bar{M}_2\langle M_2^2\rangle^2\langle M_3\rangle\langle M_4\rangle \longrightarrow \xi M_1^2\bar{M}_2[\langle M_2^2\rangle^3\langle M_3\rangle+\langle M_1^4\rangle\langle M_3\rangle^3]$$

$$2\sigma M_1M_2M_3\langle M_2^2\rangle^2\langle M_4\rangle \longrightarrow 0$$

$$2\sigma\langle M_1^4\rangle\langle M_3\rangle^2\langle M_4\rangle \longrightarrow \xi M_1^2\bar{M}_2[\langle M_3\rangle^3+\langle M_2^2\rangle\langle M_4\rangle]$$

$$2\sigma M_1^2\langle M_2^2\rangle\langle M_3\rangle^2\langle M_4\rangle \longrightarrow \xi M_1^2\bar{M}_2[\langle M_1^4\rangle\langle M_3\rangle^3+\langle M_1^4\rangle\langle M_2^2\rangle\langle M_4\rangle]$$

$$2\sigma\bar{M}_2\langle M_3\rangle^3\langle M_4\rangle \longrightarrow \xi M_1^2\bar{M}_2\langle M_2^2\rangle\langle M_3\rangle^3$$

$$2\sigma M_1M_2M_3\langle M_3\rangle^2\langle M_4\rangle \longrightarrow \xi M_1^2\bar{M}_2\langle M_1^4\rangle^2\langle M_2^2\rangle\langle M_3\rangle^2$$

$$2\sigma\langle M_1^4\rangle^3\langle M_2^2\rangle^2\langle M_3\rangle^2 \longrightarrow \xi M_1^2\bar{M}_2[\langle M_1^4\rangle^3\langle M_3\rangle^2+\langle M_1^4\rangle^2\langle M_2^2\rangle^3]$$

$$2\sigma M_1^2\langle M_1^4\rangle^2\langle M_2^2\rangle^3\langle M_3\rangle^2 \longrightarrow \xi M_1^2\bar{M}_2[\langle M_1^4\rangle^3\langle M_2^2\rangle^3+\langle M_1^4\rangle\langle M_2^2\rangle^2\langle M_3\rangle^2]$$

$$2\sigma\bar{M}_2\langle M_1^4\rangle^2\langle M_2^2\rangle^2\langle M_3\rangle^3 \longrightarrow \xi M_1^2\bar{M}_2[\langle M_1^4\rangle^3\langle M_2^2\rangle\langle M_3\rangle^2+\langle M_2^2\rangle^3\langle M_3\rangle^2]$$

$$2\sigma\langle M_1^4\rangle^3\langle M_2^2\rangle^2\langle M_4\rangle \longrightarrow \xi M_1^2\bar{M}_2[\langle M_1^4\rangle^2\langle M_2^2\rangle^2\langle M_3\rangle+\langle M_1^4\rangle^3\langle M_4\rangle]$$

$$2\sigma M_1^2\langle M_1^4\rangle^2\langle M_2^2\rangle^3\langle M_4\rangle \longrightarrow \xi M_1^2\bar{M}_2[\langle M_1^4\rangle^3\langle M_2^2\rangle^2\langle M_3\rangle+\langle M_1^4\rangle\langle M_2^2\rangle^2\langle M_4\rangle]$$

$$2\sigma\bar{M}_2\langle M_1^4\rangle^2\langle M_2^2\rangle^2\langle M_3\rangle\langle M_4\rangle \longrightarrow \xi M_1^2\bar{M}_2[\langle M_1^4\rangle^2\langle M_2^2\rangle^3\langle M_3\rangle+\langle M_1^4\rangle^3\langle M_3\rangle^3+\langle M_2^2\rangle^3\langle M_4\rangle+\langle M_1^4\rangle\langle M_3\rangle^2\langle M_4\rangle]$$

$$2\sigma M_1M_2M_3\langle M_1^4\rangle^2\langle M_2^2\rangle^2\langle M_4\rangle \longrightarrow 0$$

$$2\sigma\langle M_1^4\rangle^3\langle M_3\rangle^2\langle M_4\rangle \longrightarrow \xi M_1^2\bar{M}_2[\langle M_3\rangle^3\langle M_1^4\rangle^2+\langle M_1^4\rangle^2\langle M_2^2\rangle\langle M_4\rangle]$$

$$2\sigma M_1^2\langle M_1^4\rangle^2\langle M_2^2\rangle\langle M_3\rangle^2\langle M_4\rangle \longrightarrow \xi M_1^2\bar{M}_2[\langle M_1^4\rangle^3\langle M_3\rangle^3+\langle M_1^4\rangle^3\langle M_2^2\rangle\langle M_4\rangle+\langle M_1^4\rangle\langle M_3\rangle^2\langle M_4\rangle]$$

$$2\sigma\bar{M}_2\langle M_1^4\rangle^2\langle M_3\rangle^3\langle M_4\rangle \longrightarrow \xi M_1^2\bar{M}_2[\langle M_2^2\rangle\langle M_3\rangle^3\langle M_1^4\rangle^2+\langle M_2^2\rangle\langle M_3\rangle^2\langle M_4\rangle]$$

$$2\sigma M_1M_2M_3\langle M_1^4\rangle^2\langle M_3\rangle^2\langle M_4\rangle \longrightarrow \xi M_1^2\bar{M}_2\langle M_1^4\rangle\langle M_2^2\rangle^3\langle M_3\rangle^2$$

$$2\sigma\langle M_1^4\rangle\langle M_2^2\rangle^2\langle M_3\rangle^2\langle M_4\rangle \longrightarrow \xi M_1^2\bar{M}_2[\langle M_2^2\rangle^2\langle M_3\rangle^3+\langle M_2^2\rangle^3\langle M_4\rangle+\langle M_1^4\rangle\langle M_3\rangle^2\langle M_4\rangle]$$

$$2\sigma M_1^2\langle M_2^2\rangle^3\langle M_3\rangle^2\langle M_4\rangle \longrightarrow \xi M_1^2\bar{M}_2[\langle M_1^4\rangle\langle M_2^2\rangle^2\langle M_3\rangle^3+\langle M_1^4\rangle\langle M_2^2\rangle^3\langle M_4\rangle]$$

$$2\sigma\bar{M}_2\langle M_2^2\rangle^2\langle M_3\rangle^3\langle M_4\rangle \longrightarrow \xi M_1^2\bar{M}_2[\langle M_2^2\rangle^3\langle M_3\rangle^3+\langle M_1^4\rangle\langle M_2^2\rangle\langle M_3\rangle^2\langle M_4\rangle]$$

$$2\sigma M_1M_2M_3\langle M_2^2\rangle^2\langle M_3\rangle^2\langle M_4\rangle \longrightarrow \xi M_1^2\bar{M}_2[\langle M_2^2\rangle^3\langle M_3\rangle\langle M_4\rangle+\langle M_1^4\rangle\langle M_3\rangle^3\langle M_4\rangle]$$

$$2\sigma\langle M_1^4\rangle^3\langle M_2^2\rangle^2\langle M_3\rangle^2\langle M_4\rangle \longrightarrow \xi M_1^2\bar{M}_2[\langle M_1^4\rangle^2\langle M_2^2\rangle^2\langle M_3\rangle^3+\langle M_1^4\rangle^2\langle M_2^2\rangle^3\langle M_4\rangle+\langle M_1^4\rangle^3\langle M_3\rangle^2\langle M_4\rangle]$$

$$2\sigma M_1^2\langle M_1^4\rangle^2\langle M_2^2\rangle^3\langle M_3\rangle^2\langle M_4\rangle \longrightarrow \xi M_1^2\bar{M}_2[\langle M_1^4\rangle^3\langle M_2^2\rangle^2\langle M_3\rangle^3+\langle M_1^4\rangle^3\langle M_2^2\rangle^3\langle M_4\rangle+\langle M_1^4\rangle\langle M_2^2\rangle^2\langle M_3\rangle^2\langle M_4\rangle]$$

$$2\sigma\bar{M}_2\langle M_1^4\rangle^2\langle M_2^2\rangle^2\langle M_3\rangle^3\langle M_4\rangle \longrightarrow \xi M_1^2\bar{M}_2[\langle M_1^4\rangle^2\langle M_2^2\rangle^3\langle M_3\rangle^3+\langle M_1^4\rangle^3\langle M_2^2\rangle\langle M_3\rangle^2\langle M_4\rangle+\langle M_2^2\rangle^3\langle M_3\rangle^2\langle M_4\rangle]$$

$$2\sigma M_1M_2M_3\langle M_1^4\rangle^2\langle M_2^2\rangle^2\langle M_3\rangle^2\langle M_4\rangle \longrightarrow \xi M_1^2\bar{M}_2[\langle M_1^4\rangle^2\langle M_2^2\rangle^3\langle M_3\rangle\langle M_4\rangle+\langle M_1^4\rangle^3\langle M_3\rangle^3\langle M_4\rangle+\langle M_2^2\rangle^2\langle M_3\rangle^3\langle M_4\rangle]$$

Figure 3.4.9: d^r Boundaries

To prove (b) we summarize the boundaries of the right hand column above in the table below. The proof is analogous to the proofs of the previous theorems. ∎

ξ	1	$\langle M_1^4\rangle$	$\langle M_2^2\rangle$	$\langle M_3\rangle$	$\langle M_1^4\rangle\langle M_2^2\rangle$	$\langle M_1^4\rangle\langle M_3\rangle$	$\langle M_2^2\rangle\langle M_3\rangle$	$\langle M_1^4\rangle\langle M_2^2\rangle\langle M_3\rangle$
1	*	*	*	*	*	*	*	
M_1^2	*	*	*	*	*		*	
$\bar{M}_2$	*	*	*	*	*	*	*	
$M_1^2\bar{M}_2$	*	‡	‡	‡	‡	‡	‡	

‡ Some of these elements bound and some do not bound. The bounding elements are marked by in the table below. An entry * indicates that the box bounds. In addition, the sum of all the boxes with the same letter as entry bound.

$\xi M_1^2\bar{M}_2$	$\langle M_1^4\rangle$	$\langle M_2^2\rangle$	$\langle M_3\rangle$	$\langle M_1^4\rangle\langle M_2^2\rangle$	$\langle M_1^4\rangle\langle M_3\rangle$	$\langle M_2^2\rangle\langle M_3\rangle$
1	*	*	*	*	*	*
$\langle M_1^4\rangle^2$	*	*	*	*	*	*
$\langle M_2^2\rangle^2$	*	A	B	*	*	E
$\langle M_3\rangle^2$	A		C	*	D, E	*
$\langle M_4\rangle$	B	C		D		
$\langle M_1^4\rangle^2\langle M_2^2\rangle^2$		F	G	I	J	N
$\langle M_1^4\rangle^2\langle M_3\rangle^2$	F	*	H	K	L, N	O
$\langle M_1^4\rangle^2\langle M_4\rangle$	G	H		L		
$\langle M_2^2\rangle^2\langle M_3\rangle^2$	I	K	M	*	P	Q
$\langle M_2^2\rangle^2\langle M_4\rangle$	J	M, N		P		R
$\langle M_3\rangle^2\langle M_4\rangle$	N, M, L	O		Q	R	
$\langle M_1^4\rangle^2\langle M_2^2\rangle^2\langle M_3\rangle^2$			S		T	U
$\langle M_1^4\rangle^2\langle M_2^2\rangle^2\langle M_4\rangle$		S		T		V
$\langle M_1^4\rangle^2\langle M_3\rangle^2\langle M_4\rangle$	S			U	V	
$\langle M_2^2\rangle^2\langle M_3\rangle^2\langle M_4\rangle$	T	U	V			
$\langle M_1^4\rangle^2\langle M_2^2\rangle^2\langle M_3\rangle^2\langle M_4\rangle$						

Figure 3.4.10: Summary of d^r- Boundaries

The previous theorem has left a remnant of a few elements on the "2σ row". We see in the next theorem that these elements map monomorphicly under d^s-differentials to elements of $Z_2\theta \otimes H_*BP$ in the s+7 row. We shall show in Chapter 6 that s = 24, and in Chapter 8 we will show that this element $\theta \in \pi^S_{30}$ is the (in)famous element θ_4 of Arf invariant one.

THEOREM 3.4.10 (a) There is an s > 6 and a nonzero element $\theta \in E^s_{24-s,s+6}$ of order two such that $d^s(2\sigma M_1^{12}) = \theta$.

(b) Image $[d^{24}: E^{24}_{*,7} \longrightarrow E^{24}_{*,7}] = Z_2\theta[1, \langle M_1^4\rangle^2, \langle M_2^2\rangle^2, \langle M_3\rangle^2]$

$$\otimes Z[\langle M_1^4\rangle^4, \langle M_2^2\rangle^4, \langle M_3\rangle^4, \langle M_4\rangle^2, \{M_5\}', \ldots, \{M_N\}', \ldots].$$

(c) $E^{s+2}_{*,7} = 0$.

PROOF. By Theorem 3.4.9(b), $E^{r+2}_{24,7} = Z_2(2\sigma M_1^{12})$. Thus, there must be a nonzero d^s-differential originating on $2\sigma M_1^{12}$ hitting an element θ of order 2. Part (b) is a consequence of Theorem 3.4.9(a) and the computations in Figure 3.4.11. By Theorem 3.4.9(a), d^s is an isomorphism from $E^s_{*,7}$ to the elements listed in (b), and $E^{s+2}_{*,7} = 0$. ∎

$$
\begin{array}{lcl}
\underline{E^s_{*,7}} & & \underline{E^s_{*,s+6}} \\
2\sigma\langle M_1^4\rangle^3 & \longrightarrow & \theta \\
2\sigma M_1^2\langle M_2^2\rangle^3 & \longrightarrow & \theta\langle M_1^4\rangle^2 \\
2\sigma\bar{M}_2\langle M_3\rangle^3 & \longrightarrow & \theta\langle M_2^2\rangle^2 \\
2\sigma M_1M_2M_3\langle M_4\rangle & \longrightarrow & \theta\langle M_3\rangle^2 \\
2\sigma\bar{M}_2\langle M_3\rangle\langle M_1^4\rangle^2\langle M_3\rangle^2 & \longrightarrow & \theta\langle M_1^4\rangle^2\langle M_2^2\rangle^2 \\
2\sigma M_1M_2M_3\langle M_1^4\rangle\langle M_4\rangle & \longrightarrow & \theta\langle M_1^4\rangle^2\langle M_3\rangle^2 \\
2\sigma M_1M_2M_3\langle M_2^2\rangle\langle M_4\rangle & \longrightarrow & \theta\langle M_2^2\rangle^2\langle M_3\rangle^2 \\
2\sigma M_1M_2M_3\langle M_1^4\rangle\langle M_2^2\rangle\langle M_4\rangle & \longrightarrow & \theta\langle M_1^4\rangle^2\langle M_2^2\rangle^2\langle M_3\rangle^2
\end{array}
$$

Figure 3.4.11: Summary of d^s-Boundaries

We conclude with a complete analysis of the 8 row. The element λ mentioned in the theorem will be shown to be $\eta A[14]M_1$ in Chapter 5.

THEOREM 3.4.11 (a) $\pi_8^S = Z_2A[8] \oplus Z_2\eta\sigma$.

(b) $\eta A[8]$ and $\eta^2\sigma$ are both nonzero.

(c) The only differentials which land on the 8 row are determined by the leading differentials $d^2(\sigma M_1) = \eta\sigma$ and $d^{12}(2\nu M_1^3) = A[8]$.

(d) The only differentials which originate from the 8 row are determined by the leading differentials $d^2(\eta\sigma M_1) = \eta^2\sigma$, $d^2(A[8]M_1) = \eta A[8]$ and $d^t(A[8]M_1^2\bar{M}_2) = \lambda$.

PROOF. (a), (b), (c) $\pi_5^S = 0$ and $E^4_{*,1} = 0$. Thus the only nonzero differentials which can land in $E^r_{0,8}$ originate from the 7 row and the 3 row. These differentials were analyzed in Theorems 3.3.13 and 3.4.1. It was shown in Theorems 3.3.7 and 3.3.15 that $2A[8] = 0$ and $\eta A[8] \neq 0$. Note that $\eta\sigma M_1$ is not a d^2-boundary, $\eta\sigma M_1$ is not a d^4-boundary because $\pi_5^S = 0$, $\eta\sigma M_1$ is not a d^6-boundary because $E^6_{8,3} = 0$ and $\eta\sigma M_1$ is not a d^8-boundary because $E^4_{10,1} = 0$. Thus, $\eta^2\sigma \neq 0$.

(d) $E^4_{*,8} = Z_2A[8][M_1^2, \bar{M}_2, \ldots, \bar{M}_N, \ldots]$ and by Theorem 3.3.13(b) it follows that $E^8_{*,8} = Z_2(A[8]M_1^2\bar{M}_2) \otimes B\langle 4\rangle$. Thus there must be a nonzero differential $d^t(A[8]M_1^2\bar{M}_2) = \lambda$. Clearly d^t is a monomorphism with image $Z_2\lambda \otimes B\langle 4\rangle$. Thus, $E^{t+2}_{*,8} = 0$.∎

The following Toda bracket is not easily seen from the Atiyah-Hirzebruch spectral sequence. We therefore record it as a corollary.

COROLLARY 3.4.12 $A[8] = \langle\nu, \eta, \nu\rangle$

PROOF. Multiplication by η is a monomorphism on π_8^S. Now $\eta\langle\nu, \eta, \nu\rangle = \langle\eta, \nu, \eta\rangle\nu = \nu^2\cdot\nu = \eta A[8]$. Thus $\langle\nu, \eta, \nu\rangle = A[8]$.∎

We conclude by summarizing the notation we introduced in this chapter. This notation will be used throughout the remainder of this work.

DEFINITION 3.4.13 (a) We have the following d^2-cycles:

$$\bar{M}_2 = 3M_2 - M_1^3$$
$$\bar{M}_N = M_N - M_1M_{N-1}^2 \text{ for } N \geq 3.$$

(b) We have the following d^4-cycles:

$$\langle M_1^4\rangle = M_1^4 + 2M_1M_2$$
$$\langle M_2^2\rangle = M_2^2 + M_1^6$$
$$\langle M_3\rangle = \bar{M}_3 + 5M_1^4\bar{M}_2 + 14M_1^7$$
$$\langle M_4\rangle = \text{a polynomial generator in } H_{30}BP.$$

(c) We have the following d^8-cycles:

$\{M_N\}$, $N \geq 5$, which are polynomial generators of H_*BP.

(d) We have the following subalgebras of H_*BP:

$B\langle 2\rangle = Z_{(2)}[M_1^2, \bar{M}_2, \ldots, \bar{M}_N, \ldots]$ is a subalgebra of $E^4_{*,0}$;

$B\langle 4\rangle = Z_{(2)}[\langle M_1^4\rangle, \langle M_2^2\rangle, \langle M_3\rangle, \langle M_4\rangle, \ldots, \langle M_N\rangle, \ldots]$ is a subalgebra of $E^8_{*,0}$;

$B\langle 8\rangle = Z_{(2)}[\langle M_1^4\rangle^2, \langle M_2^2\rangle^2, \langle M_3\rangle^2, \langle M_4\rangle, \{M_5\}, \ldots, \{M_N\}, \ldots]$ is a subalgebra of $E^8_{*,0}$ such that $d^8(B\langle 8\rangle) \subset Z_8(2\sigma) \otimes H_*BP$.