

AN EQUIVARIANT TENSOR PRODUCT ON MACKEY FUNCTORS

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ABSTRACT. Let G be a cyclic p -group. We define an equivariant symmetric monoidal structure on the category of G -Mackey functors and show that the commutative ring objects in this framework are G -Tambara functors. To develop this structure, for all subgroups H in G we create norm functors that build a G -Mackey functor from an H -Mackey functor. Unlike similar functors, we define our norm functors in terms of generators and relations based solely on the intrinsic, algebraic properties of Mackey functors and Tambara functors. Thus, our construction is computationally accessible.

1. INTRODUCTION

Let G be a finite group. A commutative G -ring spectrum comes equipped with extra structure not seen in an ordinary G -spectrum. Namely, it possesses norm maps that are multiplicative versions of the transfer maps. Around a decade ago, Brun [Bru07] showed that we see the algebraic shadows of these norm maps in the zeroth stable homotopy groups of commutative G -ring spectra. More specifically, it is well known that π_0 of a G -spectrum is a Mackey functor (see for example [May96]), but Brun proved that if X is a commutative G -ring spectrum, then $\pi_0(X)$ is a Tambara functor [Bru07]. Thus, it is a Mackey functor with a ring structure (i.e. a Green functor) and an extra class of maps that are the multiplicative analogues of the transfer maps. These maps are appropriately called norm maps.

Our goal is to develop a framework under which the relationship between Mackey functors and Tambara functors mirrors the relationship between G -spectra and commutative G -ring spectra. Indeed, the category of Mackey functors is symmetric monoidal, but the commutative ring objects under the symmetric monoidal product are not Tambara functors. Notably, they do not have norm maps. Therefore, in this paper we define an equivariant symmetric monoidal structure on the category of Mackey functors under which Tambara functors are

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the commutative ring objects. Hill and Hopkins have developed an appropriate notion of equivariant symmetric monoidal, calling it *G-symmetric monoidal* [HH13]. They then call the commutative ring objects under a *G-symmetric monoidal structure* the *G-commutative monoids* [HH13]. We provide formal definitions of these concepts in Section 2.

Hill and Hopkins [HH13], Ullman [Ull13], and Hoyer [Hoy14] have independently defined *G-symmetric monoidal structures* on the category \mathcal{Mack}_G of *G-Mackey functors*. At the core of these structures are functors N_H^G for all subgroups H of G that send an *H-Mackey functor* to a *G-Mackey functor*. Hill and Hopkins and Ullman defined N_H^G by passing to an *H-spectrum* via the Eilenberg-MacLane functor, applying the Hill-Hopkins-Ravenel norm functor and then returning to *G-Mackey functors* via π_0 . In [Ull13] Ullman then develops algebraic descriptions of N_H^G and the corresponding *G-symmetric monoidal structure* and proves that Tambara functors are the *G-commutative monoids*. Hoyer defined the functors N_H^G via coends.

In this paper, for G a cyclic p -group we define a *G-symmetric monoidal structure* on \mathcal{Mack}_G without passing to spectra and without using coends. Hence, we develop a *concrete G-symmetric monoidal structure*. We construct new functors $N_H^G: \mathcal{Mack}_H \rightarrow \mathcal{Mack}_G$ for all subgroups H of G by appealing directly to the algebraic properties of Mackey and Tambara functors. We use the word “construct” deliberately. Given an *H-Mackey functor* \underline{M} we build a *G-Mackey functor* $N_H^G \underline{M}$ piece by piece using only the intrinsic properties of Mackey functors and Tambara functors. We then define the functors N_H^G via the map $\underline{M} \mapsto N_H^G \underline{M}$ and prove the theorems below to create a *G-symmetric monoidal structure* under which Tambara functors are the *G-commutative monoids*.

For H a subgroup of G let $i_H^*: \mathcal{Mack}_G \rightarrow \mathcal{Mack}_H$ be the forgetful functor. Further, let $\mathcal{S}et_G^{Fin, Iso}$ be the category whose objects are finite G -sets but whose morphisms are only isomorphisms between G -sets. Define the functor $(-) \otimes (-): \mathcal{S}et_G^{Fin, Iso} \times \mathcal{Mack}_G \rightarrow \mathcal{Mack}_G$ by the following three properties:

- $\emptyset \otimes \underline{M} := \underline{A}$, where \underline{A} is the Burnside Mackey functor,
- $G/H \otimes \underline{M} := N_H^G i_H^* \underline{M}$ for all orbits G/H of G , and
- $(X \amalg Y) \otimes \underline{M} := (X \otimes \underline{M}) \sqcup (Y \otimes \underline{M})$ for all X and Y in $\mathcal{S}et_G^{Fin, Iso}$.

Main Theorem 1. *The functor $(-) \otimes (-): \mathcal{S}et_G^{Fin, Iso} \times \mathcal{Mack}_G \rightarrow \mathcal{Mack}_G$ defined above is a *G-symmetric monoidal structure* on \mathcal{Mack}_G .*

Main Theorem 2. *Let \underline{M} be a G -Mackey functor, and endow \mathcal{Mack}_G with the G -symmetric monoidal structure defined in Main Theorem 1. Then \underline{M} is a G -commutative monoid if and only if it has the structure of a G -Tambara functor.*

Even before the emergence of Tambara functors in equivariant stable homotopy theory, we have been interested in developing a structure on \mathcal{Mack}_G that made Tambara functors into ring objects. Eleven years prior to the publication of Brun’s paper [Bru07], Dave Benson hosted a problem session at the 1996 Seattle conference on Cohomology, Representations and Actions of Finite Groups in which he invited speakers to present problems related to the field [Ben]. T. Yoshida proposed to define a tensor induction for Mackey functors that preserves tensor products of Mackey functors and satisfies Tambara’s axioms for multiplicative transfer. The G -symmetric monoidal structure that we create is such a tensor induction.

Furthermore, extending our construction to all abelian groups is an open problem. We feel strongly that it can be done but many aspects become more complicated.

We organize this paper into the following sections. In Section 2 we provide precise definitions of G -symmetric monoidal and G -commutative monoid. We also recall the diagrammatic depiction of Mackey functors and Lewis’ constructive definition of the symmetric monoidal product in \mathcal{Mack}_G . In Section 3 we provide two definitions of Tambara functors. The first is Tambara’s original definition. The second is an axiomatic and constructive definition specifically for G -Tambara functors when G is a cyclic p -group. We base the construction of the G -symmetric monoidal structure given in Main Theorem 1 on the second definition. In Section 4 we develop the functors N_H^G , before proving Main Theorems 1 and 2 in Section 5.

2. PRELIMINARIES

We begin this section with Hill and Hopkins’ definitions for a G -symmetric monoidal structure and a G -commutative monoid [HH13]. We then recall various facts about Mackey functors that are crucial to our G -symmetric monoidal construction on \mathcal{Mack}_G .

Definition 2.1. Let $(\mathcal{C}, \boxtimes, e)$ be a symmetric monoidal category. A G -symmetric monoidal structure on \mathcal{C} consists of a functor

$$(-) \otimes (-): \mathcal{S}et_G^{Fin, Iso} \times \mathcal{C} \rightarrow \mathcal{C}$$

that satisfies the following properties.

- (1) $(X \amalg Y) \otimes C = (X \otimes C) \boxtimes (Y \otimes C)$ and $X \otimes (C \boxtimes D) = (X \otimes C) \boxtimes (X \otimes D)$.
- (2) When restricted to $\mathcal{S}et^{Fin,Iso} \times \mathcal{C}$ this functor is the canonical exponentiation map given by $X \otimes \mathcal{C} = C^{\boxtimes |X|}$.
- (3) $X \otimes (Y \otimes C)$ is naturally isomorphic to $(X \times Y) \otimes C$.

Let $\mathcal{S}et_G^{Fin}$ be the category of finite G -sets. Every object C in \mathcal{C} defines a functor

$$(-) \otimes C: \mathcal{S}et_G^{Fin,Iso} \rightarrow \mathcal{C},$$

and we define a G -commutative monoid as follows.

Definition 2.2. A G -commutative monoid is an object C in \mathcal{C} together with the extension below.

$$\begin{array}{ccc} \mathcal{S}et_G^{Fin,Iso} & \xrightarrow{(-) \otimes C} & \mathcal{C} \\ \downarrow & \nearrow & \\ \mathcal{S}et_G^{Fin} & & \end{array}$$

Hence, a G -commutative monoid is an object C for which it makes sense to apply the functor $(-) \otimes C$ to maps of finite G -sets. In other words, C is a G -commutative monoid if and only if a map $X \rightarrow Y$ in $\mathcal{S}et_G^{Fin}$ induces a map $X \otimes C \rightarrow Y \otimes C$ in \mathcal{C} .

2.1. Mackey Functors and the Box Product. We will visualize the G -Mackey functor $N_H^G \underline{M}$ using the standard lattice diagram for a Mackey functor. For example, we give a C_2 -Mackey functor in Figure 1 where $res_e^{C_2}$ is the restriction map and $tr_e^{C_2}$ is the transfer map. To reduce clutter we do not draw the Weyl action.

$$\begin{array}{ccc} & \underline{M}(C_2/C_2) & \\ & \curvearrowright & \\ res_e^{C_2} & & tr_e^{C_2} \\ & \underline{M}(C_2/e) & \end{array}$$

FIGURE 1. \underline{M} is a C_2 -Mackey Functor

Further, a morphism $\phi: \underline{M} \rightarrow \underline{L}$ of Mackey functors consists of a collection of homomorphisms $\{\phi_H: \underline{M}(G/H) \rightarrow \underline{L}(G/H) : H \leq G\}$ that commute with the appropriate restriction and transfer maps. The morphism ϕ is an isomorphism if every ϕ_H is an isomorphism.

We call the symmetric monoidal product in $\mathcal{M}ack_G$ the *box product* \square . Category theoretic definitions of the box product can be found in

[Lew81] and [Ven05]. We focus on Lewis' constructive definition first published in [Lew88] and later fixed in [Shu10] because it motivates our definition of the G -Mackey functors $N_H^G \underline{M}$. For brevity we give the definition of $\underline{M} \square \underline{L}$ only for two C_p -Mackey functors. However, Definition 2.3 naturally extends to a k -fold box product for any positive integer k and to the box product of C_{p^n} -Mackey functors for any cyclic p -group C_{p^n} .

Definition 2.3. Given C_p -Mackey functors \underline{M} and \underline{L} we define their box product $\underline{M} \square \underline{L}$ by the diagram in Figure 2.

$$\begin{array}{c}
 \begin{array}{c}
 \text{Im}(tr_e^{C_p}) \\
 \overbrace{(\underline{M}(C_p/e) \otimes \underline{L}(C_p/e)) / W_{C_p}(e)} \\
 \left[\underline{M}(C_p/C_p) \otimes \underline{L}(C_p/C_p) \oplus \right] / FR
 \end{array} \\
 \begin{array}{c}
 \text{res}_e^{C_2} \left(\begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} \right) tr_e^{C_p} \\
 \underline{M}(C_p/e) \otimes \underline{L}(C_p/e)
 \end{array}
 \end{array}$$

FIGURE 2. The Box Product of C_p -Mackey Functors

The transfer map is the quotient map onto the second summand. The restriction map is induced from $res_e^{C_p} \otimes res_e^{C_p}$ on the first summand and by the trace of the Weyl action on the second summand. The Weyl action on $\underline{M}(C_p/e) \otimes \underline{L}(C_p/e)$ is the diagonal action: the generator γ of the Weyl group acts by $\gamma(m \otimes l) = \gamma m \otimes \gamma l$. Finally, FR is the *Frobenius reciprocity* submodule and is generated by all elements of the form $m' \otimes tr_e^{C_p}(l) - tr_e^{C_p}(res_e^{C_p}(m') \otimes l)$ and $tr_e^{C_p}(m) \otimes l' - tr_e^{C_p}(m \otimes res_e^{C_p}(l'))$ for all m' in $\underline{M}(C_p/C_p)$, l' in $\underline{L}(C_p/C_p)$, m in $\underline{M}(C_p/e)$, and l in $\underline{L}(C_p/e)$.

3. TAMBARA FUNCTORS

We give two definitions of Tambara functors. The first is Tambara's original definition and holds for any finite group. The second is a new definition that holds only for cyclic p -groups. However, the second definition will act as a blueprint for our construction of the functors $N_H^G: \mathcal{Mack}_H \rightarrow \mathcal{Mack}_G$. We will be able to prove Main Theorem 2 because the properties of the G -Mackey functor $N_H^G \underline{M}$ directly mirror the properties of a Tambara functor given in Definition 3.3.

Tambara first defined Tambara functors in [Tam93], originally calling them TNR-functors. TNR stands for transfer, norm and restriction. His definition relies on a few technical constructions that we recall below. These can also be found in [Tam93] or [Nak14].

Given morphisms $f: X \rightarrow Y$ and $p: A \rightarrow X$ in $\mathcal{S}et_G^{Fin}$ we define $\prod_f A$ by

$$\prod_f A = \left\{ (y, \sigma) \left| \begin{array}{l} y \in Y, \\ \sigma: f^{-1}(y) \rightarrow A \text{ is a map of sets,} \\ p \circ \sigma(x) = x \text{ for all } x \in f^{-1}(y) \end{array} \right. \right\}.$$

The group G acts on $\prod_f A$ by $\gamma(y, \sigma) = (\gamma y, \gamma \sigma)$ where $(\gamma \sigma)(x) = \gamma \sigma(\gamma^{-1} x)$.

Definition 3.1. For all morphisms $f: X \rightarrow Y$ and $p: A \rightarrow X$ in $\mathcal{S}et_G^{Fin}$ the *canonical exponential diagram* generated by f and p is the commutative diagram below, where $h(y, \sigma) = y$, the map e is the evaluation map $e(x, (y, \sigma)) = \sigma(x)$ and f' is the pullback of f by h .

$$\begin{array}{ccc} X & \xleftarrow{p} & A \xleftarrow{e} X \times_Y \prod_f A \\ f \downarrow & & \downarrow f' \\ Y & \xleftarrow{h} & \prod_f A \end{array}$$

We call a diagram in $\mathcal{S}et_G^{Fin}$ that is isomorphic to a canonical exponential diagram an *exponential diagram*.

Let $\mathcal{S}et$ be the category of non-empty sets.

Definition 3.2. [Tam93] A G -Tambara functor \underline{S} is a triple (S^*, S_*, S_\star) consisting of two covariant functors

$$S_*: \mathcal{S}et_G^{Fin} \rightarrow \mathcal{S}et$$

$$S_\star: \mathcal{S}et_G^{Fin} \rightarrow \mathcal{S}et$$

and one contravariant functor

$$S^*: \mathcal{S}et_G^{Fin} \rightarrow \mathcal{S}et$$

such that the following properties hold.

- (1) All functors have the same object function $X \mapsto \underline{S}(X)$, and each $\underline{S}(X)$ is a commutative ring.
- (2) For all morphisms $f: X \rightarrow Y$ in $\mathcal{S}et_G^{Fin}$ the map $S_*(f)$ is a homomorphism of additive monoids, $S_\star(f)$ is a homomorphism of multiplicative monoids and $S^*(f)$ is a ring homomorphism.
- (3) The pair (S^*, S_*) is a G -Mackey functor and (S^*, S_\star) is a semi- G -Mackey functor.

(4) (Distributive Law) If

$$\begin{array}{ccccc} X & \xleftarrow{p} & A & \xleftarrow{e} & X' \\ f \downarrow & & & & \downarrow f' \\ Y & \xleftarrow{h} & & & Y' \end{array}$$

is an exponential diagram then the induced diagram below commutes.

$$\begin{array}{ccccc} \underline{S}(X) & \xleftarrow{p_*} & \underline{S}(A) & \xrightarrow{e^*} & \underline{S}(X') \\ f_* \downarrow & & & & \downarrow f'_* \\ \underline{S}(Y) & \xleftarrow{h_*} & & & \underline{S}(Y') \end{array}$$

Let \mathcal{O}_G be the orbit category of a group G . We will provide a new axiomatic definition of G -Tambara functors that holds when G is a cyclic p -group. This definition is analogous to the definition of Mackey functors using axiomatic relations given in [Web00].

Definition 3.3. Let G be a cyclic p -group with generator γ . Then a G -Tambara functor \underline{S} consists of a collection of commutative rings

$$\{\underline{S}(G/H) : G/H \in \mathcal{O}_G\}$$

along with the following maps for all orbits G/H and G/K such that $H < K$: the *restriction* map $res_H^K : \underline{S}(G/K) \rightarrow \underline{S}(G/H)$, the *transfer* map $tr_H^K : \underline{S}(G/H) \rightarrow \underline{S}(G/K)$, and the *norm* map $N_H^K : \underline{S}(G/H) \rightarrow \underline{S}(G/K)$. These rings and maps satisfy the following conditions.

- (1) Every $\underline{S}(G/H)$ is equipped with a $W_G(H)$ -action.
- (2) All restriction maps are ring homomorphisms, all transfer maps are homomorphisms of additive monoids, and all norm maps are homomorphisms of multiplicative monoids.
- (3) (Transitivity) For all $H' < H < K$

$$\begin{aligned} res_{H'}^K &= res_{H'}^H res_H^K \\ tr_{H'}^K &= tr_H^K tr_{H'}^H \\ N_{H'}^K &= N_H^K N_{H'}^H. \end{aligned}$$

- (4) (Frobenius Reciprocity) If $H < K$, then for all x in $\underline{S}(G/H)$ and y in $\underline{S}(G/K)$

$$(y)tr_H^K(x) = tr_H^K(res_H^K(y)x).$$

- (5) If $H < K$, then for all γ^s in $W_K(H)$, all x in $\underline{S}(G/H)$, and all y in $\underline{S}(G/K)$

$$\begin{aligned}\gamma^s \text{res}_H^K(y) &= \text{res}_H^K(y) \\ \text{tr}_H^K(\gamma^s x) &= \text{tr}_H^K(x) \\ N_H^K(\gamma^s x) &= N_H^K(x).\end{aligned}$$

- (6) If $H < K \leq G$, then for all x in $\underline{S}(G/H)$

$$\begin{aligned}\text{res}_H^K \text{tr}_H^K(x) &= \sum_{\gamma^s \in W_K(H)} \gamma^s x \\ \text{res}_H^K N_H^K(x) &= \prod_{\gamma^s \in W_K(H)} \gamma^s x.\end{aligned}$$

- (7) (Tambara Reciprocity for Sums) If $H < K \leq G$, then for all a and b in $\underline{S}(G/H)$

$$N_H^K(a + b) =$$

$$N_H^K(a) + N_H^K(b) + \sum_{H < K' < K} \text{tr}_{K'}^{K'} \left(\sum_{k=1}^{i_{K'}} N_H^{K'} \left((\underline{ab})_k^{K'} \right) \right) + \text{tr}_H^K(g_H(a, b))$$

where $g_H(a, b)$ is a polynomial in some of the $W_K(H)$ -conjugates of a and b , and each $(\underline{ab})_k^{K'}$ is a monomial in some of the $W_K(K')$ -conjugates of a and b . These polynomials are universally determined by the group G in the sense that their exact formulas depend only on G and not on the particular Tambara functor. The integers $i_{K'}$ are similarly universally determined by G and so depend only on G .

- (8) (Tambara Reciprocity for Transfers) For all subgroups $H' < H < K$ and all x in $\underline{S}(G/H')$

$$N_H^K(\text{tr}_{H'}^H(x)) = \text{tr}_{H'}^K(f(x))$$

where $f(x)$ is a polynomial in some of the $W_K(H')$ -conjugates of x and is universally determined by the group G .

We refer to the last two properties of Definition 3.3 collectively as *Tambara reciprocity*. Moreover, we can use the proof of Theorem 3.5 to determine exact formulas for the polynomials given in Tambara reciprocity. We show how to establish these formulas for a C_4 -Tambara functor in Example 3.7.

We collect the properties of Definition 3.3 into a lattice diagram much like Figure 1. For example, a C_4 -Tambara functor is pictured in Figure 3. Further, a morphism $\phi: \underline{S} \rightarrow \underline{S}'$ of Tambara functors consists of a collection of ring homomorphisms $\{\phi_H: \underline{S}(G/H) \rightarrow \underline{S}'(G/H) \text{ for all } H \leq$

$$\begin{array}{ccc}
 \underline{S}(C_4/C_4) & & \\
 \text{res}_{C_2}^{C_4} \curvearrowright \uparrow N_{C_2}^{C_4} \curvearrowleft \text{tr}_{C_2}^{C_4} & & \\
 \underline{S}(C_4/C_2) & & \\
 \text{res}_e^{C_2} \curvearrowright \uparrow N_e^{C_2} \curvearrowleft \text{tr}_e^{C_2} & & \\
 \underline{S}(C_4/e) & &
 \end{array}$$

 FIGURE 3. \underline{S} is a C_4 -Tambara Functor

$G\}$ that commute with the appropriate restriction, transfer and norm maps.

Remark 3.4. If we forget all mention of the norm map in Definition 3.3, then the leftovers fit together to form a definition of a commutative G -Green functor. Green functors are the rings in \mathcal{Mack}_G under the box product.

To prove that Definitions 3.2 and 3.3 are equivalent for G -Tambara functors when G is a cyclic p -group we would need to check that the axioms of the first definition imply the second and vice versa. In [Web00], Webb provides a proof of an analogous result for Mackey functors. We will show that Definition 3.2 of a Tambara functor satisfies Definition 3.3 so that we can use the axioms of Definition 3.3 to define the functors $N_H^G: \mathcal{Mack}_H \rightarrow \mathcal{Mack}_G$. However, we will not include a proof of the reverse implication because it becomes messy and we will not need it in subsequent sections.

Theorem 3.5. *Let G be a cyclic p -group. A G -Tambara functor as given in Definition 3.2 satisfies the axiomatic relations of Definition 3.3.*

Proof. Let G be a cyclic p -group, and let \underline{S} be a G -Tambara functor as defined in Definition 3.2. Showing that \underline{S} satisfies properties (1)-(5) of Definition 3.3 is straightforward. Thus, it remains to use the distributive law of Definition 3.2 to show that \underline{S} satisfies Tambara reciprocity.

Let H and K be subgroups of G such that $H < K$. It suffices to derive the Tambara reciprocity formulas only for $K = G$ since the norm map N_H^K in a G -Tambara functor \underline{S} must agree with N_H^K in the K -Tambara functor $i_K^* \underline{S}$.

Let a and b be in $\underline{S}(G/H)$. We will first develop the Tambara reciprocity formula for the norm of a sum $N_H^G(a + b)$. Let $*_1 \amalg *_2$ be the disjoint union of two single point G -sets and let ∇ be the fold map.

Then the diagram below is an exponential diagram,

$$\begin{array}{ccccc} G/H & \xleftarrow{\nabla} & G/H \amalg G/H & \xleftarrow{e} & G/H \times \text{Map}_H(G, *_1 \amalg *_2) \\ \downarrow & & & & \downarrow \pi \\ G/G & \xleftarrow{h} & & & \text{Map}_H(G, *_1 \amalg *_2) \end{array}$$

and applying \underline{S} to it yields the following commutative diagram.

$$\begin{array}{ccc} \underline{S}(G/H) & \xleftarrow{\nabla_*} & \underline{S}(G/H) \oplus \underline{S}(G/H) \xrightarrow{e^*} \underline{S}(G/H \times \text{Map}_H(G, *_1 \amalg *_2)) \\ N_H^G \downarrow & & \downarrow \pi_* \\ \underline{S}(G/G) & \xleftarrow{h_*} & \underline{S}(\text{Map}_H(G, *_1 \amalg *_2)) \end{array}$$

The composition $N_H^G \nabla_*(a, b)$ equals $N_H^G(a + b)$. Thus, it remains to determine $h_* \pi_* e^*(a, b)$. We first simplify $\underline{S}(G/H \times \text{Map}_H(G, *_1 \amalg *_2))$ and $\underline{S}(\text{Map}_H(G, *_1 \amalg *_2))$ by decomposing $G/H \times \text{Map}_H(G, *_1 \amalg *_2)$ and $\text{Map}_H(G, *_1 \amalg *_2)$ into disjoint unions of G -orbits.

The decomposition of $\text{Map}_H(G, *_1 \amalg *_2)$ consists of two copies of G/G , some copies of G/K' for all subgroups K' such that $H < K' < G$, and some copies of G/H . We do not need to know the exact number of copies of every G/K' and G/H in the decomposition. Instead it suffices that these numbers are universally determined by G . Hence, $\text{Map}_H(G, *_1 \amalg *_2)$ is isomorphic to

$$G/G \amalg G/G \amalg \left(\prod_{j=1}^i (G/H)_j \right) \amalg \left[\prod_{H < K' < G} \left(\prod_{k=1}^{i_{K'}} (G/K')_k \right) \right]$$

where the integers i and $i_{K'}$ are determined by G .

Now consider $G/H \times \text{Map}_H(G, *_1 \amalg *_2)$. Whenever H is a subgroup of K , the G -set $G/H \times G/K$ is isomorphic to the disjoint union of $|G/K|$ -many copies of G/H , which we denote $\prod_{|G/K|} G/H$. Hence, $G/H \times \text{Map}_H(G, *_1 \amalg *_2)$ is isomorphic to

$$G/H \amalg G/H \amalg \left(\prod_{j=1}^i \left(\prod_{|G/H|} G/H \right)_j \right) \amalg \left[\prod_{H < K' < G} \left(\prod_{k=1}^{i_{K'}} \left(\prod_{|G/K'|} G/H \right)_k \right) \right].$$

Therefore, the ring $\underline{S}(G/H \times \text{Map}_H(G, *_1 \amalg *_2))$ is isomorphic to

$$\underline{S}(G/H) \oplus \underline{S}(G/H) \oplus \bigoplus_{j=1}^i \left(\bigoplus_{|G/H|} \underline{S}(G/H) \right)_j \oplus \bigoplus_{H < K' < G} \bigoplus_{k=1}^{i_{K'}} \left(\bigoplus_{|G/K'|} \underline{S}(G/H) \right)_k.$$

Similarly, $\underline{S}(\text{Map}_H(G, *_1 \amalg *_2))$ is isomorphic to

$$\underline{S}(G/G) \oplus \underline{S}(G/G) \oplus \bigoplus_{j=1}^i \underline{S}(G/H)_j \oplus \bigoplus_{H < K' < G} \left(\bigoplus_{k=1}^{i_{K'}} \underline{S}(G/K')_k \right).$$

Next we establish a formula for the map e^* in the above diagram: $e^*(a, b)$ is a sequence of a 's and b 's, the order of which is determined by the decomposition of $\text{Map}_H(G, *_1 \amalg *_2)$ into G -orbits. More specifically, the first summand of $\underline{S}(G/H \times \text{Map}_H(G, *_1 \amalg *_2))$ stems from the orbit in $\text{Map}_H(G, *_1 \amalg *_2)$ consisting of the constant map that sends all elements of G to $*_1$. Hence, $e^*(a, b) = a$ on that summand. Similarly, $e^*(a, b) = b$ on the next summand. To give how $e^*(a, b)$ maps into each $(\bigoplus_{|G/H|} \underline{S}(G/H))_j$ we index $(\bigoplus_{|G/H|} \underline{S}(G/H))_j$ using the elements of G/H :

$$\bigoplus_{|G/H|} \underline{S}(G/H) = \underline{S}(G/H)_e \oplus \underline{S}(G/H)_\gamma \oplus \cdots \oplus \underline{S}(G/H)_{\gamma^{|G/H|-1}}.$$

Then let the j^{th} copy of G/H in $\text{Map}_H(G, *_1 \amalg *_2)$ be the orbit of the map f_j . So, $e^*(a, b)$ is a on the γ^s -summand of $\bigoplus_{|G/H|} \underline{S}(G/H)_j$ if $f_j(\gamma^s) = *_1$ and b if $f_j(\gamma^s) = *_2$. (The choice of the element f_j in the orbit does not matter.) We denote the resulting sequence of a 's and b 's in $\bigoplus_{|G/H|} \underline{S}(G/H)_j$ by $(A, B)_j^H$. Then e^* maps (a, b) into each $(\bigoplus_{|G/K'|} \underline{S}(G/K'))_k$ analogously, and we denote the result by $(A, B)_k^{K'}$. Thus,

$$e^*(a, b) = a \oplus b \oplus \bigoplus_{j=1}^i (A, B)_j^H \oplus \bigoplus_{H < K' < G} \left(\bigoplus_{k=1}^{i_{K'}} (A, B)_k^{K'} \right).$$

Next we establish the formula for π_* . First, the map

$$(\gamma_H^t \cdot): \bigoplus_{|G/H|} \underline{S}(G/H) \rightarrow \underline{S}(G/H)$$

is given by

$$(s_e, s_\gamma, s_{\gamma^2}, \dots, s_{\gamma^{|G/H|-1}}) \mapsto s_e \gamma s_\gamma \gamma^2 s_{\gamma^2} \cdots \gamma^{|G/H|-1} s_{\gamma^{|G/H|-1}}.$$

The map $(\gamma_{K'}^t \cdot): \bigoplus_{|G/K'|} \underline{S}(G/H) \rightarrow \underline{S}(G/H)$ is analogous. Then π_* is

$$N_H^G \oplus N_H^G \oplus \bigoplus_{j=1}^i (\gamma_H^t \cdot)_j \oplus \bigoplus_{H < K' < G} \bigoplus_{k=1}^{i_{K'}} \left[N_H^{K'} (\gamma_{K'}^t \cdot)_k \right],$$

where $N_H^G: \underline{S}(G/H) \rightarrow \underline{S}(G/G)$ and $N_H^{K'}: \underline{S}(G/H) \rightarrow \underline{S}(G/K')$ are the norm maps.

Consider now the composition $\pi_* e^*(a, b)$. Under π_* , each $(A, B)_j^H$ in $e^*(a, b)$ maps to a product of some of the $W_G(H)$ -conjugates of a and b . Denote this product $(\underline{AB})_j^H$. Similarly, π_* maps each $(A, B)_k^{K'}$ to $N_H^{K'}((\underline{ab})_k^{K'})$, where $(\underline{ab})_k^{K'}$ is the product of some of the $W_G(K')$ -conjugates of a and b . Thus, $\pi_* e^*(a, b)$ equals

$$N_H^G(a) \oplus N_H^G(b) \oplus \bigoplus_{j=1}^i (\underline{AB})_j^H \oplus \bigoplus_{H < K' < G} \left(\bigoplus_{k=1}^{i_{K'}} N_H^{K'}((\underline{ab})_k^{K'}) \right).$$

Finally, the map h_* is the identity on the first two summands of $\underline{S}(\text{Map}_H(G, *_1 \amalg *_2))$, the transfer map tr_H^G on each $\underline{S}(G/H)_j$ and the transfer $\text{tr}_{K'}^G$ on each $\underline{S}(G/K')$.

Therefore, $h_* \pi_* e^*(a, b)$ equals

$$N_H^G(a) + N_H^G(b) + \text{tr}_H^G(g_H(a, b)) + \sum_{H < K' < G} \text{tr}_{K'}^G \left(\sum_{k=1}^{i_{K'}} N_H^{K'}((\underline{ab})_k^{K'}) \right)$$

where each $(\underline{ab})_k^{K'}$ and $g_H(a, b)$ are as defined in Definition 3.3.

Next, let H' be a subgroup of H , and let x be in $\underline{S}(G/H')$. To prove the Tambara reciprocity formula for the norm of a transfer $N_H^G \text{tr}_{H'}^H(x)$ consider the exponential diagram below.

$$\begin{array}{ccccc} G/H & \longleftarrow & G/H' & \xleftarrow{e} & G/H \times \text{Map}_H(G, H/H') \\ \downarrow & & & & \downarrow \pi \\ G/G & \longleftarrow & & \xleftarrow{h} & \text{Map}_H(G, H/H') \end{array}$$

As a G -set $\text{Map}_H(G, H/H')$ is isomorphic to

$$\prod_{q=1}^r (G/H')_q,$$

and so $G/H \times \text{Map}_H(G, H/H')$ is isomorphic to

$$\prod_{q=1}^r \left(\prod_{|G/H|} G/H' \right)_q.$$

We do not need to know an exact value for r , since it suffices that it is universally determined by G , H , and H' .

Thus, applying \underline{S} to the above exponential diagram results in the following commutative diagram.

$$\begin{array}{ccc}
 \underline{S}(G/H) & \xleftarrow{tr_{H'}^H} \underline{S}(G/H') & \xrightarrow{e^*} \bigoplus_{q=1}^r \left(\bigoplus_{|G/H|} \underline{S}(G/H') \right)_q \\
 \downarrow N_H^G & & \downarrow \pi_* \\
 \underline{S}(G/G) & \xleftarrow{h_*} & \bigoplus_{q=1}^r \underline{S}(G/H')_q
 \end{array}$$

The map e^* is the diagonal map. To determine a formula for π_* let the q^{th} copy of G/H' in $Map_H(G, H/H')$ be the orbit of the map f_q . (The choice of the element f_q in the orbit does not matter.) Additionally, let $\gamma^{|G/H|}$ be the generator of H in G . Since f_q is H -equivariant, it follows that $\gamma^{|G/H|} f_q(\gamma^t) = f_q(\gamma^{|G/H|+t})$ for all γ^t in G . Therefore, f_q maps $|G/H|$ -many elements of G/H' to the identity element of H/H' . Denote the elements of G/H' that map to the identity by $\gamma^{t_1}, \gamma^{t_2}, \dots, \gamma^{t_{|G/H|}}$. Then f_q induces a map

$$f_{q*}: \bigoplus_{|G/H|} \underline{S}(G/H')_q \rightarrow \underline{S}(G/H')_q$$

given by

$$f_{q*}(s_e, s_\gamma, \dots, s_{\gamma^{|G/H|-1}}) = \gamma^{t_1} s_e \gamma^{t_2} s_\gamma \cdots \gamma^{t_{|G/H|}} s_{\gamma^{|G/H|-1}},$$

and so $\pi_* = \bigoplus_{q=1}^r f_{q*}$.

The composition $\pi_* e^*(x)$ equals $\pi_* \left(\bigoplus_{i=1}^r \left(\bigoplus_{|G/H|} x \right)_q \right)$, and under π_* each $\left(\bigoplus_{|G/H|} x \right)_q$ maps to a product of some of the $W_G(H')$ -conjugates of x . We denote this product \underline{x}_q . Thus,

$$\pi_* e^*(x) = \bigoplus_{q=1}^r \underline{x}_q.$$

Finally, if $\nabla: \bigoplus_{q=1}^r \underline{S}(G/H')_q \rightarrow \underline{S}(G/H')$ is the fold map, then h_* is the composition $tr_{H'}^H \nabla$, and

$$N_H^G tr_{H'}^H(x) = h_* \pi_* e^*(x) = tr_{H'}^H(f(x))$$

where $f(x)$ is a polynomial in some of the $W_G(H')$ -conjugates of x . \square

Fact 3.6. The following seemingly innocuous fact regarding Tambara reciprocity will allow us to have some fun in Section 4.2. The monomials of $g_H(a, b)$ and $f(x)$ and the monomials $(\underline{ab})_k^{k'}$ do not contain repeated factors. That is, for every γ^m in G , the elements $\gamma^m a$ and $\gamma^m b$ appear at most once in any monomial in the formula for the norm of a sum, and it is impossible for both $\gamma^m a$ and $\gamma^m b$ to appear in the same monomial. Similarly, the element $\gamma^m x$ appears at most once in

any monomial in the formula for the norm of a transfer. These facts follow from the formulas for the π_* maps discussed in the proof of Theorem 3.5.

Example 3.7. Let \underline{S} be a C_4 -Tambara functor with a and b elements in $\underline{S}(C_4/e)$. We will develop a formula for $N_e^{C_4}(a+b)$. We start with the exponential diagram below and then decompose $Map_e(C_4, *_1 \amalg *_2)$ into G -orbits.

$$\begin{array}{ccc} C_4/e & \xleftarrow{\nabla} & C_4/e \amalg C_4/e & \xleftarrow{e} & C_4/e \times Map_e(C_4, *_1 \amalg *_2) \\ \downarrow & & & & \downarrow \pi \\ C_4/C_4 & \xleftarrow{h} & & & Map_e(C_4, *_1 \amalg *_2) \end{array}$$

Let γ be the generator of C_4 . The G -orbits in $Map_e(C_4, *_1 \amalg *_2)$ consist of two copies of C_4/C_4 , one copy of C_4/C_2 , and three copies of C_4/e . Thus, $Map_e(C_4, *_1 \amalg *_2)$ is isomorphic to

$$C_4/C_4 \amalg C_4/C_4 \amalg \left(\prod_{j=1}^3 (C_4/e)_j \right) \amalg C_4/e.$$

It follows that $C_4/e \times Map_e(C_4, *_1 \amalg *_2)$ is isomorphic to

$$C_4/e \amalg C_4/e \amalg \left(\prod_{j=1}^3 \left(\prod_{|C_4/e|} C_4/e \right)_j \right) \amalg (C_4/e \amalg C_4/e).$$

Applying \underline{S} to the above exponential diagram results in the diagram below,

$$\begin{array}{ccc} \underline{S}(C_4/e) & \xleftarrow{\nabla_*} & \underline{S}(C_4/e) \oplus \underline{S}(C_4/e) & \xrightarrow{e^*} & \underline{S}(C_4/e \times Map_e(C_4, *_1 \amalg *_2)) \\ N_e^{C_4} \downarrow & & & & \downarrow \pi_* \\ \underline{S}(C_4/C_4) & \xleftarrow{h_*} & & & \underline{S}(Map_e(C_4, *_1 \amalg *_2)) \end{array}$$

where $\underline{S}(C_4/e \times Map_e(C_4, *_1 \amalg *_2))$ is isomorphic to

$$\underline{S}(C_4/e) \oplus \underline{S}(C_4/e) \oplus \bigoplus_{j=1}^3 \left(\bigoplus_{|C_4/e|} \underline{S}(C_4/e) \right)_j \oplus (\underline{S}(C_4/e) \oplus \underline{S}(C_4/e))$$

and $\underline{S}(Map_e(C_4, *_1 \amalg *_2))$ is isomorphic to

$$\underline{S}(C_4/C_4) \oplus \underline{S}(C_4/C_4) \oplus \bigoplus_{j=1}^3 \underline{S}(C_4/e)_j \oplus \underline{S}(C_4/C_2).$$

It remains to determine $h_* \pi_* e^*(a, b)$. We use representatives of each of the G -orbits in $Map_e(C_4, *_1 \amalg *_2)$ to define e^* . The copies of C_4/C_4

in $Map_e(C_4, *_1 \amalg *_2)$ are the orbits of the two constant functions. The copy of C_4/C_2 is the orbit of the map that sends the subgroup C_2 to $*_1$ and γC_2 to $*_2$. The three copies of C_4/e are the orbits of the following three maps.

$$\left(\begin{array}{ccc} e, \gamma, \gamma^2 & \mapsto & *_1 \\ & & \gamma^3 & \mapsto & *_2 \end{array} \right), \left(\begin{array}{ccc} \gamma^3 & \mapsto & *_1 \\ e, \gamma, \gamma^2 & \mapsto & *_2 \end{array} \right), \left(\begin{array}{ccc} e, \gamma^3 & \mapsto & *_1 \\ \gamma, \gamma^2 & \mapsto & *_2 \end{array} \right)$$

Hence, we define e^* by

$$e^*(a, b) = (a, b, (a, a, a, b), (b, b, b, a), (a, b, b, a), (a, b)).$$

Then $\pi_* e^*(a, b)$ becomes

$$(N_e^{C_4}(a), N_e^{C_4}(b), a\gamma a\gamma^2 a\gamma^3 b, b\gamma b\gamma^2 b\gamma^3 a, a\gamma b\gamma^2 b\gamma^3 a, N_e^{C_2}(a\gamma b)),$$

and finally $h_* \pi_* e^*(a, b)$ results in

$$N_e^{C_4}(a) + N_e^{C_4}(b) + tr_{C_2}^{C_4}(N_e^{C_2}((\underline{ab})_1^{C_2})) + tr_e^{C_4}(g_e(a, b))$$

where $(\underline{ab})_1^{C_2} = a\gamma b$ and

$$g_e(a, b) = a\gamma a\gamma^2 a\gamma^3 b + b\gamma b\gamma^2 b\gamma^3 a + a\gamma b\gamma^2 b\gamma^3 a.$$

To develop the formula for $N_{C_2}^{C_4}(tr_e^{C_2}(x))$ we start with the following exponential diagram.

$$\begin{array}{ccccc} C_4/C_2 & \longleftarrow & C_4/e & \xleftarrow{e} & C_4/C_2 \times Map_{C_2}(C_4, C_2/e) \\ \downarrow & & & & \downarrow \pi \\ C_4/C_4 & \longleftarrow & & \xleftarrow{h} & Map_{C_2}(C_4, C_2/e) \end{array}$$

The C_4 -set $Map_{C_2}(C_4, C_2/e)$ is isomorphic to C_4/e , and we can think of it as the orbit of the map below.

$$\left(\begin{array}{ccc} e, \gamma & \mapsto & e \\ \gamma^2, \gamma^3 & \mapsto & \gamma^2 \end{array} \right)$$

Thus, applying \underline{S} to the above diagram yields the diagram below.

$$\begin{array}{ccccc} \underline{S}(C_4/C_2) & \xleftarrow{tr_e^{C_2}} & \underline{S}(C_4/e) & \xrightarrow{e^*} & \underline{S}(C_4/e) \oplus \underline{S}(C_4/e) \\ N_{C_2}^{C_4} \downarrow & & & & \downarrow \pi_* \\ \underline{S}(G/G) & \xleftarrow{h_*} & & \xleftarrow{h_*} & \underline{S}(C_4/e) \end{array}$$

Further,

$$\pi_* e^*(x) = \pi_*(x, x) = x\gamma x,$$

and $h_*(x\gamma x) = tr_e^{C_4}(x\gamma x)$. It follows that $N_{C_2}^{C_4}(tr_e^{C_2}(x)) = tr_e^{C_4}(f(x))$ where $f(x) = x\gamma x$.

We can also uncover the formula for $N_e^{C_4}(a+b)$ by decomposing $\prod_{\gamma^t \in W_{C_4}(e)} \gamma^t(a+b)$ into orbits and the formula for $N_{C_2}^{C_4}(tr_e^{C_2}(x))$ by decomposing $\prod_{\gamma^t \in W_{C_4}(C_2)} \gamma^t \left(\sum_{\gamma^{r|C_4/C_2|} \in W_{C_2}(e)} \gamma^{r|C_4/C_2|} x \right)$ into orbits.

4. CONSTRUCTING THE FUNCTORS N_H^G

From here on, let G be a cyclic p -group with generator γ and let H be a subgroup of G . In this section we define the functor $N_H^G: \mathcal{Mack}_H \rightarrow \mathcal{Mack}_G$ that is the foundation for the G -symmetric monoidal structure that we build on \mathcal{Mack}_G . In particular, given an H -Mackey functor \underline{M} , we create $N_H^G \underline{M}$ by first building a G -Mackey functor $N_H^G \underline{M}$ and then proving that the map $N_H^G: \mathcal{Mack}_H \rightarrow \mathcal{Mack}_G$ given by $\underline{M} \mapsto N_H^G \underline{M}$ is a functor. We will then prove that for all subgroups K such that $H < K < G$ the functor N_H^G is isomorphic to the composition $N_K^G N_H^K$. Not only do we need this fact to prove Main Theorems 1 and 2, but it also simplifies many of the remaining proofs in this paper. Hence, we wait until Theorem 4.10 to prove that N_H^G is strong symmetric monoidal.

4.1. Weyl Actions. Let \underline{M} be an H -Mackey functor and let K be a subgroup of G such that $H < K \leq G$. The definition of the G -Mackey functor $N_H^G \underline{M}$ involves the $|G/K|$ -fold cartesian product of $\underline{M}(H/H)$ and the $|G/H|$ -fold box product of \underline{M} . Thus, before defining $N_H^G \underline{M}$ we define an appropriate Weyl action on these products.

To define a $W_G(K)$ -action on $\underline{M}(H/H)^{\times |G/K|}$ we index $\underline{M}(H/H)^{\times |G/K|}$ by elements of G/K , and let the generator γ of $W_G(K)$ permute the factors. More specifically, denote an element in $\underline{M}(H/H)^{\times |G/K|}$ by

$$(4.1) \quad \mathbf{m}^{\times |G/K|} = m_e \times m_\gamma \times \cdots \times m_{\gamma^{|G/K|-1}}.$$

Then γ acts as follows:

$$\begin{aligned} \gamma(\mathbf{m}^{\times |G/K|}) &= \gamma(m_e \times m_\gamma \times \cdots \times m_{\gamma^{|G/K|-1}}) \\ &= m_{\gamma^{|G/K|-1}} \times m_e \times m_\gamma \times \cdots \times m_{\gamma^{|G/K|-2}}. \end{aligned}$$

Alternatively, if J is a subgroup such that $K < J < G$, then we can reindex $\underline{M}(H/H)^{\times |G/K|}$ by $(\underline{M}(H/H)^{\times |J/K|})^{\times |G/J|}$. (We will do this to prove Theorem 4.9.) We write

$$\mathbf{m}^{\times |J/K|} = m_e \times m_{\gamma^{|G/J|}} \times m_{\gamma^{2|G/J|}} \times \cdots \times m_{\gamma^{(|J/K|-1)|G/J|}}$$

for an element in $\underline{M}(H/H)^{\times |J/K|}$ and denote an element in $(\underline{M}(H/H)^{\times |J/K|})^{\times |G/J|}$ by

$$(\mathbf{m}^{\times |J/K|})^{\times |G/J|} = (\mathbf{m}^{\times |J/K|})_e \times (\mathbf{m}^{\times |J/K|})_\gamma \times \cdots \times (\mathbf{m}^{\times |J/K|})_{\gamma^{|G/J|-1}}.$$

The Weyl action still shuffles the factors of $(\mathbf{m}^{\times|J/K|})^{\times|G/J|}$, but now each $\mathbf{m}^{\times|J/K|}$ travels as a pack. Hence, if $\gamma^{|G/J|}(\mathbf{m}^{\times|J/K|})$ is the action of the generator of $W_J(K)$ on an element of $\underline{M}(H/H)^{\times|J/K|}$, then

$$(4.2) \quad \gamma(\mathbf{m}^{\times|J/K|})^{\times|G/J|} = \left[\gamma^{|G/J|}(\mathbf{m}^{\times|J/K|})_{\gamma^{|G/J|-1}} \right] \times (\mathbf{m}^{\times|J/K|})_e \times \cdots \times (\mathbf{m}^{\times|J/K|})_{\gamma^{|G/J|-2}}.$$

Now suppose that $H' \leq H$. By Definition 2.3, to define a $W_G(H')$ -action on $\underline{M}^{\square|G/H|}(H/H')$ it suffices to state how the generator γ of $W_G(H')$ acts on a simple tensor. Using notation analogous to Equation 4.1, let γ act by

$$(4.3) \quad \gamma(\mathbf{m}^{\otimes|G/H|}) = (\gamma^{|G/H|}m_{\gamma^{|G/H|-1}}) \otimes m_e \otimes m_\gamma \otimes \cdots \otimes m_{\gamma^{|G/H|-2}},$$

where $\gamma^{|G/H|}$ generates $W_H(H')$. Because we will need it to prove Theorem 4.9 we will also endow $(\underline{M}^{\square|J/H|})^{\square|G/J|}$ with a $W_G(H')$ -action:

$$\gamma(\mathbf{m}^{\otimes|J/H|})^{\otimes|G/J|} \text{ is analogous to (4.2), but now } \gamma^{|G/J|}(\mathbf{m}^{\otimes|J/H|}) = (\gamma^{|G/H|}m_{\gamma^{(|J/H|-1)|G/J|}}) \otimes m_e \otimes m_{\gamma^{|G/K|}} \otimes \cdots \otimes m_{\gamma^{(|J/H|-2)|G/J|}}.$$

The following lemmas follow directly from our choice of indexing, yet are vital in building the functors N_H^G .

Lemma 4.1. *The $W_G(K)$ -modules $\underline{M}(H/H)^{\times|G/K|}$ and $(\underline{M}(H/H)^{\times|J/K|})^{\times|G/J|}$ are equivariantly isomorphic.*

Lemma 4.2. *If \underline{M} is an H -Mackey functor, then for all subgroups H' of H , $\underline{M}^{\square|G/H|}(H/H')$ is isomorphic to $(\underline{M}^{\square|J/H|})^{\square|G/J|}(H/H')$ as $W_G(H')$ -modules.*

4.2. Fun With Tambara Reciprocity. Suppose \underline{M} is an H -Mackey functor and $H' \leq H < K \leq G$. In order to create a G -symmetric monoidal structure on \mathcal{Mack}_G under which Tambara functors are the G -commutative monoids, we need to mirror the properties of G -Tambara functors in the G -Mackey functor $N_H^G \underline{M}$. In particular, we need $N_H^G \underline{M}$ to reflect the Tambara reciprocity properties given in Definition 3.3. So, for all subgroups K of G we will force $(N_H^G \underline{M})(G/K)$ to satisfy Tambara reciprocity-like relations by quotienting out by a submodule generated by string of elements that are analogous to Tambara reciprocity. Thus, in this subsection we will create elements in $\underline{M}(H/H)^{\times|G/K|}$ and $\underline{M}(H/H')^{\otimes|G/H|}$ that are like the polynomials $(\underline{ab})_k^{K'}$, $g_H(a, b)$, and $f(x)$ defined in Tambara reciprocity.

To do this we first formally add an element 1 to $\underline{M}(H/H')$ such that the stabilizer subgroup of 1 is all of H . Define a multiplication on $\underline{M}(H/H')$ by $1m = m1 = m$ for all m in $\underline{M}(H/H')$. If neither m_1 nor m_2 is the element 1, then there is no multiplication m_1m_2 . Then for all $\gamma^j \in W_G(K)$ we can consider elements in $\underline{M}(H/H)^{\times|G/K|}$ of the form $1^{\times j} \times m_{\gamma^j} \times 1^{\times|G/K|-1-j}$ where $m_{\gamma^j} \neq 1$. We note that

$$1^{\times j} \times m_{\gamma^j} \times 1^{\times|G/K|-1-j} = \gamma^j (m_{\gamma^j} \times 1^{\times|G/K|-1}).$$

Further, if γ^k and γ^j in $W_G(K)$ are such that $j < k$ then we can multiply:

$$\begin{aligned} (1^{\times j} \times m_{\gamma^j} \times 1^{\times|G/K|-1-j}) (1^{\times k} \times m_{\gamma^k} \times 1^{\times|G/K|-1-k}) \\ = 1^{\times j} \times m_{\gamma^j} \times 1^{\times k-j-1} \times m_{\gamma^k} \times 1^{\times|G/K|-1-k}. \end{aligned}$$

Thus, we can write an element $\mathbf{m}^{\times|G/K|}$ in $M(H/H)^{\times|G/K|}$ as a (multiplicative) product over the $W_G(K)$ -action:

$$\mathbf{m}^{\times|G/K|} = \prod_{j=0}^{|G/K|-1} \gamma^j (m_{\gamma^j} \times 1^{\times|G/K|-1}).$$

Moreover, if $K' < K$, then we can embed $\mathbf{m}^{\times|G/K|}$ into $\underline{M}(H/H)^{\times|G/K'|}$ by

$$\mathbf{m}^{\times|G/K|} \mapsto \mathbf{m}^{\times|G/K|} \times 1^{\times|G/K'|-|G/K|}.$$

Letting $\gamma^{|G/K|}$ be the generator of $W_K(K')$ we define a $W_K(K')$ -action on this element by

$$\gamma^{|G/K|} \left(\mathbf{m}^{\times|G/K|} \times 1^{\times|G/K'|-|G/K|} \right) = 1^{\times|G/K|} \times \mathbf{m}^{\times|G/K|} \times 1^{\times|G/K'|-2|G/K|}.$$

We can similarly embed $\mathbf{m}^{\times|G/K|}$ into $\underline{M}(H/H)^{\otimes|G/H|}$ via

$$\mathbf{m}^{\times|G/K|} \mapsto \mathbf{m}^{\otimes|G/K|} \otimes 1^{\otimes|G/H|-|G/K|}.$$

We write $\mathbf{m}a_j^{\times|G/K|}$ (resp. $\mathbf{m}a_j^{\otimes|G/K|}$) to indicate that the element a is the γ^j -th factor of $\mathbf{m}^{\times|G/K|}$ (resp. $\mathbf{m}^{\otimes|G/K|}$), and so

$$\mathbf{m}a_j^{\times|G/K|} = m_e \times m_{\gamma} \times \cdots \times a_{\gamma^j} \times \cdots \times m_{\gamma^{|G/K|-1}}.$$

Therefore, for any a and b in $\underline{M}(H/H)$ we can consider the elements $\mathbf{m}a_j^{\times|G/K|} \times 1^{\times|G/K'|-|G/K|}$ and $\mathbf{m}b_j^{\times|G/K|} \times 1^{\times|G/K'|-|G/K|}$ in $\underline{M}(H/H)^{\times|G/K'|}$ along with their $W_K(K')$ -conjugates. Additionally, in $\underline{M}(H/H)^{\otimes|G/H|}$ we can consider $\mathbf{m}a_j^{\otimes|G/K|} \otimes 1^{\otimes|G/H|-|G/K|}$, $\mathbf{m}b_j^{\otimes|G/K|} \otimes 1^{\otimes|G/H|-|G/K|}$ and their $W_K(H)$ -conjugates. Recall from the Tambara reciprocity property for sums that each monomial $(\underline{ab})_k^{K'}$ is a product of some of the $W_K(K')$ -conjugates of a and b and $g_H(a, b)$ is a polynomial in some of the $W_K(H)$ -conjugates of a and b .

Lemma 4.3. *We can evaluate $(\underline{ab})_k^{K'}$ at the appropriate Weyl conjugates of $\mathbf{ma}_j^{\times|G/K|} \times 1^{\times|G/K'|-|G/K|}$ and $\mathbf{mb}_j^{\times|G/K|} \times 1^{\times|G/K'|-|G/K|}$ to obtain an element $(\mathbf{a}_j \mathbf{b}_j)_k^{K'}$ in $\underline{M}(H/H)^{\times|G/K'|}$ that contains no ones. Similarly, we can evaluate $g_H(a, b)$ at the appropriate Weyl conjugates of $\mathbf{ma}_j^{\otimes|G/K|} \otimes 1^{\otimes|G/H|-|G/K|}$ and $\mathbf{mb}_j^{\otimes|G/K|} \otimes 1^{\otimes|G/H|-|G/K|}$ to obtain a polynomial $g_H(\mathbf{a}_j, \mathbf{b}_j)$ in $\underline{M}(H/H)^{\otimes|G/H|}$ that contains no ones.*

Proof. This result follows directly from Fact 3.6 and from the way we defined the element 1 in $\underline{M}(H/H)$. \square

Although it is an abuse of terminology, we call the processes described in Lemma 4.3 *evaluating $(\underline{ab})_k^{K'}$ at $\mathbf{ma}_j^{\times|G/K|} \times 1^{\times|G/K'|-|G/K|}$ and $\mathbf{mb}_j^{\times|G/K|} \times 1^{\times|G/K'|-|G/K|}$ and evaluating $g_H(a, b)$ at $\mathbf{ma}_j^{\otimes|G/K|} \otimes 1^{\otimes|G/H|-|G/K|}$ and $\mathbf{mb}_j^{\otimes|G/K|} \otimes 1^{\otimes|G/H|-|G/K|}$.*

Finally, let H' be a subgroup of H and let $c = \text{tr}_{H'}^H(x)$ for some x in $\underline{M}(H/H')$. For every $\mathbf{mc}_j^{\times|G/K|}$ let r_i be the restriction $\text{res}_{H'}^H$ of the element m_i in the i^{th} factor of $\mathbf{mc}_j^{\times|G/K|}$, so r_i equals $\text{res}_{H'}^H(m_i)$. Then consider the corresponding element

$$r_e \otimes \cdots \otimes r_{\gamma^{j-1}} \otimes x_{\gamma^j} \otimes r_{\gamma^{j+1}} \otimes \cdots \otimes r_{\gamma^{|G/K|-1}} \otimes 1^{\otimes|G/H|-|G/K|}$$

in $\underline{M}(H/H')^{\otimes|G/H|}$.

Lemma 4.4. *Let H' be a subgroup of H and let x be in $\underline{M}(H/H')$. We can evaluate the polynomial $f(x)$ of the Tambara reciprocity property for transfers at*

$$r_e \otimes \cdots \otimes r_{\gamma^{j-1}} \otimes x_{\gamma^j} \otimes r_{\gamma^{j+1}} \otimes \cdots \otimes r_{\gamma^{|G/K|-1}} \otimes 1^{\otimes|G/H|-|G/K|}$$

to obtain a sum $f(\mathbf{x}_j)$ in $\underline{M}(H/H')^{\otimes|G/H|}$. This sum involves no ones.

4.3. The Functors N_H^G . Let \underline{M} be an H -Mackey functor. We shall now construct the G -Mackey functor $N_H^G \underline{M}$ and prove key properties about the map $\underline{M} \mapsto N_H^G \underline{M}$.

Definition 4.5. Given an H -Mackey functor \underline{M} define the G -Mackey functor $N_H^G \underline{M}$ as follows.

- (1) For all subgroups H' of H define

$$(N_H^G \underline{M})(G/H') := \underline{M}^{\square|G/H|}(H/H').$$

The $W_G(H')$ -action on a simple tensor $\mathbf{m}^{\otimes|G/H|}$ is as described in Section 4.1. If $H'' < H' \leq H$ then $\text{res}_{H''}^{H'}$ and $\text{tr}_{H''}^{H'}$ are the box product restriction and transfer maps given in Definition 2.3.

- (2) Let K be any subgroup of G such that $H < K \leq G$, and let K'' be the maximal subgroup of K . We define $(N_H^G \underline{M})(G/K)$ inductively by

$$(N_H^G \underline{M})(G/K) := \left(\mathbb{Z}\{\underline{M}(H/H)^{\times |G/K|}\} \oplus ((N_H^G \underline{M})(G/K'')) /_{W_K(K'')} \right) /_{TR},$$

and require that $(N_H^G \underline{M})(G/K)$ satisfies properties (a)-(d) given below.

- (a) Given $\mathbf{m}^{\times |G/K|}$ in $\underline{M}(H/H)^{\times |G/K|}$ let $N(\mathbf{m}^{\times |G/K|})$ denote the corresponding generator of the free summand $\mathbb{Z}\{\underline{M}(H/H)^{\times |G/K|}\}$. The generator γ of $W_G(K)$ acts on $N(\mathbf{m}^{\times |G/K|})$ by permuting the factors of $\mathbf{m}^{\times |G/K|}$ as described in Section 4.1.
- (b) The transfer map $tr_{K''}^K$ is the canonical quotient map onto $((N_H^G \underline{M})(G/K'')) /_{W_K(K'')}$. We will refer to $((N_H^G \underline{M})(G/K'')) /_{W_K(K'')}$ as $Im(tr_{K''}^K)$ and an element in this summand as $tr(x)$ for some x in $(N_H^G \underline{M})(G/K'')$.
- (c) We define the restriction map $res_{K''}^K$ by

$$res_{K''}^K(tr(x)) = \sum_{\gamma^s \in W_K(K'')} \gamma^s x$$

for all $tr(x)$ in $Im(tr_{K''}^K)$, and

$$res_{K''}^K(N(\mathbf{m}^{\times |G/K|})) = \begin{cases} N\left(\prod_{|K/K''|} \mathbf{m}^{\times |G/K|}\right) & \text{if } K'' > H \\ \bigotimes_{|K/K''|} \mathbf{m}^{\otimes |G/K|} & \text{if } K'' = H \end{cases}$$

for all generators $N(\mathbf{m}^{\times |G/K|})$, where $\prod_{|K/K''|} (\mathbf{m}^{\times |G/K|})$ is the $|K/K''|$ -fold Cartesian product of $\mathbf{m}^{\times |G/K|}$ and $\bigotimes_{|K/K''|} \mathbf{m}^{\otimes |G/K|}$ is the analogous tensor product.

- (d) The submodule TR is called the *Tambara reciprocity submodule*. It is generated by all elements of the following forms for all a and b in $\underline{M}(H/H)$, j such that $e \leq \gamma^j \leq \gamma^{|G/K|-1}$, and c such that $c = tr_{H'}^H(x)$ for some x in $\underline{M}(H/H')$ and for some subgroup H' of H :

$$(4.4) \quad N\left(\mathbf{m}(a+b)_j^{\times |G/K|}\right) - N\left(\mathbf{m}a_j^{\times |G/K|}\right) - N\left(\mathbf{m}b_j^{\times |G/K|}\right) \\ - \sum_{H < K' < K} tr_{K'}^K \left(\sum_k N\left((\mathbf{a}_j \mathbf{b}_j)_k^{K'}\right) \right) - tr_H^K(g_H(\mathbf{a}_j, \mathbf{b}_j))$$

and

$$(4.5) \quad N\left(\mathbf{m}c_j^{\times |G/K|}\right) - tr_{H'}^K(f(\mathbf{x}_j)).$$

We gave descriptions of $(\mathbf{a}_j \mathbf{b}_j)_k^{K'}$, $g_H(\mathbf{a}_j, \mathbf{b}_j)$, and $f(\mathbf{x}_j)$ in Lemmas 4.3 and 4.4.

Since G is a cyclic p -group, all subgroups are nested. Thus, if K'' is not maximal in K then we define $res_{K''}^K$ and $tr_{K''}^K$ using the facts that if $K'' < J < K$, then $res_{K''}^K = res_{K''}^J res_J^K$ and $tr_{K''}^K = tr_J^K tr_{K''}^J$. The Weyl equivariance of the Tambara reciprocity submodule follows directly from the definitions of $(\mathbf{a}_j \mathbf{b}_j)_k^{K'}$, $g_H(\mathbf{a}_j, \mathbf{b}_j)$, and $f(\mathbf{x}_j)$ and the way we defined the Weyl actions. In addition, the formulas for the generators of the Tambara reciprocity submodule are directly borrowed from the Tambara reciprocity formulas in Definition 3.3. Quotienting out by TR will thus guarantee that Tambara reciprocity is satisfied when we extend N_H^G to functors on Tambara functors and subsequently prove Main Theorem 2.

Example 4.6. We can think of the module $\mathbb{Z}/2$ as an $\{e\}$ -Mackey functor. We build the C_2 -Mackey functor $N_e^{C_2} \mathbb{Z}/2$. First, $(N_e^{C_2} \mathbb{Z}/2)(C_2/e) = \mathbb{Z}/2 \otimes \mathbb{Z}/2$, and hence is simply $\mathbb{Z}/2$ with trivial Weyl action. Then

$$(N_e^{C_2} \mathbb{Z}/2)(C_2/C_2) = \left(\mathbb{Z}\{N(0), N(1)\} \oplus \mathbb{Z}/2/w_{C_2}(e) \right) / TR,$$

and the TR submodule is generated by all elements of the form $N(a + b) - N(a) - N(b) - tr_e^{C_2}(a \otimes b)$. Thus, quotienting out by TR gives the following relations: $N(0) = 0$, $tr_e^{C_2}(1) = -2N(1)$, and $4N(1) = 0$, and so $(N_e^{C_2} \mathbb{Z}/2)(C_2/C_2) \cong \mathbb{Z}/4$. We give the lattice diagram for $N_e^{C_2} \mathbb{Z}/2$ is Figure 4.

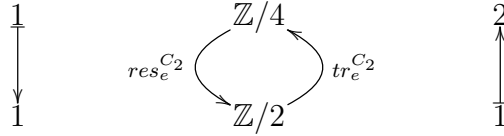


FIGURE 4. The C_2 -Mackey functor $N_e^{C_2} \mathbb{Z}/2$

Example 4.7. Using a process similar to Example 4.6 it is not difficult to see that $N_e^{C_2} \mathbb{Z}$ is the C_2 -Burnside Mackey functor \underline{A} .

Theorem 4.8. *For all subgroups H of G , the map $N_H^G: \mathcal{Mack}_H \rightarrow \mathcal{Mack}_G$ given by $\underline{M} \mapsto N_H^G \underline{M}$ is a functor.*

Proof. Given a morphism $\phi: \underline{M} \rightarrow \underline{L}$ in \mathcal{Mack}_H we define the associated morphism $N_H^G(\phi): N_H^G \underline{M} \rightarrow N_H^G \underline{L}$ in \mathcal{Mack}_G as follows. For all subgroups H' in H define $N_H^G(\phi)_{H'}$ to be $\phi_{H'}^{\square|G/H|}$. If K is a subgroup such that $H < K \leq G$, then we inductively define $N_H^G(\phi)_K$ so that it is

compatible with the appropriate restriction and transfer maps. More specifically, let K'' be the maximal subgroup of K . Then for all $tr(x)$ in the transfer summand of $(N_H^G \underline{M})(G/K)$ we define $N_H^G(\phi)_K(tr(x))$ to be $tr_{K''}^K(N_H^G(\phi)_{K''}(x))$. If $N(\mathbf{m}^{\times |G/K|})$ is a generator of the free summand of $(N_H^G \underline{M})(G/K)$, define $N_H^G(\phi)_K(N(\mathbf{m}^{\times |G/K|}))$ to be

$$N(\phi_H(m_e) \times \phi_H(m_\gamma) \times \cdots \times \phi_H(m_{\gamma^{|G/K|-1}})).$$

The map $N_H^G(\phi)_K$ preserves the Tambara reciprocity relations because the polynomials $(\mathbf{a}_j \mathbf{b}_j)_k^{K'}$, $g_H(\mathbf{a}_j, \mathbf{b}_j)$, and $f(\mathbf{x}_j)$ are universally determined by the group G . Further, by definition, the maps $\{N_H^G(\phi)_K : K \leq G\}$ form a natural transformation of G -Mackey functors, and the assignment $\phi \mapsto N_H^G(\phi)$ is functorial. \square

We next prove that the functors $N_H^G: \mathcal{Mack}_H \rightarrow \mathcal{Mack}_G$ are composable in the sense that N_H^G is isomorphic to $N_K^G N_H^K$ whenever K is such that $H < K < G$. We prove this fact now because it will make the proofs of subsequent theorems regarding N_H^G significantly easier by allowing us to prove the statements only for the case when H is the maximal subgroup in G .

Theorem 4.9. *For all subgroups $H < K < G$ the functor N_H^G is isomorphic to the composition of functors $N_K^G N_H^K$.*

Proof. Let \underline{M} be an H -Mackey functor, and let J be the maximal subgroup of G . Via induction on the size of G it suffices to show that N_H^G is naturally isomorphic to $N_J^G N_H^J$. We will construct an isomorphism $\Phi: N_J^G N_H^J \underline{M} \rightarrow N_H^G \underline{M}$ by building a collection of isomorphisms

$$\{\Phi_P: (N_J^G N_H^J \underline{M})(G/P) \rightarrow (N_H^G \underline{M})(G/P) \text{ for all } P \leq G\}$$

that commute with the appropriate restriction and transfer maps.

For any subgroup H' of H we let $\Phi_{H'}$ be the isomorphism of Lemma 4.2, since $(N_H^G \underline{M})(G/H') = \underline{M}^{\square |G/H'|}(H/H')$, and

$$\begin{aligned} (N_J^G N_H^J \underline{M})(G/H') &= (N_H^J \underline{M})^{\square |G/J|}(J/H') \\ &= (\underline{M}^{\square |J/H'|})^{\square |G/J|}(H/H'). \end{aligned}$$

Next let K be a subgroup of G such that $H < K < G$, and let K'' be the maximal subgroup of K . Before defining Φ_K we will first simplify $(N_J^G N_H^J \underline{M})(G/K)$. This module equals $(N_H^J \underline{M})^{\square |G/J|}(J/K)$, and by Definition 2.3, this box product is equal to the following module:

$$((N_H^J \underline{M})(J/K))^{\otimes |G/J|} \oplus (N_J^G N_H^J \underline{M})(G/K'')/_{W_K(K'')} /_{FR}.$$

The summand $(N_J^G N_H^J \underline{M})(G/K'')/_{W_K(K'')}$ is the image of the transfer map $tr_{K''}^K$, and thus we denote it $Im(tr_{K''}^K)$. Via Frobenius reciprocity we identify all transfer elements of each $(N_H^J \underline{M})(J/K)$ with elements

in $Im(tr_{K''}^K)$. Then since $\bigotimes_i \mathbb{Z}\{M\}$ is isomorphic to $\mathbb{Z}\{\prod_i M\}$, the module $(N_J^G N_H^J \underline{M})(G/K)$ is isomorphic to the module given below:

$$(\widetilde{N_J^G N_H^J \underline{M}})(G/K) = \left[\mathbb{Z} \left\{ (\underline{M}(H/H)^{\times |J/K|})^{\times |G/J|} \right\} \oplus Im(tr_{K''}^K) \right] / \widetilde{FR}.$$

A generator of the free summand of $(\widetilde{N_J^G N_H^J \underline{M}})(G/K)$ is given by $N \left((\mathbf{m}^{\times |J/K|})^{\times |G/J|} \right)$ and $W_G(K)$ acts on $(\mathbf{m}^{\times |J/K|})^{\times |G/J|}$ as described in Equation 4.2. The submodule \widetilde{FR} stems from combining Frobenius reciprocity with the Tambara reciprocity relations of each $(N_H^J \underline{M})(J/K)$. It is generated by the following two classes of elements for all a, b, j, r , and for all c such that $c = tr_{H'}^H(x)$ for any x and any subgroup H' of H :

$$\begin{aligned} & N \left((\mathbf{m}^{\times |J/K|})_e \times \cdots \times (\mathbf{m}(a+b)_j^{\times |J/K|})_{\gamma^r} \times \cdots \times (\mathbf{m}^{\times |J/K|})_{\gamma^{|G/J|-1}} \right) \\ & - N \left((\mathbf{m}^{\times |J/K|})_e \times \cdots \times (\mathbf{m}a_j^{\times |J/K|})_{\gamma^r} \times \cdots \times (\mathbf{m}^{\times |J/K|})_{\gamma^{|G/J|-1}} \right) \\ & - N \left((\mathbf{m}^{\times |J/K|})_e \times \cdots \times (\mathbf{m}b_j^{\times |J/K|})_{\gamma^r} \times \cdots \times (\mathbf{m}^{\times |J/K|})_{\gamma^{|G/J|-1}} \right) \\ & - \sum_{H < K' < K} tr_{K'}^K \left(N \left(\prod_{|K/K'|} (\mathbf{m}^{\times |J/K|})_e \times \cdots \right. \right. \\ & \quad \left. \left. \times \left(\sum_i (\mathbf{a}_j \mathbf{b}_j)_i^{K'} \right)_{\gamma^r} \times \cdots \times \prod_{|K/K'|} (\mathbf{m}^{\times |J/K|})_{\gamma^{|G/J|-1}} \right) \right) \\ & - tr_H^K \left(\bigotimes_{|K/H|} (\mathbf{m}^{\otimes |J/K|})_e \otimes \cdots \otimes g_H(\mathbf{a}_j, \mathbf{b}_j)_{\gamma^r} \otimes \cdots \otimes \bigotimes_{|K/H|} (\mathbf{m}^{\otimes |J/K|})_{\gamma^{|G/J|-1}} \right) \end{aligned}$$

and

$$\begin{aligned} & N \left((\mathbf{m}^{\times |J/K|})_e \times \cdots \times (\mathbf{m}c_j^{\times |J/K|})_{\gamma^r} \times \cdots \times (\mathbf{m}^{\times |J/K|})_{\gamma^{|G/J|-1}} \right) \\ & - tr_H^K \left(\bigotimes_{|K/H|} (\mathbf{m}^{\otimes |J/K|})_e \otimes \cdots \otimes tr_{H'}^H(f(\mathbf{x}_j))_{\gamma^r} \otimes \cdots \otimes \bigotimes_{|K/H|} (\mathbf{m}^{\otimes |J/K|})_{\gamma^{|G/J|-1}} \right). \end{aligned}$$

Thus, it suffices to use induction to build isomorphisms

$$\Phi_K: (\widetilde{N_J^G N_H^J \underline{M}})(G/K) \rightarrow (N_H^G \underline{M})(G/K)$$

for all K . For the base case assume K is the subgroup \tilde{K} in which H is maximal. We define $\Phi_{\tilde{K}}$ by defining two maps, one on the transfer

summand

$$\phi_{\tilde{K}} : \left(\underline{M}^{\square|J/H|} \right)^{\square|G/J|} (H/H) /_{W_{\tilde{K}}(H)} \rightarrow \underline{M}^{\square|G/H|} (H/H) /_{W_{\tilde{K}}(H)}$$

and one on the free summand

$$\phi'_{\tilde{K}} : \mathbb{Z} \left\{ \left(\underline{M}(H/H)^{\times|J/\tilde{K}|} \right)^{\times|G/J|} \right\} \rightarrow \mathbb{Z} \left\{ \underline{M}(H/H)^{\times|G/\tilde{K}|} \right\}.$$

Let $\phi_{\tilde{K}}$ be the isomorphism induced by Φ_H , and let $\phi'_{\tilde{K}}$ be the isomorphism of Lemma 4.1 that rearranges the indices of a generator $N \left(\left(\underline{m}^{\times|J/\tilde{K}|} \right)^{\times|G/J|} \right)$. We then have the diagram of short exact sequences shown below, and thus $\Phi_{\tilde{K}}$ is an isomorphism by the Five Lemma.

$$\begin{array}{ccccc} \text{Im}(\text{tr}_{\tilde{K}}^{\tilde{K}}) \hookrightarrow & (N_J^G N_H^J \underline{M})(G/\tilde{K}) & \twoheadrightarrow & (N_J^G N_H^J \underline{M})(G/\tilde{K}) /_{\text{Im}(\text{tr})} \\ \cong \downarrow \phi_{\tilde{K}} & \downarrow \Phi_{\tilde{K}} & & \cong \downarrow \phi'_{\tilde{K}} \\ \text{Im}(\text{tr}_{\tilde{K}}^{\tilde{K}}) \hookrightarrow & (N_H^G \underline{M})(G/\tilde{K}) & \twoheadrightarrow & (N_H^G \underline{M})(G/\tilde{K}) /_{\text{Im}(\text{tr})} \end{array}$$

The definition of Φ_K and proof that Φ_K is an isomorphism for any subgroup K such that $H < K < G$ is analogous.

It remains only to define Φ_G . The module $(N_J^G N_H^J \underline{M})(G/G)$ is

$$(\mathbb{Z}\{(N_J^G \underline{M})(J/J)\} \oplus (N_J^G N_H^J \underline{M})(G/J) /_{W_G(J)}) /_{TR},$$

which equals

$$\left(\mathbb{Z}\{[\mathbb{Z}\{\underline{M}(H/H)\} \oplus \text{Im}(\text{tr})] /_{TR}\} \oplus \text{Im}(\text{tr}_J^G) \right) /_{TR}.$$

Via TR we can identify all elements in the transfer summand of $(N_H^J \underline{M})(J/J)$ with elements in $\text{Im}(\text{tr}_J^G)$. Thus,

$$(N_J^G N_H^J \underline{M})(G/G) \cong (\mathbb{Z}\{\mathbb{Z}\{\underline{M}(H/H)\}\} \oplus \text{Im}(\text{tr}_J^G)) /_{TR}.$$

Further, we derive the generators of the Tambara reciprocity submodule directly from the formulas for Tambara reciprocity for sums and transfers in Definition 3.3. Therefore, by Property 3 of Definition 3.3, $(N_J^G N_H^J \underline{M})(G/G)$ is isomorphic to $(\mathbb{Z}\{\underline{M}(H/H)\} \oplus \text{Im}(\text{tr}_J^G)) /_{TR}$. We then define Φ_G by defining two maps ϕ_G and ϕ'_G as we did above. It is clear that these maps will pass to a well-defined isomorphism. \square

Indeed, we now have composable functors $N_H^G : \mathcal{Mack}_H \rightarrow \mathcal{Mack}_G$ for all subgroups H of G . However, we need these functors to be strong symmetric monoidal if we want to use them to build a G -symmetric monoidal structure on \mathcal{Mack}_G .

Theorem 4.10. *The functor $N_H^G: \mathcal{Mack}_H \rightarrow \mathcal{Mack}_G$ is strong symmetric monoidal for all subgroups H of G .*

Proof. By Theorem 4.9 it suffices to let H be maximal in G , in which case the definition of $N_H^G \underline{M}$ simplifies nicely. Indeed, if H' is a proper subgroup of G , then $(N_H^G \underline{M})(G/H') = \underline{M}^{\square|G/H'|}(H/H')$, and so the only interesting module is $(N_H^G \underline{M})(G/G)$, which is

$$\left(\mathbb{Z}\{\underline{M}(H/H)\} \oplus \underline{M}^{\square|G/H|}(H/H)/_{W_G(H)} \right) /_{TR}.$$

Further, the Tambara reciprocity submodule of $(N_H^G \underline{M})(G/G)$ is relatively simple. Since there are no subgroups between H and G and only one copy of $\underline{M}(H/H)$ in the free summand of $(N_H^G \underline{M})(G/G)$, the generators of TR are elements of the forms

$$N(a+b) - N(a) - N(b) - tr_H^G(g_H(\mathbf{a}, \mathbf{b}))$$

and

$$N(tr_{H'}^H(x)) - tr_{H'}^G(f(\mathbf{x}))$$

for all a and b in $\underline{M}(H/H)$, x in $\underline{M}(H/H')$ and $H' < H$. The polynomial $g_H(\mathbf{a}, \mathbf{b})$ is the polynomial $g_H(a, b)$ of Property 7 in Definition 3.3 evaluated at $a \otimes 1^{\otimes|G/H|-1}$ and $b \otimes 1^{\otimes|G/H|-1}$ (as described in Lemma 4.3). Similarly, $f(\mathbf{x})$ is the polynomial $f(x)$ of Property 8 in Definition 3.3 evaluated at $x \otimes 1^{\otimes|G/H|-1}$.

To show that N_H^G is strong symmetric monoidal we will build an isomorphism $\Psi: N_H^G \underline{M} \square N_H^G \underline{L} \rightarrow N_H^G(\underline{M} \square \underline{L})$. To begin, we define $\Psi_{H'}$ for all subgroups H' of H . By definition,

$$(N_H^G \underline{M} \square N_H^G \underline{L})(G/H') = (\underline{M}^{\square|G/H'|} \square \underline{L}^{\square|G/H'|})(H/H'),$$

and consists of simple tensors of the form $\mathbf{m}^{\otimes|G/H'|} \otimes \mathbf{l}^{\otimes|G/H'|}$. The generator γ of $W_G(H')$ acts by $\gamma(\mathbf{m}^{\otimes|G/H'|}) \otimes \gamma(\mathbf{l}^{\otimes|G/H'|})$. On the other hand, $N_H^G(\underline{M} \square \underline{L})(G/H') = (\underline{M} \square \underline{L})(H/H')$ with simple tensors

$$(\mathbf{m} \otimes \mathbf{l})^{\otimes|G/H'|} = m_e \otimes l_e \otimes m_\gamma \otimes l_\gamma \otimes \cdots \otimes m_{\gamma|G/H'-1} \otimes l_{\gamma|G/H'-1},$$

and γ acts by

$$\begin{aligned} & \gamma(\mathbf{m} \otimes \mathbf{l})^{\otimes|G/H'|} = \\ & (\gamma^{|G/H'|} m_{\gamma|G/H'-1} \otimes \gamma^{|G/H'|} l_{\gamma|G/H'-1}) \otimes m_e \otimes l_e \otimes \cdots \otimes m_{\gamma|G/H'-2} \otimes l_{\gamma|G/H'-2}. \end{aligned}$$

Thus, we let $\Psi_{H'}$ be an isomorphism analogous to that of Lemmas 4.1 and 4.2 that rearranges the indices.

It remains to define $\Psi_G: (N_H^G \underline{M} \square N_H^G \underline{L})(G/G) \rightarrow N_H^G(\underline{M} \square \underline{L})(G/G)$. We simplify $(N_H^G \underline{M} \square N_H^G \underline{L})(G/G)$ using a process akin to the simplification of $(N_J^G N_H^J \underline{M})(G/K)$ in the proof of Theorem 4.9. Thus,

$(N_H^G \underline{M} \square N_H^G \underline{L})(G/G)$ equals

$$\left((N_H^G \underline{M})(G/G) \otimes (N_H^G \underline{L})(G/G) \oplus \text{Im}(\text{tr}_H^G) \right) /_{FR},$$

which is isomorphic to

$$\left(\mathbb{Z}\{\underline{M}(H/H) \times \underline{L}(H/H)\} \oplus \text{Im}(\text{tr}_H^G) \right) /_{\widetilde{FR}}.$$

The submodule \widetilde{FR} stems from combining the Tambara reciprocity submodules of $(N_H^G \underline{M})(G/G)$ and $(N_H^G \underline{L})(G/G)$ with Frobenius reciprocity. If we let $\bigotimes_{|G/H|} m$ denote the $|G/H|$ -fold tensor product of an element m , then \widetilde{FR} is generated by

$$N((a+b) \times l) - N(a \times l) - N(b \times l) - \text{tr}_H^G \left(g_H(\mathbf{a}, \mathbf{b}) \otimes \bigotimes_{|G/H|} l \right),$$

$$N(m \times (y+z)) - N(m \times y) - N(m \times z) - \text{tr}_H^G \left(\bigotimes_{|G/H|} m \otimes g_H(\mathbf{y}, \mathbf{z}) \right),$$

$$N(\text{tr}_{H'}^H(d) \times l) - \text{tr}_H^G \left(\text{tr}_{H'}^H(f(\mathbf{d})) \otimes \bigotimes_{|G/H|} l \right),$$

and

$$N(m \times \text{tr}_{H'}^H(x)) - \text{tr}_H^G \left(\bigotimes_{|G/H|} m \otimes \text{tr}_{H'}^H(f(\mathbf{x})) \right)$$

for all a, b , and m in $\underline{M}(H/H)$, d in $\underline{M}(H/H')$, y, z , and l in $\underline{L}(H/H)$, x in $\underline{L}(H/H')$ and subgroups H' of H .

Further,

$$N_H^G(\underline{M} \square \underline{L})(G/G) = \left(\mathbb{Z}\{(\underline{M} \square \underline{L})(H/H)\} \oplus \text{Im}(\text{tr}_H^G) \right) /_{TR},$$

and after combining the Frobenius reciprocity submodule of $(\underline{M} \square \underline{L})(H/H)$ with Tambara reciprocity,

$$N_H^G(\underline{M} \square \underline{L})(G/G) \cong \left(\mathbb{Z}\{\underline{M}(H/H) \otimes \underline{L}(H/H)\} \oplus \text{Im}(\text{tr}_H^G) \right) /_{\widetilde{TR}}.$$

To define \widetilde{TR} let $g_H(\mathbf{a} \otimes \mathbf{y}, \mathbf{b} \otimes \mathbf{z})$ be the polynomial $g_H(a, b)$ of Property 7 in Definition 3.3 evaluated at $a \otimes y \otimes (1 \otimes 1)^{\otimes |G/H|-1}$ and $b \otimes z \otimes (1 \otimes 1)^{\otimes |G/H|-1}$, and let $f(\mathbf{d} \otimes \text{res}_{H'}^H(\mathbf{y}))$ and $f(\text{res}_{H'}^H(\mathbf{a}) \otimes \mathbf{x})$ be the polynomial of Property 8 of Definition 3.3 evaluated at $d \otimes \text{res}_{H'}^H(y) \otimes (1 \otimes 1)^{\otimes |G/H|-1}$ and $\text{res}_{H'}^H(a) \otimes x \otimes (1 \otimes 1)^{\otimes |G/H|-1}$, respectively. Then the following elements generate \widetilde{TR} for all $a \otimes y$ and $b \otimes z$ in

$\underline{M}(H/H) \otimes \underline{L}(H/H)$, d in $\underline{M}(H/H')$, x in $\underline{L}(H/H')$, and subgroups H' of H :

$$\begin{aligned} N(a \otimes y + b \otimes z) - N(a \otimes y) - N(b \otimes z) - \text{tr}_H^G(g_H(\mathbf{a} \otimes \mathbf{y}, \mathbf{b} \otimes \mathbf{z})), \\ N(\text{tr}_{H'}^H(d) \otimes y) - \text{tr}_{H'}^G(f(\mathbf{d} \otimes \text{res}_{H'}^H(\mathbf{y}))), \\ N(a \otimes \text{tr}_{H'}^H(x)) - \text{tr}_{H'}^G(f(\text{res}_{H'}^H(\mathbf{a}) \otimes \mathbf{x})). \end{aligned}$$

We now define the isomorphism Ψ_G by employing a strategy similar to the method used in Theorem 4.9 to define $\Phi_{\tilde{K}}$. Thus, we define Ψ_G by defining two maps, ψ_G on $\text{Im}(\text{tr}_H^G)$ and

$$\psi'_G: \mathbb{Z}\{\underline{M}(H/H) \times \underline{L}(H/H)\} \rightarrow \mathbb{Z}\{\underline{M}(H/H) \otimes \underline{L}(H/H)\}.$$

Let ψ_G be the isomorphism induced from Ψ_H , and define $\psi'_G(N(a \times y))$ to be $N(a \otimes y)$. Then the map ψ'_G will pass to an isomorphism

$$(N_H^G \underline{M} \square N_H^G \underline{L})(G/G)/_{\text{Im}(\text{tr})} \rightarrow N_H^G(\underline{M} \square \underline{L})(G/G)/_{\text{Im}(\text{tr})}$$

because the module $N_H^G(\underline{M} \square \underline{L})(G/G)/_{\text{Im}(\text{tr})}$ is isomorphic to $\underline{M}(H/H)/_{\text{Im}(\text{tr})} \otimes \underline{L}(H/H)/_{\text{Im}(\text{tr})}$, and

$$(N_H^G \underline{M} \square N_H^G \underline{L})(G/G)/_{\text{Im}(\text{tr})} \cong \mathbb{Z}\{\underline{M}(H/H)/_{\text{Im}(\text{tr})} \times \underline{L}(H/H)/_{\text{Im}(\text{tr})}\}/_Q,$$

where Q is the submodule generated by the elements $N((a+b) \times l) - N(a \times l) - N(b \times l)$ and $N(m \times (c+d)) - N(m \times c) - N(m \times d)$ for all a, b , and m in $\underline{M}(H/H)/_{\text{Im}(\text{tr})}$ and c, d , and l in $\underline{L}(H/H)/_{\text{Im}(\text{tr})}$. Hence, ψ'_G is an isomorphism by definition of the tensor product. We then build a diagram of short exact sequences analogous to short exact sequence in the proof of Theorem 4.9 and use the Five Lemma to show that Ψ_G is an isomorphism. \square

4.4. The Composition $N_H^G i_H^* \underline{M}$. We now let \underline{M} be a G -Mackey functor. In Main Theorem 1 we define the G -symmetric monoidal structure on Mack_G by letting $G/H \otimes \underline{M}$ be $N_H^G i_H^* \underline{M}$. So, we do not just use the functor N_H^G , but rather the composition $N_H^G i_H^*$. The Mackey functor $i_H^* \underline{M}$ is an H -Mackey functor that maintains the Weyl action defined on \underline{M} . Hence, for all subgroups H' of H , $(i_H^* \underline{M})(H/H')$ is isomorphic to $\underline{M}(G/H')$. We can define a new Weyl action on $N_H^G i_H^* \underline{M}$ that is isomorphic to the action in Definition 4.5 but also remembers the Weyl action defined on \underline{M} . This new action will be useful because it more transparently ties to Tambara functors. Moreover, with this new action it is easy to see that the composition $N_H^G i_H^* \underline{M}$ is the multiplicative form of $\text{Ind}_H^G \text{Res}_H^G \underline{M}$.

We define the new action below. Then in Theorem 4.3 we show that it is isomorphic to the original Weyl action given in Definition 4.5. We define the Weyl action on $N_H^G i_H^* \underline{M}$ as follows.

- For all subgroups H' of H , the generator γ of $W_G(H')$ acts on a simple tensor of $(N_H^G i_H^* \underline{M})(G/H')$ by permuting the factors of the tensor product *and* by acting on each factor. Thus,

$$(4.6) \quad \gamma(\mathbf{m}^{\otimes |G/H|}) = \gamma m_{\gamma|G/H|-1} \otimes \gamma m_e \otimes \gamma m_\gamma \otimes \cdots \otimes \gamma m_{\gamma|G/H|-2}.$$

- Let $r = |G/H| - |G/K| + 1$. For all subgroups K such that $H < K \leq G$ the generator γ of $W_G(K)$ acts on a generator $N(\mathbf{m}^{\times |G/K|})$ of the free summand of $(N_H^G i_H^* \underline{M})(G/K)$ by

$$(4.7) \quad \gamma(\mathbf{m}^{\times |G/K|}) = \gamma^r m_{\gamma|G/K|-1} \times \gamma m_e \times \gamma m_\gamma \times \cdots \times \gamma m_{\gamma|G/K|-2}.$$

Further, the restriction maps of $N_H^G i_H^* \underline{M}$ must remain compatible with the Weyl action. So, let $(\gamma^j \mathbf{m})^{\otimes |G/H|}$ (or $(\gamma^j \mathbf{m})^{\times |G/K|}$) denote γ^j acting on each factor of the product:

$$(\gamma^j \mathbf{m})^{\otimes |G/H|} = \gamma^j m_e \otimes \gamma^j m_\gamma \otimes \cdots \otimes \gamma^j m_{\gamma|G/H|-1}.$$

If K is a subgroup such that $H < K \leq G$ and K'' is maximal in K , then

$$\text{res}_{K''}^K(N(\mathbf{m}^{\times |G/K|})) = \begin{cases} N \left(\prod_{j=0}^{|K/K''|-1} (\gamma^j |G/K| \mathbf{m})^{\times |G/K|} \right) & \text{if } K'' > H \\ \bigotimes_{j=0}^{|K/K''|-1} (\gamma^j |G/K| \mathbf{m})^{\otimes |G/K|} & \text{if } K'' = H \end{cases}.$$

To prove that the two Weyl actions are isomorphic let $U i_H^* \underline{M}$ denote the underlying H -Mackey functor of $i_H^* \underline{M}$. So, $U i_H^* \underline{M}$ does not remember the Weyl action from \underline{M} . Hence, the Weyl action on $N_H^G U i_H^* \underline{M}$ is as defined in Definition 4.5.

Theorem 4.11. *Given a G -Mackey functor \underline{M} , the composition $N_H^G i_H^* \underline{M}$ is isomorphic to $N_H^G U i_H^* \underline{M}$.*

Proof. We will define an isomorphism $\chi: N_H^G U i_H^* \underline{M} \rightarrow N_H^G i_H^* \underline{M}$ by defining a collection of isomorphisms

$$\{\chi_P: (N_H^G U i_H^* \underline{M})(G/P) \rightarrow (N_H^G i_H^* \underline{M})(G/P) \text{ for all } P \leq G\}.$$

First, if $H' \leq H$, then define

$$\chi_H: (i_H^* \underline{M})^{\square |G/H|}(G/H') \rightarrow (U i_H^* \underline{M})^{\square |G/H|}(H/H')$$

to be the identity on the image of the transfer map and $1 \otimes \gamma \otimes \cdots \otimes \gamma^{|G/H|-1}$ on the tensor summand. Then for subgroups K such that $H < K \leq G$ let χ_K be the identity on the image of the transfer and $1 \times \gamma \times \cdots \times \gamma^{|G/K|-1}$ on the free summand. \square

5. PROOFS OF MAIN THEOREMS 1 AND 2

We now have enough tools to prove Main Theorem 1.

Proof of Main Theorem 1. Let \underline{M} be a G -Mackey functor. We will show that the functor $(-) \otimes (-): \mathcal{S}et_G^{Fin, Iso} \times \mathcal{M}ack_G \rightarrow \mathcal{M}ack_G$ defined in Main Theorem 1 satisfies the properties of Definition 2.1. First, it satisfies Property 1 by definition and Theorem 4.10. Further, if X is a finite set, then we can regard it as a disjoint union of $|X|$ -many copies of the G -orbit G/G . Thus, $(-) \otimes (-)$ satisfies Property 2 of Definition 2.1 because

$$X \otimes \underline{M} = (G/G \otimes \underline{M})^{\square|X|} = \underline{M}^{\square|X|}.$$

Finally, to show that Property 3 of Definition 2.1 holds it suffices to show that $(G/K \times G/H) \otimes \underline{M} \cong G/K \otimes (G/H \otimes \underline{M})$ for all orbits G/H and G/K of G . Without loss of generality we assume that H is a subgroup of K , so $G/K \times G/H$ is isomorphic to $\amalg_{|G/K|} G/H$. Then

$$(G/K \times G/H) \otimes \underline{M} \cong (G/H \otimes \underline{M})^{\square|G/K|} \cong N_H^G i_H^* (\underline{M}^{\square|G/K|}).$$

Furthermore, $G/K \otimes (G/H \otimes \underline{M}) = N_K^G i_K^* N_H^G i_H^* \underline{M}$. By Theorem 4.9, $N_H^G \cong N_K^G N_H^K$, and hence, $i_K^* N_H^G i_H^* \underline{M}$ is isomorphic to $(N_H^K i_H^* \underline{M})^{\square|G/K|}$, which in turn is isomorphic to $N_H^K i_H^* (\underline{M}^{\square|G/K|})$ via Theorem 4.10. Therefore,

$$G/K \otimes (G/H \times \underline{M}) \cong N_K^G N_H^K i_H^* (\underline{M}^{\square|G/K|}) \cong N_H^G i_H^* (\underline{M}^{\square|G/H|}).$$

□

Let $\mathcal{A}b$ be the category of abelian groups, and let $ev_{H/H}: \mathcal{M}ack_H \rightarrow \mathcal{A}b$ and $ev_{G/G}: \mathcal{M}ack_G \rightarrow \mathcal{A}b$ be evaluation functors. We then have a natural transformation $ev_{H/H} \rightarrow ev_{G/G} N_H^G$, and further, since the evaluation map is lax monoidal, it preserves Green functors. Thus, the above natural transformation is built to be compatible with both the box product and the Tambara reciprocity submodule.

Definition 5.1. For all \underline{M} in $\mathcal{M}ack_H$ let $N: \underline{M}(H/H) \rightarrow (N_H^G \underline{M})(G/G)$ be the morphism defined via the natural transformation $ev_{H/H} \rightarrow ev_{G/G} N_H^G$.

Hence, given an H -Mackey functor \underline{M} , for all m in $\underline{M}(H/H)$, $N(m)$ is the corresponding generator in the free summand $\mathbb{Z}\{\underline{M}(H/H)\}$ of $(N_H^G \underline{M})(G/G)$.

We will use the morphism N in the proof of Main Theorem 2. Let $\mathcal{T}amb_G$ be the category of G -Tambara functors and let \underline{M} be in $\mathcal{M}ack_G$. To prove Main Theorem 2 we need to show that \underline{M} is a Tambara functor if and only if a map $X \rightarrow Y$ of G -sets induces a map $X \otimes \underline{M} \rightarrow Y \otimes \underline{M}$

in \mathcal{Mack}_G . We will prove the forward implication by extending the functors $N_H^G: \mathcal{Mack}_H \rightarrow \mathcal{Mack}_G$ to functors $\mathcal{N}_H^G: \mathcal{Tamb}_H \rightarrow \mathcal{Tamb}_G$ and then building an adjunction between \mathcal{N}_H^G and the forgetful functor $i_H^*: \mathcal{Tamb}_G \rightarrow \mathcal{Tamb}_H$. Given an H -Tambara functor \underline{S} , we will use Definition 5.1 to define the internal norm maps in $\mathcal{N}_H^G \underline{S}$.

Lemma 5.2. *For all subgroups H of G the functor $N_H^G: \mathcal{Mack}_H \rightarrow \mathcal{Mack}_G$ extends to a functor $\mathcal{N}_H^G: \mathcal{Tamb}_H \rightarrow \mathcal{Tamb}_G$.*

Proof. We need to show that for all subgroups H of G , if \underline{S} is an H -Tambara functor then $\mathcal{N}_H^G \underline{S}$ is a G -Tambara functor. However, by Theorem 4.9 it suffices to only show this when H is the maximal subgroup in G . Since N_H^G is a strong symmetric monoidal functor it naturally extends to a functor $\mathcal{Green}_H \rightarrow \mathcal{Green}_G$. Here \mathcal{Green}_G is the category of G -Green functors. Hence, if \underline{S} is an H -Tambara functor, then we need only to define the internal norm maps $N_{K'}^K: (\mathcal{N}_H^G \underline{S})(G/K') \rightarrow (\mathcal{N}_H^G \underline{S})(G/K)$ for all subgroups $K' < K$ in G .

Since the box product is the coproduct in \mathcal{Tamb}_H [Str12, Prop 9.1], if both H' and H'' are subgroups of H with $H'' < H'$, then we define $N_{H''}^{H'}$ to be the $|G/H|$ -fold box product of the norm map $N_{H''}^{H'}: \underline{S}(H/H'') \rightarrow \underline{S}(H/H')$ in \underline{S} . To define the norm $N_H^G: \underline{S}^{\square|G/H|}(H/H) \rightarrow (\mathcal{N}_H^G \underline{S})(G/G)$ let $m: \underline{S}^{\square|G/H|} \rightarrow \underline{S}$ be the multiplication map of \underline{S} and let N_H^G be the composition

$$\underline{S}^{\square|G/H|}(H/H) \xrightarrow{m} \underline{S}(H/H) \xrightarrow{N} (\mathcal{N}_H^G \underline{S})(G/G).$$

If H' is a proper subgroup of H , then define the norm map $N_{H'}^G$ by the composition $N_H^G N_{H'}^H$. We have constructed the functor $\mathcal{N}_H^G: \mathcal{Mack}_H \rightarrow \mathcal{Mack}_G$ so that the above maps satisfy all properties of the norm maps of a Tambara functor. \square

Lemma 5.3. *The functor $\mathcal{N}_H^G: \mathcal{Tamb}_H \rightarrow \mathcal{Tamb}_G$ is left adjoint to the restriction functor $i_H^*: \mathcal{Tamb}_G \rightarrow \mathcal{Tamb}_H$.*

Proof. Since we can compose adjunctions in a natural fashion [ML98], by Theorem 4.9, it suffices to let H be maximal in G . Let \underline{R} be in \mathcal{Tamb}_G and \underline{S} be in \mathcal{Tamb}_H . Further, let $\mathcal{Tamb}_H(\underline{S}, i_H^* \underline{R})$ be the set of morphisms from \underline{S} to $i_H^* \underline{R}$ in \mathcal{Tamb}_H . We will show that $\mathcal{Tamb}_H(\underline{S}, i_H^* \underline{R})$ is in natural bijective correspondence with $\mathcal{Tamb}_G(\mathcal{N}_H^G \underline{S}, \underline{R})$ by showing that every morphism in $\mathcal{Tamb}_G(\mathcal{N}_H^G \underline{S}, \underline{R})$ determines and is determined by a morphism in $\mathcal{Tamb}_H(\underline{S}, i_H^* \underline{R})$.

Consider a morphism Ω in $\mathcal{Tamb}_G(\mathcal{N}_H^G \underline{S}, \underline{R})$, and note that every element in $(\mathcal{N}_H^G \underline{S})(G/G)$ is either in the image of the transfer map or is a sum of elements in the image of the norm map. (Indeed, every generator $N(s)$ in $\mathbb{Z}\{\underline{S}(H/H)\}$ is the norm of the element $s \otimes 1^{\otimes |G/H|-1}$

in $(\mathcal{N}_H^G \underline{\mathcal{S}})(G/H)$.) Thus, the ring homomorphism Ω_G is completely determined by Ω_H , and since H is maximal in G , the morphism Ω is completely determined by the collection of ring homomorphisms

$$\{\Omega_{H'} : \underline{\mathcal{S}}^{\square|G/H|}(H/H') \rightarrow \underline{R}(G/H') : H' \leq H\}.$$

By properties of the box product [Lew81, Shu10] the above collection of maps determines and is determined by a collection of maps

$$\{\theta_{H'} : \underline{\mathcal{S}}(H/H')^{\otimes|G/H|} \rightarrow \underline{R}(G/H') : H' \leq H\}$$

that satisfies the following properties.

- Each $\theta_{H'}$ is $W_G(H')$ -equivariant.
- If $H'' < H'$, then

$$(5.1) \quad \begin{aligned} \theta_{H''} \circ (\text{res}_{H''}^{H'})^{\otimes|G/H|} &= \text{res}_{H''}^{H'} \circ \theta_{H'}, \\ \theta_{H'} \circ (N_{H''}^{H'})^{\otimes|G/H|} &= N_{H''}^{H'} \circ \theta_{H''}. \end{aligned}$$

- If $H'' < H'$ then the following diagram commutes for all $1 \leq i \leq |G/H|$, where $\mathcal{S}^i = \underline{\mathcal{S}}(H/H')^{\otimes i-1} \otimes \underline{\mathcal{S}}(H/H'') \otimes \underline{\mathcal{S}}(H/H')^{\otimes|G/H|-i}$, $T^i = \text{id}^{\otimes i-1} \otimes \text{tr}_{H''}^{H'} \otimes \text{id}^{\otimes|G/H|-i}$, and $R^i = (\text{res}_{H''}^{H'})^{\otimes i-1} \otimes \text{id} \otimes (\text{res}_{H''}^{H'})^{\otimes|G/H|-i}$.

$$\begin{array}{ccc} & \underline{\mathcal{S}}(H/H')^{\otimes|G/H|} & \xrightarrow{\theta_{H'}} \underline{R}(G/H') \\ & \nearrow T^i & \uparrow \text{tr}_{H''}^{H'} \\ \mathcal{S}^i & & \\ & \searrow R^i & \\ & \underline{\mathcal{S}}(H/H'')^{\otimes|G/H|} & \xrightarrow{\theta_{H''}} \underline{R}(G/H'') \end{array}$$

But we can write every $\mathfrak{s}^{\otimes|G/H|}$ in $\underline{\mathcal{S}}(H/H')^{\otimes|G/H|}$ as a product over the $W_G(H')$ -action. Thus, each $\theta_{H'}$ determines and is determined by a $W_G(H')$ -equivariant homomorphism $\Lambda_{H'} : \underline{\mathcal{S}}(H/H') \rightarrow \underline{R}(G/H')$ because we can write $\theta_{H'}(\mathfrak{s}^{\otimes|G/H|})$ as the following product.

$$\begin{aligned} \theta_{H'}(\mathfrak{s}^{\otimes|G/H|}) &= \theta_{H'} \left[\prod_{j=0}^{|G/H|-1} \gamma^j (s_{\gamma^j} \otimes 1^{\otimes|G/H|}) \right] \\ &= \prod_{j=0}^{|G/H|-1} \gamma^j \theta_{H'} (s_{\gamma^j} \otimes 1^{\otimes|G/H|-1}) \\ &= \prod_{j=0}^{|G/H|-1} \gamma^j \Lambda_{H'}(s_{\gamma^j}). \end{aligned}$$

□

We conclude with a proof of Main Theorem 2.

Proof of Main Theorem 2. Let \underline{M} be a G -Mackey functor. We need to show that the functor $(-) \otimes \underline{M}: \mathcal{S}et_G^{Fin, Iso} \rightarrow \mathcal{M}ack_G$ extends to a functor on $\mathcal{S}et_G^{Fin}$ if and only if \underline{M} is a G -Tambara functor. First, suppose \underline{M} is a Tambara functor. It suffices to show that a map of orbits $G/H \rightarrow G/K$ induces a map $G/H \otimes \underline{M} \rightarrow G/K \otimes \underline{M}$. Consider the K -Tambara functor $i_K^* \underline{M}$. By Lemma 5.3 there is an adjunction between \mathcal{N}_H^K and i_H^* , and hence a counit map $\mathcal{N}_H^K i_H^* i_K^* \underline{M} \rightarrow i_K^* \underline{M}$. We define $G/H \otimes \underline{M} \rightarrow G/K \otimes \underline{M}$ by applying \mathcal{N}_K^G to the above counit map.

Now, assume $(-) \otimes \underline{M}$ extends to a functor $\mathcal{S}et_G^{Fin} \rightarrow \mathcal{M}ack_G$. We will first show that \underline{M} is a commutative G -Green functor by showing that \underline{M} satisfies the categorical definition of a Green functor as given in [Lew81] or [Ven05]. We will then show that the codomain of $(-) \otimes \underline{M}$ is $\mathcal{G}reen_G$. We need the latter fact so that the internal norm maps that we will define to make \underline{M} into a Tambara functor are multiplicative. Let $*$ be the orbit G/G in $\mathcal{S}et_G^{Fin}$. The projection map $p: * \amalg * \rightarrow *$ induces a multiplication map $\underline{M} \square \underline{M} \rightarrow \underline{M}$ on \underline{M} , and the inclusion map $i: \emptyset \hookrightarrow *$ induces a unit map $\underline{A} \rightarrow \underline{M}$. Applying $(-) \otimes \underline{M}$ to the following three diagrams in $\mathcal{S}et_G^{Fin}$ results in the commutative diagrams in $\mathcal{M}ack_G$ needed to make \underline{M} a G -Green functor.

$$\begin{array}{ccc}
 * \amalg * \amalg * & \xrightarrow{id \amalg p} & * \amalg * \\
 p \amalg id \downarrow & & \downarrow p \\
 * \amalg * & \xrightarrow{p} & *
 \end{array}$$

$$\begin{array}{ccc}
 \emptyset \amalg * & \xrightarrow{i \amalg id} & * \amalg * & \xleftarrow{i \amalg id} & \emptyset \amalg * \\
 & \searrow = & \downarrow & \swarrow = & \\
 & & * & &
 \end{array}$$

$$\begin{array}{ccc}
 * \amalg * & \xrightarrow{\tau} & * \amalg * \\
 p \searrow & & \swarrow p \\
 & & *
 \end{array}$$

To show that the codomain of $(-) \otimes \underline{M}$ is $\mathcal{G}reen_G$ we note that $G/H \otimes \underline{M}$ is a commutative Green functor for all orbits of G because \underline{M} is a commutative Green functor and both functors \mathcal{N}_H^G and i_H^* are strong symmetric monoidal. Then given a map $f: G/H \rightarrow G/K$ in $\mathcal{S}et_G^{Fin}$ we show that the induced map $G/H \otimes \underline{M} \rightarrow G/K \otimes \underline{M}$ is a morphism in $\mathcal{G}reen_G$ by applying $(-) \otimes \underline{M}$ to the diagrams below.

$$\begin{array}{ccc}
 G/H \amalg G/H & \xrightarrow{p} & G/H \\
 f \amalg f \downarrow & & \downarrow f \\
 G/K \amalg G/K & \xrightarrow{p} & G/K
 \end{array}$$

$$\begin{array}{ccc}
 \emptyset & \xrightarrow{i} & G/H \\
 & \searrow i & \downarrow f \\
 & & G/K
 \end{array}$$

To show that \underline{M} is a Tambara functor it remains to define norm maps $N_H^K: \underline{M}(G/H) \rightarrow \underline{M}(G/K)$ for all subgroups $H < K \leq G$.

However, we need only construct the norm maps N_H^G since we can subsequently build every N_H^K by applying the process below to $i_K^* \underline{M}$. Let $\pi^*: N_H^G i_H^* \underline{M} \rightarrow \underline{M}$ be the map induced from $\pi: G/H \rightarrow G/G$ and recall from Section 4.4 that $(i_H^* \underline{M})(H/H)$ is isomorphic to $\underline{M}(G/H)$. Then we define the norm map N_H^G by the composition

$$\underline{M}(G/H) \xrightarrow{N} (N_H^G i_H^* \underline{M})(G/G) \xrightarrow{\pi_G^*} \underline{M}(G/G).$$

(We defined N in Definition 5.1.) This composition satisfies Property 3 of Definition 3.3 by Theorem 4.9 and Tambara reciprocity by construction of the functor $N_H^G: \mathcal{Mack}_H \rightarrow \mathcal{Mack}_G$.

To show that the norm map $\pi_G^* N$ factors through the Weyl action (i.e. that $\pi_G^* N$ satisfies Property 5 of Definition 3.3) recall that automorphisms of G/H are given by multiplication by γ^j for some γ^j in $W_G(H)$, and these automorphisms induce the Weyl action on $G/H \otimes \underline{M}$. Hence, the commutative diagram of G -sets on the left below induces the commutative diagram of Mackey functors on the right.

$$\begin{array}{ccc} G/H & \xrightarrow{\gamma^j} & G/H \\ & \searrow \pi & \downarrow \pi \\ & & G/G \end{array} \qquad \begin{array}{ccc} G/H \otimes \underline{M} & \xrightarrow{(\gamma^j \cdot)^*} & G/H \otimes \underline{M} \\ & \searrow \pi^* & \downarrow \pi^* \\ & & \underline{M} \end{array}$$

It follows that $\pi_G^*(N(\gamma^j x)) = \pi_G^*(N(x))$ for all x in $\underline{M}(G/H)$.

Finally, we show that $\text{res}_H^G \pi_G^* N(x) = \prod_{\gamma^j \in W_G(H)} \gamma^j x$ for all x in $\underline{M}(G/H)$. Using the discussion in Section 4.4 and properties of morphisms of Mackey functors we have

$$\text{res}_H^G \pi_G^* N(a) = \pi_H^* \text{res}_H^G N(a) = \pi_H^* (a \otimes \gamma a \otimes \cdots \otimes \gamma^{|G/H|-1} a).$$

Since the G -symmetric monoidal structure is compatible with the forgetful functor $i_H: \mathcal{S}et_G^{Fin} \rightarrow \mathcal{S}et_H^{Fin}$, it follows that π_H^* is induced from $i_H \pi$, which is the fold map $\coprod_{|G/H|} H/H \rightarrow H/H$. Therefore,

$$\pi_H^* (a \otimes \gamma a \otimes \cdots \otimes \gamma^{|G/H|-1} a) = a \gamma a \cdots \gamma^{|G/H|-1} a.$$

□

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