ON c. s. s. COMPLEXES.*

By Daniel M. Kan.

1. Introduction. It was indicated in [3] how the usual notions of homotopy theory may be defined for cubical complexes which satisfy a certain extension condition. In the same manner (see [9]) these notions may be defined for complete semi-simplicial (c. s. s.) complexes which satisfy the following c. s. s. version of the extension condition. The notation used will be that of [2] except that the face and degeneracy operators will be denoted by ϵ^i and η^j (instead of $\epsilon_n{}^i$ and $\eta_n{}^j$).

Definition (1.1). A c. s. s. complex K is said to satisfy the *extension condition* if for every pair of integers (k,n) with $0 \leqq k \leqq n$ and for every n $(n-1)$-simplices $\sigma_0, \cdots, \sigma_{k-1}, \sigma_{k+1}, \cdots, \sigma_n \in K$ such that $\sigma_i\epsilon^{j-1} = \sigma_j\epsilon^i$ for $i < j$ and $i \neq k \neq j$, there exists an *n-simplex* $\sigma \in K$ such that $\sigma\epsilon^i = \sigma_i$ for $i = 0, \cdots, \hat{k}, \cdots, n$.

Let $\mathcal{S}$ be the category of c. s. s. complexes and c. s. s. maps and let $\mathcal{S}_E$ be its full subcategory generated by the c. s. s. complexes which satisfy the extension condition.

Many interesting c. s. s. complexes do not satisfy the extension condition; for example the finite c. s. s. complexes (finite = with only a finite number of non-degenerate simplices). The definitions of some homotopy notions, such as the homology groups, apply to all c. s. s. complexes, but the definition of the homotopy groups of [9], for instance, cannot be carried over to c. s. s. complexes which are not in $\mathcal{S}_E$.

In order to extend the definitions of all homotopy notions defined on the category $\mathcal{S}_E$ to the whole category $\mathcal{S}$ one needs what will be called an H-pair, i. e., a pair (Q, q) consisting of

(i) a functor $Q\colon \mathcal{S} \to \mathcal{S}_E$,

(ii) a natural transformation $q\colon E \to Q$ (where $E\colon \mathcal{S} \to \mathcal{S}$ denotes the identity functor), satisfying the following conditions:

(a) The functor Q maps homotopic maps into homotopic maps.

(b) Let $K \in \mathcal{S}_E$, then the map $qK\colon K \to QK$ is a homotopy equivalence.

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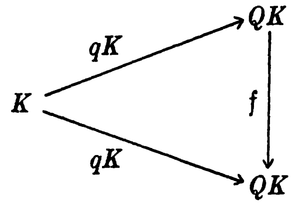
(c) Let $K \in \mathcal{S}$ and let $f\colon QK \to QK$ be a map such that commutativity holds in the diagram



Then f is a homotopy equivalence.

In view of condition (a) every homotopy notion on the category $\mathcal{S}_E$ yields by composition with the functor Q a homotopy notion on the whole category $\mathcal{S}$. Condition (b) implies that on the category $\mathcal{S}_E$ the homotopy notions induced by the functor Q coincide with the original ones. Condition (c) essentially ensures the uniqueness of the homotopy notions induced by Q; if (R, r) is another H-pair, then Q and R induce the same homotopy notions. In particular QK and RK have the same homotopy type, even if K does not satisfy the extension condition.

An example of an H-pair is the following. Let $S|\ |\colon \mathcal{S} \to \mathcal{S}_E$ be the functor which assigns to a c. s. s. complex K the simplicial singular complex $S|K|$ of the geometrical realization $|K|$ of K and let $j\colon E \to |\ |$ be the natural transformation which assigns to a c. s. s. complex K the natural embedding $jK\colon K \to S|K|$. Then it is readily seen that the pair $(S|\ |, j)$ is an H-pair.

Although the existence of an H-pair is sufficient in order to do homotopy theory on the whole category $\mathcal{S}$, it is sometimes convenient to have an H-pair which (unlike the pair $(S|\ |, j)$) may be defined in terms of c. s. s. complexes and c. s. s. maps only. Such an H-pair $(\mathrm{Ex}^\infty, \mathrm{e}^\infty)$ will be defined in this paper. A useful property of the functor $\mathrm{Ex}^\infty\colon \mathcal{S} \to \mathcal{S}_E$ is that it preserves fibre maps.

The main tool used in the definition of the functor Ex^∞ is what we call the *extension* $\mathrm{Ex}\,K$ of a c. s. s. complex K, which is in a certain sense dual to the *subdivision* $\mathrm{Sd}\,K$ of K. More precisely: let K and L be c. s. s. complexes, then there exists (in a natural way) a one-to-one correspondence between the c. s. s. maps $\mathrm{Sd}\,K \to L$ and the c. s. s. maps $K \to \mathrm{Ex}\,L$. In the terminology of [6] this means that the functor Ex is a right adjoint of the functor Sd.

The simplicial approximation theorem may be generalized to c. s. s. complexes roughly as follows: let $K, L \in \mathcal{S}$, K finite, then every continuous map $f\colon |K| \to |L|$ is homotopic with the geometrical realization of a c. s. s.

map $g\colon \mathrm{Sd}^n K \to L$ for some n. Using the adjointness of the functors Sd and Ex a dual theorem may be obtained which involves a c. s. s. map $h\colon K \to \mathrm{Ex}^n L$ instead of $g\colon \mathrm{Sd}^n K \to L$. This dual theorem may be strengthened as follows: let $K \in \mathcal{S}$ and $L \in \mathcal{S}_E$, then every continuous map $f\colon |K| \to |L|$ is homotopic with the geometrical realization of a c. s. s. map $h\colon K \to L$. It is essentially because of this property that, as far as homotopy theory is concerned, the c. s. s. complexes which satisfy the extension condition "*behave like topological spaces.*"

The paper is divided into two chapters. In Chapter I the definitions and results are stated; most of the proofs are given in Chapter II.

The results of this paper were announced in [5].

Chapter I. Definitions and results.

2. The standard simplices and their subdivision. For each integer $n \geqq 0$ let $[n]$ denote the *ordered set* $(0, \cdots, n)$. By a map $\alpha\colon [m] \to [n]$ we mean a *monotone function*, i. e., a function such that $\alpha(i) \leqq \alpha(j)$ for $0 \leqq i \leqq j \leqq m$

For each integer $n \geqq 0$ the *standard n-simplex* $\Delta[n]$ is the c. s. s. complex defined as follows. A q-simplex of $\Delta[n]$ is a map $\sigma\colon [q] \to [n]$. For each map $\beta\colon [p] \to [q]$ the p-simplex $\sigma\beta$ is defined as the composite map

$$[p] \xrightarrow{\beta} [q] \xrightarrow{\sigma} [n].$$

For each map $\alpha\colon [m] \to [n]$ let $\Delta\alpha\colon \Delta[m] \to \Delta[n]$ be the c. s. s. map which assigns to a q-simplex $\tau \in \Delta[m]$ the composite map

$$[q] \xrightarrow{\tau} [m] \xrightarrow{\alpha} [n].$$

The *subdivision* of $\Delta\ [n]$ is the c. s. s. complex $\Delta'[n]$ defined as follows. A q-simplex of $\Delta'[n]$ is a sequence $(\sigma_0, \cdots, \sigma_q)$ where the σ_i are *non-degenerate* simplices of $\Delta[n]$ (i. e., the map $\sigma_i\colon [\dim \sigma_i] \to [n]$ is a monomorphism) and σ_i *lies on* σ_{i+1} (i. e.. $\sigma_i = \sigma_{i+1}\alpha$ for some α) for all i. For each map $\beta\colon [p] \to [q]$ we have $(\sigma_0, \cdots, \sigma_q)\beta = (\sigma_{\beta(0)}, \cdots, \sigma_{\beta(p)})$.

The *subdivision* of $\Delta\alpha$ is the c. s. s. map $\Delta'\alpha\colon \Delta'[m] \to \Delta'[n]$ given by $\Delta'\alpha(\tau_0, \cdots, \tau_q) = (\sigma_0, \cdots, \sigma_q)$, where σ_i is the unique non-degenerate simplex of $\Delta[n]$ for which (see [2]) there exist an epimorphism $\gamma_i\colon [\dim \tau_i] \to [\dim \sigma_i]$ such that commutativity holds in the diagram

$$\begin{array}{ccc} [\dim \tau_i] & \xrightarrow{\tau_i} & [m] \\ \downarrow \gamma_i & & \downarrow \alpha \\ [\dim \sigma_i] & \xrightarrow{\sigma_i} & [n] \end{array} \tag{2.1}$$

For each integer $n \geqq 0$ let $\delta[n]: \Delta'[n] \to \Delta[n]$ be the c. s. s. map which assigns to a q-simplex $(\sigma_0, \cdots, \sigma_q) \in \Delta'[n]$ the q-simplex $\sigma \in \Delta[n]$, i. e., the map $\sigma: [q] \to [n]$, given by $\sigma(i) = \sigma_i(\dim \sigma_i)$, $0 \leqq i \leqq q$.

LEMMA (2.2). *For each map $\alpha: [m] \to [n]$ commutativity holds in the diagram*

$$\begin{array}{ccc} \Delta[m] & \xrightarrow{\Delta\alpha} & \Delta[n] \\ \uparrow \delta[m] & & \uparrow \delta[n] \\ \Delta'[m] & \xrightarrow{\Delta'\alpha} & \Delta'[n] \end{array} \tag{2.2a}$$

Proof. It follows from the definitions that for every q-simplex $(\tau_0, \cdots, \tau_q) \in \Delta'[m]$ and each integer i with $0 \leqq i \leqq q$,

$$(\Delta\alpha \circ \delta[m])(\tau_0, \cdots, \tau_q)(i) = \alpha\tau_i(\dim \tau_i),$$

$$(\delta[n] \circ \Delta'\alpha)(\tau_0, \cdots, \tau_q)(i) = \delta[n](\sigma_0, \cdots, \sigma_q)(i) = \sigma_i(\dim \sigma_i),$$

where σ_i is the unique non-degenerate simplex of $\Delta[n]$ for which there exists an epimorphism γ_i such that commutativity holds in diagram (2.1). Because γ_i is onto,

$$\alpha\tau_i(\dim \tau_i) = \sigma_i\gamma_i(\dim \tau_i) = \sigma_i(\dim \sigma_i).$$

Hence commutativity holds in diagram (2.2a).

3. The extension of a c. s. s. complex.

The *extension* of a c. s. s. complex K is the c. s. s. complex $\operatorname{Ex} K$ defined as follows. An n-simplex of $\operatorname{Ex} K$ is a c. s. s. map $\sigma: \Delta'[n] \to K$. For each map $\alpha: [m] \to [n]$ the m-simplex $\sigma\alpha$ is the composite map

$$\Delta'[m] \xrightarrow{\Delta'\alpha} \Delta'[n] \xrightarrow{\sigma} K.$$

Similarly the *extension* of a c. s. s. map $f: K \to L$ is the c. s. s. map $\operatorname{Ex} f$: $\operatorname{Ex} K \to \operatorname{Ex} L$ which assigns to every n-simplex $\sigma \in \operatorname{Ex} K$ the composite map

$$\Delta'[n] \xrightarrow{\sigma} K \xrightarrow{f} L.$$

Clearly the function Ex so defined is a covariant functor Ex: $\mathcal{S} \to \mathcal{S}$. By Ex^n we shall mean the functor Ex applied n times.

For c. s. s. complex K define a monomorphism $eK: K \to \mathrm{Ex}\, K$ as follows. For every n-simplex $\sigma \in K$, $(eK)\sigma$ is the composite map

$$\Delta'[n] \xrightarrow{\delta[n]} \Delta[n] \xrightarrow{\phi\sigma} K,$$

where $\phi\sigma: \Delta[n] \to K$ is the unique map such that $\phi\sigma\alpha = \sigma\alpha$ for all $\alpha \in \Delta[n]$. It follows from Lemma (2.2) that the function e is a natural transformation $e: E \to \mathrm{Ex}$ (where $E: \mathcal{S} \to \mathcal{S}$ denotes the identity functor), i. e., for every c. s. s. map $f: K \to L$ commutativity holds in the diagram

$$\begin{array}{ccc} K & \xrightarrow{f} & L \\ \downarrow{\scriptstyle eK} & & \downarrow{\scriptstyle eL} \\ \mathrm{Ex}\, K & \xrightarrow{\mathrm{Ex}\, f} & \mathrm{Ex}\, L \end{array}$$

We shall denote by $e^n K: K \to \mathrm{Ex}^n K$ the composite monomorphism

$$K \xrightarrow{eK} \mathrm{Ex}\, K \xrightarrow{e(\mathrm{Ex}\, K)} \cdots \xrightarrow{e(\mathrm{Ex}^{n-1} K)} \mathrm{Ex}^n K.$$

LEMMA (3.1). *The functor* Ex: $\mathcal{S} \to \mathcal{S}$ *maps homotopic maps into homotopic maps.*

The proof will be given in Section 9.

An important property of the functor Ex is that if it is twice applied to a c. s. s. complex K, then the resulting complex $\mathrm{Ex}^2 K$ partially satisfies the extension condition; if $\rho_0, \cdots, \rho_{k-1}, \rho_{k+1}, \cdots, \rho_n \in \mathrm{Ex}^2 K$ are n $(n-1)$-simplices which "match" and which are in the image of $\mathrm{Ex}\, K$ under the map $e(\mathrm{Ex}\, K): \mathrm{Ex}\, K \to \mathrm{Ex}^2 K$, then there exists an n-simplex $\rho \in \mathrm{Ex}^2 K$ (not necessarily in the image of $\mathrm{Ex}\, K$) such that $\rho\epsilon^i = \rho_i$ for $i \neq k$. An exact formulation is given in the following lemma.

LEMMA (3.2). *Let* $K \in \mathcal{S}$. *Then for every pair of integers* (k, n) *with* $0 \leqq k \leqq n$ *and for* n $(n-1)$*-simplices* $\tau_0, \cdots, \tau_{k-1}, \tau_{k+1}, \cdots, \tau_n \in \mathrm{Ex}\, K$ *such that* $\tau_i\epsilon^{j-1} = \tau_j\epsilon^i$ *for* $i < j$ *and* $i \neq k \neq j$, *there exists an* n*-simplex* $\rho \in \mathrm{Ex}^2 K$ *such that* $\rho\epsilon^i = (e(\mathrm{Ex}\, K))\tau_i$ *for* $i = 0, \cdots, \hat{k}, \cdots, n$.

The proof will be given in Section 10.

Another useful property of the functor Ex is that it preserves fibre maps. This is stated in Lemma (3.4).

Definition (3.3). A c.s.s. map $f\colon K \to L$ is called a *fibre map* if for each pair of integers (k, n) with $0 \leqq k \leqq n$, for every n $(n-1)$-simplices $\tau_0, \cdots, \tau_{k-1}, \tau_{k+1}, \cdots, \tau_n \in K$ such that $\tau_i \epsilon^{j-1} = \tau_j \epsilon^i$ for $i < j$ and $i \neq k \neq j$ and for every n-simplex $\rho \in L$ such that $\rho \epsilon^i = f \tau_i$ for $i = 0, \cdots, \hat{k}, \cdots, n$, there exists an n-simplex $\tau \in K$ such that $f\tau = \rho$ and $\tau \epsilon^i = \tau_i$ for $i = 0, \cdots, \hat{k}, \cdots, n$. Let $\phi \in L$ be a 0-simplex. Then the counter image of ϕ and its degeneracies is called *the fibre of f over ϕ*. It is denoted by $F(f, \phi)$.

LEMMA (3.4). *Let $f\colon K \to L$ be a fibre map and let $\phi \in L$ be a 0-simplex. Then* $\mathrm{Ex} f\colon \mathrm{Ex} K \to \mathrm{Ex} L$ *is a fibre map and* $\mathrm{Ex}(F(f, \phi)) = F(\mathrm{Ex} f, (eL)\phi)$.

The proof will be given in Section 11.

Let $f\colon K \to \Delta[0]$ be a fibre map, then it follows readily from the fact that $\Delta[0]$ has only one simplex in every dimension that $K \in \mathcal{S}_E$. Conversely $K \in \mathcal{S}_E$ implies that the (unique) map $f\colon K \to \Delta[0]$ is a fibre map. As $\mathrm{Ex}\,\Delta[0] \approx \Delta[0]$ Lemma (3.4) thus implies

COROLLARY (3.5). *If $K \in \mathcal{S}_E$, then* $\mathrm{Ex} K \in \mathcal{S}_E$.

The following lemmas relate the homology groups of K and $\mathrm{Ex} K$ and, if $K \in \mathcal{S}_E$, their homotopy types.

LEMMA (3.6). *Let $K \in \mathcal{S}$. Then the map $eK\colon K \to \mathrm{Ex} K$ induces isomorphisms of the homology groups, i.e.,* $(eK)_*\colon H_*(K) \approx H_*(\mathrm{Ex} K)$.

The proof will be given in Section 12.

LEMMA (3.7). *Let $K \in \mathcal{S}_E$. Then the map $eK\colon K \to \mathrm{Ex} K$ is a homotopy equivalence.*

The proof will be given in Section 13.

4. The functor Ex^∞. Let K be a c.s.s. complex. Consider the sequence

$$K \xrightarrow{eK} \mathrm{Ex} K \xrightarrow{e(\mathrm{Ex} K)} \mathrm{Ex}^2 K \xrightarrow{e(\mathrm{Ex}^2 K)} \mathrm{Ex}^3 K \to \cdots$$

and let $\mathrm{Ex}^\infty K$ be the direct limit of this sequence. The n-simplices of $\mathrm{Ex}^\infty K$ then are the pairs (σ, q) where $\sigma \in \mathrm{Ex}^q K$ is an n-simplex; two n-simplices (σ, q) and $(\tau, p+q)$ are considered equal if and only if $(e^p(\mathrm{Ex}^q K))\sigma = \tau$. For each map $\alpha\colon [m] \to [n]$, $(\sigma, q)\alpha = (\sigma\alpha, q)$. Similarly for a c.s.s. map $f\colon K \to L$ let $\mathrm{Ex}^\infty f\colon \mathrm{Ex}^\infty K \to \mathrm{Ex}^\infty L$ be the induced map given by $f(\sigma, q) = (f\sigma, q)$. Clearly the function Ex^∞ so defined is a covariant functor.

For a c. s. s. complex K denote by $e^\infty K\colon K \to \mathrm{Ex}^\infty K$ the limit monomorphism

$$K \xrightarrow{eK} \mathrm{Ex}\, K \xrightarrow{e(\mathrm{Ex}\, K)} \cdots \to \mathrm{Ex}^\infty K$$

i. e., $(e^\infty K)\sigma = ((eK)\sigma, 1)$ for every simplex $\sigma \in K$. Naturality of the function e^∞ follows immediately from the naturality of e.

THEOREM (4.1). *The functor* Ex^∞ *maps homotopic maps into homotopic maps.*

The proof is similar to that of Lemma (3.1) (see Section 9), using Ex^∞ and e^∞ instead of Ex and e.

An important property of the functor Ex^∞ is:

THEOREM (4.2). $\mathrm{Ex}^\infty K \in \mathcal{S}_E$ *for all objects* $K \in \mathcal{S}$, i. e., Ex^∞ *is a functor* $E^\infty\colon \mathcal{S} \to \mathcal{S}_E$.

This follows immediately from Lemma (3.2) and the definition of Ex^∞.

Another useful property of the functor Ex^∞ is that it preserves fibre maps.

THEOREM (4.3). *Let* $f\colon K \to L$ *be a fibre map and let* $\phi \in L$ *be a* 0-*simplex. Then* $\mathrm{Ex}^\infty f\colon \mathrm{Ex}^\infty K \to \mathrm{Ex}^\infty L$ *is a fibre map and* $\mathrm{Ex}^\infty(F(f, \phi)) = F(\mathrm{Ex}^\infty f, (e^\infty L)\phi)$.

This follows immediately from Lemma (3.4).

We shall now relate the homology groups of K and $\mathrm{Ex}^\infty K$ and, if $K \in S_E$, their homotopy types.

THEOREM (4.4). *Let* $K \in \mathcal{S}$. *Then the map* $e^\infty K\colon K \to \mathrm{Ex}^\infty K$ *induces isomorphisms of the homology groups, i. e.,* $(e^\infty K)_*\colon H_*(K) \approx H_*(\mathrm{Ex}^\infty K)$.

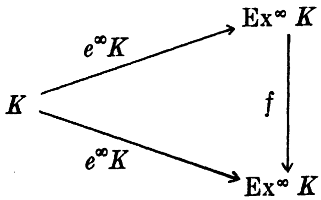
This follows immediately from Lemma (3.6).

Similarly, Lemma (3.7) implies.

THEOREM (4.5). *Let* $K \in \mathcal{S}_E$. *Then the map* $eK\colon K \to \mathrm{Ex}^\infty K$ *is a homotopy equivalence.*

Let K be a c. s. s. complex which does *not* satisfy the extension condition. Then the homotopy type of $\mathrm{Ex}^\infty K$ cannot be related to the homotopy type of K because the latter has (not yet) been defined. However the homotopy type of $\mathrm{Ex}^\infty K$ may be related to K as follows:

THEOREM (4.6). *Let $K \in \mathcal{S}$ and let $f\colon \mathrm{Ex}^\infty K \to \mathrm{Ex}^\infty K$ be a c.s.s. map such that commutativity holds in the diagram*



Then f is a homotopy equivalence.

The proof will be given in Section 14.

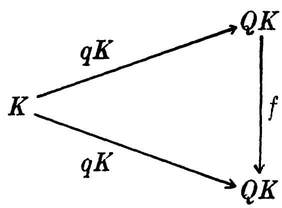
5. Homotopy notions induced on $\mathcal{S}$.

Definition (5.1). A pair (Q, q) where $Q\colon \mathcal{S} \to \mathcal{S}_E$ is a covariant functor and $q\colon E \to Q$ a natural transformation (E denotes the identity functor $E\colon \mathcal{S} \to \mathcal{S}$), is called an *H-pair* if the following conditions are satisfied.

(a) The functor $Q\colon \mathcal{S} \to \mathcal{S}_E$ maps homotopic maps into homotopic maps

(b) Let $K \in \mathcal{S}_E$. Then the map $qK\colon K \to QK$ is a homotopy equivalence

(c) Let $K \in \mathcal{S}$ and let $f\colon QK \to QK$ be a c.s.s. map such that commutativity holds in the diagram



Then f is a homotopy equivalence.

Example (5.2). The pair $(\mathrm{Ex}^\infty, e^\infty)$ is an H-pair; this follows directly from Theorems (4.1), (4.5) and (4.6).

A more exact formulation of the statements about H-pairs made in the introduction will be given in Theorems (5.4), (5.5) and (5.8).

Definition (5.3). By a *homotopy notion* on the category $\mathcal{S}$ (resp. $\mathcal{S}_E$) with values in a category $\mathcal{Z}$ we mean a functor $N\colon \mathcal{S} \to \mathcal{Z}$ (resp. $N\colon \mathcal{S}_E \to \mathcal{Z}$) such that for two maps $f, g \in \mathcal{S}$ (resp. $\mathcal{S}_E$) $f \simeq g$ implies $Nf = Ng$.

THEOREM (5.4). *Let* $N\colon \mathcal{S}_E \to \mathfrak{Z}$ *be a homotopy notion on* $\mathcal{S}_E$ *and let* (Q, q) *be an H-pair. Then the composite functor*

$$\mathcal{S} \xrightarrow{Q} \mathcal{S}_E \xrightarrow{N} \mathfrak{Z}$$

is a homotopy notion on $\mathcal{S}$.

This is an immediate consequence of condition (5.1a).

Let $J\colon \mathcal{S}_E \to \mathcal{S}$ be the inclusion functor and let $N\colon \mathcal{S}_E \to \mathfrak{Z}$ be a homotopy notion on $\mathcal{S}_E$. We then want to compare the composite functor

$$\mathcal{S}_E \xrightarrow{J} \mathcal{S} \xrightarrow{Q} \mathcal{S}_E \xrightarrow{N} \mathfrak{Z}$$

i.e., the restriction to $\mathcal{S}_E$ of the homotopy notion on $\mathcal{S}$ induced by the functor Q, with the original homotopy notion N on $\mathcal{S}_E$. The following theorem then asserts that these functors differ only by a natural equivalence.

THEOREM (5.5). *Let* $N\colon \mathcal{S}_E \to \mathfrak{Z}$ *be a homotopy notion on* $\mathcal{S}_E$ *and let* (Q, q) *be an H-pair. Then the function* $Nq\colon N \to NQJ$ *is a natural equivalence.*

This follows immediately from condition (5.1b).

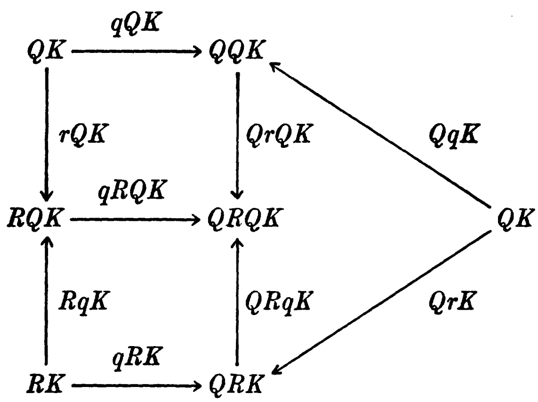
In order to prove the uniqueness of the homotopy notions on $\mathcal{S}$ induced by an H-pair (Q, q) we need the following lemma

LEMMA (5.6). *Let* (Q, q) *and* (R, r) *be H-pairs and let* $K \in \mathcal{S}$. *Then the maps* $QrK\colon QK \to QRK$ *and* $RqK\colon RK \to RQK$ *are homotopy equivalences.*

The proof will be given in Section 15; use will be made of condition (5.1c).

Let (Q, q) and (R, r) be H-pairs and consider the following commutative diagram

(5.7)



It follows from Lemma (5.6) and condition (5.1b) that all maps involved in diagram (5.7) are homotopy equivalences; application of a homotopy notion $N: \mathcal{S}_E \to \mathcal{Y}$ to this diagram thus yields a diagram in $\mathcal{Y}$ consisting only of equivalences. If we put $Q = R$ and $q = r$ then it follows from the commutativity of diagram (5.7) that

$$(NQqK)^{-1} \circ NqQK = (NqQK)^{-1} \circ NQqK \circ (NQqK)^{-1} \circ NqQK = i_{NQK}.$$

Consequently

$$(NRqK)^{-1} \circ NrQK = (NqRK)^{-1} \circ NQrK \circ (NQqK)^{-1} \circ NqQK$$
$$= (NqRK)^{-1} \circ NQrK.$$

Hence the following uniqueness theorem holds.

THEOREM (5.8). *Let $N: \mathcal{S}_E \to \mathcal{Y}$ be a homotopy notion on $\mathcal{S}_E$ and let (Q, q) and (R, r) be H-pairs. Then the function $h: NQ \to NR$ given by*

$$hK = (NRqK)^{-1} \circ NrQK = (NqRK)^{-1} \circ NQrK$$

is a natural equivalence.

6. The simplicial singular complex of the geometrical realization. We shall now use the results of Section 5 in order to compare the simplicial singular complex of the geometrical realization of a c. s. s. complex K with $\text{Ex}^\infty K$.

Let $\mathcal{A}$ be the category of topological spaces and continuous maps and let $|\;|: \mathcal{S} \to \mathcal{A}$ be the *geometrical realization functor* which assigns to a c. s. s. complex K its geometrical realization $|K|$ in the sense of J. Milnor (see [8]); $|K|$ is a CW-complex of which the n-cells are in one-to-one correspondence with the non-degenerate n-simplices of K.

Let $S: \mathcal{A} \to \mathcal{S}_E$ be the *simplicial singular functor* which assigns to a topological space X its simplicial singular complex SX (see [2]); an n-simplex of SX is any continuous map $\sigma: |\Delta[n]| \to X$ and for every map $\alpha: [m] \to [n]$ the n-simplex $\sigma\alpha$ is the composite map

$$|\Delta[m]| \xrightarrow{|\Delta\alpha|} |\Delta[n]| \xrightarrow{\sigma} X.$$

The functor S maps homotopic maps into homotopic maps.

For every c. s. s. complex K let $jK: K \to S|K|$ be the natural monomorphism which assigns to an n-simplex $\sigma \in K$ the simplex $|\phi_\sigma|: |\Delta[n]|$

$\to |K|$ of $S|K|$, where $\phi_\sigma\colon \Delta[n] \to K$ is the unique c. s. s. map such that $\phi_\sigma\alpha = \sigma\alpha$ for all $\alpha \in \Delta[n]$.

The following results are due to J. Milnor ([8]).

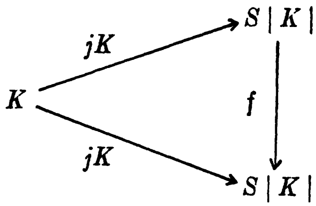
THEOREM (6.1). *The functor* $|\ |\colon \mathcal{S} \to \mathcal{A}$ *maps homotopic maps into homotopic maps.*

COROLLARY (6.2). *The functor* $S|\ |\colon \mathcal{S} \to \mathcal{S}_E$ *maps homotopic maps into homotopic maps.*

THEOREM (6.3). *Let* $K \in \mathcal{S}_E$. *Then the map* $jK\colon K \to S|K|$ *is a homotopy equivalence.*

It is also readily verified that

THEOREM (6.4). *Let* $K \in \mathcal{S}$ *and let* $f\colon S|K| \to S|K|$ *be a c. s. s. map such that commutativity holds in the diagram*



Then f *is a homotopy equivalence.*

It follows from Corollary (6.2) and Theorems (6.3) and (6.4) that the pair $(S|\ |, j)$ is an H-pair. Application of Lemma (5.6) and Theorem (5.8) now yields

LEMMA (6.5). *Let* $K \in \mathcal{S}$. *Then the maps*

$$S|jK|\colon S|K| \to S|S|K\|, \qquad S|e^\infty K|\colon S|K| \to S|\operatorname{Ex}^\infty K|,$$

$$\operatorname{Ex}^\infty jK\colon \operatorname{Ex}^\infty K \to \operatorname{Ex}^\infty S|K|, \qquad \operatorname{Ex}^\infty e^\infty K\colon \operatorname{Ex}^\infty K \to \operatorname{Ex}^\infty \operatorname{Ex}^\infty K$$

are homotopy equivalences.

THEOREM (6.6). *Let* $N\colon \mathcal{S}_E \to \mathcal{G}$ *be a homotopy notion on* $\mathcal{S}_E$. *Then the function* $h\colon N\operatorname{Ex}^\infty \to NS|\ |$ *given by*

$$hK = (NS|e^\infty K|)^{-1} \circ Nj \operatorname{Ex}^\infty K = (Ne^\infty S|K|)^{-1} \circ N \operatorname{Ex}^\infty jK$$

is a natural equivalence.

Theorem (6.6) asserts that the homotopy notions on $\mathcal{S}$ induced by the functor Ex^{∞} are equivalent with these induced by the functor $S|\ |$. In particular we have

COROLLARY (6.7). *Let* $K \in \mathcal{S}$. *Then* $\mathrm{Ex}^{\infty} K$ *and* $S|K|$ *have the same homotopy type.*

7. Extension and subdivision. The *subdivision* of a c.s.s. complex K is a c.s.s. complex $\mathrm{Sd}\, K$ defined as follows. Let $\bar{K}$ denote the c.s.s. complex of which the q-simplices are pairs (σ, ξ) such that $\sigma \in K$, $\xi \in \Delta'[\dim \sigma]$ and $\dim \xi = q$, while for a map $\gamma: [p] \to [q]$ the p-simplex $(\sigma, \xi)\gamma$ is given by $(\sigma, \xi)\gamma = (\sigma, \xi\gamma)$. Define a relation on $\bar{K}$ by calling two simplices $(\sigma, \xi), (\tau, \rho) \in \bar{K}$ equivalent if there exists a map $\alpha: [\dim \tau] \to [\dim \sigma]$ such that $\tau = \sigma\alpha$ and $\xi = \Delta'\alpha(\rho)$ and let $\sim$ denote the resulting equivalence relation. Then $\mathrm{Sd}\, K$ is the collapsed complex $\mathrm{Sd}\, K = \bar{K}/(\sim)$.

A c.s.s. map $f: K \to L$ clearly induces a c.s.s. map $\bar{f}: \bar{K} \to \bar{L}$ (given by $\bar{f}(\sigma, \xi) = (f\sigma, \xi))$ which is compatible with the relation $\sim$. The *subdivision* of f then is defined as the collapsed map $\mathrm{Sd}\, f: \mathrm{Sd}\, K \to \mathrm{Sd}\, L$. Clearly the function $\mathrm{Sd}: \mathcal{S} \to \mathcal{S}$ so defined is a covariant functor. By $\mathrm{Sd}^n: \mathcal{S} \to \mathcal{S}$ we shall mean the functor Sd applied n times.

The functors Ex and Sd are closely related. With a c.s.s. map f: $\mathrm{Sd}\, K \to L$ we may associate a c.s.s. map $\beta f: K \to \mathrm{Ex}\, L$ as follows. Let $\sigma \in K$ be an n-simplex and let $c: \bar{K} \to \mathrm{Sd}\, K$ be the collapsing map. Then $(\beta f)\sigma$ is the n-simplex of $\mathrm{Ex}\, L$, i.e., the c.s.s. map $(\beta f)c: \Delta'[n] \to L$, given by $((\beta f)\sigma)\xi = (f \circ c)(\sigma, \xi)$. The function β is natural, i.e., for every two maps $a: K' \to K$ and $b: L \to L'$

$$\beta(b \circ f \circ \mathrm{Sd}\, a) = \mathrm{Ex}\, b \circ \beta f \circ a.$$

An important property of the function β is

LEMMA (7.1). *Let* $K, L \in \mathcal{S}$. *Then the function* β *establishes a one-to-one correspondence between the c.s.s. maps* $\mathrm{Sd}\, K \to L$ *and the c.s.s. maps* $K \to \mathrm{Ex}\, L$.

Lemma (7.1) is an immediate consequence of the results of [7]. It can also be verified by a straightforward computation

For every c.s.s. complex K define an epimorphism $dK: K \to K$ as follows. Let $\bar{d}K: \bar{K} \to K$ be the map given by

$$\bar{d}K(\sigma, \xi) = (\phi_\sigma \circ \delta[\dim \sigma])\xi,$$

where $\phi_\sigma: \Delta[\dim \sigma] \to K$ is the (unique) map such that $\phi_\sigma \alpha = \sigma\alpha$ for all

$\alpha \in \Delta[\dim \sigma]$. Then $\bar{d}K$ maps equivalent simplices of $\bar{K}$ into the same simplex of K and dK: $\mathrm{Sd}\, K \to K$ is defined as the map obtained by collapsing $\bar{d}K$. By $d^n K$: $\mathrm{Sd}^n K \to K$ we shall mean the composite epimorphism

$$\mathrm{Sd}^n K \xrightarrow{d(\mathrm{Sd}^{n-1} K)} \mathrm{Sd}^{n-1} K \to \cdots \to \mathrm{Sd}\, K \xrightarrow{dK} K$$

It is readily verified that the function d is a natural transformation $d: Sd \to E$.

The natural transformations $e: E \to \mathrm{Ex}$ and d: $\mathrm{Sd} \to E$ are also closely related. In fact a simple computation yields

LEMMA (7.2). *Let $K \in \mathcal{S}$. Then $\beta(dK) = eK$.*

Remark (7.3). Lemma (7.1) states that, in the terminology of [6], the functor Sd is a left adjoint of the functor Ex.

Remark (7.4). The ordered sets $[n]$ and the maps α: $[m] \to [n]$ form a category which will be denoted by $\mathfrak{V}$. The subdivided standard simplices $\Delta'[n]$ and the maps $\Delta'\alpha$: $\Delta'[m] \to \Delta'[n]$ now may be considered as the images of the objects $[n]$ and maps $\alpha: [m] \to [n]$ of the category $\mathfrak{V}$ under a covariant functor Δ': $\mathfrak{V} \to \mathcal{S}$. It then may be verified that the functors Sd and Ex may be obtained by the general method of [7], Section 3 by putting $\mathfrak{Y} = \mathcal{S}$ and $\Sigma = \Delta'$.

Let $K \in \mathcal{S}$. A q-simplex of $\mathrm{Ex}^\infty K$ is a pair (σ, n) where $\sigma \in \mathrm{Ex}^n K$ is a q-simplex. As $\mathrm{Ex}^n K = \mathrm{Ex}^{n-1}(\mathrm{Ex}\, K)$ it follows that the pair $(\sigma, n-1)$ is a q-simplex of $\mathrm{Ex}^\infty(\mathrm{Ex}\, K)$. It is readily verified that this correspondence yields an isomorphism i: $\mathrm{Ex}^\infty K \to \mathrm{Ex}^\infty(\mathrm{Ex}\, K)$ such that commutativity holds in the diagram

$$(7.3) \qquad \begin{array}{ccc} K & \xrightarrow{e^\infty K} & \mathrm{Ex}^\infty K \\ \downarrow{\scriptstyle eK} & & \downarrow{\scriptstyle i} \\ \mathrm{Ex}\, K & \xrightarrow{e^\infty(\mathrm{Ex}\, K)} & \mathrm{Ex}^\infty(\mathrm{Ex}\, K) \end{array}$$

In view of Lemma (6.5) the maps $S\,|e^\infty K|$ and $S\,|e^\infty(\mathrm{Ex}\, K)|$ are homotopy equivalences. Consequently the maps $|e^\infty K|$ and $|e^\infty(\mathrm{Ex}\, K)|$ are homotopy equivalences and it follows from the commutativity in diagram (7.3) that

LEMMA (7.4). *Let $K \in \mathcal{S}$. Then the continuous map $|eK|: |K| \to |\mathrm{Ex}\, K|$ is a homotopy equivalence.*

The following can be shown using standard methods.

LEMMA (7.5). *Let $K \in \mathcal{S}$. Then the continuous map $|dK|: |\operatorname{Sd} K| \to |K|$ is a homotopy equivalence.*

8. C. s. s. approximation theorems. We shall now give an exact formulation of the c. s. s. approximation theorems mentioned in the introduction.

THEOREM (8.1). *Let $K \in \mathcal{S}$ and let $M \in \mathcal{S}_E$. Then for every continuous map $f: |K| \to |M|$ there exists a c.s.s. map $h: K \to M$ such that $|h| \simeq f$.*

Let $L \in \mathcal{S}$ and let $M = \operatorname{Ex}^\infty L$. Then Theorem (8.1) implies

COROLLARY (8.2). *Let $K, L \in \mathcal{S}$. Then for every continuous map $f: |K| \to |L|$ there exists a c.s.s. map $h: K \to \operatorname{Ex}^\infty L$ such that the diagram*

$$\begin{array}{ccc} |K| & \xrightarrow{\ f\ } & |L| \\ & \searrow^{|h|} & \big\downarrow |e^\infty L| \\ & & |\operatorname{Ex}^\infty L| \end{array}$$

is commutative up to homotopy, i. e., $|h| \simeq |e^\infty L| \circ f$.

Proof of Theorem (8.1). Let $\bar{\jmath}M: S|M| \to M$ be a homotopy inverse of the map $jM: M \to S|M|$. Consider the diagram

$$\begin{array}{ccccc} |K| & \xrightarrow{\ |jK|\ } & |S|K|| & \xleftarrow{\ |jK|\ } & |K| \\ \big\downarrow f & & \big\downarrow |Sf| & & \big\downarrow |h| \\ |M| & \xrightarrow{\ |jM|\ } & |S|M|| & \xleftarrow{\ |jM|\ } & |M| \end{array}$$

where $h: K \to M$ is the composite map

$$K \xrightarrow{\ jK\ } S|K| \xrightarrow{\ Sf\ } S|M| \xrightarrow{\ \bar{\jmath}M\ } M.$$

Clearly commutativity holds in the rectangle at the left and the definition of h implies that the rectangle at the right is commutative up to homotopy. It follows from Lemma (6.6) that the maps $S|jK|$ and $S|jM|$ and therefore the maps $|jK|$ and $|jM|$ are homotopy equivalences. Hence $|h| \simeq f$.

A c. s. s. complex K is called *finite* if it has only a finite number of non-degenerate simplices.

THEOREM (8.3). *Let $K, L \in \mathcal{S}$ and let K be finite. Then for every continuous map $f\colon |K| \to |L|$ there exists an integer $n > 0$ and a c.s.s. map $h\colon K \to \mathrm{Ex}^n L$ such that the diagram*

$$\begin{array}{ccc} |K| & \xrightarrow{f} & |L| \\ & \searrow^{|h|} & \downarrow |e^n L| \\ & & |\mathrm{Ex}^n L| \end{array}$$

is commutative up to homotopy, i.e., $|h| \simeq |e^n L| \circ f$.

Proof. Application of Corollary (8.2) yields a c.s.s. map $h'\colon K \to \mathrm{Ex}^\infty L$ such that $|h'| \simeq |e^\infty L| \circ f$. As K is finite only a finite number of non-degenerate simplices of $\mathrm{Ex}^\infty L$ are in the image of K under h'. Hence there exists an integer n such that the map $h'\colon K \to \mathrm{Ex}^\infty L$ may be factorized

$$K \xrightarrow{h} \mathrm{Ex}^n L \xrightarrow{b} \mathrm{Ex}^\infty L$$

where b is the embedding map which assigns to a simplex $\sigma \in \mathrm{Ex}^n L$ the simplex $(\sigma, n) \in \mathrm{Ex}^\infty L$. By an argument similar to that used in the proof of Lemma (7.4) it follows that $|b|$ is a homotopy equivalence. The theorem now follows from the fact that the map $e^\infty L\colon L \to \mathrm{Ex}^\infty L$ may be factorized

$$L \xrightarrow{e^n L} \mathrm{Ex}^n L \xrightarrow{b} \mathrm{Ex}^\infty L.$$

In order to obtain the dual theorem, involving the functor Sd instead of Ex, we need the following lemma

LEMMA (8.4). *Let $K, L \in \mathcal{S}$. Then for every c.s.s. map $h\colon K \to \mathrm{Ex}\, L$ the diagram*

$$\begin{array}{ccc} |K| & \xrightarrow{|h|} & |\mathrm{Ex}\, L| \\ \uparrow |dK| & & \uparrow |eL| \\ |\mathrm{Sd}\, K| & \xrightarrow{|\beta^{-1}h|} & |L| \end{array}$$

is commutative up to homotopy, i.e., $|eL| \circ |\beta^{-1}h| \simeq |h| \circ |dK|$.

The proof will be given in Section 16.

Applying Lemma (8.4) n times to Theorem (8.3) we get

THEOREM (8.5). *Let $K, L \in \mathcal{S}$ and let K be finite. Then for every continuous map $f\colon |K| \to |L|$ there exists an integer $n > 0$ and a c.s.s. map $g\colon Sd^n K \to L$ such that the diagram*

$$\begin{array}{ccc} |K| & \xrightarrow{\quad f \quad} & |L| \\ \uparrow {\scriptstyle |d^n K|} & \nearrow {\scriptstyle |g|} & \\ |\mathrm{Sd}^n K| & & \end{array}$$

is commutative up to homotopy, i.e., $|g| \simeq f \circ |d^n K|$.

Chapter II. Proofs.

9. Proof of Lemma (3.1). Let $f_0, f_1\colon K \to L \in \mathcal{S}$ be maps such that $f_0 \simeq f_1$. Using the terminology of [4] this means that there exists a c.s.s. map $f_1\colon I \times K \to L$ such that $f_I \circ \epsilon K = f_\epsilon$ $(\epsilon = 0, 1)$. It is readily verified that the functor Ex commutes with the cartesian product, i.e., that for every two c.s.s. complexes A and B

$$\operatorname{Ex}(A \times B) = (\operatorname{Ex} A) \times (\operatorname{Ex} B).$$

Straightforward computation shows that commutativity holds in the diagram

$$\begin{array}{ccc} \operatorname{Ex} K & \xrightarrow{\epsilon(\operatorname{Ex} K)} & I \times (\operatorname{Ex} K) \\ \downarrow {\scriptstyle \operatorname{Ex}(\epsilon K)} & & \downarrow {\scriptstyle eI \times i_{\operatorname{Ex} K}} \\ \operatorname{Ex}(I \times K) & \xrightarrow{\quad i \quad} & (\operatorname{Ex} I) \times (\operatorname{Ex} K) \end{array}$$

where i is the identity. Hence

$$(\operatorname{Ex} f_I) \circ (eI \times i_{\operatorname{Ex} K}) \circ \epsilon(\operatorname{Ex} K) = (\operatorname{Ex} f_I) \circ (\operatorname{Ex}(\epsilon K)) = \operatorname{Ex}(f_I \circ \epsilon K) = \operatorname{Ex} f_\epsilon,$$

i.e., $(\operatorname{Ex} f_I) \circ (eI \times i_{\operatorname{Ex} K})\colon \operatorname{Ex} f_0 \simeq \operatorname{Ex} f_1$.

10. Proof of Lemma (3.2). We shall first investigate the structure of $\operatorname{Ex} K$.

A map $\alpha\colon [m] \to [n]$ was defined as a monotone function. By a *function* $\zeta\colon [m] \to [n]$ we shall mean merely a function which thus need not be monotone. A *permutation* $\pi\colon [m] \to [m]$ is a function which is one-to-one onto.

Let $\pi: [m] \to [m]$ be a permutation. Then π induces an automorphism $\pi': \Delta'[m] \to \Delta'[m]$ as follows. For each map $\sigma: [q] \to [m]$ let $\sigma^\pi: [q] \to [m]$ be a map and let $\phi: [q] \to [q]$ be a permutation such that commutativity holds in the diagram

$$\begin{array}{ccc} [q] & \xrightarrow{\sigma} & [m] \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\pi} \\ [q] & \xrightarrow{\sigma^\pi} & [m] \end{array}$$

Clearly such a map σ^π and permutation ϕ exist. It is easily seen that

(a) σ^π is unique;

(b) if σ is a monomorphism, then so is σ^π;

(c) if σ lies on τ, then σ^π lies on τ^π.

We now define the automorphism $\pi': \Delta'[m] \to \Delta'[m]$ by

$$\pi'(\sigma_0, \cdots, \sigma_q) = (\sigma_0{}^\pi, \cdots, \sigma_q{}^\pi).$$

Let $\zeta: [m] \to [n]$ be a function. Then ζ induces a c. s. s. map ζ': $\Delta'[m] \to \Delta'[n]$ as follows. There clearly exists a permutation $\pi: [m] \to [m]$ and a unique map $\alpha: [m] \to [n]$ such that commutativity holds in the diagram

$$\begin{array}{ccc} [m] & & \\ \downarrow{\scriptstyle\pi} & \searrow{\scriptstyle\zeta} & \\ [m] & \xrightarrow{\alpha} & [n] \end{array}$$

The c. s. s. map $\zeta': \Delta'[m] \to \Delta'[n]$ is now defined as the composite map

$$\Delta'[m] \xrightarrow{\pi'} \Delta'[m] \xrightarrow{\Delta'\alpha} \Delta'[n].$$

It is readily verified that

(a) the c. s. s. map ζ' is independent of the choice of the permutation π;

(b) if ζ is a permutation, then this definition of ζ' coincides with the above one;

(c) if ζ is a map, then $\zeta' = \Delta'\zeta$;

(d) if $\vartheta: [l] \to [m]$ is a function, then $(\zeta\vartheta)'$ is the composite map;

$$\Delta'[l] \xrightarrow{\vartheta'} \Delta'[m] \xrightarrow{\zeta'} \Delta'[n].$$

$\operatorname{Ex} K$ is a c. s. s. complex. This means that for every n-simplex $\sigma \in \operatorname{Ex} K$ and every *map* $\alpha: [m] \to [n]$ there is given an m-simplex $\sigma\alpha \in \operatorname{Ex} K$ such that

(i) $\sigma\epsilon_n = \sigma$ where $\epsilon_n: [n] \to [n]$ is the identity;

(ii) if $\beta: [l] \to [m]$ is a map, then $(\sigma\alpha)\beta = \sigma(\alpha\beta)$.

Now let $\sigma \in \operatorname{Ex} K$ be an n-simplex and let $\zeta: [m] \to [n]$ be a *function.* Then the composite map

$$\Delta'[m] \xrightarrow{\zeta'} \Delta'[n] \xrightarrow{\sigma} K$$

is an m-simplex of $\operatorname{Ex} K$ which will be denoted by $\sigma\zeta$. If $\vartheta: [l] \to [m]$ is also a function, then clearly $(\sigma\zeta)\vartheta = \sigma(\zeta\vartheta)$. Thus $\operatorname{Ex} K$ *has more structure than a c. s. s. complex.* It is this additional structure which will be used in the proof of Lemma (3.2).

Proof of Lemma (3.2). Let $\Lambda \subset \Delta[n]$ be the subcomplex generated by the non-degenerate $(n-1)$-simplices $\epsilon^0, \cdots, \epsilon^{k-1}, \epsilon^{k+1}, \cdots, \epsilon^n$ and let $\lambda: \Lambda \to \operatorname{Ex} K$ be the c. s. s. map such that $\lambda\epsilon^i = \tau_i$. Then we must define a c. s. s. map $\rho: \Delta'[n] \to \operatorname{Ex} K$ such that for each $i \neq k$ commutativity holds in the diagram

$$\begin{array}{ccccc} \Delta'[n-1] & \xrightarrow{\Delta'\epsilon^i} & \Delta'[n] & & \\ \Big\downarrow{\scriptstyle \delta[n-1]} & & & \searrow^{\rho} & \\ & & & & \operatorname{Ex} K \\ \Delta[n-1] & \xrightarrow{\Delta\epsilon^i} & \Lambda & \nearrow_{\lambda} & \end{array} \tag{10.1}$$

For each simplex $(\sigma_0, \cdots, \sigma_q) \in \Delta'[n]$ define a function $\zeta(\sigma_0, \cdots, \sigma_q):$ $[q] \to [n]$ by

$$\zeta(\sigma_0, \cdots, \sigma_q)(i) = \sigma_i(\dim \sigma_i), \qquad \sigma_i \neq \epsilon^k \text{ or } \epsilon_n$$

$$\zeta(\sigma_0, \cdots, \sigma_q)(i) = k, \qquad \sigma_i = \epsilon^k \text{ or } \epsilon_n.$$

Then there exists a permutation $\phi: [q] \to [q]$ and a unique map $\sigma: [q] \to [n]$ such that commutativity holds in the diagram

$$\begin{array}{ccc} [q] & & \\ \Big\downarrow{\scriptstyle \phi} & \searrow^{\zeta(\sigma_0, \cdots, \sigma_q)} & \\ [q] & \xrightarrow{\sigma} & [n] \end{array}$$

It is easily seen that $\sigma \in \Lambda$. We now define $\rho(\sigma_0, \cdots, \sigma_q) = (\lambda\sigma)\phi$. It may be verified by direct computation that this definition is independent of the choice of the permutation ϕ.

We now show that the function $\rho\colon \Delta'[n] \to \operatorname{Ex} K$ so defined is a c. s. s. map. Let $\beta\colon [p] \to [q]$ be a map. Then there exists a permutation $\psi\colon [p] \to [p]$ and a unique map $\gamma\colon [p] \to [q]$ such that commutativity holds in the diagram

$$\begin{array}{ccccc} [p] & \xrightarrow{\beta} & [q] & & \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\phi} & \searrow^{\zeta(\sigma_0, \cdots, \sigma_q)} & \\ [p] & \xrightarrow{\gamma} & [q] & \xrightarrow{\sigma} & [n] \end{array}$$

The function $\zeta((\sigma_0, \cdots, \sigma_q)\beta)$ is the composite function

$$[p] \xrightarrow{\beta} [q] \xrightarrow{\zeta(\sigma_0, \cdots, \sigma_q)} [n]$$

and consequently $\rho((\sigma_0, \cdots, \sigma_q)\beta) = (\lambda(\sigma\gamma))\psi$. As commutativity also holds in the diagram

$$\begin{array}{ccccc} \Delta'[p] & \xrightarrow{\Delta'\beta} & \Delta'[q] & & \\ \downarrow{\scriptstyle\psi'} & & \downarrow{\scriptstyle\phi'} & & \\ \Delta'[p] & \xrightarrow{\Delta'\gamma} & \Delta'[q] & \xrightarrow{\lambda\sigma} & K \end{array}$$

it follows that

$$\lambda(\sigma\gamma))\psi = \lambda\sigma \circ \Delta'\gamma \circ \psi' = \lambda\sigma \circ \phi' \circ \Delta'\beta = ((\lambda\sigma)\pi)\beta$$

i. e., the function $\rho\colon \Delta'[n] \to \operatorname{Ex} K$ is a c. s. s. map.

It thus remains to show that commutativity holds in diagram (10.1). Let $(\tau_0, \cdots, \tau_q) \in \Delta'[n-1]$. Then

$$\Delta'\epsilon^i(\tau_0, \cdots, \tau_q) = (\epsilon^i\tau_0, \cdots, \epsilon^i\tau_q).$$

If $i \neq k$, then clearly $\epsilon^i\tau_j \neq \epsilon^k$ and $\epsilon^i\tau_j \neq \epsilon_n$ for all j and it follows from the definitions of the maps ρ and $\delta[n]$ that

$$(\rho \circ \Delta'\epsilon^i)(\tau_0, \cdots, \tau_q) = (\lambda \circ \delta[n] \circ \Delta'\epsilon^i)(\tau_0, \cdots, \tau_q).$$

Application of Lemma (2.2) now yields

$$(\rho \circ \Delta'\epsilon^i)(\tau_0, \cdots, \tau_q) = (\lambda \circ \Delta\epsilon^i \circ \delta[n-1])(\tau_0, \cdots, \tau_q).$$

This completes the proof.

11. Proof of Lemma (3.4). Let k be an integer with $0 \leqq k \leqq n$, let $\tau_0, \cdots, \tau_{k-1}, \tau_{k+1}, \cdots, \tau_n \in \operatorname{Ex} K$ be n $(n-1)$-simplices such that $\tau_i\epsilon^{j-1} = \tau_j\epsilon^i$ for $i < j$ and $i \neq k \neq j$ and let $\rho \in \operatorname{Ex} L$ be an n-simplex such that $(\operatorname{Ex} f)\tau_i = \rho\epsilon^i$ for $i = 0, \cdots, \hat{k}, \cdots, n$. Then in order to prove the first part of Lemma (3.4) we must show that there exists a c.s.s. map $\tau\colon \Delta'[n] \to K$ such that for each integer $i \neq k$ commutativity holds in the diagram

(11.1)
$$\begin{array}{ccc} \Delta'[n-1] & \xrightarrow{\ \tau_i\ } & K \\ {\scriptstyle \Delta'\epsilon^i}\downarrow & \nearrow{\scriptstyle \tau} & \downarrow{\scriptstyle f} \\ \Delta'[n] & \xrightarrow{\ \rho\ } & L \end{array}$$

For each simplex $(\sigma_0, \cdots, \sigma_q) \in \Delta'[n]$ for which there exists an integer $i \neq k$ and a simplex $(\sigma_0{}^i, \cdots, \sigma_q{}^i) \in \Delta'[n-1]$ such that $\Delta'\epsilon^i(\sigma_0{}^i, \cdots, \sigma_q{}^i) = (\sigma_0, \cdots, \sigma_q)$ define

$$\tau(\sigma_0, \cdots, \sigma_q) = \tau_i(\sigma_0{}^i, \cdots, \sigma_q{}^i).$$

This definition is independent of the choice of i. If j is another such integer and $i < j$ then there exists a simplex $(\sigma_0{}^{ij}, \cdots, \sigma_q{}^{ij}) \in \Delta'[n-2]$ such that $\Delta'\epsilon^{j-1}(\sigma_0{}^{ij}, \cdots, \sigma_q{}^{ij}) = (\sigma_0{}^i, \cdots, \sigma_q{}^i)$ and $\Delta'\epsilon^i(\sigma_0{}^{ij}, \cdots, \sigma_q{}^{ij}) = (\sigma_0{}^j, \cdots, \sigma_q{}^j)$.

Hence

$$\begin{aligned} \tau_i(\sigma_0{}^i, \cdots, \sigma_q{}^i) &= \tau_i(\Delta'\epsilon^{j-1}(\sigma_0{}^{ij}, \cdots, \sigma_q{}^{ij})) = \tau_i\epsilon^{j-1}(\sigma_0{}^{ij}, \cdots, \sigma_q{}^{ij}) \\ &= \tau_j\epsilon^i(\sigma_0{}^{ij}, \cdots, \sigma_q{}^{ij}) = \tau_j(\Delta'\epsilon^i(\sigma_0{}^{ij}, \cdots, \sigma_q{}^{ij})) = \tau_j(\sigma_0{}^j, \cdots, \sigma_q{}^j). \end{aligned}$$

It is readily verified that the function τ so defined on all simplices of $\Delta'[n]$ which are in the image of $\Delta'[n-1]$ under a map $\Delta'\epsilon^i$ with $i \neq k$, (i.e., those simplices $(\sigma_0, \cdots, \sigma_q) \in \Delta'[n]$ for which $\sigma_q \neq \epsilon_n$ or ϵ^k), commutes with all operators $\beta\colon [p] \to [q]$ and is such that commutativity holds in the upper left triangle of diagram (11.1).

It thus remains to show that τ can be extended over all of $\Delta'[n]$ (i.e., over the simplices $(\sigma_0, \cdots, \sigma_q) \in \Delta'[n]$ for which $\sigma_q = \epsilon_n$ or ϵ^k) to a c.s.s.

map in such a manner that commutativity also holds in the lower right triangle of diagram (11.1). For each non-degenerate simplex $(\sigma_0,\cdots,\sigma_q)$ with $\sigma_q=\epsilon_n$ let $T(\sigma_0,\cdots,\sigma_q)$ denote the triple (l,m,q) where l is the smallest integer such that $\sigma_l(i)=k$ for some i and $m=\dim\sigma_l$. Order these triples lexicographically. It is readily verified that

(i) if $T(\sigma_0,\cdots,\sigma_q)=(l,m,q)$ and $\dim_{l-1}<m-1$ or $l=0, m>0$, then there exists a simplex $(\sigma_0',\cdots,\sigma_{q+1}')\in\Delta'[n]$ such that $(\sigma_0',\cdots,\sigma_{q+1}')\epsilon^l=(\sigma_0,\cdots,\sigma_q)$ and $T(\sigma_0',\cdots,\sigma_{q+1}')=(l,m-1,q+1)<(l,m,q)$.

(ii) if $T(\sigma_0,\cdots,\sigma_q)=(l,m,q)$ and $\dim\sigma_{l-1}=m-1, l<q$ or $l=m=0$, then (a) $T((\sigma_0,\cdots,\sigma_q)\epsilon^i)<(l,m,q)$ for $i\neq l,q$, (b) $\sigma_{q-1}\neq\epsilon^k$ and hence $\tau((\sigma_0,\cdots,\sigma_q)\epsilon^q)$ has already been defined, (c) $T((\sigma_0,\cdots,\sigma_q)\epsilon^l)>(l,m,q)$ and (d) if $T(\sigma_0',\cdots,\sigma_q')\leqq(l,m,q)$, then $(\sigma_0,\cdots,\sigma_q)\epsilon^l$ is not a face of $(\sigma_0',\cdots,\sigma_q')$.

(iii) if $T(\sigma_0,\cdots,\sigma_q)=(q,n,q)$ and $\dim\sigma_{l-1}=n-1$, then (a) $T((\sigma_0,\cdots,\sigma_q)\epsilon^i)<(q,n,q)$ for $i\neq q$, (b) $\sigma_{q-1}=\epsilon^k$ and (c) if $T(\sigma_0',\cdots,\sigma_q')\leqq(q,n,q)$, then $(\sigma_0,\cdots,\sigma_{q-1})$ is not a face of $(\sigma_0',\cdots,\sigma_q')$.

We now extend τ as follows. Let (l,m,q) be a triple and suppose that τ has already been extended over all non-degenerate simplices $(\sigma_0,\cdots,\sigma_{q-1},\epsilon_n)$ and their faces for which $T(\sigma_0,\cdots,\sigma_{q-1},\epsilon_n)<(l,m,q)$ and over some non-degenerate simplices $(\sigma_0,\cdots,\sigma_{q-1},\epsilon_n)$ and their faces for which $T(\sigma_0,\cdots,\sigma_{q-1},\epsilon_n)=(l,m,q)$ in such a manner that τ commutes with all face operators and that commutativity holds in the lower right triangle of diagram (11.1). Let $(\sigma_0,\cdots,\sigma_{q-1},\epsilon_n)$ be a non-degenerate simplex such that $T(\sigma_0,\cdots,\sigma_{q-1},\epsilon_n)=(l,m,q)$ and on which τ has not yet been defined. It then follows from (i) that $\dim\sigma_{l-1}=m-1$ or $l=m=0$ and from (ii) or (iii) that τ already has been defined on all faces of $(\sigma_0,\cdots,\sigma_{q-1},\epsilon_n)$ except $(\sigma_0,\cdots,\sigma_{q-1},\epsilon_n)\epsilon^l$. Because f is a fibre map there exists a q-simplex $\psi\in K$ such that

$$\rho(\sigma_0,\cdots,\sigma_{q-1},\epsilon_n)=f\psi,\qquad \tau((\sigma_0,\cdots,\sigma_{q-1},\epsilon_n)\epsilon^i)=\psi\epsilon^i\qquad(i\neq k).$$

Now define

$$\tau(\sigma_0,\cdots,\sigma_{q-1},\epsilon_n)=\psi,\qquad \tau((\sigma_0,\cdots,\sigma_{q-1},\epsilon_n)\epsilon^l)=\psi\epsilon^l.$$

It is readily verified that the function τ so extended commutes with all face operators and is such that commutativity holds in the lower right triangle of diagram (11.1). Thus using induction on the triples (l,m,q) τ may be extended over all non-degenerate simplices $(\sigma_0,\cdots,\sigma_{q-1},\epsilon_n)\in\Delta'[n]$ and their faces. As every non-degenerate simplex $(\sigma_0,\cdots,\sigma_{q-2},\epsilon^k)\in\Delta'[n]$ is a face of a non-degenerate simplex $(\sigma_0,\cdots,\sigma_{q-2},\epsilon^k,\epsilon_n)$ it follows that τ may be extended over all non-degenerate simplices of $\Delta'[n]$ in such a manner that τ commutes with all face operators and that commutativity holds in diagram (11.1). Extensions of τ over the degenerate simplices of $\Delta'[n]$ (which is always possible in a unique way) now yields the desired c.s.s. map τ: $\Delta'[n]\to K$.

The second part of Lemma (3.4) is obvious.

12. Proof of Lemma (3.6). We shall use the theory of acyclic models of Eilenberg-MacLane (see [1]). The models will be the complexes $\Delta[n]$ and $\Delta'[n]$. Let C_a: $\mathcal{S}\to\partial\mathcal{G}$ be the augmented chain functor. As the map eK: $K\to \operatorname{Ex} K$ induces a one-to-one correspondence between the 0-simplices of K and those of $\operatorname{Ex} K$ it is sufficient to prove that

(a) the functor C_a: $\mathcal{S}\to\partial\mathcal{G}$ is representable in dimension >0,

(b) the composite functor $\mathcal{S}\xrightarrow{\operatorname{Ex}}\mathcal{S}\xrightarrow{C_a}\partial\mathcal{G}$ is representable in dimension >0, and

(c) for every integer $n\geqq 0$,

$$H_*(\Delta[n])=H_*(\operatorname{Ex}\Delta[n])=0,\qquad H_*(\Delta'[n])=H_*(\operatorname{Ex}\Delta'[n])=0.$$

Let $K\in\mathcal{S}$, for every n-simplex $\sigma\in K$ let ϕ_σ: $\Delta[n]\to K$ be the unique c.s.s. map such that $\phi_\sigma\alpha=\sigma\alpha$ for all $\alpha\in\Delta[n]$ and let ϵ_n' be the generator of $C_a\Delta[n]$ corresponding to the identity map ϵ_n: $[n]\to[n]$, i.e., the only non-degenerate n-simplex of $\Delta[n]$. Then it is easily seen that the function $\sigma\to(\sigma,\epsilon_n')$ yields a representation of the functor C_a.

Let $K\in\mathcal{S}$, let τ: $\Delta'[n]\to K$ be an n-simplex of $\operatorname{Ex} K$ and let ι_n' be the generator of $C_a\operatorname{Ex}\Delta'[n]$ corresponding with the identity map ι_n: $\Delta'[n]\to\Delta'[n]$. Then it is easily seen that the function $\tau\to(\tau,\iota_n')$ yields a representation of the functor $C_a\operatorname{Ex}$.

For every integer $n\geqq 0$ the (unique) map $\Delta[n]\to\Delta[0]$ is a homotopy equivalence in $\mathcal{S}$. Combining this with Lemma (3.1) and the fact that $\Delta[0]\approx\operatorname{Ex}\Delta[0]$ and $H_*(\Delta[0])=0$ we get $H_*(\Delta[n])=H_*(\operatorname{Ex}\Delta[n])=0$. If for each integer $n\geqq 0$ the map $\delta[n]$: $\Delta'[n]\to\Delta[n]$ is a homotopy equivalence, then $H_*(\Delta'[n])=H_*(\Delta[n])=0$, and Lemma (3.1) implies $H_*(\operatorname{Ex}\Delta'[n])=H_*(\operatorname{Ex}\Delta[n])=0$. It thus remains to show that $\delta[n]$ is a homotopy equivalence.

For each integer i with $0\leqq i\leqq n$ let β_i: $[i]\to[n]$ be the map given by $\beta_i(j)=j$, $0\leqq j\leqq i$. Define a function $\delta'[n]$: $\Delta[n]\to\Delta'[n]$ by $\delta[n]\sigma=(\beta_{\sigma(0)},\cdots,\beta_{\sigma(q)})$, $\dim\sigma=q$. As for every map α: $[p]\to[q]$,

$$(\delta'[n]\sigma)\alpha = (\beta_{\sigma(0)}, \cdots, \beta_{\sigma(q)}\alpha = (\beta_{\sigma\alpha(0)}, \cdots, \beta_{\sigma\alpha(p)}) = \delta'[n](\sigma\alpha),$$

it follows that $\delta'[n]$ is a c. s. s. map. The composite map

$$\Delta[n] \xrightarrow{\delta'[n]} \Delta'[n] \xrightarrow{\delta[n]} \Delta[n]$$

is the identity because for $\sigma \in \Delta[n]$ and $0 \leqq i \leqq \dim \sigma$

$$(\delta[n]\delta'[n]\sigma)(i) = \beta_{\sigma(i)}(\sigma(i)) = \sigma(i).$$

It thus remains to prove that the composite map

$$\Delta'[n] \xrightarrow{\delta[n]} \Delta[n] \xrightarrow{\delta'[n]} \Delta'[n]$$

is homotopic with the identity $\iota_n \colon \Delta'[n] \to \Delta'[n]$.

For each simplex $\sigma \in \Delta[n]$, let $\bar{\sigma} = \beta_{\sigma(\dim \sigma)}$. Define a function h: $\Delta[1] \times \Delta'[n] \to \Delta'[n]$ by

$$h(\epsilon^0\eta^0 \cdots \eta^{q-1}, (\sigma_0, \cdots, \sigma_q)) = (\bar{\sigma}_0, \cdots, \bar{\sigma}_q),$$

$$h(\epsilon^1\eta^0 \cdots \eta^{q-1}, (\sigma_0, \cdots, \sigma_q)) = (\sigma_0, \cdots, \sigma_q),$$

$$h(\epsilon_1\eta^0 \cdots \eta^{i-1}\eta^{i+1} \cdots \eta^{q-1}, (\sigma_0, \cdots, \sigma_q)) = (\sigma_0, \cdots, \sigma_i, \bar{\sigma}_{i+1}, \cdots, \bar{\sigma}_q).$$

A straightforward computation shows that the function h so defined is a c. s. s. map. It is now easily verified that h is the required homotopy.

13. Proof of Lemma (3. 7). Use will be made of the following c. s. s. analogues of two theorems of J. H. C. Whitehead ([10]).

THEOREM (13. 1). *Let $K, L \in \mathcal{S}_E$ be connected and let $\phi \in K$ be a 0-simplex. Then a c. s. s. map $f \colon K \to L$ is a homotopy equivalence if and only if f induces isomorphisms of all homotopy groups, i. e., $f_* \colon \pi_n(K;\phi) \approx \pi_n(L; f\phi)$, $n \geqq 1$.*

THEOREM (13. 2). *Let $K, L \in \mathcal{S}_E$ be simply connected. Then a c. s. s. map $f \colon K \to L$ is a homotopy equivalence if and only if f induces isomorphisms of all homology groups, i. e., $f_* \colon H_*(K) \approx H_*(L)$.*

We also need the following lemma

LEMMA (13. 3). *Let $K \in \mathcal{S}_E$ and let $\phi \in K$ be a 0-simplex. Then $(eK)_* \colon \pi_1(K;\phi) \approx \pi_1(\operatorname{Ex} K; (eK)\phi)$.*

Proof of Lemma (3. 7). In this proof we shall freely use the results of [9] Clearly K may be supposed to be minimal. Let $\pi = \pi_1(K)$. Then there exists a fibre map $p: K \to K(\pi, 1)$ with simply connected fibre F. Let $q: F \to K$ be the inclusion map, then it follows from the naturality of e that commutativity holds in the diagram

$$\begin{array}{ccccc} F & \xrightarrow{q} & K & \xrightarrow{p} & K(\pi,1) \\ \downarrow{\scriptstyle eF} & & \downarrow{\scriptstyle eK} & & \downarrow{\scriptstyle e(K(\pi,1))} \\ \operatorname{Ex} F & \xrightarrow{\operatorname{Ex} q} & \operatorname{Ex} K & \xrightarrow{\operatorname{Ex} p} & \operatorname{Ex} K(\pi,1) \end{array}$$

By Lemma (3.4) $\operatorname{Ex} p$ is a fibre map with $\operatorname{Ex} F$ as a fibre. Hence in order to prove that eK is a homotopy equivalence it is, in view of the exactness of the homotopy sequence of a fibre map, the "five lemma" and Theorem (13.1), sufficient to prove that eF and $e(K(\pi,1))$ are homotopy equivalences.

As F is simply connected, so is $\operatorname{Ex} F$ (Lemma (13.3)). Hence it follows from Lemma (3.6) and Theorem (13.2) that eF is a homotopy equivalence.

There exists a fibre map $t: W(K(\pi,0)) \to K(\pi,1)$ with $K(\pi,0)$ as fibre and, as above, in order to prove that $e(K(\pi,1))$ is a homotopy equivalence it suffices to prove that $e(W(K(\pi,0)))$ and $e(K(\pi,0))$ are so. As $W(K(\pi,0))$ is contractible and a fortiori simply connected the argument applied to F yields that $e(W(K(\pi,0)))$ is a homotopy equivalence. It is also readily verified that $e(K(\pi,0))$ is an isomorphism. Hence $e(K(\pi,1))$ is a homotopy equivalence.

This completes the proof of Lemma (3.7).

Proof of Lemma (13.3). For a definition of the fundamental group see [9].

Let $\sigma \in \Delta[n]$ be a non-degenerate q-simplex, i.e., the map $\sigma: [q] \to [n]$ is a monomorphism. Then σ is completely determined by the set $(\sigma(0), \cdots, \sigma(q))$, the image of $[q]$ under σ. We shall often write $(\sigma(0), \cdots, \sigma(q))$ instead of σ.

We first prove that $(eK)_*: \pi_1(K;\phi) \to \pi_1(\operatorname{Ex} K; (eK)\phi)$ is a monomorphism. Let $a \in \pi_1(K;\phi)$ be such that $(eK)_* a = 1$ and let $\tau \in a$. Then there exists a 2-simplex $\rho \in \operatorname{Ex} K$ such that $\rho\epsilon^2 = (eK)\tau$ and $\rho\epsilon^0 = \rho\epsilon^1 = (eK)\phi\eta^0$. Iterated application of the extension condition yields 4 3-simplices $\tau_1, \tau_3, \tau_2, \tau_4 \in K$ such that

$$\tau_1\epsilon^1 = \rho((1),(0,1),(0,1,2)); \quad \tau_1\epsilon^2 = \rho((1),(1,2),(0,1,2)); \quad \tau_1\epsilon^3 = \phi\eta^0\eta^0$$
$$\tau_2\epsilon^0 = \tau_1\epsilon^0; \quad \tau_2\epsilon^2 = \rho((2),(1,2),(0,1,2)); \quad \tau_2\epsilon^3 = \phi\eta^0\eta^0$$

$$\tau_3\epsilon^1 = \tau_2\epsilon^1; \quad \tau_3\epsilon^2 = \rho((2),(0,2),(0,1,2)); \quad \tau_3\epsilon^3 = \phi\eta^0\eta^0$$

$$\tau_4\epsilon^0 = \tau_3\epsilon^0; \quad \tau_4\epsilon^1 = \rho((0),(0,1),(0,1,2)); \quad \tau_4\epsilon^2 = \rho((0),(0,2),(0,1,2)).$$

Then

$$\tau_4\epsilon^3\epsilon^0 = \tau_4\epsilon^0\epsilon^2 = \tau_3\epsilon^0\epsilon^2 = \tau_3\epsilon^3\epsilon^0 = \phi\eta^0$$

$$\tau_4\epsilon^3\epsilon^1 = \tau_4\epsilon^0\epsilon^2 = \rho((0),(0,1)) = \sigma$$

$$\tau_4\epsilon^3\epsilon^2 = \tau_4\epsilon^2\epsilon^2 = \rho((0),(0,2)) = \phi\eta^0.$$

Consequently $a = 1$.

We now show that $(ek)_*: \pi_1(K;\phi) \to \pi_1(\operatorname{Ex} K;(eK)\phi)$ is an epimorphism. Let $\psi \in b \in \pi_1(\operatorname{Ex} K;(eK)\phi)$. Define a c. s. s. map $\rho: \Delta'[2] \to K$ by $\rho((0),(0,1)) = \psi((0),(0,1))$, $\rho((1),(0,1)) = \psi((1),(0,1))$,

$$\rho((1),(1,2),(0,1,2)) = \rho((2),(0,2),(0,1,2)) = \rho((2),(1,2),(0,1,2)) = \phi\eta^0\eta^0,$$

and extend ρ over $((0),(0,1),(0,1,2))$, $((0),(0,2),(0,1,2))$ and $((1),(0,1),(0,1,2))$ by iterated application of the extension condition. Then

$$\rho\epsilon^0 = (eK)\phi\eta^0, \qquad \rho\epsilon^1 = (eK)\rho((0),(0,2)), \qquad \rho\epsilon^2 = \tau$$

Consequently there exists an element $a \in \pi_1(K,\phi)$ such that $\rho((0),(0,2)) \in a$ and $(eK)_*a = b$.

14. Proof of Theorem (4.6). Clearly K may suppose to be connected. Let $\phi \in \operatorname{Ex}^\infty K$ be a 0-simplex, then in view of Theorem (13.1) it suffices to prove that $f_*: \pi_n(\operatorname{Ex}^\infty K;\phi) \approx \pi_n(\operatorname{Ex}^\infty K; f\phi)$ for all $n \geqq 1$. We shall only give a proof for $n = 1$. The proof for $n > 1$ is similar although more complicated.

Let $a \in \pi_1(\operatorname{Ex}^\infty K;\phi)$ and let τ be a representant of a. Suppose there exists a 2-simplex $\rho \in \operatorname{Ex}^\infty K$ such that

$$(14.1) \qquad \rho\epsilon^0 = \tau\epsilon^0\eta^0, \qquad \rho\epsilon^1 = \tau, \qquad \rho\epsilon^2 = f\tau.$$

Then clearly $f_*a = a$. Hence it suffices to show that for every 1-simplex $\tau \in \operatorname{Ex}^\infty K$ there exists a 2-simplex $\rho \in \operatorname{Ex}^\infty K$ satisfying condition (14.1).

Let $\tau \in \operatorname{Ex}^\infty K$ be a 1-simplex and let n be the smallest integer $n \geqq 0$ such that $\tau = (\psi, n)$ (by $\tau = (\psi, 0)$ we mean $\tau = (e^\infty K)\psi$). If $n = 0$, then by hypothesis $\rho = \tau\eta^1$ is the desired 2-simplex. Now suppose it has already been proved that if $n < m$, then there exists a 2-simplex ρ satisfying (14.1a). Then we must show that this is also the case if $n = m$.

Define, using the notation of Section 13, a 2-simplex $\vartheta \in \operatorname{Ex}^n K$ as follows.

$$\vartheta((0),(0,1),(0,1,2))=\vartheta((0),(0,2),(0,1,2))=\psi((0),(0,1))\eta^1$$
$$\vartheta((1),(0,1),(0,1,2))=\vartheta((1),(1,2),(0,1,2))=\psi((1),(0,1))\eta^1$$
$$\vartheta((2),(0,2),(0,1,2))=\vartheta((2),(1,2),(0,1,2))=\psi((0,1))\eta^0\eta^1.$$

Then it is readily verified that

$$\vartheta\epsilon^0=(e(\mathrm{Ex}^{n-1}K))\psi((1),(0,1)),\qquad \vartheta\epsilon^1=(e(\mathrm{Ex}^{n-1}K))\psi((0),(0,1)),\qquad \vartheta\epsilon^2=\psi.$$

By the induction hypothesis there exist 2-simplices $\rho_0,\rho_1\in\mathrm{Ex}^\infty K$ such that

$$\rho_0\epsilon^0=(\psi((0,1))\eta^0,n-1),\qquad \rho_0\epsilon^1=(\vartheta\epsilon^0,n),\qquad \rho_0\epsilon^2=f(\vartheta\epsilon^0,n)$$
$$\rho_1\epsilon^0=(\psi((0,1))\eta^0,n-1),\qquad \rho_1\epsilon^1=(\vartheta\epsilon^1,n),\qquad \rho_1\epsilon^2=f(\vartheta\epsilon^1,n).$$

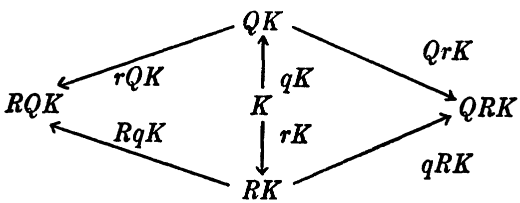
Application of the extension condition then yields 3-simplices $\kappa,\lambda\in\mathrm{Ex}^\infty K$ such that

$$\kappa\epsilon^0=\rho_0,\quad \kappa\epsilon^1=\rho_1,\quad \kappa\epsilon^3=f(\vartheta,n),$$
$$\lambda\epsilon^0=(\vartheta\epsilon^0\eta^0,n),\quad \lambda\epsilon^1=(\vartheta,n),\quad \lambda\epsilon^2=\kappa\epsilon^2.$$

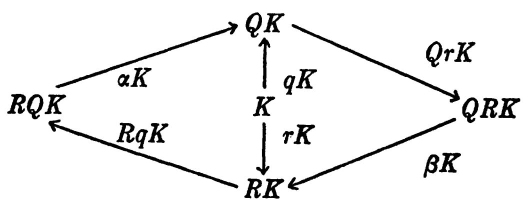
It then follows by direct computation that $\lambda\epsilon^3$ is the desired 2-simplex, i.e.,

$$\lambda\epsilon^3\epsilon^0=\tau\epsilon^0\eta^0,\quad \lambda\epsilon^3\epsilon^1=\tau,\quad \lambda\epsilon^3\epsilon^2=f\tau.$$

15. Proof of Lemma (5.7). Consider the commutative diagram



It follows from Definition (5.1b) that the maps rQK and qRK are homotopy equivalences. Let αK (resp. βK) be a homotopy inverse of rQK (resp. qRK). Then the following diagram is commutative up to homotopy



i. e., $qK \simeq \alpha K \circ RqK \circ rK$ and $rK \simeq \beta K \circ QrK \circ qK$. Consequently

$$qK \simeq (\alpha K \circ RqK) \circ (\beta K \circ QrK) \circ qK,$$
$$rK \simeq (\beta K \circ QrK) \circ (\alpha K \circ RqK) \circ rK.$$

Application of the homotopy extension theorem (which holds for objects of $\mathcal{S}_B$; see [9]) yields c. s. s. maps $s\colon QK \to QK$, $t\colon RK \to RK$ such that

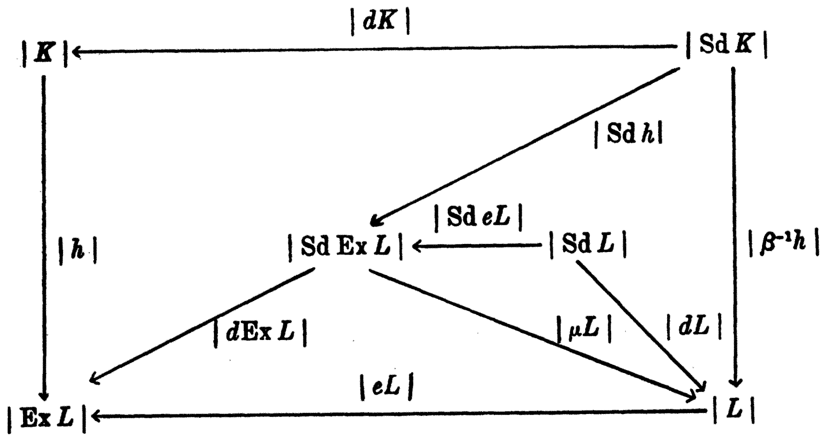
$$s \simeq (\alpha K \circ RqK) \circ (\beta K \circ QrK), \qquad t \simeq (\beta K \circ QrK) \circ (\alpha K \circ RqK)$$

and

$$s(qK)\sigma = (qK)\sigma, \qquad t(rK)\sigma = (rK)\sigma$$

for every simplex $\sigma \in K$. It then follows from condition (5.1c) that s and t are homotopy equivalences. Thus $\alpha K \circ RqK$ and $\beta K \circ QrK$ are homotopy equivalences and because αK and βK are also homotopy equivalences, so are RqK and QrK.

16. Proof of Lemma (8.4). Let $i_{\mathrm{Ex}\,L}\colon L \to \mathrm{Ex}\,L$ be the identity map and let $\mu L = \beta^{-1} i_{\mathrm{Ex}\,L}$. Consider the diagram



In view of the naturality of d commutativity holds in the upper left triangle and the trapezium and because of the naturality of β and the fact that (Lemma (7.2)) $dL = \beta^{-1}(eL)$, commutativity also holds in both triangles which have $|\mu L|$ as lower edge. It follows from Lemma (7.4) and (7.5) that the maps $|dL|$, $|eL|$ and $|d\,\mathrm{Ex}\,L|$ are homotopy equivalences. The commutativity in the trapezium and the smallest triangle involving $|\mu L|$

therefore implies that the maps $|Sd\,eL|$ and $|\mu L|$ are also homotopy equivalences. Consequently the lower triangle is commutative up to homotopy and

$$|h|\circ|dK| = |d\mathrm{Ex}\,L|\circ|\mathrm{Sd}\,h| \simeq |eL|\circ|\mu L|\circ|\mathrm{Sd}\,h| \simeq |eL|\circ|\beta^{-1}h|.$$

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