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BP Operations and Morava's Extraordinary K-Theories

David Copeland Johnson and W. Stephen Wilson*

Introduction

In a series of papers [17–19] Morava uses an infinite sequence of extraordinary K-theories to give an elegant structure theorem for the complex cobordism of a finite complex. Much of Morava's theory is embedded in a rather sophisticated algebraic setting. In our attempt to understand his work, we have found more conventional algebraic topological proofs of many of his results. Also, our approach has yielded new contributions to the general Morava program. We hope this paper will help make Morava's work more accessible and ease the transition between standard algebraic topology and Morava's exposition.

Morava is forced by his algebraic setting to work throughout with complex cobordism, $MU^*(\ )$. We can work directly with Brown-Peterson homology where many of the phenomena we are studying are more transparent. BP denotes the Brown-Peterson spectrum at a fixed prime p [1, 7, 21]. This spectrum gives a multiplicative homology theory, $BP_*(\ )$, with coefficient ring $BP_* = \mathbb{Z}_{(p)}[v_1, \ldots, v_n, \ldots]$. ($\mathbb{Z}_{(p)}$ is the ring of integers localized at the prime p. The dimension of the polynomial generator v_n is $2(p^n - 1)$.) The operation ring for BP, $BP^*(BP)$, operates on $BP_* = BP_*(S^0)$. One of the first benefits of our approach was an easy direct proof of the invariant prime ideal theorem.

(1.10) **Corollary** (Landweber [16], Morava [17, 18]). *If I is a prime ideal of BP_* which is invariant under the action of $BP^*(BP)$, then I is one of the following ideals:* (0), (p), $(p, v_1), \ldots, (p, v_1, \ldots, v_n), \ldots, (p, \ldots, v_n, v_{n+1}, \ldots)$.

Using the Baas-Sullivan theory of manifolds with singularities, we can construct homology theories $P(n)_*(\ )$ with coefficient modules $P(n)_* = BP_*/(p, v_1, \ldots, v_{n-1}) \cong \mathbb{F}_p[v_n, v_{n+1}, \ldots]$ for which we can compute and use the operations. These homology theories are interlocked in the following exact triangle where f_n acts as multiplication by v_n.

$$\begin{tikzcd} P(n)_*(X) \arrow[r, "f_n"] & P(n)_*(X) \arrow[r, "g_n"] & P(n+1)_*(X) \arrow[ll, bend left, "h_n"] \end{tikzcd}$$

$P(0)_*(\ )$ is thus Brown-Peterson homology and $P(1)_*(\ )$ is Brown-Peterson homology with mod p coefficients. The above exact triangle can be used to study those classes of $P(n)_*(X)$ which are annihilated by multiples of v_n. These classes constitute the T_n torsion part of $P(n)_*(X)$ where T_n is the multiplicative set $\{1, v_n, v_n^2, \ldots\}$. If we localize with respect to T_n, we obtain a periodic homology

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theory $B(n)_*(\ )=T_n^{-1}P(n)_*(\ )$. $B(n)_*=\mathbb{F}_p[v_n^{-1}, v_n, v_{n+1}, \ldots]$. We prove:

(3.1) **Theorem.** *If X is a finite complex, then $B(n)_*(X)$ is a free $B(n)_*$ module.*

N.B. Morava's Theorem (5.1) of [17] can be roughly translated as saying that $B(n)_*(X)$ is projective over $B(n)_*$.

Morava studies extraordinary K-theories $K(n)_*(\ )$ with $K(n)_*\cong\mathbb{F}_p[v_n^{-1}, v_n]$. (Thus $K(n)_*(\ )$ is periodic with period $2(p^n-1)$. Every non-zero element of $K(n)_*$ is invertible.) Let $k(n)_*(\ )$ be the connective homology theory associated to $K(n)_*(\ )$. ($k(n)_*=\mathbb{F}_p[v_n]$.) Actually, we construct $k(n)_*(\ )$ using the Baas-Sullivan technique and then we define $K(n)_*(\ )=T_n^{-1}k(n)_*(\ )$. From this construction, there are natural Thom homomorphisms: $P(n)_*(\ )\to k(n)_*(\ )\to H_*(\ ;\mathbb{F}_p)$. By applying the functor T_n^{-1} – to the first morphism, we have a natural homomorphism $B(n)_*(\ )\to K(n)_*(\ )$. A second part of our Theorem (3.1) says that this induces a natural isomorphism:

$$B(n)_*(X)\oplus_{B(n)_*} K(n)_*\cong K(n)_*(X).$$

In § 4, we develop a spectral sequence of the general Atiyah-Hirzebruch-Dold type relating $k(n)_*(X)$ to $P(n)_*(X)$.

(4.8) **Theorem.** *There is a natural spectral sequence for finite complexes*

$$E^2_{*,*}(X)=k(n)_*(X)\oplus\mathbb{F}_p[v_{n+1}, v_{n+2}, \ldots]\Rightarrow P(n)_*(X).$$

The spectral sequence collapses if and only if $P(n)_*(X)\to k(n)_*(X)$ is epic. Its differentials are T_n torsion valued.

As corollary to this theorem, we prove (4.16) that $k(n)_*(X)\to H_*(X;\mathbb{F}_p)$ epic implies that $k(n+1)_*(X)\to H_*(X;\mathbb{F}_p)$ is also epic (i.e. if the Atiyah-Hirzebruch spectral sequence for $k(n)_*(X)$ collapses then so does the one for $k(n+1)_*(X)$). $H^*(k(n);\mathbb{F}_p)\cong A/AQ_n$, where A is the mod p Steenrod algebra. (This implies that if all the higher order cohomology operations arising from $(Q_n)^2=0$ vanish, then so do all those arising from $(Q_{n+1})^2=0$. We shall defer our discussion of this and related matters to a future note written jointly with F. P. Peterson.)

Theorems (3.1) and (4.8) depend on our knowledge of $P(n)$ operations. Recall that $BP^*(BP)$ is isomorphic to $BP^*\hat{\otimes}R$, where R is a connected coalgebra free over $\mathbb{Z}_{(p)}$. The basis elements of R, r_E, are indexed over exponent sequences $E=(e_1, e_2, \ldots)$.

(2.12) **Lemma** (Morava). *$P(n)^*(P(n))\cong P(n)^*\hat{\otimes}R\otimes E[Q_0, \ldots, Q_{n-1}]$ as left $P(n)^*$ modules.*

Nearly all of the results in the paper depend on the technical ability to handle operations modulo the ideal $(p, v_1, \ldots, v_{n-1})$. Our computations in § 1 are motivated by and improve on earlier work by Stong, Smith and Hansen [25, 23, 11]. The following innocuous looking lemma is the distilled version of our main technical result.

(1.9) **Lemma.** *Let $n>0$. If $0\neq y\in BP_*\setminus(p, v_1, \ldots, v_{n-1})$, then there is an exponent sequence $F=(p^n e_{n+1}, p^n e_{n+2}, \ldots)$ such that $r_F(y)=u(v_n)^t$ modulo $(p, v_1, \ldots, v_{n-1})$. Here u is a unit of $\mathbb{Z}_{(p)}$ and $t=e_{n+1}+e_{n+2}+\cdots$. (The exponent sequence F depends on y in an easily computed fashion.)*

We should point out that the real richness of Morava's approach comes from his computation of the $K(n)$ operations in important cases. This computation seems to require his algebraic setting and is something which we have yet to handle from our point of view.

The organization of the paper is as follows:

§ 1. *BP* Operations Modulo $(p, v_1, \ldots, v_{n-1})$.

§ 2. $P(n)$ and its Operations.

§ 3. The Relationship between $B(n)_*(X)$ and $K(n)_*(X)$.

§ 4. The Relationship between $P(n)_*(X)$ and $k(n)_*(X)$.

§ 5. An Expository Summary.

Appendix: A Proof of (2.4).

1. *BP* Operations Modulo $(p, v_1, \ldots, v_{n-1})$

Let BP be the Brown-Peterson spectrum for the fixed prime p. It is a ring spectrum which represents the homology theory $BP_*(\ )$ constructed in [1, 7, 21]. $H^*(BP;\mathbb{F}_p)\cong A/(Q_0)$ where A is the mod p Steenrod algebra and (Q_0) is the two-sided ideal generated by the Bockstein. $H_*(BP;\mathbb{Z}_{(p)})\cong \mathbb{Z}_{(p)}[m_1, \ldots, m_s, \ldots]$ where the generator m_s has degree $2(p^s-1)$. The Hurewicz homomorphism $h\colon BP_* = \pi_*(BP)\to H_*(BP, \mathbb{Z}_{(p)})$ is a monomorphism. We identify BP_* with the subring $h(BP_*)$ of $H_*(BP, \mathbb{Z}_{(p)})$. $BP_*\cong \mathbb{Z}_{(p)}[v_1, \ldots, v_s, \ldots]$ where the generators v_s are defined inductively by (1.1).

(1.1) $$h(v_s)=p m_s-\sum_{j=1}^{s-1} m_j h(v_{s-j})^{p^j} \quad [12].$$

(1.2) $$p^s m_s\in h(BP_*), \quad \text{but} \quad p^{s-1} m_s\notin h(BP_*).$$

In $H_*(BP;\mathbb{Z}_{(p)})$, $p\,|\,h(v_s)$, but $p^2\nmid h(v_s)$.

Let $E=(e_1, \ldots, e_n)=(e_1, \ldots, e_n, 0, 0, \ldots)$ be an exponent sequence of non-negative integers with all but finitely many are zero. We define

$$|E|=\sum_{i=1} 2(p^i-1)\, e_i.$$

Thus if we define $v^E=v_1^{e_1}, \ldots, v_n^{e_n}$, $|E|$ is the degree of v^E. $0=(0, 0, \ldots)$ and $v^0=1$. We add exponent sequences termwise and if m is a positive integer mE represents the m-fold sum of E's. Δ_s represents the exponent sequence $E=(e_1, e_2, \ldots)$ with $e_i=0$, $i\neq s$, and $e_s=1$. There is a connected free $\mathbb{Z}_{(p)}$ coalgebra of BP operations, $R\subset BP^*(BP)$. A $\mathbb{Z}_{(p)}$ basis of R is given by operations r_E of degree $|E|$. r_0 is the identity operation.

(1.3) The coproduct of R is given by

$$\psi(r_E)=\sum_{F+G=E} r_F\otimes r_G \quad [21, 29].$$

(1.4) The action of R on the generators of $H_*(BP; \mathbb{Z}_{(p)})$ is

$$r_E(m_s) = \begin{cases} m_{s-i} & \text{if } E = p^{s-i}\Delta_i \\ 0 & \text{if } E \neq p^{s-i}\Delta_i \end{cases} \qquad [21, 29].$$

(1.5) The Hurewicz homomorphism $h\colon BP_s \to H_s(BP; \mathbb{Z}_{(p)})$ has the form:

$$h(y) = \sum_{|E|=s} t^E r_E(y) \quad \text{for some elements } t^E \in H_s(BP; \mathbb{Z}_{(p)}) \qquad [1].$$

BP operations commute with the Hurewicz homomorphism; so (1.1), (1.3) and (1.4) allow one to effectively compute $r_F(v^E)$. R is not a subalgebra of $BP^*(BP)$; but we do have $BP^*(BP) \cong BP^* \hat{\otimes} R$ [21, 29]. (Note that as an element of $BP^* \cong BP_{-*}$, v^E has degree $-|E|$.)

(1.6) **Lemma.** (a) *If* $|F| > |E|$, *then* $r_F(v^E) = 0$.

(b) *If* $|F| = |E|$, *then* $r_F(v^E) \equiv 0$ *modulo* (p).

Proof. In (a), $r_F(v^E)$ has negative degree and thus is zero. Now suppose $|F| = |E| = m$. If the composition

$$S^m \xrightarrow{v^E} BP \xrightarrow{r_F} S^m BP$$

were essential modulo p, then $H^*(r_F \circ v^E; \mathbb{F}_p) \not\equiv 0$. But $H^0(v^E; \mathbb{F}_p) \equiv 0$ for dimensional reasons. As $H^*(BP; \mathbb{F}_p)$ is a cyclic A-module, this implies that $H^*(v^E; \mathbb{F}_p) \equiv 0$. $\square$

The following lemma is a strong version of propositions due to Stong, Smith, and Hansen [25, 23, 11]. It does not hold in the $n=0$ case [24].

(1.7) **Lemma.** *Let* $I_n = (p, v_1, \ldots, v_{n-1})$, $n > 0$.

(a) *If* $|E| > 2(p^s - p^n)$, *then* $r_E(v_s) \equiv 0$ *modulo* I_n.

(b) *If* $|E| = 2(p^2 - p^n)$, *then*

$$r_E(v_s) \equiv \begin{cases} v_n \text{ modulo } I_n & E = p^n \Delta_{s-n} \\ 0 \text{ modulo } I_n & E \neq p^n \Delta_{s-n}. \end{cases}$$

Proof. If $|v_s| > |E| > |p^n \Delta_{s-n}|$, then $0 < |r_E(v_s)| < |v_n|$ and $r_E(v_s) \in (v_1, \ldots, v_{n-1}) \subseteq I_n$ for dimensional reasons. If $|v_s| \leqq |E|$, then $r_E(v_s) \in (p) \subseteq I_n$ by (1.6). Thus (a) is established.

If $|E| = 2(p^s - p^n)$, $r_E(v_s) \equiv a v_n$ modulo $(v_1, \ldots, v_{n-1})$ for some $a \in \mathbb{Z}_{(p)}$, again for dimensional reasons. By (1.2) and (1.1),

$$h(v_1, \ldots, v_{n-1}) \subseteq (pm_1, \ldots, pm_{n-1})$$

and

$$h(v_n) \equiv pm_n \qquad \text{modulo } (p^p m_1, \ldots, p^p m_{n-1}) \subseteq (p^2).$$

So

$$r_E(v_s) \notin I_n \Leftrightarrow h(r_E(v_s)) \notin (p^2) \Leftrightarrow p r_E(m_s) \notin (p^2) \Leftrightarrow E = p^n \Delta_{s-n}$$

(since $|E| = 2(p^s - p^n)$). When $E = p^n \Delta_{s-n}$, $p r_E(m_s) = pm_n$ implying $r_E(v_s) \equiv v_n$ modulo I_n as required. $\square$

We need a shift-like operator to act on exponent sequences. If $E=(e_1, e_2, \ldots)$, we define $\sigma E=(pe_2, pe_3, \ldots)$ and $\sigma^n E=\sigma(\sigma^{n-1}E)=(p^n e_{n+1}, p^n e_{n+2}, \ldots)$, $n>0$. We interpret Lemma (1.7) as saying: if $|F|\geqq|\sigma^n \Delta_s|$, then $r_F(v^{\Delta_s})\equiv v_n$ or 0 modulo I_n as to whether $F=\sigma^n\Delta_s$ or not. Observe that if $|F|\geqq|E|$, $E=E_1+E_2$, $F=F_1+F_2$, and if $E\neq F$, then $|F_i|\geqq|E_i|$ with $F_i\neq E_i$ holds for i equal to at least one of the numbers 1 and 2. This observation and (1.3) then imply the following corollary to (1.7).

(1.8) **Corollary.** *Let $n>0$. Let E and F be two exponent sequences such that*

(a) $E=(0, \ldots, 0, e_n, e_{n+1}, \ldots)$ *and* $t=e_n+e_{n+1}+\cdots$.

(b) $|F|\geqq|\sigma^n E|$.

Then

$$r_F(v^E)\equiv\begin{cases} v_n^t & \text{modulo } I_n \quad \text{if } F=\sigma^n E \\ 0 & \text{modulo } I_n \quad \text{if } F\neq\sigma^n E. \end{cases} \qquad \square$$

(1.9) **Lemma.** (a) *Let $n>0$. If $0\neq y\in BP_s$ and if $y\notin(p, v_0, \ldots, v_{n-1})$, then there is an exponent sequence E and a unit $u\in\mathbb{Z}_{(p)}$ such that $r_{\sigma^n E}(y)=uv_n^t$ modulo I_n where $t(2p^n-2)=s-|\sigma^n E|$.*

(b) *If $0\neq y\in BP_s$, then there is an exponent sequence E with $|E|=s$ such that $0\neq r_E(y)=up^t$ for some $t>0$ and some unit $u\in\mathbb{Z}_{(p)}$.*

Proof. Suppose $y=\sum a_F v^F\notin(p, v_0, \ldots, v_{n-1})$, $a_F\in\mathbb{Z}_{(p)}$, then there is an exponent sequence E, $|E|=s$ and $E=(0, \ldots, 0, e_n, e_{n+1}, \ldots)$, such that a_E is a unit of $\mathbb{Z}_{(p)}$. Of such sequences, we pick one with $|\sigma^n E|$ maximal. By (1.8), $r_{\sigma^n E}(y)\equiv a_E r_{\sigma^n E}(v^E)\equiv a_E v_n^t$ modulo I_n. More generally: if $0\neq y=\sum a_F v^F$, $h(y)\neq 0$ implies that there is an exponent sequence E, $|E|=s$, such that $r_E(y)\neq 0$ (1.5). But by (1.6) $0\neq r_E(y)\in BP_0\cap(p)\cong\mathbb{Z}_{(p)}\cap(p)$. (b) then follows. $\square$

Since (1.7), I_n has been the BP_* ideal, $(p, v_1, \ldots, v_{n-1})$. We define $I_0=(0)$ and $I_\infty=(p, v_1, \ldots, v_{n-1}, v_n, \ldots)$. Thus we have an ascending tower of BP_* ideals:

$$0=I_0\subseteq I_1\subseteq I_2\subseteq\cdots\subseteq I_n\subseteq I_{n+1}\subseteq\cdots\subseteq I_\infty.$$

Obviously, each of these is prime. By (1.6) and the now familiar dimensional considerations, *BP* operations preserve each of these ideals.

(1.10) **Corollary.** (Invariant Prime Ideal Theorem of Landweber [16] and Morava [17, 18].) *If $I\subseteq BP_*$ is a prime ideal which is invariant under the action of $BP^*(BP)$, then $I=I_n$ for some n, $n=0, 1, 2, \ldots$, or ∞.*

Proof. $I_0\subseteq I$. Suppose $I_{n-1}\subseteq I$, $n\geqq 1$. If $I=I_{n-1}$, good! If $0\neq y\in I\setminus I_{n-1}$, then by (1.9) there is a *BP* operation r_F such that $r_F(y)\equiv uv_{n-1}^t$ for some $t>0$ and some unit $u\in\mathbb{Z}_{(p)}\cdot(v_0=p)$. Since I is invariant, $v_{n-1}^t\in I$. Since I is prime, $v_{n-1}\in I$ and thus $I_n\subseteq I$. $\square$

Observe that multiplication by v_n^s induces a homomorphism of $BP^*(BP)$ modules

$$BP^*\twoheadrightarrow BP^*/I_n\xrightarrow{v_n^s} BP^*/I_n.$$

Lemma (1.9) shows that these are the only such $BP^*(BP)$ homomorphisms in the following sense.

(1.11) **Corollary** (Landweber [16]).

$$\mathbb{F}_p[v_n] \to \mathrm{Hom}_{BP^*(BP)}(BP^*, BP^*/I_n) \cong \mathrm{Ext}^{0,*}_{BP^*(BP)}(BP^*, BP^*/I_n)$$

is an isomorphism. □

Suppose $n>0$ and $s \geqq 0$ are integers. We may write $s=a2(p^n-1)+b$ where $a \geqq 0$ and $0 \leqq b < 2(p^n-1)$. Let $M(n,s)$ be the free $\mathbb{Z}_{(p)}$ submodule of BP_s with basis the elements v^E with $|E|=s$ and E of form $E=(0,\ldots,0,e_n,e_{n+1},\ldots)$. Any b-dimensional BP operation $\theta\colon BP \to S^b BP$ induces a $\mathbb{Z}_{(p)}$-homomorphism

$$\theta_\#\colon M(n,s) \to \mathbb{Z}_{(p)}$$

given by the following composition:

$$\begin{aligned}\theta_\#\colon M(n,s) \subset BP_s &\xrightarrow{\theta} BP_{s-b} \to (BP_*/(v_1,\ldots,v_{n-1},v_{n+1},\ldots))_{s-b}\\ &\cong (\mathbb{Z}_{(p)}[v_n])_{s-b} \cong \mathbb{Z}_{(p)}.\end{aligned}$$

Note that $M(1,s)=BP_s$. The $n=1$ case of the following lemma is essentially one of Stong's approaches to the Stong Hattori Theorem [8].

(1.12) **Lemma.** *Any $\mathbb{Z}_{(p)}$ homomorphism $h\colon M(n,s) \to \mathbb{Z}_{(p)}$ can be realized by a b-dimensional BP operation θ in that $h \equiv \theta_\#$. Furthermore, θ has the form*

$$\theta = \sum_E a_E v_n^{b_E} r_{p^n E}$$

where the sum is finite, $a_E \in \mathbb{Z}_{(p)}$, and $b_E \cdot 2(p^n-1)+b = p^n|E|$.

Proof of (1.12). Order the dimension s exponent sequences of form $E=(0,\ldots,0,e_n,e_{n+1},\ldots)$: $E_1, E_2, \ldots, E_u$ such that $|\sigma^n E_1| \leqq |\sigma^n E_2| \leqq \cdots \leqq |\sigma^n E_u|$. By (1.8), $r_{\sigma^n E_i}(v^{E_j}) \equiv 0$ modulo I_n if $i>j$ and $r_{\sigma^n E_i}(v^{E_i}) \not\equiv 0$ modulo I_n. So a basis for $\mathrm{Hom}_{\mathbb{Z}_{(p)}}(M(n,s),\mathbb{Z}_{(p)}) \otimes \mathbb{F}_p$ is given by elements of form $v_n^{b_E} r_{p^n E} \otimes 1$. (1.12) then follows from Nakayama's lemma. □

2. $P(n)$ and its Operations

We have seen that the only prime ideals of BP_* which are invariant under the action of $BP^*(BP)$ are: $I_0=(0)$, $I_1=(p), \ldots, I_n=(p,v_1,\ldots,v_{n-1}), \ldots,$ and $I_\infty=(p,v_1,\ldots,v_n,\ldots)$. A natural extension of the classical idea of working with homology with modulo (p) coefficients would be to consider Brown-Peterson homology with I_n coefficients. One can use the Sullivan-Baas technique of defining bordism theories with singularities [4, 5, 26] to construct homology theories $P(n)_*(\ )$, $n=0,1,2,\ldots$, (and ∞) which are represented by the CW spectra $P(n)$, $n=0,1,2,\ldots$ and which have the following properties.

(2.1) $P(0)=BP$, $\pi_*(P(n))=P(n)_* \cong BP_*/I_n \cong \mathbb{F}_p[v_n,v_{n+1},\ldots]$, $0<n<\infty$. $P(\infty)=H\mathbb{F}_p$.

(2.2) $P(n)$ is a left module spectrum over the ring spectrum BP. (There are pairings $m_n\colon BP \wedge P(n) \to P(n)$ such that $m_n \circ (m_0 \wedge 1) = m_n \circ (1 \wedge m_n)$, etc.)

(2.3) $P(n+1)$ is related to $P(n)$ by a stable cofibration.

$$S^{2(p^n-1)}P(n) \xrightarrow{f_n} P(n) \xrightarrow{g_n} P(n+1) \xrightarrow{h_n} S^{2p^n-1}P(n).$$

The maps indicated are morphisms of BP module spectrums. The cofibration induces an exact triangle of (BP_*-module) homology theories.

$$\begin{tikzcd} P(n)_*(X) \arrow[r, "f_n"] & P(n)_*(X) \arrow[r, "g_n"] & P(n+1)_*(X) \arrow[ll, bend left, "h_n"] \end{tikzcd}$$

f_n acts as multiplication by v_n. When X is a sphere, g_n is onto.

(2.4) For $0 \leqq i < n$, $v_i y = 0$ for any element $y \in P(n)_*(X)$. (This follows from a geometric result of Morava's [20]. A proof is sketched in the Appendix.)

(2.5) *Remarks.* (a) It follows from (2.4), that $P(n)_*(X)$ is a module over

$$\mathbb{F}_p[v_n, v_{n+1}, \ldots] \cong P(n)_*.$$

(b) By [6] or by similar techniques to those in [28], one can compute that $H^*(P(n); \mathbb{F}_p) \cong A/A(Q_n, Q_{n+1}, \ldots)$.

(c) Suppose for the fixed prime p, stable complexes $V(i)$, $i = -1, 0, \ldots, n-1$ exist such that $BP_*(V(i)) \cong BP_*/I_{i+1}$. (The existence of such complexes has been studied by Smith [22, 23] and Toda [27].) Then $P(n)$ is equivalent to $BP \wedge V(n-1)$. This assertion is proved inductively beginning with $BP \wedge V(-1) = BP \wedge S^0 \cong BP$ and using the fact that $V(i)$ is constructed as the cofibre of a stable map

$$S^{2^i-2}V(i-1) \to V(i-1)$$

which realizes multiplication by v_i in $BP_*(V(i-1))$.

(d) When the fixed prime p is 2, $P(1)^*(\ )$ has no commutative admissible multiplication in the sense of Araki-Toda [2]. (See Corollary 4.2 of [15]). Thus in general, we cannot expect the $P(n)$'s to be nice ring spectra.

Let $T_n = \{1, v_n, v_n^2, \ldots\}$ be the multiplicative set of non-negative powers of v_n, $n > 0$. As in [14], we may localize with respect to T_n. We form the periodic homology theory $B(n)_*(X) = T_n^{-1}P(n)_*(X)$. Note that $B(n)_* \cong \mathbb{F}_p[v_n, v_n^{-1}, v_{n+1}, v_{n+2}, \ldots]$. $T_0 = \{1, p, p^2, \ldots\}$ and $B(0)_*(X) = T_0^{-1}P(0)_*(X) = BP_*(X) \otimes \mathbb{Q}$. (For mnemonic purposes, note that "B" is the union of "P" and an inverted "P".)

Before proceeding to compute $P(n)$ operations, let us use the Sullivan-Baas technique further. We kill the generators $v_{n+1}, v_{n+2} \ldots$ of $P(n)_*$ (thus we are killing the generators, $p, v_1, \ldots, v_{n-1}, v_{n+1}, v_{n+2}, \ldots$ of BP_*) to construct the homology theory $k(n)_*(\ )$ represented by the BP-module spectrum $k(n)$. $k(n)_* \cong \mathbb{F}_p[v_n]$.

(2.6) *Remarks.* (a) The Sullivan-Baas method of constructing $k(n)$ gives a BP-module morphism of spectra $\lambda_n\colon P(n) \to k(n)$ such that the induced homomorphism $\lambda_n\colon P(n)_* \to k(n)_*$ sends the v_i, $i \neq n$, to 0 and $\lambda_n(v_n) = v_n$.

(b) Let $\phi_n\colon S^{2p^n-2}k(n) \to k(n)$ represent multiplication by v_n and let $\gamma_n\colon k(n) \to H\mathbb{F}_p$ be the resulting map to ϕ_n's cofibre (which is computed to be an $H\mathbb{F}_p$)

we define $\mu_n = \gamma_n \circ \lambda_n$: $P(n) \to H\mathbb{F}_p$. We now have a commutative diagram of BP module spectra and BP-module spectra morphisms (2.7).

(2.7)
$$\begin{tikzcd}
S^{2p^n-2}P(n) \arrow[r, "f_n"] \arrow[d, "\lambda_n"] & P(n) \arrow[r, "g_n"] \arrow[d, "\lambda_n"] \arrow[dr, "\mu_n"] & P(n+1) \arrow[r, "h_n"] \arrow[d, "\mu_{n+1}"] & S^{2p^n-1}P(n) \arrow[d, "\lambda_n"] \\
S^{2p^n-2}k(n) \arrow[r, "\phi_n"] & k(n) \arrow[r, "\gamma_n"] & H\mathbb{F}_p \arrow[r, "\eta_n"] & S^{2p^n-1}k(n)
\end{tikzcd}$$

(c) The bottom row is a cofibration sequence and induces an exact triangle of (BP-module) homology theories. The spectral sequence arising from this exact triangle (couple) may be identified with the usual Atiyah-Hirzebruch-Dold one.

$$\begin{tikzcd}
k(n)_*(X) \arrow[r, "\phi_n"] & k(n)_*(X) \arrow[r, "\gamma_n"] & H_*(X:\mathbb{F}_p) \arrow[ll, bend left, "\eta_n"]
\end{tikzcd}$$

We may localize $k(n)_*(X)$ with respect to the multiplicative set T_n also. We gain a periodic homology theory $K(n)_*(\ ) = T_n^{-1}k(n)_*(\ )$ with $K(n)_* \cong \mathbb{F}_p[v_n, v_n^{-1}]$. The dual cohomology $K(n)^*(\ )$ is one of Morava's extraordinary K-theories. λ_n: $P(n) \to k(n)$ induces a morphism of BP-module homology theories $T_n^{-1}\lambda_n$: $B(n)_*(\ ) \to K(n)_*(\ )$. We shall refer to $T_n^{-1}\lambda_n$ as λ_n in the sequel.

(2.8) **Lemma.** *Suppose $j > n$. The cofibration of* (2.3) *induces short exact sequences* (a) *and* (b)

(a) $0 \to P(j)^*(S^{2p^n-1}P(n)) \xrightarrow{h_n^*} P(j)^*(P(n+1)) \xrightarrow{g_n^*} P(j)^*(P(n)) \to 0$,

(b) $0 \to k(j)^*(S^{2p^n-1}P(n)) \xrightarrow{h_n^*} k(j)^*(P(n+1)) \xrightarrow{g_n^*} k(j)^*(P(n)) \to 0$.

Proof. Let v_n: $S^{2p^n-2} \to BP$ represent the homotopy class of the same name. Then f_n is given by the composition

$$S^{2p^n-2} \wedge P(n) \xrightarrow{v_n \wedge 1} BP \wedge P(n) \xrightarrow{m_n} P(n).$$

By (1.10), $BP^*(v_n)$: $BP^*(BP) \to BP^*(S^{2p^n-2})$ has image contained in the invariant prime ideal I_{n+1}. If $W = S^{2p^n-2}$ or BP, we may identify

$$P(j)^*(W \wedge P(n)) \cong BP^*(W) \hat{\otimes}_{BP^*} P(j)^*(P(n)).$$

With this identification, image $P(j)^*(f_n) \subseteq \text{image}\, P(j)^*(v_n \wedge 1) \subseteq I_{n+1} \hat{\otimes}_{BP^*} P(j)^*(P(n))$. By (2.4) this last module is zero when $j > n$. Similarly, $k(j)^*(f_n) \equiv 0$ for $j > n$. $\square$

(2.9) **Corollary.** *Given a $P(n-1)$ operation θ_{n-1}: $P(n-1) \to S^m P(n-1)$, there is a (non-unique) operation θ_n: $P(n) \to S^m P(n)$ such that $g_{n-1} \circ \theta_{n-1} = \theta_n \circ g_{n-1}$. In particular, if we are given a BP operation θ: $BP \to S^m BP$, there are $P(n)$ operations θ_n: $P(n) \to S^m P(n)$, $n = 0, 1, 2, \ldots$ such that $\theta_0 = \theta$ and $g_{n-1} \circ \theta_{n-1} = \theta_n \circ g_{n-1}$.* $\square$

Recall from §1 that we have an identification $BP^* \hat{\otimes} R \to BP^*(BP)$ which we now label Φ_0. Note that the objects here are locally finitely generated free topologized (l.f.g.f.t) $\mathbb{Z}_{(p)}$ modules [29]. When its range object (e.g. $BP^*(BP)$) is Hausdorff a continuous homomorphism (e.g. Φ_0) is determined by its restriction to a dense submodule of its domain (e.g. $BP^* \otimes R$). Thus Φ_0 is determined by the rule: $\Phi_0(v^A \otimes r_B) = v^A r_B$. Of course the analogous observations hold for l.f.g.f.t $\mathbb{F}_p$ modules.

By (2.9), the basis elements r_B of R give rise to (non-unique) operations $(r_B)_n$: $P(n) \to S^{|B|} P(n)$. Fix the choice of these such that $g_{n-1} \circ (r_B)_{n-1} = (r_B)_n \circ g_{n-1}$, $(r_B)_0 = r_B$. Since the maps g_i are morphisms of *BP* module spectra, we may (and shall) choose $(v^A r_B)_n$ to be $v^A (r_B)_n$. Let $C = (c_0, \ldots, c_{n-2})$ be an exponent sequence consisting of zeros and ones. Q^C will denote the $\mathbb{F}_p$ basis element $Q_0^{c_0} \ldots Q_{n-2}^{c_{n-2}}$ of the $\mathbb{F}_p$ exterior algebra $E_{n-1} = E[Q_0, \ldots, Q_{n-2}]$. The degree of Q_i is $2p^i - 1$. If we have constructed an element $(Q^C)_{n-1}$ corresponding to Q^C in $P(n-1)^*(P(n-1))$, then we use (2.9) to construct $(Q^C)_n$ in $P(n)^*(P(n))$ such that $(Q^C)_n \circ g_{n-1} = g_{n-1} \circ (Q^C)_{n-1}$. This accounts for half of the basis elements in $E_n = E(Q_0, \ldots, Q_{n-2}, Q_{n-1})$. We define $(Q^C Q_{n-1})_n \equiv (Q^C)_n \circ g_{n-1} \circ h_{n-1} \in P(n)^*(P(n))$ and let it correspond to $Q^C Q_{n-1}$. Now we can define Φ_n: $P(n)^* \hat{\otimes} R \otimes E_n \to P(n)^*(P(n))$ by (2.10) which gives its values on a $\mathbb{F}_p$ basis of the dense submodule $P(n)^* \otimes R \otimes E_n$ of the domain. ($P(n)^*(P(n))$ is Hausdorff in the skeletal filtration.)

(2.10) $$\Phi_n(v^A \otimes r_B \otimes Q^C) \equiv v^A (r_B)_n \circ (Q^C)_n.$$

(2.10) defines Φ_n to be a left $P(n)^*$ module homomorphism. Beginning with Φ_0, we assume inductively that Φ_{n-1} is an isomorphism of left $P(n-1)^*$ modules. Since the short exact sequence

$$0 \to P(n-1)^* \xrightarrow{v_{n-1}} P(n-1)^* \to P(n)^* \to 0$$

remains exact when decorated with $-\hat{\otimes} R \otimes E_{n-1}$, we see that the isomorphism Φ_{n-1} induces an isomorphism Φ': $P(n)^* \hat{\otimes} R \otimes E_{n-1} \to P(n)^*(P(n-1))$ (proof by the five lemma). Right multiplication by Q_{n-1} induces a short exact sequence of $\mathbb{F}_p$ modules

$$0 \to E_{n-1} \xrightarrow{Q_{n-1}} E_n \to E_{n-1} \to 0.$$

This induces the left (short exact) column of commutative diagram (2.11).

(2.11)
$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
P(n)^* \hat{\otimes} R \otimes E_{n-1} \arrow[r, "\Phi'"] \arrow[d] & P(n)^*(P(n-1)) \arrow[d, "h^*_{n-1}"] \\
P(n)^* \hat{\otimes} R \otimes E_n \arrow[r, "\Phi_n"] \arrow[d] & P(n)^*(P(n)) \arrow[d, "g^*_{n-1}"] \\
P(n)^* \hat{\otimes} R \otimes E_{n-1} \arrow[r, "\Phi'"] \arrow[d] & P(n)^*(P(n-1)) \arrow[d] \\
0 & 0
\end{tikzcd}$$

The exactness of the right column of (2.11) follows from (2.8). Since Φ' is an isomorphism, Φ_n is also. Our induction is completed and we have following computation of Morava [19].

(2.12) **Lemma** (Morava).

$$\Phi_n\colon\ \mathbb{F}_p[v_n, v_{n+1}, \ldots]\hat{\otimes} R\otimes E[Q_0, \ldots, Q_{n-1}]\to P(n)^*(P(n))$$

is an isomorphism of left $P(n)^*$ modules. □

(2.13) *Remark.* The pairing (2.2), $m_n\colon BP\wedge P(n)\to P(n)$ gives us a homomorphism $m_n^*\colon P(n)^*(P(n))\to P(n)^*(BP\wedge P(n))\cong BP^*(BP)\hat{\otimes}_{BP^*}P(n)^*(P(n))$ making $P(n)^*(P(n))$ into a $BP^*(BP)$ comodule. Unfortunately our naive analysis does not show that Φ_n is a morphism of $BP^*(BP)$ comodules.

(2.14) *Remark.* If $n<2p-2\equiv q$, then $P(n)^{iq}(P(n))=\Phi_n((P(n)^*\hat{\otimes} R)^{iq})$. In this case, the choices of the elements $(r_B)_n$ are unique. $\Phi_n(P(n)^*\hat{\otimes} R)$ inherits its $BP^*(BP)$ comodule structure from $BP^*(BP)$ via the projection

$$BP^*\hat{\otimes} R\to P(n)^*\hat{\otimes} R.$$

(2.15) *Remark.* When $n=\infty$, $P(\infty)=H\mathbb{F}_p$ and we recover the mod p Steenrod algebra.

3. The Relationship Between $B(n)_*(X)$ and $K(n)_*(X)$

Recall that $B(n)_*(\ )$ and $K(n)_*(\ )$ are periodic homology theories with coefficient modules $B(n)_*\cong\mathbb{F}_p[v_n, v_n^{-1}, v_{n+1}, v_{n+2}, \ldots]$ and $K(n)_*\cong\mathbb{F}_p[v_n, v_n^{-1}]$ $(n>0)$. Both $B(n)_*(X)$ and $K(n)_*(X)$ are modules over $\mathbb{F}_p[v_n, v_{n+1}, \ldots]$, hence over $B(n)_*$ also. By the construction of these homology theories, there is a natural homomorphism of $B(n)_*$ modules, $\lambda_n(X)\colon B(n)_*(X)\to K(n)_*(X)$. Morava [19] proves that $B(n)_*(X)$ is a projective $B(n)_*$ module (X a finite complex). This leads one to suspect that $B(n)_*(X)\otimes_{B(n)_*}K(n)_*$ determines a homology theory which is isomorphic to $K(n)_*(X)$. This suspicion is confirmed in the following strengthened form of Morava's result.

(3.1) **Theorem.** *Let X be a finite complex.*

(a) *$B(n)_*(X)$ is a free $B(n)_*$ module.*

(b) *$\lambda_n(X)$ induces a natural isomorphism*

$$\tilde{\lambda}_n(X)\colon\ B(n)_*(X)\otimes_{B(n)_*}K(n)_*\to K(n)_*(X).$$

(c) *There is an unnatural isomorphism*

$$B(n)_*(X)\cong K(n)_*(X)\otimes\mathbb{F}_p[v_{n+1}, v_{n+2}, \ldots].$$

The proof of (3.1) *will be given in Lemma* (3.5) *and Corollary* (3.9).

(3.2) *Remark.* In Section 4 we construct a natural isomorphism

$$B(n)_*(X)\cong K(n)_*(X)\otimes\mathbb{F}_p[v_{n+1}, v_{n+2}, \ldots]\qquad\text{for } n<2(p-1).$$

(3.3) **Lemma.** *Let $f\colon S^m\to X$ be a map of a sphere into a complex. The induced homomorphism $B(n)_*(f)\colon B(n)_*(S^m)\to B(n)_*(X)$ is either monic or is trivial.*

Proof. Let $\rho\colon BP_*(X)\to P(n)_*(X)$ be the natural reduction homomorphism (induced by $g_{n-1}\circ\cdots\circ g_0\colon BP=P(0)\to P(n)$). Let $\iota_0\in BP_m(S^m)$ be a generator and let $\iota_n=\rho(\iota_0)\in P(n)_m(S^m)$. Let $x_j=P(j)_*(\iota_j)\in P(j)_m(X)$, $j=0, n$. Suppose there is an element $0\neq y\in\mathbb{F}_p[v_n, v_{n+1}, \ldots]\cong BP_*/I_n$ such that $y\cdot x_n=0$. We may consider y as an element of BP_* such that $y \notin I_n = (p, v_1, \ldots, v_{n-1})$. By (1.9), there is a BP operation θ such that $\theta(y) = v_n^t$ modulo I_n, $t > 0$. By (2.9), there is a $P(n)$ operation θ_n such that $\theta_n \circ \rho = \rho \circ \theta$ holds. We compute:

$$\begin{aligned} 0 &= \theta_n(y \circ x_n) = \theta_n(\rho(y \circ x_0)) = \rho\,\theta(y \circ x_0) = \rho\,\theta(BP_*(f)(y \circ \iota_0)) \\ &= \rho\, BP_*(f)(\theta(y)\iota_0) = P(n)_*(f)(v_n^t \iota_n) = v_n^t \circ x_n. \end{aligned}$$

Thus when we localize, x_n passes to $B(n)_*(f)(\iota_n) = 0$. We conclude that either $x_n = P(n)_*(f)(\iota_n)$ has no annihilators or $B(n)_*(f) \equiv 0$. $\square$

(3.4) *Remark.* If X is a finite complex, then $K(n)_*(X)$ is a finitely generated free $\mathbb{F}_p[v_n, v_n^{-1}] = K(n)_*$ module. This follows from the fact that every non-zero element is invertible ($K(n)_*$ is a "graded field").

(3.5) **Lemma.** *If X is a finite complex, then the following assertions hold.*

(a) *$\lambda_n(X)$: $BP(n)_*(X) \to K(n)_*(X)$ is onto.*

(b) *$\lambda_n(X)$ induces a natural isomorphism*

$$\tilde{\lambda}_n(X)\colon\; B(n)_*(X) \otimes_{B(n)_*} K(n)_* \to K(n)_*(X).$$

(c) *Let $\{b_i\}$ be a finite set of elements of $B(n)_*(X)$ such that $\{\lambda_n(X)(b_i)\}$ forms a $K(n)_*$ basis for $K(n)_*(X)$. Then $B(n)_*(X)$ is a free $B(n)_*$ module with basis $\{b_i\}$.*

Proof. Note that (a) follows from (b). Both (b) and (c) hold if X is a sphere or if $H_*(X : \mathbb{F}_p)$ is trivial. We now prove (b) and (c) by induction on the $\mathbb{F}_p$ dimension of $H_*(X : \mathbb{F}_p)$. If X is a finite complex with the $\mathbb{F}_p$ dimension of $H_*(X : \mathbb{F}_p)$ equal to $q > 0$, then we may use the Hurewicz theorem (Serre class version) to construct a cofibration

$$S^m \xrightarrow{\;f\;} X \xrightarrow{\;g\;} Y \xrightarrow{\;h\;} S^{m+1}$$

with $H_*(f; \mathbb{F}_p)$ monic. Thus (b) and (c) hold for the complex Y. By (3.3), we need only consider two cases.

Case 1. $B(n)_*(f) \equiv 0$. The cofibration induces the top short exact sequence of commutative diagram (3.6)

(3.6)
$$\begin{tikzcd}[column sep=tiny]
0 \arrow[rr] \arrow[dr] & & B(n)_*(X) \arrow[rr, "g_*"] \arrow[dd, "\lambda_X"'] \arrow[dr] & & B(n)_*(Y) \arrow[rr, "h_*"] \arrow[dd, "\lambda"'] \arrow[dr] & & B(n)_*(S^{m+1}) \arrow[rr] \arrow[dd, "\lambda"'] \arrow[dr] & & 0 & \\
& T \arrow[rr] & & B(n)_*(X) \otimes K(n)_* \arrow[rr] \arrow[dl, "\tilde{\lambda}_X"] & & B(n)_*(Y) \otimes K(n)_* \arrow[rr] \arrow[dl, "\tilde{\lambda}_Y"] & & B(n)_*(S^{m+1}) \otimes K(n)_* \arrow[rr] \arrow[dl, "\tilde{\lambda}_S"] & & 0 \\
0 \arrow[rr] & & K(n)_*(X) \arrow[rr, "g_\#"] & & K(n)_*(Y) \arrow[rr, "h_\#"] & & K(n)_*(S^{m+1}) \arrow[rr] & & 0 &
\end{tikzcd}$$

$\lambda_W = \lambda_n(W)$ and $\tilde{\lambda}_W = \tilde{\lambda}_n(W)$. The tensor product in the middle sequence (which is induced by the top sequence) is over $B(n)_*$. $T = \operatorname{Tor}_{1,*}^{B(n)_*}(B(n)_*(S^{m+1}), K(n)_*)$ which is zero since $B(n)_*(S^{m+1})$ is $B(n)_*$ projective. $\lambda_S \circ h_* = h_\# \circ \lambda_Y$ is epic; so the bottom sequence is short exact as shown. By our induction, $\tilde{\lambda}_Y$ is an isomorphism (and $\tilde{\lambda}_S$ is also); so $\tilde{\lambda}_X$ is an isomorphism by the five lemma. It is immediate that $B(n)_*(X)$ is $B(n)_*$ projective, but we must show it is free. Let $\{b_i\}$ be a (finite) set of elements of $B(n)_*(X)$ such that $\{\lambda_X(b_i)\}$ is a basis for $K(n)_*(X)$. Let $a \in B(n)_*(Y)$ be an element so that $h_*(a) \neq 0$ i.e. $h_\# \lambda_Y(a)$ generates $K(n)_*(S^{m+1})$. Then $\{\lambda_Y g_*(b_i), \lambda_Y(a)\}$ forms a $K(n)_*$ basis for $K(n)_*(Y)$. By induction, $\{g_*(b_i), a\}$ forms a $B(n)_*$ free basis for $B(n)_*(Y)$. Let $F(A)$ denote the free $B(n)_*$ module on the graded set A. Then we have the commutative diagram (3.7) where the unlabeled morphism are the obvious ones.

(3.7)
$$\begin{tikzcd}
0 \arrow[r] & F(\{b_i\}) \arrow[r] \arrow[d] & F(\{g_*(b_i)\}) \oplus F(\{a\}) \arrow[r] \arrow[d, "\cong"] & F(\{h_*(a)\}) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & B(n)_*(X) \arrow[r, "g_*"] & B(n)_*(Y) \arrow[r, "h_*"] & B(n)_*(S^{m+1}) \arrow[r] & 0
\end{tikzcd}$$

By the five lemma, the left vertical morphism is an isomorphism thus confirming the lemma for Case 1.

Case 2. *$B(n)_*(f)$ is Monic.* Our cofibration gives us short exact sequence (3.8).

(3.8)
$$0 \to B(n)_*(S^m) \xrightarrow{f_*} B(n)_*(X) \xrightarrow{g_*} B(n)_*(Y) \to 0.$$

Both $B(n)_*(S^m)$ and $B(n)_*(Y)$ are $B(n)_*$ free; thus $B(n)_*(X) \cong B(n)_*(S^m) \oplus B(n)_*(Y)$ is also. Identification of a basis is routine. Proof of (b) in this second case is by a five lemma argument using a diagram similar to (3.6) induced by the short exact sequence (3.8) □

(3.9) **Corollary.** *Let X be a finite complex. Then there is an (unnatural) isomorphism*

$$A\colon\ B(n)_*(X) \to K(n)_*(X) \otimes \mathbb{F}_p[v_{n+1}, v_{n+2}, \ldots].$$

Proof. Let $\{b_i\}$ be a $B(n)_*$ free basis for $B(n)_*(X)$ such that $\{\lambda_X(b_i)\}$ is a $K(n)_*$ free basis for $K(n)_*(X)$ (as in (3.5)). A typical $\mathbb{F}_p$ basis element of $B(n)_*(X)$ is $v_n^m v^E b_i$ where $m \in \mathbb{Z}$ and $v^E \in \mathbb{F}_p[v_{n+1}, v_{n+2}, \ldots]$. We define $A(v_n^m v^E b_i) = v_n^m \lambda_X(b_i) \otimes v^E$. □

4. The Relationship Between $P(n)_*(X)$ and $k(n)_*(X)$

An ideal approach to the results of the last section would be to have a spectral sequence of the form (4.1)

(4.1)
$$E^2_{*,*}(X) = k(n)_*(X) \otimes \mathbb{F}_p[v_{n+1}, v_{n+2}, \ldots] \Rightarrow P(n)_*(X)$$

which would collapse when localized with respect to the multiplicative set $T_n = \{1, v_n, v_n^2, \ldots\} \subset \mathbb{F}_p[v_n]$. In this section we shall develop a spectral sequence of form (4.1); unfortunately, we cannot prove it is a spectral sequence of $\mathbb{F}_p[v_n]$ modules. Although we cannot localize the spectral sequence with respect to T_n, we can prove that its differentials are killed by high multiples of v_n. This property leads to some insight into the relationship between the connective theories $P(n)_*(\ )$ and $k(n)_*(\ )$.

For the fixed prime p and for the fixed positive integer n, let $\mathscr{E}$ be the collection of all exponent sequences E of form $E = (0, \ldots, 0, e_{n+1}, e_{n+2}, \ldots)$. Note that an $\mathbb{F}_p$ basis for $\mathbb{F}_p[v_{n+1}, v_{n+2}, \ldots]$ is given by $\{v^E\colon E \in \mathscr{E}\}$. Given E, recall that we defined $\sigma^n E$ to be the exponent sequence $\sigma^n E = (p^n e_{n+1}, p^n e_{n+2}, \ldots)$. We assume that for every exponent sequence F (e.g. $F=\sigma^n E$), a $P(n)$ operation $(r_F)_n$: $P(n)\to S^{|F|}P(n)$ with $(r_F)_n \circ \rho = \rho \circ r_F$ has been chosen and fixed. ($\rho = g_{n-1} \circ \cdots \circ g_0$: $BP \to P(n)$. See (2.9).) Given $E \in \mathscr{E}$, let $q=|E|=2(p^n-1)\,b+a$ where $0 \leqq a < 2(p^n-1)$. Let $c=b-(e_{n+1}+e_{n+2}+\cdots)$ and note that $|\sigma^n E| = c\,2(p^n-1)+a$. Define $s_E \in k(n)^a\,(P(n))$ to be the composition (4.2).

$$\text{(4.2)} \qquad s_E\colon\ P(n) \xrightarrow{(r_F)_n} S^{c\,2(p^n-1)+a}\,P(n) \xrightarrow{v_n^c} S^a\,P(n) \xrightarrow{\lambda_n} S^a\,k(n)$$

$$F=\sigma^n E.$$

Now display the finite set $\{E\colon E\in\mathscr{E}, |E|=q\}$ as an ordered set $\{E_1, \ldots, E_v\}$, ($v=\mathbb{F}_p$ dimension of $(\mathbb{F}_p[v_{n+1}, v_{n+2}, \ldots]_q)$). The ordering here is irrelevant, but for sake of definiteness let us suppose it is the reverse-lexiographic ordering. When $E=E_u\in\{E\colon E\in\mathscr{E}, |E|=q\}$, let us denote s_E by s_u.

(4.3) **Lemma.** *For each integer q, there is a cofibration of spectra* (4.4) *satisfying conditions* (a) *through* (e).

$$\text{(4.4)} \qquad D(q) \xrightarrow{i(q)} D(q-1) \xrightarrow{j(q)} E(q) \xrightarrow{k(q)} D(q).$$

(a) *The degrees of $i(q)$, $j(q)$, and $k(q)$ are* 0, 0, *and* -1, *respectively.*

(b) *Let v be the $\mathbb{F}_p$ dimension of $(\mathbb{F}_p[v_{n+1}, v_{n+2}, \ldots])_q$, then*

$$E(q)=S^q\,k(n)\times\cdots\times S^q(n),$$

v many factors.

(c) $D(q)=P(n), q<0$. $D(2t(p-1)+u)=D(2t(p-1))$ *for* $0\leqq u<2p-2$. $D(2t(p-1))$ *is* $2t(p-1)+2p-3$ *connected for* $t\geqq 0$.

(d) (4.4) *induces the short exact sequence* (4.5).

$$\text{(4.5)} \qquad 0\to\pi_*(D(q)) \xrightarrow{i(q)_*} \pi_*(D(q-1)) \xrightarrow{j(q)_*} \pi_*(E(q)) \to 0.$$

(e) *Let* $i(-1, q-1)=i(0)\circ\cdots\circ i(q-1)\colon D(q-1)\to D(q-2)\to\cdots\to D(-1)=P(n)$. *Then diagram* (4.6) *commutes.*

$$\text{(4.6)} \qquad \begin{array}{ccc} D(q-1) & \xrightarrow{j(q)} & E(q)=S^q\,k(n)\times\cdots\times S^q\,k(n) \\ \Big\downarrow{\scriptstyle i(-1,q-1)} & & \Big\downarrow{\scriptstyle v_n^b\times\cdots\times v_n^b} \\ D(-1)=P(n) & \xrightarrow{(s_1,\ldots,s_v)} & S^a\,k(n)\times\cdots\times S^a\,k(n) \end{array}$$

($s_1, \ldots, s_v$, *b, and a as in the preceding discussion*).

Proof. For $q<0$, we define $D(q)=P(n)$ and $E(q)=*$. We assume the construction is complete through the $(q-1)$-st stage. If $q\not\equiv 0$ modulo $2(p-1)$, we define $D(q)=D(q-1)$ and $E(q)=*$. If $q\equiv 0$ modulo $2(p-1)$, we note that $D(q-1)$ is $q-1$ connected. By (2.6c), we have an isomorphism:

$$v_n^b\colon\ k(n)^q(D(q-1))\to k(n)^{q-2(p^n-1)}(D(q-1))\to\cdots\to k(n)^a(D(q-1)).$$

We define $E(q)=S^q\,k(n)\times\cdots\times S^q(n)$, v times, as in (b). By the above mentioned isomorphism, the composition $(s_1, \ldots, s_v)\circ i(-1, q-1)$ in (4.6) lifts uniquely to

$j(q)$: $D(q-1) \to E(q)$. This map induces the cofibration (4.4) satisfying (a). It remains to confirm (c) and (d).

By induction (d and e), we may identify $\pi_*(D(q-1))$ with the intersection of the kernels of the homomorphisms

$$s_F\colon \pi_*(P(n)) \to \pi_*(k(n)), \qquad F \in \mathscr{E},\ |F| < q.$$

By Corollary (1.12), the functions:

$$\{s_F\colon \pi_q(P(n)) \to \pi_{q-a}(k(n)) \cong \mathbb{F}_p\colon F \in \mathscr{E}, |F| \leqq q, |F| \equiv q \equiv a \text{ modulo } 2(p^n-1)\}$$

form a basis of $\mathrm{Hom}(\pi_q(P(n))\colon \mathbb{F}_p)$. Let

$$\{y^E \in \pi_q(P(n))\colon E \in \mathscr{E}, |E| \leqq q, |E| \equiv q \text{ modulo } 2(p^n-1)\}$$

be a dual basis. Then $\{y^E\colon E \in \mathscr{E}, |E| = q\}$ forms a basis of $\pi_q(D(q-1)) \subseteq \pi_q(P(n))$ and this basis is dual to the subspace of $\mathrm{Hom}(\pi_q(P(n))\colon \mathbb{F}_p)$ with basis $\{s_1, \ldots, s_v\}$. Examination of diagram (4.6) shows that $\{j(q)_*(y^E)\colon E \in \mathscr{E}, |E| = q\}$ gives a basis of $\pi_q(E(q))$: thus $j(q)_*\colon \pi_q(D(q-1)) \to \pi_q(E(q))$ is an isomorphism (proving assertion (c)). Note that $\pi_*(D(q-1))$ is preserved under multiplication by v_n since $s_E(y \cdot v_n) = s_E(y)\, v_n$. Thus $\{j(q)_*(y^E v_n^t)\colon E \in \mathscr{E}, |E| = q\}$ is onto in all dimensions. The establishes assertion (d). □

(4.7) *Remark.* This lemma describes a Postnikov decomposition of $P(n)$ with Postnikov factors products of suspensions of $k(n)$'s (instead of the usual Eilenberg-MacLane spectra) and Postnikov fibres the $D(q)$'s.

(4.8) **Theorem.** *There is a natural spectral sequence $\{E^r_{s-q,q}(X), d^r(X)\}$ for any finite complex X. It has the following properties.*

(a) $E^2_{s-q,q}(X) = \pi_s(E(q) \wedge X) \cong k(n)_{s-q}(X) \otimes (\mathbb{F}_p[v_{n+1}, v_{n+2}, \ldots])_q$

(b) $E^\infty_{s-q,q}(X) \cong F_{s-q+1}\, P(n)_s(X)/F_{s-q}\, P(n)_s(X)$ *where*

$$F_{s-q}\, P(n)_s(X) = \mathrm{Image}\{\pi_s(D(q) \wedge X) \to \pi_s(P(n) \wedge X)\}.$$

(c) *The spectral sequence collapses if and only if*

$$\lambda_n(X)\colon\ P(n)_*(X) \to k(n)_*(X)$$

is epic.

(d) *The differentials in the spectral sequence are $T_n = \{1, v_n, v_n^2, \ldots\}$ torsion valued in the following sense. $E^r_{*,q}(X)$ is a subquotient of*

$$E^2_{*,q}(X) \cong k(n)_*(X) \otimes (\mathbb{F}_p[v_{n+1}, v_{n+2}, \ldots])_q$$

which has a left $\mathbb{F}_p[v_n]$ multiplication. If $z \in E^2_{s-q-r,q+r-1}(X)$ represents $d^r(y)$ for $y \in E^r_{s-q,q}(X)$, then $v_n^t z = 0$ for t satisfying $t(2p^n-2) \geqq q+r$.

(4.9) *Remark.* $k(n)_*(X)$ is said to be T_n torsion free if no member of $T_n = \{1, v_n, v_n^2, \ldots\}$ annihilates a nonzero element of $k(n)_*(X)$. From (2.6c), we have the exact sequence.

$$\cdots \to k(n)_{i-2(p^n-1)}(X) \xrightarrow{v_n} k(n)_i(X) \xrightarrow{\gamma_n} H_i(X; \mathbb{F}_p) \xrightarrow{\eta_n} k(n)_{i-(2p^n-1)}(X).$$

Thus $k(n)_*(X)$ is T_n torsion free if and only if γ_n is epic. In this case, (4.8d) tells us that the spectral sequence collapses.

Comments on the Proof of (4.8). This theorem is a direct analog of Theorem (4.4) of [13] as Lemma (4.3) was to Proposition (4.1) of [13]. The proofs of (a), (b), and (d) are exactly as the proofs of the corresponding parts of [13, 4.4]. The proof of (c) will follow the pattern of that of [13, 4.4(iii)] once we have demonstrated the following lemma (which is the obvious analog of a trick of Atiyah's [3]).

(4.10) **Lemma.** *Given a finite complex X, there is a finite complex A and a stable map $f: A \to X$ such that $k(n)_*(A)$ is T_n torsion free and $P(n)_*(f): P(n)_*(A) \to P(n)_*(X)$ is epic.*

Proof Outline. (a) $P(n)_*(X)$ is a coherent BP_* module and thus is finitely generated over BP^* and $P(n)_*$. This is proved by cellular induction using the techniques of [9, Section 1].

(b) If DX is the Spanier-Whitehead dual of X, realization of the $P(n)_*$ generators of $P(n)^*(DX)$ gives a map $g: DX \to VS^m P(n) = Y$ such that the wedge sum is finite and $P(n)^*(g)$ is epic. We may assume g is skeletal.

(c) v_n: $k(n)^*(P(m)) \to k(n)^*(P(m))$ is monic when $m=0$ $(P(0)=BP)$. By an induction using two copies of the short exact sequence of (2.8 b) and applying the five lemma, we see that it is monic when $m=1, 2, \ldots, n$ also. Thus $k(n)^*(Y)$ is T_n torsion free.

(d) Let Y^k be the k-skeleton of Y; then $k(n)^*(Y^k)$ is T_n torsion free also. $\gamma_n(Y^k)$ is seen to be epic by diagram (4.11), since either $k(n)^{i+2p^n-1}(Y^k)$ or $H^{i+1}(Y/Y^k: \mathbb{F}_p)$ is zero for any given i.

$$\begin{tikzcd}
k(n)^i(Y) \arrow[r] \arrow[d, "\gamma_n(Y)"] & k(n)^i(Y^k) \arrow[d, "\gamma_n(Y^k)"] & \\
H^i(Y; \mathbb{F}_p) \arrow[r] \arrow[d] & H^i(Y^k; \mathbb{F}_p) \arrow[r] \arrow[d] & H^{i+1}(Y/Y^k; \mathbb{F}_p) \\
0 & k(n)^{i+2p^n-1}(Y^k). &
\end{tikzcd} \tag{4.11}$$

(e) Now assume k is sufficiently large so that $g(DX) \subseteq Y^{k-1} \subseteq Y^k$. $H_*(Y^k: \mathbb{Z}_{(p)})$ is finitely generated and so there is a finite complex F and a stable map $h: F \to Y^k$ such that $H_*(h; \mathbb{Z}_{(p)})$ is an isomorphism. Thus

$$h_* \otimes 1: [DX, F] \otimes \mathbb{Z}_{(p)} \to [DX, Y^k] \otimes \mathbb{Z}_{(p)}$$

is an isomorphism. So there is a unit u of $\mathbb{Z}_{(p)}$ such that $u \cdot g = h \circ f^{\#}$, for some map $f^{\#}$: $DX \to F$. Let A be the Spanier-Whitehead dual of F and let f: $A = DF \to DDX = X$ be dual to $f^{\#}$. One checks that f and A satisfy the requirements of the lemma. □

(4.12) *Remark.* Recall that $s_E \in k(n)^{a_E}(P(n))$ where $q = |E| = b_E(2p^n - 2) + a_E$, $0 \leqq a_E < 2p^n - 2$. There is a well defined natural homomorphism given by the composition:

$$P(n) \xrightarrow{s_E} S^{a_E} k(n) \longrightarrow S^{a_E} K(n) \xrightarrow{v_n^{-b_E}} S^q K(n).$$

These induce a natural homomorphism of the Chern-Dold type for any finite complex X.

$$\hat{\Lambda}(X)\colon\ P(n)_*(X) \to K(n)_*(X) \otimes \mathbb{F}_p[v_{n+1}, v_{n+2}, \dots],$$
$$\hat{\Lambda}(X)(y) = \sum_{E \in \mathscr{E}} v_n^{-b_E} s_E(y) \otimes v^E.$$

If $k(n)_*(X)$ is T_n torsion free, Theorem (4.8) shows that $\hat{\Lambda}(X)$ is a monomorphism.

(4.13) *Remark.* In general, we do not know that $\hat{\Lambda}(X)$ is a homomorphism of $\mathbb{F}_p[v_n]$ modules since each s_E had an $(r_F)_n$, $F = \sigma^n E$, in its defining composition. By (2.14), we know that $(r_F)_n$ is canonically defined if $n < 2p-2$. Thus each s_E: $P(n)_*(X) \to k(n)_*(X)$ is a $\mathbb{F}_p[v_n]$ homomorphism (for $r_E(y \cdot v_n) = r_E(y) v_n$ modulo I_n). So in this case, $\hat{\Lambda}(X)$'s domain of definition may be extended to be $B(n)_*(X) = T_n^{-1} P(n)_*(X)$. Now we apply the uniqueness theorem for homology theories to obtain the following theorem.

(4.14) **Proposition.** *Let $n < 2p-2$ and let X be a finite complex, then there is a natural isomorphism induced by $\hat{\Lambda}(X)$.*

$$\Lambda(X)\colon\ B(n)_*(X) \to K(n)_*(X) \otimes \mathbb{F}_p[v_{n+1}, v_{n+2}, \dots]. \quad \square$$

Diagram (2.7) induces commutative diagram (4.15).

$$\begin{array}{ccccc} P(n)_*(X) & & \xrightarrow{\quad g_n(X) \quad} & & P(n+1)_*(X) \\ \downarrow{\scriptstyle \lambda_n(X)} & & & & \downarrow{\scriptstyle \lambda_{n+1}(X)} \\ k(n)_*(X) & \xrightarrow{\gamma_n(X)} & H_*(X; \mathbb{F}_p) & \xleftarrow{\gamma_{n+1}(X)} & k(n+1)_*(X) \end{array} \tag{4.15}$$

(4.16) **Theorem.** *Let X be a finite complex. If $\gamma_n(X)$ is epic, then all four other homomorphisms in* (4.15) *are also epic.*

Proof. $\gamma_n(X)$ epic $\Rightarrow$ $k(n)_*(X)$ is T_n torsion free (4.9) $\Rightarrow$ our spectral sequence collapses $\Rightarrow \lambda_n(X)$ epic (4.8) $\Rightarrow \gamma_{n+1}(X) \circ \lambda_{n+1}(X) \circ g_n(X)$ epic (4.15) $\Rightarrow \gamma_{n+1}(X)$ epic $\Rightarrow \lambda_{n+1}(X)$ epic. Since $\mu_k(X) = \gamma_k(X) \circ \lambda_k(X)$: $P(k)_*(X) \to H_*(X : \mathbb{F}_p)$ are epic for $k = n$ and $n+1$, the spectral sequences

$$E^2_{*,*}(X) = H_*(X : \mathbb{F}_p) \otimes P(k)_* \Rightarrow P(k)_*(X)$$

collapse for $k = n$ and $n+1$. g_n induces an epimorphism on the E^2 terms and thus $g_n(X)$ is epic by induction over filtrations. $\square$

(4.17) *Remark.* In the spirit of (4.14), we may prove that if $n < 2p-2 = q$, there are natural isomorphisms:

$$K(n)^*(X) \xrightarrow{\tilde{\mu}} \operatorname{Hom}_{B(n)_*}(B(n)_*(X), K(n)_*) \xleftarrow{\lambda_n^{\#}} \operatorname{Hom}_{K(n)_*}(K(n)_*(X), K(n)_*)$$

$\lambda_n^{\#}$ is an isomorphism by Theorem (3.1). $(\lambda_n^{\#})^{-1}\circ\tilde{\mu}$ is a natural transformation of homology theories which is an isomorphism when $X=S^0$ – provided that $\tilde{\mu}$ can be defined. Note that if $n<q$, we may identify $\pi_{iq}(k(n)\wedge P(n))$ with $\pi_{iq}(k(n)\wedge BP)$. The pairing $\pi_{iq}(k(n)\wedge BP)\to\pi_{iq}(k(n))$ extends to the second factor of $\hat{\mu}$:

$$\hat{\mu}\colon\ k(n)^*(X)\otimes P(n)_*(X)\to\pi_*(k(n)\wedge P(n))\to\pi_*(k(n)).$$

$\hat{\mu}$ is compatible with the appropriate BP_* actions. Upon localization, it induces

$$\mu\colon\ K(n)^*(X)\otimes B(n)_*(X)\to K(n)_*$$

which induces $\tilde{\mu}$ in turn.

(4.18) *Remark.* Observe that there is an invariant of a finite complex X given by the least integer n such that $k(n)_*(X)\to H_*(X;\mathbb{F}_p)$ is epic. This invariant differs radically from the invariant $\operatorname{hom\,dim}_{BP_*} BP_*(X)$ studied in [14]. For example: if $X=\mathbb{R}P(2^n)$, Q_n is non-zero in $H^*(X;\mathbb{F}_2)$ and $k(n)_*(X)\to H_*(X;\mathbb{F}_2)$ fails to be epic; yet $\operatorname{hom\,dim}_{BP_*} BP_*(X)=1$. On the other hand, we may form a three-cell complex $Y=S^0\cup_8 e^1\cup_{\bar{v}} e^5$ such that $\operatorname{hom\,dim}_{BP_*} BP_*(Y)=2$ [10], but $k(1)_*(Y)\to H_*(Y;\mathbb{F}_2)$ is epic. ($\bar{v}\colon S^4\to S^0\cup_8 e^1$ is a coextension of the Hopf invariant one element $v\in\pi_3^S$.)

5. An Expository Summary

The classical prototype for Morava's and our efforts is the description of the integral homology of a finite simplicial complex by its Betti numbers and by its torsion coefficients. If we localize this antecedent, the $\mathbb{Z}_{(p)}$ module structure of $H_*(X;\mathbb{Z}_{(p)})$ is determined by data displayed in (5.1).

(5.1)
$$\begin{tikzcd}
H_*(X;\mathbb{Z}_{(p)}) \arrow[dd] & & \\
 & H_*(X;\mathbb{Z}_{(p)}) \arrow[ul, "p"'] \arrow[r, dashed] & H_*(X;\mathbb{Q}) \\
H_*(X;\mathbb{F}_p) \arrow[ur] & &
\end{tikzcd}$$

Here the dashed horizontal map is rational localization into $H_*(X;\mathbb{Q})$ which gives the Betti numbers. The kernel of the localization map – the p torsion part of $H_*(X;\mathbb{Z}_{(p)})$ – can be computed by knowing $H_*(X;\mathbb{F}_p)$ and the behavior of the Bockstein exact triangle which indeed forms the triangular part of (5.1).

Morava's structure theorem for $BP_*(X)$ is schematically described by (5.2). Again the dashed horizontal arrow represents localization: the n-th one is T_n localization where $T_n=\{1, v_n, v_n^2, \ldots\}$. The T_n torsion-free part of $P(n)_*(X)$ passes monomorphically to $B(n)_*(X)$ and so is largely determined by $K(n)_*(X)$. ($K(n)_*(X)$ can be described by some "extraordinary Betti numbers.") The T_n torsion part of $P(n)_*(X)$ is given by $P(n+1)_*(X)$ and the behavior of the n-th Bockstein triangle (the n-th triangle of (5.2)). For the structure of $P(n+1)_*(X)$, one considers its T_{n+1} torsion-free part and its T_{n+1} torsion part .... This is a finite process! There is an n (e.g. if the cellular dimension of X is less than $2p^n-1$) such that if $m\geqq n$, then

$P(m)_*(X) \cong H_*(X; \mathbb{F}_p) \otimes P(m)_*$ and the m-th exact triangle collapses (its vertical morphism is epic).

(5.2)

$$
S_n = \mathbb{Z}_{(p)}[v_{n+1}, v_{n+2}, \ldots]
$$

$$
\begin{tikzcd}
BP_*(X) \arrow[d, equal] & & \\
P(0)_*(X) \arrow[dd] & & \\
 & P(0)_*(X) \arrow[ul, "p"'] \arrow[r, dashed] & B(0)_*(X) \cong K(0)_*(X) \otimes S_0 \cong H_*(X; Q) \otimes S_0 \\
P(1)_*(X) \arrow[ur] \arrow[dd] & & \\
 & P(1)_*(X) \arrow[ul, "v_1"'] \arrow[r, dashed] & B(1)_*(X) \cong K(1)_*(X) \otimes S_1 \\
P(2)_*(X) \arrow[ur] \arrow[d] & & \\
\cdots\cdots \arrow[d] & & \\
P(n)_*(X) \arrow[dd] & & \\
 & P(n)_*(X) \arrow[ul, "v_n"'] \arrow[r, dashed] & B(n)_*(X) \cong K(n)_*(X) \otimes S_n \\
P(n+1)_*(X) \arrow[ur] \arrow[d] & & \\
\cdots\cdots & &
\end{tikzcd}
$$

N.B. Unless $n < 2p - 2$, the isomorphisms displayed may not be natural.

Appendix: A Proof of (2.4)

The purpose of this appendix is to provide a proof of assertion (2.4). We assume the definitions and notation of Baas [5]. Let Y be a finite complex. Recall that $P(m)_*(Y)$ is constructed as a direct limit of homology modules $MU(S_n)_*(Y)$ where $S_n = \{* = P_0, P_1, \ldots, P_n\}$ is a singularity set of closed unitary manifolds such that $[P_j]$ is not a zero divisor of $MU_*/([P_1], \ldots, [P_{j-1}])$. Without loss of generality, we may assume $Y = X^+$. (Our $MU(S_n)_*(X^+)$ is Baas's $MU(S_n)_*(X, \phi)$.) Thus to prove (2.4), it suffices to confirm:

(A.1) For any element $[A, \alpha, f]$ of $MU(S_n)_*(X^+)$ and for any $i = 1, \ldots, n$, $[A, \alpha, f] \cdot [P_i] = 0$.

The proof which we sketch is reconstructed from one told us by Morava. However, we are certain (A.1) was also known to Sullivan. Any temptation to extend this proof to argue that $MU(S_n)$ is a nice ring spectrum should be dampened by observing that the proof involves an uncanonical choice of the manifold E. (Also see (2.5d).)

The Proof. For notational convenience, let us assume $i = 1$. We are given the following data. For each subset $\omega \subset \{0, 1, \ldots, n\}$, we have a unitary manifold with boundary $A(\omega)$ and a continuous map $f(\omega)$: $A(\omega) \to X$. $A(\phi) = A$ and $f(\phi) = f$.

The boundary of $A(\omega)$ is decomposed into a union of manifolds $\partial_j A(\omega)$, where $\partial_j A(\omega)=\phi$ if $j\in\omega$. Also $\partial_0 A(\omega)=\phi$. For $j\in\{0,1,\ldots,n\}\smallsetminus\omega$, there is an equivalence of unitary manifolds $\alpha(\omega,j)\colon \partial_j A(\omega)\to A(\omega,j)\times P_j$. All of these manifolds, maps, and homeomorphisms satisfy coherence conditions given in §2 of [5].

To show $[A,\alpha,f]\cdot[P_1]=0$, we must construct an appropriately coherent system of manifolds, homeomorphisms, and maps: $\{B(\omega),\beta(\omega,j),g(\omega)\}$ satisfying:

(A.2) $\partial_0 B(\omega)=B(\omega,0)=A(\omega)\times P_1$;

(A.3) $\beta(\omega,0,j)\colon \partial_j A(\omega)\times P_1\to A(\omega,j)\times P_1\times P_j$ is defined by $\beta(\omega,0,j)(a,y)=(b,y,x)$ for $(a,y)\in\partial_j A(\omega)\times P_1$ and where $\alpha(\omega,j)(a)=(b,x)$. (Warning: this must hold even when $j=1$.)

(A.4) $g(\omega,0)\colon A(\omega)\times P_1\to X$ is defined by $g(\omega,0)(a,x)=f(\omega)(a)$ for $(a,x)\in A(\omega)\times P_1$.

Let $D=P_1\times P_1\times[0,1]$. We define $\partial_0 D=\phi$, $\partial_1 D=P_1\times P_1\times\{0,1\}$, and $D(1)=P_1\times 0\cup P_1\times 1$. We build an important twist into D's P_1 structure by defining $\delta(1)\colon\partial_1 D\to D(1)\times P_1$ by $\delta(1)(x,y,0)=(y,0,x)$ and $\delta(1)(x,y,1)=(x,1,y)$ for $x,y\in P_1$. Thus $\{D,\delta(1)\}$ defines a P_1 manifold of odd dimension. It then gives a trivial class in $MU(\{P_1\})_*\cong MU_*/([P_1])$ and it bounds some P_1 manifold $\{E(\omega),\varepsilon(\omega,j)\}$ which satisfies: $\partial_0 E=E(0)=D$; $E(0,1)=D(1)$; and $\varepsilon(0,1)=\delta(1)$. We consider $\{E(\omega),\varepsilon(\omega,j)\}$ as an S_n manifold by defining $\partial_j E(\omega)=\phi$ for $j\neq 0,1$.

We form B from the union of $A\times P_1\times[0,1]$ and $A(1)\times E$ by the identification:

$$\partial_1 A\times P_1\times[0,1]\xrightarrow{\alpha(1)\times 1\times 1}A(1)\times P_1\times P_1\times[0,1]=A(1)\times\partial_0 E.$$

The topological boundary of B is the union of: $\partial_0 B=A\times P_1\times 0$; $\partial_1 B=A\times P_1\times 1\cup A(1)\times\partial_1 E$; and $\partial_j B=\partial_j A\times P_1\times[0,1]\cup\partial_j A(1)\times E, j\neq 0,1$ (with identifications as above).

In the definitions which follow, let $\mu\subset\{2,\ldots,n\}$ and $j\in\{2,\ldots,n\}\smallsetminus\mu$.

(A.5) We define:

$B(\mu)=A(\mu)\times P_1\times[0,1]\cup A(\mu,1)\times E$ with $\partial_1 A(\mu)\times P_1\times[0,1]$ identified with $A(\mu,1)\times\partial_0 E$;

$B(\mu,1)=A(\mu)\times 1\cup A(\mu,1)\times E(1)$ with $\partial_1 A(\mu)\times 1$ identified with part of $A(\mu,1)\times\partial_0 E(1)$;

$B(\mu,0)=A(\mu)\times P_1\times 0$; and $B(\mu,1,0)=A(\mu,1)\times P_1\times 0$.

(A.6) We define:

$$\beta(\mu,j)\colon \partial_j A(\mu)\times P_1\times[0,1]\cup\partial_j A(\mu,1)\times E\to\big(A(\mu,j)\times P_1\times[0,1]\cup A(\mu,j,1)\times E\big)\times P_j$$

by $\beta(\mu,j)(a,y,t)=(b,y,t,x)$ for $(a,y,t)\in A(\mu)\times P_1\times[0,1]$ and where $\alpha(\mu,j)(a)=(b,x)$ and $\beta(\mu,j)(a,e)=(b,e,x)$ for $(a,e)\in\partial_j A(\mu,1)\times E$ and where $\alpha(\mu,1,j)(a)=(b,x)$;

$\beta(\mu,1,j)$ and $\beta(\mu,0,j)$ to be restrictions of $\beta(\mu,j)$;

$$\beta(\mu,1)\colon\ A(\mu)\times P_1\times 1\cup A(\mu,1)\times\partial_1 E\to\big(A(\mu)\times 1\cup A(\mu,1)\times E(1)\big)\times P_1$$

by $\beta(\mu,1)(a,x,1)=(a,1,x)$ for $(a,x,1)\in A(\mu)\times P_1\times 1$ and $\beta(\mu,1)(a,e)=(a,\varepsilon(1)(e))$ for $(a,e)\in A(\mu,1)\times\partial_1 E$;

$\beta(\mu,0)$ and $\beta(\mu,1,0)$ to be the appropriate identity maps.

(A.7) We define g: $B \to X$ by $g(a, x, t) = f(a)$ for $(a, x, t) \in A \times P_1 \times [0, 1]$ and $g(a, e) = f(1)(a)$ for $(a, e) \in A(1) \times E$. The map g induces the other maps $g(\omega)$ by Definition 2.3 (ii) of [5].

Definitions (A.5), (A.6), and (A.7) organize a singular S_n manifold in X: $\{B(\omega), \beta(\omega, j), g(\omega)\}$. This is seen to satisfy conditions (A.2), (A.3), and (A.4) as required.

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Professor D.C. Johnson
Department of Mathematics
University of Kentucky
Lexington, Kentucky 40506, USA

Professor W.S. Wilson
Department of Mathematics
Princeton University
Princeton, New Jersey 08540, USA
and
The Institute of Advanced Study
Princeton, New Jersey 08540, USA

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